The Dissertation Committee for Hai Viet Bui certifies that this is the approved version of the following dissertation:

Time Asymmetric Quantum Theory and its Applications in Non-Relativistic and Relativistic Quantum Systems

Committee:

____

Arno Bohm, Supervisor

____

Duane Dicus

____

Thomas Chen

____

Christina Markert

____

Can Kilic

Time Asymmetric Quantum Theory and its Applications in Non-Relativistic and Relativistic Quantum Systems

by

Hai Viet Bui, B.S. Phys.

Dissertation

Presented to the Faculty of the Graduate School of

The University of Texas at Austin

in Partial Fulfillment

of the Requirements

for the degree of

Doctor of Philosophy

The University of Texas at Austin

August, 2016

Acknowledgments

I would like to express my appreciation to my supervisor, Professor Arno Bohm for his ultimate patient, invaluable advisor and assistance, and tireless guidance throughout research for this dissertation. Without his enthusiasm and endless encouragement throughout my Ph.D studies, this dissertation can not be completed.

I would like to thank my committee members, Professors- Duane Dicus, Thomas Chen, Christina Markert, and Can Kilic for their supports to grant my Ph.D extension so i can finish this dissertation and for their services in my Ph.D committee.

I wish to thank my mom, my family, my wife, and my children. Without their love, support, and encouragement, this dissertation would never have been finished.

Hai Viet Bui

The University of Texas At Austin
August, 2016

Time Asymmetric Quantum Theory and its Applications in Non-Relativistic and Relativistic Quantum Systems

Hai Viet Bui, Ph.D.
The University of Texas at Austin, 2016

Supervisor: Arno Bohm

The exact relation $\tau = \hbar/\Gamma$ between the width Γ of a resonance and the lifetime τ for a decay of this resonance could not be obtained in the conventional quantum theory based on the Hilbert space as well as on the Schwartz space in which the spaces of the states and of the observables are described by the same space. Furthermore, both dynamical evolution of the states (in Schrödinger picture) and observables (in Heisenberg picture) are symmetrically in time, given by an unitary group with time extending over $-\infty < t < +\infty$. This time symmetric evolution is a mathematical consequence of Von Neumann theorem for the dynamical differential equations, Schrödinger equation for the state or Heisenberg equation for the observable, under either the Hilbert space or Schwartz space boundary condition. However, this unitary group evolution violates causality. In order to get a quantum theory in which the exact relation and causality are obtained, one has to replace the Hilbert space or the Schwartz space boundary condition by a new boundary condition based on the Hardy space axioms in which the space of the states and the space of the observables are described by two distinguishable Hardy spaces. As a consequence of the new Hardy space axiom, one obtains, instead of the time symmetric evolution for the states and the observables, time asymmetrical evolutions for the states and observables which are described by two semi-groups. The time asymmetrical evolution predicts an finite beginning of time $t_0(= 0)$ for quantum system which can be experimental observed by directly measuring the lifetime of the decaying state as the beginning of time of the ensemble in scattering experiments. A resonance obeying the exponential time evolution can then be described by a Gamow vector, which is defined as superposition of the exact out-plane wave states with exact Breit-Wigner energy distribution.

Contents

List of Figures

Chapter One: Introduction

Resonance and decaying state phenomena have been studied for a long time in scattering experiments. Although resonances and decaying states are studied in two difference processes, resonances in the scattering process and decaying states in the decay process, they are believed to be two aspects of the same phenomenon. It means that the lifetime of the decaying state is mathematically inverse width of the resonance. This belief came first in non-relativistic scattering theory [1] using the Hilbert space axiom for which an approximate relation between lifetime of the decaying state and the width of resonance was obtained. Khalfin [2] showed that in the standard quantum theory based on Hilbert space axiom, there exists no vector with the exponential time evolution, i.e. it is impossible to obtain the exact relation between the width of the resonance and the lifetime of the decaying state in the quantum theory using the Hilbert space axiom. Such a vector, called Gamow vector, with an exponential time evolution property was introduced by Gamow [3]. However, the Gamow vector has not been favored by physicists since there are no complex eigenvalues for a self-adjoint Hamiltonian of physical system in the conventional quantum physics.

In a scattering experiment, for example, the preparation consists of the acceleration and the collimation of the projectile, which interacts with the target and then perhaps forming a resonance. The registration consists of the detection of scattered particles, e.g. the decay products of the resonance which decays into different channels. To distinguish what is prepared in the preparation process from what is detected in the registration process, one experimentally defines: state for what is prepared and observable for what is detected or registered by a detector. Despite this experimental distinction between prepared state and detected observable, conventional quantum mechanics based on the Hilbert space does usually not distinguish in the mathematical description between space of states and spaces of observables. Mathematically, it means that any vector in the Hilbert Space can either represent a state or an observable.

In contrast to the mathematical description of states and observables in conventional quantum theory, experimentally, an observable is defined by a registration apparatus, e.g. a detector or counter, and a state is defined by a preparation apparatus, e.g. an accelerator. Thus, observables and states are experimentally different physical concepts. Therefore, they should be distinguished in their mathematical descriptions.In other words, the set of observables and the set of states can not be the same Hilbert space. They could came from different subspaces of the Hilbert space which could even be dense in the same Hilbert space.

The time evolution of the solutions of the dynamical equations , Schrödinger equation for the state in the Schrödinger picture or Heisenberg equation for the observable in the Heisenberg picture, extends over $-\infty < t < \infty$ which is discussed in Chapter 2. This time-symmetric evolution is a result of the famous mathematical theorem, Stone-von Neumann theorem [4], for the Hilbert space. The time-symmetric evolution of the observable (in the Heisenberg picture) or the state (in the Schrödinger picture) leads to the result, that the Born probability to detect the observable in the state exists for all time, $-\infty < t < \infty$. This means that the detector can detect the observable, e.g, products of decay, for all the time, even before the apparatus are turned on, e.g, an accelerator in the scattering experiment. This is in contrast to the causality condition for the scattering experiment since the detector can not detect anything until the experiment is prepared and the apparatus are turned on.

In order to obtain the unified theory of resonances and decaying states, one has to modify the Hilbert space and also the Schwartz space boundary condition for the space of the states and the space of the observables, and use a new boundary condition, called Hardy space axioms, which distinguishes between the space of the states and the space of the observables [5]. The space of states and of observables are mathematically described by two different Hardy spaces, Hardy space of the lower and of the upper complex energy plane on second sheet of the S-matrix, respectively. The result of the Hardy space axioms is that, we can get an exact relation between the width of the resonance and the lifetime of corresponding decaying state. Furthermore, we predicts that the probability to detect an observable in a state exists only after a finite time, beginning of time, $t_0(= 0)$ at which the state has been prepared.

We will derive that the dynamical evolution for the state and for the observable are not described by time-symmetric group evolution, but by time-asymmetric semigroup evolution under the Hardy space boundary condition in non-relativistic regime in Chapter 3 and in relativistic regime in Chapter 4. Mathematically, instead of extending over $-\infty < t < +\infty$ in time-symmetric group evolution, the time parameter just extends over $t_0(= 0) \leq t < +\infty$ for the state and over $-\infty < t \leq t_0(= 0)$ for the observable in non-relativistic regime, respectively. In relativistic regime, the states and the observables can be represented by the two different semi-groups, one is the forward light cone semi-group for the observables and another is the backward light cone semi-group for the states.

The resonance corresponding to the decaying state in the scattering process can be represented as the Gamow state (or vectors) which is associated with a first-order pole of the S-matrix located in second sheet of Riemann surface. This Gamow vector is defined as continuous linear superposition of the exact out-plane waves of the exact Hamiltonian with the exact Breit-Wigner energy distribution which energy spectrum is over the whole energy values $-\infty < E < +\infty$. The dynamical evolution of the Gamow state is also governed as same as the time-asymmetric semi-group for the state. In chapter 5, we will discuss about the beginning of time in the Rigged Hilbert space formalism and how they can observed in the experiment. The observation of beginning of time $t_0(= 0)$ has been experimentally observed as beginning of time of the ensemble by directly measuring the lifetime of the decaying state represented by a Gamow vector in quantum jump for non-relativistic regime and in the kaon physics for relativistic regime.

Chapter Two: Time symmetric Quantum Mechanics

2.1 The Hilbert Space Boundary Condition

Time evolution in quantum mechanics is described in various ways, called pictures. In the Schrödinger picture, the time evolution is described as the dynamical evolution of the state vector $\phi^+(t)$. The differential equation is given by the Schrödinger equation for the state vector $\phi^+(t)$:

$$i\,\frac{\partial}{\partial t}\phi^+(t) = H\phi^+(t)\,, \tag{2.1a}$$

and the observable A is chosen as time independent ($A(t){=}A_{t=0} = A$). The Hamiltonian H of the system is a self-adjoint or essentially self-adjoint operator ($H^\dagger = H$). It represents the energy operator or Hamiltonian of the quantum mechanical system. We are using the natural units in which $\hbar = c = 1$ in the entire dissertation.

For the more complicated mixed-state, represented by the state operator $\rho(t)$ (also called density operator with the property $\rho(t) = \rho^\dagger(t)$, $Tr\,(\rho(t)) = 1$) (or density "matrix"), the dynamical equation is given by the von-Neumann equation:

$$i\,\frac{\partial\rho(t)}{\partial t} = [H, \rho(t)]\,, \tag{2.1b}$$

which yields to (2.1a) in special case that $\rho(t) = |\phi^+(t)\rangle\langle\phi^+(t)\,|$ is a pure state.

In the Heisenberg picture, the time evolution is described as the time dependence of the observables which are represented by the (essentially self-adjoint) operator $\Lambda(t)$ ($\Lambda^\dagger = \Lambda$), fulfilling the Heisenberg equation:

$$i\frac{\partial}{\partial t}\Lambda(t) = -[H, \Lambda(t)]\,. \tag{2.2a}$$

For the special case $\Lambda(t) = |\psi^-(t)\rangle\langle\psi^-(t)\,|$, i.e. when $\Lambda(t)$ is represented by the vector up to a phase, the time evolution of the Heisenberg equation for this observable vector $\psi^-(t)$ is:

$$i\frac{\partial}{\partial t}\psi^-(t) = -H\psi^-(t)\,. \tag{2.2b}$$

The label $^+$ for the state vector $\phi^+(t)$ and the label $^-$ for the observable $|\psi^-(t)\rangle\langle\psi^-(t)|$ have been chosen here such that they will agree with the standard notation of physicists for the Lippmann-Schwinger kets [6]. Those notations are also used to denote the in-state and out-observable in the time asymmetric quantum mechanics in chapter 3 and 4.

There is another picture which is called interaction picture, which is not discussed in this dissertation. In interaction picture, both time evolutions of state and of observable are time dependent with respect to the free Hamiltonian. The state and observable in any these pictures can be expressed in other pictures.

The theoretical quantities that are be compared with the experimental data, are the Born probabilities $\mathcal{P}_\rho(\Lambda)$ for the observable $\Lambda(t)$ in the state ρ defined by

$$\mathcal{P}_\rho(\Lambda) \equiv \mathrm{Tr}(\Lambda(t)\rho) = \mathrm{Tr}(\Lambda\rho(t))\,. \tag{2.3}$$

The Born probabilities (2.3) are the mathematical predictions that are compared with the experimental data, e.g, the counting rates $\frac{N(t)}{N}$ of detector. For the simplest case that the observable in (2.3) is $\Lambda(t) = |\psi^-(t)\rangle\langle\psi^-(t)|$ and the state in (2.3) is $\rho(t) = |\phi^+(t)\rangle\langle\phi^+(t)|$, the Born probabilities are given by:

$$\mathcal{P}_\rho(\Lambda(t)) \equiv \mathcal{P}_{\phi^+}(|\psi^-(t)\rangle\langle\psi^-(t)|) = \mathrm{Tr}(|\psi^-(t)\langle\psi^-(t)|\phi^+\rangle\langle\phi^+|\rangle\rangle) \tag{2.4a}$$

$$= |\langle\psi^-(t)|\phi^+\rangle|^2 \quad \text{(in Heisenberg picture)} \tag{2.4b}$$

$$= |\langle\psi^-|\phi^+(t)\rangle|^2 \quad \text{(in Schrödinger picture)}\,. \tag{2.4c}$$

These theoretical Born probabilities $|\langle\psi^-(t)|\phi^+\rangle|^2 = |\langle\psi^-|\phi^+(t)\rangle|^2$ are calculated in the quantum theory by solving the Schrödinger equations (2.1a) for the state $\phi^+(t)$, or by solving the Heisenberg equation (2.2b) for the observable $\Lambda(t) = |\psi^-(t)\rangle\langle\psi^-(t)|$. The calculated probabilities can then be checked by comparing them with the experimental counting rates of detectors $\frac{N(t)}{N}$, $N = \sum N(t)$. The comparison between the calculated Born probabilities $\mathcal{P}_{\rho(t)}(A) = \mathcal{P}_\rho(A(t))$ of the theory and the experimentally measured counting rates $\frac{N(t)}{N}$ of the detectors is given in the general case by:

$$\mathcal{P}_\rho(A)(t) = \mathcal{P}_{\rho(t)}(A) \simeq \frac{N(t)}{N}\,, \tag{2.5a}$$

and for the special case (2.4), this comparison is given by:

$$|\langle\psi^-(t)|\phi^+\rangle|^2 = |\langle\psi^-|\phi^+(t)\rangle|^2 \simeq \frac{N(t)}{N}\,. \tag{2.5b}$$

To solve the differential equations (2.1) and (2.2), one needs to choose a *boundary condition*. The boundary condition for a differential equation specifies what *kind* of state-vectors ϕ^+ should be admitted as possible solutions of the differential equation (2.1a), and what kind of observable-vectors ψ^- should be admitted as possible solutions of equation (2.2b). In other words, the boundary condition specifies what should be the allowed set $\{\phi^+\}$ for the state-vectors $\phi^+(t)$ that fulfill the Schrödinger equation (2.2a) in Schrödinger picture. Similarly, in the Heisenberg picture, the boundary condition specifies what is the allowed set $\{\psi^-\}$ for the observable vectors ψ^-, which fulfill the Heisenberg equation (2.2b).

The boundary condition in the conventional quantum theory is the Hilbert space boundary condition, originally introduced by von Neumann, for the equations (2.1a):

$$\begin{array}{ccc} \text{Set of states: } \{\phi^+\} & \text{with} & \phi^+ \in \text{Hilbert space}\,\mathcal{H}\,, \\ \text{(all possible solutions of (2.1a))} & & \end{array} \tag{2.6a}$$

and for the equation (2.2b):

$$\begin{array}{ccc} \text{Set of observables: } \{\psi^-\} & \text{with} & \psi^- \in \text{Hilbert space}\,\mathcal{H}\,. \\ \text{(all possible solutions of (2.2b))} & & \end{array} \tag{2.6b}$$

In other words, the set of observables is same as the set of states which is the Hilbert space:

$$\text{Set of states } \{\phi^+\} = \text{set of observables } \{\psi^-\} \equiv \mathcal{H} \text{ (Hilbert space)}\,. \tag{2.6c}$$

Following a theorem of Stone and von Neumann [4], all solutions of the Schrödinger equation (2.1a) for the state ϕ^+ under the boundary condition (2.6a) are generally given by:

$$\phi^+(t) = U^\dagger(t,t_0)\,\phi^+(t_0) = \mathrm{e}^{-iH(t-t_0)}\,\phi^+(t_0)\,, \tag{2.7a}$$

where $-\infty < t_0 < +\infty$ is a finite initial time of the state. Instead of finite initial time t_0, one could mathematically choose $t_0 = 0$ and $\phi^+(t_0 = 0) = \phi^+$ is the initial in-state at the initial time $t_0 = 0$. Thus, the time evolution of the state ϕ^+ in (2.7a) can be rewritten as:

$$\phi^+(t) = U^\dagger(t,0)\phi^+ = U^\dagger(t)\phi^+ = \mathrm{e}^{-iHt}\phi^+ \qquad \text{with} \qquad -\infty < t < +\infty\,. \tag{2.7b}$$

Similarly, by the same theorem, all solutions of the Heisenberg equation (2.2b) for observable vector ψ^- under the boundary condition (2.6b) are given by:

$$\psi^-(t) = U(t,t_0)\psi^-(t_0) = \mathrm{e}^{iH(t-t_0)}\psi^-(t_0), \qquad \text{with } -\infty < t < +\infty. \tag{2.7c}$$

Or if one could mathematically choose $t_0 = 0$ and $\psi^- = \psi^-(t_0 = 0)$,

$$\psi^-(t) = U(t,0)\psi^- = U(t)\psi^- = \mathrm{e}^{iHt}\psi^-, \qquad \text{with } -\infty < t < +\infty. \tag{2.7d}$$

The equations (2.7a) or (2.7b) and (2.7c) or (2.7d) describe the unitary group evolution given by the unitary operator $U^\dagger(t,t_0) = \mathrm{e}^{-iH(t-t_0)}$ in general or $U^\dagger(t) = U^\dagger(t,0) = \mathrm{e}^{-iHt}$ in case $t_0 = 0$ for the state ϕ^+, and by $U(t,t_0) = \mathrm{e}^{iH(t-t_0)}$ in general or $U(t,t_0) = U(t) = \mathrm{e}^{iHt}$ for the observable ψ^-, respectively. These operators form a one-parameter group of unitary operators with property $U^\dagger(t) = U(-t) = U^{-1}(t)$ and $U(t_1,t_2)U(t_2,t_3) = U(t_2,t_3)$.

The solutions of the equation (2.2a) for the operator Λ and of the (2.1b) for the density operator ρ are also given by the unitary group:

$$\Lambda(t) = U(t,t_0)\Lambda_0\, U^\dagger(t,t_0) = \mathrm{e}^{iH(t-t_0)}\Lambda_0\,\mathrm{e}^{-iH(t-t_0)}\,, \quad \text{with} \quad -\infty < t < +\infty\,, \tag{2.8a}$$

and

$$\rho(t) = U^\dagger(t,t_0)\rho_0\, U(t,t_0)) = \mathrm{e}^{-iH(t-t_0)}\rho_0\,\mathrm{e}^{iH(t-t_0)}\,, \quad \text{with} \quad -\infty < t < +\infty\,. \tag{2.8b}$$

Here, Λ_0 and ρ_0 are the observable Λ and density operator ρ at any finite time t_0, respectively. Mathematically, one can choose finite time t_0 as time at original, i.e, $t_0 = 0$ as chosen in (2.7b) and (2.7d).

The Hilbert space $\mathcal{H}$ is a linear and complete scalar product space in which the scalar products are defined by *Lebesgue* integral:

$$(\phi_1|\phi_2) \;=\; \underset{\text{Lebesgue}}{\int_0^\infty} dE\, \overline{\phi_1(E)}\, \phi_2(E)\,. \tag{2.9a}$$

Here, one are working in the energy representation in which $\phi_1(E)$ and $\phi_2(E)$ represent the energy wave functions of the state ϕ_1 and ϕ_2, respectively, but the same kind of integration in (2.9a) is also assumed for the position wave functions $\phi(x)$ in the position representation, the momentum wave functions $\phi(p)$ in the momentum representation, or the function of any continuous variables which represent the physical property of the system.

The Hilbert space $\mathcal{H}$ is a complete space. This means all Cauchy sequences in the Hilbert space $\mathcal{H}$ have a limit point in this space $\mathcal{H}$. The norm in the Hilbert space $\mathcal{H}$ is defined through the scalar product (2.9a) as:

$$||\phi||^2 = (\phi|\phi) \;=\; \underset{\text{Lebesgue}}{\int_0^\infty} dE\, |\phi(E)|^2\,, \tag{2.9b}$$

and the convergence is defined with respect to the norm as:

$$\phi_n \to \phi \qquad \text{iff} \quad ||\phi_n - \phi|| \to 0 \quad \text{for } n \to \infty\,. \tag{2.9c}$$

However, in order that $\mathcal{H}$ is complete, the integration in the norm (2.9c) and in the scalar product (2.9a) need to be defined in terms of *Lebesgue* integrals, not Riemann integrals. Since most physicists do not work with Lebesgue integrals but Riemann integrals, the Hilbert space $\mathcal{H}$ is hardly used by physicists.

The Lebesgue integrals need to be chosen for complete Hilbert space $\mathcal{H}$, because the space of Riemann square-integrable functions is not a complete space. Thus the energy wave function $\phi_1(E) = \langle E|\phi_1\rangle$ in (2.9a) must be a Lebesgue square-integrable functions. However, the Hilbert space boundary condition (2.6) with (2.9a) is not the way physicists think of the Schrödinger or Heisenberg equations and their solutions since physicists use the Riemann integrals in the scalar product as well as in the norm. Thus, the Hilbert space axiom (2.6) with the norm and scalar product properties (2.9) are in-intuitive boundary conditions for the dynamical differential equations which describes the dynamics of the physics system.

2.2 The Schwartz Space Boundary Conditions and Dirac Formalism

Fortunately, it is known that the Hilbert space boundary conditions (2.6) are not the only possible boundary conditions for the dynamical equations (2.1a) and (2.2b). The Hilbert space $\mathcal{H}$ is not the most suitable space to use for the theory of quantum physics since physicists use linear scalar product spaces called pre-Hilbert spaces in which the scalar product is defined by Riemann integrals

$$(\phi_1,\, \phi_2)_{\text{Riemann}} = \int_{0\ \text{Riemann}}^{\infty} dE \langle \phi_1|E\rangle\langle E|\phi_2\rangle = \int_{0\ \text{Riemann}}^{\infty} dE\, \overline{\langle E|\phi_1\rangle}\, \langle E|\phi_2\rangle\,. \tag{2.10}$$

Here $\langle E \mid \phi_2\rangle = \phi(E)$ is the energy wave function as in (2.9a) and $\overline{\langle E|\phi_1\rangle}$ is the complex conjugate of the energy wave function $\langle E|\phi_1\rangle$.

These pre-Hilbert spaces are not complete with respect to the norm-convergence of the Hilbert space defined in (2.9a). A scalar product space (or linear topological space) is complete if every Cauchy sequence has a limiting element in the space. This is not the case if norm and scalar product are defined by Riemann integrals and convergence is defined with respect to the norm; i.e.

$$
\begin{aligned}
&\phi_n \to \phi \qquad \text{iff} \quad ||\phi_n - \phi||_{\text{Riemann}} \to 0 \quad \text{for } n \to \infty\,, \\
\text{where} \quad & \\
&||\phi||^2_{\text{Riemann}} = (\phi, \phi)_{\text{Riemann}} = \int_{\text{Riemann}} dE\, \overline{\phi(E)}\, \phi(E)\,.
\end{aligned}
\tag{2.11}
$$

In order to keep using Riemann integrals for the scalar product $(\cdot\,,\cdot)$ in (2.11), one can not define the meaning of convergence by one norm or scalar product, it has to be defined in a different way in term of Dirac's kets and bras as anti-linear functionals on the functional space [7]. The Dirac kets which can not be defined in Hilbert space. Dirac's energy kets $|E\rangle$, which are the eigen-vectors or -kets of the total Hamiltonian H of the physics system with eigenvalues E, have been shown the way towards spaces which are complete spaces and in which the scalar product can be defined by Riemann integrals without using the Hilbert space and Lebesgue integral in the norm definition. The Dirac's energy kets $|E\rangle$ have been used to write the energy wave function $\phi_1(E)$ as the bra-ket product $\langle E \mid \phi_1\rangle$ and the complex conjugate $\overline{\phi_2(E)}$ of energy wave function $\phi_2(E)$ as $\langle \phi_2 \,|E\rangle = \overline{\langle E\,|\phi_2\rangle}$ and treated the integral in the scalar product as Riemann integral as in (2.10).

There was no clear at that time that what kind of space in which the Dirac kets are defined and what are their mathematical meaning until Schwartz theory of distributions [8]. In the theory of distributions, Dirac kets $|E\rangle$ were defined as continuous anti-linear functionals on the Schwartz space $F_E(\phi) = \langle \phi\,|E\rangle = \overline{\langle E\,|\phi\rangle}$. In the Schwartz space, usually denoted by Φ, the convergence of vectors is defined not by one scalar product as in (2.11), but by a *countable number* of scalar products [9] which will be defined below.

According to the Dirac formalism, the Hamiltonian H of the system has a system of both discrete and continuous eigenvectors:

$$H \mid E_n) = E_n \mid E_n) \quad \text{for discrete eigenvalue } E_n\,, \tag{2.12a}$$

$$H\,|E\rangle = E\,|E\rangle \qquad \text{for continuous eigenvalue } E\,. \tag{2.12b}$$

The discrete eigenvectors $\mid E_n)$ in (2.12a) represent bound states of the system.

Generally, every vector $\phi \in \Phi$ can be expanded with respect to the energy kets (2.12) or with respect to eigenkets of other observables A. The Dirac basis vector expansion of the vector $\phi \in \Phi$ with respect to eigenket of Hamiltonian H is:

$$\phi = \sum_{n=\text{integer}} |\, E_n)(E_n\, |\phi\rangle + \int_0^\infty dE\, |E\rangle\langle E\,|\phi\rangle\,, \tag{2.13a}$$

or, if there are *no* discrete eigenvectors $|\, E_n)$ the basis vector expansion (2.13a) is rewritten as:

$$\phi = \int_0^\infty dE\, |E\rangle\langle E\,|\phi\rangle\,. \tag{2.13b}$$

The eigenvectors $|\, E_n)$ for discrete eigenvalues E_n fulfill the orthonormality condition:

$$(|\, E_n), |\, E_m)) \equiv (E_n \mid E_m) = \delta_{nm}. \tag{2.14}$$

The eigenvectors $|E\rangle$ for the continuous eigenvalue expansion (2.13b) were postulated to fulfill the new orthogonality condition called Dirac orthogonality condition [7]:

$$\langle E\,|E'\rangle = \delta(E - E')\,, \tag{2.15}$$

where $\delta(E - E')$ is defined as the mathematical entity which fulfills the identity:

$$\int_{-\infty}^{+\infty} dE' \langle E\,|E'\rangle\langle E'\,|\phi\rangle = \langle E\,|\phi\rangle\,, \tag{2.16a}$$

or equivalently,

$$\int_{-\infty}^{+\infty} dE' \delta(E - E')\phi(E') = \phi(E)\,, \tag{2.16b}$$

for the set $\{\phi(E)\}$ of well-behaved energy wave function $\phi(E) = \langle E\,|\phi\rangle$.

Well-behaved function mathematically means that energy wave function $\phi(E)$ is smooth, infinitely differentiable and rapidly decreasing for increasing energy variable $|E|$ than any polynomial function of energy $|E|$. This set of functions is defined as the Schwartz function space, $\{\phi(E)\} \equiv S$, of rapidly decreasing and infinitely differentiable functions [8].

This Schwartz function space S is a dense subspace of the space L^2 of *Lebesgue* square integrable function: $S \subset L^2$. Mathematically, this means that starting with abstract Schwartz space S and adjoining it all limit points of Cauchy sequences with respect to the Hilbert space convergence, one obtains Lebesgue square space L^2. Hence, all functions $\phi(E) \in S$ are members of the subset of some *classes* of L^2-functions, i.e., $\phi(E) \in L^2$. In other words, the element of L^2 is not a function but a class of Lebesgue square integrable functions. Some of these classes contain a continuous rapidly decreasing function $\phi(E)$ which is an element of S. But in addition to these classes with $\phi(E) \in S$, there are sets of functions $\{h(E)\} \in L^2$ which contain no Schwartz space function. Thus $S \subset L^2$. Since according to the Fréchet-Riesz theorem: $L^2 = (L^2)^\times$, where $(L^2)^\times$ denotes the space of anti-linear Hilbert space-continuous functionals on L^2. It follows that one has the triplet of function spaces [10][11][12]:

$$\{\phi(E)\} = S \subset L^2 = (L^2)^\times \subset S^\times \,. \tag{2.17}$$

Here $(L^2)^\times$ and $S^\times$ denote the linear spaces of *continuous* anti-linear functionals on L^2 and on S, respectively. The triplet (2.17) is the Rigged Hilbert Space (RHS) of Schwartz space functions. It gives a mathematical meaning to the Dirac kets $|E\rangle \in \Phi^\times$, as continuous anti-linear functionals on abstract Schwartz space Φ. Therefore, the Dirac δ-function, $\delta(E-E')$, is not a function, like a well-behaved function $\phi(E) \in S$, but a distribution defined by its property (2.16) for all well-behave function in the Schwartz function space $\phi(E) \in S$.

The abstract Schwartz space Φ is the set of vectors $\{\phi\}$ of (2.13b) for which their energy wave functions are elements of the Schwartz space of functions, $\langle E\,|\phi\rangle \in S$. According to (2.17), the abstract Schwartz space Φ associated to the Schwartz space functions S is the dense subspace of the abstract Hilbert space $\mathcal{H}$: $\{\phi\} \equiv \Phi \subset \mathcal{H}$. The Schwartz space Φ has a stronger definition of convergence, τ_Φ, than the Hilbert space convergence $\tau_\mathcal{H}$. Mathematically, it can be stated as [13]:

$$\phi_\nu \xrightarrow{\tau_\Phi} \phi \quad \Rightarrow \quad \phi_\nu \xrightarrow{\tau_\mathcal{H}} \phi \,, \quad \text{but} \quad \phi_\nu \xrightarrow{\tau_\mathcal{H}} \phi \quad \not\Rightarrow \quad \phi_\nu \xrightarrow{\tau_\Phi} \phi \,. \tag{2.18}$$

Consequently, the space of continuous functionals on the Hilbert space $\mathcal{H}^\times$ is the dense subspace of the space of continuous functionals on the abstract Schwartz space $\mathcal{H}^\times \subset \Phi^\times$.

Therefore, in correspondence to the triplet of the Schwartz space functions (2.17), one obtains the triplet of the abstract Schwartz space, called the Rigged Hilbert Space (RHS) or Gelfand Triplet [14] [15] [16]:

$$\{\phi\} = \Phi \subset \mathcal{H} = \mathcal{H}^\times \subset \Phi^\times \,. \tag{2.19}$$

Here, the abstract Hilbert space $\mathcal{H}$ is the same of its corresponding continuous functional space $\mathcal{H}^\times$ since the Hilbert space is a complete space, i.e, $\mathcal{H} = \mathcal{H}^\times$.

The Schwartz space Φ is a nuclear space. This means that for the Schwartz space, the Dirac basis vector expansion (2.13b) hold as the nuclear spectral theorem

[16]. This nuclear spectral theorem states that every vector $\phi \in \Phi$ can be uniquely expanded with respect to a complete set of generalized eigenvectors $|E\rangle \in \Phi^\times$

$$\phi = \int dE \; |E\rangle\langle E\,|\phi\rangle \qquad \text{for every } \phi \in \Phi \,, \tag{2.20a}$$

$$\text{and } \phi = 0 \quad \Longleftrightarrow \quad \phi(E) = \langle E\,|\phi\rangle = 0 \text{ for all } E \,. \tag{2.20b}$$

There exists a complete set of eigenkets $|E\rangle \in \Phi^\times$ which are generalized eigenvectors of the Hamiltonian H with continuous eigenvalues $E \in \mathbb{R}$, i.e,

$$H^\times \; |E\rangle = E \; |E\rangle \qquad \text{where } |E\rangle \in \Phi^\times \,, \tag{2.21a}$$

precisely in term of a scalar product

$$\langle H\phi|E\rangle \equiv \langle \psi| \; H^\times \; |E\rangle = E\langle \phi|E\rangle \quad \text{for all } \phi \in \Phi \,. \tag{2.21b}$$

Here, the operator H is essentially self-adjoint with property $H^\times \supset H^\dagger = \overline{H} \supset H$ and $H^\times$ is the unique extension of $H^\dagger$ to the conjugate space $\Phi^\times$ which is the space of all anti-linear continuous functionals of the Schwartz space Φ. It means that every $\phi \in \Phi$ can be expanded with respect to the eigenkets $|E\rangle$ as in (2.20). The kets $|E\rangle$ are functionals of the functional space on the abstract Schwartz space.

The abstract Schwartz space Φ is a linear topological space with the convergence defined by a countable number of norms: $||\phi||_p$, $p = 1\,,2\,,\cdots$.

$$||\phi||_p \;=\; (\phi|(\Delta+1)^p|\phi)\,, \qquad \text{and} \tag{2.22a}$$

$$\phi_\nu \to \phi \text{ for } \nu \to \infty \quad \Longrightarrow \quad ||\phi_\nu - \phi||_p \to 0 \text{ for } \nu \to \infty\,, \forall\, p\,. \tag{2.22b}$$

Here Δ is the Nelson operator of the quantum system, e.g, for the harmonic oscillator, the Nelson operator is the number operator $N = \frac{1}{\omega}(H - 1/2)$. These countable norms are chosen such that the algebra of observables called $\mathcal{A}$ is represented by *continuous* operators in all of the space Φ [13]. In case of the harmonic oscillator, the algebra of the observables is generated by the momentum operator P, position operator Q, and the Hamiltonian operator H. However, the momentum P and position Q of the oscillator are not continuous operators in Hilbert space $\mathcal{H}$.

