A COMPARISON OF COMPUTATIONAL MODELS FOR THE SATELLITE RELATIVE POSITION PROBLEM

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I. Introduction

The problem of accurately modeling the relative positions of two orbiting vehicles has become increasingly important with the advent of missions whose success depend upon this information. Three instances of such missions are:

1) Missions involving rendezvous
2) Missions which attempt to study certain physical phenomena by observing their relative effects on two orbiting vehicles.
3) Inexpensive modeling of a set of satellites to check for possible collision.

One obvious approach to determining the relative position of two vehicles is simply to difference the position vectors of the two vehicles. Thus, if the position vectors, $\overline{r}_1$ and $\overline{r}_2$ of the two vehicles are given as:

$$\overline{r}_1 = x_1 \hat{i} + y_1 \hat{j} + z_1 \hat{k} \tag{1}$$

and

$$\overline{r}_2 = x_2 \hat{i} + y_2 \hat{j} + z_2 \hat{k} .$$

the relative vector, $\overline{r}$, would be computed as

$$\overline{r} = \overline{r}_1 - \overline{r}_2 + (x_1 - x_2)\hat{i} + (y_1 - y_2) \hat{j} + (z_1 - z_2) \hat{k}. \tag{2}$$

Clearly, when $\overline{r}_1$ approaches $\overline{r}_2$, a serious loss of significant digits may result from the subtractions involved due to the finite word length of the computer being used.

There are obviously other limiting factors upon the accuracy with which the relative vector can be obtained. The most visible limiting factor is the method used to obtain the positions of each satellite, in the case of numerical integration this is a factor of the accuracy of the integrator, the stepsize used and the length of time integrated.

The use of time transformations have recently been applied to the equations governing the motion of space vehicles in attempts to more accurately predict their positions. It is hoped that these methods may be benificial in the case of relative positions between two satellites. These studies have shown that the value of the exponent used in the time transformations, n, can have dramatic effect on the global integration error. These investigations have generally been concerned with applying the transformations to a single orbiting body. Velez[10] has shown that the transformations with $1 \leq n \leq 2$ enhance the stability of the system in the sense of Liapunov, and that as n is increased, the numerical stability of the numerical integration algorithms increases. In addition, Velez[10,11] Feagin and Mikkilineni[5], and Nacozy[9] have studied the effects of these methods on truncation errors. Nacozy[9] gives an excellent survey of these investigations. One major result of these studies is that the optimal value of n is dependant upon the model employee

Other studies such as Dunning[4] and Meuller[6,7] have developed specialized equations of motion based upon approximations which are valid only when the vehicles are in close proximity. For this reason, when using these rendezvous type equations it is necessary to alternate between a set of equations which are valid only for near approaches, and a set valid for large separations. Such switching can introduce a number of complications. One primary complication introduced is the determination of when to switch, which increases the complexity of the required coding. Other complications result from questions inherent in the alteration, such as how to restart the integration if a multi-step integrator is used, and how to control loss in accuracy resulting from required transformations.

This study compares eight formulations of the equations of motion for two bodies orbiting a central body. All equations compared will be valid for both close approaches and large separations. As well as avoiding the problems mentioned above, this will allow a more careful study of the regions in which the individual equations are most applicable.

CHAPTER II

Formulations Studied

Eight equation sets are studied which represent several philosophies for improving computational accuracy in the relative motion problem. Some of the formulations attempt to increase precision by coupling the equations. This is accomplished by integration of one satellite in earth centered coordinates and integration of the relative vector joining the satellites. Another concept employed attempts to make the equations of motion more stable through the use of a new independent variable of integration in order to reduce the propagated error. Several studies have shown that time smoothed equations enhance the stability of the equations, [Beaudet[2,3]], [Velez[10,11]]. Still other formulations are based upon a more direct approach and attempt to reduce the errors resulting from subtraction of the two nearly equal quantities which appear while computing relative acceleration. The formulations studied are listed below.

1. Standard Cowell
2. Cowell equations modified for relative motion
3. Independently time smoothed
4. Time smoothing of one satellite and the relative vector, based upon smoothed satellite
5. Time smoothing of one satellite and the realtive vector, based up upon both satellites
6. Cowell equations modified for relative motion employing Nacozy-Szebehely method

7. Nacozy-Szebehely method applied to equation set Four
8. Nacozy-Szebehely method applied to equation set Five

The most commonly used formulation of the equations of motion in space dynamics is the Cowell set. Two versions of these equations were included in the comparisons to serve as reference.

2.1 The Cowell type equations

Equation Set 1 (Standard Cowell equations)

The classical Cowell equations were applied independently to each satellite as follows:

$$\ddot{\bar{r}}_1 = -\mu \frac{\bar{r}_1}{r_1^3} + \bar{P}(\bar{r}_1, \dot{\bar{r}}_1)$$
$$\ddot{\bar{r}}_2 = -\mu \frac{\bar{r}_2}{r_2^3} + \bar{P}(r_2, \dot{\bar{r}}_2) \tag{3}$$

where the first term on the right is Newtonian Force and the $\bar{P}$'s represent the perturbing forces.

The relative vector $\bar{r}$ is determined at desired output points by differencing

$$\bar{r} = \bar{r}_1 - \bar{r}_2 \tag{4}$$

In this formulation the equations are very easy to implement, however, the problem of lost significant digits in the computation of $\bar{r}$ from (4) can lead to difficulties in obtaining the relative vector accurately when $\bar{r}_1$ and $\bar{r}_2$ are very close to one another. In an attempt to reduce this source of error, it is shown in the next section that the equations can be coupled by altering them for integration of one satellite referenced to the central body, and the other satellite position relative to the first.

Equation Set 2. (Cowell Equations modified for relative motion)

Differentiation of equation (4), and substitution of equations (3) into the result yields

$$\ddot{\bar{r}} = \ddot{\bar{r}}_1 - \ddot{\bar{r}}_2 = -\mu \left(\frac{\bar{r}_1}{r_1^3} - \frac{\bar{r}_2}{r_2^3} \right) + \bar{P}_1(\bar{r}_1, \dot{\bar{r}}_1) - \bar{P}_2(\bar{r}_2, \dot{\bar{r}}_2) \qquad (5)$$

In this formulation, one of equations (3) and equation (5) are integrated simultaneously. In the evaluation of equation (5), equation (4) must be used to determine the position vector of the satellite which is not integrated in earth centered coordinates. It should be noted that as $\bar{r}_1$ approaches $\bar{r}_2$, the subtraction required by equation (5) may still result in a loss of significant digits. Thus, the original problem has not been eliminated. Rather than having the loss of significant digits appear directly in the calculation of $\bar{r}$, as $\bar{r} = \bar{r}_1 - \bar{r}_2$, it now appears in the acceleration of $\bar{r}$. There are other techniques which can be applied to the problem which may allow the solution to be computed with greater precision. One such technique is the use of time smoothing to enhance the stability of the equation.

2.2 The Time Smoothed Equations

The time smoothed equations are all based on a transformation of the independent variable of integration from time to a new independent variable s. This transformation generally requires that another differential equation, relating t and s, be added to the equation set. In general, the relation can be given as

$$ds = f\ dt, \tag{6}$$

where f is a function of the magnitudes of the position vectors $\bar{r}_1$ and/or $\bar{r}_2$. Detailed derivations of the time smoothed equations are presented in the Appendix. The most obvious use of these equations is to apply them independently to each satellite.

<u>Equation Set 3</u>: (Independently smoothed equations for each satellite)

For this set f_1 and f_2 are defined as

$$f_1 = \frac{ds_1}{dt} = \frac{1}{a_1 r_1{}^{n_1}} \tag{7}$$

and

$$f_2 = \frac{ds_2}{dt} = \frac{1}{a_2 r_2{}^{n_2}} \tag{8}$$

The resulting equations become

$$\begin{aligned} \bar{r}_1{}'' &= \frac{n_1}{r^1} \frac{dr_1}{ds_1} \frac{d\bar{r}_1}{ds_1} - \mu\, a_1^2\, r_1{}^{2n_1-3}\, \bar{r}_1 + \bar{P}(\bar{r}_1, \dot{\bar{r}}_1)\, a_1^2\, r_1{}^{2n_1} \\ \bar{r}_2{}'' &= \frac{n_2}{r_2} \frac{dr_2}{ds_2} \frac{dr_2}{ds_2} - \mu\, a_2^2\, r_2{}^{2n_2-3}\, \bar{r}_2 + \bar{P}(\bar{r}_2, \dot{\bar{r}}_2)\, a_2^2\, r_2{}^{2n_2} \end{aligned} \tag{9}$$

where primes denote differentiation with respect to s, the new independent variable of integration. The relations between s_1, s_2, and t are

$$t_1 = \int_{s_{10}}^{s_{1t}} a_1\, r_1{}^{n_1}\, ds_1 \tag{10}$$

and

$$t_2 = \int_{s_{20}}^{s_{2t}} a_2 r_2{}^{n_2}\, ds_2$$

Thus, in order to obtain information about the relative vector at a desired output point it is first necessary to match t_1 and t_2. This time matching is extremely difficult since the relations are both highly non-linear, and get even more so as the values of n_1 and n_2 increase. For this reason a formulation which eliminates the need for this time matching is desirable. One method for accomplishing this is integration of the relative vector directly.

Equation Set 4: (Time smoothing based on one satellite's distance from the central body applied to relative vector.)

One satellite and the relative vector are smoothed, the smoothing function f is defined as

$$f = \frac{ds}{dt} = \frac{1}{a_2 r_2^{n_2}} \tag{11}$$

The resulting equations of motion are:

$$\bar{r}_2'' = \frac{n_2}{r_2} \frac{dr_2}{ds} \frac{d\bar{r}_2}{ds} - \mu\, a_2^2\, r_2^{2n-3} \bar{r}_2 + \bar{P}\,(\bar{r}_2, \dot{\bar{r}}_2) a_2^2\, r_2^{2n_2} \tag{12}$$

$$\bar{r}'' = \frac{n_2^2}{r_2} \frac{dr}{ds} \frac{d\bar{r}}{ds} + a_2^2\, r_2^{2n_2} \left[P\,(\bar{r}_1, \dot{\bar{r}}_1) - P(\bar{r}_2, \dot{\bar{r}}_2) + \mu \left(\frac{\bar{r}_2}{r_2^3} - \frac{\bar{r}_1}{r_1^3} \right) \right] \tag{13}$$

In order to evaluate (13) the relation

$$\bar{r}_1 = \bar{r}_2 + \bar{r} \tag{14}$$

must be used. For this formulation, there is only one relation between t and s,

$$t = \int_{s_0}^{s_f} \frac{ds}{f} \tag{15}$$

This eliminates the major drawback inherent in equation set 3, the required matching of t_1 and t_2. However, only information about one of the satellites is being incorporated into the smoothing function. If the smoothing function is based upon r_1 and r_2, it might adapt the equations to the satellites even more accurately.

Equation Set 5: (Time smoothing based on both satellites applied to the relative vector)

Another time smoothing formulation, suggested by Dr. V. G. Szebehely[8], also smooths one satellite and the relative vector, but this time information about both satellites is supplied to the smoothing function which is now defined as

$$f = \frac{ds}{dt} = (r_1 r_2)^{-n} \tag{16}$$

For this case the equations of motion emerge as:

$$\begin{aligned} \bar{r}_2'' &= \frac{n}{r_1 r_2}(r_1 r_2' + r_2 r_1')\,\bar{r}_2' - (r_2 r_1)^{2n} \mu \frac{\bar{r}_2}{r_2^3} + \bar{P}(\bar{r}_2, \dot{\bar{r}}_2)(r_1 r_2)^{2n} \\ \bar{r}'' &= \frac{n}{r_1 r_2}(r_1 r_2' + r_2 r_1')\bar{r}' - \mu (r_1 r_2)^{2n}\left[\frac{\bar{r}_1}{r_1^3} - \frac{\bar{r}_2}{r_2^3}\right] + \left[\bar{P}(\bar{r}_1, \dot{\bar{r}}_1) - \bar{P}(\bar{r}_2, \dot{\bar{r}}_2)\right](\bar{r}_1 \bar{r}_2)^{2n} \end{aligned} \tag{17}$$

The relationship between t and s is

$$t = \int_{s_0}^{s_f} (r_1 r_2)^n \, ds \tag{18}$$

In each of the time smoothed equation sets, the exponent in the time smoothing equation, denoted n, is arbitrary. The value of n can affect the methods to a great extent, as will be shown. Thus, one necessary consideration of the present study is to determine the most advantageous choice for n for each time smoothed equation set. In this study, each equation set was tested at n=1.0, 1.5, and 2.0.

It can be seen that in each of the formulations involving integration of the relative vector, there is a term arising in the acceleration of $\bar{r}$ due to the two body forces of the form

$$\left[\frac{\bar{r}_2}{r_2^3} - \frac{\bar{r}_1}{r_1^3}\right]$$

As discussed earlier, this subtraction can lead to numerical difficulties. The Nacozy-Szebehely method described below attempts to relieve these difficulties by computing this term in a more suitable fashion.

2.3 The Nacozy-Szebehely method for computing relative acceleration.

Nacozy and Szebehely[8] have recently developed methods for accurately calculating the acceleration of the relative vector. Nacozy and Szebehely apply Encke's method as modified by Potter to the problem of non-interacting space vehicles.

Expressing the relative vector, $\bar{r}$ as

$$\bar{r} = \bar{r}_1 - \bar{r}_2 \tag{19}$$

the relative acceleration can be written in the following form.

$$\ddot{\bar{r}} = \mu \left[\frac{\bar{r}_2}{r_2^3} - \frac{\bar{r}_1}{r_1^3}\right] + \bar{P}(\bar{r}_1,\dot{\bar{r}}_1) - \bar{P}\,(\bar{r}_2,\dot{\bar{r}}_2) \tag{20}$$

The first term on the right hand side is the acceleration due to the Keplerian forces, while $\bar{P}(\bar{r}_1,\dot{\bar{r}}_1)$ and $\bar{P}\,(\bar{r}_2,\dot{\bar{r}}_2)$ are the perturbing forces. If the perturbing forces are small, compared to the Keplerian forces, the errors incurred in calculating the Keplerian acceleration can be expected to dominate. In cases where the perturbing forces are large, a similar technique for calculating their differences as described below for the two body forces may be employed.

Observing that in equations (20), the Keplerian term is similar to a term arising in Encke's method for special perturbations, Nacozy and Szebehely applied a technique analogous to Encke's to that term. Rewriting the subtraction in equation (20) as

$$\frac{\bar{r}_2}{r_2^3} - \frac{\bar{r}_1}{r_1^3} = \frac{-1}{r_2^3}\left[\left(\frac{r_2^3}{r_1^3} - 1\right)(\bar{r}_2 + \bar{r}) + \bar{r}\right] \tag{21}$$

a quantity q is defined as

$$q = \left[\frac{r_2}{r_1}\right] - 1$$

as in the classical Encke method. It should be noted that when q is small (the magnitude of $\bar{r}_1$ is close to the magnitude of $\bar{r}_2$), a substraction of two nearly equal numbers is still involved. A more accurate means of computing q can be obtained by substitution of equation (19) into the above expression. This results in

$$q = \frac{-(2\bar{r}_2+\bar{r}) \cdot \bar{r}}{(\bar{r}_2+r)\ (\bar{r}_2+\bar{r})} \tag{22}$$

The classical approach at this point is to define

$$f(q) = \left[\left(\frac{r_2}{r_1}\right)^2 - 1\right] = (1 + q)^{3/2} - 1 \tag{23}$$

and expand $f(q)$ in a power series in q, or to use tabulated values.

Battin[1], however, derives a closed form expression for $f(q)$ that provides accurate computations of $f(q)$ for small q.

Nacozy and Szebehely adopt this formulation, which is designed for mutually attracting bodies to the relative motion on non-attracting bodies. Multiplying equation (23) by

$$\frac{1 + (1 + q)^{3/2}}{1 + (1 + q)^{3/2}}$$

equation (23) becomes

$$f(q) = q \left(\frac{3 + 3q + q^2}{1 \quad (1 + q)^{3/2}}\right) \tag{24}$$

Substitution into equation (19) for the relative acceleration yields:

$$\bar{r} = \frac{-\mu}{r_1^3}\left[f(q)\bar{r}_1 + (1 + f(q))\bar{r}\right] + \bar{P}(\bar{r}_1,\dot{\bar{r}}_1) - \bar{P}(\bar{r}_2,\dot{\bar{r}}_2) \tag{25}$$

This method for computing the relative acceleration was incorporated in the Cowell and Time smoothed equations involving integration of the relative vector.

Equation Set 6: (Nacozy-Szebehely (N-S) method applied to the modified Cowell Formulation)

For this formulation, the equations for Set 2 are modified to use the (N-S) technique for computing the relative acceleration. The equations become

$$\ddot{\bar{r}}_2 = -\mu\frac{\bar{r}_2}{r_2^3} + \bar{P}(\bar{r}_2, \dot{\bar{r}}_2)$$

$$\ddot{\bar{r}} = \frac{-\mu}{r_1^3}\left[f(q)\,\bar{r}_1 + (1 + f(q))\bar{r}\right] + \bar{P}(\bar{r}_1,\dot{\bar{r}}_1) - \bar{P}(\bar{r}_2,\dot{\bar{r}}_2) \tag{26}$$

The implementation of the N-S method into the Time smoothed equation sets is straight forward. The N-S formulation for the acceleration of the relative vector can be substituted directly where it is required.

Equation Set 7: (Nacozy-Szebehely method applied to Equation set 4).

When the N-S method is substituted into equation set 4, (equations (12) and (13)), the results are.

$$\bar{r}_2{}'' = \frac{n_2}{r_2}\frac{dr_2}{ds}\frac{dr_2}{ds} - \mu\, a_2^2 r_2^{2n-3}\, r_2 + \bar{P}(\bar{r}_1,\dot{\bar{r}}_1) - \bar{P}(\bar{r}_1,\dot{\bar{r}}_2)$$

$$\bar{r}'' = \frac{n^2}{r_2}\frac{dr}{ds}\frac{d\bar{r}}{ds} + a_2^2 r_2^{2n}\left[\bar{P}(\bar{r}_1,\dot{\bar{r}}_1) - \bar{P}(\bar{r}_2,\dot{\bar{r}}_2)\right] + a_2^2 r_2^{2n}\left[\frac{-\mu}{r_1^3}\left[f(q)\bar{r}_1 + 1 + f(q)\bar{r}\right]\right] \tag{27}$$

The Nacozy-Szebehely method can be incorporated into equation set 5 in the same manner:

Equation Set 8: (N-S method applied to equation set 5.)

After introduction of the N-S technique into equation set 5, the equations can be written as

$$\bar{r}_2{}'' = \frac{n}{r_1 r_2}(r_1 r_2' + r_2 r_1')\bar{r}_2' - (r_2 r_1)^{2n}\,\mu\,\frac{r_2}{r_2^3} + P(\bar{r}_2,\dot{\bar{r}}_2)(r_2 r_1)^{2n}$$

$$\bar{r}'' = \frac{n}{r_1 r_2}(r_1 r_2' + r_2 r_1')\bar{r}' + (r_2 r_1)^{2n}\left[\bar{P}(\bar{r}_1,\dot{\bar{r}}_1) - \bar{P}(\bar{r}_2,\dot{\bar{r}}_2)\right] + (r_2 r_1)^{2n}\left[\frac{-\mu}{r_1^3}\left[f(q)\bar{r}_1 + (1 + f(q))\bar{r}\right]\right] \tag{28}$$

Eight equation sets are now available for comparison, the two Cowell formulations, the three time smoothed equation sets, and the Nacozy-Szebehely method applied to Modified Cowell formululation, and the two time smoothed eauations involving integration of the relative vector.

The next consideration is how the comparison was carried out, the first aspect being which satellite pairs were modeled. This is described in the next chapter on Computational Procedures.

CHAPTER III

Computational Procedures

Obviously the eight equation sets cannot be compared for every conceivable pair of satellites. Thus, a representative group was chosen which includes many different situations. The effect which varying the altitudes of one or both satellites has on the relative efficencies of the equation sets can be studied as well as the effects of the eccentricities of the orbits.

Due to the force model chosen (which will be described in the next section of this chapter), the absolute inclination, (I) ascending node (ω), and argument of perigee (Ω) have no effect on the models. Only the relative inclination (i), eccentricity (e), and semi-major axis (a) of the two orbits will affect them. In all cases inclination ω and Ω were chosen as 0. In all pairs, closest approach is at perigee and the initial time 2,000 seconds after the start of the integration. This allows the first small separation to have an effect on the integration. In an attempt to observe the effects of the altitudes of the orbit on the equation sets, very high, moderately high and near earth orbits were modeled. The effect of the subtraction on the equation sets was studied both by using orbits which were similar and dissimilar as well as by using orbits which were close for the entire integration time and those which were close only for small periods of time.

3.1 Satellite Pairs Modeled

The eight satellite pairs modeled are diverse enough to cover a large variety of possible orbits while they have enough similarities to allow comparisons of the effects of changes in certain parameters on the relative efficiencies of the methods.

Pair A.