Using the Schwartz space (2.19) and Dirac's bra-ket formalism, the set of vector-states $\{\phi^+\}$ fulfilling (2.1a) and the set of vector-observables $\{\psi^-\}$ fulfilling (2.2b) are both described by the same Schwartz space Φ which is a dense subspace of the

Hilbert space $\mathcal{H}$. Precisely, the abstract Schwartz space Φ differs from the Hilbert space $\mathcal{H}$ by limit points of Hilbert space Cauchy sequences. Mathematically, one can now seek for all solutions either of the in-states $\phi^+(t)$ fulfilling the Schrödinger equation (2.1a) or of the observable $\psi^-(t)$ fulfilling the Heisenberg equation (2.2b) under the Schwartz space boundary condition:

$$\text{Set of state vectors } \{\phi^+\} = \Phi = \text{Schwartz space} \subset \mathcal{H} \subset \Phi^\times \,, \tag{2.23a}$$

$$\text{Set of observable vectors } \{\psi^-\} = \Phi = \text{Schwartz space} \subset \mathcal{H} \subset \Phi^\times \,. \tag{2.23b}$$

The Schwartz space boundary conditions (2.23a) and (2.23b) for the dynamical equation (2.1a) and (2.2b), respectively, mean that only the in-state vectors $\phi^+ \in \Phi \subset \mathcal{H}$, i.e, not all vectors of the Hilbert space $\mathcal{H}$, represent physical states prepared by a preparation apparatus, e.g, an accelerator in a scattering experimental setup. In analogy, only observable vectors in the same Schwartz space, $\psi^- \in \Phi \subset \mathcal{H}$, not the whole Hilbert space, represent the physical observables detected by the registration apparatus, e.g, a detector.

Using the Schwartz space boundary conditions (2.23a)(2.23b) as the boundary condition axioms replaced the Hilbert space axiom for the solutions for the Schrödinger equation (2.1a) and for the Heisenberg equation (2.2b), one obtains by a same mathematical theorem [16] like the Stone-von Neumann theory for the Hilbert space that the time evolutions of the in-state ϕ^+ and observable ψ^- are given by unitary groups. These unitary groups are same as the unitary group in (2.7a) and (2.7d) for the Hilbert space $\mathcal{H}$ but restricted to the subspace of Hilbert space, the Schwartz space $\Phi \subset \mathcal{H}$. Mathematically, the time evolution of the solutions of the Schrödinger equation (2.1a) under the boundary conditions (2.23a) is given by

$$\phi^+(t) = U_\Phi^\dagger(t)\ \phi^+ = \mathrm{e}^{-iHt}\phi^+ \quad -\infty < t < +\infty \,. \tag{2.24a}$$

Similarly, the time evolution of the solutions for the Heisenberg equation (2.2b) under the boundary conditions (2.23b) is given by

$$\psi^-(t) = U_\Phi(t)\ \psi^- = \mathrm{e}^{iHt}\psi^-, \quad \text{with } -\infty < t < +\infty \,. \tag{2.24b}$$

In (2.24a) and (2.24b), the operators $U_\Phi^\dagger(t)$ and $U_\Phi(t)$ denote the restriction of the unitary operator $U^\dagger(t)$ in (2.7a) and of $U(t)$ in (2.7d) for the Hilbert space $\mathcal{H}$ to the dense Schwartz - subspace Φ of the Hilbert space $\mathcal{H}$, $U^\dagger_\Phi(t) = U^\dagger(t)|_\Phi$ and $U_\Phi(t) = U(t)|_\Phi$, respectively.

For the Dirac eigenkets $|E\rangle$ of the Hamiltonian H, the time evolution is given by

$$|E\,;t\rangle = \mathrm{e}^{-iH^\times t}\,|E\rangle = \mathrm{e}^{-iEt}\,|E\rangle, \quad \text{with } -\infty < t < +\infty\,. \tag{2.25}$$

The Born probabilities, $\mathcal{P}_\rho(\Lambda(t))$, to measure an observable $\Lambda(t)$ in a state ρ under the Schwartz space axiom are thus again predicted for all t:

$$\mathcal{P}_W(\Lambda(t)) = Tr(W\,\Lambda(t)) = Tr(W(t)\,\Lambda) \quad \text{for all } -\infty < t < +\infty\,. \tag{2.26}$$

For the special or simplest case, in which the state ρ is a pure state $\rho = |\phi^+\rangle\langle\phi^+|$ and the observable is given by $\Lambda = |\psi^-\rangle\langle\psi^-|$, this probability is written as

$$\begin{aligned}\mathcal{P}_{\phi^+}(|\psi^-\rangle\langle\psi^-|) &= Tr(|\psi^-\rangle\langle\psi^-|\phi^+(t)\rangle\langle\phi^+(t)|) \\ &= |\langle\psi^-|\phi^+(t)\rangle|^2 = |\langle\psi^-(t)|\phi^+\rangle|^2 \quad \text{for all } -\infty < t < +\infty\,.\end{aligned} \tag{2.27}$$

This means that the theory based on the Hilbert space boundary condition (2.6a)(2.6b) as well as the theory based on the Schwartz space boundary condition (2.23a)(2.23b) predict a certain probability $|\langle\psi^-|\phi^-(t)\rangle|^2$ to detect the observable $\Lambda = |\psi^-\rangle\langle\psi^-|$ in the state $\phi^+(t)$ for any arbitrary negative times $t < 0$, i.e. even for time before the state $\phi^+(t)$ had been prepared at the time $t = t_0 = 0$ in the scattering experiment.

2.3 Causality Condition for Quantum Mechanics

If one use the Hilbert space axiom (2.6) or the Schwartz space axiom (2.23) as the boundary condition to find the solutions of the Schrödinger equation (2.1) for the states ϕ^+ or the Heisenberg equation (2.2) for the observables ψ^- in quantum theory of scattering and decay, the time parameter t in the time evolution operators needs to extend from $-\infty < t < +\infty$ as in (2.24). It means that one can find a certain probability to detect the observable ψ^- in the state $\phi^+(t)$ even before the time t_0 at which the state $\phi^+(t)$ had been prepared as calculated in (2.27).

In contrast to the mathematical prediction (2.27) for the Hilbert space boundary condition (2.6) as well as for the Schwartz space boundary condition (2.23), the situation in the laboratory is quite different because of the causality principle. This empirical causality principle states that:

$$\begin{array}{l}\text{A state } \phi^+ \text{ needs to be prepared first at a time } t_0 \textit{ before} \text{ an observable} \\ |\psi^-(t)\rangle\langle\psi^-(t)\,| \text{ can be measured in that state } \phi^+ \text{ with the probability} \\ \mathcal{P}_{\phi^+}(|\psi^- i(t)\rangle\langle\psi^-(t)\,|) \text{ in the scattering experiment.}\end{array} \tag{2.28}$$

The causality principle (2.28) means that the observable $|\psi^-(t)\rangle\langle\psi^-(t)\,|$ or any observables $A(t)$ can only be detected in the state ϕ^+ at the times t after the time t_0 which is the time at which the state ϕ^+ is prepared $t > t_0$, but not at any arbitrary time $t < t_0$ in the distant past. Therefore, the time symmetric group evolution (2.7a) and (2.7d) as well as (2.24a) and (2.24b) predicted by von Newman theorem for the Hilbert space boundary condition (2.6a), (2.6b) and also by similar theory for the Schwartz space boundary condition (2.23a), (2.23b) is in contradiction with the causality principle (2.28).

The causality principle (2.28) means that an observable ψ^- can not be detected in a state ϕ^+ *before* this state ϕ^+ exits, i.e, before it has been prepared by a preparation apparatus , e.g, an accelerator in the scattering experiment. Hence, the Born probabilities $\mathcal{P}_\rho(\Lambda(t))$ to measure observables $\Lambda(t)$ in states ρ make *experimentally* sense only for the time t after the preparation time t_0 of the state ρ, i.e, $t \geq t_0$.

Therefore, a new boundary condition is needed in place of the Hilbert space (2.23a)(2.23b) as well as the Schwartz space boundary condition (2.6a)(2.6b) for which the Born probabilities:

$$\begin{aligned}\mathcal{P}_{\phi^+}(|\psi^-\rangle\langle\psi^-\,|) &= Tr(|\psi^-\rangle\langle\psi^-\,|\phi^+(t)\rangle\langle\phi^+(t)\,|) \\ &= |\langle\psi^-\,|\phi^+(t)\rangle|^2 = |\langle\psi^-(t)\,|\phi^+\rangle|^2\end{aligned} \tag{2.29}$$

exit *only* for $t > t_0$.

Here t_0 denotes a *finite time* at which the state $\phi^+(t)$ had been prepared and after which an observable $|\psi^+\rangle\langle\psi^+\,|$ can be measured in this state.

Therefore, a theory in order to describe the scattering experiment phenomenon needs to agrees with the causality principle. In other words, the Hilbert space axiom as well as the Schwartz axiom needs to be replace by a new axiom in which the time evolution of solutions of the Schrödinger equations (2.1a) for the state ϕ^+ or of the Heisenberg equation (2.2b) for the observable exist only for $t \geq t_0$, where t_0 is finite time which is mathematically be chosen as zero, $t_0 = 0$. This means that the time evolution group is expected to be semi-group instead of the unitary group time evolution in order to describe the scattering experiment phenomena.

Chapter Three: Non-relativistic Time Asymmetric Quantum Mechanics

3.1 Resonances and Decaying states in Scattering Experiments and Boundary Condition Beyond the Hilbert or Schwartz Space

In the scattering experiments, resonances R_i, $i = 1\,, 2, \cdots$ are generally defined as intermediate states in the scattering processes, e.g,:

$$a + b \to R_1 + R_2 + \cdots \to c + d + \cdots \tag{3.1}$$

In the scattering process (3.1) setup, two beams of particles, i.e, a and b, are accelerated through an accelerator and guided to collide together, for example, head-on collisions in electron-positron or proton-anti proton experiments, or just one beam of particles a is accelerated by the accelerator and hits the target b which is at rest. The resonances R_1, $R_2 \cdots$ are formed after the collisions and then decay into the products c, $d \cdots$ which are registered by detectors.

A resonance R is characterized by a mass M_R and a width Γ_R. In non-relativistic scattering experiment, the mass M_R and width Γ_R of the resonance R are experimentally measured by fitting the partial cross section $\sigma_j(E)$ to the Breit-Wigner or Lorentzian energy distribution with a slowly varying background $B(E)$ with respect to energy E:

$$\sigma_j(E) \sim \left|a_j^{BW}(E)\right|^2 = \left|\frac{r}{E - (E_R - i\Gamma_R/2)} + B(E)\right|^2 \quad \text{with} \quad 0 < E < \infty\,. \tag{3.2}$$

A decaying state D appears in decay process:

$$D \to \eta + \eta' \cdots \tag{3.3}$$

In the decay experiment, the decay products $\eta\,, \eta'\,, \cdots$, are counted by detectors which may be constructed to observe some specific particles.

A decaying state D is characterized by a mass M_D and a lifetime τ_D. The lifetime τ_D of the decaying state D is experimentally obtained by fitting the counting rate $N_\eta/N(t)$ of the product η in the time interval Δt_i to the exponential law for a partial decay rate of product η, $R_\eta(t)$:

$$R_\eta(t) = R_\eta e^{-Rt} = R_\eta e^{-\frac{t}{\tau_D}} \approx \frac{1}{N}\frac{\Delta N_\eta(t_i)}{\Delta t_i}\,. \tag{3.4}$$

Here, N is the total number of decay products of the decaying state D, i.e, it is sum of all partial decay product $\eta\,,\ \eta' \cdots$ which are counted by the detectors:

$$N = \sum_{\text{all } \eta \text{ decay products}} N_\eta\,, \tag{3.5}$$

and R is total decay rate of decaying state D and inverse of the lifetime τ_D:

$$R = R_{total} = \sum R_\eta = \frac{1}{\tau_D}\,. \tag{3.6}$$

The partial decay rate R_η for the product η can be calculated in the non-relativistic regime by the rate of the Born probabilities $\mathcal{P}_D(t)$ for the decaying state $\psi^D(t)$ decaying to the product η or final observable $\Lambda_\eta = |\psi_\eta^-\rangle\langle\psi_\eta^-|$:

$$R_\eta = \frac{d\,\mathcal{P}_\eta(t)}{dt} = \frac{d}{dt}\left(\text{Tr}\left(\Lambda_\eta\,|\psi^D(t)\rangle\langle\psi^D(t)\,|\right)\right) = \frac{d}{dt}\left(|\langle\psi^D(t)\,|\psi_\eta^-\rangle|^2\right)\,. \tag{3.7}$$

The observed partial decay rate $\frac{1}{N}\frac{\Delta N_\eta(t_i)}{\Delta t_i}$ in (3.4) is fitted to the exponential time evolution $e^{-\frac{t}{\tau}}$. From this fit, the lifetime τ_D of the decaying state ψ^D is then determined. Physicists believe that the resonance and decaying state are two different aspects of the same entities. In other words, they are two different aspects of the same scattering phenomenon and the lifetime τ can be calculated as inverse width Γ, and vice versa:

$$\Gamma = R_{total} = \frac{1}{\tau}\,. \tag{3.8}$$

The relation (3.8) between the lifetime τ and the width Γ was first justified in non-relativistic quantum physics in the zero order of probability to detect the observable ψ^- in the state $\phi^+(t)$ using Weisskopf-Wigner methods [17] by Goldberger and Watson [1]:

$$\begin{aligned} |\langle\psi^-\,|\phi^+(t)\rangle|^2 = \mathcal{P}_{\phi^+(t)}(\psi^-) \quad &\sim e^{-\Gamma t}(1 + \tfrac{\Gamma t}{2} + \tfrac{3\Gamma^2 t^2}{28} + \tfrac{\Gamma^3 t^3}{84} + ...) \\ &\sim e^{-\Gamma t} = e^{-\frac{t}{\tau}} \quad \text{where} \quad \tau \approx 1/\Gamma\,. \end{aligned} \tag{3.9}$$

There exists a problem in the standard quantum theory based on the Hilbert space axiom that the exact relation between the lifetime τ and the width Γ in (3.8) can not be obtained because the exponential time evolution is not possible in the Hilbert space by the Khalfin's theorem [2]. It means that there exists no vector in the Hilbert space $\psi^D \in \mathcal{H}$ for which its decay rate is obey the exponential law (3.4).

In contrast to Khalfin's theorem on the Hilbert space, experimental data show that the decay rate of quantum systems is in good agreement with the exponential law, if the decaying state can be isolated from the background, e.g, in K_S^0-decay experiment in the relativistic regime [18].

Another problem in standard quantum theory is that Born's probability to detect an observable $\Lambda(t) = |\psi^-(t)\rangle\langle\psi^-(t)\,|$ in the state ϕ^+:

$$\begin{aligned} \mathcal{P}_\phi(\Lambda(t)) &= \mathrm{Tr}(|\psi^-(t)\rangle\langle\psi^-(t)\,||\phi^+\rangle\langle\phi^+)\,|) \\ &= |\langle\psi^-(t)|\phi^+\rangle|^2 = |\langle \mathrm{e}^{-iHt}\psi^-(0)|\phi^+\rangle|^2, \end{aligned} \tag{3.10}$$

is theoretically predicted for all time $-\infty < t < +\infty$. This prediction (3.10) is a mathematical consequence, by Stone-von Neumann theorem [4], of the Hilbert space boundary condition for the solutions of the Schrödinger equation for the state ϕ^+ or of the Heisenberg equation for the observable ψ^-. However, the prediction (3.10) violates the causality principle (2.28), which states that a state ϕ^+ must be prepared first, e.g, by an accelerator, before the observable $\psi^-(t)$ for decay products can be measured, e.g, by detectors, in the scattering experiment. Therefore, the theory based on the Hilbert space (2.6) or Schwartz space boundary condition (2.24) is mathematically not sufficient to describe the resonance and decaying state in the scattering experiment phenomena. An expected theory to describe the resonance and decaying state phenomena needs to be mathematically able both to derive the exact relation (3.8) between the width Γ of a resonance and the lifetime τ of associated decaying state and to reflect the causality principle (2.28) .

3.2 In-states and Out-observables in Quantum Mechanics

In scattering experiments, the in-states ϕ^{in} are controlled or prepared in the remote past and the out-states ψ^{out} controlled or detected in the distance future. The in-state ϕ^{in}, as well as the out-states ψ^{out} are thought to obey the Schrödinger equation

(2.1a) under the Hilbert space boundary condition (2.6a) for the states. However, the controlled or out-state vectors ψ^{out} are controlled by a registration apparatus, e.g, a detector in the scattering experiment. This means that the controlled out-vectors represent the observables registered by the detector. Therefore, the so called "out-state" in the standard quantum mechanics is really an observable which should be governed by the Heisenberg equation (2.2b) with the solutions given by (2.7d) or (2.24b), not governed by the Schrödinger equation (2.7a).

The in-state vectors $\phi^{in}(t)$ evolve in time according to the free Hamiltonian K as $\phi^{in}(t) = e^{iKt}\phi^{in}$ in the Schrödinger picture. When the particles reach their interaction regions, the free in-state vector ϕ^{in} changes into an exact state vector ϕ^{+}.The time evolution of exact state $\phi^{+}(t)$ is governed by the exact Hamiltonian $H = K + V$, where V is the interaction potential. After interacting, the particles move apart from the their interaction regions, the exact state vector $\phi^{+}(t)$ changes into the free out-state vector ϕ^{out}. These changes, from the free in-state ϕ^{in} to the exact state ϕ^{+} and from the exact state ϕ^{+} to the free out-state vector ϕ^{out}, can be described by the Moeller wave operator $\Omega^{\pm}$ [19]:

$$\phi^{+}(t) \equiv \Omega^{+}\phi^{in} = e^{-iHt}\phi^{+} = \Omega^{-}\phi^{out}\,. \tag{3.11}$$

Here t is the time in the center-of-mass of the projectile and target or of two beams of particles. The free out-vector ϕ^{out} in the further future thus describes a state vector which is determined by the preparation of ϕ^{in} in the further past and by the dynamics of the scattering process as the asymptotic states. The dynamical transformation of the asymptotic in-state ϕ^{in} into an asymptotic out- state ϕ^{out} is described by the operator S: [20] [21]

$$\phi^{out} = S\,\phi^{in}, \qquad S = \Omega^{-\dagger}\Omega^{+}\,. \tag{3.12}$$

Both states vectors ϕ^{in} and ϕ^{+} are determined by the preparation apparatus, i.e, an accelerator, only. Thus, the states vectors ϕ^{+} and ϕ^{in} represent apparatus controlled states. The preparation of out-state ϕ^{out} is determined by the preparation apparatus and the interaction (or dynamics) described by interaction potential V or operator S so it does not represent neither controlled states nor observables .

The vector ψ^{out} represents the asymptotically free out-observables, usually also called out-states.The out-vectors ψ^{out} also evolve in time, in the Heisenberg picture, according to the free Hamiltonian K. However, they evolve by the adjoint (or conjugate) operator e^{iKt} since they are observables and solutions of the Heisenberg equation (2.2b) as $\psi^{out}(t) = e^{iKt}\psi^{out}$.

In a scattering experiment, a detector, or the registration apparatus, registers or detects an observable $|\psi^{out}\rangle\langle\psi^{out}|$ outside the interaction region. This observable vector ψ^{out} comes from a vector ψ^{-} in the interaction region which is given by:

$$\psi^{-} = \Omega^{-}\psi^{out} . \tag{3.13}$$

The state ψ^{out} is in the asymptotic region, which is far away interaction region and usually after the target, while the observable ψ^{-} is in the interaction region, and ϕ^{in} is in the asymptotic region which becomes ϕ^{+} in the interaction region. The vectors ϕ^{+} fulfill the Schrödinger equation with the exact Hamiltonian H and therefore the time evolution of the exact in-states ϕ^{+} are given by:

$$\phi^{+}(t) = e^{-iHt}\phi^{+} . \tag{3.14a}$$

While the vectors $\psi^{-}(t)$ fulfill the Heisenberg equation of motion and hence the time evolution of the observable state ψ^{-} is given by:

$$\psi^{-}(t) = e^{iHt}\psi^{-} . \tag{3.14b}$$

The Born probability amplitude (ψ^{-}, ϕ^{+}) to detect the observable ψ^{-} in the state ψ^{-} is expressed using the standard notions of scattering theory (2.4) and can be expressed in terms of the S-operator [20]:

$$(\psi^{-}, \phi^{+}) = (\Omega^{-}\psi^{out}, \Omega^{+}\phi^{in}) = (\psi^{out}, S\,\phi^{in}) = (\psi^{out}, \phi^{out}) . \tag{3.15}$$

This matrix element (3.15) is the probability amplitude for the observable ψ^{-} in the state ϕ^{+}. It can also be given in terms of the asymptotic states as the probability amplitude for the observable ψ^{out} in the state ϕ^{out}.

The observable ψ^{out} is clearly not same as the state ϕ^{out}, since ϕ^{out}, like ϕ^{+} and ϕ^{in}, is prepared by preparation apparatus (e.g, the accelerator) and ψ^{out} as well as ψ^{-} is defined (or controlled) entirely by the registration apparatus (e.g, the detector). Thus, the set of the observable vectors $\{\psi^{-}\}$ could be distinct from the set of in-state vectors $\{\psi^{-}\}$. This distinction between the set of the observable vectors $\{\psi^{-}\}$ and the set of in-state vectors $\{\psi^{-}\}$ can be expressed as the new boundary condition which will be discussed in section 3.3 in this chapter.

Under the standard boundary condition (2.6b) and (2.23b), the solutions of the Heisenberg equation for the observables are predicted for all time t which extends $-\infty < t < +\infty$. This is in conflict with the causality principle (2.29): According to (2.29) the observable $|\psi^-\rangle\langle\psi^+|$ in the state ϕ^+ can be predicted only for times $t > t_0$ where t_0 is the time at which ϕ^+ has been prepared. To avoid violation of the causality principle, we need to find boundary conditions for the solutions of the dynamical equations (2.1) and (2.2), which will be different from the Hilbert space boundary condition (2.6) and also different from the Schwartz space boundary condition (2.23).

These new boundary conditions need to use different representation spaces than the Hilbert space $\mathcal{H}$ or the Schwartz space Φ. These new spaces are:

$$\Phi_- \text{ for the solutions of the Schrödinger equation of the states } \{\phi^+\}. \tag{3.16a}$$

$$\Phi_+ \text{ for the solutions of the Heisenberg equation of the observables } \{\psi^-\}. \tag{3.16b}$$

This means, we need to modify the Hilbert space axiom (2.6a)(2.6b). Similarly, the Schwartz space axiom of the Dirac formalism has to be modified. In other words, we replace the axiom (2.23a)(2.23b) (or (2.6a)(2.6b)), in which the space of states and of observables are the same, for the dynamical equations by a new axiom that *distinguishes mathematically* between:

> The prepared states which are represented by the set of prepared in-state vectors $\{\phi^+\}$, obeying the Schrödinger equation (2.1a), and the detected observables which are represented by the set of registered out-observables $\{\psi^-\}$, obeying the Heisenberg equations (2.2b).

If we define the $|E, j, j_3, \eta^\pm\rangle$ are the energy basis kets or energy eigenvector of the space $\Phi_\mp$ of (3.16) in the energy representation where j, j_3, and η denote the quantum numbers of angular momentum , its third component, and the particle label quantum number, e.g., charge operators, respectively. Then, the Dirac basis vector expansionof in-state vectors $\phi^+ \in \Phi_-$ is given by

$$\phi^+ = \sum_{j,j_3,\eta} \int_0^\infty dE |E, j, j_3, \eta^+\rangle\langle^+E, j, j_3, \eta|\phi^+\rangle = \int_0^\infty dE |E^+\rangle\langle^+E|\phi^+\rangle\,. \tag{3.17a}$$

And for the out-observable vector ψ^-, the Dirac basis vector expansion is given by

$$\psi^- = \sum_{j,j_3,\eta} \int_0^\infty dE|E,j,j_3,\eta^-\rangle\langle^- E,j,j_3,\eta|\psi^-\rangle = \int_0^\infty dE|E^-\rangle\langle^- E|\psi^-\rangle \,. \qquad (3.17b)$$

The second terms in (3.17a) and (3.17b) are the short notation when we ignore the quantum numbers of the angular momentum j, its third component j_3, and the species quantum number η, which do not affect the vector expansions in general.

The energy eigenkets $|E^+\rangle \in \Phi_-^\times$ are continuous anti-linear functionals on the space Φ_- of prepared states. They fulfill the eigen equation for the exact Hamiltonian H with the eigenvalues E:

$$\langle H\phi^+|E^+\rangle = \langle\phi^+|\, H^\times\, |E^+\rangle = E\langle\phi^+|E^+\rangle \quad \text{for all} \quad \phi^+ \in \Phi_- \,. \qquad (3.18a)$$

Similarly, the $|E^-\rangle \in \Phi_+^\times$ of (3.17b) are continuous anti-linear functions on Φ_+ and fulfill the eigen equation for the exact Hamiltonian H with the eigenvalues E::

$$\langle H\psi^-|E^-\rangle = \langle\psi^-|\, H^\times\, |E^-\rangle = E\langle\psi^-|E^-\rangle \quad \text{for all} \quad \psi^- \in \Phi_+ \,. \qquad (3.18b)$$

The kets $|E^\pm\rangle$ have been used extensively for a long time in scattering theory though the spaces $\Phi_\pm$ had not been mathematically defined. These kets $|E^\pm\rangle$ had been introduced in the phenomenological scattering theory as the in- and out- plane wave kets $|E^\pm\rangle$ which fulfill the Lippmann-Schwinger equation [6]:

$$|E^\pm\rangle = |E \pm i\epsilon^\pm\rangle = |E\rangle + \frac{1}{E - H \pm i\epsilon} V|E\rangle = \Omega^\pm|E\rangle, \quad \epsilon \to +0 \,. \qquad (3.19)$$

The kets $|E^\pm\rangle$ are eigenkets of the "exact" Hamiltonian $H = K + V$,

$$H\,|E,j,j_3,\eta^\pm\rangle = H\,|E^\pm\rangle = E\,|E^\pm\rangle = E\,|E,j,j_3,\eta^\pm\rangle \,. \qquad (3.20)$$

The kets $|E\rangle$ or $|E,j,j_3,\eta\rangle$ in (3.19) are the eigenkets of the free-interaction Hamiltonian K: $K\,|E\rangle = E\,|E\rangle$. The operator $V = H - K$ is the interaction Hamiltonian or perturbation Hamiltonian, and $\Omega^\pm$ are the Möller operators [19].

The $+i\epsilon$ in the ket $|E^+\rangle = |E + i\epsilon^+\rangle$ of the Lippmann-Schwinger equation (3.19) suggested that the energy wave functions of the in-states ϕ^+ of (3.17a),

$$\phi^+(E) \equiv \langle^+ E|\phi^+\rangle = \overline{\langle\phi^+|E^+\rangle} \,, \qquad (3.21a)$$

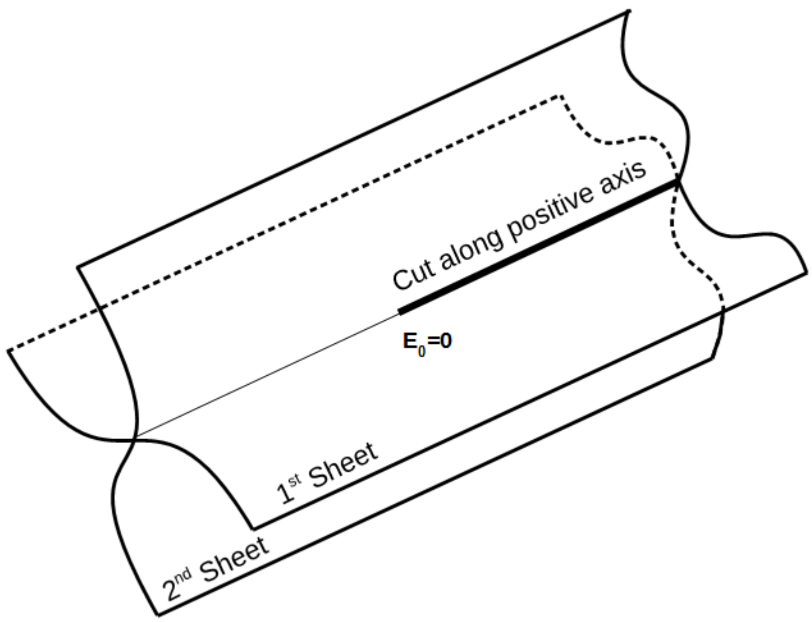

Figure 3.1: Two sheeted Riemann Surfaces with a cut along the energy positive axis.

are the boundary value of an analytic function in the lower complex energy semi-plane $\mathbb{C}_-$ for complex energy $z = \overline{E + i\epsilon} = E - i\epsilon$ which is immediately below the positive real axis on the second sheet of the $\mathcal{S}$-matrix, as shown in Fig 3.1

Similarly, the $-i\epsilon$ in the ket $|\,E^-\rangle\ = |\,E - i\epsilon\rangle$ indicates that the energy wave functions of the observables $|\psi^-\rangle\langle\psi^-|$ in (3.17b),

$$\psi^-(E) \equiv \langle^- E\,|\psi^-\rangle = \overline{\langle\psi^-\,|E^-\rangle}\,, \tag{3.21b}$$

are the boundary value of an analytic function on the upper complex energy semi-plane $\mathbb{C}_+$ for complex energy $z = \overline{E - i\epsilon} = E + i\epsilon$ which is above the real axis on the second sheet of the $\mathcal{S}$-matrix. Consequently, complex conjugate $\overline{\psi^-(E)} = \langle\psi^-\,|E^-\rangle$ of the energy wave function of the observable $\langle^+E\,|\phi^+\rangle$ and complex conjugate $\overline{\phi^+(E)} = \langle\phi^+\,|E^+\rangle$ of the energy wave function of the state $\langle^-E\,|\psi^-\rangle$ are analytic functions on the lower and upper complex energy semi-plane on second sheet of $\mathcal{S}$-matrix, respectively.

3.3 The Hardy Space Boundary Condition

The general eigenkets $|E^{\pm}\rangle$ suggested by the phenomenological Lippmann-Schwinger equation (2.24) can then be mathematically defined by the complete spectral theorem for the exact Hamiltonian H for the in-states ϕ^+ in the space Φ_- as (3.17a) and for the observables ψ^- in the space Φ_+, if one choose the wave functions $\langle^- E\,|\psi^-\rangle = \psi^-(E)$ and $\langle^+ E\,|\phi^+\rangle = \phi^+(E)$ to be the analytic functions in the upper and lower complex energy plane on the second sheet of the S-matrix, respectively. This means that the space $\{\phi^+\} = \Phi_-$ of the states and the space $\{\psi^-\} = \Phi_+$ of the observables will be mathematically defined as the space of the lower and upper complex semi plane, respectively. These spaces $\Phi_\mp$ are known as the Hardy space of of the lower and upper complex semi plane which are also called the lower and upper Hardy space, respectively. Therefore, the basis eigenkets $|E^{\pm}\rangle$ will now be defined as Hardy space functionals of the upper and lower Hardy space, $|\,E^{\pm}\rangle \in \Phi_\mp^\times$. The new axiom replacing the Hilbert space axiom (2.6a)(2.6b), or replacing the Schwartz space axiom (2.23a)(2.23b) for the Dirac formulation, is now described as the new *Hardy space axiom* of quantum mechanics [23] [22]:

The set of prepared (in-)states obeying Schrödinger equation $\{\phi^+\}$ is mathematically represented by Φ_-, the Hardy space of the lower complex energy plane of the second sheet of the $\mathcal{S}$-matrix:

$$\{\phi^+\} \equiv \Phi_- \,. \tag{3.22a}$$

The set of detected or registered observables obeying Heisenberg equation $\{\psi^-\}$ is mathematically represented by Φ_+, the Hardy space of the upper complex energy plane of the second sheet of the $\mathcal{S}$-matrix:

$$\{\psi^-\} \equiv \Phi_+ \,. \tag{3.22b}$$

This Hardy space axiom means that the energy wave functions $\phi^+(E) = \langle^+ E\,|\phi^+\rangle$ and $\psi^-(E) = \langle^- E\,|\psi^-\rangle$ in the Dirac basis vector expansion (3.17a) and (3.17b) are not just functions of the Schwartz space, but that $\phi^+(E)$ can also be analytically continued into the lower complex energy plane and $\psi^-(E)$ can be analytically continued into the upper complex plane on second sheet of the $\mathcal{S}$-matrix.

In terms of the energy wave functions of the Dirac basis vector expansions (3.17a) and (3.17b), the Hardy space axioms (3.22a) and (3.22b) are also stated as:

$$\phi^+(E) = \langle^+ E\,|\phi^+\rangle \in (\mathcal{H}^2_- \cap S)\,|_{\mathbb{R}_+} \subset\; L^2(\mathbb{R}_+) \;\subset\; (\mathcal{H}^2_- \cap S)^\times\,|_{\mathbb{R}_+}\,, \tag{3.23a}$$

$$\psi^-(E) = \langle^- E|\psi^-\rangle \in (\mathcal{H}^2_+ \cap S)\,|_{\mathbb{R}_+} \subset\ L^2(\mathbb{R}_+)\ \subset\ (\mathcal{H}^2_+ \cap S)^\times\,|_{\mathbb{R}_+}),. \qquad (3.23b)$$

Here $(\mathcal{H}^2_\pm \cap S)_{\mathbb{R}_+}$ denotes the space of Hardy classes $\mathcal{H}^2_\pm$ intersected with the Schwartz space function S and then restricted to the positive semi-axis $\mathbb{R}_+$, where the cut of the $\mathcal{S}$-matrix is located as depicted in Figure 3.1. $L^2(\mathbb{R}_+)$ is the space of the Lebesgue square integrable functions of energy which is restricted to the positive value. Thus, the Hardy space axiom in the energy representation (3.23a)(3.23b) thus says that the energy wave functions are very well-behaved functions-Schwartz functions that can also be analytically continued into the complex energy plane second sheet of the $\mathcal{S}$-matrix.

The new Hardy space axioms (3.22a)(3.22b) conjectured from the phenomenological Lippmann-Schwinger equation (3.19), suggests that the energy wave functions of the prepared in-state $\phi^+(E)$ and the energy wave functions of the detected out-observable $\psi^-(E)$ are not only smooth, rapidly decreasing and infinitely differentiable functions on the real axis, as they would be under the Schwartz space axiom (2.23), but $\phi^+(E)$ and $\psi^-(E)$ are also analytic in the lower and upper complex energy semi-plane on the *second* sheet of the $\mathcal{S}$-matrix, where resonance poles of the $\mathcal{S}$-matrix are located. These functions are by axiom (3.23a),(3.23b) *postulated* to be *smooth* Hardy functions $\mathcal{H}^2_\mp \cap S$ of the lower and upper complex plane second sheet of the $\mathcal{S}$-matrix, restricted to the positive real axis. These energy wave functions of the in-state $\phi^+(E) = \langle^+ E|\phi^+\rangle$ and of the out-observable $\psi^-(E) = \langle^- E|\psi^-\rangle$ are elements of the spaces which are an intersection of two spaces: the Schwartz space S and the lower and upper Hardy class space $\mathcal{H}^2_\mp$, respectively:

$$\langle^+ E|\phi^+\rangle \in (\mathcal{H}^2_- \cap S)\,|_{\mathbb{R}_+} \quad \text{or} \quad \overline{\langle^+ E|\phi^+\rangle} = \langle\phi^+|E^+\rangle \in (\mathcal{H}^2_+ \cap S)\,|_{\mathbb{R}_+}\,, \qquad (3.24a)$$

$$\langle^- E|\psi^-\rangle \in (\mathcal{H}^2_+ \cap S)\,|_{\mathbb{R}_+} \quad \text{or} \quad \overline{\langle^- E|\psi^-\rangle} = \langle\psi^-|E^-\rangle \in (\mathcal{H}^2_- \cap S)\,|_{\mathbb{R}_+}\ . \qquad (3.24b)$$

The energy wave function conditions (3.24a)(3.24b) , which in the vector notation are the conditions (3.22a)(3.22b), constitute an axiom called the Hardy space axiom. Like the Hilbert space axiom (2.6a)(2.6b), these kinds of axioms can only be justified by its success with experimental data. The smooth Hardy space wave functions of (3.24a)(3.24b) posses many properties needed for the analytic $\mathcal{S}$-matrix and the phenomenological theory of resonances and decay.

The Hardy space condition (3.24) had been used by Lax-Phillips [24] for the analysis the scattering problems involving resonance in the scattering of acoustic waves off compact obstacles. The generalization of the Lax-Phillip scattering theory to the scattering theory has been developed in [25][26]. However, the boundary condition based on the Lax-Phillip scattering theory or associated theories is little difference from the Hardy space axiom developed by Bohm and his colleagues [27][28][29][30]. In Lax-Phillip scattering theory, the out-observable and in-state are the elements of two distinguished subspaces or below Hardy space $\mathcal{D}_-$ and upper Hardy space $\mathcal{D}_+$ of the Hilbert space $\mathcal{H}$, respectively. These Hardy spaces $\mathcal{D}_-$ and $\mathcal{D}_+$ are orthogonal to each other. In other words, all elements of two Hardy spaces are orthogonal each others in terms of Hilbert space scalar product. The family of the time evolution operators of elements of two Hardy spaces are formed semi-group of one continuous parameter t called Lax-Phillip semi-group, one is negative value in parameter t which is associated to the time evolution of the in-coming waves and another is positive value in t which is associated to the time evolution of the out-going waves. In the Bohm and colleagues' scattering theory, two Hardy spaces $\Phi_\pm$ representing the space of observables and in-states are overlap each other for some non-zero functions. In the mathematic language, it means that there exits a non-zero vector or associated non-zero energy wave function in the energy representation which belongs to the union space between two Hardy spaces.

3.4 Decaying States, Gamow Vector and Growing States in The Scattering Phenomenon

Working in the energy representation, there are two sets of the complete system of commutative observable (c.s.c.o) which are corresponding to the set of the eigenkets of the free-interaction Hamiltonian K, angular momentum J^2, its third component J_3, and the the particle label quantum number operator N, e.g, the species quantum number operator or the charge operator and the set of the eigenkets of the interaction Hamiltonian $H = K + V$, angular momentum J^2, its third component J_3, and the operator of the particle label quantum number N:

$$H\,,J^2\,,J_3\,,N \quad \text{and} \quad K\,,J^2\,,J_3\,,N\,. \tag{3.25}$$

These eigenkets, free-interaction kets $|E, j, j_3, \eta\rangle$ and exact kets $|E, j, j_3, \eta^\pm\rangle$, are related by the Lippmann-Schwinger equation (3.19). They are fulfilling the identity:

$$\langle^{-}E', j', j_3', \eta' | E, j, j_3, \eta^{+}\rangle = \langle E', j', j_3', \eta' |\ S\ | E, E, j, j_3, \eta\rangle\,, \tag{3.26}$$

where S is the S-operator defined as in (3.12). This matrix element (3.26) can be used as the normalization condition of the exact eigenkets in terms of the free-interaction eigentkets. Using energy and angular momentum conservation, we can further simplify (3.12) in term of reduced S-matrix elements $S_j^{\eta'\eta}(E)$:

$$\langle^{-}E', j', j_3', \eta' | E, j, j_3, \eta^{+}\rangle = \delta(E'-E)\delta_{j_3' j_3}\delta_{j' j} S_j^{\eta'\eta}(E)\,. \tag{3.27}$$

In general, from Hardy space boundary conditions (3.22) or in term of energy wave function conditions (3.23), any in-state vectors $\phi^{+} \in \Phi_{-}$ can generally be expanded, as similar to the Dirac basis expansion for any vectors $\phi \in \phi$ in Dirac's formalism, in terms of the discrete basis $|E_n, j, j_3, \eta)$ which describes the bounded states and the continuous basis $|E_n, j, j_3, \eta^{+}\rangle$ as

$$\phi^{+} = \sum_{n,j,j_3} |E_n, j, j_3, \eta)(E_n, j, j_3, \eta|\phi^{+}) + \sum_{j,j_3,\eta} \int_0^\infty dE |E, j, j_3, \eta^{+}\rangle\langle^{+}E, j, j_3, \eta|\phi^{+}\rangle\,, \tag{3.28a}$$

similarly, any the out-observable vector $\psi^{-} \in \Phi_{+}$ can be expanded as

$$\psi^{-} = \sum_{n,j,j_3,\eta} |E_n, j, j_3, \eta)(E_n, j, j_3, \eta|\psi^{-}) + \sum_{j,j_3,\eta} \int_0^\infty dE |E, j, j_3, \eta^{-}\rangle\langle^{-}E, j, j_3, \eta|\psi^{-}\rangle\,. \tag{3.28b}$$

The bounded states are usually be ignored when describing the resonances and decaying phenomena since the bounded states have no contribution in resonance or scattering process. If there are no bounded states, the Dirac's expansion for the in-state vectors $\phi^{+} \in \Phi_{-}$ and for the out-observable vector $\psi^{-} \in \Phi_{+}$ expressed in (3.28a) and (3.28b), respectively, can be reduced to:

$$\phi^{+} = \sum_{j,j_3,\eta} \int_0^\infty dE |E, j, j_3, \eta^{+}\rangle\langle^{+}E, j, j_3, \eta|\phi^{+}\rangle\,, \tag{3.29a}$$

$$\psi^{-} = \sum_{j,j_3,\eta} \int_0^\infty dE |E, j, j_3, \eta^{-}\rangle\langle^{-}E, j, j_3, \eta|\psi^{-}\rangle\,. \tag{3.29b}$$

In this dissertation, we consider the general scattering process (3.1) in which there are n resonances R_n are created in the scattering experiment.

The Born probability amplitude to detect an observable ψ^- in the in-state ϕ^+:

$$
\begin{aligned}
&(\psi^{\rm out}, S\,\phi^{\rm in}) = (\psi^- \mid \phi^+) = \\
&= \sum_{j,j_3}\sum_{j',j'_3}\int_0^\infty dE\,dE'\langle\psi^-\,|E',j',j'_3,\eta'^-\rangle\langle^-E',j',j'_3,\eta'\,|E,j,j_3,\eta^+\rangle\langle^+E,j,j_3,\eta\,|\phi^+\rangle \\
&= \sum_{j,j_3}\int_0^\infty dE\langle\psi^-\,|E,j,j_3,\eta^-\rangle S_j^{\eta'\eta}(E)\langle^+E,j,j_3,\eta\,|\phi^+\rangle\,.
\end{aligned}
\tag{3.30}
$$

The first expansion in (3.30) is obtained by inserting the Dirac basis expansions, (3.29a) for the in-state ϕ^+ and (3.29b) for the observable ψ^-. The second expansion in (3.30) is obtained by using the the identity (3.26) and the normalization (3.27) of the exact eigenkets and then integrating with respect to the energy variable E' and using the properties of Kronecker delta-δ with respect to angular momentums j' and its third angular momentum j'_3. The reduced S- matrix elements $S_j^{\eta'\eta}(E)$ depends on the total angular momentum j of the system in the central of mass and on the particle quantum numbers η, e.g, particle species label and channel number. In the scattering experiment, the prepared or in-states ψ^+ are usually independent on the spins as well as the observable. If there is no degeneracy in particle quantum number η, then the reduced S-matrix element relates to the the scattering amplitude by:

$$
S_j^{\eta\,\eta'}(E) = \begin{cases} 2ia_j(E) + 1 & \text{for elastic scattering } \;\eta' = \eta\,, \\ 2ia_j^{\eta}(E) & \text{for inelastic scattering from}\,\eta' \to \eta\,. \end{cases}
\tag{3.31}
$$

Here, $a_j(E)$ is the j-th partial scattering amplitude for the elastic scattering, $a_j^{\eta}(E)$ is the j-partial scattering amplitude for the inelastic scattering from channel η' to channel η, and the channel number η means the particle species and channel quantum number reflecting the characterization of the detectors which are designed to detect specific particle family. The reduced S-matrix element $S_j(E)$ with the angular momentum j is defined as an analytic function of energy on a Riemann energy surface with a cut along the real energy positive axis $0 \leq E < \infty$ as depicted in Figure 3.1.

In the scattering process, the S-operator reflects interaction between the particles in the the scattering process [31]. If there is no interaction, the S-operator is the identity operator $\boldsymbol{I}$. Then, the S-operator can be defined in term of the interaction operator T:

$$
S = \boldsymbol{I} + i\,T\,.
\tag{3.32}
$$

Here, the T operator reflects the interaction part of the scattering process and fulfills the Lippmann-Schwinger equation for the operator:

$$T = V + \frac{1}{E - K + i\epsilon} V T, \qquad \epsilon \longrightarrow 0\,. \tag{3.33}$$

Or the T-operator can explicitly be expressed in term of the free Hamiltonian K and interaction potential V:

$$\begin{aligned} T &= V\left(1 + \tfrac{1}{E-K+i\epsilon}V\right)^{-1}, \qquad \epsilon \longrightarrow 0 \\ &= V + V\tfrac{1}{E-K+i\epsilon}V + V\tfrac{1}{E-K+i\epsilon}V\tfrac{1}{E-K+i\epsilon}V + \cdots \end{aligned} \tag{3.34}$$

The scattering amplitude $a_j(E)$ expresses the conversation of energy and is related to the T-matrix by:

$$a_j(E) = -\pi i \delta(E - E')\langle E, j, j_3, \eta|\, T\, |E', j', j_3', \eta'\rangle \tag{3.35}$$

In the scattering process, the scattering amplitude $a_j(E)$ can be calculated as in (3.35) and hence the reduced S-matrix elements as in (3.31) unless we know about the interaction potential V.

In the S-matrix description of bounded state and resonance, a resonance with a spin j is defined by a first-order pole, at $z_R = E_R - i\Gamma/2$, of the j-th partial S-matrix element $S_j(E)$ on the lower complex energy semi-plane in the second sheet of the S-matrix on Riemann Surfaces, i.e, unphysical sheet. A bounded states is defined by a pole of the S-matrix on the negative energy axis in the first sheet of the S-matrix Riemann Surface, i.e, physical physics. Experimentally, a resonance is usually associated with a fixed value of total angular momentum j or pair of angular momentum j and the parity P, j^P. This means, in the partial wave analysis in which the experimental data is fitted by the differential cross section $\sigma_j(E)$, a resonance is identified by a Breit-Wigner amplitudes (3.2) in a partial wave amplitude $a_j(E)$. The value j for this partial wave amplitude is considered as the spin of the resonance. Therefore, a resonance is assigned to one partial wave with fixed value of angular momentum j in theoretical calculation and experimental analysis. Thus, the Born probability density (3.30) can now be written in sum over terms which are dependent on the fixed values of angular momentum j:

$$\begin{aligned} (\psi^-, \phi^+) &= \sum_{j, j_3} \int_0^\infty dE \langle \psi^- |E, j, j_3, \eta^-\rangle\; S_j^{\eta'\eta}(E)\; \langle^+ E, j, j_3, \eta|\phi^+\rangle \\ &= \sum_j (\psi^-, \phi^+)_j\,. \end{aligned} \tag{3.36}$$

The sum over the total angular momentum j in (3.36) means that we sum over all possible resonances with different values in the total angular momentum j created in the scattering process. The $(\psi^- \mid \phi^+)_j$ is the j-th partial probability density which is corresponding to the j-th partial S-matrix element $S_j(E)$ and also corresponds to the resonances:

$$(\psi^-, \phi^+)_j = \sum_{j_3} \int_0^\infty dE \langle \psi^- | E, j, j_3, \eta^- \rangle \; S_j^{\eta' \eta}(E) \; \langle ^+E, j, j_3, \eta | \phi^+ \rangle \, . \tag{3.37a}$$

The sum over the third component of the spin j_3 reflects all possibilities of the third component of the spin j_3 of fixed angular momentum j.

According to the standard analyticity assumptions of the j-th partial S-matrix, $S_j(E)$ is an analytic function of energy with fixed total angular momentum j on the sheets of the S-matrix excepts at some singularities on energy axis or at some singularities in both the upper and lower complex energy semi-planes in the energy representation. Furthermore, the S-matrix is either vanished and bounded or grows slower than a polynomials of any power n as $|E| \to \infty$.

There may be poles on the real axis for negative energy values $E \leq 0$ which are corresponding to the bounded states. Here, we just consider the resonance and scattering phenomena so we ignore such bound states if they exist in the system. The two sheeted Riemann surface has a cut along the real energy axis that starts at $E = 0$ as depicted in Figure 3.1. In order to reach the second sheet of the S-matrix, we burrow through the cut, $E \geq 0$. The contour of integration in (3.37a) is right above the cut which extends along the lower edge of the first sheet, $E + i\epsilon$, $\epsilon \to 0$. The j-th partial S-matrix in (3.37a) can be written as $S_j^{\eta' \eta}(E) = S_j^{\eta' \eta}(E + i\epsilon)$ since by the Lebesgue theorem, (3.37b) converges to (3.37a) as $\epsilon \to 0$. For the simplest in analysis of the S-matrix, we can drop the fixed angular momentum index j and the channel indexes $\eta' \eta$ in $S_j^{\eta' \eta}$ and then (3.37) can be rewritten as:

$$(\psi^-, \phi^+)_j = \sum_{j_3} \int_0^\infty dE \langle \psi^- | E, j, j_3, \eta^- \rangle \; S(E + i\epsilon) \; \langle ^+E, j, j_3, \eta | \phi^+ \rangle \, . \tag{3.37b}$$

The j-th partial S-matrix element $S_j(E)$ can have singularities in the second sheet. We generally consider the case that there are N first-order poles of the S-matrix which are corresponding to N resonances created in the scattering process.

The higher order-pole of the S-matrix element can be treated in a similar way and lead to higher order Gamow states. [32]

According to the Hardy space boundary conditions (3.24a) (3.24b), the energy wave function of the state $\phi^+(E) = \langle^+E|\phi^+\rangle$ and the conjugate energy wave function of the out-observable $\overline{\psi^-(E)} = \langle\psi^-|E^-\rangle$ are the elements of the space which is the intersection of the lower Hardy class and the Schwartz space of functions. Thus, the integrand in (3.37b) along the cut of the first sheet of the S-matrix can be analytically continued into the lower complex energy semi-plane in the second sheet of the S-matrix. The contour of integration in (3.37b) extends along the lower edge of the first sheet, right above the cut along $0 \leq E < \infty$. Since $S(E+i\epsilon) = S(E-i\epsilon_{II})$ as $\epsilon \to 0$, where $E - i\epsilon_{II}$ is on the second sheet and $E + i\epsilon$ is on the first sheet, we can as well extend the integration along the upper edge of the second sheet which is just below the cut. Therefore, the contour of integration parallel to the real axis in the second sheet over $0 \leq E < \infty$ in (3.37b) can now be deformed into the contour shown in Figure 3.2: an integral along the C_- from 0 to $-\infty_{II}$, then along the infinite semi-circle C_∞, along the L_1^i which is a path parallel to the imagine axis and connected infinite semicircle C_∞ with the the contour C_i around the poles z_{R_i}, the clockwise oriented circles C_i which contain the first-order poles at z_{R_i}, along the L_2^i which is a path parallel to the imagine axis and connected the contour C_i around the poles z_{R_i} with the infinite semicircle C_∞, and again along the infinite semicircle C_∞. The contour lines L_1 and L_2 can be chosen as just one line which connect the infinite semicircle C_∞ to the contour around the poles C_i but with opposite directions. Figure 3.2 show L_1^i and L_2^i are separated with zero-limit separation.

After this contour deformation, the integral in (3.37) becomes:

$$
\begin{aligned}
(\psi^-, \phi^+)_j &= \sum_{j_3} \int_0^{-\infty_{II}} dz \langle\psi^-|z, j, j_3, \eta^-\rangle \; S_{II}(z) \; \langle^+z, j, j_3, \eta|\phi^+\rangle \\
&+ \sum_{j_3} \int_{C_\infty} dz \langle\psi^-|z, j, j_3, \eta^-\rangle \; S_{II}(z) \; \langle^+z, j, j_3, \eta|\phi^+\rangle \\
&+ \sum_{j_3} \int_{L_1^i} dz \langle\psi^-|z, j, j_3, \eta^-\rangle \; S_{II}(z) \; \langle^+z, j, j_3, \eta|\phi^+\rangle \\
&+ \sum_{j_3} \int_{L_2^i} dz \langle\psi^-|z, j, j_3, \eta^-\rangle \; S_{II}(z) \; \langle^+z, j, j_3, \eta|\phi^+\rangle \\
&+ \sum_{j_3} \oint_{C_i} dz \langle\psi^-|z, j, j_3, \eta^-\rangle \; S_{II}(z) \; \langle^+z, j, j_3, \eta|\phi^+\rangle
\end{aligned}
\tag{3.38}
$$

Here, the II subscript in S_{II} denotes the S-matrix in the second sheet. We ignore

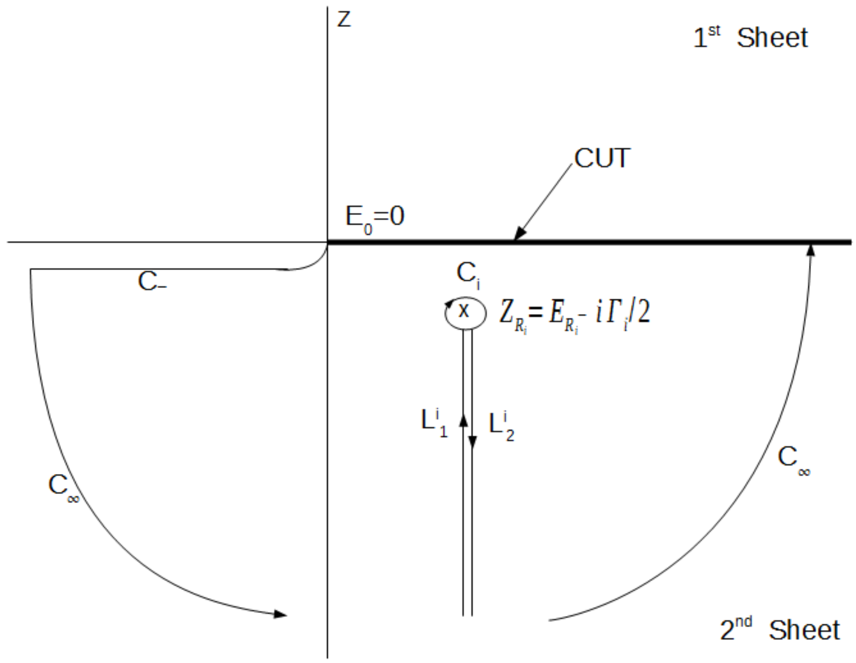

Figure 3.2: Contour of integration in the complex energy plane with first-order poles $z_{R_i} = E_{R_i} - i\Gamma_i/2$ located on the second sheet.

the total angular momentum j index and the particle quantum numbers $\eta\,\eta'$ in the partial S-matrix element $S_j^{\eta\,\eta'}(E)$ without affecting the dynamics of the system or the mathematical calculation in the theory since the resonance which is associated to the first-order pole in the second sheet is assigned with fixed total angular momentum j.

The S-matrix is well-defined on the whole Riemann surface and is bounded by a polynomial. The energy wave functions $\langle\psi^- \,|z, j, j_3, \eta^-\rangle$ and $\langle^+ z, j, j_3, \eta\,|\phi^+\rangle$ are analytic functions on lower half plane in second sheet based on the Hardy boundary condition (3.24). Thus, the integrand $\langle\psi^- \,|z, j, j_3, \eta^-\rangle\; S_{II}(z)\; \langle^+ z, j, j_3, \eta\,|\phi^+\rangle$ is an analytic and well defined function on lower half plane in second sheet except for the singularities. Therefore, The integral around the the infinite semicircle C_∞ vanishes. Each pair of integrals along L_1^i and L_2^i, which connect the infinite semicircle C_∞ to the contour around each pole C_i, cancels each other since these integrals have the same integrand which are integrated along the same line but in opposite directions. The first integral in the contour deformation (3.38) is along the axis which is just below the negative real energy axis in the second sheet and does not involve in any poles which are corresponding to resonances. This is non-resonant term and hence expressed as j-partial background term, BG_j . Since it integration paths is along the axis which is just below the negative real energy axis, it is same as a integral of the real variable E in limit $\epsilon \to 0$:

$$\begin{aligned} BG_j &\equiv \langle \psi^- | \phi^{BG} \rangle \\ &= \sum_{j_3} \int_0^{-\infty_{II}} dz \langle \psi^- | z, j, j_3, \eta^- \rangle \; S_{II}(z) \; \langle ^+z, j, j_3, \eta | \phi^+ \rangle \\ &= \sum_{j_3} \int_0^{-\infty_{II}} dE \langle \psi^- | E, j, j_3, \eta^- \rangle \; S_{II}(E) \; \langle ^+E, j, j_3, \eta | \phi^+ \rangle \,. \end{aligned} \tag{3.39}$$

This expresses the interaction between the observable ψ^- with the j-partial generalized background vector ϕ^{BG} which is defined by

$$\phi_j^{BG} = \sum_{j_3} \int_0^{-\infty_{II}} dE \; |E, j, j_3, \eta^- \rangle \; S_{II}(E) \; \langle ^+E, j, j_3, \eta | \phi^+ \rangle \,. \tag{3.40}$$

If there are N resonances in the scattering process (3.1) which are corresponding to N first-order poles in the lower half plane of the second sheet of the S-matrix and there are no higher order poles in the lower half plane in the second sheet than the first-order pole, then we can expand the reduced S-matrix element $S_j(E)$ around first-order poles at complex energies $z_{R_i} = E_i - i\Gamma_i/2$, $i = 1, 2, \cdots N$ by using the Laurent expansion:

$$S_{II}(z) = \frac{r^i_{-1}}{z - z_{R_i}} + + r^i_0 + r^i_1 (z - z_{R_i}) + \cdots \tag{3.41}$$

Here, the index i is assumed to be summed over a range $\{1 \cdots N\}$ in the expansion (3.38) if there are N resonances created through the collision. The r^i_{-1} are the residues of the $S_{II}(z)$ in the second sheet at the first-order poles $z_{R_i} = E_i - i\Gamma_i/2$. The r^i_0, $r^i_1, \cdots$ are the constants corresponding to the positive powers of polynomial around the poles z_{R_i} Laurent expansion.

The last integral in (3.38) around the circles C_i can then be performed around the poles z_{R_i} using the expansion (3.41). This integral just involves the first-order poles or singularities of the S-matrix so we call it the pole term:

$$\begin{aligned} (\psi^-, \phi^+)_{poles} &= \sum_{j_3} \oint_{C_i} dz \langle \psi^- | z, j, j_3, \eta^- \rangle \; S_{II}(z) \; \langle ^+z, j, j_3, \eta | \phi^+ \rangle \\ &= \sum_{j_3} \oint_{C_i} dz \langle \psi^- | z, j, j_3, \eta^- \rangle \; \tfrac{r^i_{-1}}{z - z_{R_i}} \; \langle ^+z, j, j_3, \eta | \phi^+ \rangle \,. \end{aligned} \tag{3.42}$$

The second integral in (3.42) is understood that there are i integrals around i first-order poles z_{R_i}, $i = 1,\ 2 \cdots N$. Each integral around pole z_{R_i} is then the residue of the integrand in second integral of (3.42) at $z_{R_i} = E_i - i\Gamma_i/2$:

$$\begin{aligned}(\psi^-, \phi^+)_{poles} &= \sum_{j_3} \oint_{C_i} dz \langle \psi^- | z, j, j_3, \eta^- \rangle \, \tfrac{r^i_{-1}}{z - z_{R_i}} \, \langle^+ z, j, j_3, \eta | \phi^+ \rangle \\ &= -2\pi i \, r^i_{-1} \sum_{j_3} \langle \psi^- | z_{R_i}, j, j_3, \eta^- \rangle \ \langle^+ z_{R_i}, j, j_3, \eta | \phi^+ \rangle \,. \end{aligned} \tag{3.43}$$

The negative sign before the coefficients in the residue in (3.43) represents that the integral is calculated in clock-wise while the resides is defined as the integral in counter-clockwise around the pole.

The integral around each pole z_{R_i} in (3.42) can be analyzed in different way using the Cauchy theorem in which the each contour of integration C_i around the pole z_{R_i} in clockwise direction is deformed into path along the real axis from $-\infty_{II} < E < \infty$ and along the infinite semicircle which vanishes because of the Hardy class property:

$$\begin{aligned}(\psi^-, \phi^+)_{poles} &= \sum_{j_3} \oint_{C_i} dz \langle \psi^- | z, j, j_3, \eta^- \rangle \, \tfrac{r^i_{-1}}{z - z_{R_i}} \, \langle^+ z, j, j_3, \eta | \phi^+ \rangle \\ &= \sum_{j_3} \int_{-\infty_{II}}^{\infty} dE \langle \psi^- | E, j, j_3, \eta^- \rangle \, \tfrac{r^i_{-1}}{E - z_{R_i}} \, \langle^+ E, j, j_3, \eta | \phi^+ \rangle \,. \end{aligned} \tag{3.44}$$

The integral (3.44) extends along the line which is parallel to the negative real axis in the second sheet $\infty_{II} < E \leq 0$ to the positive axis $0 \leq E < \infty$ which can be in either first or second sheet. However, the major contribution to the integral (3.44) comes from integration along the physical values $0 \leq E < \infty$. The distribution $\frac{r^i_{-1}}{E - z_{R_i}}$ is same as the Breit-Wigner energy distribution [33] but the difference is: the range of the energy is from whole line of energy axis $-\infty_{II} < E \leq \infty$ than from just half positive energy axis $0 \leq E < \infty$ as in the standard Breit-Wigner distribution. We call this energy distribution is generalized or exact Breit-Wigner energy distribution:

$$a^{BW}_j = \frac{r_{-1}}{E - z_R} \quad \text{for} \quad -\infty_{II} < E < \infty \,. \tag{3.45}$$

Therefore, the pole terms $(\psi^-, \phi^+)_{poles}$ can be calculated into two methods: one is the sum of residues of the integrand $\langle \psi^- | z, j, j_3, \eta^- \rangle \langle^+ z, j, j_3, \eta | \phi^+ \rangle$ around each pole at z_{R_i}; another is sum of integrals of the same integrand but with the real variable $\langle \psi^- | E, j, j_3, \eta^- \rangle \langle^+ E, j, j_3, \eta | \phi^+ \rangle$ along real axis from $-\infty_{II} < E < \infty$ with the exact Breit-Wigner energy distribution (3.45) for each resonance corresponding to each pole at z_{R_i}.

From the equations (3.43) and (3.44), we have the equality:

$$-2\pi i \, \langle \psi^- | z_{R_i}, j, j_3, \eta^- \rangle \langle^+ z_{R_i}, j, j_3, \eta | \phi^+ \rangle = \int_{-\infty_{II}}^{\infty} dE \, \frac{\langle \psi^- | E, j, j_3, \eta^- \rangle \langle^+ E, j, j_3, \eta | \phi^+ \rangle}{E - z_{R_i}} \,. \tag{3.46}$$

This equality (3.46) can also be justified by using the Titchmarsh theorem for Hardy class functions $\langle \psi^- | z_{R_i}, j, j_3, \eta^- \rangle \langle^+ z_{R_i}, j, j_3, \eta | \phi^+ \rangle \in \mathcal{H}^2_- \cap S$. From the equality(3.46), each resonance corresponding to each pole at $z_{R_i} = E_{R_i} - i\Gamma_i/2$ can be associated with the exact Breit-Wigner energy distribution (3.45). The equality(3.46) can be rewritten as:

$$-\, 2\pi i \, \langle \psi^- | z_{R_i}, j, j_3, \eta^- \rangle = \int_{-\infty_{II}}^{\infty} dE \, \frac{\langle \psi^- | E, j, j_3, \eta^- \rangle}{E - z_{R_i}} \frac{\langle^+ E, j, j_3, \eta | \phi^+ \rangle}{\langle^+ z_{R_i}, j, j_3, \eta | \phi^+ \rangle} \,. \tag{3.47}$$

The integral on the l.h.s of (3.47) can be further analyzed by using the Cauchy theorem to deform the contour of integration along $-\infty_{II} < E < \infty$ into the integration around the pole s_{R_i}. By this deformation, we can get rid of the $\frac{\langle^+ E, j, j_3, \eta | \phi^+ \rangle}{\langle^+ z_{R_i}, j, j_3, \eta | \phi^+ \rangle}$ in the integrand of (3.47) and rewrite (3.47) in the simpler form:

$$\langle \psi^- | z_{R_i}, j, j_3, \eta^- \rangle = -\frac{1}{2\pi i} \int_{-\infty_{II}}^{\infty} dE \, \frac{\langle \psi^- | E, j, j_3, \eta^- \rangle}{E - z_{R_i}} \,. \tag{3.48}$$

Since the vector $\psi^- \in \Phi_+$ is arbitrary, we can omit in the (3.48) and get a ket $|z_{R_i}, j, j_3, \eta^- \rangle$ which is associate with the exact Breit-Wigner energy distribution as in (3.45):

$$|z_{R_i}, j, j_3, \eta^- \rangle = -\frac{1}{2\pi i} \int_{-\infty_{II}}^{\infty} dE \, \frac{|E, j, j_3, \eta^- \rangle}{E - z_{R_i}} \,. \tag{3.49}$$

We define non-relativistic Gamow ket or decaying Gamow ket ψ^G associated with each first-order pole of the S-matrix and hence each resonance by:

$$\psi^G_i = \sqrt{2\pi\Gamma} \, |z_{R_i}, j, j_3, \eta^- \rangle = i\sqrt{\frac{\Gamma}{2\pi}} \int_{-\infty_{II}}^{\infty} dE \, \frac{|E, j, j_3, \eta^- \rangle}{E - z_{R_i}} \,. \tag{3.50}$$

This means that each Gamow ket (3.50) associated with a resonance is a continuous linear superposition of the exact out-plane wave or Lippmann-Schwinger kets

$|E,j,j_3,\eta^-\rangle$ with the exact Breit-Wigner energy distribution (3.24). Each Gamow ket in (3.50) are defined as the functional on the Hardy space $\psi^- \in \Phi_+^\times$ as same as the Lippmann-Schwinger kets are continuously anti-linear functionals on the Hardy space Φ_+, $|E,j,j_3,\eta^-\rangle \in \Phi_+^\times$. The coefficient $\sqrt{2\pi\Gamma}$ in (3.50) is conventionally used in order that the Gamma ket ψ^G can be normalized.