This first pair was inspired by the International Sun-Earth Explorers (ISEE) A B satellites which will be launched to study the structure of the solar wind. This pair models two high altitude orbits which have large eccentricities. The orbital parameters are:

	Satellite 1	Satellite 2
Semi-major axis	76,000km	76,000km
Eccentricity	.913	.78142
Inclination	0°	0°
Perifocus passage	2,000 sec	2,000 sec
Argument of perigee	0°	0°
Ascending node	0°	0°
Final integration time	150,000 sec	

Figure 1a shows these orbits, while Figure 1b displays the time history of the relative distance between the two satellites. The minimum separation for this pair occur at apogee and perigee and are on the order of 10,000 km. Apogee for satellite one is 145 380km, and for the second satellite is 135 380 km, so for this pair, the subtraction of $\bar{r}_1 - \bar{r}_2$ even at its worst case involves only minimal loss in significance. This allows a comparison of the equation sets when the subtraction does not introduce a significant error.

The next pair modeled is more representative of the ISEE A B pair, with much smaller separations at apogee and perigee to determine the relative effects on the equation sets of the close approach.

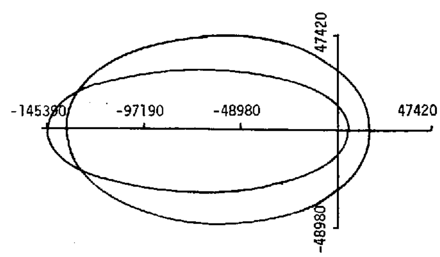


Figure 1a
x vs. y for Pair A
(Kilometers)

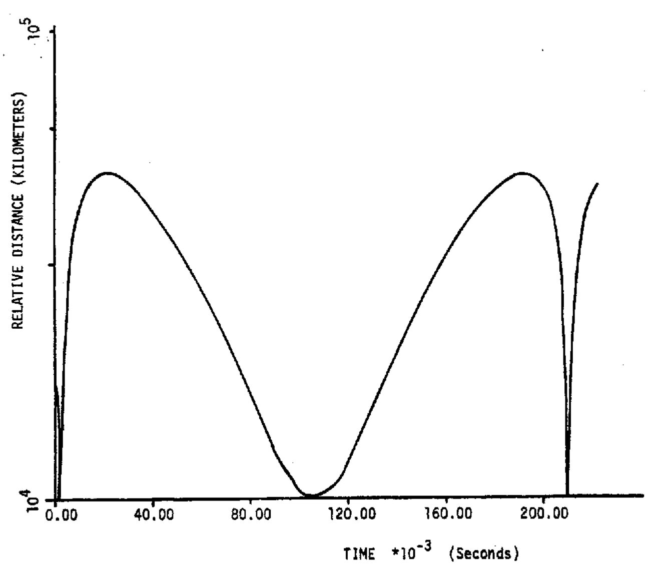


Figure 1b
TIME HISTORY OF RELATIVE DISTANCE
FOR PAIR A

Pair B

The orbital parameters for this pair are:

	Satellite 1	Satellite 2
Semi-major axis	76,000km	76,000km
Eccentricity	.913	.9129342105
Inclination	0°	0°
Perifocus passage	2,000 sec	2,000 sec
Argument of perigee	0°	0°
Ascending node	0°	0°

Final integration time 150,000 sec.

These parameters describe orbits which have minimum separations of only 5 kilometers, thus at agogee and perigee, as many as five significant digits can be lost in the subtraction of $\bar{r}_1 - \bar{r}_2$.

Figure 2a shows the similarity between the two orbits (one cannot distinguish between them at the scale used) while 2b displays the relative separation as a function of time. The close approach is the primary difference between this and the previous pair, and will allow us to draw conclusions as to the effect of a very small separation for this type of orbit.

The effects of these large eccentricities on the equation sets is also of interest. One way to study this is to use another pair model with much smaller eccentricities.

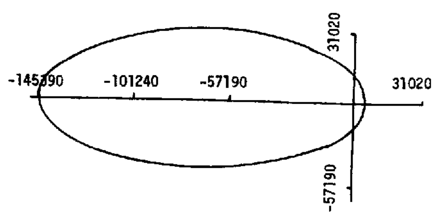


Figure 2a
x vs. y for Pair B
(Kilometers)

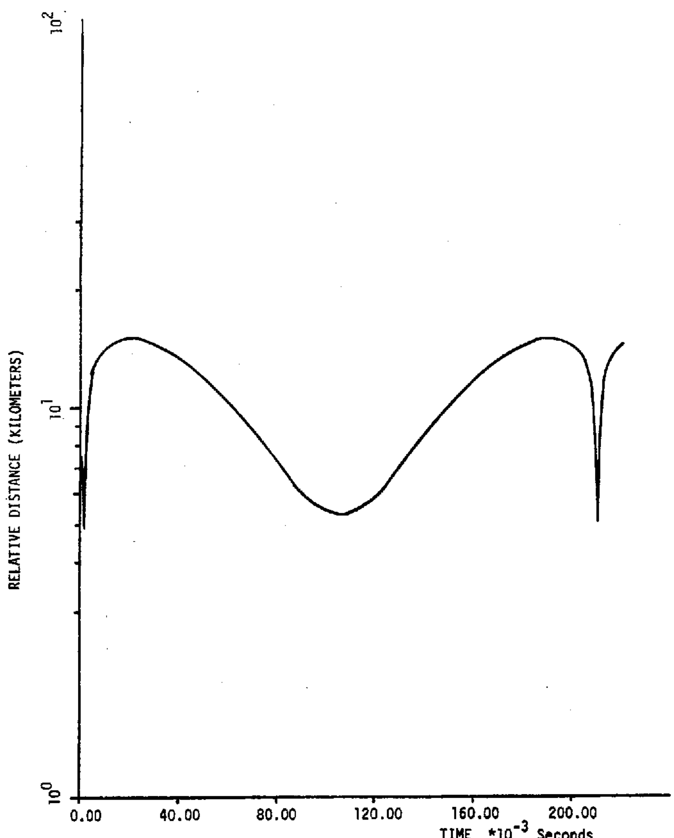


Figure 2b
TIME HISTORY OF RELATIVE DISTANCE
FOR PAIR B

Pair C

These satellites are two high altitude orbits, similar to the previous set, but one is circular, while the other is very nearly circular.

Orbital Parameters

	Satellite 1	Satellite 2
Semi-major axis	42,162.8km	42,162.8km
Eccentricity	0	1.0E-6
Inclination	0°	0°
Perifocus Passage	2,000 sec	2,000 sec
Argument of Perigee	0°	0°
Ascending Node	0°	0°
Final Intigration time	86,160 sec	

As can be seen from Figure 3a, these orbits are very similar, as were Pair B. The minimum separations (from Figure 3b) are approximately 4.216×10^{-2}km., and again up the five digits can be lost in the subtraction of $\bar{r}_2 - \bar{r}_1$. Thus, the major difference between Pairs B and C are the eccentricities of the orbit.

Up to this point, only very similar orbit pairs have been described but the comparisons of the equation sets for dissimilar orbits should also be investigated.

Pair D

For this case two very dissimilar orbits are modeled, one circular and the other moderately eccentric with close approach at perigee. The period of the second satellite is three times that of the first, so that every three revolutions of the first satellite there is another close

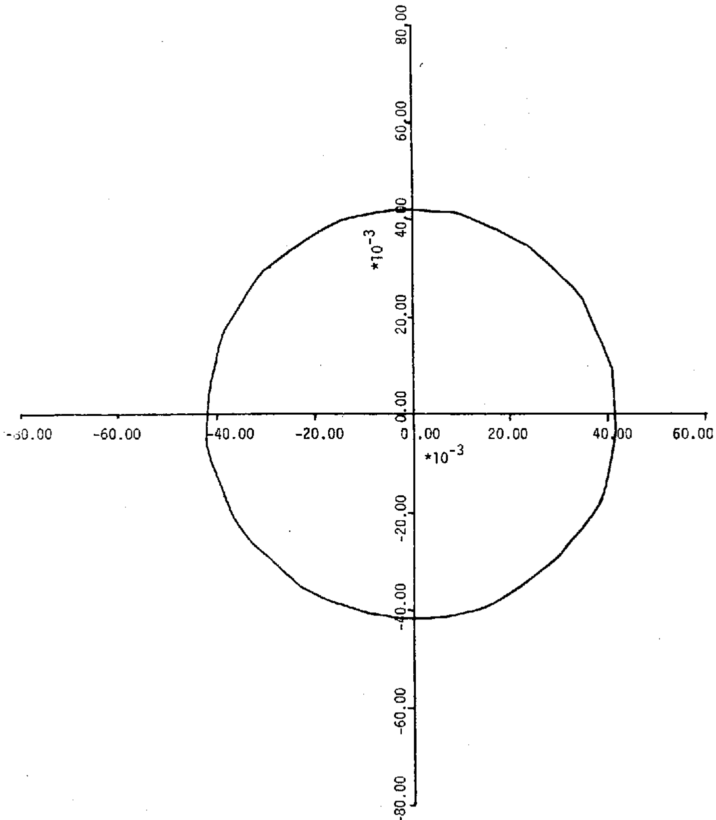


Figure 3a
x vs. y for Pair C
(kilometers)

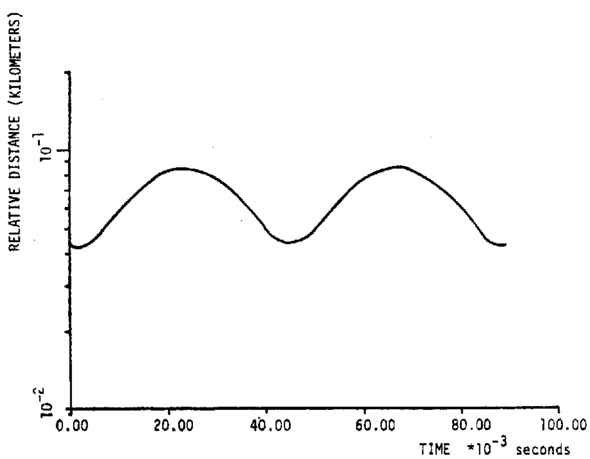


Figure 3b
TIME HISTORY OF RELATIVE DISTANCE FOR
PAIR C

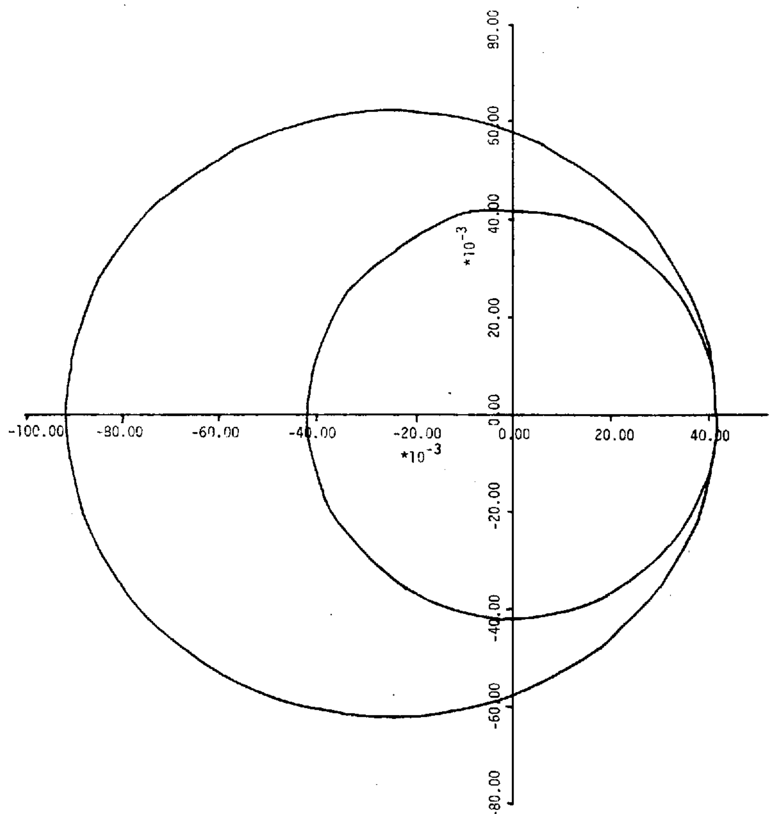


Figure 4a
x vs. y for Pair D
(kilometers)

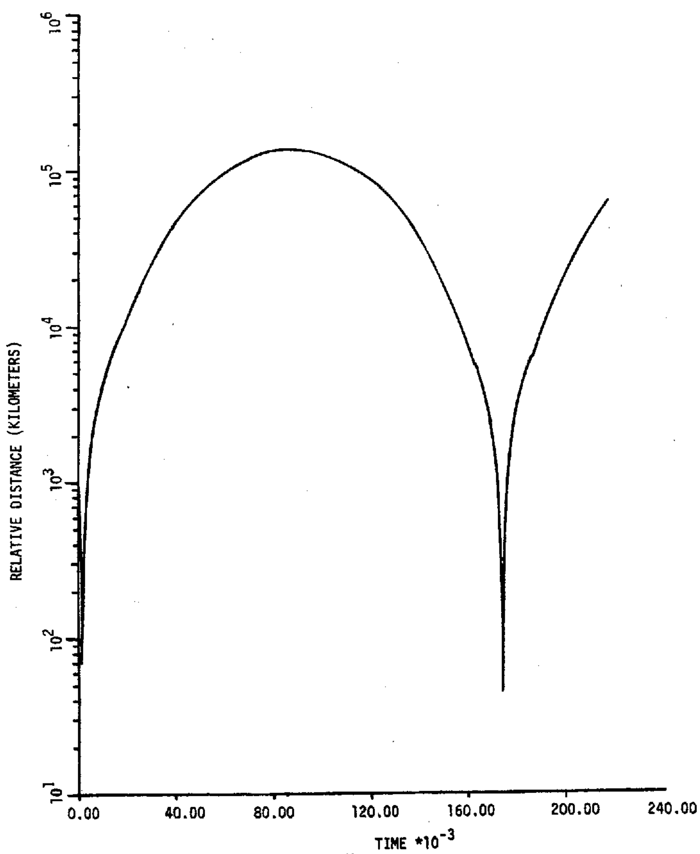


Figure 4b
TIME HISTORY OF RELATIVE DISTANCE
FOR PAIR D

approach. The orbits and their relative time histories are shown in Figures 4a and 4b.

Orbital Parameters

	Satellite 1	Satellite 2
Semi-major axis	42,162.8 km	66,929.37106 km
Eccentricity	0	.37004010125
Inclination	0°	0°
Perifocus passage	2,000 sec	2,000 sec
Argument of perigee	0°	0°
Ascending node	0°	0°

Final integration time = 86,160 sec

At the point of minimum separation, up to five significant digits may be lost in computing $\bar{r}_1 - \bar{r}_2$. Since this close separation occurs only once every three revolutions of Satellite 1, we can attempt to observe the propagated effects of the subtraction of the two nearly equal numbers. Another objective of the study is to compare the relative efficiencies of the equation sets as the altitudes of the orbits vary.

Pair E

This pair models two near earth low eccentricity orbits which are very similar and have small separation for the entire orbit. The only difference between this and Pair C are the altitudes of the orbits

The orbits are indistinguishable at the scale shown as can be seen in Figure 5a. The relative distance time history is shown in Figure 5b.

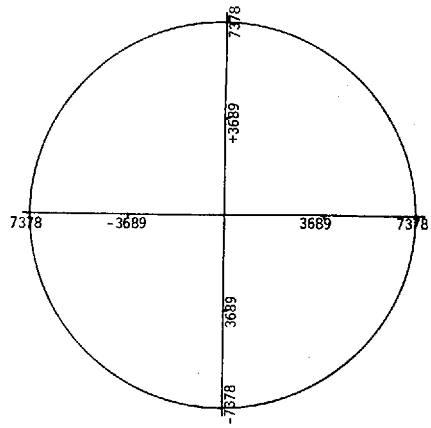


Figure 5a
x vs. y for Pair E
(kilometers)

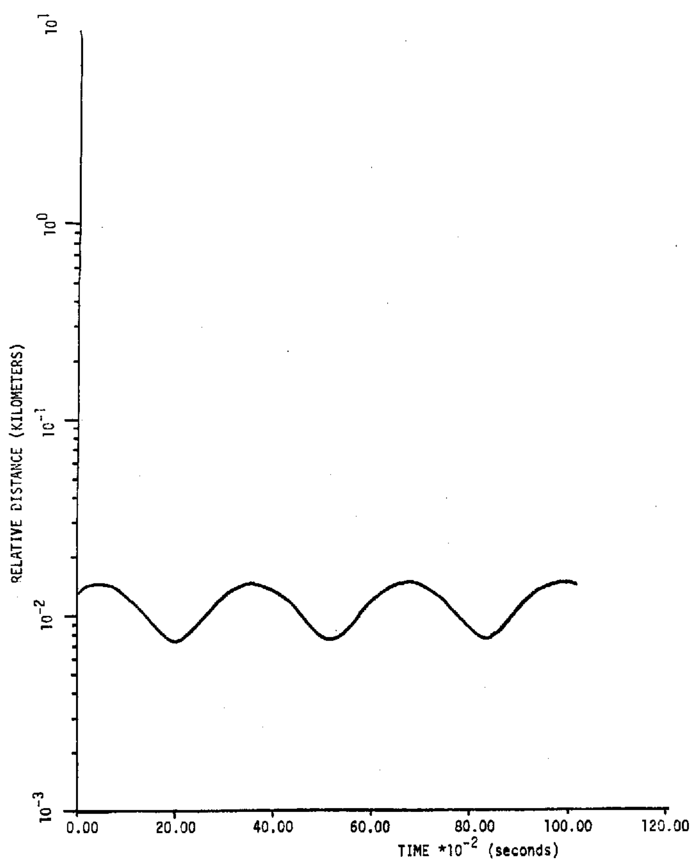


Figure 5b
TIME HISTORY OF RELATIVE DISTANCE
FOR PAIR E

Orbit Parameters

	Satellite 1	Satellite 2
Semi-major axis	7378.165km	7378.165 km
Eccentricity	0	1.0E-6
Inclination	0°	0°
Perifocus passage	2,000 sec	2,000 sec
Argument of perigee	0°	0°
Ascending Node	0°	0°

Final Integration Time = 10,000 sec

At the two points of minimum separation approximately 5 digits are lost in the subtraction of the two position vectors.

To further study the effects of eccentricities on the equation sets a situation similar to Pair E is used, but this time with a greater eccentricity for both satellites.

Pair F

Pair F represents two low altitude orbits which are very similar, but of moderate eccentricity. The orbit shape and relative vector time history are shown in Figure 6a and 6b, respectively.

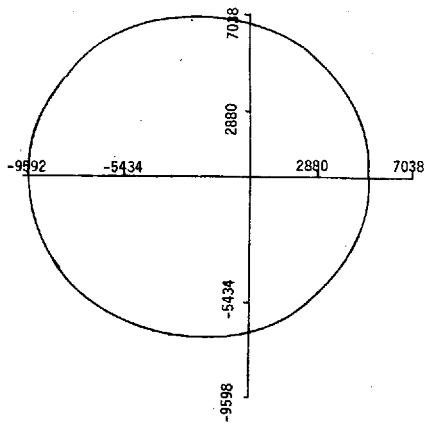


Figure 6A
x vs. y for Pair F
(Kilometers)

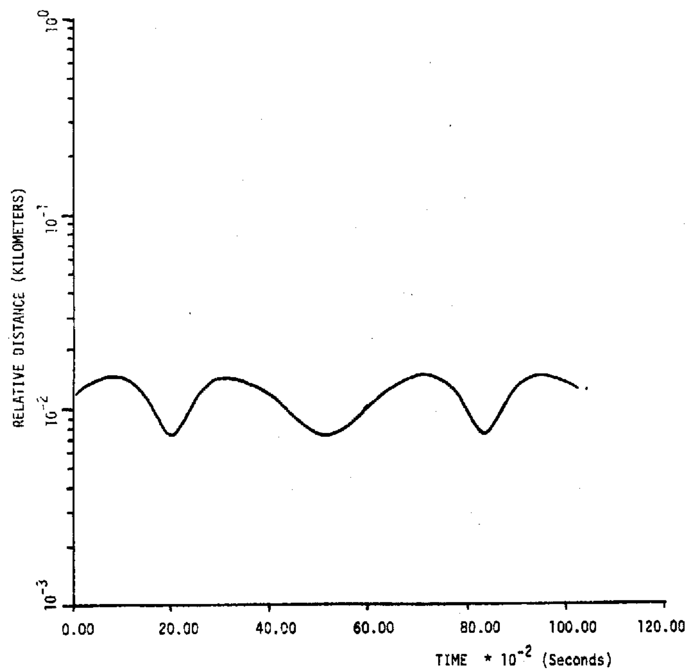


Figure 6b
TIME HISTORY OF RELATIVE DISTANCE
FOR PAIR F

Orbit Parameters

	Satellite 1	Satellite 2
Semi-major axis	7378.165km	7378.165km
Eccentricity	.3	.300001
Inclination	0°	0°
Perifocus Passage	2,000sec	2,000sec
Argument of Perigee	0°	0°
Ascending Node	0°	0°

Final Integration Time = 10,000 sec

Again the subtraction of $\bar{r}_1 - \bar{r}_2$ can result in a loss of up to 5 significant digits. The primary differences between this and the previous Pair is the larger eccentricities. Thus enabling another look at the effects of a greater eccentricity on the orbits. Two more pairs were considered, both pairs were very non-similar, the first one is presented below.

Pair G

In this case, two near earth orbits which are very dissimilar are modeled. The semi-major axis of satellite 2 is chosen such that the ratio of the periods of the two satellite is 1/3 resulting in a small separation every 3 revolutions of satellite 1. This provides another chance to study the propagation of the errors incurred in the first subtraction.