The j-th partial Born probability (3.37) can then be expanded into two terms, one is background term (3.39) which is not related to any poles or resonance and another is related to poles which are represented by Gamow kets ψ_i^G or equivalent ket $|z_{R_i},j,j_3,\eta^-\rangle$ as in (3.50):

$$(\psi^-,\phi^+)_j = BG_{\ j} - 2\pi i \sum_{i=1}^{N} r_{-1}^i \sum_{j_3} \langle\psi^-|z_{R_i},j,j_3,\eta^-\rangle\ \langle^+ z_{R_i},j,j_3,\eta|\phi^+\rangle\,, \quad (3.51a)$$

or in terms of the integral of the integrand with the exact Breit-Wigner energy distribution:

$$(\psi^-,\phi^+)_j = BG_{\ j} + \sum_{j_3} \int_{-\infty_{II}}^{\infty} dE\langle\psi^-|E,j,j_3,\eta^-\rangle\ \frac{r_{-1}^i}{E-z_{R_i}}\ \langle^+E,j,j_3,\eta|\phi^+\rangle\,. \quad (3.51b)$$

According to hermitian analyticity property of the S-matrix, if there is a first-order pole at the complex position $z_R = E_R + i\Gamma/2$ then there must also be another first-order pole at the complex conjugate position, $z_R^* = E_R + i\Gamma/2$, on the second sheet reached by burrowing through the cut, $0 \le E < \infty$ on the physical sheet. Thus, a scattering resonance is defined by a pair of poles on the second sheet of the analytically continued S-matrix located at positions that are complex conjugates of each other. Therefore, associated with the complex conjugate positions at $z_{R_i}^* = E_{R_i} + i\Gamma_i/2$, we consider the complex conjugate of (3.26) which represents j-th partial Born probability to detect the in-state ϕ^+ in the out observable ψ^-:

$$\begin{aligned}(\psi^-,\phi^+)_j^* = (\phi^+,\psi^-)_j \ &= \sum_{j_3}\int_0^\infty dE\langle\phi^+|E,j,j_3,\eta^+\rangle\ S^*(E+i\epsilon)\ \langle^-E,j,j_3,\eta|\psi^-\rangle \\ &= \sum_{j_3}\int_0^\infty dE\langle\phi^+|E,j,j_3,\eta^+\rangle\ S(E-i\epsilon)\ \langle^-E,j,j_3,\eta|\psi^-\rangle\,.\end{aligned} \quad (3.52)$$

The second equality in (3.52) is obtained from the hermitian property of the S-matrix, $S^*(E+i\epsilon) = S(E-i\epsilon)$. The energy wave functions $\langle E,j,j_3,\eta^-|\psi^-\rangle$ and $\langle\phi^+|^+E,j,j_3,\eta\rangle$ are analytic function on the upper half complex energy plane in the second sheet $\mathcal{H}^2_- \cap S$. The matrix $S(E - i\epsilon)$ on the first sheet possesses the same continuously analytic property as the matrix $S(E + i\epsilon_{II})$ on the second sheet in the limit $\epsilon \to 0$. This $S(E + i\epsilon_{II}) \equiv S_{II}(E)$ matrix is continuously analytic function on the whole second sheet except for some singularities. Thus, we can deform the contour of integration in (3.51) into the upper half-plane in the second sheet in a similar way as we did before for the lower half-plane, as depicted in Figure 3.3: an integral along the C_+ from 0 to $-\infty_{II}$ and the counter-clockwise oriented circles C_i^* which contain the first-order poles at $z^*_{R_i}$ in the first sheet. The integral along the infinite semicircle C^*_∞ and the two opposite direction line integral $L_1'^i$ and $L_2'^i$ which vanish because of the analyticity of the energy wave functions and the S-matrix:

$$\begin{aligned}(\phi^+, \psi^-)_j \;&= \sum_{j_3} \int_0^{-\infty_{II}} dz \langle \phi^+ | z, j, j_3, \eta^+ \rangle \; S_{II}(z) \; \langle ^- z, j, j_3, \eta | \psi^- \rangle \\ &+ \sum_{j_3} \oint_{C_i^*} dz \langle \phi^+ | E, j, j_3, \eta^+ \rangle \; S_{II}(z) \; \langle ^- E, j, j_3, \eta | \psi^- \rangle \,.\end{aligned} \tag{3.53}$$

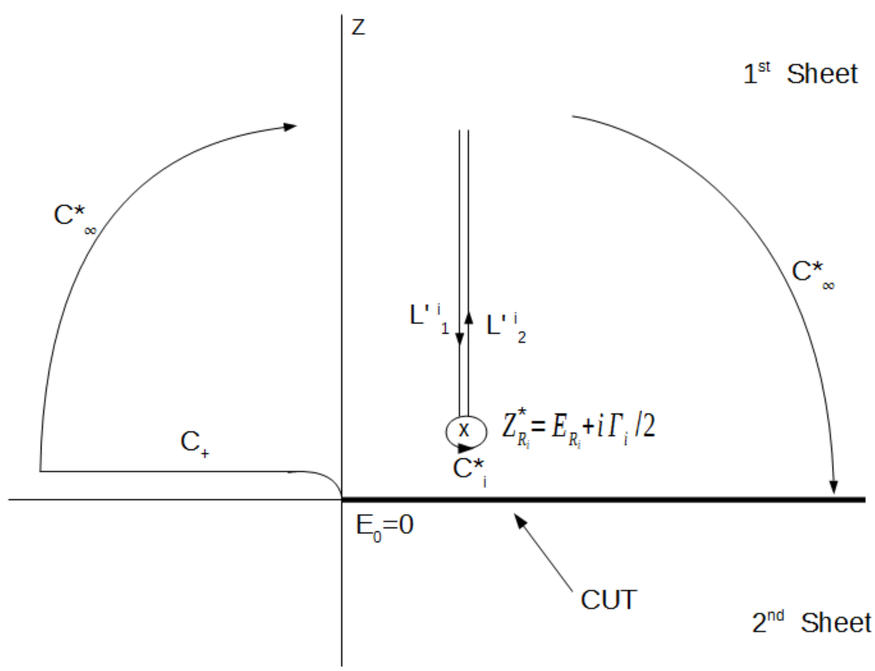

Figure 3.3: Contour of integration in the complex energy plane with first-order poles $z^*_{R_i} = E_{R_i} + i\Gamma_i/2$ located on the first sheet.

Again, the first integral in the contour deformation (3.53) is along the axis which is just above the negative real energy axis in the second sheet and does not involve in any poles which are corresponding to anti-resonances, which obey the growing exponential law. This is non anti-resonant term and hence expressed as j-partial background term, BG_j^* which is conjugate of the background term (3.39):

$$\begin{aligned} BG_j^* &\equiv \langle\phi^+|\phi^{BG*}\rangle = \overline{\langle\psi^-|\phi^{BG}\rangle} \\ &= \sum_{j_3}\int_0^{-\infty_{II}} dz\langle\phi^+|z,j,j_3,\eta^+\rangle\, S_{II}(z)\,\langle^- z,j,j_3,\eta|\psi^-\rangle \\ &= \sum_{j_3}\int_0^{-\infty_{II}} dE\langle\phi^+|E,j,j_3,\eta^+\rangle\, S_{II}(z)\,\langle^- E,j,j_3,\eta|\psi^-\rangle\,. \end{aligned} \tag{3.54}$$

This expresses the interaction between the in-state ϕ^+ with the generalized background vector ϕ^{BG*} which is defined by:

$$\phi_j^{BG*} = \sum_{j_3}\int_0^{-\infty_{II}} dE\,|E,j,j_3,\eta^+\rangle\, S_{II}(z)\,\langle^- E,j,j_3,\eta|\psi^-\rangle\,. \tag{3.55}$$

Corresponding to N first-order poles at $z_{R_i} = E_{R_i} - i\Gamma_i/2$, $i = 1, 2, \cdots N$ on the lower half plane of the second sheet of the S-matrix, there are N first-order poles at the complex conjugate position, $z^*_{R_i} = E_{R_i} + i\Gamma_i/2$, on the upper half plane of second sheet. Therefore, the the reduced S-matrix element $S_{II}(E)$ can be expanded around these poles using the Laurent expansion:

$$S_{II}(z) = \frac{r'^i_{-1}}{z - z^*_{R_i}} + +r'^i_0 + r'^i_1(z - z^*_{R_i}) + \cdots \tag{3.56}$$

Here, the r'^i_{-1} are the residues of the $S_{II}(z)$ on the upper half plane of the second sheet at the first-order poles $z^*_{R_i} = E_i + i\Gamma_i/2$. The r'^i_0, $r'^i_1, \cdots$ are the constants corresponding to the positive powers of polynomial around the poles $z^*_{R_i}$ in Laurent expansion.

With same analysis for second integral in (3.53) on the upper half plane of the second sheet as we did above on the lower half plane of the second sheet, we can express the integral around the poles at $z^*_{R_i} = E_{R_i} + i\Gamma_i/2$ as the pole terms $(\phi^+, \psi^-)_{poles}$ which can further be either calculated as sum of the residues of the integrand in the last integral of (3.53) at $z^*_{R_i}$ or deformed into the integral along the real axis from

$-\infty_{II} < E < \infty$:

$$\begin{aligned}(\phi^+,\psi^-)_{pole} &= \sum_{j_3}\oint_{C_i} dz\langle\phi^+|z,j,j_3,\eta^+\rangle \tfrac{r'^i_{-1}}{z-z^*_{R_i}} \langle\phi^+|z,j,j_3,\eta^+\rangle \\ &= 2\pi i\, r'^i_{-1} \sum_{j_3}\langle\phi^+|z_{R_i},j,j_3,\eta^+\rangle \ \langle^- z_{R_i},j,j_3,\eta|\psi^-\rangle \\ &= -\sum_{j_3}\int_{-\infty_{II}}^{\infty} dE\langle\psi^-|E,j,j_3,\eta^-\rangle \tfrac{r'^i_{-1}}{E-z^*_{R_i}} \langle^+E,j,j_3,\eta|\phi^+\rangle\,.\end{aligned} \tag{3.57}$$

The last integral in (3.57) also extends along the line which is parallel to the negative real axis in the second sheet $\infty_{II} < E \leq 0$ to the positive axis $0 \leq E < \infty$. The distribution $\frac{r'^i_{-1}}{E-z^*_{R_i}}$ is exact Breit-Wigner energy distribution as same as (3.45) but corresponding to poles at $z^*_{R_i}$:

$$a'^{BW}_j = \frac{r'^i_{-1}}{E-z^*_{R_i}} \quad \text{for} \quad -\infty_{II} < E < \infty\,. \tag{3.58}$$

From the equality between the last two terms in (3.57), we can define a ket $|\, z^*_{R_i},j,j_3,\eta^-\rangle$ which is associate with the exact Breit-Wigner energy distribution (3.58):

$$|z^*_{R_i},j,j_3,\eta^+\rangle = -\frac{1}{2\pi i}\int_{-\infty_{II}}^{\infty} dE\ \frac{|E,j,j_3,\eta^+\rangle}{E-z^*_{R_i}}\,. \tag{3.59}$$

Now, we can define non-relativistic growing Gamow kets as the way we defined the Gamow kets, by:

$$\psi^{GR} = \sqrt{2\pi\Gamma}\ |z^*_{R_i},j,j_3,\eta^+\rangle = i\sqrt{\frac{\Gamma}{2\pi}}\int_{-\infty_{II}}^{\infty} dE\ \frac{|E,j,j_3,\eta^+\rangle}{E-z^*_{R_i}}\,. \tag{3.60}$$

This means that the growing Gamow kets (3.60) associated with a anti-resonance is a continuous linear superposition of the exact out-plane wave or Lippmann-Schwinger kets $|E,j,j_3,\eta^+\rangle$ with the exact Breit-Wigner energy distribution (3.58). Each growing ket in (3.60) is defined as a continuously anti-linear functional on the Hardy space $\Phi^\times_-$.

The j-th partial Born probability (3.51) can then expand into two terms, one is background term (3.54) which is not related to any poles or anti-resonance and another is related to poles which are represented by growing vectors ψ^{GR} or equivalent ket $|z_{R_i},j,j_3,\eta^+\rangle$ as in (3.60):

$$(\phi^+,\psi^-)_j = BG^*_j + 2\pi i\sum_{i=1}^{N} r^i_{-1} \ \sum_{j_3}\langle\phi^+|E,j,j_3,\eta^+\rangle \ \langle^-E,j,j_3,\eta|\psi^-\rangle\,, \tag{3.61a}$$

or in terms of the integral of the integrand with the exact Breit-Wigner energy distribution:

$$(\phi^+,\psi^-)_j = BG_j^* - \sum_{j_3} \int_{-\infty_{II}}^{\infty} dE\langle\phi^+|E,j,j_3,\eta^+\rangle \, \frac{r'^{\,i}_{-1}}{E-z^*_{R_i}} \, \langle^-E,j,j_3,\eta|\psi^-\rangle \,. \quad (3.61b)$$

The Born probability (3.36) to detect the observable ψ^- in the state ϕ^+ can be expressed in terms of the background or non-exponential term and the resonances which are associated to the first-order poles of the S-matrix on the lower half plane of the second sheet:

$$\begin{aligned}(\psi^-,\phi^+) &= \sum_j BG_j - 2\pi i \sum_{i=1}^{N} \, r^i_{-1} \, \sum_{j,j_3} \langle\psi^-|z_{R_i},j,j_3,\eta^-\rangle \;\; \langle^+z_{R_i},j,j_3,\eta|\phi^+\rangle \\ &= BG - 2\pi i \sum_{i=1}^{N} \, r^i_{-1} \, \sum_{j,j_3} \langle\psi^-|z_{R_i},j,j_3,\eta^-\rangle \;\; \langle^+z_{R_i},j,j_3,\eta|\phi^+\rangle \,,\end{aligned} \quad (3.62)$$

Here, the *background* term, BG, is the sum of j-partial background (3.39) over the spin j:

$$\begin{aligned} BG &= \sum_j BG_j = \sum_j \langle\psi^-|\phi_j^{BG}\rangle \\ &\equiv \sum_{j\, j_3} \int_0^{-\infty_{II}} dE\langle\psi^-|E,j,j_3,\eta^-\rangle \; S_{II}(E) \; \langle^+E,j,j_3,\eta|\phi^+\rangle \\ &= \langle\psi^-|\phi^{BG}\rangle \,. \end{aligned} \quad (3.63)$$

The background term (3.63) can be explained as the interaction between the observable ψ^- with the generalized background vector ϕ^{BG} defined by sum of the j-partial background vector ϕ_j^{BG} (3.39) over the spin j:

$$\phi^{BG} = \sum_j \phi_j^{BG} = \sum_{j\,,j_3} \int_0^{-\infty_{II}} dE \; |E,j,j_3,\eta^-\rangle \; S_{II}(E) \; \langle^+E,j,j_3,\eta|\phi^+\rangle \,. \quad (3.64)$$

Similarly, the complex conjugate of (ψ^-,ϕ^+) can be expressed as

$$\begin{aligned}(\phi^+,\psi^-) &= \sum_j BG_j^* + 2\pi i \sum_{i=1}^{N} \, r^i_{-1} \, \sum_{j\,,j_3} \langle\phi^+|E,j,j_3,\eta^+\rangle \;\; \langle^-E,j,j_3,\eta|\psi^-\rangle \\ &= BG^* + 2\pi i \sum_{i=1}^{N} \, r^i_{-1} \, \sum_{j\,,j_3} \langle\phi^+|E,j,j_3,\eta^+\rangle \;\; \langle^-E,j,j_3,\eta|\psi^-\rangle \,,\end{aligned} \quad (3.65)$$

Where, the *background* * term, BG^*. is the complex conjugate of *background* term, BG in (3.63), explicitly,

$$\begin{aligned} BG^* &= \sum_j BG_j^* = \sum_j \langle \phi^+ | \phi_j^{BG\,*} \rangle \\ &\equiv \sum_{j\,,j_3} \int_0^{-\infty_{II}} dE \langle \phi^+ | E, j, j_3, \eta^+ \rangle \; S_{II}(z) \; \langle ^-E, j, j_3, \eta | \psi^- \rangle \\ &= \langle \phi^+ | \phi^{BG^*} \rangle \, . \end{aligned} \tag{3.66}$$

This expresses the interaction between the in-state ϕ^+ with the generalized background vector ϕ^{BG^*} which is defined by:

$$\phi^{BG^*} = \sum_{j,j_3} \int_0^{-\infty_{II}} dE \, |E, j, j_3, \eta^+ \rangle \; S_{II}(z) \; \langle ^-E, j, j_3, \eta | \psi^- \rangle \, . \tag{3.67}$$

If we omit the arbitrary vectors $\psi^- \in \Phi_+$ in (3.62), we get the expansion of the state ϕ^+ in terms of the background kets ϕ^{BG} and the Gamow kets or equivalent $|E, j, j_3, \eta^- \rangle$:

$$\begin{aligned} \phi^+ &= \sum_{j\,,j_3} \int_0^{-\infty_{II}} dE \; |E, j, j_3, \eta^- \rangle \; S_{II}(E) \; \langle ^+E, j, j_3, \eta | \phi^+ \rangle \\ &\qquad\qquad - 2\pi i \sum_{i=1}^{N} r_{-1}^i \sum_{j,j_3} |z_{R_i}, j, j_3, \eta^- \rangle \;\; \langle ^+z_{R_i}, j, j_3, \eta | \phi^+ \rangle \\ &= \phi^{BG} - 2\pi i \sum_{i=1}^{N} r_{-1}^i \sum_{j,j_3} |z_{R_i}, j, j_3, \eta^- \rangle \;\; \langle ^+z_{R_i}, j, j_3, \eta | \phi^+ \rangle \, . \end{aligned} \tag{3.68}$$

It means that the in-state ϕ^+ can be expanded in the kets $|z_{R_i}, j, j_3, \eta^- \rangle$ which are associated to the resonances plus the extra *background* term ϕ^{BG} which is not related to the resonances and maybe describes the non-exponential decay. This expansion (3.68) is similar with Dirac's basis expansion in the Hilbert or Schwartz space but for the Hardy space in the energy representation. This expansion (3.68) is not true for other representations, e.g, position representation.

Similarly, if we omit the arbitrary vector $\phi^+ \in \Phi_-$ in (3.65), we get the expansion of the observable ψ^- in term of growing Gamow kets $|z_{R_i}^*, j, j_3, \eta^+ \rangle$ and an extra term *background* ϕ^{BG*} :

$$
\begin{aligned}
\psi^- &= \sum_{j,j_3} \int_0^{-\infty_{II}} dE\, |E,j,j_3,\eta^+\rangle\, S_{II}(E)\, \langle^- E,j,j_3,\eta|\psi^-\rangle \\
&\qquad +2\pi i \sum_{i=1}^{N} r_{-1}^i \sum_{j\,,j_3} |z_{R_i}^*,j,j_3,\eta^+\rangle\ \langle^- z_{R_i}^*,j,j_3,\eta|\psi^-\rangle \\
&= \phi^{BG*} + 2\pi i \sum_{i=1}^{N} r_{-1}^i \sum_{j\,,j_3} |z_{R_i}^*,j,j_3,\eta^+\rangle\ \langle^- z_{R_i}^*,j,j_3,\eta|\psi^-\rangle\,.
\end{aligned} \tag{3.69}
$$

The ket $|z_{R_i}^*,j,j_3,\eta^+\rangle$ and $|z_{R_i},j,j_3,\eta^-\rangle$ are defined as continuously anti-linear functionals on the space Φ_- and Φ_+, respectively. Following (3.68) and (3.69), it shows that ϕ^+ and ψ^- are also continuously anti-linear functionals of functional spaces $\Phi_-^\times$ and $\Phi_+^\times$ on the Hardy spaces Φ_- and Φ_+, respectively.

The $|E,j,j_3,\eta^\pm\rangle$ are the generalized eigenkets of the total Hamiltonian H with the eigenvalue E to the power p, for $p=0,1,\cdots$. They fulfill

$$
H^{\times p}\,|E,j,j_3,\eta^\pm\rangle = E^p\,|E,j,j_3,\eta^\pm\rangle \quad \text{for every } \ |E,j,j_3,\eta^\pm\rangle \in \Phi_\mp^\times\,. \tag{3.70}
$$

These generalized eigenkets $|E,j,j_3,\eta^\pm\rangle$ can also obtained from the eigenkets of the free-interaction Hamiltonian K using the Lippmann-Schwinger equation (3.19).

From Hardy space boundary condition (3.24) and the analyticity of the S-matrix element, $S_j(E)$, it shows $\langle\psi^-|z_R,j,j_3,\eta^-\rangle \in (\mathcal{H}_-^2\cap S)\,|_{\mathbb{R}_+}$ and $\langle\phi^+|z_R^*,j,j_3,\eta^+\rangle \in (\mathcal{H}_+^2\cap S)\,|_{\mathbb{R}_+}$. It means that the $\langle\psi^-|z_R,j,j_3,\eta^-\rangle$ are the Hardy class function on the lower half plane of the second sheet. Since Hamiltonian H is the continuous operator on Φ_-, using (3.48) with replacement $H^p\psi^-$ of ψ^-:

$$
\begin{aligned}
\langle H^p\psi^-|\psi^G\rangle &= \sqrt{2\pi\Gamma}\langle H^p\psi^-|z_R,j,j_3,\eta^-\rangle = \sqrt{2\pi\Gamma}\langle\psi^-|\ H^{\times p}\ |z_R,j,j_3,\eta^-\rangle \\
&= -\tfrac{\sqrt{2\pi\Gamma}}{2\pi i}\int_{-\infty_{II}}^{\infty} dE\ \tfrac{\langle H^p\psi^-|E,j,j_3,\eta^-\rangle}{E-z_{R_i}} \\
&= -\tfrac{\sqrt{2\pi\Gamma}}{2\pi i}\int_{-\infty_{II}}^{\infty} dE\ \tfrac{\langle\psi^-|H^{\times p}|E,j,j_3,\eta^-\rangle}{E-z_{R_i}} \\
&= -\tfrac{\sqrt{2\pi\Gamma}}{2\pi i}\int_{-\infty_{II}}^{\infty} dE\ E^p\tfrac{\langle\psi^-|E,j,j_3,\eta^-\rangle}{E-z_{R_i}} \\
&= z_R^p\langle\psi^-|\psi^G\rangle \qquad\qquad \text{for all } \psi^-\in\Phi_+\,.
\end{aligned} \tag{3.71}
$$

Hence, omitting the arbitrary vector $\psi^-\in\Phi_+$, we get

$$
H^{\times p}\ |\psi^G\rangle = z_R^p\ |\psi^G\rangle = (E_R - i\Gamma_R/2)^p\ |\psi^G\rangle\,, \tag{3.72a}
$$

or equivalently,

$$H^{\times p}\,|z_R, j, j_3, \eta^-\rangle = z_R^p\,|z_R, j, j_3, \eta^-\rangle = (E_R - i\Gamma_R/2)^p\,|z_R, j, j_3, \eta^-\rangle\,. \tag{3.72b}$$

Similarly, since the $\langle\phi^+|z_R, j, j_3, \eta^-\rangle$ are the Hardy class function on the upper half plane of the second sheet, we get:

$$H^{\times p}\,|\psi^{GR}\rangle = z_R^p\,|\psi^{GR}\rangle = (E_R + i\Gamma_R/2)^p\,|\psi^{GR}\rangle\,, \tag{3.73a}$$

or equivalently

$$H^{\times p}\,|z_R^*, j, j_3, \eta^+\rangle = z_R^{*p}\,|z_R^*, j, j_3, \eta^+\rangle = (E_R + i\Gamma_R/2)^p\,|z_R^*, j, j_3, \eta^+\rangle\,. \tag{3.73b}$$

Therefore, the Gamow kets $\psi^G = \sqrt{2\pi\Gamma}\,|z_R, j, j_3, \eta^-\rangle$ defined by (3.50) or its equivalence $|z_R, j, j_3, \eta^-\rangle$ and the growing Gamow kets $\psi^{GR} = \sqrt{2\pi\Gamma}\,|z_R^*, j, j_3, \eta^+\rangle$ defined by (3.60) or its equivalence $|z_R^*, j, j_3, \eta^+\rangle$ are generalized eigenkets of the total Hamiltonian H^p for $p = 0, 1, 2, \cdots$ with the complex eigenvalues z_R^p and z_R^{*p}, respectively.

The energy density distribution of the Gamow vector ϕ^G is

$$\begin{aligned}\langle^-E|\phi^G\rangle &= \sqrt{2\pi\Gamma}\langle^-E|z_R^-\rangle = \\ &= -\frac{\sqrt{2\pi\Gamma}}{2\pi i}\int_{-\infty_{II}}^{\infty} dE'\,\frac{\langle^-E|E', j, j_3, \eta^-\rangle}{E - z_{R_i}} \\ &= i\sqrt{\frac{\Gamma}{2\pi}}\,\frac{1}{E-(E_R - i\Gamma/2)} \qquad \text{for } -\infty_{II} < E < \infty\,,\end{aligned} \tag{3.74}$$

which is the exact Breit-Wigner energy distribution for energy from negative values in the second sheet $-\infty_{II} < E \leq 0$ to positive value in the first sheet $0 \leq< E < \infty$. Similarly, the growing state ϕ^{GR} also have the exact Breit-Wigner energy distribution:

$$\langle^+E|\phi^{GR}\rangle = i\sqrt{\frac{\Gamma}{2\pi}}\,\frac{1}{E-(E_R + i\Gamma/2)} \qquad \text{for } -\infty_{II} < E < \infty\,. \tag{3.75}$$

3.5 Non-Relativistic Time Asymmetric Evolution

There are two basis vector expansions for the vectors in the Hardy space formalism, i.e, quantum mechanics under Hardy space boundary conditions. The first expansion uses the nuclear spectral theorem to justify the Dirac's basis expansion in terms of the discrete basis $|E_n, j, j_3, \eta)$ describing the bounded states and the continuous basis, $|E_n, j, j_3, \eta^+\rangle$ for any in-state vectors $\phi^+ \in \Phi_-$ as (3.17a) and $|E_n, j, j_3, \eta^-\rangle$ for any the out-observable vector $\psi^- \in \Phi_+$ as (3.17b), respectively. The second expansion uses the complex basis vector expansion which are associated with the resonances corresponding to the first-order poles of the S-matrix elements $S(E)$ plus the extra term called the background term and bounded states (ignored if we consider the scattering phenomenon), $|z_R, j, j_3, \eta^-\rangle$ defined by (3.49) and ϕ^{BG} defined by (3.64) for any in-state vectors $\phi^+ \in \Phi_-$ as (3.68) and $|z_R^*, j, j_3, \eta^+\rangle$ defined by (3.59) and ϕ^{BG*} defined by (3.67) for any observable vectors $\psi^- \in \Phi_+$ as (3.69). In both basis expansion (3.68) for the observable ψ^- and (3.69) for the state ϕ^+, the Gamow states $|z_R, j, j_3, \eta^-\rangle$ or ψ^G and growing Gamow states ψ^{GR} or $|z_R^*, j, j_3, \eta^-\rangle$ appear as discrete basis as stationary or bounded state $\mid E_n\,)$ in the usual basis vector expansion for a system with discrete energy eigenvalues E_n in Hilbert space $\mathcal{H}$.

The time evolution of the vector states are solutions of the differential equations under specific boundary conditions which specify the properties that these solutions will fulfill. The time evolution of the observable ψ^- and of the in-state ϕ^+ is described by the unitary group evolution (2.7a), (2.7d) for the solutions of the dynamical equations (2.1a) and (2.2b) under the Hilbert space boundary conditions (2.6) or under the Schwartz space boundary conditions (2.24) by the Stone-von Neumann theorem.

From the Hardy space boundary condition (3.23) for the wave energy functions, it follow that $E\langle^- E\,|\psi^-\rangle \in (\mathcal{H}^2_+ \cap S)\mid_{\mathbb{R}_+}$ or its complex conjugate $E\langle\psi^-\,|E^-\rangle \in (\mathcal{H}^2_- \cap S)\mid_{\mathbb{R}_+}$ and $E\langle^+ E\,|\phi^+\rangle \in (\mathcal{H}^2_- \cap S)\mid_{\mathbb{R}_+}$ or its complex conjugate $E\langle\phi^+\,|E^+\rangle \in (\mathcal{H}^2_+ \cap S)\mid_{\mathbb{R}_+}$, respectively. The operator e^{iHt} is continuously operator on Φ_+ for $t > 0$ and on Φ_- for $t < 0$. Equivalently, the adjoint operator $(e^{iHt})^\times = e^{-iH^\times t}$ is continuously operator on anti-linear functional space $\Phi_+^\times$ for $t > 0$ and on $\Phi_-^\times$ for $t < 0$ [16]. Explicitly,

$$\langle^- E\,|\; e^{-iH^\times t}\;|\psi^-\rangle = \langle^- E\; e^{iHt}|\psi^-\rangle = e^{iEt}\langle^- E\,|\psi^-\rangle \quad \text{for} \quad t > 0\,, \tag{3.76a}$$

similarly,

$$\langle^+ E\,|\; e^{-iH^\times t}\;|\phi^+\rangle = \langle^+ E\; e^{iHt}|\phi^+\rangle = e^{iEt}\langle^+ E\,|\phi^+\rangle \quad \text{for} \quad t < 0\,. \tag{3.76b}$$

We can omit the general vector $\langle^- E\,| \in \Phi_+$ in (3.76a) and the time evolution of the observable ket $|\psi^-\rangle \in \Phi_+^\times$ is described by the *semi-group* of operator $U_+^\times(t)$ restricted on the functional space $\Phi_+^\times$ of the Hard space of the upper complex energy plane of the second sheet Φ_+, $U_+^\times(t) \equiv U_{\Phi_-}^\times(t) = e^{-iH^\times t}$ for $t \geq 0$. Equivalently, the time evolution of the observable vector $\psi^- \in \Phi_+$, which are the solutions of the Heisenberg equation

(2.2b), is described by the *semi-group* of operator $U_+(t)$ restricted on the Hard space of the upper complex energy plane of the second sheet Φ_+, $U_+(t) \equiv U_{\Phi_-}(t) = e^{iHt}$:

$$\psi^-(t) = U_+(t)\,\psi^-(0) = e^{iHt}\,\psi^- \quad \text{with } 0 \le t < \infty \quad \text{for} \quad \psi^- \in \Phi_+\,. \qquad (3.77a)$$

Similarly, we omit the general vector $\langle^+E\,| \in \Phi_-$ in (3.76b) and the state ket $|\phi^+\rangle \in \Phi_-^\times$ is described by the *semi-group* of operator, $e^{-iH^\times t}$ for $t \le 0$, or $U_-^\times(t)$ restricted on the functional space $\Phi_-^\times$ of the Hard space of the lower complex energy plane of the second sheet Φ_-, $U_-^\times(t) \equiv (U_{\Phi_-}^\dagger)^\times(t) = e^{iH^\times t}$ for $t \ge 0$. Equivalently, the time evolution of the state $\phi^+ \in \Phi_-$,, which are the solutions of the Schrödinger equation (2.1a) under the new Hardy space boundary condition (3.22a), is described by by the *semi-group* of operator $U_-(t)$ restricted on the Hard space of the lower complex energy plane of the second sheet Φ_-, $U_-(t) \equiv U_{\Phi_-}^\dagger(t) = e^{-iHt}$:

$$\phi^+(t) = U_-(t)\phi^+(0) = e^{-iHt}\,\phi^+ \quad \text{with } 0 \le t \le \infty \quad \text{for} \quad \phi^+ \in \Phi_-\,. \qquad (3.77b)$$

Here, $\psi^- = \psi^-(0)$ and $\phi^+ = \phi^+(0)$ are the the observable and state at the initial time or beginning of time $t_0 = 0$, respectively. The operators $U_\pm(t)$ form semi-groups since their inverse operators $U_\pm^{-1}(t)$ do not exit.

In general, the time evolution of the observable $\psi^- \in \Phi_+$ and of the state $\phi^+ \in \Phi_+$ at the arbitrary time t when the initial condition of the observable and of the state are given at the any finite beginning of time t_0 are given by:

$$\psi^-(t) = U_{\Phi_+}(t)\,\psi^-(t_0) = U_+(t)\,\psi^-(t_0) = e^{iH(t-t_0)}\,\psi^-(t_0) \quad \text{with} \quad t_0 \le t < \infty\,, \qquad (3.78a)$$

and

$$\phi^+(t) = U_{\Phi_-}^\dagger(t)\,\phi^+(t_0) = U_-(t)\phi^+(t_0) = e^{-iH(t-t_0)}\,\phi^+(t_0) \quad \text{with} \quad t_0 \le t < \infty\,. \qquad (3.78b)$$

Mathematically, we can choose $t_0 = 0$. Then, the time evolutions of the observable ψ^- and of the state ϕ^+ in (3.36) are the special case of (3.78) as the finite time t_0 is chosen as $t_0 = 0$. The value of $t_0 = 0$ indicates the initial time or the beginning of time of the physics process in the scattering experiment at which the resonances or decaying state are created through interaction.

The Born probabilities to detect the observable $\psi^-(t)$ in the state ϕ^+ under Hardy space boundary conditions (3.22) are given by:

$$\begin{aligned}\mathcal{P}_{\phi^+}(\psi^-(t)) \;&= |\langle\psi^-(t)\,|\phi^+\rangle|^2 = |\langle\psi^-\,|\phi^+(t)\rangle|^2 \\ &= |\langle e^{iHt}\psi^-\,|\phi^+\rangle|^2 = |\langle\psi^-\,|e^{-iH^\times t}\phi^+\rangle|^2 \quad \textit{for only } t \ge t_0 = 0\end{aligned} \qquad (3.79)$$

This means that the Born probabilities (3.79) are predicted under the Hardy space axiom (3.22) only for $t \geq t_0 = 0$, i.e. only for a time t after the *finite* time $t_0 = 0$ at which the resonances has been created. This prediction is in agreement with the causality principle (2.28) and (2.29). The time t_0 is chosen as the finite time $t_0 = 0$. It represents the time at which the resonances has been created, before which the state ϕ^+ has been prepared, e.g. by an accelerator beam and target, and after which the observable ψ^- can be registered, e.g. by a detector with the counting rates $N(t)/N$ proportional to the probability (3.79).

The time evolution of the Gamow vector ψ^G or equivalent vector $|z_R, j, j_3, \eta^-\rangle$ can be obtained by considering the the time evolution of functions $\langle\psi^-(t)\,|z_R, j, j_3, \eta^-\rangle = \langle\psi^- \,|\, e^{-iH^\times t}\,|z_R, j, j_3, \eta^-\rangle$. From (3.76a) and the expansion of the $|z_R, j, j_3, \eta^-\rangle$ on the exact basis vectors $|E, j, j_3, \eta^-\rangle \in \Phi_+$, $\langle\psi^-(t)\,|\,z_R, j, j_3, \eta^-\rangle \in (\mathcal{H}^2_- \cap S)\,|_{\mathbb{R}_+}$ for any $\psi^- \in \Phi_+$ is defined for only only $t > 0$, explicitly:

$$\begin{aligned}\langle\psi^- \,|\, e^{-iH^\times t}\,|z_R, j, j_3, \eta^-\rangle &= -\frac{1}{2\pi i}\int_{-\infty_{II}}^{\infty} dE\ \frac{\langle\psi^-|e^{-iH^\times t}|E, j, j_3, \eta^-\rangle}{E - z_{R_i}} \\ &= -\frac{1}{2\pi i}\int_{-\infty_{II}}^{\infty} dE\ \frac{e^{-iEt}\langle\psi^-|E, j, j_3, \eta^-\rangle}{E - z_{R_i}} \\ &\qquad\text{for all } \psi^- \in \Phi_+ \text{ and for only } t \geq 0\,.\end{aligned} \tag{3.80}$$

On the other hands, since $|z_R, j, j_3, \eta^-\rangle$ can be defined as a functional on the space Φ_+, the l.h.s of (3.80) can be written as and hence defined for only $t \geq 0$:

$$\begin{aligned}\langle\psi^- \,|\, e^{-iH^\times t}\,|z_R, j, j_3, \eta^-\rangle &= e^{-iz_R t}\langle\psi^-\,|z_R, j, j_3, \eta^-\rangle \\ &\qquad\text{for only } t \geq 0 \text{ and any } \psi^- \in \Phi_+\,.\end{aligned} \tag{3.81}$$

Since $\psi^- \in \Phi_+$ is an arbitrary vector, we can omit it in(3.81) and hence get the time evolution of the $|z_R, j, j_3, \eta^-\rangle$ is:

$$e^{-iH^\times t}\,|z_R, j, j_3, \eta^-\rangle = e^{-iz_R t}\,|z_R, j, j_3, \eta^-\rangle \quad \text{for only } t \geq 0\,. \tag{3.82}$$

Analogically, the time evolution of the vector $|z^*_R, j, j_3, \eta^+\rangle$ can be obtained by considering the time evolution of the functions $\langle\phi^+(t)\,|\,z^*_R, j, j_3, \eta^+\rangle = \langle\phi^+ \,|\, e^{-iH^\times t}\,|\,z^*_R, j, j_3, \eta^+\rangle$ and is given by:

$$e^{-iH^\times t}\,|z^*_R, j, j_3, \eta^+\rangle = e^{-iz^*_R t}\,|z^*_R, j, j_3, \eta^+\rangle \quad \text{for only } t \leq 0\,. \tag{3.83}$$

Therefore, the time evolution of the Gamow ket ψ^G is given as (3.82) by:

$$e^{-iH^\times t}\,|\psi^G\rangle = e^{-iz_R t}\,|\psi^G\rangle = e^{-iEt}e^{-\Gamma t/2}\,|\psi^G\rangle \quad \text{for only } t \geq 0\,. \tag{3.84}$$

And, the time evolution of the growing Gamow ket ψ^{GR} is given as (3.83) by:

$$e^{-iH^{\times}t}\,|\psi^{GR}\rangle = e^{-iz*_R t}\,|\psi^{GR}\rangle = e^{-iEt}e^{\Gamma t/2}\,|\psi^{GR}\rangle \quad \text{for only } t \leq 0\,. \tag{3.85}$$

The equations (3.84) and (3.85) describe the exponential decaying Gamow state ψ^G and the exponential growing Gamow state ψ^{GR}, respectively. The time evolution of decaying Gamow vector ψ^G is only defined into the forward time direction, $t \geq 0$, in which the time t must be larger than initial time or beginning of time $t_0 = 0$. Similarly, The time evolution of growing Gamow vector ψ^{GR} is only defined into the backward time direction, $t \leq 0$, in which the time t must be less than initial time or beginning of time $t_0 = 0$.

The Born probability for an observable $\psi^-(t)$, i.e, the decay products, in a decaying state $|z_R, j, j_3, \eta^-\rangle$ corresponding to a resonance at any arbitrary time $t \geq 0$ is:

$$\begin{aligned}
\mathcal{P}_{z_R^-}(\psi^-(t)) \;&= |\langle\psi^-(t)\,|z_R, j, j_3, \eta^-\rangle|^2 = |\langle e^{iHt}\,\psi^-\,|E_R - i\Gamma/2, j, j_3, \eta^-\rangle|^2 \\
&= |\langle\psi^-\,|\; e^{-iH^{\times}t}\;|E_R - i\Gamma/2, j, j_3, \eta^-\rangle|^2 \\
&= |e^{-iE_R t}\; e^{-(\Gamma/2)t}\langle\psi^-\,|E_R - i\Gamma/2, j, j_3, \eta^-\rangle|^2 \\
&= e^{-\Gamma t}\;|\langle\psi^-\,|z_R, j, j_3, \eta^-\rangle|^2 \quad \text{for} \quad \psi^- \in \Phi_+ \;\text{ and }\; t \geq 0 \;\text{ only.}
\end{aligned} \tag{3.86}$$

This probability (3.86) describes the exact exponential law for the decaying states which corresponds to the resonance defined as the first-order pole of the S-matrix at the position $z_R = E - i\Gamma/2$. Hence, we get an exact exponential decay for the Gamow state with a lifetime τ which is exactly given by the width $\Gamma = \Gamma_R$ of the corresponding scattering resonance:

$$\tau = \frac{1}{\Gamma} \quad \text{or} \quad \tau = \frac{\hbar}{\Gamma}\,. \tag{3.87}$$

Therefore, the resonances in the non-relativistic scattering process or corresponding unstable, decaying states are described by the Gamow kets ψ^G defined in (3.50). These Gamow kets ψ^G are associated with the first-order poles of the S-matrix at $s_{R_i} = E_{R_i} - i\Gamma_i/2$. The time evolution of the Gamow kets ψ^G is not given by an unitary group for which the time t extends for all time $-\infty < t < \infty$, but by the semi-group (3.84) for which the time t extends over $0 \leq t < \infty$.

Chapter Four: Relativistic Time Asymmetric Quantum Mechanics

4.1 Representation of Poincaré group in 4-Velocity Basis

A relativistic particle of the invariant mass squared $s = m_p^2$ and spin j_p is described by the unitary irreducible representation (UIR) of the Poincaré group $\mathcal{P}$ (Wigner's representations) labeled by invariant mass squared $s = m_p^2$ and spin j_p, $(s,\, j_p)$ and other quantum numbers η_p, e.g, species quantum number, isospin, lepton number, etc... [35] [36]:

$$\boldsymbol{\mathcal{P}} = \{(x, \Lambda) \mid \Lambda \in \overline{SO(3,1)},\ \det\Lambda = +1,\ \Lambda_0^0 \geq 0\}\,. \tag{4.1}$$

Here Λ is the Lorentz transformation. The standard basis vectors are chosen as the eigenkets of the complete set of commuting observables (c.s.c.o.) of the the Poincaré group $\mathcal{P}$, $|p\, j_3[sj], \eta\rangle$:

$$M^{\times}\ |p\, j_3[sj], \eta\rangle = \sqrt{s}\ |p\, j_3[sj], \eta\rangle\,, \tag{4.2a}$$

$$P_\mu^{\times}\ |p\, j_3[sj], \eta\rangle = p_\mu\ |p\, j_3[sj], \eta\rangle\,, \tag{4.2b}$$

$$-w^2\ |p\, j_3[sj], \eta\rangle = j(j+1)\ |p\, j_3[sj], \eta\rangle\,, \tag{4.2c}$$

$$S_3 \equiv U(L(p)) w_3 U^{-1}(L(p))\ |p\, j_3[sj], \eta\rangle = j_3\ |p\, j_3[sj], \eta\rangle\,. \tag{4.2d}$$

Here, $M^2 = P^\mu P_\mu$ is the mass squared operator with the eigenvalue $s = p^\mu p_\mu$, which is the mean-squared invariant mass squared. For the scattering process (3.1), $M = P^\mu P_\mu = (P_a + P_b)^\mu (P_a + P_b)_\mu$ and $s = p^\mu p_\mu = (p_a + p_b)^\mu (p_a + p_b)_\mu$. Both operators $P_\mu^\times$ and $M^\times$ are conjugate operators of P_μ and M in the dual Schwartz space $\Phi^\times$, i.e, continuous anti-linear functionals. They are unique extensions of the adjoint operators $P_\mu^\dagger$ and $M^\dagger$ of P_μ and M in the Hilbert space to the dual Schwartz space $\Phi^\times$. The operator $-w^2 = -w^\mu w_\mu$ is the squared relativistic spin operator with eigenvalue $j(j+1)$, w^μ is the Pauli-Lubanski operator defined as $w^\mu = \epsilon^{\mu\nu\rho\sigma}\, P_\nu\, J_{\rho\sigma}$, and $S_3 \equiv U(L(p)) w_3 U^{-1}(L(p))$ is defined as third component of spin operator with its eigenvalue j_3, respectively. $U(L(p))$ is the representation of the rotation-free Lorentz boost $L(p)$ which depends on the eigenvalues of the 4-momentum operator p_μ. The matrix $L(p)$

$$L^\mu_{\ \nu}(p) = \begin{pmatrix} \frac{p^0}{m} & -\frac{p_n}{m} \\ \frac{p^m}{m} & \delta^m_n - \frac{\frac{p^m}{m}\frac{p_n}{m}}{1+\frac{p^0}{m}} \end{pmatrix} \text{for} \quad m,n = 1,2,3\,, \tag{4.3}$$

is the transformation between the rest frame $\boldsymbol{p} = 0$ and the laboratory frame:

$$\begin{pmatrix} m \\ \boldsymbol{p} \end{pmatrix} = L^\mu_{\ \nu}(p) \begin{pmatrix} m \\ \boldsymbol{0} \end{pmatrix} \quad \text{or} \quad L^{-1}(p)^\mu_{\ \nu} \begin{pmatrix} m \\ \boldsymbol{p} \end{pmatrix} = \begin{pmatrix} m \\ \boldsymbol{0} \end{pmatrix}. \tag{4.4}$$

Here, $\boldsymbol{p}$ is the three-momentum vector. The rotation-free Lorentz boost (4.3) does not depend on the 4-momentum p^μ but on the 4-velocity $\widehat{p}^\mu = p^\mu/m$. Therefore, instead of using the momentum eigenvector $|\boldsymbol{p}\, j_3[sj], \eta\rangle$ as basis vectors of the irreducible representation space of [s,j], we can as well use the three space-components of the 4-velocity $\widehat{\boldsymbol{p}} = \boldsymbol{p}/\sqrt{s}$ and label the eigenvectors as $|\widehat{\boldsymbol{p}}\, j_3[s\, j], \eta\rangle$. These labels can also be used for the basis vectors of semi-group which will be discussed later. The 4-velocity kets $|\widehat{\boldsymbol{p}}\, j_3[sj], \eta\rangle$ are the eigenkets of [51]:

$$M^\times\ |\widehat{\boldsymbol{p}}\, j_3[sj], \eta\rangle = \sqrt{s}\ |\widehat{\boldsymbol{p}}\, j_3[sj], \eta\rangle\,, \tag{4.5a}$$

$$\widehat{P}^\times_\mu\ |\widehat{\boldsymbol{p}}\, j_3[sj], \eta\rangle = \widehat{p}_\mu\ |\widehat{\boldsymbol{p}}\, j_3[sj], \eta\rangle\,, \tag{4.5b}$$

$$-\widehat{w}^2\ |\widehat{\boldsymbol{p}}\, j_3[sj], \eta\rangle = j(j+1)\ |\widehat{\boldsymbol{p}}\, j_3[sj], \eta\rangle\,, \tag{4.5c}$$

$$\mathcal{U}(L(\widehat{p}))\widehat{w}_3\, \mathcal{U}^{-1}(L(\widehat{p}))\ |\widehat{\boldsymbol{p}}\, j_3[sj], \eta\rangle = j_3\ |\widehat{\boldsymbol{p}}\, j_3[sj], \eta\rangle\,. \tag{4.5d}$$

Here $\widehat{p}_\mu = p_\mu/\sqrt{s}$ is 4-velocity with $\widehat{p}^0 = \widehat{E}(\widehat{\boldsymbol{p}}) = \sqrt{1+\widehat{\boldsymbol{p}}^2}$, and space-components $\widehat{\boldsymbol{p}} \in \mathcal{R}^3$, the modified Pauli-Lubanski operator is $\widehat{w} = \epsilon^{\mu\nu\rho\sigma}\ \widehat{P}_\nu\, J_{\rho\sigma}$, and η is particle species number.

The 4-velocity kets $|\,\widehat{\boldsymbol{p}}\, j_3[sj], \eta\rangle$ are elements of the dual space $\Phi^\times$. As in the non-relativistic case, the Schwartz space Φ is a dense subspace of Hilbert space $\mathcal{H}$ and then one obtains the Schwartz space triplet (or Rigged Hilbert Space (RHS)): $\Phi \subset \mathcal{H} \subset \Phi^\times$.

The 4-velocity kets $|\widehat{\boldsymbol{p}}\, j_3[sj], \eta\rangle$ transform under inhomogeneous Lorentz transformation (Λ, x) as

$$\mathcal{U}^\times(\Lambda, x)\ |\widehat{\boldsymbol{p}}\, j_3[sj], \eta\rangle = e^{-i\,p.x} \sum_{j_3'} D^j_{j_3\, j_3'}(W(\Lambda^{-1}, p))\ |\Lambda^{-1}\widehat{\boldsymbol{p}}\, j_3[sj], \eta\rangle\,. \tag{4.6}$$

Explicitly, under the translation (x, I) and Lorentz boost $(0, \Lambda)$ transformation as

$$\mathcal{U}^\times(x, I)\,|\widehat{\boldsymbol{p}}\,j_3[sj], \eta\rangle = e^{-ip.x}\,|\widehat{\boldsymbol{p}}\,j_3[sj], \eta\rangle\,, \tag{4.7a}$$

$$\mathcal{U}^\times(0, \Lambda)\,|\widehat{\boldsymbol{p}}\,j_3[sj], \eta\rangle = \sum_{j_3'} D^j_{j_3\,j_3'}(W(\Lambda^{-1}, p))\,|\Lambda^{-1}\widehat{\boldsymbol{p}}\,j_3[sj], \eta\rangle\,. \tag{4.7b}$$

Here, $\mathcal{U}^\times$ is the extension of the operator $\mathcal{U}$ in the dual space $\Phi^\times$ of the Schwartz space Φ. W is the Wigner rotation which depends on the Lorentz transformation Λ and 4-velocity $\widehat{p}$ since the transformation $L(P)$ only depends on 4-velocity $\widehat{p}$:

$$W(\Lambda, p) = L^{-1}(\Lambda p)\Lambda L(p)\,, \tag{4.8}$$

and $D^j_{j_3\,,j_3'}(W(\Lambda, p))$ is the rotation matrix corresponding to the j-th angular momentum. The transformation of the 4-velocity $|\widehat{\boldsymbol{p}}\,j_3[sj], \eta\rangle$ between the laboratory and the rest frame $\widehat{\boldsymbol{p}} = 0$ is given by

$$\mathcal{U}^\times(L(\widehat{\boldsymbol{p}}))\,|\widehat{\boldsymbol{p}} = 0\,, j_3[sj], \eta\rangle = |\widehat{\boldsymbol{p}}\,j_3[sj], \eta\rangle\,. \tag{4.9}$$

The normalization of free 4-velocity eigenkets is chosen as

$$\langle\widehat{\boldsymbol{p}}\,j_3[sj], \eta\,|\widehat{\boldsymbol{p}}'\,j_3'[s'j'], \eta'\rangle = 2\,\widehat{E}(\widehat{\boldsymbol{p}})\,\delta(\widehat{\boldsymbol{p}} - \widehat{\boldsymbol{p}}')\,\delta(s - s')\,\delta_{j_3 J_3'}\,\delta_{jj'}\delta_{\eta,\eta'}\,, \tag{4.10}$$

where $\widehat{E}(\widehat{\boldsymbol{p}}) \equiv E(\boldsymbol{p})/\sqrt{s} = \sqrt{1 + \widehat{\boldsymbol{p}}^2} = \gamma = \frac{1}{\sqrt{1 - \boldsymbol{v}^2}}$.

If the relativistic particle system consists of 2-particles, which each of which is represented by an unitary irreducible representation (UIR) of Poincaré group labeled by the mass squared s_i, spin j_i, $i = 1, 2$, and by the other quantum numbers collectively denoted by quantum number η- including particle species numbers, the direct product space of the 2-particle system $\mathcal{H}_{12} = \mathcal{H}_1 \otimes \mathcal{H}_2$, can be reduced into a direct sum of two UIR spaces of different mass and spin [37] [38] [39]:

$$\mathcal{H}_{12} = \mathcal{H}_1^\eta(s_1, j_1) \oplus \mathcal{H}_2^\eta(s_2, j_2)\,. \tag{4.11}$$

Any vector $\phi \in \Phi \subset \mathcal{H} \subset \Phi^\times$ can be expanded as a continuous linear superposition of the 4-velocity kets $|\widehat{\boldsymbol{p}}\,j_3[sj], \eta\rangle \in \Phi^\times$ which is analogy to the Dirac's basis vector expansion for momentum kets by Nuclear Spectral Theorem for Rigged Hilbert Space (RHS) [16]:

$$\phi = \sum_{jj_3\eta} \int_{s_0}^{\infty} ds \int \frac{d^3\widehat{\boldsymbol{p}}}{2\widehat{p}^0}\,|\widehat{\boldsymbol{p}}\,j_3[sj], \eta\rangle\langle\widehat{\boldsymbol{p}}\,j_3[sj], \eta\,|\phi\rangle\,. \tag{4.12}$$

Here, the total invariant mass squared s has the lower bound which is the total mass squared of the system in the center of mass $s_0 = (m_1 + m_2 + \cdots + m_n)^2$. In the center of mass of 2 particles system of masses m_1 and m_2 as in the scattering experiment (3.1), $s_0 = (m_1 + m_2)^2$.

In the interacting picture, the interacting in- and out-states basis kets $|\widehat{\boldsymbol{p}}\, j_3\, [sj]\eta, n^{\pm}\rangle$ are obtained from the free-interaction basis kets by the Möller wave operators $\Omega^{\pm}$ [31]:

$$|\widehat{\boldsymbol{p}}\, j_3[sj], \eta^{\pm}\rangle = \Omega^{\pm}\, |\widehat{\boldsymbol{p}}\, j_3[sj], \eta\rangle\,. \tag{4.13}$$

In the rest frame of the system or center of mass frame ($\widehat{\boldsymbol{p}} = 0$), the interacting in- and out-kets $|\,\widehat{\boldsymbol{p}} = 0\, j_3\, [sj]\eta, n^{\pm}\rangle$ can be described as formal solutions of the Lippmann-Schwinger equation in analogy to the Lippmann-Schwinger equation in non-relativistic scattering. Therefore, the solutions of the Lippmann-Schwinger equation [6] in the relativistic case are:

$$|\mathbf{0}\, j_3\, [sj]\eta, n^{\pm}\rangle = \left(1 + \frac{1}{s - H \pm i\epsilon} V\right)\, |\mathbf{0}\, j_3\, [sj]\eta, n\rangle\,. \tag{4.14}$$

The in- and out-kets in the lab frame can then be obtained from the rest frame kets by the rotation-free Lorentz boost transformation, $\mathcal{U}(L(\widehat{p}))$:

$$|\widehat{\boldsymbol{p}}\, j_3[sj], \eta^{\pm}\rangle = (\mathcal{U}^{\dagger}(L(\widehat{p}))^{\times}\, |\mathbf{0}\, j_3\, [sj]\eta, n^{\pm}\rangle\,. \tag{4.15}$$

Here $L(\widehat{p})$ is the rotation-free Lorentz boost which is a function of the real 4-velocity $\widehat{p}^{\mu}$. The conjugate operator $(\mathcal{U}^{\dagger}(L(\widehat{p}))^{\times}$ is the extension of $\mathcal{U}^{\dagger}$ acts on the dual space $\Phi_{\pm}^{\times}$.

4.2 Hardy Space Boundary Condition of Relativistic Quantum Mechanics

The empirical causality principle (2.28) suggests that in the scattering experiment in both relativistic and non-relativistic regimes, the detectors are only able to detect the observables, e.g, decay products, after the state has been prepared, i.e, after the accelerator is turned on and the beam has hit the fixed target, or two accelerated beams are guided to make head-on collision. It means that detectors can not detect the observables which are the decaying products of the resonances before the resonances are created. In the relativistic scattering experiment, detectors are designed to detect some special decaying products of particular resonances. In analogy to the Hardy space axiom (3.22) in non-relativistic regime, the the Hardy space axiom in relativistic regime states that:

The space of in-states $\{\phi^+\}$ is the Hardy space Φ_- of the lower complex s-plane $\mathbb{C}_-$ on second sheet of S-matrix:

$$\{\phi^+\} = \Phi_- \subset \mathcal{H} \subset \Phi_-^\times . \tag{4.16a}$$

Space of observables $\{\psi^-\}$ is the Hardy space Φ_+ of the upper complex s-plane $\mathbb{C}_+$ on second sheet of S-matrix:

$$\{\psi^-\} = \Phi_+ \subset \mathcal{H} \subset \Phi_+^\times . \tag{4.16b}$$

These Hardy spaces $\Phi_\pm$ in (4.16) are mathematically represented by the smooth Hardy functions rather than the L^2-integrable functions of the invariant mass $s = p^\mu p_\mu$ used in the Hilbert space. The Hardy space axiom requires that the functions of invariant mass $s = p^\mu p_\mu$ in the upper and lower half-plane of the second sheet of the s-surface are smooth Hardy functions [40]:

$$\text{for the observable } \psi^- \in \Phi^+ \iff \psi^-(s) = \langle^- \hat{\boldsymbol{p}}\, j_3[sj], \eta|\psi^-\rangle \in \tilde{\mathcal{S}} \cap \mathcal{H}_+^2|_{\mathbb{R}_{s_0}} \otimes \tilde{\mathcal{S}}(\mathbb{R}^3)\,, \tag{4.17a}$$

$$\text{for the in-state } \phi^+ \in \Phi^- \iff \phi^+(s) = \langle^+ \hat{\boldsymbol{p}}\, j_3[sj], \eta|\phi^+\rangle \in \tilde{\mathcal{S}} \cap \mathcal{H}_-^2|_{\mathbb{R}_{s_0}} \otimes \tilde{\mathcal{S}}(\mathbb{R}^3)\,. \tag{4.17b}$$

Here, the space $\tilde{\mathcal{S}} \cap \mathcal{H}_\pm^2|_{\mathbb{R}_{s_0}}$ are not the spaces of L^2-integrable energy wave functions but are the intersection between the Schwartz function space $\tilde{\mathcal{S}}$ and Hardy class function $\mathcal{H}_\pm^2$ of the lower and upper complex s plane of the S-matrix, respectively. $\mathbb{R}_{s_0}$ means that the value of $s = p_\mu p^\mu$ is restricted to the physical values, $s_0 = (m_1 + m_2)^2 < s < \infty$ for the scattering experiment (3.1). The space $\tilde{\mathcal{S}}(\mathbb{R}^3)$ is the Schwartz function space of the space-components of the 4-velocity vector $\hat{\boldsymbol{p}} = \boldsymbol{p}/\sqrt{s}$.

The another causality needed to be implemented into the theory in the relativistic regime is that the velocity of the unstable particles or resonances created after collision have to be less than the velocity of the light. In other words, if a resonance or an unstable particle is created at an initial space-time position $(t_i, \boldsymbol{x_i})$ and decays at a final space-time position $(t_f, \boldsymbol{x_f})$ in the laboratory frame, the relativistic causality implies that the space-distance $|\boldsymbol{x_f} - \boldsymbol{x_i}|$ for the unstable particle traveling during the time interval $t_f - t_i$ should be less then the space-distance $(t_f - t_i)$ for the light traveling during the same time interval, mathematically:

$$(t_f - t_i) \geq |\boldsymbol{x_f} - \boldsymbol{x_i}| \quad \text{or} \quad t \geq |\boldsymbol{x}| \ . \tag{4.18}$$

Here, $\boldsymbol{x} = \boldsymbol{x_f} - \boldsymbol{x_i}$ and $t = t_f - t_i$ are the space and time intervals of the unstable particle or resonance, respectively. In terms of the space-time interval or distance, the relativistic causality (4.18) implies that :

$$x^2 = (t)^2 - |\boldsymbol{x}|^2 \geq 0 \, . \tag{4.19}$$

This means that all the events in the relativistic scattering experiment are time-like events. This relativistic causality (4.19) can be obtained as the result of Hardy boundary condition (4.16) that allows the wave functions of mass squared s can be analytically into the upper or lower complex s-plane.

From the Hardy space axiom for functions of the invariant mass squared s, these functions can be analytically continued into the upper and lower of the s-complex plane. It means that the variable s can be analytically extended from the real to complex values. Since $s = p_\mu p^\mu$, p^μ can also be analytically extended to complex value and may not keep the same value during this analytic extension. Furthermore, the Lorentz boost (4.3) only depends on the real 4-velocity $\widehat{p} = p/\sqrt{s}$. Therefore, the 4-velocity basis $|\widehat{\boldsymbol{p}}\, j_3[sj], \eta\rangle$ is good choice for basis vector than the momentum basis $|p\, j_3[sj], \eta\rangle$. We restrict the representation space of $[s,\ j]$ to the case in which 4-velocity $\widehat{p}_\mu = p_\mu/\sqrt{s}$ is real while the mass $\sqrt{s}$ and therewith 4-momentum p^μ are complex. This is called minimally complex representation for the 4-velocity ket [37]. The minimally complex representation is subclass of semi-group representations of Poincaré group. In the minimal complex representation, the unitary representations of the space-time translations $\mathcal{U}(I, x)$ turn into semi-group representations while the representations of homogeneous Lorentz transformations $\mathcal{U}(\Lambda, 0)$ keep the same as in Wigner's representations [41].

In the minimally complex representation, the kets $|\widehat{\boldsymbol{p}}\, j_3[sj], \eta^{\pm}\rangle$ are eigenkets of the conjugate operator $\widehat{P}_\mu^{\times}$ of the 4 velocity operator $\widehat{P}_\mu = P_\mu\, M^{-1}$ with real eigenvalue $\widehat{p}_\mu = p_\mu/\sqrt{s}$ and the conjugate operator $M^{\times}$ of the mass operator $M = (P_\mu\, P^\mu)^{1/2}$ with the complex eigenvalue $\sqrt{s}$, respectively:

$$M^{\times}\ |\widehat{\boldsymbol{p}}\, j_3[sj], \eta^{\pm}\rangle = \sqrt{s}\ |\widehat{\boldsymbol{p}}\, j_3[sj], \eta^{\pm}\rangle \ , \tag{4.20}$$

$$\widehat{P}_\mu^\times\;|\widehat{\boldsymbol{p}}\,j_3[sj],\eta^\pm\rangle = \widehat{p}_\mu\;|\widehat{\boldsymbol{p}}\,j_3[sj],\eta^\pm\rangle\,. \tag{4.21}$$

The operator $M^\times$ and $\widehat{P}_\mu^\times$ are conjugate operators of M and $\widehat{P}_\mu$ in the dual spaces $\Phi_\pm^\times$ of the Hardy spaces $\Phi_\pm$, respectively, which are unique extensions of $M^\dagger$ and $\widehat{P}_\mu^\dagger$ in the Hilbert space $\mathcal{H}$ to the dual spaces $\Phi_\pm^\times$. The in-coming kets $|\widehat{\boldsymbol{p}}\,j_3[sj],\eta^+\rangle$ and out-going ket $|\widehat{\boldsymbol{p}}\,j_3[sj],\eta^-\rangle$ are elements of dual spaces: $|\widehat{\boldsymbol{p}}\,j_3[sj],\eta^\pm\rangle \in \Phi_\mp^\times$ of the Hardy spaces $\Phi_\mp$, respectively.

The branch of the complex variable s is chosen by the principle values of $Arg(s)$. It is usually to be chosen as

$$-\pi \leq Arg(s) < \pi\,. \tag{4.22}$$

This $Arg(s)$ is not a continuous function. It has a discontinuity along the positive real axis. The equation (4.22) is to hold up to integer multiples of 2π.

The kets $|\widehat{\boldsymbol{p}}\,j_3[sj],\eta^\pm\rangle$ are chosen as basis kets of the space $\Phi_\mp^\times$. Like in the non-relativistic case, any vector $\phi^+ \in \Phi_-$ can be expanded in the basis $|\widehat{\boldsymbol{p}}\,j_3[sj],\eta^+\rangle \in \Phi_-^\times$ as:

$$\phi^+ = \sum_{j\,j_3}\int_{s_0}^\infty ds \int \frac{d^3\widehat{\boldsymbol{p}}}{2\widehat{p}^0}\;\;|\widehat{\boldsymbol{p}}\,j_3[sj]\eta^+\rangle\langle^+\widehat{\boldsymbol{p}}\,j_3[sj]\eta'\,|\phi^+\rangle\,. \tag{4.23}$$

Similarly, any vector $\psi^- \in \Phi_+$ can be expanded in the basis $|\widehat{\boldsymbol{p}}\,j_3[sj],\eta^-\rangle \in \Phi_+^\times$:

$$\psi^- = \sum_{j\,j_3}\int_{s_0}^\infty ds \int \frac{d^3\widehat{\boldsymbol{p}}}{2\widehat{p}^0}\;\;|\widehat{\boldsymbol{p}}\,j_3[sj]\eta^-\rangle\langle^-\widehat{\boldsymbol{p}}\,j_3[sj]\eta'\,|\psi^-\rangle\,. \tag{4.24}$$

As a consequence of (4.20) and (4.21), the kets $|\widehat{\boldsymbol{p}}\,j_3[sj],\eta^\pm\rangle$ are also the eigenkets of the conjugate operators $(P^\mu)^\times$ of the exact 4-momentum operator P^μ with the eigenvalue $\widehat{p}^\mu\sqrt{s}$:

$$(P^\mu)^\times\;|\widehat{\boldsymbol{p}}\,j_3[sj],\eta^\pm\rangle = \widehat{p}^\mu\sqrt{s}\;|\widehat{\boldsymbol{p}}\,j_3[sj],\eta^\pm\rangle, \quad \text{where} \quad \widehat{p}^\mu = (\widehat{p}^0,\widehat{\boldsymbol{p}}) = (\gamma,\gamma\boldsymbol{v}). \tag{4.25a}$$

Or equivalently,

$$H^\times\;|\widehat{\boldsymbol{p}}\,j_3[sj],\eta^\pm\rangle = \gamma\sqrt{s}\;|\widehat{\boldsymbol{p}}\,j_3[sj],\eta^\pm\rangle\,, \tag{4.25b}$$

$$P_i^\times\;|\widehat{\boldsymbol{p}}\,j_3[sj],\eta^\pm\rangle = \widehat{p}_i\sqrt{s}\;|\widehat{\boldsymbol{p}}\,j_3[sj],\eta^\pm\rangle \text{ for } i = 1,2,3\,. \tag{4.25c}$$

Here, P_i for $i = 1,2,3$ are the momentum operators in space coordinates. The operator $H^\times$ and $P_i^\times$ are conjugate operators of Hamiltonian H and momentum operators P_i in the dual spaces $\Phi_\pm^\times$ of the Hardy spaces $\Phi_\pm$ which are unique extensions of $H^\dagger$ and $P_i^\dagger$ in the Hilbert space $\mathcal{H}$ to the dual spaces $\Phi_\pm^\times$, respectively.

4.3 Relativistic Decaying and Growing Gamow Vector

Analogy to non-relativistic scattering experiment, a relativistic resonance R is also characterized by a mass M_R and a width Γ_R. The mass M_R and width Γ_R of the resonance R are experimentally measured by fitting the partial cross section $\sigma_j(s)$ to the Breit-Wigner line shape with a slowly varying background $B(s)$.