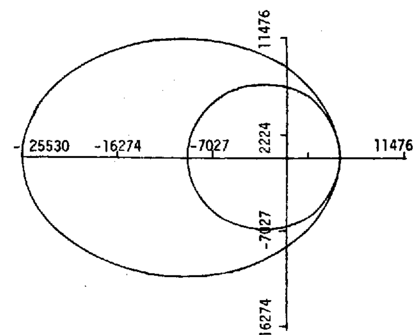


Figure 7a
x vs. y for Pair 6
(KILOMETERS)

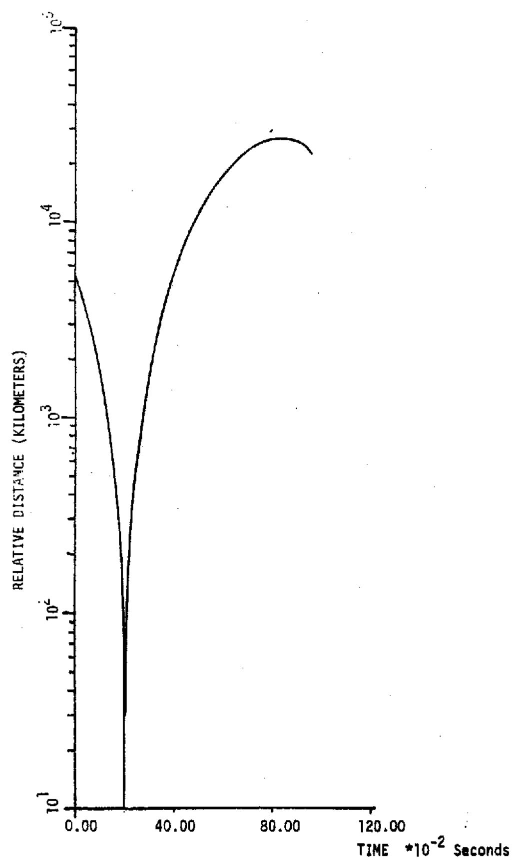


Figure 7b
TIME HISTORY OF RELATIVE DISTANCE
FOR PAIR G

Orbital Parameters

	Satellite 1	Satellite 2
Semi-major axis	7378.165km	15,347.2km
Eccentricity	.3	.6634883367
Inclination	0°	0°
Perifocus Passage	2,000sec	2,000sec
Argument of Perifocus	0°	0°
Ascending Node	0°	0°

Final Integration Time = 10,000sec

The minimum separation occurs at perigee and approximately 5 significant digits can be lost in the subtraction. The pair model resembles Pair D, except that the eccentricities are greater for both satellites. The orbit shapes and relative vector time history are shown in Figures 7c and 7b. The last pair model is again a nonsimilar pair, but they are more similar than G or D.

Pair H

These again model two near earth orbits, but which are more similar than Pair G. The ratio of the periods is now 1/2; and there is one close approach every two revolutions of the first satellite. The orbit shapes and relative position vector time history is shown in Figures 8a and 8b respectively.

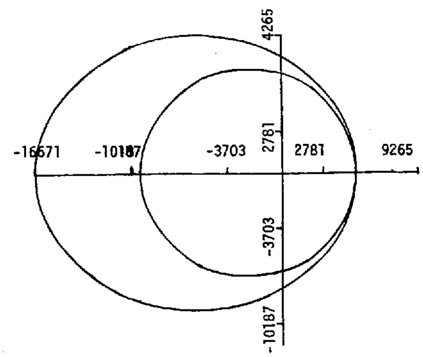


Figure 8a
x vs. y for Pair H
(Kilometers)

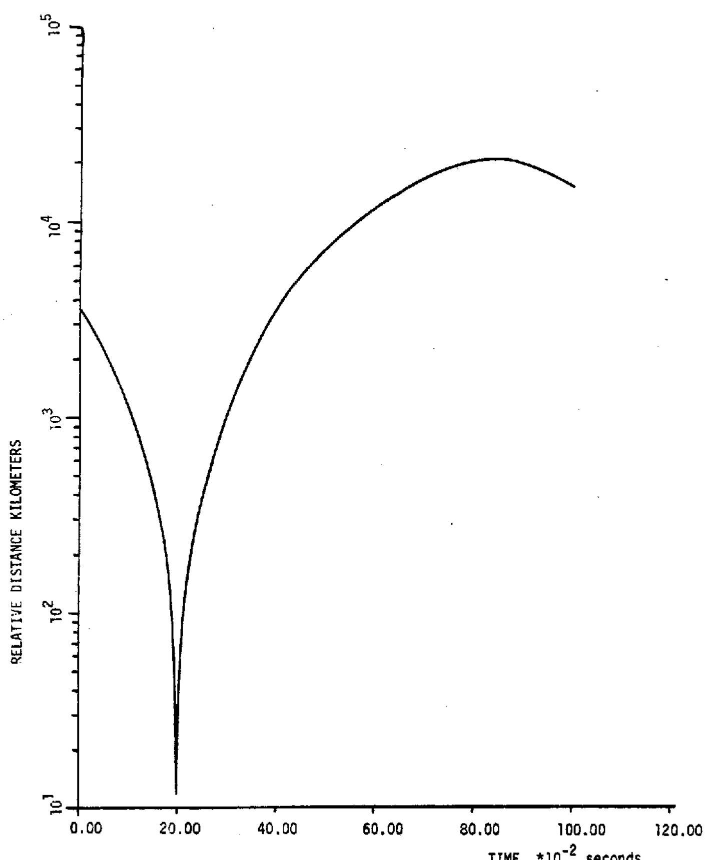


Figure 8b
TIME HISTORY OF RELATIVE DISTANCE FOR
PAIR H

Orbit Parameter

	Satellite 1	Satellite 2
Semi-major axis	7378.165km	10,917.672km
Eccentricity	.3	.526949
Inclination	0°	0°
Perifocus passage	2,000sec	2,000sec
Argument of Perigee	0°	0°
Ascending node	0°	0°

Final Integration Time = 10,000sec

The subtraction of $\bar{r}_1 - \bar{r}_2$ can again result in a loss of up to 5 significant digits.

Once the satellite pairs to be modelled have been decided upon, the next step is to choose the comparison method and criteria for comparison.

3.2 Testing Procedure

The first step in evaluating the relative merits of the various formulations is to develop viable comparison criteria. Since, in theory, it should be possible in any integration to achieve any reasonable accuracy, given a small enough step size, the criterion chosen was efficiency. The efficiency of a method can be based either on the total amount of computing time required to achieve a certain accuracy, or upon the number of function evaluations (calls to the derivative subroutine) required to achieve a certain accuracy. The total integtation time required by each method is obviously going to be dependent on many factors such as:

1) differences in the coding of the method

2) amount of information calculated at each step for each method,

3) relative difficulties involved in obtaining certain types of

information from each method.

One instance of the third problem is that if acceleration, velocity, and position information for each of the satellites and the relative vector are desired in the coordinate system where time is the independent variable at each output point, then transformations from s to t are required for each of the time smoothed equation sets, while the information is readily available for the Cowell formulations. Since the type and frequence of information required is very dependent upon needs of the particular user, the number of function evaluations required to achieve a disired accuracy was chosen as an efficiency criterion. For the particular force model chosen (two body only) the function evaluations are relatively inexpensive; but for more complicated force models they can become the dominant user of computing time. It is felt that trends apparent in this study will be reflected in those with more complicated force models.

For any comparison, a method of determining relative accuracy is a necessary consideration. Three approaches for determining relative accuracy were examined.

In the first approach, perturbed motion of the two satellites was integrated backwards in time from some reference state to some initial state using an integrator which had control over truncation error. The satellites were then integrated forward to the reference state, and a bound on the integration error was obtained by comparing the final solution to the reference state.

This approach had two major drawbacks. First, a considerable amount of setup time was required for each problem, to determine the "true" or reference solution. Second this method would only examine the effects of a particular perturbation history. It was felt that it would be more appropriate to examine the basic behavior of the equation sets, so this procedure was abandoned.

Another possibility is to transform the problem into one in which the initial conditions could be recaptured at a given time. The integrated solution would then be compared with the initial conditions. This recapturing of initial conditions severly restricts the type of problems which could be studied, and was discarded in favor of the following method.

A model was developed with strict two body motion. Such a model allowed use of the known integrals of the two body equations. Given the initial states for each satellite, reference state vectors for each satellite could then be computed using the known two body solution at any time. These reference states were computed in double precision to obtain the true (reference) states as accurately as possible.

A program was written to compare the accuracies and efficiencies of the methods tested. The logic flow within the program is diagramed in Figure 9. The desired initial conditons for each satellite, either in rectangular Cartesian or orbital elements are first read in, along with the desired central body, (earth or moon), the unit of length to be used (kilometers, earth radii, or lunar radii), and the output desired (Table 1). All required transformations necessary to get the initial states in the proper form for each of the methods are then carried out. These transformations are described in the Appendix.

After all input and initialization is completed, the actual integration is carried out. Because of the required time matching for the third equation set, and the complications this entails, the third equation set is treated first, then the others all follow the same pattern, with the exception of the time matching required for set 3. Equation set 3 is first integrated to the first desired output point. This is done by integration of the first satellite until the time has passed the first output time, then the second satellite is integrated until t_2 passes t_1, the integration is then iterated until the two times are within 10^{-6} second of one another. It was found that a more restrictive match than this often resulted in a stepsize in the s system which was so small as to be numerically zero. The time state was then calculated and all transformations required to compute desired output information was calculated. The output information was then computed and stored.

At this point a note on the method chosen to acheive time matching for equation set 3 is in order. The process of iterating on stepsize requires a great number of added function evaluations, and the efficiency curves for the method make the formulation look totally unfeasible. Other methods may be devised for accomplishing the necessary time matching more economically, but at this stage, the iteration method seems to be a fair approach. The added expense of interpolation would not be visible in the comparison, and errors incurred in an interpolation process would be difficult to isolate from the errors inherent in the integration.

After the required time matching has been completed, the system of equations is integrated to the next output point. The process is repeated until the final time has been reached.

The integration method employed was a Runge-Kutta (7)8 with Fehlberg's coefficients. This code employs variable step size and attempts to keep the relative local truncation error at each step less than the desired tolerance. Each case was run at various tolerances between 10^{-3} and 10^{-8} in order to generate the data for the efficency plots.

The rest of the equation sets are then processed in the same manner sequentially. Although it would be more efficient to integrate the equations simultaneously and make use of some of the intermediate information which is common to two or more equation sets, it was felt that totally isolating each equation set from the others would prevent any possible interaction by the equation sets. One such interaction is that since a variable step Runge-Kutta method was employed to carry out the integration; the smallest step size required for any of the equations would be used for all equation sets.

After the integration of all eight equation sets has been completed, the desired tables and plots are produced. This included the description of the comparison methods employed. The next chapter will present the results of these tests.

OUTPUT INFORMATION

WHICH CAN BE SELECTED.

TABULAR DATA

1. CP Time required by each method
2. Function evaluations for each method

PLOTTED DATA

(Line Printed, Calcomp, or Both)

Magnitude of Vector vs. Time	Available For Satellite 1	Satellite 2	Relative Vector
Position	x	x	x
Velocity	x	x	x
Acceleration	x	x	x
Angular Momenta	x	x	
Error in Position	x	x	x
Error in Velocity	x	x	x
Error in Acceleration	x	x	x
Error in Angular Momentum	x	x	
x vs. y	x	x	
y vs. z	x	x	
Function Evaluations vs. Error			x

TABLE 1

LOGIC FLOW WITHIN COMPARISON ROUTINE

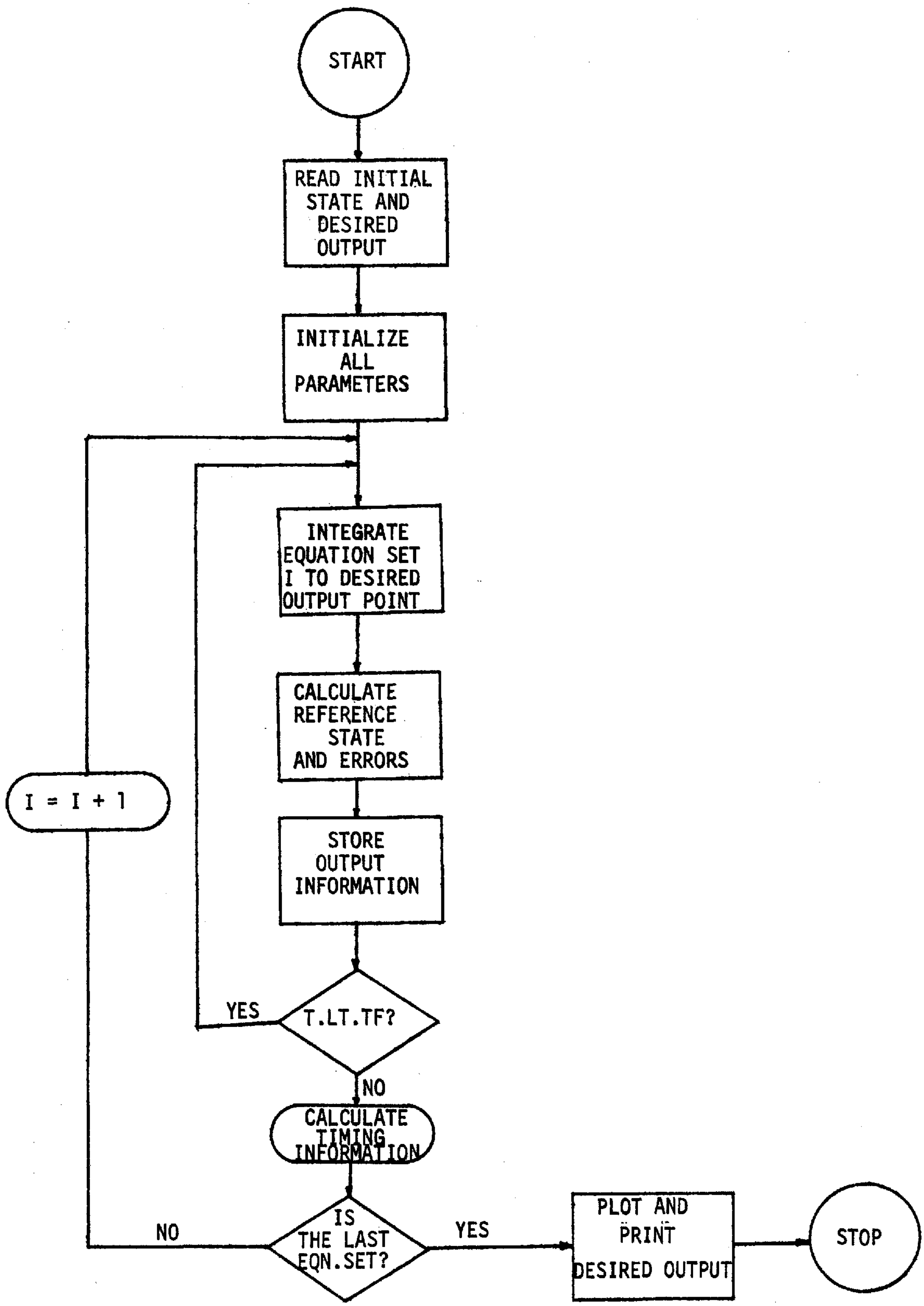


Figure 9

CHAPTER IV

Results of Numerical Studies

Each of the eight satellites pair models described in Chapter III were integrated using the eight equation of motion formulations. The information from these runs was first examined in order to determine the most efficient choice of n, the time smoothing exponent for equation sets incorporating time smoothing. Efficiency plots were made for each of the time smoothed equations, the normalized error being plotted against the number of functions evaluations for various values of n. The most efficient value of n is then the one which lies closest to the origin, (the one which yields the best accuracy with the fewest function evaluations). To determine the effects of the Nacozy-Szebehely method, methods involving it were plotted against their counterparts separately. Thus, Equation sets 4 and 7 are plotted separately, as are 5 and 8, and 2 and 6. Efficiency information is presented for the independently time smoothed equation set 3 in the first pair model only. Due to the large number of function evaluations required for time matching, it was not competative with the other methods from an efficiency standpoint. It was found that inclusion of set 3 on the comparison graph caused the scale to become so large as to make the efficiency information on the other equation sets difficult to discern. Thus, set 3 data was omitted for all pairs except pair A. In all of the plots, all errors plotted are the normalized L_2 norms of the errors at the final integration time, i.e., error in

$$\bar{r} = \frac{(x_{true}-x_{calculated})^{2}+(y_{true}-y_{calculated})^{2}+(z_{true}-z_{calculated})}{|\bar{r}_{true}|}$$

In the efficiency plots, each of the lines is composed of three data points, the lines connecting them are just to show trends, and do not denote imbedded data points.

The results will be broken down into eight sections, one for each satellite pair. The first portion of each section will deal with a determination of the best choice for the smoothing exponent in the time smoothing equation sets. Once this has been found, the Nacozy-Szebehely version of the relative motion equations will be compared with their standard counterparts. Finally, the most efficient form of each equation set will be presented on the final graph of each section.

As this comparison is based only upon the errors at the final time, it is necessary to determine how the equation sets behave as a function of time for each Pair tested. A plot depicting the time history of the normalized error in distance of the various equation sets is included for each pair.

4.1 Satellite Pair A

From Figures 10.1 a and b, the appropriate values of n for equation set 4, and its Nacozy-Szebehley counterpart equations, set 7, are seen to be 1.0 and 1.5 respectively. With these values a comparison of the two equation sets (Figure 10.1c) show that the N-S formulation does help the efficiency over a large region. Figures 10.1d, and 10.1e show that the value $n=1$ is most efficient for both sets 5 and 8. The N-S method appears to make equation set 8 slightly more efficient than set 5 at high accuracies. Figure 10.1g shows that set 6, (the N-S version of set 2) is again slightly more efficent. In general for this pair, the N-S method does not appear to make as large difference in the efficiencies of the methods. This is probably due to the fact that the smallest separations which the two satellites experience is very large, and the number of significant digits lost in the subtraction of $\bar{r}_1 - \bar{r}_2$ is small. Looking at