In relativistic regime, the partial cross section $\sigma_j(s)$ is fitted to either relativistic Breit-Wigner or Lorentzian energy distribution of the S-matrix pole $s_R = (M_R - i\Gamma_R/2)^2$:

$$\sigma_j(s) \sim \left| \frac{r_\eta}{s - s_R} + B(s) \right|^2 \quad \text{with} \quad m_0^2 < s < \infty \, . \tag{4.26}$$

Or relativistic Breit-Wigner or Lorentzian energy distribution with energy dependent width $\Gamma(s)$ of the on-shell renormalization scheme:

$$\sigma_j(s) \sim \left| \frac{r_\eta}{s - (M_R^2 - i\frac{s}{M_R}\Gamma_R)} + B(s) \right|^2 \quad \text{with} \quad m_0^2 < s < \infty \, . \tag{4.27}$$

Here, m_0 is the rest mass of the system, e.g, for the scattering process (3.1), $m_0 = m_a + m_b$ and $s = (p_a^\mu + p_b^\mu)^2$.

The width Γ_R of a resonance R can theoretically be related by the decay rate Γ of resonance R to final states f using quantum field theory [42]:

$$\Gamma = \frac{1}{2\,m} \sum_f \int d\Pi_f \, |\mathcal{M}(R \to f)|^2 \ . \tag{4.28}$$

Here, $d\Pi_f$ is the relativistically invariant phase space of the final states f and $\mathcal{M}(R \to f)$ is a scattering amplitude of resonance R which decays into the final states f. The accuracy of the value of decay rate depends on the approximation used in calculation of scattering amplitude $\mathcal{M}(R \to f)$, i.e, the order in the Feynman diagrams used to calculate the matrix $\mathcal{M}(R \to f)$.

Using the basis expansion for the state $\phi^+ \in \Phi_-$ as (4.23) and for the observable $\psi^- \in \Phi_+$ as (4.24), the probability density to detect the out-observable ψ^- in the in-state ϕ^+ can be given in term of S-matrix element $S_{jj_3}^{\eta\,\eta'}(s,p)$:

$$(\psi^-, \phi^+) = \sum_{jj'J_3J_3'} \int_{s_0}^\infty \int_{s_0}^\infty ds ds' \int \int \frac{d^3\widehat{\boldsymbol{p}}}{2\widehat{p}^0} \frac{d^3\widehat{\boldsymbol{p}}}{2\widehat{p'}^0} \, \langle \psi^- | \widehat{\boldsymbol{p}'} \, j_3'[s'j']\eta'^- \rangle S_{jj'}^{\eta\,\eta'}(s,p) \langle ^+\widehat{\boldsymbol{p}} \, j_3[sj]\eta | \phi^+ \rangle \, . \tag{4.29}$$

Here, the S-matrix element $S_{jj'}^{\eta\,\eta'}(s,p)$ is defined as the scalar product of the in-coming and our-going kets $|\widehat{p}\,j_3[sj]\eta^{\pm}\rangle$:

$$\langle^{-}\widehat{\boldsymbol{p}}'\,j_3'[s'j']\eta'\,|\widehat{\boldsymbol{p}}\,j_3[sj]\eta^{+}\rangle = \langle\widehat{\boldsymbol{p}}'\,j_3'[s'j']\eta'\,|\;S\;|\widehat{\boldsymbol{p}}\,j_3[sj]\eta\rangle = S_{jj_3}^{\eta\,\eta'}(s,p)\,. \tag{4.30}$$

In relativistic regime, the S-matrix is invariant under the poincaré transformation or inhomogeneous Lorentz transformation (x,Λ). In particular, the S operator is invariant under the homogeneous Lorentz transformation $(0,\Lambda)$. In forms of the operators, it can be expressed as

$$U^{\dagger}(\Lambda)\;S\;U(\Lambda) = S\,. \tag{4.31}$$

If we choose $\Lambda = L^{-1}(\widehat{p})$, or $U(\Lambda) = U(L^{-1}(\widehat{p}))$, the quantity in (4.31) involving in the S-operator is invariant under the transformation (4.32):

$$\begin{aligned} S_{jj_3}^{\eta\,\eta'}(s,p) &= \langle\widehat{\boldsymbol{p}}'\,j_3'[s'j']\eta'\,|\;S\;|\widehat{\boldsymbol{p}}\,j_3[sj]\eta\rangle \\ &= \langle\widehat{\boldsymbol{p}}'\,j_3'[s'j']\eta'\,|\;U^{\dagger}(L^{-1}(\widehat{p}))SU(L^{-1}(\widehat{p}))\;|\widehat{\boldsymbol{p}}\,j_3[sj]\eta\rangle \\ &= \langle\widehat{\mathbf{0}}'\,j_3'[s'j']\eta'\,|\;S\;|\widehat{\mathbf{0}}\,j_3[sj]\eta\rangle \\ &= \langle\langle j_3'[s'j']\eta'|S|j_3[sj]\eta\rangle\rangle \equiv S_{jj_3}^{\eta\,\eta'}(s)\,. \end{aligned} \tag{4.32}$$

The second equation in (4.32) is obtained by using the property of the operator $U(L^{-1}(\widehat{p}))$ in (4.9) which transforms the ket $|\widehat{\boldsymbol{p}}\,j_3[sj]\eta\rangle$ in the laboratory into the ket in the center of mass frame $|\widehat{\mathbf{0}}\,j_3[sj]\eta\rangle$. The expression (4.32) shows that the S-matrix does not depends on the the 4-velocity parameters, i.e, on the frame of reference, laboratory or center of mass frame. The S-operator is invariant with respect to the rotation $\mathcal{R}$ in the center of mass frame for the discrete quantum number j_3 and hence do not depends on j_3. Since the Poincaré transformation does not change the variable s and j, the S-matrix element $S_{jj_3}^{\eta\,\eta'}(s)$ still depends on s and j. Furthermore, the S-matrix in (4.32) must express the conversations of energy and momentum, therefore the S-matrix $S_{jj_3}^{\eta\,\eta'}(s)$ can be expressed in term of the reduced S-matrix $S_{j}^{\eta\,\eta'}(s)$:

$$S_{jj_3}^{\eta\,\eta'}(s) = 2\widehat{p}_0\;\delta(s-s')\;\delta(\widehat{\boldsymbol{p}}'-\widehat{\boldsymbol{p}})\delta_{jj'}\delta_{j_3j_3'}S_{j}^{\eta\,\eta'}(s)\,. \tag{4.33}$$

Using the expression (4.33), the probability density (4.29) can be reduced more in term of reduced or j-th partial S-matrix element $S_{j}^{\eta\,\eta'}(s)$:

$$\begin{aligned} (\psi^{-},\phi^{+}) &= \sum_{j\,J_3}\int_{s_0}^{\infty} ds \int \tfrac{d^3\widehat{\boldsymbol{p}}}{2\widehat{p}^0}\langle\psi^{-}\,|\widehat{\boldsymbol{p}}\,j_3[sj]\eta^{-}\rangle S_{j}^{\eta\,\eta'}(s)\langle^{+}\widehat{\boldsymbol{p}}\,j_3[sj]\eta\,|\phi^{+}\rangle \\ &= \sum_{j}(\psi^{-},\phi^{+})_j\,. \end{aligned} \tag{4.34}$$

The reduced S-matrix $S_j^{\eta\,\eta'}(s)$ in (4.34) is then connected to the scattering amplitude of the j-th partial wave $a_j(s)$ of the scattering process which is analogous with (3.31). The j-th partial probability density $(\psi^-,\phi^+)_j$ responds to the reduced S-matrix $S_j^{\eta\,\eta'}(s)$ with fixed total angular momentum j as:

$$(\psi^-,\phi^+)_j = \int_{s_0}^{\infty} ds \sum_{j_3} \int \frac{d^3\widehat{\boldsymbol{p}}}{2\widehat{p}^0}\, \langle \psi^-\,|\widehat{\boldsymbol{p}}\, j_3[sj]\eta^-\rangle S_j^{\eta\,\eta'}(s)\langle^+\widehat{\boldsymbol{p}}\, j_3[sj]\eta'\,|\phi^+\rangle\,. \tag{4.35}$$

We consider the general case of n resonances created in the collision of the scattering process (3.1). These resonances correspond to the first-order poles of j-th partial S-matrix $S_j(s)$ at the position $s_{R_i}=(M_{R_i}-i\Gamma_i/2)^2$, $i=1,2\cdots n$. Then, we can expand the j-th partial S-matrix $S_j(s)$ as a series of the first-order poles $s_{R_i}=(M_{R_i}-i\Gamma_i/2)^2$:

$$S_j(s)=\sum_{i=1}^{n}\frac{R_i}{s-s_{R_i}}+R_0^i+R_1^i(s-s_{R_i})+\cdots \tag{4.36}$$

According to the relativistic Hardy space axiom (4.15) or corresponding wave function (4.16), the wave functions $\langle\psi^-\mid\widehat{\boldsymbol{p}}\, j_3[sj]\eta^-\rangle$ and $\langle^+\widehat{\boldsymbol{p}}\, j_3[sj]\eta'\mid\phi^+\rangle$ can be analytically continued into the lower complex plane of the s-plane. With the same analytical process , we can deform the contour integral of j-th partial probability density $(\psi^-,\phi^+)_j$ in (4.35) as we did in the non-relativistic case. However, the cut along the positive real axis is started at the physical value $s_0=(m_a+m_b)^2$ in relativistic case instead of origin in non-relativistic case. Therefore, the j-th partial S-matrix of the probability can be expressed in terms of a contour integral C around the pole s_R of $S_j(s)$ and a background which does not depend on the pole s_R:

$$\begin{aligned}(\psi^-,\phi^+)_j &= \int_{s_0}^{\infty} ds \sum_{j_3}\int \tfrac{d^3\widehat{\boldsymbol{p}}}{2\widehat{p}^0}\, \langle\psi^-\,|\widehat{\boldsymbol{p}}\, j_3[sj]\eta^-\rangle S_j^{\eta\,\eta'}(s)\langle^+\widehat{\boldsymbol{p}}\, j_3[sj]\eta'\,|\phi^+\rangle\\ &=\sum_{j_3}\int \tfrac{d^3\widehat{\boldsymbol{p}}}{2\widehat{p}^0}\int_{s_0}^{-\infty_{II}} ds\,\langle\psi^-\,|\widehat{\boldsymbol{p}}\, j_3[sj]\eta^-\rangle S_{II}(s)\langle^+\widehat{\boldsymbol{p}}\, j_3[sj]\eta'\,|\phi^+\rangle+\\ &\qquad+\sum_{j_3}\int \tfrac{d^3\widehat{\boldsymbol{p}}}{2\widehat{p}^0}\oint_{C_i} ds\,\langle\psi^-\,|\widehat{\boldsymbol{p}}\, j_3[sj]\eta^-\rangle \tfrac{R_i}{s-s_{R_i}}\langle^+\widehat{\boldsymbol{p}}\, j_3[sj]\eta'\,|\phi^+\rangle\\ &=\langle\psi^-\,|\phi_j^{bg}\rangle+\sum_{j_3}\int \tfrac{d^3\widehat{\boldsymbol{p}}}{2\widehat{p}^0}\int_{-\infty_{II}}^{\infty} ds\,\langle\psi^-\,|\widehat{\boldsymbol{p}}\, j_3[sj]\eta^-\rangle \tfrac{R}{s-s_R}\langle^+\widehat{\boldsymbol{p}}\, j_3[sj]\eta'\,|\phi^+\rangle\,.\end{aligned} \tag{4.37}$$

Here, the C_i in the second integral in (4.37) the term indicates that there are N integrals around N first-order poles at $z_{R_i}=(M_{R_i}-i\Gamma_i/2)^2$. The ket $|\phi_j^{bg}\rangle$ is the background term which does not relate to the resonance pole s_R and therefore represents the non-resonant background term in the scattering experiment:

$$|\phi_j^{bg}\rangle = \sum_{j_3} \int \frac{d^3\widehat{\boldsymbol{p}}}{2\widehat{p}^0} \int_{s_0}^{-\infty_{II}} ds\ |\widehat{\boldsymbol{p}}\, j_3[sj]\eta^-\rangle S_{II}(s) \langle^+ \widehat{\boldsymbol{p}}\, j_3[sj]\eta'\, |\phi^+\rangle\,. \tag{4.38}$$

We can then define a relativistic Gamow ket or 4-velocity eigenket at a pole $s = s_{R_i}$ as superposition of the exact out-plane waves $|\widehat{\boldsymbol{p}}\, j_3[s\, j]\eta^-\rangle$ with the Breit-Wigner energy distribution $\frac{1}{s-s_{R_i}}$ as we did in the non-relativistic case:

$$\psi^G_{s_{R_i}} = \sqrt{2\pi\Gamma}\ |\widehat{\boldsymbol{p}}\, j_3[s_{R_i}\, j]\eta^-\rangle = i\sqrt{\frac{\Gamma}{2\pi}} \int_{-\infty_{II}}^{\infty} ds\ |\widehat{\boldsymbol{p}}\, j_3[sj]\eta^-\rangle \frac{1}{s - s_{R_i}}\,. \tag{4.39}$$

The exact relativistic Breit-Wigner energy distribution:

$$a_j^{BW}(s) = \frac{R_i}{s - s_{R_i}} \quad \text{with} \quad -\infty_{II} < s < \infty\,, \tag{4.40}$$

is the generalization of the the exact non-relativistic Breit-Winger distribution (3.45) for the relativistic case which extends to negative axis $-\infty_{II} < s < 0$ in the second (unphysical) sheet. The kets $|\widehat{\boldsymbol{p}}\, j_3[s_{R_i}\, j]\eta^-\rangle$ are anti-linear functionals on the Hardy space Φ_+. In other words, the kets $|\widehat{\boldsymbol{p}}\, j_3[s_{R_i}\, j]\eta^-\rangle$ are elements of the dual space $\Phi_+^\times$ of the Hardy space Φ_+, $|\widehat{\boldsymbol{p}}\, j_3[s_{R_i}\, j]\eta^-\rangle \in \Phi_+^\times$.

Similarly, we can consider the complex conjugate of the Born probability density $(\phi^+, \psi^-) = \overline{(\psi^-, \phi^-)}$ or equivalent the j-th partial Born probability density $(\phi^+, \psi^-)_j = \overline{(\psi^-, \phi^-)_j}$ in which the wave functions $\langle\phi^+\, |\widehat{\boldsymbol{p}}\, j_3[sj]\eta^+\rangle = \overline{\langle\psi^-\, |\widehat{\boldsymbol{p}}\, j_3[sj]\eta^-\rangle}$ and $\overline{\langle^- \widehat{\boldsymbol{p}}\, j_3[sj]\eta'\, |\psi^-\rangle} = \langle^- \widehat{\boldsymbol{p}}\, j_3[sj]\eta'\, |\psi^-\rangle$ can be analytically continued into the upper complex plane of the s-plane. The complex conjugate of the S-matrix $S_j^*(s)$ is the analytic functions of the s except for some poles in the upper complex plane in the second sheet of the S-matrix. Therefore, we can expand it as a series of the first -order poles at $s^*_{R_i} = (M_{R_i} + i\Gamma_i/2)^2$:

$$S_j^*(s) = \sum_{i=1}^{n} \frac{R'_i}{s - s^*_{R_i}} + R_0'^i + R_1'^i(s - s^*_{R_i}) + \cdots \tag{4.41}$$

Then we can deform the integral contour along the the cut $s_0 \le s < \infty$ into the same path we did in the non-relativistic case in section 3.4, and get:

$$(\phi^+, \psi^-)_j = \langle\phi^+\, |\phi_j^{*bg}\rangle + \sum_{j_3} \int \tfrac{d^3\widehat{\boldsymbol{p}}}{2\widehat{p}^0} \int_{-\infty_{II}}^{\infty} ds\, \langle\phi^+\, |\widehat{\boldsymbol{p}}\, j_3[sj]\eta^+\rangle \tfrac{R'_i}{s-s^*_{R_i}} \langle^- \widehat{\boldsymbol{p}}\, j_3[sj]\eta'\, |\psi^-\rangle\,. \tag{4.42}$$

The ket $|\phi_j^{*bg}\rangle$ is the background term which does not relate to the anti-resonances poles s_R^*:

$$|\phi_j^{*bg}\rangle = \sum_{j_3} \int \frac{d^3\hat{\boldsymbol{p}}}{2\hat{p}^0} \int_{s_0}^{-\infty_{II}} ds \; |\hat{\boldsymbol{p}}\, j_3[sj]\eta^+\rangle S_{II}(s) \langle^- \hat{\boldsymbol{p}}\, j_3[sj]\eta' \, |\psi^-\rangle \, . \tag{4.43}$$

A relativistic growing Gamow ket or 4-velocity eigenket at a pole $s = s_{R_i}^* = (M_{R_i} + i\Gamma_i/2)^2$ can be defined as superposition of the exact in-plane waves $|\hat{\boldsymbol{p}}\, j_3[s\, j]\eta^+\rangle$ with the exact relativistic Breit-Wigner energy distribution $\frac{1}{s-s_{R_i}^*}$:

$$\psi^G_{s^*_{R_i}} = \sqrt{2\pi\Gamma} \;\; |\hat{\boldsymbol{p}}\, j_3[s^*_{R_i}\, j]\eta^+\rangle = i\sqrt{\frac{\Gamma}{2\pi}} \int_{-\infty_{II}}^{\infty} ds \;\; |\hat{\boldsymbol{p}}\, j_3[sj]\eta^+\rangle \frac{1}{s - s^*_{R_i}} \, . \tag{4.44}$$

The exact Breit-Wigner energy distribution $a_j^{*BW}(s)$ corresponds to the first-order poles in the upper complex s-plane at $s^*_{R_i} = (M_{R_i} + i\Gamma_i/2)^2$ for the whole s spectrum $-\infty_{II} < s < \infty$:

$$a_j^{*BW}(s) = \frac{R'_i}{s - s^*_{R_i}} \quad \text{with} \quad -\infty_{II} < s < \infty \, . \tag{4.45}$$

The Born probability density (ψ^-, ϕ^+) to detect the out-observable ψ^- in the in-state ϕ^+ can then be expressed in terms of the non-resonant term or background term and the term related to the resonances or decaying Gamow vector:

$$(\psi^-, \phi^+) = \langle \psi^- \, |\phi^{bg}\rangle - 2\pi i \sum_{i=1}^{N} \; R_i \; \sum_j \langle \psi^- \, |\hat{\boldsymbol{p}}\, j_3[s_{R_i} j]\eta'^- \rangle \langle^+ \hat{\boldsymbol{p}}\, j_3[s_{R_i} j]\eta' \, |\phi^+\rangle \, . \tag{4.46}$$

Here, ϕ^{bg} is the total background term which is sum of the j-th partial background term ϕ_j^{bg} as (4.38) over the spin j:

$$\phi^{bg} = \sum_j \phi_j^{bg} \, . \tag{4.47}$$

Since the vector $\psi^- \in \Phi_+$ in (4.46) is arbitrary, we can omit it and express the ket $|\phi^+\rangle$ in terms of the background term ϕ^{bg} and the superposition of the Gamow kets $|\hat{\boldsymbol{p}}\, j_3[s_{R_i} j]\eta'^- \rangle$:

$$\phi^+ = \phi^{bg} - 2\pi i \sum_{i=1}^{N} \; R_i \; \sum_j \; |\hat{\boldsymbol{p}}\, j_3[s_{R_i} j]\eta'^- \rangle \langle^+ \hat{\boldsymbol{p}}\, j_3[s_{R_i} j]\eta' \, |\phi^+\rangle \, . \tag{4.48}$$

Similarly, we can express the vector ψ^- in terms of the background term ϕ^{*bg} and sum of the growing Gamow kets $|\widehat{\boldsymbol{p}}\, j_3[s_{R_i}j]\eta'^{+}\rangle$:

$$\psi^- = \phi^{*bg} + 2\pi i \sum_{i=1}^{N} R'_i \sum_j |\widehat{\boldsymbol{p}}\, j_3[s_{R_i}j]\eta'^{+}\rangle\langle^{-}\widehat{\boldsymbol{p}}\, j_3[s_{R_i}j]\eta'\,|\psi^-\rangle\,. \tag{4.49}$$

Where the background ϕ^{*bg} is the total background term which is sum of the j-th partial background term ϕ_j^{*bg} as (4.43) over the spin j:

$$\phi^{*bg} = \sum_j \phi_j^{*bg}\,. \tag{4.50}$$

The expression (4.48) and (4.49) are the generalization of the expansions (3.68) of the in-state ϕ^+ and (3.69) of the observable ψ^- into the relativistic regime in term of the relativistic Gamow vector $|\,\widehat{\boldsymbol{p}}\, j_3[s_{R_i}j]\eta'^{-}\rangle$ and growing Gamow vector $|\widehat{\boldsymbol{p}}\, j_3[s_{R_i}j]\eta'^{+}\rangle$, respectively.

By considering the action of the operator P^μ to the vector ψ^- in the wave function $\langle\psi^-\,|\,\widehat{\boldsymbol{p}}\, j_3[s_R\, j]\eta^-\rangle \in \widetilde{\mathcal{S}} \cap \mathcal{H}^2_-$, we can show that the relativistic Gamow kets $\psi^G_{s_{R_i}}$ or equivalent $|\,\widehat{\boldsymbol{p}}\, j_3[s_{R_i}\, j]\eta^-\rangle$ are eigenvectors of the 4-momentum operators $P^\mu = (P^0, P^i) = (H, P^i)$ with complex eigenvalues $\sqrt{s_R}\;\widehat{p}_\mu$:

$$P^\times_\mu\;\; |\widehat{\boldsymbol{p}}\, j_3[s_R\, j]\eta^-\rangle = \sqrt{s_R}\;\widehat{p}_\mu\, |\widehat{\boldsymbol{p}}\, j_3[s_R\, j]\eta^-\rangle \quad \text{with} \quad \widehat{p}_\mu = (\widehat{p}^0, \widehat{\mathbf{p}}) = (\gamma, \gamma\boldsymbol{v})\,. \tag{4.51a}$$

The (4.51a) can be obtained by following the fact that the $s^{n/2}f(s)$ for any continuous functions $f(s) \in \widetilde{\mathcal{S}} \cap \mathcal{H}^2_-$ are still continuous functions in $\widetilde{\mathcal{S}} \cap \mathcal{H}^2_-$ for any integer numbers n. With the same analysis of considering the actions of the square mass operator $M^2 = P^\mu P_\mu$, total angular momentum operator $-\widehat{w}^2$, and third component of total angular momentum S_3 to to the vector ψ^- in the wave function $\langle\psi^-\,|\,\widehat{\boldsymbol{p}}\, j_3[s_R\, j]\eta^-\rangle \in \widetilde{\mathcal{S}} \cap \mathcal{H}^2_-$, the relativistic Gamow kets $\psi^G_{s_{R_i}}$ or equivalent $|\widehat{\boldsymbol{p}}\, j_3[s_{R_i}\, j]\eta^-\rangle$ can be showed to be eigenvectors of these operators with the eigenvalues, $s = p^\mu p_\mu$, $j(j+1)$ and j_3, respectively:

$$M^2\;\; |\widehat{\boldsymbol{p}}\, j_3[s_R\, j]\eta^-\rangle = s\;\; |\widehat{\boldsymbol{p}}\, j_3[s_R\, j]\eta^-\rangle\,, \tag{4.51b}$$

$$-\widehat{w}^2\;\; |\widehat{\boldsymbol{p}}\, j_3[s_R\, j]\eta^-\rangle = j(j+1)\; |\widehat{\boldsymbol{p}}\, j_3[s_R\, j]\eta^-\rangle\,, \tag{4.51c}$$

$$S_3\;\; |\widehat{\boldsymbol{p}}\, j_3[s_R\, j]\eta^-\rangle = j_3\;\; |\widehat{\boldsymbol{p}}\, j_3[s_R\, j]\eta^-\rangle\,. \tag{4.51d}$$

The equation (4.51a)- (4.51d) show that the spaces of Gamow vectors $|\widehat{\boldsymbol{p}}\, j_3[s_R\, j]\eta^-\rangle$ labeled by a pair numbers $[s, j]$ representing the resonances or unstable particles is completely analogy to the Wigner space $[m^2, j]$ for stable particles as in (4.23). The difference is that the generalized eigenvalue $\sqrt{s}$ of the mass operator for the relativistic Gamow vectors is a complex number, instead of real number.

4.4 Relativistic Time Asymmetric Evolution

We have defined the relativistic decaying Gamow kets $|\widehat{\boldsymbol{p}}\, j_3[s_{R_i}\, j]\eta^-\rangle$ in (4.39) and growing Gamow kets $|\,\widehat{\boldsymbol{p}}\, j_3[s^*_{R_i}\, j]\eta^+\rangle$ in (4.44) as the superpositions of the exact incoming-plane wave $|\widehat{\boldsymbol{p}}\, j_3[s\, j]\eta^-\rangle$ and exact out-going plane wave $|\widehat{\boldsymbol{p}}\, j_3[s\, j]\eta^+\rangle$ with the exact relativistic Breit-Wigner energy distributions (4.40) and (4.45) in which the invariant mass squared s extends into the unphysical sheet $\infty_{II} < s < \infty$. These Gamow kets $|\widehat{\boldsymbol{p}}\, j_3[s_{R_i}\, j]\eta^\pm\rangle$ are associated with the first-order poles of the S-matrix in the lower and upper s complex plane in the second sheet of the S-matrix which are responding to the resonances, or decaying states and the anti-resonances, or growing states.

The space-time evolution of the observable ψ^- in the relativistic regime is described by operator $\mathcal{U}^\times_+(I,x)$ of the space-time translation (I,x) :

$$\psi^-(x) = \mathcal{U}_+(I,x)\psi^- = e^{iP.x}\psi^- \,. \tag{4.52}$$

Similarly, the space-time evolution of the in-state ϕ^+ is relativistically described by operator $\mathcal{U}(I,x)$:

$$\phi^+(x) = \mathcal{U}_-(I,x)\phi^+ = e^{-iP.x}\phi^+ \,. \tag{4.53}$$

Here we use the subscript $\pm$ to describe the operators corresponding to the space-time translation of the observable ψ^- and of the state ϕ^+ which are also used to represent the operators corresponds to the forward light cone and to backward light cone of semi-groups of the Poincaré transformation.

Since the operator e^{iHt} is a continuously operator on the space Φ_+ for the time $t \geq 0$, the time evolution of the observable e^{iHt} for $\psi^- \in \Phi_+$ is valid only for $t \geq 0$. We consider the time evolution of the wave function $\langle\psi^-(x)\,|\widehat{\boldsymbol{p}}\, j_3[s_R\, j]\eta^-\rangle \in \widetilde{\mathcal{S}} \cap \mathcal{H}^2_-$

in the center of mass frame in which $\widehat{\boldsymbol{p}} = 0$

$$\begin{aligned}\langle\psi^-(x)\,|\widehat{\boldsymbol{p}}=0,j_3[s_R\,j]\eta^-\rangle &= \langle\mathcal{U}_+(I,x)\psi^-\,|\widehat{\boldsymbol{p}}=0,j_3[s_R\,j]\eta^-\rangle\\ &= \langle\psi^-\,|\,\mathcal{U}_+^\times(I,x)\;|\widehat{\boldsymbol{p}}=0,j_3[s_R\,j]\eta^-\rangle\\ &= \langle\psi^-\,|\,e^{-iH^\times t}\;|\widehat{\boldsymbol{p}}=0\,,j_3[s_R\,j]\eta^-\rangle\\ &= e^{-i\gamma\sqrt{s_R}t}\langle\psi^-\,|\widehat{\boldsymbol{p}}=0\,,j_3[s_R\,j]\eta^-\rangle \text{ for only } t\geq 0\,.\end{aligned} \tag{4.54}$$

Omitting the arbitrary vector $\psi^-\in\Phi_+$ in (4.54), we get the time evolution of the ket $|\widehat{\boldsymbol{p}}=0,j_3[s_R\,j]\eta^-\rangle$ or the Gamow ket $\psi^G_{s_R}$ in the center of mass frame $\widehat{\boldsymbol{p}}=0$:

$$\begin{aligned}\mathcal{U}_+^\times(I,x)\;|\widehat{\boldsymbol{p}}=0,j_3[s_R\,j]\eta^-\rangle &= e^{-iH^\times t}\;|\widehat{\boldsymbol{p}}=0\,,j_3[s_R\,j]\eta^-\rangle\\ &= e^{-i\gamma\sqrt{s_R}t}\;|\widehat{\boldsymbol{p}}=0\,,j_3[s_R\,j]\eta^-\rangle \text{ for only } t\geq 0\,.\end{aligned} \tag{4.55}$$

The wave function $\langle\psi^-(x)\,|\widehat{\boldsymbol{p}}\,j_3[s_R\,j]\eta^-\rangle\in\widetilde{\mathcal{S}}\cap\mathcal{H}^2_-$ can be obtained from wave function in the center of mass frame $\langle\psi^-(x)\,|\widehat{\boldsymbol{p}}=0,j_3[s_R\,j]\eta^-\rangle$ by using the equation (4.15). The condition $t\geq 0$ does not changed when we move from the center of mass frame to laboratory frame since $(\mathcal{U}^\dagger(L(\widehat{p}))^\times$ operator commutes with the the operator $\mathcal{U}_+^\times(I,x)$ responding to time-space translations. Therefore, the time-space translation of the wave function $\langle\psi^-(x)\,|\widehat{\boldsymbol{p}}\,j_3[s_R\,j]\eta^-\rangle\in\widetilde{\mathcal{S}}\cap\mathcal{H}^2_-$ in the laboratory is given by:

$$\begin{aligned}\langle\psi^-(x)\,|\widehat{\boldsymbol{p}},j_3[s_R\,j]\eta^-\rangle &= \langle\mathcal{U}_+(I,x)\psi^-\,|\widehat{\boldsymbol{p}},j_3[s_R\,j]\eta^-\rangle\\ &= \langle\psi^-\,|\,\mathcal{U}_+^\times(I,x)\;|\widehat{\boldsymbol{p}},j_3[s_R\,j]\eta^-\rangle\\ &= \langle\psi^-\,|\,e^{-iP\cdot x}\;|\widehat{\boldsymbol{p}}\,,j_3[s_R\,j]\eta^-\rangle\\ &= e^{-i\gamma\sqrt{s_R}(t-\boldsymbol{x}\cdot\boldsymbol{v})}\langle\psi^-\,|\widehat{\boldsymbol{p}}\,,j_3[s_R\,j]\eta^-\rangle \text{ for only } t\geq 0\,.\end{aligned} \tag{4.56}$$

However, $t\geq 0$ is not enough guarantee for $e^{-i\gamma\sqrt{s_R}(t-\boldsymbol{x}\cdot\boldsymbol{v})}\langle\psi^-\,|\widehat{\boldsymbol{p}}\,,j_3[s_R\,j]\eta^-\rangle\in\widetilde{\mathcal{S}}\cap\mathcal{H}^2_-$. With analogous condition for the parameter $t\geq 0$ in exponent in (4.54), it requires that

$$t-\boldsymbol{x}\cdot\boldsymbol{v}\geq 0 \quad\Longrightarrow\quad t\geq\boldsymbol{x}\cdot\boldsymbol{v} \quad\text{or}\quad 1\geq\boldsymbol{v}^2\,, \tag{4.57}$$

in order for the wave function $e^{-i\gamma\sqrt{s_R}(t-\boldsymbol{x}\cdot\boldsymbol{v})}\langle\psi^-\,|\widehat{\boldsymbol{p}}\,,j_3[s_R\,j]\eta^-\rangle\in\widetilde{\mathcal{S}}\cap\mathcal{H}^2_-$ [41]. Furthermore, the condition (4.57) implies that the velocity $\boldsymbol{v}$ is always less then the speed of light c, i.e, $\boldsymbol{v}\leq 1$. As a consequence of condition (4.57), the space-time interval x^2 of the 4-vector $(t,\boldsymbol{x})$ fulfills constraint condition:

$$x^2=t^2-\boldsymbol{x}^2\geq 0\,. \tag{4.58}$$

Therefore, the space-time translation of the decaying Gamow ket ϕ^G or equivalent $|\widehat{\boldsymbol{p}}, j_3[s_R\, j]\eta^-\rangle$ is given by

$$\mathcal{U}_+^\times(I,x)\, |\widehat{\boldsymbol{p}}, j_3[s_R\, j]\eta^-\rangle = e^{-i\gamma\sqrt{s_R}(t-\boldsymbol{x}\cdot\boldsymbol{v})}\, |\widehat{\boldsymbol{p}}\,, j_3[s_R\, j]\eta^-\rangle \text{ for only } t \geq 0\,, x^2 \geq 0. \quad (4.59)$$

The transformations of the 4-velocity kets $|\widehat{\boldsymbol{p}}\, j_3[sj], \eta^\mp\rangle \in \Phi_\pm^\times$ under inhomogeneous Lorentz transformation (Λ, x) are given by:

$$\mathcal{U}_+^\times(\Lambda,x)\, |\widehat{\boldsymbol{p}}\, j_3[sj], \eta^-\rangle = e^{-i\,p.x} \sum_{j_3'} D^j_{j_3\, j_3'}(W(\Lambda^{-1},p))\, |\Lambda^-1\widehat{\boldsymbol{p}}\, j_3[sj], \eta^-\rangle \text{ for } t \geq 0, x^2 \geq 0\,, \quad (4.60)$$

With analogous analysis for the wave function $\langle\phi^+(x)\,|\widehat{\boldsymbol{p}}\, j_3[s_R\, j]\eta^+\rangle \in \widetilde{\mathcal{S}} \cap \mathcal{H}_+^2$, the constraint conditions for time t and the space-time interval for 4-vector $(t, \boldsymbol{x})$ are:

$$x^2 = t^2 - \boldsymbol{x}^2 \geq 0 \qquad \text{and} \qquad t \leq 0\,. \quad (4.61)$$