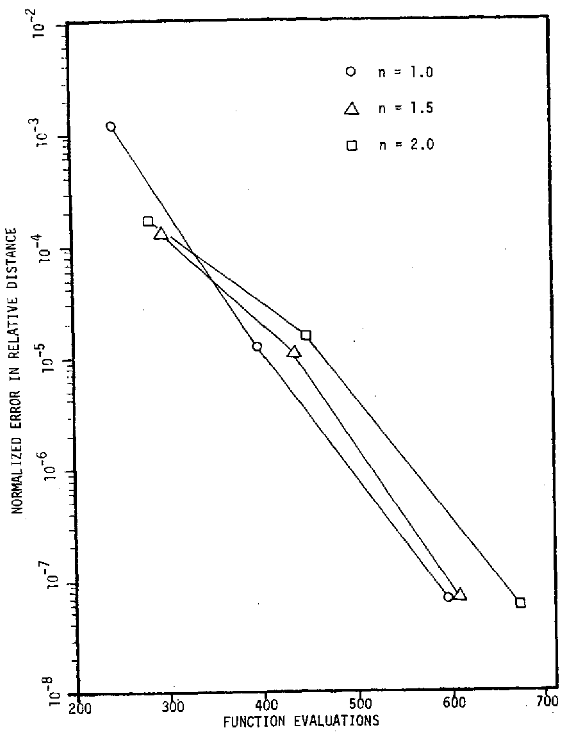


Figure 10.1a
Equation Set 4

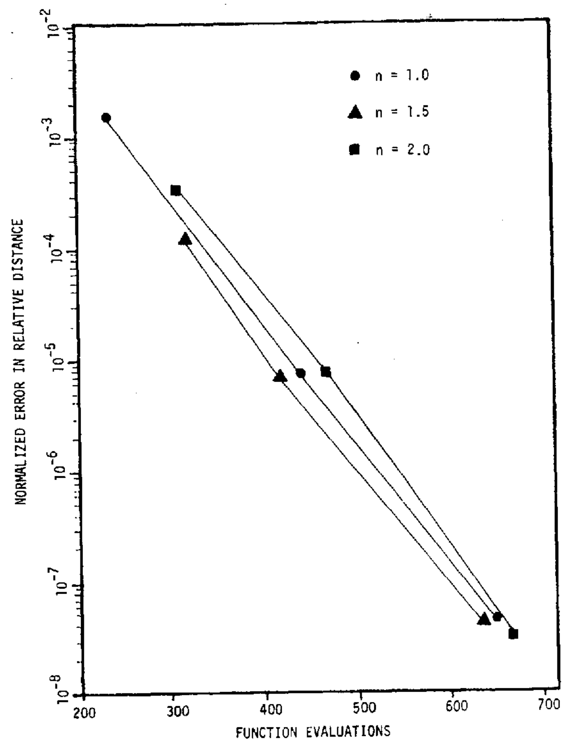


Figure 10.1b
Equation Set 7

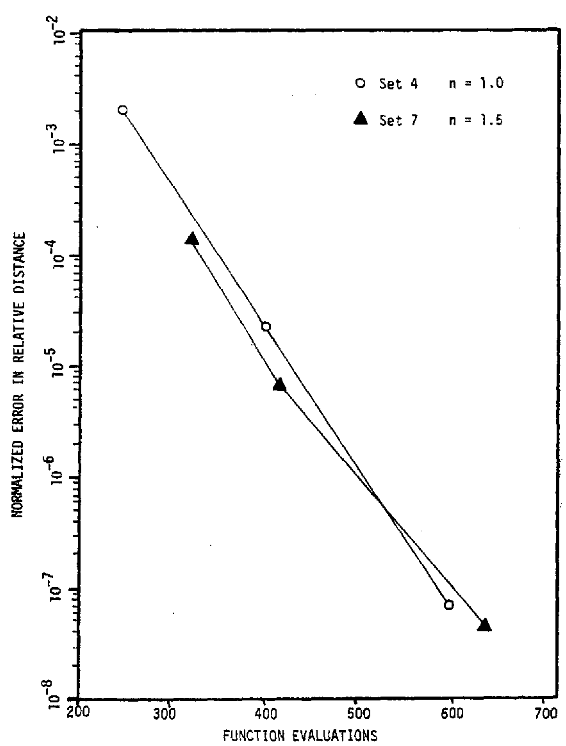


Figure 10.1c
Equation Set 4 vs. Equation Set 7

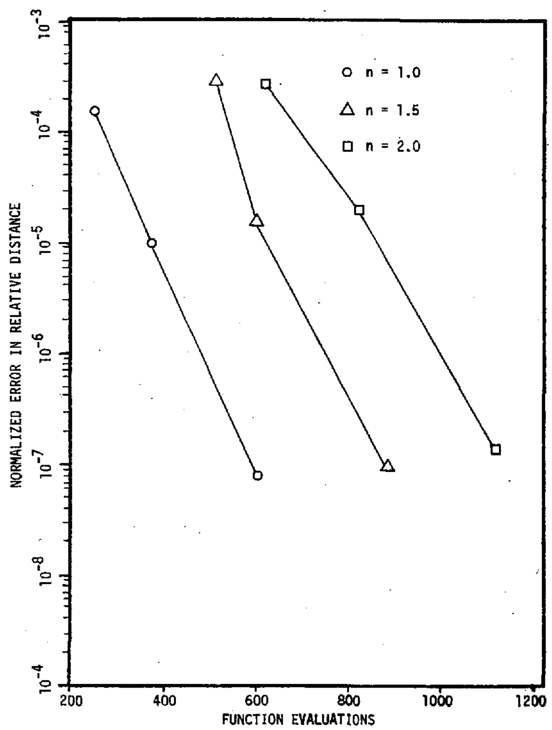


Figure 10.1d
Equation Set 5

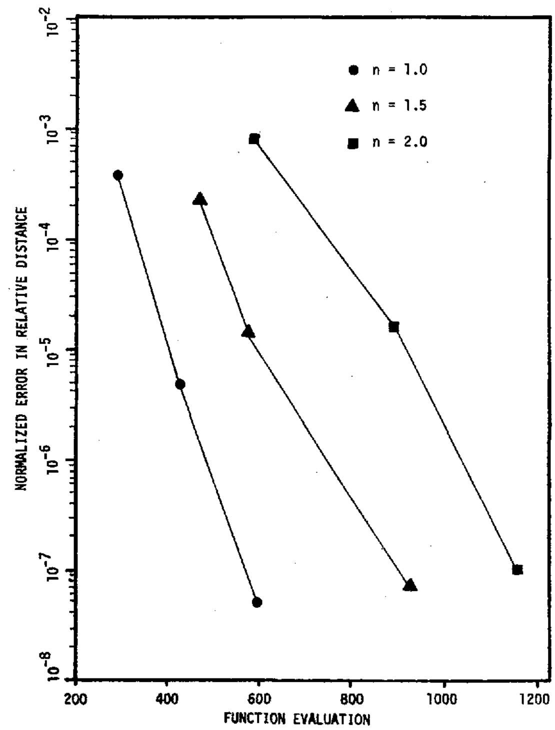


Figure 10.1e
Equation Set 8

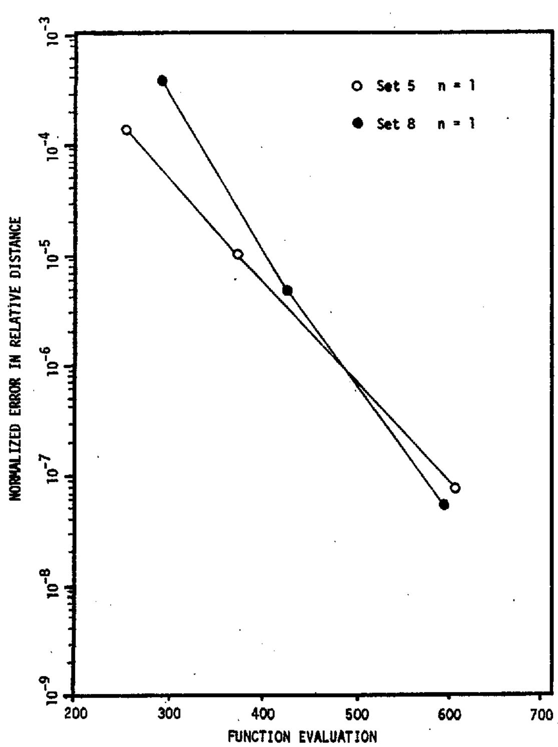


Figure 10.1f
Equation Set 5 vs. Equation Set 8

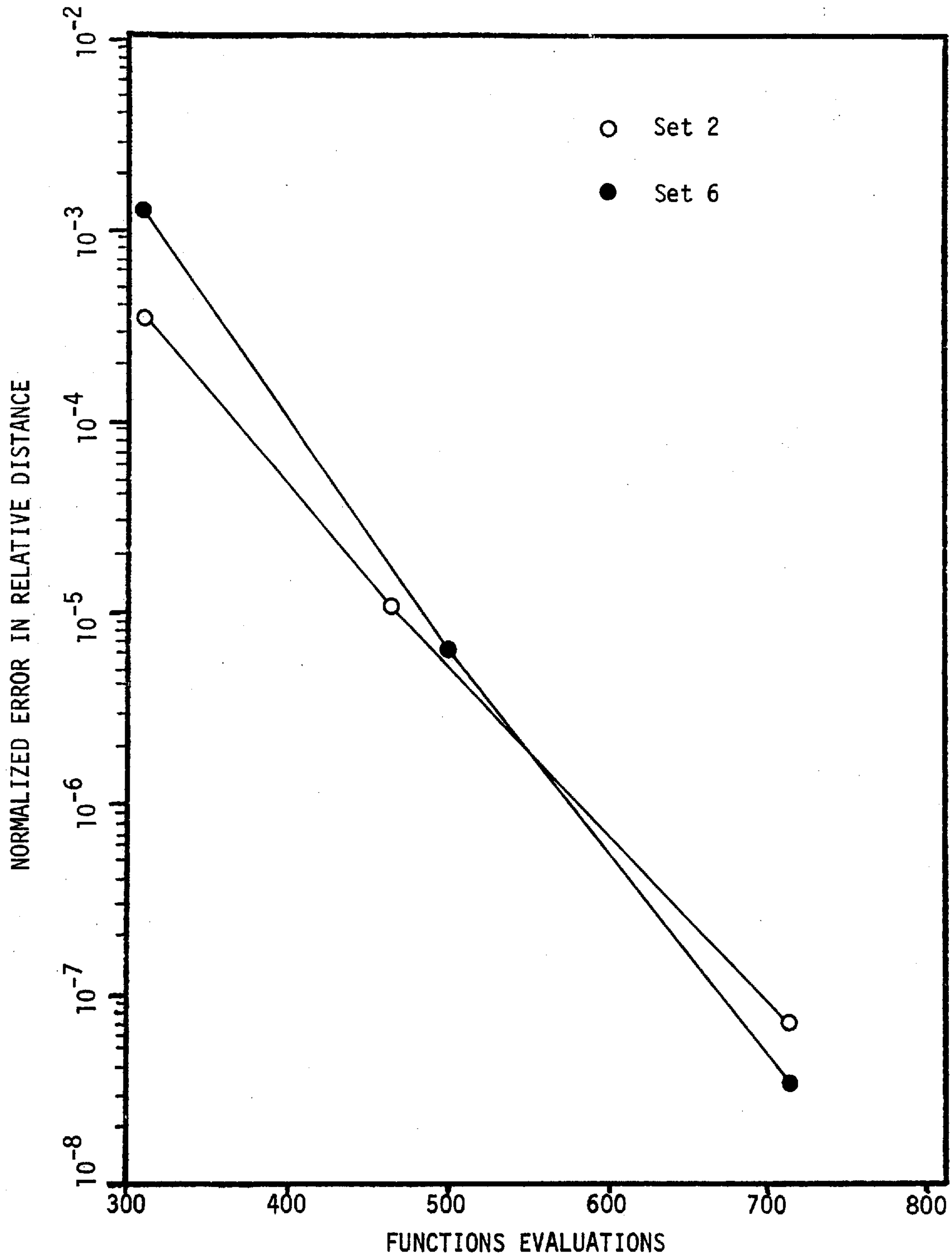


Figure 10.1g
Equation Set 2 vs. Equation Set 6

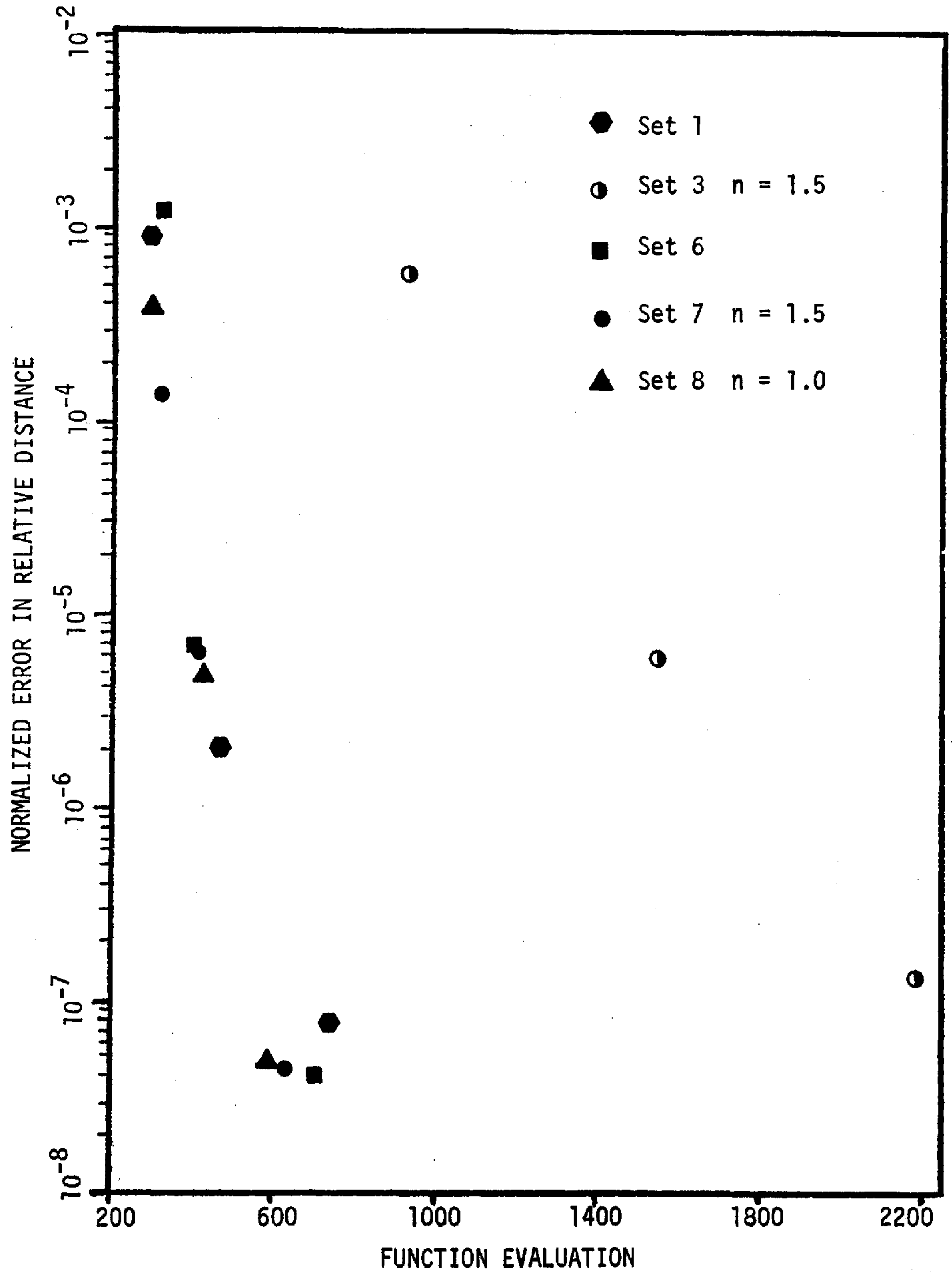
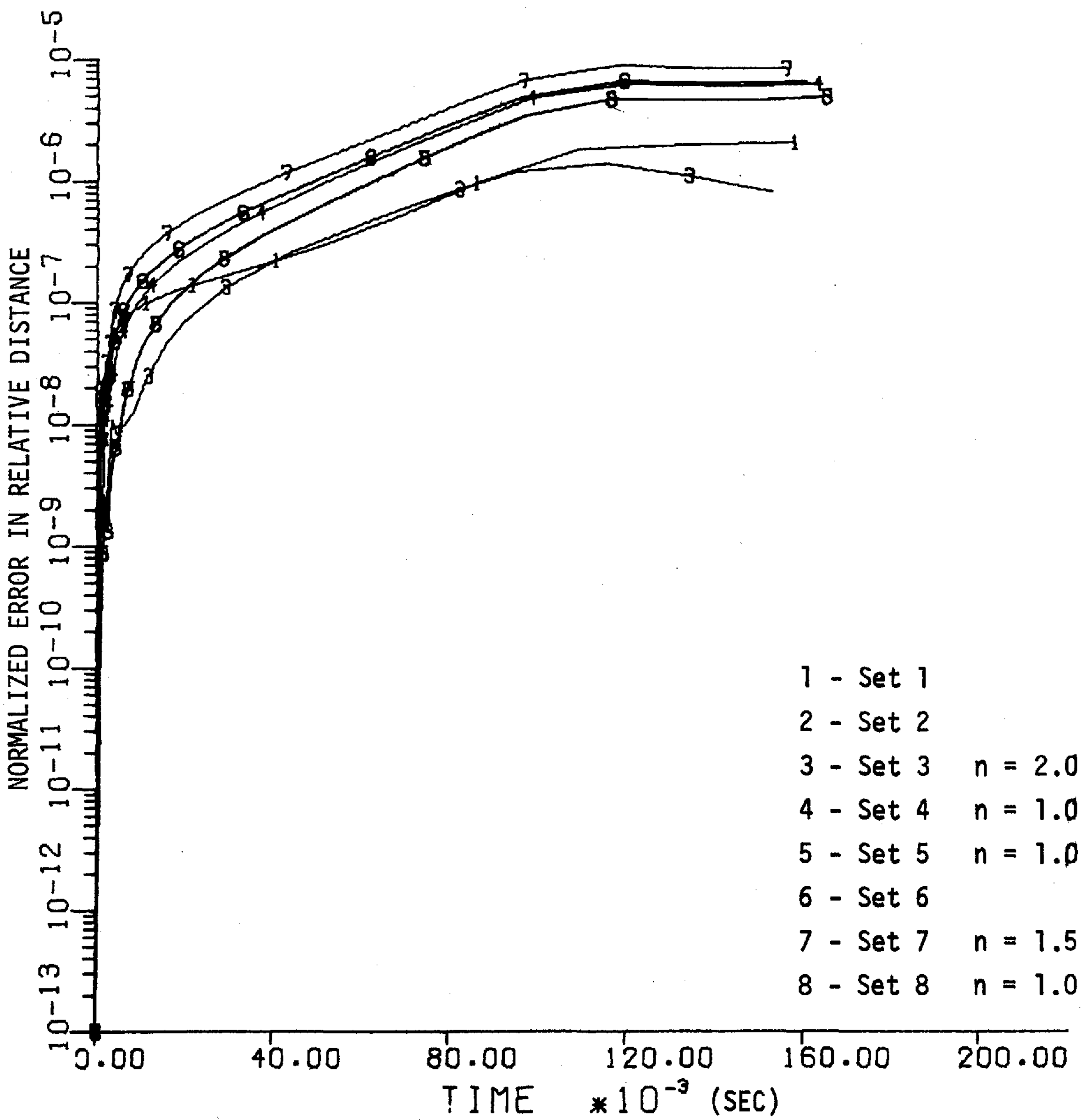


Figure 10.1h
Comparison of Equations for Satellite Pair A



TIME HISTORY OF ERROR PAIR A

Figure 10.1i

Figure 10.1h, it can be seen that equation set 3, the independently smoothed equation set was very inefficient as previously mentioned. Equation set 6 appears appreciably more efficient than the standard Cowell equations, especially at higher accuracies, and the time smoothed equation set are even more economical for higher accuracies.

If the time histories of the equation sets (Figure 10.1i) are now compared, it can be seen that the various equation sets appear to hold their positions relative to one another for the time displayed. From this observation one might be justified in the assumption that the efficiency comparisons are valid over a reasonable neighborhood of time around the point at which the comparison was actually made.

4.2 Satellite Pair B

The satellites modelled in this pair have much smaller separations at apogee and perigee and the orbits are more similar than those of Pair A. Figure 10.2a shows the value of n to have a more pronounced effect upon the efficiencies of equation set 4 than was observed in satellite Pair A (Figure 10.1a). The most efficient choice of n for this equation set is not clear cut and apparently depends upon the accuracies required, but for most cases the choice $n=1.5$ appears optimum. Figure 10.2b is much more definitive in showing $n=1$ to be most efficient for equation set 7. However comparison with Figure 10.1b shows the equation set to be more drastically affected by the choice of n for this satellite pair than the previous one. The effect of the Nacozy-Szebehely method upon equation set 4 is much more pronounced for this pair, as can be seen from Figure 10.2c. An accuracy of 10^{-7} can be achieved by Equation set 7 with almost 13% fewer function evaluations than with its standard subtraction counterpart, Equation set 4.

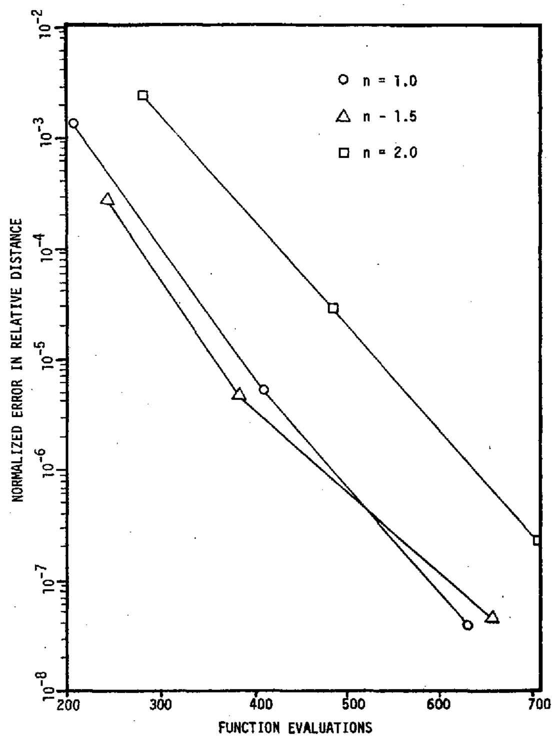


Figure 10.2a
Equation Set 4

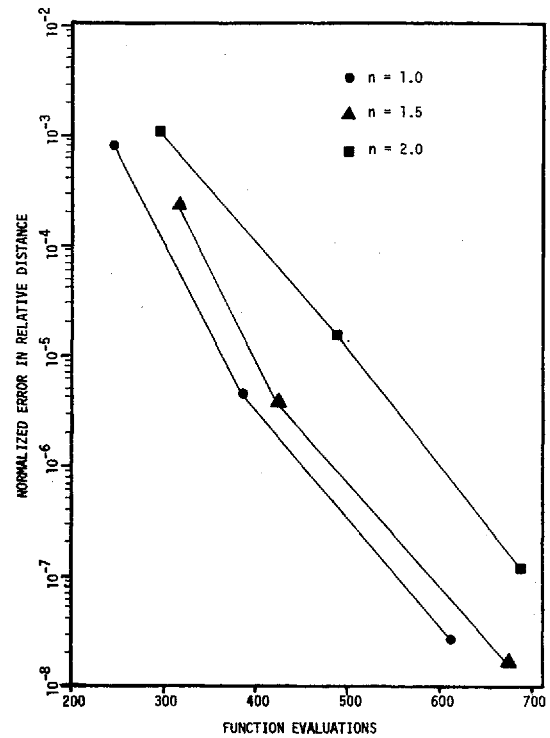


Figure 10.2b
Equation Set 7

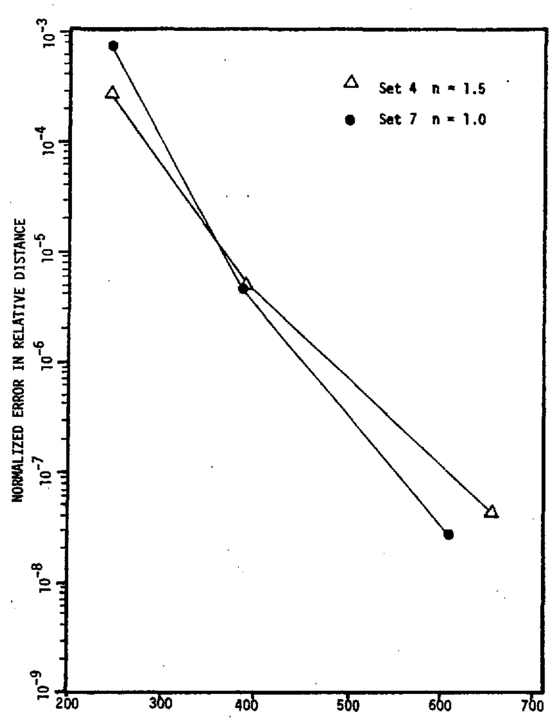


Figure 10.2c
Equation Set 4 vs. Equations Set 7

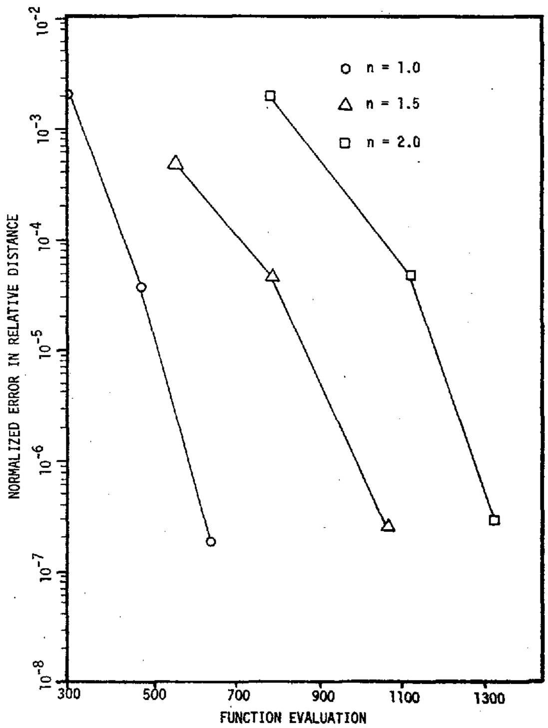


Figure 10.2d
Equation Set 5

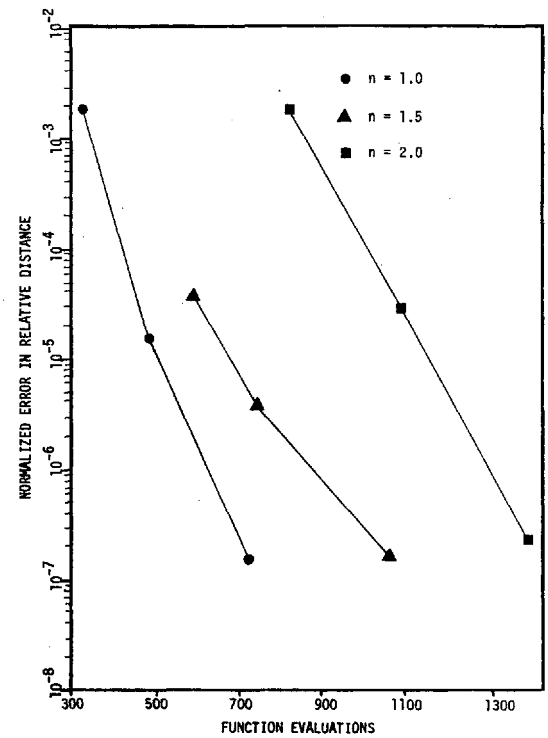


Figure 10.2e
Equation Set 8

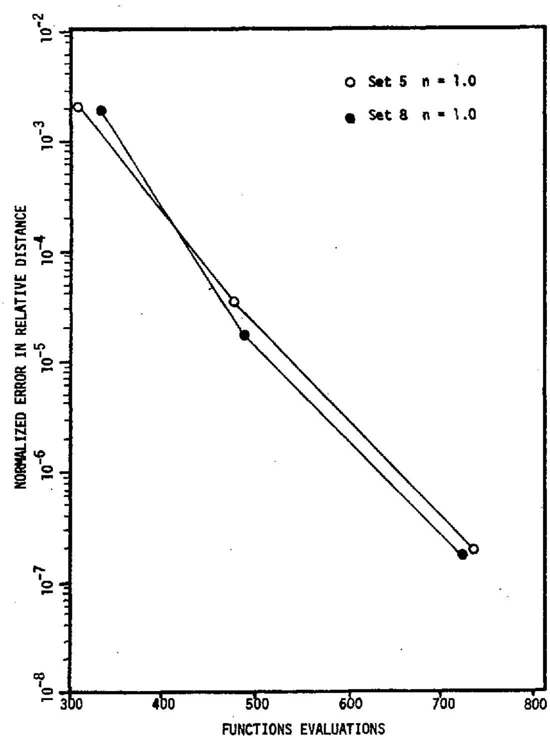


Figure 10.2f
Equation Set 5 vs. Equation Set 8

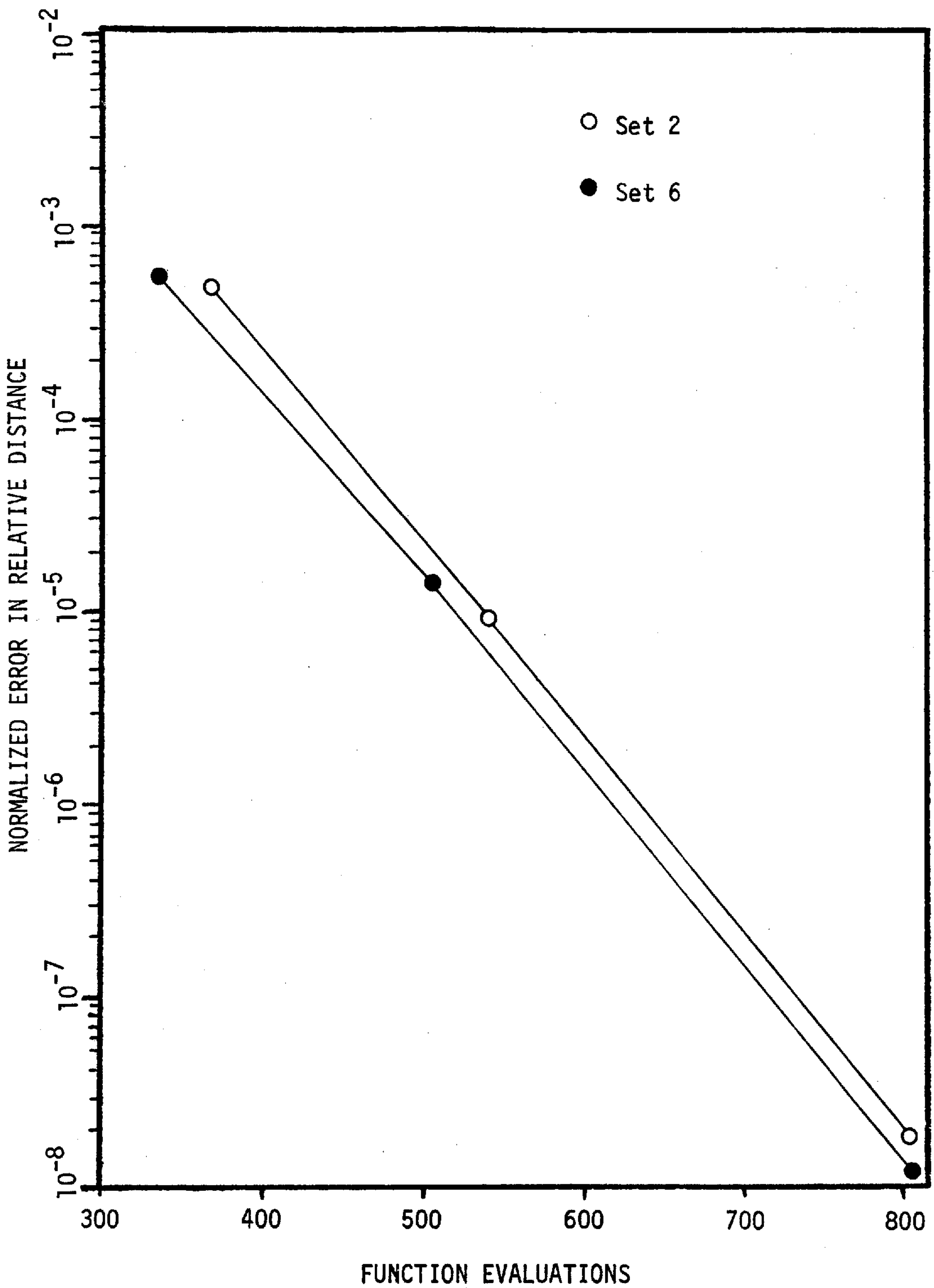


Figure 10.2g
Equation Set 2 vs. Equation Set 6

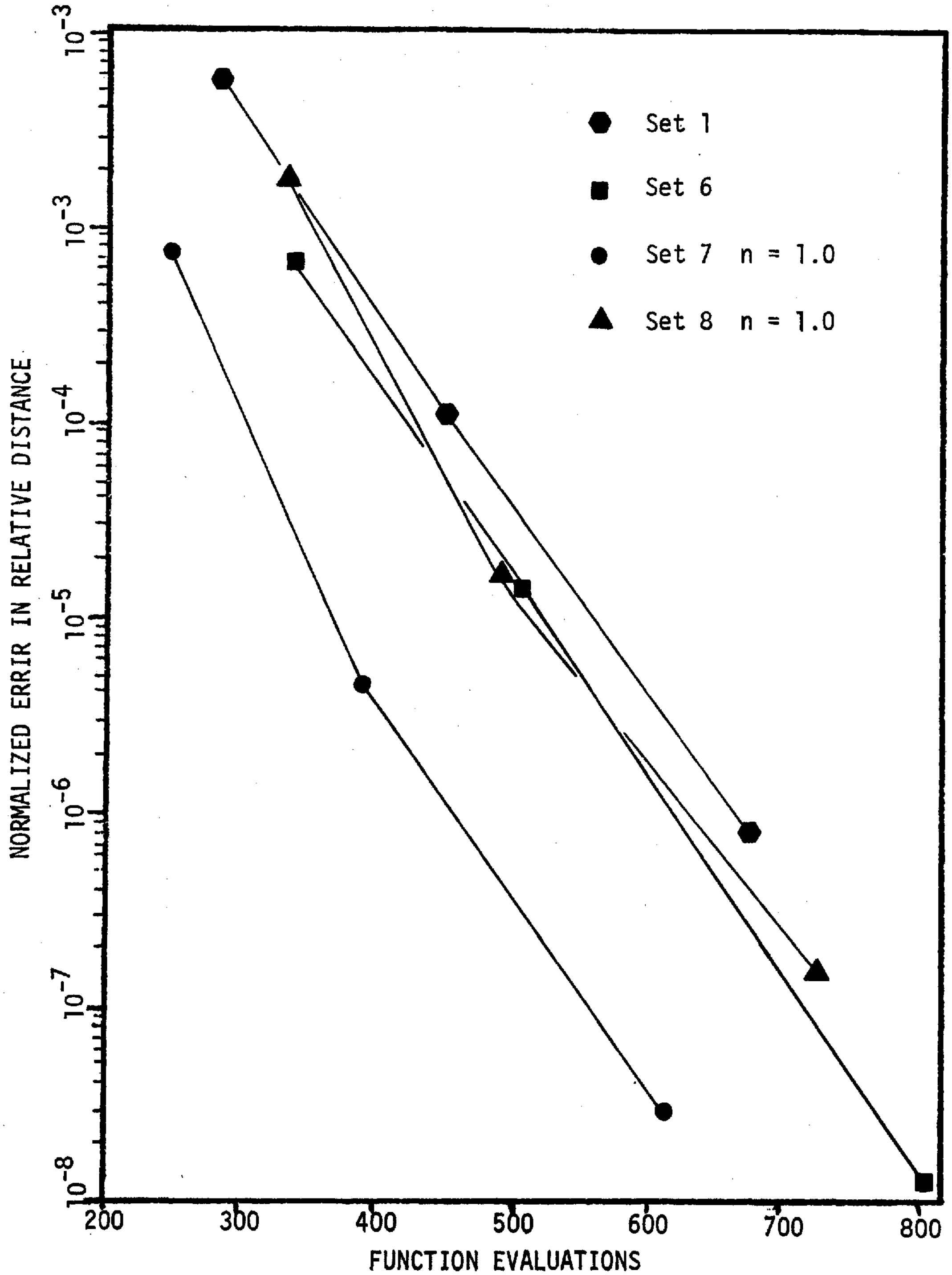
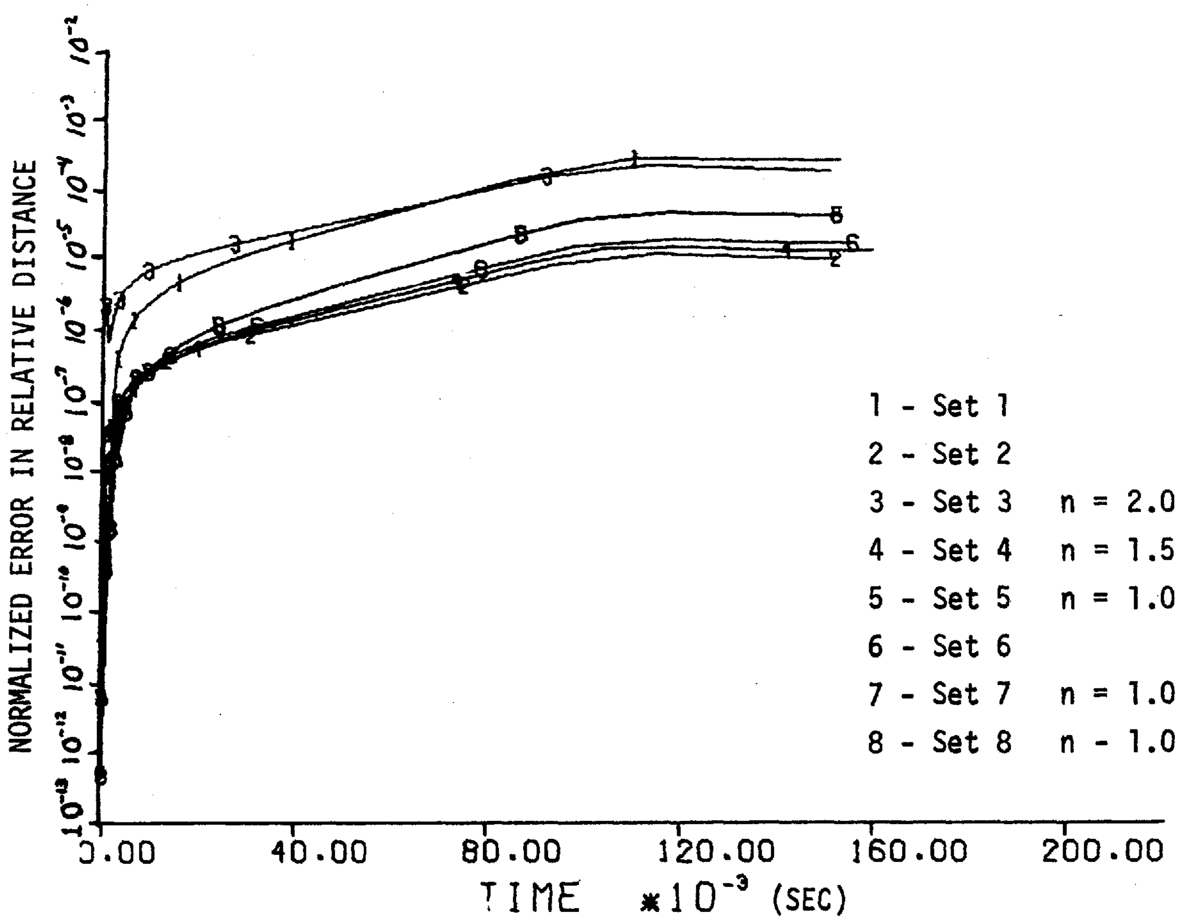


Figure 10.2h
Comparison of Equation for Satellite Pair B



TIME HISTORY OF ERROR PAIR B

Figure 10.2i

Comparison of Figures 10.2d and 10.1d show that this pair's closer approach did not seem to affect the choice of n for equation set 5, but merely moved the points to the left and up. The same conclusion can be reached for equation set 8 by observing the differences in Figures 10.2e and 10.1e. The N-S technique does not appear to have a large effect upon equation set 5, as can be seen from Figure 10.2f, but has a better effect than it did on Pair A (Figure 10.1f) resulting in a savings of almost 6% in function evaluations for most of the graph. Figure 10.2g shows set 6, (the N-S version of equation set 2) to be consistently more efficient than set 2, although this increase in efficiency is limited to about 6%. Comparison of this to the same graph for Pair A (Figure 10.1g) shows that the N-S method definitely helps set 2 for Pair B to a greater extent than Pair A.

The comparison of the best version of each of the equation sets on Figure 10.2h shows that a dramatic increase in efficiency can be realized by Equation set 7. The same accuracy as the next best set can be obtained with 22% fewer function evaluations. The other equation sets seem pretty much clustered with one another. For Pair B, then, the N-S technique improves the efficiencies of relative equations and the effect of n on sets 4 and 7 is more pronounced than for Pair A.

A glance at Figure 10.2i shows that once again the equations sets all appear to vary in the same fashion, with time. This observation seems to support the belief that the efficiency comparisons could be accurately extrapolated to the region around the chosen comparison time.

The reader should recall that the primary difference between Pair A and Pair B is that the orbits in Pair B are much more similar and have a