The transformation of the growing Gamow ket $\phi^G_{s_R^*}$ or equivalent $|\widehat{\boldsymbol{p}}, j_3[s_R^*\, j]\eta^+\rangle$ under space-time translation (I, x) is given by:

$$\mathcal{U}_-^\times(I,x)\, |\widehat{\boldsymbol{p}}, j_3[s_R^*\, j]\eta^+\rangle = e^{-i\gamma\sqrt{s_R^*}(t-\boldsymbol{x}\cdot\boldsymbol{v})}\, |\widehat{\boldsymbol{p}}\,, j_3[s_R^*\, j]\eta^+\rangle \text{ for only } t \leq 0\,, x^2 \geq 0\,, \quad (4.62)$$

and under inhomogeneous Lorentz transformation (Λ, x) is given by

$$\mathcal{U}_-^\times(\Lambda,x)\, |\widehat{\boldsymbol{p}}\, j_3[sj], \eta^+\rangle = e^{-i\,p.x} \sum_{j_3'} D^j_{j_3\, j_3'}(W(\Lambda^{-1},p))\, |\Lambda^-1\widehat{\boldsymbol{p}}\, j_3[sj], \eta^+\rangle \text{ for } t \leq 0, x^2 \geq 0\,. \quad (4.63)$$

Corresponding to the two distinct Hardy spaces $\Phi_\pm$ for the out-observables ψ^- and in-states ϕ^+ stated in the Hardy space axiom (2.14), there are two semi-groups of the Poincaré transformation. One is into the forward light cone ($t \geq 0$) for the Hardy space Φ_+ of the upper complex s-plane $\mathbb{C}_+$ on second sheet of S-matrix [41]:

$$\boldsymbol{\mathcal{P}}_+ = \{(\Lambda, x) \mid \Lambda \in \overline{SO(3,1)},\ \det\Lambda = +1,\ \Lambda^0_0 \geq 0,\ x^2 = t^2 - \mathbf{x} \geq 0, t \geq 0\}\,. \quad (4.64a)$$

Another semi-group of the Poincaré transformation is into the backward light cone ($t \leq 0$) for the Hardy space Φ_- of the lower complex s-plane $\mathbb{C}_+$ on second sheet of S-matrix:

$$\boldsymbol{\mathcal{P}}_- = \{(\Lambda, x) \mid \Lambda \in \overline{SO(3,1)},\ \det\Lambda = +1,\ \Lambda^0_0 \geq 0,\ x^2 = t^2 - \mathbf{x} \geq 0, t \leq 0\}\,. \quad (4.64b)$$

The time t is dilated by a factor γ when we move from the laboratory frame to the center of mass frame $\widehat{\boldsymbol{p}} = 0$, $t' = \gamma t$ which is a proper time. The time evolution of the relativistic Gamow kets $\psi^G_{s_R}(t')$ at time $t' = t\gamma$ with $\sqrt{s_R} = M_R - i\Gamma/2$ in the center of mass frame (or at the time t in the laboratory frame) is given by

$$\begin{aligned}
\psi^G_{s_R}(t') &= e^{-iH^\times t'}\sqrt{2\pi\Gamma}\ |\hat{\mathbf{p}} = 0\,, j_3\,[s_R\,j]\eta^-\rangle \\
&= \sqrt{2\pi\Gamma} e^{-i\sqrt{s_R}\,t'}\ |\hat{\mathbf{p}} = 0\,, j_3\,[s_R\,j]\eta^-\rangle \\
&= \sqrt{2\pi\Gamma} e^{-iM_R t'} e^{-(\Gamma_R/2)t'}\ |\hat{\mathbf{p}} = 0\,, j_3\,[s_R\,j]\eta^-\rangle \\
&= e^{-iM_R t'} e^{-(\Gamma_R/2)t'}\ \psi^G_{s_R}(\hat{\mathbf{p}} = 0)\,.
\end{aligned} \tag{4.65}$$

and is valid only for $t' \geq 0$. The probability to detect an observable ψ^- in the Gamow ket $\psi^G_{s_R}(t')$ in the center of mass frame is then given by:

$$\begin{aligned}
|\langle\psi^-|\psi^G_{s_R}(t')\rangle|^2 &= |e^{-iM_R t'} e^{-(\Gamma_R/2)t'}\langle\psi^-|\psi^G_{s_R}(\hat{\mathbf{p}} = 0)\rangle|^2 \\
&= e^{-\Gamma_R\,t'}\ |\langle\psi^-|\hat{\mathbf{p}} = 0\,, j_3\,[s_R\,j]\eta^-\rangle|^2\,,
\end{aligned} \tag{4.66}$$

and exists only for $t' \geq 0$. This probability (4.66) follows the exact exponential decay for the decaying state which corresponds to the resonance defined as the first-order pole of the S-matrix. Again, we get an exact exponential decay for the Gamow state with a lifetime τ which is exactly given by the width $\Gamma = \Gamma_R$ of the corresponding scattering resonance in the relativistic regime:

$$\tau = \frac{1}{\Gamma} \quad \text{or} \quad \tau = \frac{\hbar}{\Gamma}\,. \tag{4.67}$$

Therefore, the resonances in the relativistic scattering process or unstable, decaying states in relativistic regime are described by the relativistic Gamow kets $\psi^G_{s_R}$ defined in (4.39). These Gamow kets $\psi^G_{s_R}$ are associated with the first-order poles of the S-matrix at $s_{R_i} = (M_{R_i} - i\Gamma_i/2)^2$. As a result of (4.65), the time evolution of the relativistic Gamow kets $\psi^G_{s_R}$ is not given by an unitary group for which the time t extends for all time $-\infty < t < \infty$, but by the semi-group of the Poincaré transformation (4.64a) for which the time t extends over $0 \leq t < \infty$.

Chapter Five: Beginning of Time in the Hardy Space Formalism and Experimental Observations

5.1 Beginning of Time in Hardy Space Quantum Mechanics

In Hardy space formalism, an observable $A(t) =|\psi^-(t)\rangle\langle\psi^-(t)\,|$ cannot be measured in a state ϕ^+ at a time $t < t_0$, where t_0 is the (finite) time at which the state ϕ^+ has been prepared (in the laboratory): A state ϕ^+ needs to be prepared first, before an observable $|\,\psi^-(t)\rangle\langle\psi^-(t)\,|$ can be detected in ϕ^+ at a time $t > t_0$ which can mathematically be chosen as zero. The observable $\psi^-(t)$ can be registered in this state by detector counts $\frac{N(t)}{N}$ only for $t > t_0$. E.g, a detector for the observable $|\,\psi^-(t)\rangle\langle\psi^-(t)\,|$ *cannot* count the decay products of a decaying state $\phi^G(t)$, before the decaying quantum systems $\phi^G(t)$ has been prepared (in a resonance scattering experiment). The decay products of a decaying state (e.g. a Gamow state ϕ^G) can only be counted at a time t *after* the time t_0, i.e., for $t > t_0$ [43].

The finite time $t_0(=0)$ is defined as the preparation time of the state ϕ^-– by preparation apparatus. It reflects the initial time of the decaying state ψ^G of a single scattering process, i.e, *beginning of time* of the decaying process: $\psi^G(t) \longrightarrow \psi^-$. This finite time $t_0 = 0$ does not exist in the conventional quantum theory based on the Hilbert or Schwartz space. This is a result of solving the dynamical equation under the Hardy space boundary condition. This is in agreement with the intuitive idea: that a state ϕ^+ must be prepared first by a time $t_0 = 0$, before the observable $\psi^-(t)$ can be detected in the state ϕ^+ for times $t > t_0(= 0)$ which the decaying state ψ^G has been created, as well causality condition: the detection time t of the decay products $\psi^-(t)$ must be later than the preparation time $t_0 = 0$ of the decaying state ψ^G, $t_0(= 0) < t < \infty$. This means that the time evolution in the real world or quantum system is time asymmetric [44].

The initial or beginning of time t_0 can experimentally be justified by directly measuring the lifetime of the decaying state ψ^G in the scattering experiment. In principle, if a decaying state ψ^G is formed at the time t_0 and decays at the time t on the laboratory's clock in non-relativistic regime, then a lifetime τ_{ψ^G} of the decaying state ψ^G would be measured as time interval Δt:

$$\tau_{\psi^G} = \Delta t = t - t_0 \,. \tag{5.1}$$

Experimentally, one prepare an ensemble of arbitrary large number N identical micro-systems (e.g., electrons of the molecules or beam of particles in scattering experiment) in the same state ϕ^+ in a laboratory and performing measurements under the same condition. In principle, there are two different but completely equivalent ways to prepare this ensemble. The first way is that one can prepare all of the N micro-systems or decaying events simultaneously but at different places at one instant of time. This kind of preparation can be done in the scattering experiment in which the ensemble is beams of same particles accelerated via accelerator. The beginnings of time of individual decaying events will be at t_0^i. The set of times $\{t_0^1, t_0^2, ..., t_0^N\}$ is referred as the ensemble of beginnings of time. Since this ensemble describes the times at which the same state ϕ^+ is prepared, all of these times must be exactly the same at t_0 which in general can be set as zero. Therefore, the ensemble of beginnings of time for the ensemble of N-decaying events is

$$\{t_0^1, t_0^2, ..., t_0^N\} = t_0 = 0 \,. \tag{5.2}$$

Each decay state of the ensemble then decays into decaying products ψ^- which are detected at detectors at the time $t^i > t_0$. If each of time t_i can be measured, then the lifetime of each decaying state can be calculated as

$$\tau^i = \Delta t^i = t^i - t_0^i \,. \tag{5.3}$$

In principle, decaying states in the ensemble are represented same decaying state and hence the lifetime τ^i will be same,

$$\tau = \tau^i = t^1 - t_0^1 = ... = t^N - t_o^N \,. \tag{5.4}$$

In realistic experiment, because of the systematic errors in measurement, the decaying times t^i will be slightly different. Therefore, the lifetime τ of the decaying states will be calculated as the average value of the lifetimes τ^i:

$$\tau = \text{average value of } \{t^1, t^2, ..., t^N\} \,. \tag{5.5}$$

An example of this preparation is the KLOE (K Long Experiment) experiment which will be discussed in detail in framework of Rigged Hilbert space in section 5.3. The KLOE experiment justifies the prediction of beginning of time of relativistic quantum mechanics formalism based on the Hardy spaces in chapter 4.

The another way to prepare decaying experiments is that one can prepare a single micro-system in the unstable state at a time and repeat its preparation N times at different preparation time t_0^i. An example of this preparation is the quantum jump of single ion which will be discussed in section 5.2. In this experiment, decaying events are different from each other and take place in chronological order as $t_0^1 < t_0^1 < ... < t_0^N$. The ensemble of the beginnings of time is

$$\{t_0^1, t_0^2, ..., t_0^N\} \neq \{0\}\,, \quad \text{with} \quad t_0^1 < t_0^1 < ... < t_0^N\,. \tag{5.6}$$

The lifetime of unstable state in $i-th$ micro-system in the ensemble will be calculated as

$$\tau^i = t^i - t_0^i,\,, \quad \text{for} \quad i = 1, 2, ..., N\,. \tag{5.7}$$

Theoretically, these lifetime τ_i would have the same value since they are the lifetime of the same unstable-states which are just prepared in different time. However, real measurement in the laboratory will be yielded to slightly different values due to the systematic and random errors during the experiments. The average of the lifetime of the unstable state can be calculated as in (5.5).

5.2 Quantum Jump and Non-relativistic Experimental Observation of Beginning of Time

The quantum jump experiments [45][46][47][48][49] on metastable states of single ions in the Paul trap have been done using Dehmelts proposal of shelving the single ion on a metastable level for a sensitive optical double-resonance detection method. The Dehmelt's idea can be simplified by considering three energy levels as in Fig. 5.1

The level $|g\rangle$ is the ground state of single ion. The level $|e\rangle$ is the excited state with a short lifetime, i.e, high spontaneous decay rate than the levels $|m\rangle$ which is the long-lifetime metastable state. A laser source is applied to the single ion which

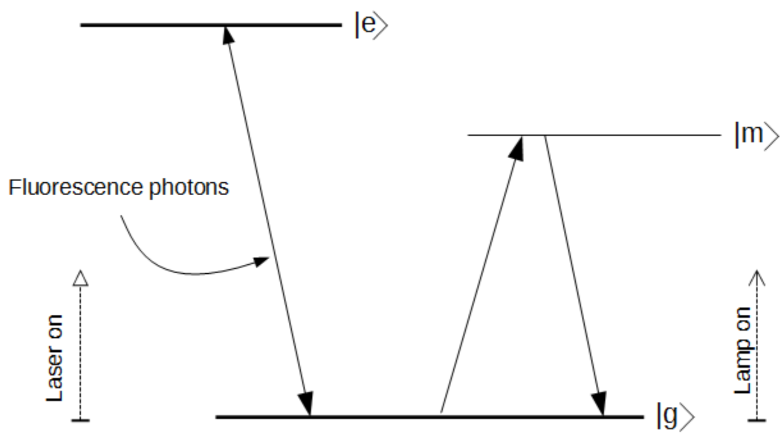

Figure 5.1: Simplified energy level diagram of a three-level ion

initially is in the ground state $|g\rangle$ to drive the ion from the ground state $|g\rangle$ to the excited state $|e\rangle$. The ion in the excited state $|e\rangle$ then spontaneously decay back to the ground state $|e\rangle$ and emit a fluorescent photon which can be monitored by detectors. The second source such as lamp will be turned on to attempt to drive the ion from the ground state $|g\rangle$ to the metastable state $|m\rangle$. The ion will remain in the metastable state for a lifetime of the metastable state. The ion is then shelved in the metastable state $|m\rangle$ and can no longer be driven to the excited state $|e\rangle$ by the resonant laser.

The transition of the ion from ground state $|g\rangle$ to metastable state $|m\rangle$ is then detected by the absence of many fluorescence photons or dark periods in fluorescent monitor as in Figure 5.2 for single ion Ba^+ in the Paul trap. The duration of the shelf-time spent in the metastable state $|m\rangle$, i.e, the lifetime of the metastable state, is therefore observed as a dark period in the fluorescence. In the experiment [45], here were $N = 203$ such dark periods, three were shown in the Figure 5.2. The fluorescence photon is emitted by transition from excited state $|e\rangle = 6^2\ P_{3/2}$ to the ground state $|e\rangle = 6^2\ S_{1/2}$ and the dark period is the duration when the ion Ba^+

jumps from the background state $|g\rangle = 6^2\ S^{1/2}$ to the metastable state $|m\rangle = 5^2\ D_{5/2}$ through the intermediate state $6^2\ P_{3/2}$.

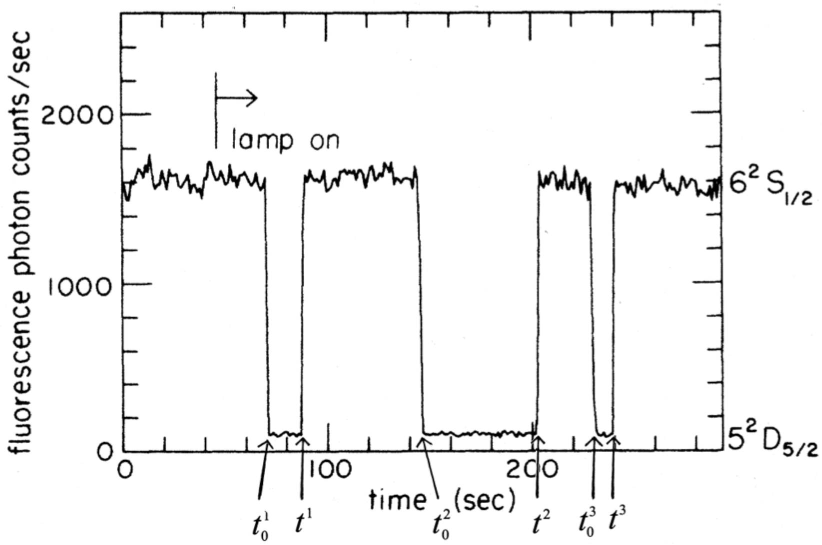

Figure 5.2: Amplification of single quantum jumps by the fluorescence of $S_{1/2} \leftrightarrow P_{1/2}$. The three (out of 203) shown onset times t_0^1, t_0^2, t_0^3 of dark fluorescence are the preparation times of single "$D_{5/2}$"-quantum systems. (This figure is taken from [45])

The onset of a dark period at t_0^i indicates that at these times t_0^i the ion jumps to metastable state $|m\rangle$ and has been shelved in this metastable state $|m\rangle$ during the natural lifetime of metastable state which were shown in the fluorescence monitor as the dark periods. The ion then decays back to the ground state $|g\rangle$ at the dwell time t^i which were shown in the fluorescence monitor as the return of fluorescence. The the dwell time or lifetime τ^i of $i-th$ metastable state $|m\rangle$ is measured as the duration of the $i-th$ dark period:

$$\tau^i = t^i - t_0^i\,, \quad \text{for} \quad i = 1, ..., M\,. \tag{5.8}$$

The duration of the length of the dark periods in the single ion quantum jump experiments have justified the initial time or beginning of time for quantum system predicted in the Rigged Hilbert Space formalism in chapter 3 in non-relativistic regime. Therefore The ensemble of the beginnings of time and of the dwell time are $\{t_0^i\}$ and $\{\tau^i\}$ for $i = 1, 2, ..., M$, respectively.

In this experiment [45], the state vector ϕ^+ describes an ensemble of single particles in Paul traps, thus the preparation time t_0 of the state ϕ^+ or the beginning of time of the decaying state ψ^G which later decays, is detected as an ensemble of times $t_0^{(i)}$ on the clocks in the laboratory and also t is an ensemble of times $t^{(i)}$ on the clock in the laboratory. The lifetime of a decaying state ϕ^G is then the weighted average of the individual lifetimes $(t^{i)} - t_0^{(i)})$ of the dark periods shown in Figure 5.2 as the durations of the (ensemble of) dark periods in the experiment with single ions in the Paul trap [50].

The decaying state representing metastable state is Gamow state $\psi^G = |5^2\ D_{5/2}\ ^-\rangle$ and the observable is $\psi^- = |5^2\ S_{1/2}\rangle$. Each metastable state $i-th$ is created at the time t_0^i and decays after τ^i time. The decay state $\psi^G = |\,5^2\ D_{5/2}\ ^-\rangle$ follows the exponential decay laws as

$$\rho(t) = e^{-\Gamma t} \quad \text{and} \quad \tau = 1/\Gamma \text{ is the mean lifetime}\,. \tag{5.9}$$

The equation (5.9) is the =the survival probability for a single unstable particle which is created at $t = 0$. In the quantum jump experiment, equation (5.9) is the survival probability of an ion in a metastable state $\psi^G = |5^2\ D_{5/2}\ ^-\rangle$ at the a given duration in time from when it was initially prepared. Therefore, the time parameter t is is the time evolution parameter to which the duration of each dwell time τ^i is compared. Theoretically, it is the probability for a single ion at the time t is still in the prepared the metastable state $|m\rangle$:

$$\mathcal{P}_{|m\rangle}(|m(t)\rangle) = |\langle m\,|m(t)\rangle|^2 = e^{-t/\tau}\,. \tag{5.10}$$

Where the time evolution of the metastable $|m(t)\rangle$ is described by the semi group as in (3.84):

$$U(t) = e^{-iH^\times t} \quad \text{for} \quad 0 \le t < \infty\,, \tag{5.11}$$

and

$$H^\times\ |m\rangle = (E - i\Gamma/2)\ |m\rangle\,. \tag{5.12}$$

The number of the metastable states $N_{|m\rangle(t)}$ at the time t is the number of the metastable states of single ion for which their dwell times τ^i is greater than evolution parameter t [52]:

$$N_{|m\rangle(t)} \equiv \text{Number of } \ |m\rangle \text{ for which } \tau^i > t\,. \tag{5.13}$$

The experimental survive probability is then calculated as the counting rate of the number of the metastable state $| \, m\rangle(t)$ over the total number of dwell times (dark periods) measured in the experiment M which is then compared with the theoretical Born probability (5.10):

$$\frac{N_{|m\rangle(t)}}{M} = \mathcal{P}_{|m\rangle}(|m(t)\rangle) = e^{-t/\tau}\,. \tag{5.14}$$

By the comparison between the theoretical Born probability $\mathcal{P}_{|m\rangle}(|\, m(t)\rangle$ and the experimental counting rate, i.e, fitting the history of dwell times of duration with the exponential law in [45] or the logarithmic plot of the the experimental number of dwell times of duration greater than the time parameter t [52] , one can determine the beginning of time $t_0 = 0$ of the semi-group (5.10). Therefore, there is correspondence between beginning of time $t = 0$ of the time parameter, t which parametrizes the evolution of the decaying state or metastable state $\psi^G = |m\rangle$ with the ensemble of M preparation times measured by clocks in the laboratory, $\{t_0^i\}$, for $i = 1, 2, .., M$, at which corresponding single ion in the ensemble of single ions to be prepared in the metastable state $|m\rangle$:

$$\{t_0^i : \; i = 1, , 2..., M\} \Longleftrightarrow t_0 = 0\,. \tag{5.15}$$

The correspondence (5.15) justifies not only the beginning of time for each individual decaying state but also the ensemble of the beginnings of time, which are predicted by the Rigged Hilbert space formalism. This also verifies that the time evolution in the decaying phenomena is time-asymmetric, not symmetric.

5.3 Kaon Physics and Relativistic Observation of Beginning of Time

Kaons have a relatively long lifetime because they decay only through the weak interaction. As a result, studies of their decays provide key insights into the behavior of the weak interaction under the three fundamental symmetry operators C, P, and T. In this dissertation, we discuss the Kaon phenomena with the KLOE detector at DAΦNE collider. At DAΦNE collider, Kaons are created from the decay of Φ mesons produced in e^+e collisions around 1020 MeV, the mass of the Φ-meson. The Φ-meson, lifetime $\tau_\Phi = 1.55 \times 10^{-22}\ s$, decays dominantly to charged kaon pairs

$K^{\pm}$ (49%), neutral kaon pairs K_S and K_L (34%), $\rho\ \pi$ (15%), and $\eta\gamma$ (1.3%). A KLOE detector was designed to detect the decay products from Kaon and especially to minimize $K_L \rightarrow K_S$ regenerations. The $K_L \rightarrow K_S$ regenerations is process in which K_S generated from the strong interaction of a K_L with the traversed medium. These regenerations might simulate CP violating decays which make the test of CP violating be less accuracy [53].

5.3.1 Kaon Physics in Hardy Space

The neutral K mesons K_0 and $\overline{K}_0$ are eigenstates of the parity operator P and charge conjugate to each other under the charge operator C. Therefore, they transform one into another under the action of CP operator :

$$CP\,|K_0\rangle = -\,|\overline{K}_0\rangle\,, \quad CP\,|\overline{K}_0\rangle = -\,|K_0\rangle\,. \tag{5.16}$$

The linear combinations K_1 and K_2 of K_0 and $\overline{K}_0$ are the eigenvectors of the CP operator:

$$|K_1\rangle = \frac{1}{\sqrt{2}}(|K_0\rangle - |\overline{K}_0\rangle) \qquad \text{and} \qquad CP\,|K_1\rangle = -\,|K_1\rangle\,, \tag{5.17a}$$

$$|K_2\rangle = \frac{1}{\sqrt{2}}(|K_0\rangle + |\overline{K}_0\rangle) \qquad \text{and} \qquad CP\,|K_2\rangle = +\,|K_2\rangle\,. \tag{5.17b}$$

When CP is conserved, the allowed decay for K_1 is only into 2π while the K_2 can decay into $3\pi\,,\ \pi\,e\nu,..$ The lifetime of the K_1 kaon is short ($\tau_S\ \approx 8.92\times10^{11}s$), while the lifetime of the K_2 kaon is quite longer ($\tau_L\ \approx 5.17\times10^{8}s$). CP-symmetry can be violated which was discovered by Christenson et al. [54]. This means that the long-lived Kaon can also decay to 2π. The CP symmetry is slightly violated (by a factor of 103) by weak interactions. Therefore, the CP eigenstates K_1 and K_2 are not exact eigenstates of the decay interaction. Let us denote K_S (S means short-lived) and K_L (L means long-lived) are the eigenstates of the decay interaction, then they can express in the term of the K_1 and K_2 or equivalently in terms of K_0 and $\overline{K}_0$ as

$$|K_S\rangle = \frac{1}{\sqrt{1+|\epsilon|^2}}(|K_1\rangle + \epsilon\,|K_2\rangle = \frac{1}{\sqrt{2(1+|\epsilon|^2)}}((1+\epsilon)\,|K_0\rangle + (1-\epsilon)\,|\overline{K}_0\rangle)\,, \tag{5.18a}$$

$$|K_L\rangle = \frac{1}{\sqrt{1+|\epsilon|^2}}(\epsilon\,|K_1\rangle + |K_2\rangle = \frac{1}{\sqrt{2(1+|\epsilon|^2)}}((1+\epsilon)\,|K_0\rangle - (1-\epsilon)\,|\overline{K}_0\rangle)\,. \tag{5.18b}$$

Here, ϵ is a CP violation parameter and a complex number in general, $|\epsilon| \ll 0$.

The neutral Kaon pairs originate from Φ-meson at rest. In the rest frame of Φ-meson (or the lab's frame), two neutral Kaons flight in opposite direction and have the same momentum $\boldsymbol{p}_S = \boldsymbol{p}_L = \boldsymbol{p}$. The K_S and K_L are defined to be Gamow vectors $| K_S^- \rangle = | K_S, \boldsymbol{p}, s_S^- \rangle$ and $| K_L^- \rangle = | K_S, -\boldsymbol{p}, s_L^- \rangle$ of the complex exact Hamiltonian (including the violate CP term) with the complex eigenvectors $\sqrt{S_{R_{S(L)}}} = \left(m_{S(L)} - i\frac{\Gamma_{S(L)}}{2} \right)$, respectively:

$$H^{\times} |K_S^-\rangle = \left(m_S - i\frac{\Gamma_S}{2} \right) |K_S^-\rangle \,, \tag{5.19a}$$

$$H^{\times} |K_L^-\rangle = \left(m_L - i\frac{\Gamma_L}{2} \right) |K_L^-\rangle \,. \tag{5.19b}$$

Here, $H = H_0 + H_w + H_{sw} = H_0 + H_{int}$ is the exact Hamiltonian; H_w is weak Hamiltonian; H_{sw} is the violate CP Hamiltonian and $H_{int} = H_w + H_{sw}$ is the interaction Hamiltonian.

The space-time evolutions of the Gamow vectors $|K_{S,L}^-\rangle$ are described by the semigroup of the Poincare transformation into the forward light cone (4.64a) by (4.57):

$$\mathcal{U}_+^{\times}(I, x)\, |K_S^-\rangle = e^{-i\gamma_S \sqrt{s_S}(t - \boldsymbol{x}\cdot\boldsymbol{v}_S)}\, |K_S^-\rangle \,, \quad t \geq 0\,, x^2 \geq 0\,, \tag{5.20a}$$

$$\mathcal{U}_+^{\times}(I, x)\, |K_L^-\rangle = e^{-i\gamma_L \sqrt{s_L}(t - \boldsymbol{x}\cdot\boldsymbol{v}_L)}\, |K_L^-\rangle \,, \quad t \geq 0\,, x^2 \geq 0\,. \tag{5.20b}$$

Since $\frac{\Gamma_S}{m_S} \approx 1 \times 10^{-14}$ and $\frac{\Gamma_L}{m_L} \approx 1 \times 10^{-17}$, $\Gamma_{S(L)}$ is negligible compared with $m_{S(L)}$, then the momentum of the neutral Kaon pairs can approximately be calculated as $\pm\boldsymbol{p} = \pm\sqrt{s_{R_{S(L)}}}\ \hat{\mathbf{p}}_{S(L)} \approx \pm m_{S(L)}\ \hat{\mathbf{p}}$. In other words, the velocities of Kaon pairs are well determined by their mass and momentum.

A detector, i.e, registration apparatus has been built such that it counts 2π pairs which are coming from the position of decay vertex at $\boldsymbol{x} = (0, 0, z)$. The observable $|\psi^-\rangle = |f\rangle$ registered by the detector is the projection operator:

$$\Lambda(x) = |\psi^-(x,t)\rangle\langle\psi^-(x,t)| = |f,t\rangle\langle f,t| \ . \tag{5.21}$$

Where f are decaying products which originate from the the decay vertex's space-time position at $x = (t, \boldsymbol{x}) = (t, 0, 0, z)$. The decaying particle has been created at position $(t_0 = 0, 0, 0, 0)$ and decay at position $(t, 0, 0, z = t\boldsymbol{v})$ where $\boldsymbol{v} = \boldsymbol{p}/m$ is the velocity of decaying particle, i.e, neutral Kaon, along the z-axis. The space-time parameters $x=(t,\boldsymbol{x})$ must fulfill $\frac{\boldsymbol{x}}{t}=\frac{\hat{\mathbf{p}}}{\gamma}=\boldsymbol{v}$. Therefore, we have

$$x=(t,\boldsymbol{x})=(t,0,0,z=t\boldsymbol{v})=(\gamma\tau,0,0,z=\gamma t^{*}\boldsymbol{v})\,. \tag{5.22}$$

Here, t^* is the proper time of the decaying particle in its rest frame. The term in the exponential of (5.20) can be rewritten as

$$\gamma\sqrt{s}(t-\boldsymbol{x}\cdot\boldsymbol{v})=\gamma\sqrt{s}(t-zv)=\gamma\sqrt{s}(1-v^{2})t=\sqrt{s}\frac{t}{\gamma}=\sqrt{s}t^{*}\,. \tag{5.23}$$

Here, $\gamma=\frac{1}{\sqrt{1-v^2}}$ is Lorentz coefficient, $t^*=t/\gamma$ is the proper time in the rest frame of decaying particle. Hence, the time evolution of the Gamow vectors $|K_S^-\rangle=|K_S,\boldsymbol{p}\,,s_S^-\rangle$ and $|K_L^-\rangle=|K_S,-\boldsymbol{p}\,,s_L^-\rangle$ can be written as

$$e^{-iH^{\times}t}\;|K_S^-\rangle=e^{-i\sqrt{s_S}\frac{t}{\gamma_S}}\;|K_S^-\rangle=e^{-im_S t^*}e^{-\frac{\Gamma_S}{2}t^*}\;|K_S^-\rangle\,,\quad t^*\geq 0\,, \tag{5.24a}$$

$$e^{-iH^{\times}t}\;|K_L^-\rangle=e^{-i\sqrt{s_L}\frac{t}{\gamma_S}}\;|K_L^-\rangle=e^{-im_L t^*}e^{-\frac{\Gamma_L}{2}t^*}\;|K_L^-\rangle\,,\quad t^*\geq 0\,. \tag{5.24b}$$

Therefore, the Gamow vectors $|K_{S,L}\,^-\rangle$ evolve irreversibly and obey the exact exponential decay law (4.65) in their rest frame.

According to the Rigged Hilbert Space formalism in chapter 4, every prepared state $\phi^+\in H^2_-$ can be expanded in terms of Gamow vectors which are associated with the first-order poles of the S-matrix in the lower complex invariant mass $s_R=M_R-i\Gamma_R/2$ and the background term as (4.46). Therefore, in the double resonance systems,$|K_{S,L}^-\rangle$ or equivalent $|K^{\pm-}\rangle$, prepared state $\phi^+\in H^2_-$ can be expanded as

$$\phi^{+}=\frac{1}{\sqrt{2}}(b_s\;|K_S^-\rangle+b_L\;|K_L^-\rangle+\frac{1}{\sqrt{2}}\phi^{BG}\,. \tag{5.25}$$

Here, b_S and b_L are the complex expansion coefficients. ϕ^{BG} is the background term which represents the interaction of the in-state with the background in the scattering experiment:

$$\begin{aligned}|\phi^{bg}\rangle &= \sum_{j,j_3,\eta}\int\frac{d^3\widehat{p}}{2\widehat{p}^0}\int_{s_0}^{-\infty_{II}}ds\;\;|\widehat{p}\,j_3[sj]\eta^-\rangle S_{II}(s)\langle^+\widehat{p}\,j_3[sj]\eta'\,|\phi^+\rangle\\ &= \sum_{j,j_3,\eta}\int\frac{d^3\widehat{p}}{2\widehat{p}^0}\;|\widehat{p}\,j_3[sj]\eta^-\rangle b_\eta(s)\,.\end{aligned} \tag{5.26}$$

Where, $|\widehat{p}\,j_3[sj]\eta^-\rangle$ are the generalized eigenvector of the exact Hamiltonian H. Since the background term ϕ^{BG} is not related to the poles of the S-matrix, and represents the non-resonant background term in the scattering experiment, it is usually small compared to the decaying signal. As a result, (5.25) can be approximated as the superposition of K_S and K_L Gamow vectors:

$$\phi^+ \approx \frac{1}{\sqrt{2}}\left(b_s\,|K_S^-\rangle + b_L\,|K_L^-\rangle\right)\,. \tag{5.27}$$

This approximation (5.27) is the result of the Lee-Oehme-Yang theory [55], which is based on the Weisskopf-Wigner approximation [56].