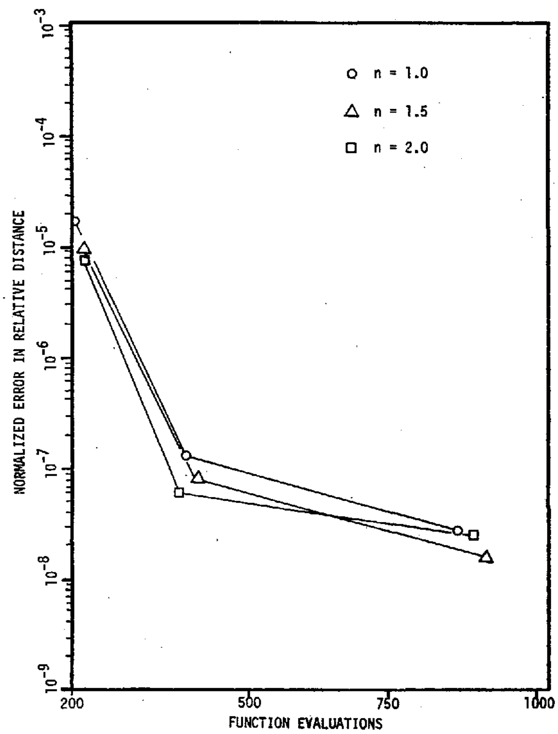


Figure 10.3a
Equation Set 4

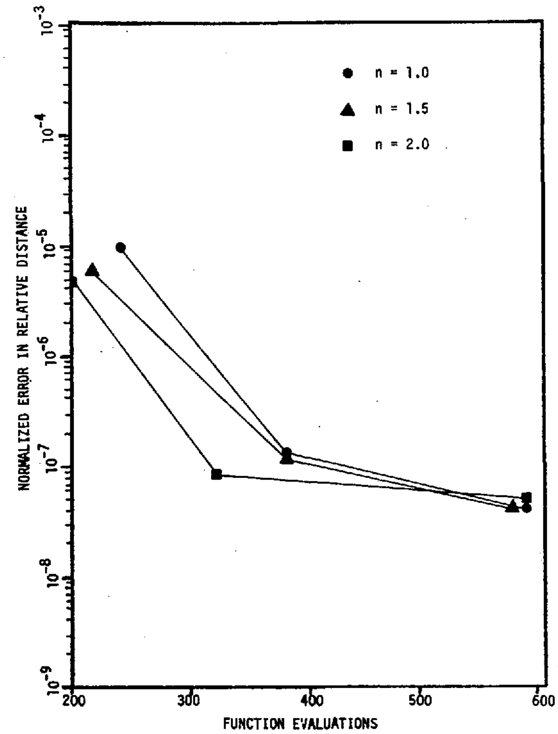


Figure 10.3b
Equation Set 7

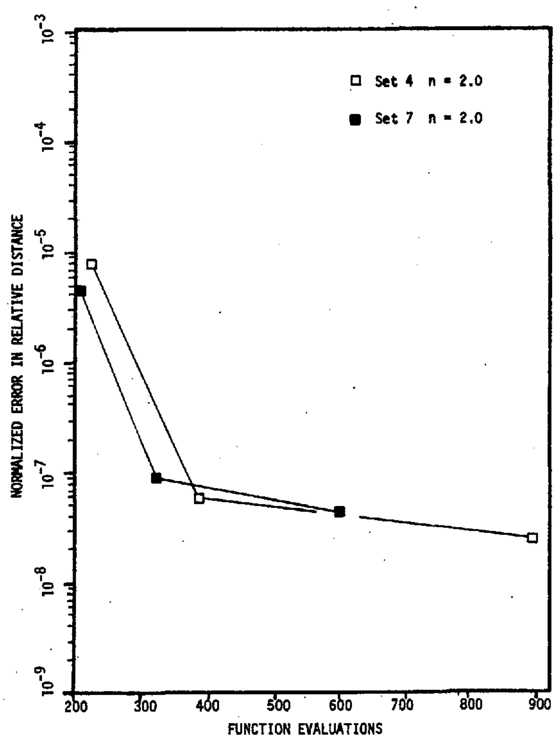


Figure 10.3c
Equation Set 4 vs. Equation Set 7

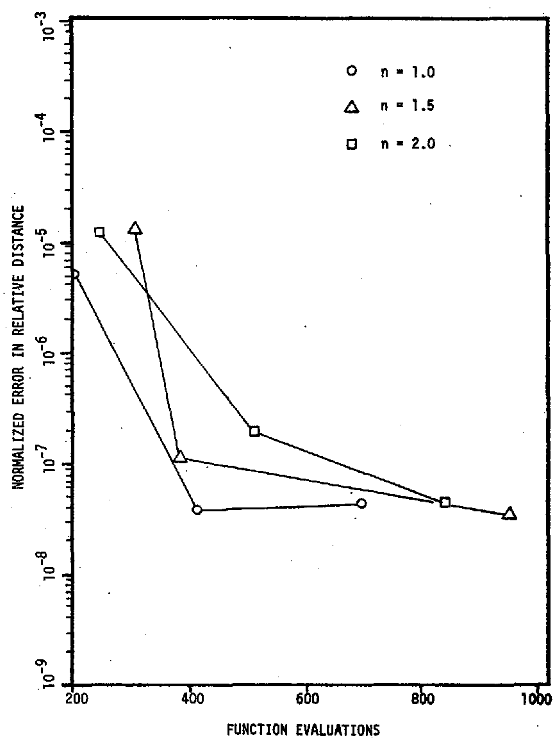


Figure 10.3d
Equation Set 5

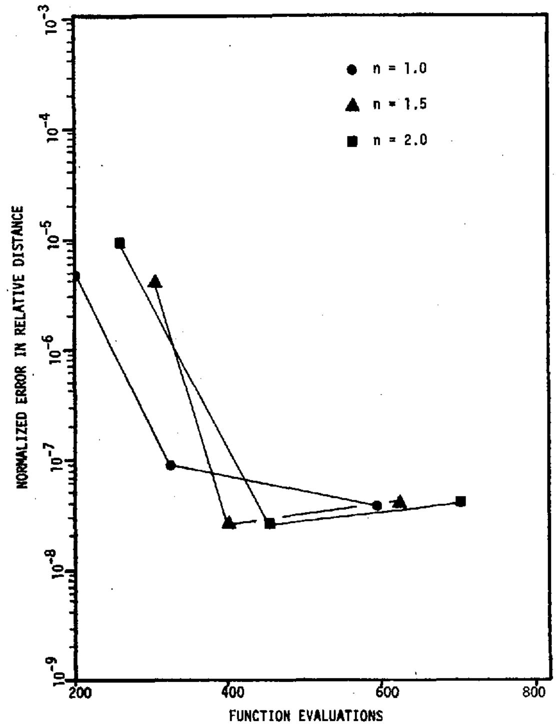


Figure 10.3e
Equations Set 8

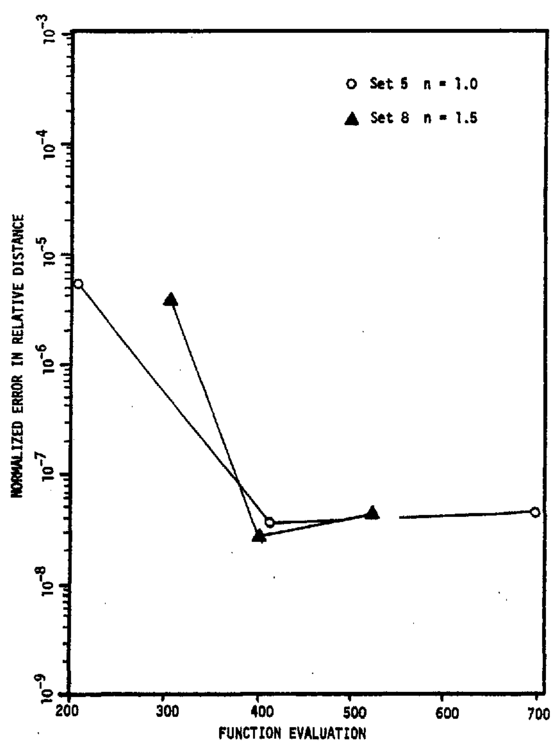


Figure 10.3f
Equation Set 5 vs. Equation Set 8

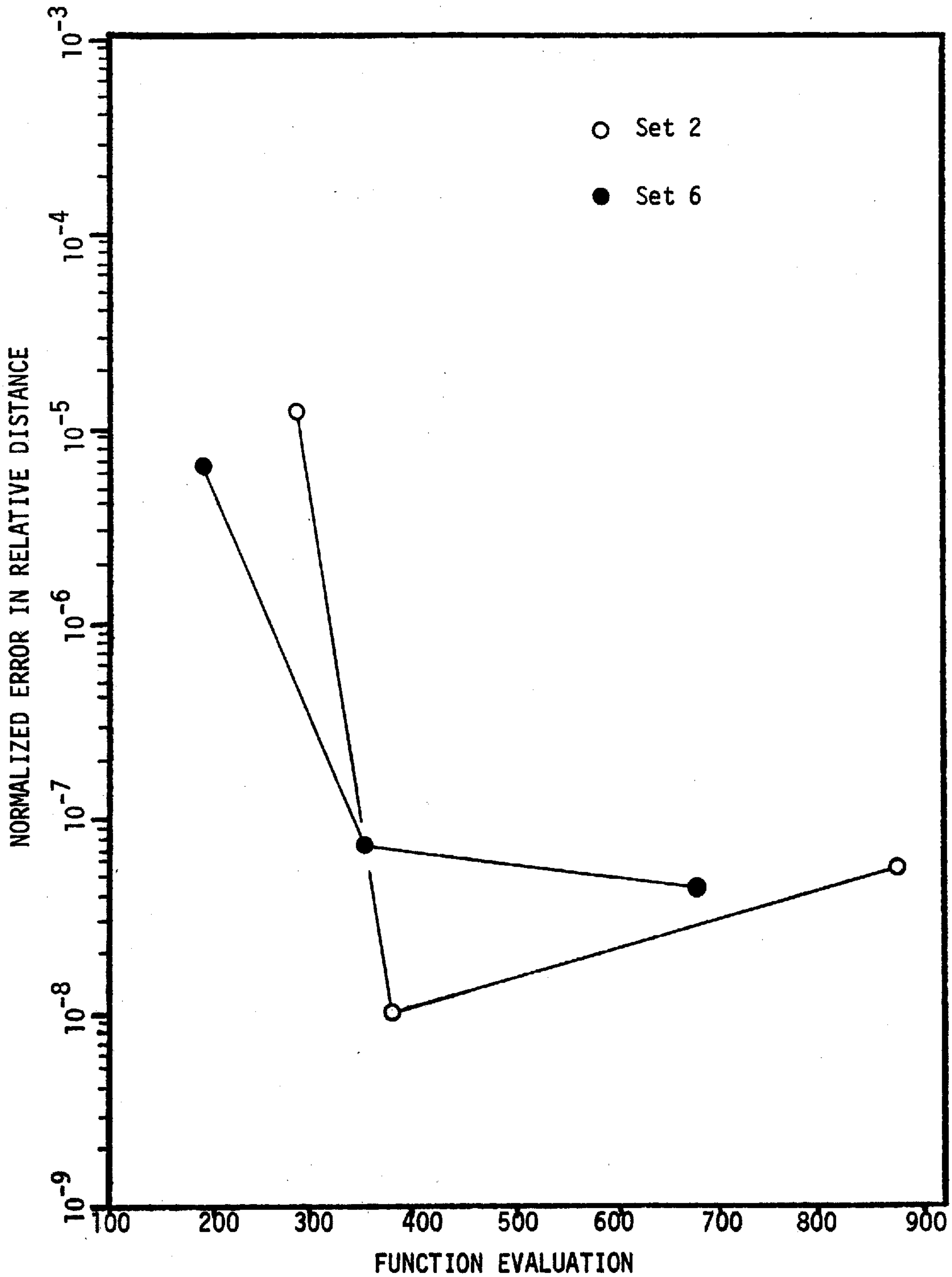


Figure 10.3g
Equation Set 2 vs. Equation Set 6

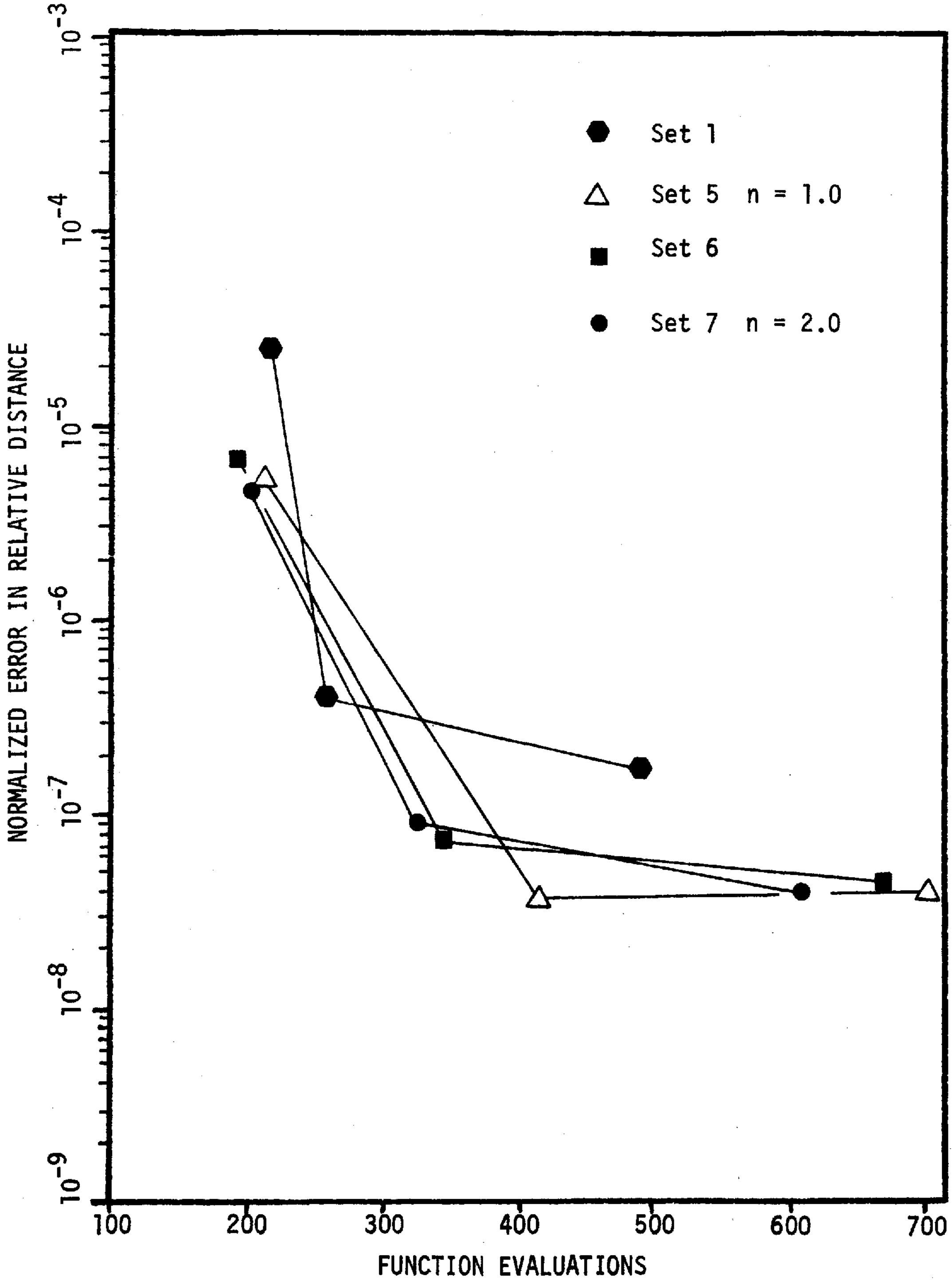
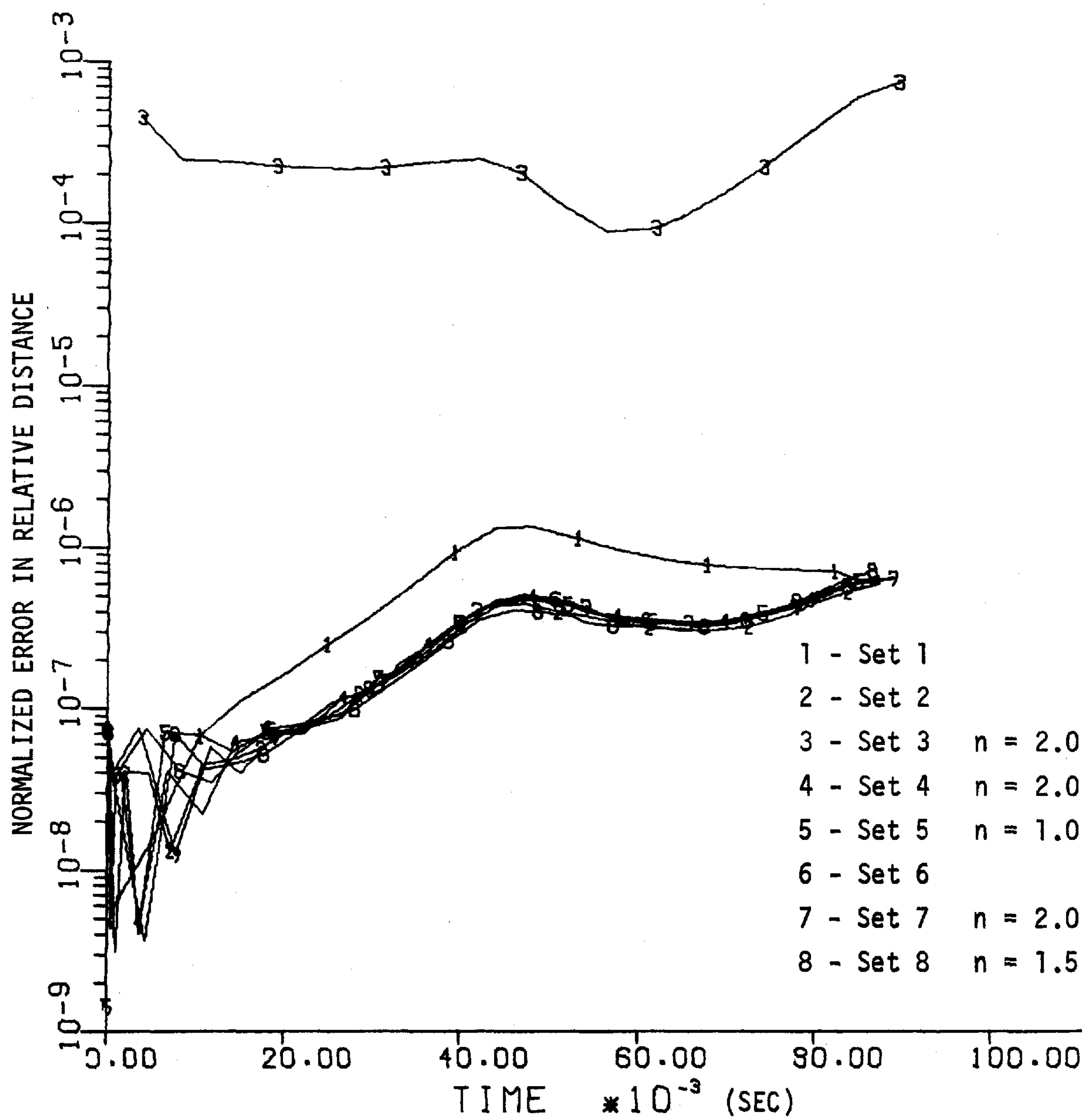


Figure 10.3h
Comparison of Equations for Satellite Pair C



TIME HISTORY OF ERROR PAIR C

Figure 10.3i

smaller separation. The orbits in the next pair, Pair C, are also very similar and have small separation distances. Pair C is a high altitude Pair with both satellites having very small accentricities.

4.3 Satellite Pair C

The efficiency plots show rather erratic behavior for the methods in high accuracy regimes for this satellite pair. It is felt that this is due to the effects of entering the region of round-off error.

Figure 10.3a, when compared with Figure 10.2a, shows the effect of n to be much less on the efficiency of equation set 4, but the best choice for n appears to be 2.0. In Figure 10.3b, the same type of behavior is exhibited for set 7, with $n = 2.0$ appearing to be the most efficient value. The comparison of equation sets 4 and 7 in Figure 10.3c shows the N-S method to be very advantageous for this equation set. Note that a savings of up to 25% in function evaluations can be realized over most of the range of accuracies. The determination of the best n for sets 5 and 8 from Figures 10.3d and 10.3e, yields $n = 1$ to be the best but it should be noted that these equation sets are not as clearly affected by the value of n as were equation sets 4 and 7.

Figures 10.3f and 10.3g show little benefit from the application of the N-S technique to sets 5 and 2. This is a major change from the previous pair. The comparison of the most efficient version of each equation set, presented in Figure 10.3h, does not show any one set to be most efficient for all accuracy ranges of accuracies, but shows set 7 to be the best over the largest range of accuracies with the other sets clustered near it.

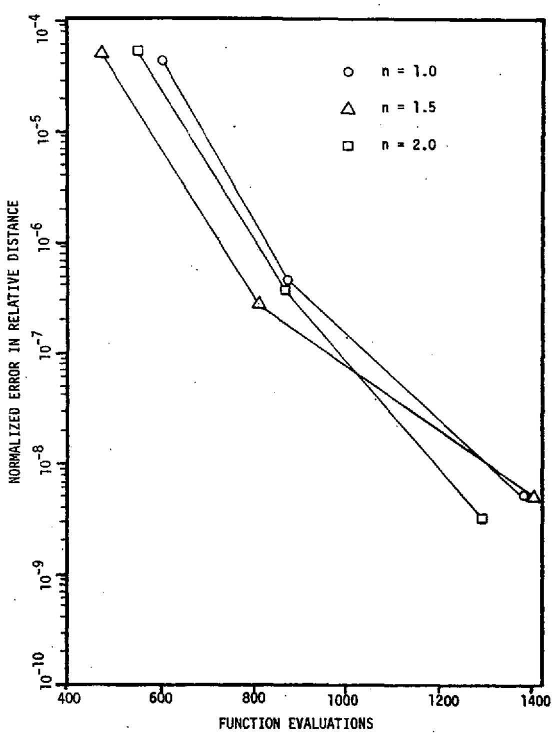


Figure 10.4a
Equation Set 4

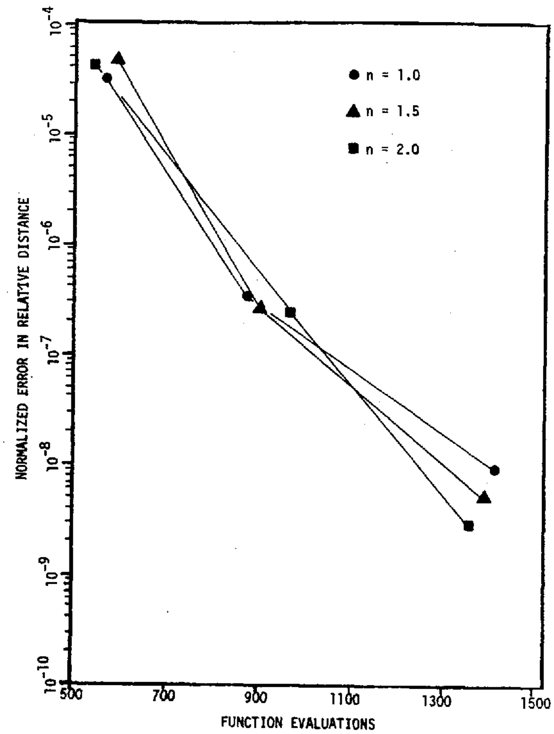


Figure 10.4b
Equations Set 7

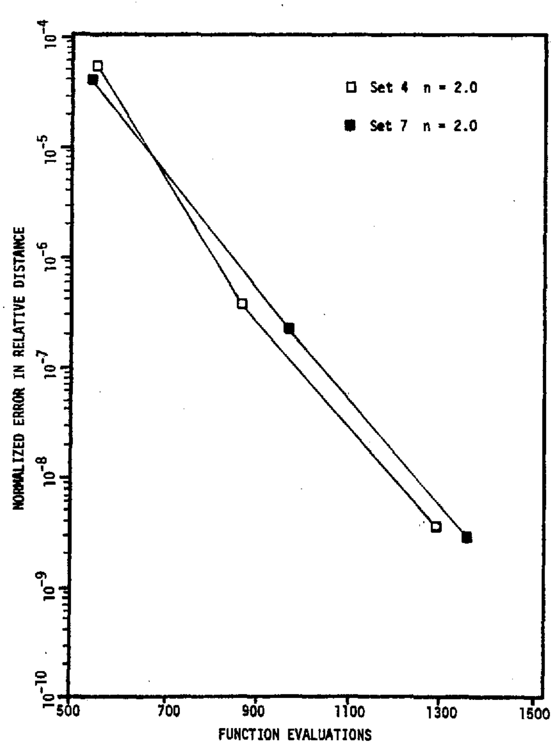


Figure 10.4c
Equation Set 4 vs. Equation Set 7

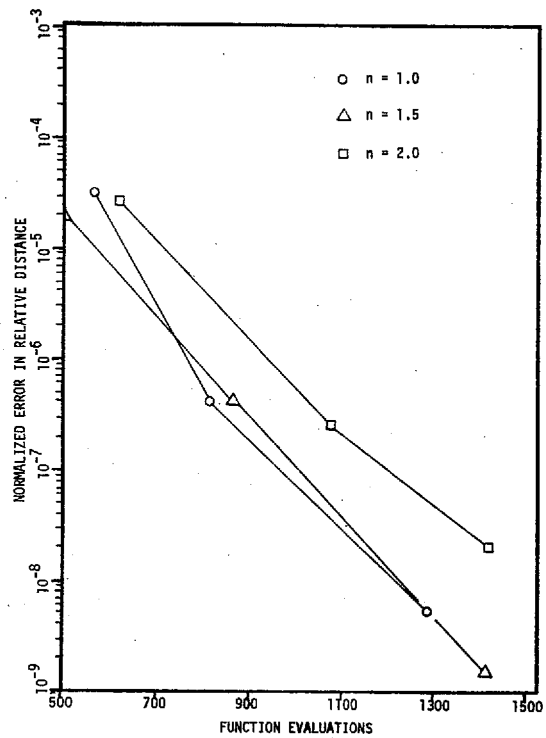


Figure 10.4d
Equation Set 5

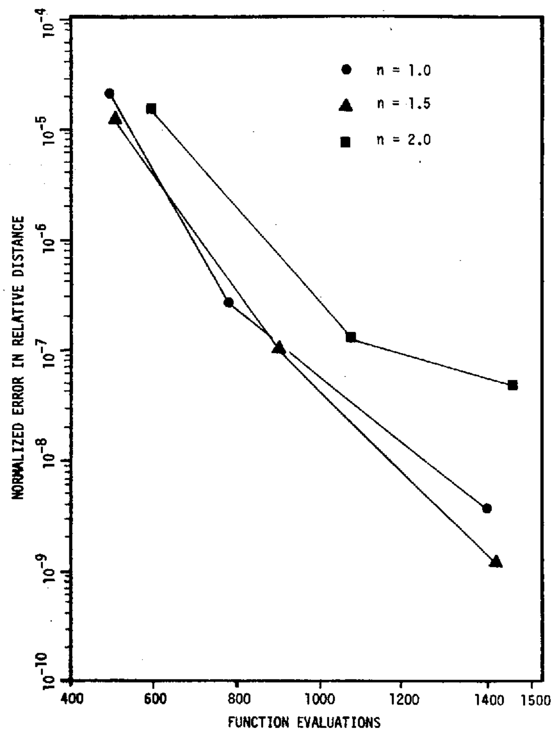


Figure 10.4e
Equation Set 8

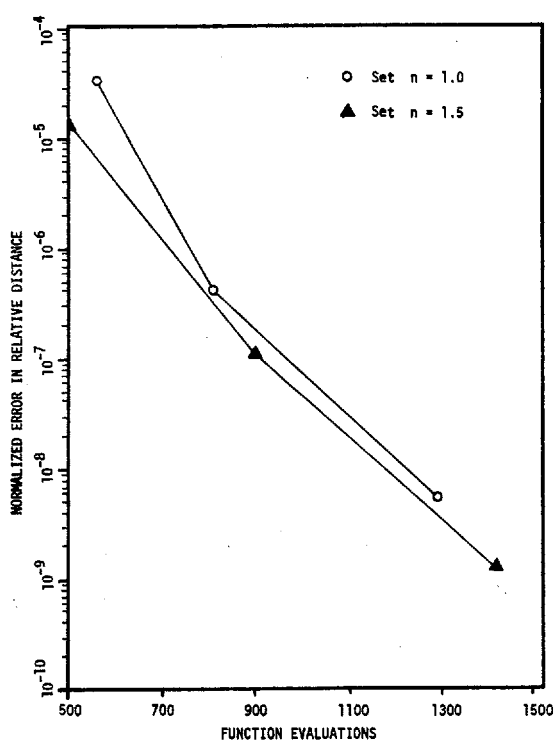


Figure 10.4f
Equation Set 5 vs. Equation Set 8

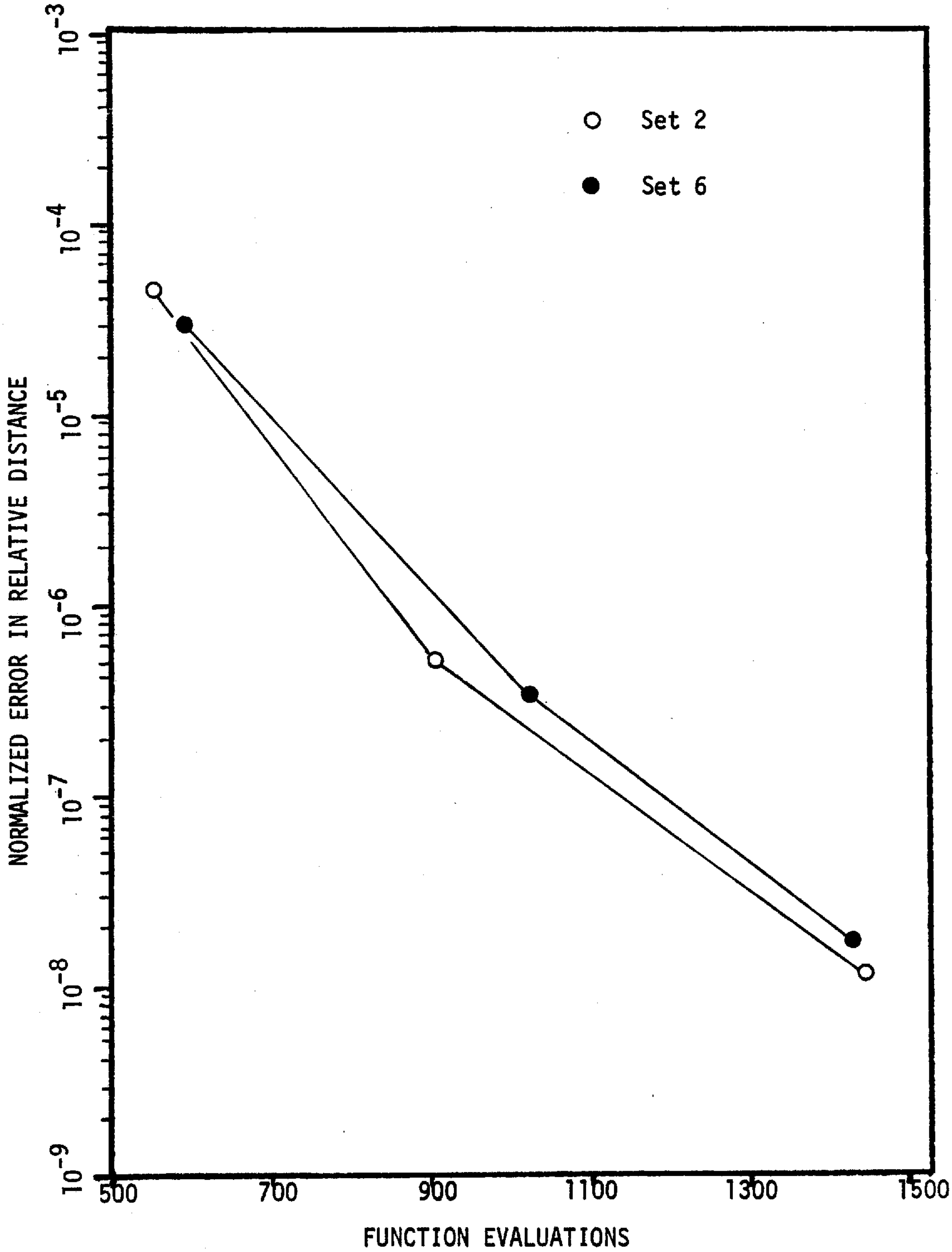


Figure 10.4g
Equation Set 2 vs. Equation Set 6

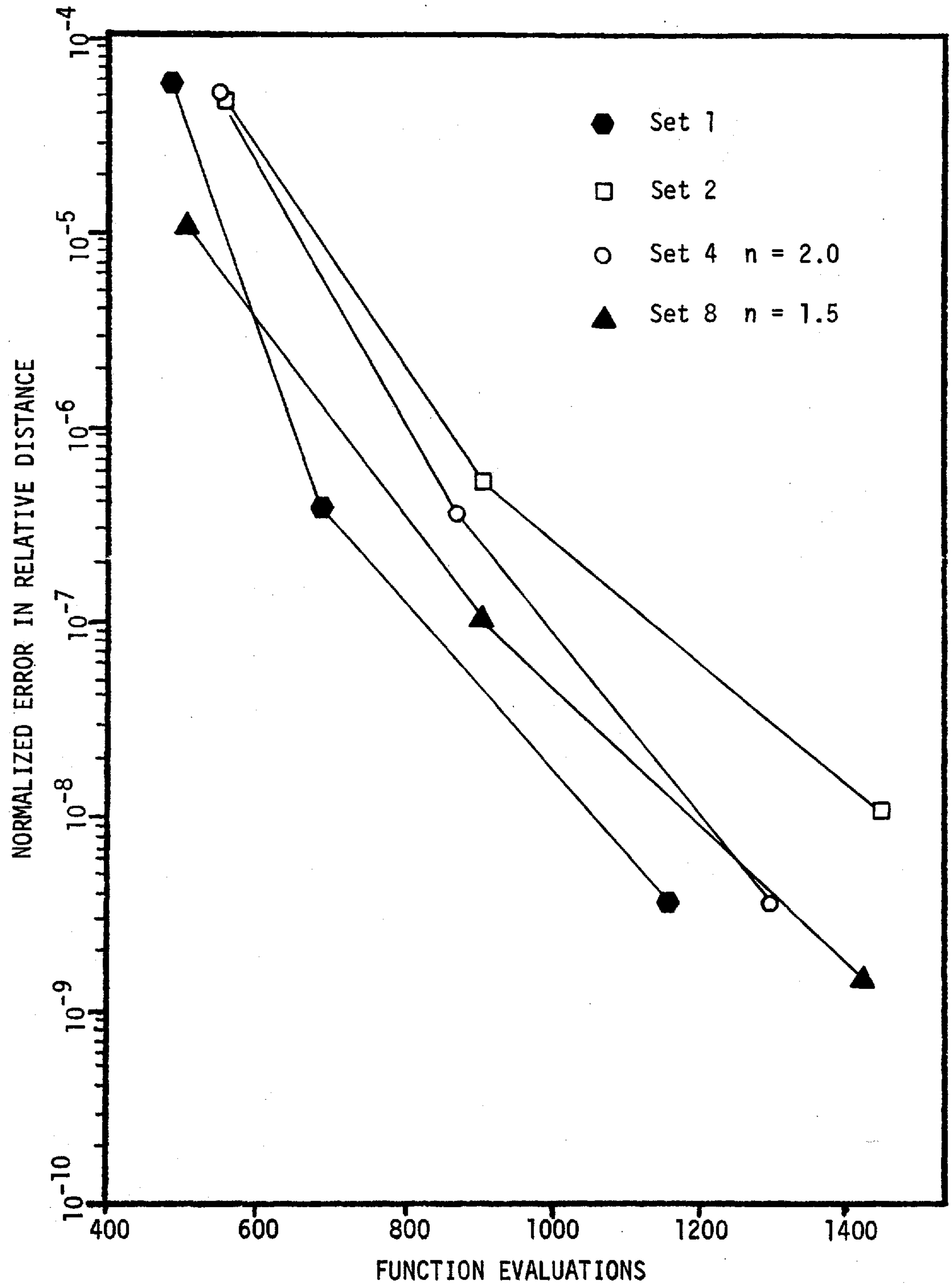
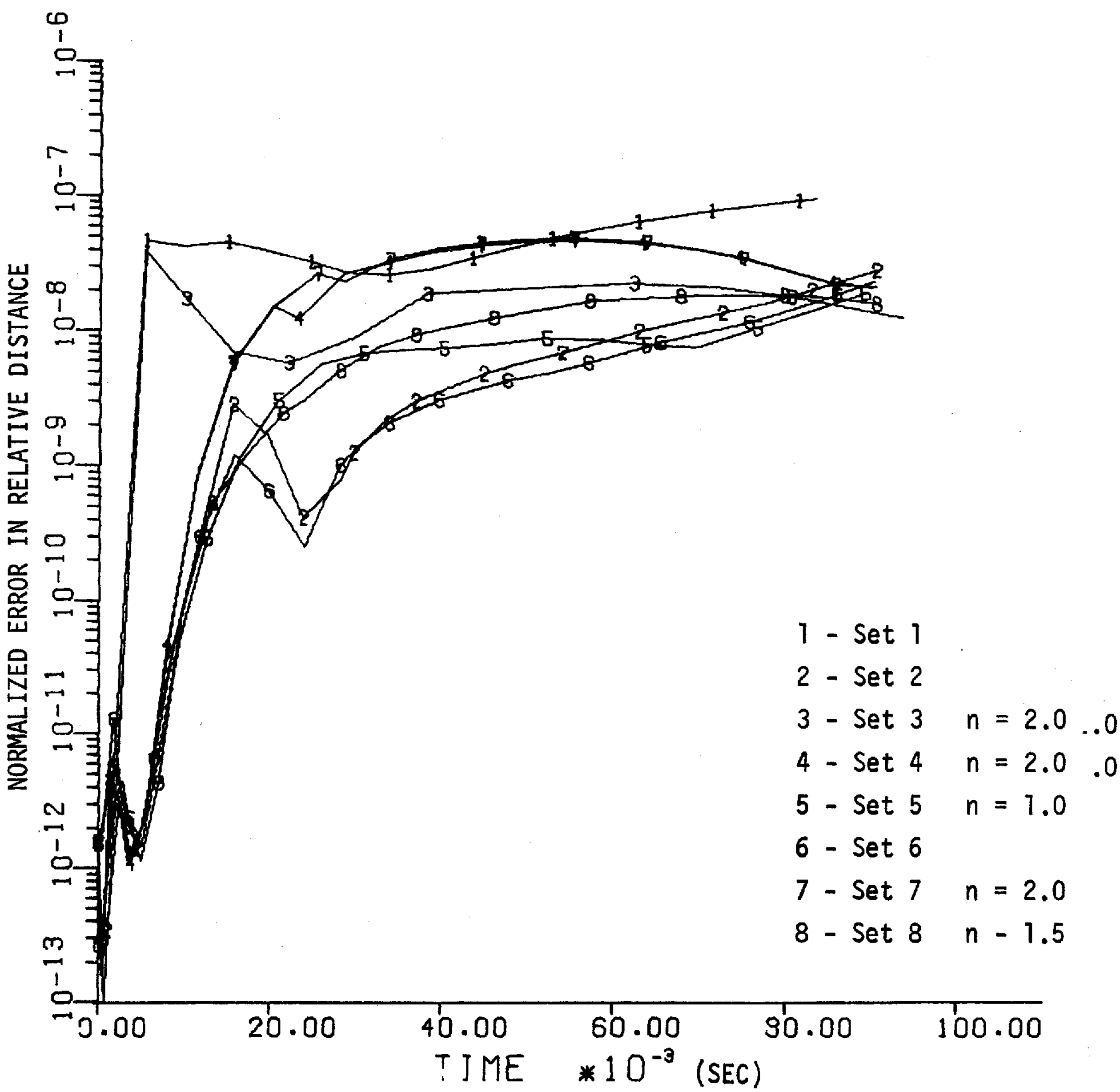


Figure 10.4h
Comparison of Equation for Satellite Pair D



TIME HISTORY OF ERROR PAIR D

Figure 10.4i

In summary, for Pair C, which primarily differed only in eccentricity from the Pair B, the N-S method does not seem to help as much, and n not affect the time smoothed equations as drastically as before, although for sets 5 and 8 the exponent n=1 now appears best.

Figure 10.3i shows the equation sets to vary little relative to one another as the time increases. The calculated efficiencies at the final time the, would appear to be representative for this pair.

The next Pair models two very dissimilar orbits, which differ even more than Pair A.

4.4 Satellite Pair D

Figures 10.4a, b, and c are very similar to their counterparts for Pair A, in that the value of n affects the efficiencies of equation sets 4 and 7 to a lesser degree than in pair B. Again, for set 4, the most efficient choice of n appears to oscillate between 1 and 1.5, with n = 1.5 being better over a larger region than for n = 1.0. The value of n for set 7 also has small effect again, but this time choice of the most efficient value is not clear cut. However, n = 2 seems to be a good compromise, due to its high efficiency at greater accuracies. Figure 10.4c shows the N-S implementation in set 4 does not improve matters but is very close to set 7, as in Pair A.

Figures 10.4d, e, and f show sets 5 and 8 to be relatively insensitive to the value of n, but the N-S implementation seems to have helped slightly. Figure 10.4g displays the change in efficiencies of set 2 with and without the N-S technique, and again it doesn't affect the accuracy or efficiency much.

Figure 10.4h compares the most efficient forms of each of the sets, and set 1 appears much better than the others. In summary, for these very dissimilar orbits, the N-S method does not appear in general to be as effective as the striaght subtraction, (standard Cowell). In an attempt to understand why set 1 is so much more efficient, it may be advisable to look at the relative distance vs. time plot for this pair (Figure 4b). As the satellites approach the point of close separation, the subtractions needed by the methods which require integration of the relative vector can cause a very abrupt loss in precision. This presents problems for the other methods and they must reduce step size

problem and can continue at a larger step size. Another important aspect of this pair can be seen from Figure 4b. Due to the final time chosen, the satellites have a large separation when the relative vector is calculated for equation set 1, and a minimal loss of significant digits occurs in the subtraction of $\bar{r}_1 - \bar{r}_2$. From this standpoint, the fact that the time smoothed equations which integrate the relative vector competed as well as they did is quite impressive.

Figure 10.4i shows the equation sets to be reasonably well behaved, however there is quite a bit of range in the relative efficiencies of the methods at different time points, and no extrapolation to times around the comparison points can safely be made.

The effect of the altitudes on the equation will be studied next by comparing the results of Pairs C and E which except for their altitudes are almost indentical.

4.5 Satellite Pair E

Figures 10.5a, b, and c are almost identical with those from Pair C (for Equation Sets 4 and 7) in shape. However, when Figure 10.5c is