Since the Gamow vectors $|K_{S,L}^-\rangle$ and $|\widehat{p}\,j_3[sj]\eta^-\rangle$ are the generalized eigenvector of the exact Hamiltonian H, the time evolution operator of the prepared state $e^{-iH^\times t}$ can be diagonalized. Therefore, the time evolution of the prepared state $\phi^+(t)$ which is expanded in basis vectors of the the Gamow vectors $|K_{S,L}^-\rangle$ and $|\widehat{p}\,j_3[sj]\eta^-\rangle$ can be expressed in term of each time evolution of these basis vectors:

$$\begin{aligned}\phi^+(t) = e^{-iH^\times t}\phi \;&= \tfrac{1}{\sqrt{2}}\left(b_S e^{-iH^\times t}\,|K_S^-\rangle + b_L e^{-iH^\times t}\,|K_L^-\rangle\right) + \tfrac{1}{\sqrt{2}}e^{-iH^\times t}\phi^{BG}\\ &= \tfrac{1}{\sqrt{2}}\left(b_S e^{-im_S t^*}e^{-\frac{\Gamma_S}{2}t^*}\,|K_S^-\rangle + b_L e^{-im_L t^*}e^{-\frac{\Gamma_L}{2}t^*}\,|K_L^-\rangle\right) + \\ &\quad + \tfrac{1}{\sqrt{2}}\sum_{j,j_3,\eta}\int\frac{d^3\widehat{p}}{2\widehat{p}^0}e^{-i\gamma^2\sqrt{s}t^*}\,|\widehat{p}\,j_3[sj]\eta^-\rangle b_\eta(s)\,.\end{aligned} \tag{5.28}$$

Similarly, in the double resonances system of $|K^{\pm}\;{}^-\rangle$, the prepared state $\phi^+(t)$ can be expanded as (5.28) in complex vectors basis $|K^{\pm\,-}\rangle$ which are eigenkets of the total Hamiltonian H:

$$\begin{aligned}\phi^+(t) = e^{-iH^\times t}\phi \;&= \tfrac{1}{\sqrt{2}}\left(b_+ e^{-iH^\times t}\,|K^{+\,-}\rangle + b_- e^{-iH^\times t}\,|K^{-\,-}\rangle\right) + \tfrac{1}{\sqrt{2}}e^{-iH^\times t}\phi^{BG}\\ &= \tfrac{1}{\sqrt{2}}\left(b_+ e^{-im_{K^+}t^*}e^{-\frac{\Gamma_{K^+}}{2}t^*}\,|K^{+\,-}\rangle + b_- e^{-im_{K^-}t^*}e^{-\frac{\Gamma_{K^-}}{2}t^*}\,|K^{-\,-}\rangle\right) + \\ &\quad + \tfrac{1}{\sqrt{2}}\sum_{j,j_3,\eta}\int\frac{d^3\widehat{p}}{2\widehat{p}^0}e^{-i\gamma^2\sqrt{s}t^*}\,|\widehat{p}\,j_3[sj]\eta^-\rangle b_\eta(s)\,.\end{aligned} \tag{5.29}$$

The charged Kaons, $K^{\pm}$ are the anti-particles of each other. Hence, $m_{K^-} = m_{K^=} = m_K$ and $\tau_- = \tau_+ = \tau_K$. The comparison of K^+ and K lifetimes τ_+/τ_- is used as a confirmation of CPT invariance which requires the equality of the decay lifetimes for particle and antiparticle $\tau_+/\tau_- = 1$ under CPT transformation [57].

The time evolution of the approximation for the Gamow vectors $|K_S\,{}^-\rangle$ and $|K_L\,{}^-\rangle$ in (5.27), therefore, is

$$\begin{aligned}\phi^+(t) = e^{-iH^\times t}\phi \;&= \tfrac{1}{\sqrt{2}}\left(b_S e^{-iH^\times t}\,|K_S^-\rangle + b_L e^{-iH^\times t}\,|K_L^-\rangle\right)\\ &= \tfrac{1}{\sqrt{2}}\left(b_S e^{-im_S t^*}e^{-\frac{\Gamma_S}{2}t^*}\,|K_S^-\rangle + b_L e^{-im_L t^*}e^{-\frac{\Gamma_L}{2}t^*}\,|K_L^-\rangle\right)\,.\end{aligned} \tag{5.30}$$

The time evolution of the Gamow vectors $|K_S{}^-\rangle$ and $|K_L{}^-\rangle$ (similarly $|K^{-\,-}\rangle$ and $|K^{+\,-}\rangle$ is given by either (5.20a) and (5.20b), respectively, as the eigenkets of the exact Hamiltonian H with the constrain condition for the space-time parameter that $t \geq 0$ and $x^2 \geq 0$ in the laboratory or (5.20a) and (5.20b) in the their rest frames, respectively, with the constrain $t^* \geq 0$. We have obtained the standard phenomenological description of the neutral Kaon system with CP violation as an approximation of the exact theory in the Rigged Hilbert Space.

The probability density amplitude to detect the observable or decaying products ψ^- in (5.21) in the prepared state ϕ^+ which is expanded in the complex basis vectors $|K_{K,L}{}^-\rangle$ as (5.28) at the time t:

$$\begin{aligned}\langle\psi^-|\phi^+(t)\rangle &= \tfrac{1}{\sqrt{2}}\left(b_S e^{-im_S\tau_S}e^{-\frac{\Gamma_S}{2}\tau_S}\langle\psi^-|K_S^-\rangle + b_L e^{-im_L\tau_L}e^{-\frac{\Gamma_L}{2}\tau_L}\langle\psi^-|K_L^-\rangle\right) + \\ &\quad + \tfrac{1}{\sqrt{2}}\sum_{j,j_3,\eta}\int\frac{d^3\widehat{p}}{2\widehat{p}^0}e^{-i\gamma\sqrt{s}t}\langle\psi^-|\widehat{p}\,j_3[sj]\eta^-\rangle b_\eta(s)\,, \\ &= \tfrac{1}{\sqrt{2}}\left(b_S e^{-im_S\tau_S}e^{-\frac{\Gamma_S}{2}\tau_S}\langle^-f|K_S^-\rangle + b_L e^{-im_L\tau_L}e^{-\frac{\Gamma_L}{2}\tau_L}\langle^-f|K_L^-\rangle\right) + \\ &\quad + \tfrac{1}{\sqrt{2}}\sum_{j,j_3,\eta}\int\frac{d^3\widehat{p}}{2\widehat{p}^0}e^{-i\gamma\sqrt{s}t}\langle^-f|\widehat{p}\,j_3[sj]\eta^-\rangle b_\eta(s)\,.\end{aligned} \tag{5.31}$$

The coefficient $b_\eta(s) \sim \langle^+\widehat{p}\,j_3[sj]\eta|\phi^+\rangle$ is the scalar product of the prepared state ϕ^+ with the eigenket or outgoing plane-wave $|\widehat{p}\,j_3[sj]\eta^+\rangle$ of the exact Hamiltonian H, and therefore the time dependence of the background term in (5.31) depends upon the preparation of the state ϕ^+, i.e, the setup of the experiment, and thus might vary from experiment to experiment. The background term in (5.31) is not related to the poles of the S-matrix, therefore its time evolution is non-exponential. This background term represents either the interaction of the prepared state ϕ^+ with the background in the scattering experiment or the non-exponential scattering phenomena which is not discussed in this dissertation. This background signal is usually small and be removed from the experimental data analysis. Hence, the probability density amplitude (5.31) can approximately take after isolating the background term as

$$\langle\psi^-|\phi^+(t)\rangle \approx \frac{1}{\sqrt{2}}\left(b_S e^{-im_S\tau_S}e^{-\frac{\Gamma_S}{2}\tau_S}\langle^-f|K_S^-\rangle + b_L e^{-im_L\tau_L}e^{-\frac{\Gamma_L}{2}\tau_L}\langle^-f|K_L^-\rangle\right)\,. \tag{5.32}$$

This approximation (5.32) is used as the standard counting rate probability expression to fit the the neutral Kaon experimental data. The theoretical description of the Kaon phenomena using the Hardy Space has been discussed in [58] in both relativistic and non-relativistic regime.

5.3.2 Measurement of Kaon Lifetime with KLOE Detector and Relativistic Observation of Beginning of Time

The neutral kaon pair from decaying process of Φ meson: $\Phi \rightarrow K_S + K_L$ is in a pure $J^{PC} = 1^{--}$ state and about 34 %. The decay process to charged Kaon pair: $\Phi \rightarrow K^+ + K^-$ is about 49%. Data were collected with the KLOE detector at DAΦNE, the Frascati Φfactory. DAΦNE is an e^+e collider operating at a center of mass energy $\sqrt{s} \sim 1020\ MeV$, the Φ-meson mass. The center of mass energy or Φ-meson mass $\sqrt{s}$, Φ-meson momentum p_Φ and the average position of the beams interaction point P are measured using Bhabha scattering events.

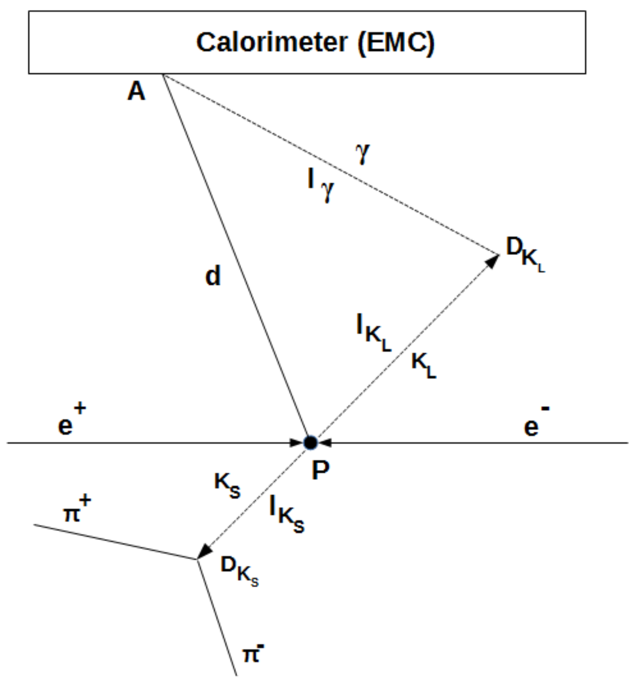

Figure 5.3: Geometrical diagram of the process: $\Phi \rightarrow K_L + K_S$, $K_S \rightarrow \phi^+ + \phi^-$, $K_L \rightarrow 2\,\phi^0 \rightarrow 4\gamma$. K_L and K_S are created at interaction point P and decay at decay vertex D_{K_L} and D_{K_S}, respectively. The K_L path is obtained from the time for each photon γ to arrive at the calorimeter. (Diagram is taken from [60])

About 20×10^6 $K_S \to \phi^+ + \phi^-$ events has been collected to calculate the lifetime of K_S [59] corresponding to an $e + e$ integrated luminosity of 0.4 fb^{-1}. The K_S-meson vector momentum p_{K_S}, the kaon production point or interaction point of the e^+e^- beam P and its decay point D_{K_S} are determined event by event $K_S \to \pi^+\pi^-$. $K_S \to \pi^+\pi^-$ decays are reconstructed from two pions opposite sign tracks $\pi^+\pi^-$ which must intersect at a point P with radius $r_P < 10$ cm and $|z_P|< 20$ cm, where x=y=z=0 (coordinate of interaction point P) is the e^+e average collision point and e^+e beams are directed along the z-direction. The kaon momentum p_{K_S} can be obtained from the sum of the pion momenta ($p_{\pi^+} + p_{\pi^-}$) and also from the kaon direction with respect to known, fixed Φ-meson momentum p_Φ.

The K_L mesons travel in the opposite direction of the K_S mesons and decay at the decaying vertex D_{K_L}. Productions of a K_L is tagged by the observation of $K_S \to \pi^+\pi^-$ decay, i.e, the $K_S \to \pi^+\pi^-$ decay provides measurement of K_L momentum, $p_{K_L} = p_\Phi - p_{K_S}$, where p_Φ is the Φ-meson momentum determined from Bhabha scattering events. About 15×10^6 $K_L \to 3\pi^0$ decay events are selected tagged by $K_S \to \pi^+\pi^-$ events [60]. The decaying vertex D_{K_L}, for example, can be constructed from the decay $K_L \to \pi^0\pi^0 \to 4\gamma$ from photon arrival times at the EMC as in Figure 5.3. Each photon defines a time-of-fight triangle PAD_{K_L}, where l_{K_L} is the K_L path length which is created at interaction point P and decay at the decaying vertex D_{K_L}, l_γ is the distance of the photon traveling from the decay vertex D_{K_L} to the the calorimeter cluster registering photon A at EMC and d is the distance from the the calorimeter cluster to the interaction point P. The decaying vertex D_{K_L} can be determined by:

$$d^2 + l_{K_L}^2 - 2\,\mathbf{d}.\mathbf{l}_{K_L} = l_\gamma^2\,, \tag{5.33a}$$

$$l_{K_L}/\beta_{K_L} + l_\gamma = ct_\gamma\,. \tag{5.33b}$$

Here, β_{K_L} is the K_L velocity and t_γ is the cluster time.

The $K_{S(L)}$ proper time or lifetime for i-th event, $\tau^i_{K_L}$, is obtained by dividing the decay length $l^i_{K_L}$ by $\beta^i\gamma^i$ of the $K_{S(L)}$ in the laboratory:

$$\tau^i_{K_{S(L)}} = \frac{t^i_{K_{S(L)}}}{\gamma_i} = \frac{1}{\gamma^i}\,\frac{l^i_{K_{S(L)}} m_{K_{S(L)}}}{p_{K_{S(L)}}} = \frac{l^i_{K_{S(L)}}}{\beta^i_{K_{S(L)}}\gamma^i_{K_L}}\,. \tag{5.34}$$

Here, $p_{K_{S(L)}}$, $\beta^i_{K_{S(L)}} = \frac{p_{K_{S(L)}}}{m_{K_{S(L)}}}$, and $t^i_{K_{S(L)}}$ are the momentum, velocity and time of the Kaon $K_{S(L)}$ traveling in the laboratory, for i-th event, respectively. The measured lifetime of K_S and K_L with the statistical and systematic errors are: $\tau_{K_S} = (89.562\pm, 0.029_{\text{stat}} \pm 0.043_{syst})\ ps$ and $\tau_{K_L} = (50.92\pm, 0.17_{\text{stat}} \pm 0.25_{syst})\ ns$, respectively.

For the measurement of the lifetime of the charged Kaon $K^{\pm}$, two methods are used to analysis data from different kind of detectors [57]. One is the Kaon decay length method, in which decay vertex of the kaon for neutral Kaon is reconstructed using the using a data control sample given by cylindrical drift chamber information only, with a sample of 15×10^6 tagged kaon decays. The charged Kaons are tagged using the decay process: $K^{\pm} \rightarrow \mu^{\pm}\overline{\nu}_{\mu}(K_{\mu 2})$ which corresponds to about 63% of charged Kaon decay width [61]. Another method is Kaon decay time method using lead-scintillating fibers calorimeter (EMC) information only for the signal side by selecting the event with a π^0 in the final state: $K^{\pm} \rightarrow X + \pi^0 \rightarrow X + 2\gamma$. The average of $K^{\pm}$ lifetime using two methods is: $\tau_{+} = (12.325 \pm 0.038)\ ns$ and $\tau_{-} = (12.374 \pm 0.040)\ ns$.

The direct measurement of the lifetimes of the decaying Kaons ($K_{L(S)}$ and $K^{\pm}$) justifies the the time asymmetric evolution of the state and observable ψ^{-} in the scattering experiment which states that the observable can be detected after the time $t_0 = 0$ at which the in-state ϕ^{+} has been prepared. It also verifies the beginning of time $t_0 = 0$ at which the decaying state ψ_K^G is created and then decays at the decay time t in the laboratory or t^* in their rest frame. In the scattering experiment, an ensemble of kaon pairs is created at various times $t_n = 0$, where n is number of sample using in analysis, in the laboratory at the position P with $z = 0$ or the interaction point at which the Φ-meson is created and decay spontaneously ($\tau_{\Phi} = 1.55 \times 10^{-}22$ s). The $n - th$ pair of neutral Kaons $K_{S(L)}$ or charged Kaon $K^{\pm}$ moves in opposite directions during the time interval $t^n_{K_S} - t^n_0$ for the K_S Kaon and $t^n_{K_L} - t^n_0$ for the K_L Kaon (or $t^n_{K^{\pm}} - t^n_0$ for the charged Kaons $K^{\pm}$ Kaon) and decays at $t^n_{K_{S(L)}}$ for neutral Kaons (or $t^n_{K^{\pm}}$ for charged kaon $K^{\pm}$) after they has moved the distance:

$$\begin{aligned} l_K &= v\,(t^n - t^n_0) = \tfrac{p_K}{m_K}\,(t^n - t^n_0) \\ &= \tfrac{p_K}{m_K}\Delta t^n = v\gamma t^*_K = \beta\gamma t^*_K\,. \quad (\text{in } c = 1 \text{ unit }\ , v \equiv \beta)\,. \end{aligned} \tag{5.35}$$

The decaying length (5.35) is applied for any Kaons with v is the velocity of kaon, p_K is the momentum of kaon, γ is the Lorentz coefficient, Δt is the time of kaon traveling in the laboratory and $t^* = \Delta t/\gamma$ is proper time or lifetime of kaon in its rest frame.

The ensemble of kaon pairs ($K_{L(S)}$ or $K^{\pm}$) created at these different times $t_0^n =$ at the interaction point P is described by the the Gamow vectors $\psi_K^G = |K^-\rangle$ (K is denoted for $K_{L(S)}$ or $K^{\pm}$) which time evolution is given by exact formula as (5.20)

$$\psi_K^G(t) = \mathcal{U}_+^{\times}(I, x = (t,0,0,z))\,|K^-\rangle = e^{-i\gamma\sqrt{s}(t-\boldsymbol{z}\cdot\boldsymbol{v_z})}\,|K^-\rangle\,, \quad t \geq 0\,, x^2 \geq 0\,, \tag{5.36a}$$

or approximately as (5.24):

$$\psi_K^G(t) = e^{-i\sqrt{s}\frac{t}{\gamma_K}}\,|K^-\rangle = e^{-im_K\tau_K}e^{-\frac{\Gamma_K}{2}t^*}\,|K^-\rangle\,, \quad t^* \geq 0\,. \tag{5.36b}$$

The set of the beginnings of time $\{t_0^1, t_0^2, .., t_0^n\} = t_0 = 0$ represents the beginning of time of ensemble which at which ψ_K^G is created and after which the decay products are detected at the detector, i.e, the beginning of time $t_0^n = 0$ for of each individual kaon is identified with the mathematical semi-group time $t_0 = 0$ of the semi-poincaré group into the forward light cone (4.64a). Therefore, the Gamow ket $\psi_K^G(t)$ for the Kaons with the condition $t \geq 0$ describes an ensemble of individual Kaon (for each K_S and K_L or each charged Kaons $K^{\pm}$) under the same conditions. These Kaons have a well-defined lifetime τ_K which is measured as average of the lifetime $\tau_K =$ average of τ^n, where τ^n is determined by (5.34).

Since the kaon states is represented by relativistic Gamow vectors ψ_K^G which defined from the first-order poles of S-matrix $s_K = (m_K - i\Gamma_K/2)^2$, the inverse of $_K$ is exactly the lifetime τ_K in the rest frame, $\tau_K = 1/\Gamma_K$. The Born probability to detect the final decaying products or observable $\psi^- = |f^-\rangle$ in the Gamow vectors ψ_K^G at the time t in the laboratory

$$\begin{aligned}\mathcal{P}_{|f\rangle}(\psi_K^G(t)) &= \left|\langle^- f\,|\psi_K^G(t)\rangle\right|^2 = \left|e^{-i\sqrt{s}\frac{t}{\gamma_K}}\langle^- f\,|K^-\rangle\right|^2 \\ &= \left|e^{-im_K\frac{t}{\gamma_K}}e^{-\frac{\Gamma_K}{2}\frac{t}{\gamma_K}}\langle^- f\,|K^-\rangle\right|^2 = e^{-\Gamma_K\frac{t}{\gamma_K}}\left|\langle^- f\,|K^-\rangle\right|^2\,, \quad t \geq 0\,.\end{aligned} \tag{5.37a}$$

or at the proper time t^* in the rest frame of the Kaon ψ_K^G

$$\begin{aligned}\mathcal{P}_{|f\rangle}(\psi_K^G(t^*)) &= \mathcal{P}_{|f\rangle}(\psi_K^G(t)) = \left|e^{-im_Kt^*}e^{-\frac{\Gamma_K}{2}t^*}\langle^- f\,|K^-\rangle\right|^2 \\ &= e^{-\Gamma_Kt^*}\left|\langle^- f\,|K^-\rangle\right|^2 = e^{-\frac{t^*}{\tau_K}}\left|\langle^- f\,|K^-\rangle\right|^2\,, \quad t^* \geq 0\,.\end{aligned} \tag{5.37b}$$

The theoretical probability (5.37) is used to compare with the counting rate of event in the decay $\psi_K^G \to f$ in the proper time t^* as:

$$\frac{N_f}{N} = \mathcal{P}_{|f\rangle}(\psi_K^G(t)) \sim e^{-\frac{t^*}{\tau_K}}\,, \quad t^* \geq 0\,. \tag{5.38}$$

The expression (5.38) has been used to fit the proper time t^* distribution of the decay events of Kaons with an exponential decay in [57][59][60], to get the lifetime of the Kaons over an ensemble of n decays. The difference is that the Heaviside function $\theta(t)$ is impose into the exponential time, $\theta(t)e^{-t^*/\tau}$ when fitting the proper time t^* distribution of Kaons which is automatically put into as the consequence of time evolution of the relativistic Gamow vectors $\psi_K^G(t)$ under the semi-group of the Poincaré group into the forward light cone (4.64a).

Chapter Six: Conclusion

By replacing the Hilbert space boundary condition, in which the states and observable are described by the same Hilbert space. by the Hardy space boundary condition, in which the states and observable are described by a pair of distinguishable Hardy spaces, we get the exact mathematical quantum theory which unites the resonance and decaying state phenomena. The time evolution of the state ϕ^+ and of the observable ψ^- are asymmetric, i.e, dynamical evolutions are described by semi-group operators, $U_+(t) = e^{iH(t-t_0)}$ with $t_0(=0) \leq t < \infty$ for the observable ψ^- and $U_-(t) = e^{-iH(t-t_0)}$ with $-\infty < t \leq t_0(=0)$ for the state ϕ^+ in non-relativistic regime, or space-time translation $\mathcal{U}_+(I, x) = e^{-iP.x}$ with $t \geq 0, x^2 \geq 0$ for the observable ψ^- and $\mathcal{U}_-(I, x) = e^{iP.x}$ with $t \leq 0, x^2 \geq 0$ for the state ϕ^+ in the relativistic regime.

The non-relativistic resonance associate with the first-order pole of the S matrix at $Z_R = E_R - i\Gamma/2$ in the second sheet can be represented by the Gamow vector ϕ^G obeying the exact time exponential decay. The time evolution of the Gamow vector is also described by the semi-group $U_+(t) = e^{iH(t-t_0)}$ with $t_0(=0) \leq t < \infty$ as the consequence of the Hardy space boundary condition. The state ϕ^+ and observable ϕ^- can either be expanded in terms of background and discrete decaying Gamow vectors ϕ^G as (3.69) and the growing Gamow vectors ϕ^{GR} (3.68) or in terms of continuous exact in-plane wave states $|E^+\rangle$ as (3.17a) and out-plane wave states $|E^-\rangle$ as (3.17b) of the exact Hamiltonian H of the physics system, respectively.

The resonances in the relativistic scattering process or unstable, decaying states in relativistic regime are described by the relativistic Gamow kets $\psi^G_{s_R}$ defined in (3.49). These Gamow kets $\psi^G_{s_R}$ are associated with the first-order pole of the S-matrix at $s_R = M_R - i\Gamma_R/2$. The time evolution of the relativistic Gamow kets $\psi^G_{s_R}$ is given by semi-group $\boldsymbol{\mathcal{P}}_+$ of the Poincaré transformation (4.64a) into the forward light cone $(t \geq 0)$. These the relativistic Gamow kets $\psi^G_{s_R}$ obey the exact exponential decay in their rest frame.

The finite quantum mechanical beginning of time t_0 of the semi-group is experimentally observed as an ensemble of finite times in the quantum jump for the non-relativistic or in Kaon experiment (KLOE) for the relativistic case. In the experiments with a single ion in the Paul trap, ensemble of the beginnings of time

$\{t_0^{(i)}\}$ which are the onset times of dark periods, i.e, the beginning of time on the meta-stable state described by a Gamow vector, responds to the beginning of time of semi-group time evolution $t_0 = 0$ observed in the lab: $t_0 \leftrightarrow \{t_0^{(i)}\}$. In the Kaon experiment, the set of the beginnings of time $\{t_0^i\} = 0$ at which individual kaon is created represents the beginning of time t_0 of ensemble described by Gamow vector ψ_K^G. In other words, the beginning of time $t_0^n = 0$ for of each individual kaon is identified with the mathematical semi-group time $t_0 = 0$ of the semi-poincaré group $\boldsymbol{\mathcal{P}}_+$ of the Poincaré transformation into the forward light cone (4.64a): $\{t_0^i\} = t_0 = 0$.

Bibliography

[1] M. Goldberger and K. Watson, *Collision theory*, New York: Wiley (1964).

[2] L. A. Khalfin *Sov. Phys. JETP* **6** (1958), 1053-63

[3] G. Gamow, *Z. Phys.* **51** (1928), 204-12.

[4] M. H. Stone, *Ann. Math.* 33(3) (1932), 643 (1932). J. von Neumann, *Ann. Math.* 33(3) (1932), 567.

[5] A. Bohm, *Lett. Math. Phys.* **3** (1978), 455.

[6] B. A. Lippmann and J. Schwinger, *Phys. Rev.* **79** (1950), 469-480.

[7] P. Dirac, *Proc, Roy. Soc.* 113 (1932), 621-641; P. Dirac, *The Principles of Quantum Mechanics*, 4th Edition, Oxford University Press, Oxford, 1958.

[8] L. Schwartz, *Thèories des distributions*, 1st edition, Hermann, Paris, 1950.

[9] A. Bohm, *The Rigged Hilbert Space and Quantum Mechanics*, Springer Lecture Notes in Physics, vol.78 (1978).

[10] S. L. Sobolev, *Math. Sbornik* 1(43) 1936, 39-72; S. L. Sobolev, *Some Applications of Functional Analysis in Mathematical Physics*, Leningrad, 1950;

[11] A. Grothendieck, *Memoirs Amer. Math Soc., Nr. 16*, Providence, RI 1966;

[12] J. P. Antoine and C. Trapani, *Partial Inner Product Spaces*, Lecture Notes in Mathematics, Springer Verlag Berlin, 2009

[13] A. Bohm, *The Rigged Hilbert Space and Quantum Mechanics*, Springer Lecture Notes in Physics, vol.78 (1978).

[14] I. M. Gelfand and N. Ya. Vilenkin, *Generalized functions*, Vol. 4, 1st edition, Academic Press, New York (1964).

[15] K. Maurin, *Generalized Eigenfunction Expansion and Unitary Representations of Topological Groups*, 1st edition, Polish Scientific Publishers, Warszawa (1969).

[16] A. Bohm and M. Gadella, *Dirac Kets, Gamow Vectors and Gelfand Triplet*, Lecture Note in Physics, vol. 348, Springer, Berlin, 1989.

[17] V. Weisskopf and E. P. Wigner, *Z. Phys.* **65** (1930) 18.

[18] L. K. Gibbon, *Phys. Rev. D* **55** (1997), 6625.

[19] R. G. Newton, *Scattering Theory of Waves and Particles*, 2nd edition, Springer, New York (1982).

[20] R. J. Eden, P. V. Landshoff , D. I. Olive and J. C. Polkinghorne , *The Analytic S-matrix*,Cambridge University Press: Cambridge (1966).

[21] J. R. Taylor, *Scattering Theory*, John Wiley & Sons, Inc (1972).

[22] A. Bohm, *J. Math. Phys.* 22 (1981) 2813.

[23] A. Bohm, S. Maxson, M. Loewe, and M. Gadella, *Physica A 236* (1997) 485.

[24] P. D. Lax and R. S. Phillips, *Scattering theory*, 1st edition, Academic Press, New York (1967).

[25] Strauss Y, Horwitz L P and Eisenberg E 2000 *J. Math. Phys.* **41** No. 12 8050.

[26] Strauss Y, Horwitz L P and Volovick A 2006 *J. Math. Phys.* **47**, 123505.

[27] H. Baumgärtel, *Int. J. Theor. Phys.* **46** (2007), 1959.

[28] H. Baumgärtel, *J. Math. Phys.* **51** (2010), 113508.

[29] A. Bohm, M. Gadella, and G. B. Mainland, *Am. J. Phys.* **57** (1989), 1103.

[30] A. Bohm, I. Antoniou and P. Kielanowski, *Phys. Lett. A* **189** (1994), 442.

[31] S. Weinberg, *The Quantum Theory of Field*, Vol 1, Cambridge University Press (1995).

[32] A. Bohm, M. Loewe, S. Maxson, P. Patuleanu, C. Puntmann and M. Gadella, *J. Math. Phys.* **38** (1997), 6072.

[33] G. Breit and E. Wigner, *Phys. Rev.* **49** (1936), 519.

[34] R. Paley and N. Wiener, *Fourier Transform in the Complex Domain*, American Mathematical Society, New York (1934).

[35] E. Wigner, *Ann. Math* **40** (1939), 149.

[36] V. Bargmann and E. Wigner, *Proc. Natl. Acad. Sci. USA* **34** (1984), 211.

[37] A. Bohm and H. Kaldass, *Phys. Rev.* A **60** (1999), 4606.

[38] H. Joos, *Fortschr. Physik* **10** (1962) 65.

[39] A. J. Macfarland , *Rev. Mod. Phys.* **34** (1962), 41.

[40] A. Bohm, H. Kaldass, and S. Wickramasekara, *Fortschr. Phys.* **51** (2003), 569.

[41] A. Bohm, H. Kaldass, S. Wickramasekara, *Fortschr. Phys.* **51** (2003), 604.

[42] M. E Peskin and D. V. Schroeder D V, *An Introduction to Quantum Field Theory*, Addison Wesley (1995).

[43] A. Bohm, P. Kielanowski and S. Wickramasekara, *Ann. Phys.* **321**(2006), 2299.

[44] A. Bohm and H. V. Bui, *J. Phys.: Conf. Ser.* **428** (2013) 012016.

[45] W. Nagourney, J. Sandberg and H. Dehmelt *Phys. Rev. Lett.* **56**, 2797 (1986).

[46] Th. Sauter, W. Neuhauser, R.Blatt and P. E. Toschek, *Phys.Rev. Lett.* **57**, 1696 (1986).

[47] J. C. Bergquist, Randall G. Hulet, Wayne M. Itano and D. J. Wineland, *Phys. Rev. Lett.* **57**, 1699 (1986).

[48] E. Peik, G. Hollemann and H.Walther, *Phys. Rev.* A **49**, 402 (1994).

[49] X. Zhao, N. Yu, H. Dehmelt and W. Nagourney, *Phys. Rev.* A **51**, 4483 (1995).

[50] A. Bohm, P. Kielanowski and H. V. Bui, *J. Russ. Laser Res.* **35** (2014), 14.

[51] H. V. Bui, *J. Phys.: Conf. Ser.* **597** (2015) 012020.

[52] A. Bohm and B. W. Bryant, *Int. J. of Theor. Phys.* **50** (2011), 2094.

[53] F. Bossi, E. De Lucia, J. Lee-Franzini, S. Miscetti, M. Palutan and KLOE Collaboration, *Riv. Nuovo Cim.* **31** (2008) 531-623. arXiv:0811.1929v2 [hep-ex].

[54] J. H. Christenson , J. W. Cronin J. W., V. L. Fitch et Turlay R, *Phys. Rev. Lett.*, **13** (1964), 138

[55] T. D. Lee, R. Oehme and C. N. Yang, *Phys. Rev.* **106**, 340 (1957).

[56] V. Weisskopf and E. P. Wigner, *Z. f. Physik* **63**, 54 (1930); {65, 18 (1930).

[57] The KLOE collaboration, *Measurement of the charged kaon lifetime with the KLOE detector*, *JHEP* 0801:073,2008. arXiv:0712.1112v2 [hep-ex].

[58] A. Bohm, *Int. J. Theor. Phys.* **36** (1997) 2239-2270.

[59] The KLOE collaboration, *Eur. Phys. J .C* **71**, 1604 (2011). arXiv:1011.2668v3 [hep-ex].

[60] The KLOE collaboration, *Phys. Lett. B* **626** (2005) 1523.

[61] W. M. Yao et al., *Journal of Physics* **G33**, 1 (2006).

Vita

Hai Viet Bui was born on Central Highlands, Viet Nam on 20th Feb, 1982. He came to the United States in 2003 and started his higher eduction at the Houston community college. He then transfered to the University of Texas at Austin and received the degree of Bachelor of Science in Physics in May, 2008. After finishing the degree of Bachelor of Science in Physics, he continued Ph.D degree in physics at The University of Texas at Austin in September, 2008.

Permanent Address: 2501 Lake Austin Blvd Apt#H102
Austin, TX 78703.