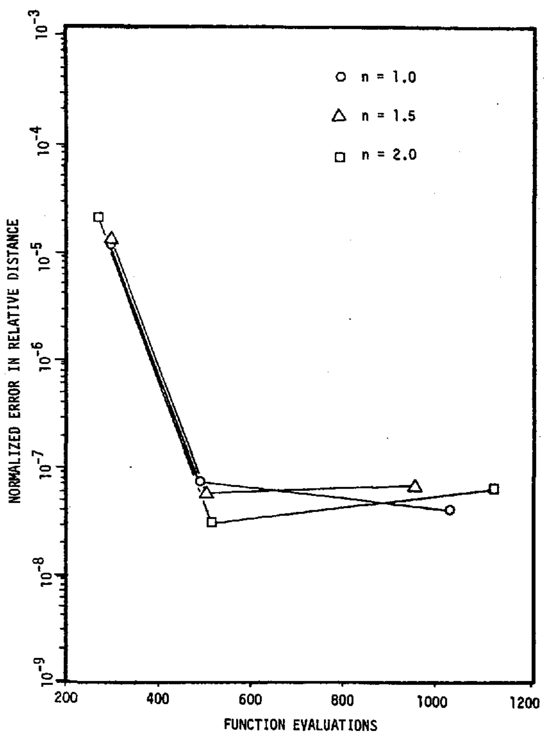


Figure 10.5a
Equation Set 4

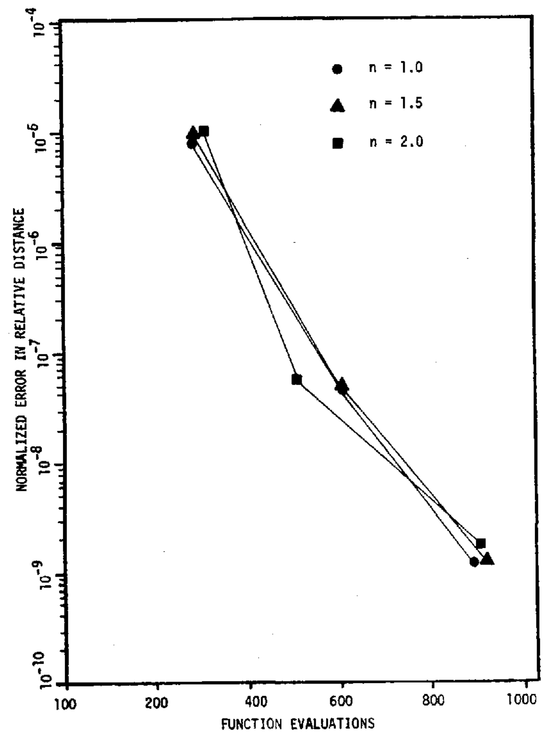


Figure 10.5b
Equation Set 7

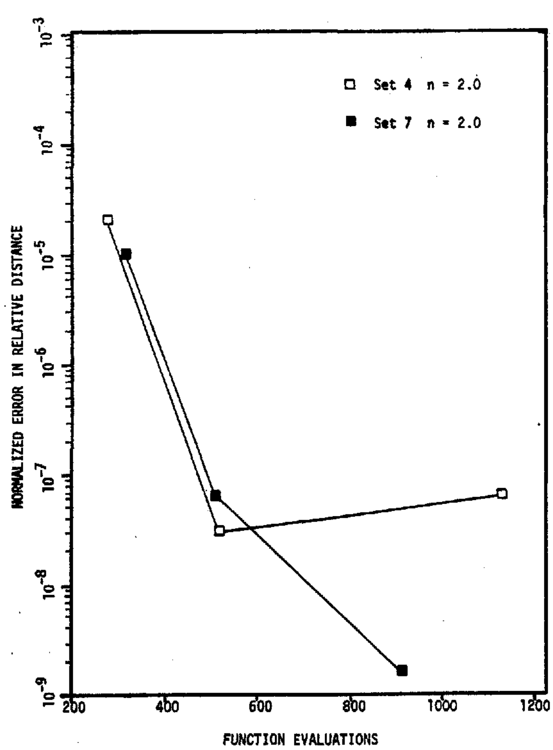


Figure 10.5c
Equation Set 4 vs. Equation Set 7

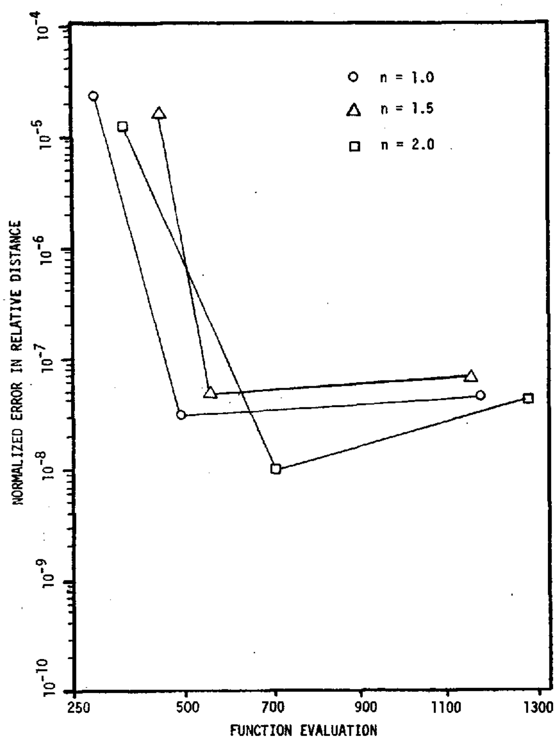


Figure 10.5d
Equation Set 5

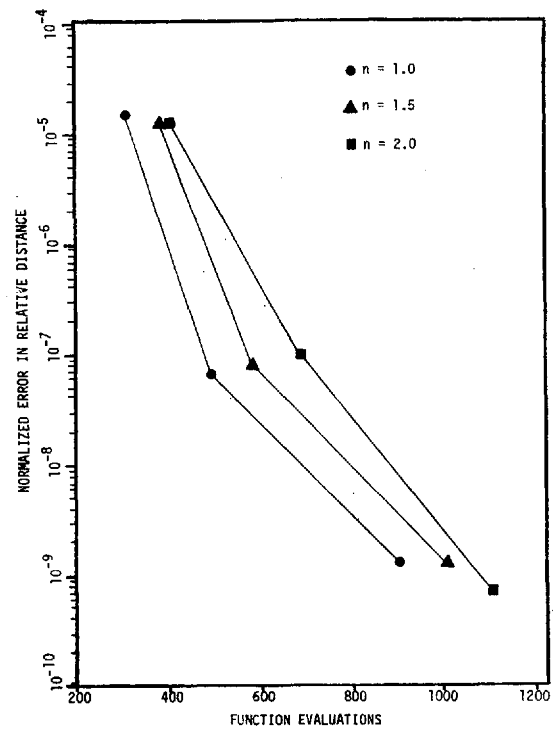


Figure 10.5e
Equation Set 8

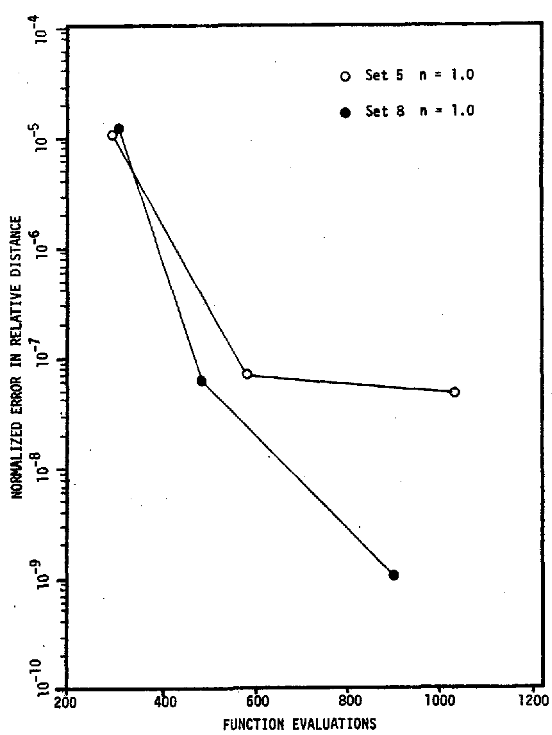


Figure 10.5f
Equation Set 5 vs. Equation Set 8

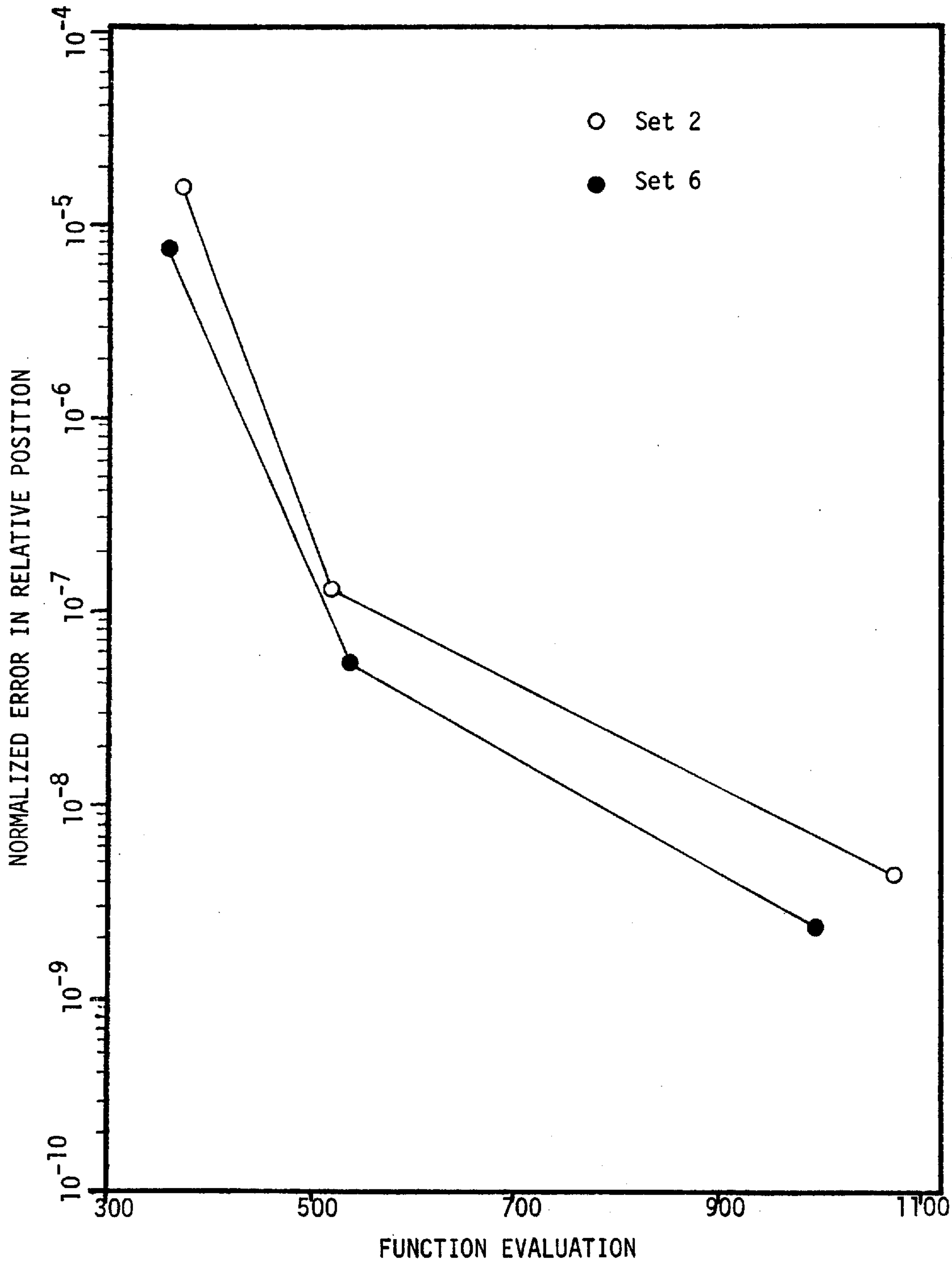


Figure 10.5g
Equation Set 2 vs. Equation Set 6

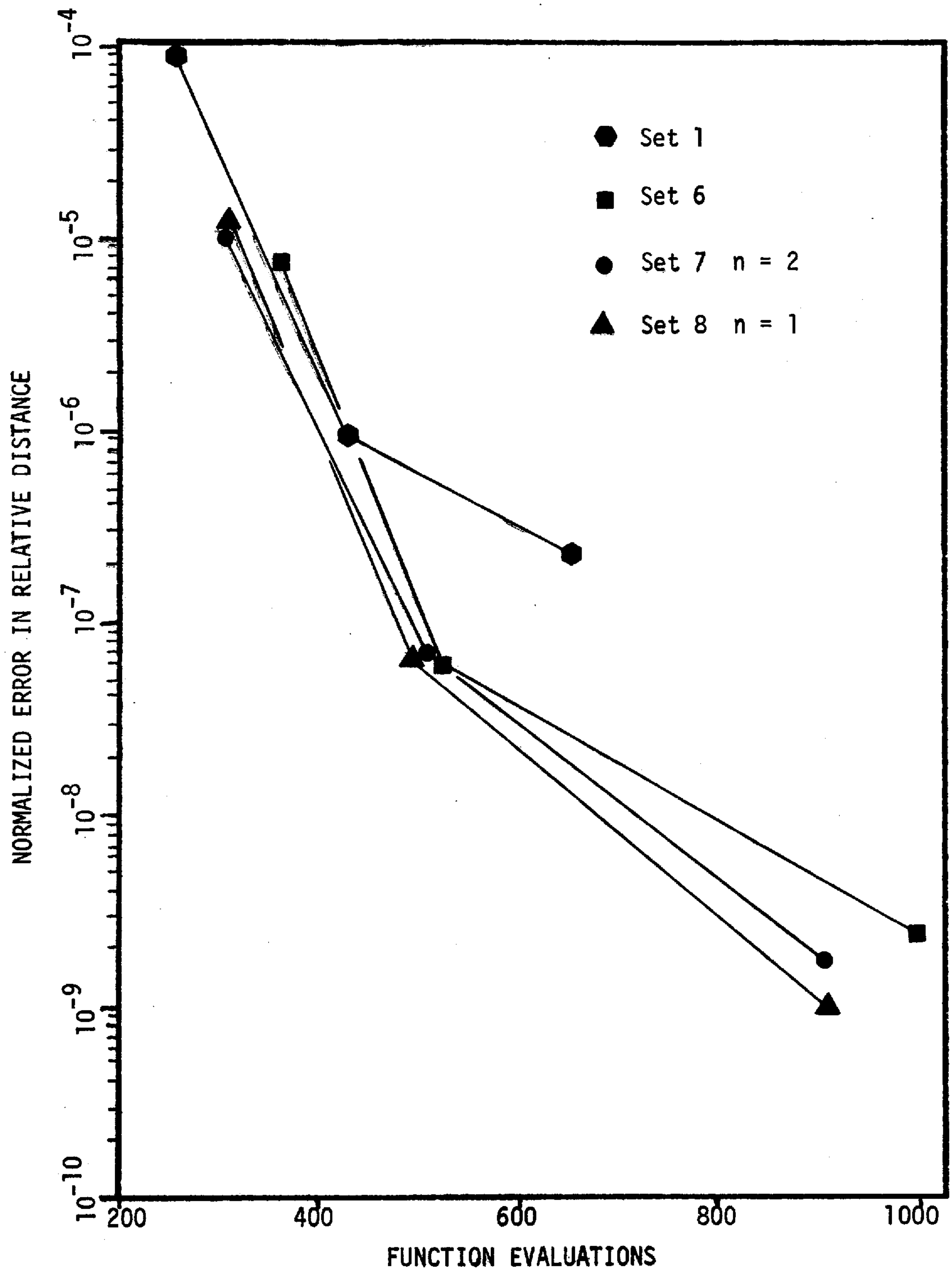
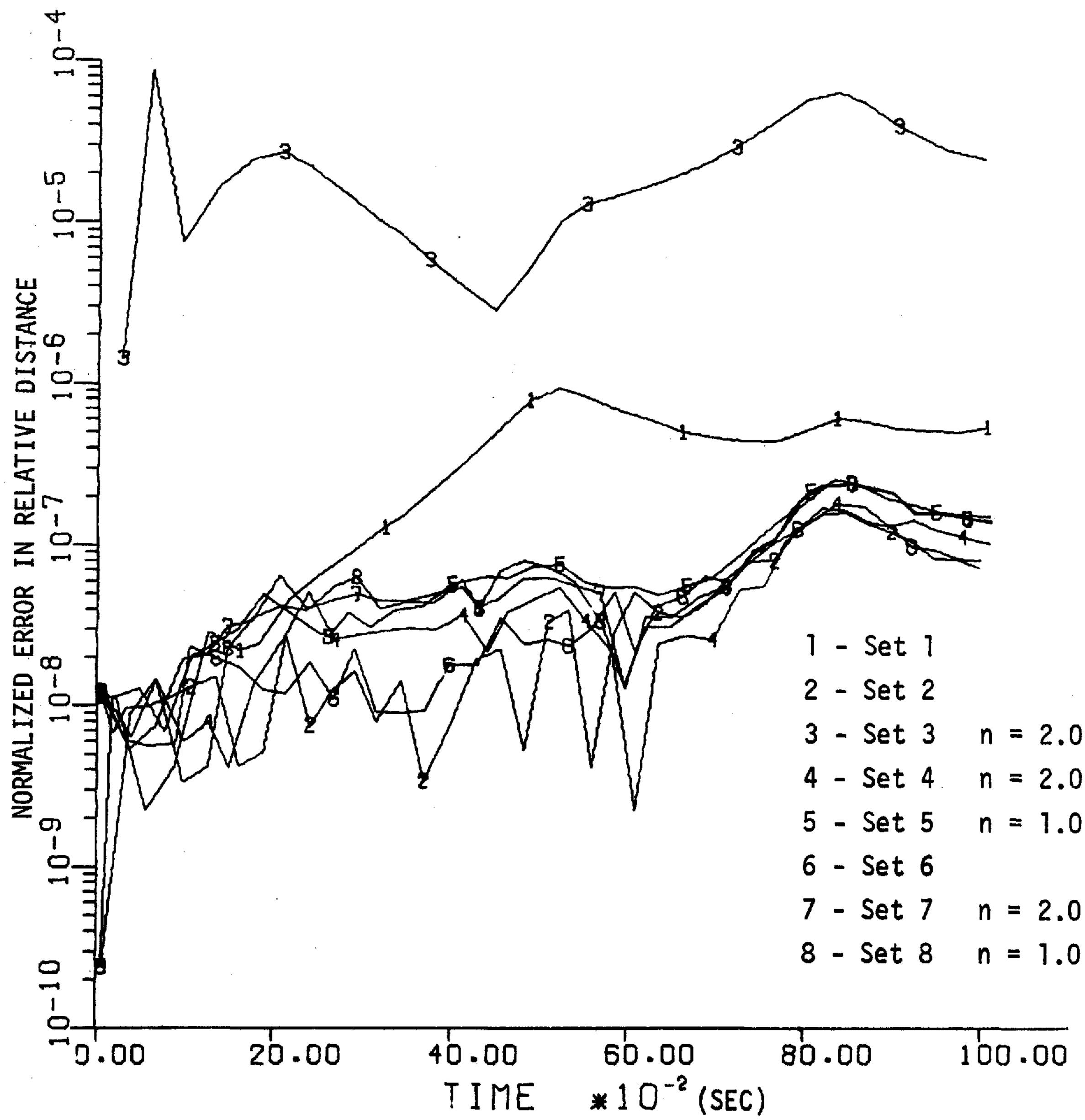


Figure 10.5h
Comparison of Equations for Satellite Pair E



TIME HISTORY OF ERROR PAIR E

Figure 10.5i

compared to 10.3c it can be seen that the relative effectiveness of the N-S method on set 4 has decreased with decrease in altitude.

Figure 10.5d is very similar to its equivalent in Pair C, while for equation set 8, Figure 10.5e, the effect of n on the methods is much more pronounced. In general, from Figure 10.5f the efficiency of equation set 5 is greatly improved by the N-S technique, while for Pair C there was not much effect.

Equation set 2 also is greatly aided by the N-S technique, as can be seen from Figure 10.5g. The comparison of the best form of each equation set, Figure 10.5h, shows set 8 to be the most efficient over most of the range.

The major effect of the decreased altitude between Pairs C and E was to increase the amount the N-S technique helped equation set 5, while it did not seem to increase the efficiency of equation set 4.

In an attempt to see if the efficiency comparisons made at this particular time can be extrapolated for other time periods within the the integration, a study of Figure 10.5i is in order. The equation sets oscillate drastically in the early part of the integration, then become more smooth near the comparison point. Even near the comparison point there is some movement of the sets relative to one another, and it is apparent that the relative efficiency comparisons are only valid at the point compared.

A comparison of Pairs E and F will help to show the effect of changing the eccentricities on the equation sets.

4.6 Pair F

A comparison of Figures 10.6a and b with 10.5a and b tend to show that an increase in eccentricity has only a small effect on the relative efficiencies of equation sets 5 and 7 at various values of n. In contrast, Figure 10.6c shows a dramatic increase in efficiency realized from the Nacozy-Szebehely method applied to equation set 4. However, this efficiency increase is not as drastic as it was for Pair E. Figures 10.6d and e vs. 10.5d and e again show little change in the effects of the value of n for Pairs E and F, sets 5 and 8. Again, the increase in efficiency provided by the N-S technique has increased for set 5 with the increased eccentricity.

Figure 10.6g shows a dramatic gain in efficiency when the N-S method is applied to set 2, (set 6), especially at high accuracy regions.

Figure 10.6h shows the two time smoothed equations compared to their performance in Figure 10.5h have improved a great deal relative to the nonsmoothed equations. However, set 6 has gotten a little less efficient although it is still better than set 1 over most of the range.

Figure 10.6i displays the time history of the error norm used for each equation set. The equation sets are all well behaved, and do not cross one another, this tends to show that the exact time chosen for comparison would not critically affect the conclusions drawn.

4.7 Pair G

Pair G, like Pair D, approaches the point of close separation abruptly. The N-S method does not increase the efficiency of set 4 (Figure 10.7c), while it is competative for set 5 vs. set 8, (Figure 10.7f) and offers

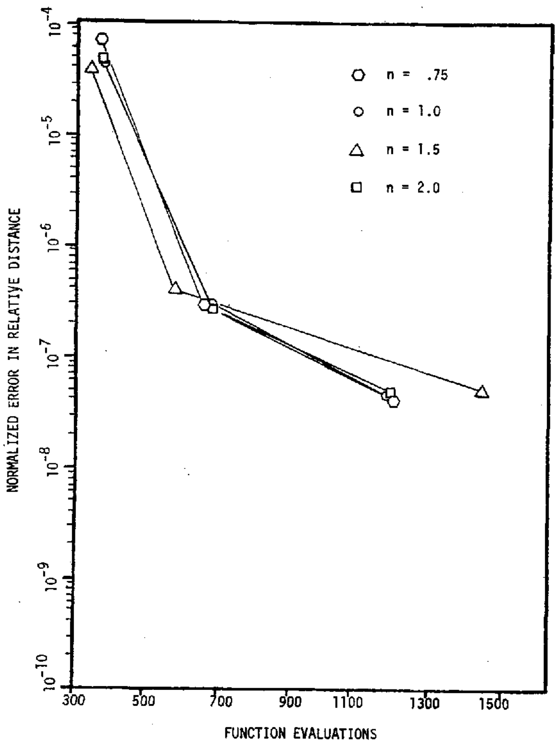


Figure 10.6a
Equation Set 4

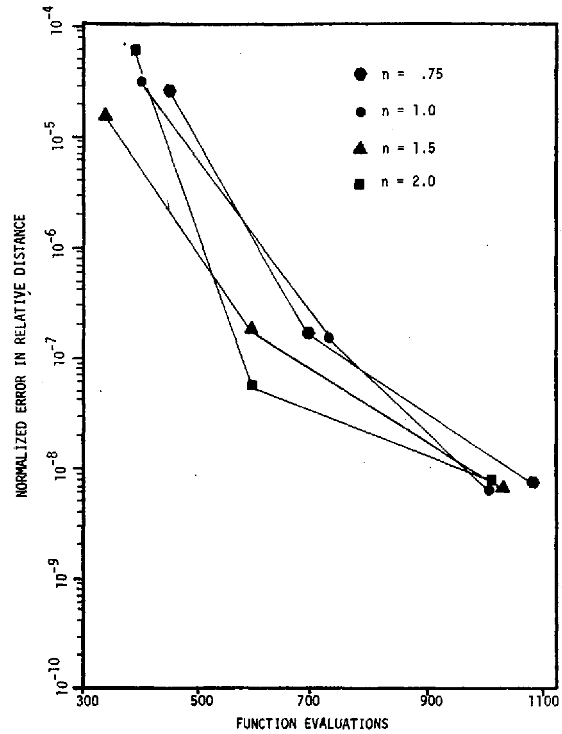


Figure 10.6b
Equation Set 7

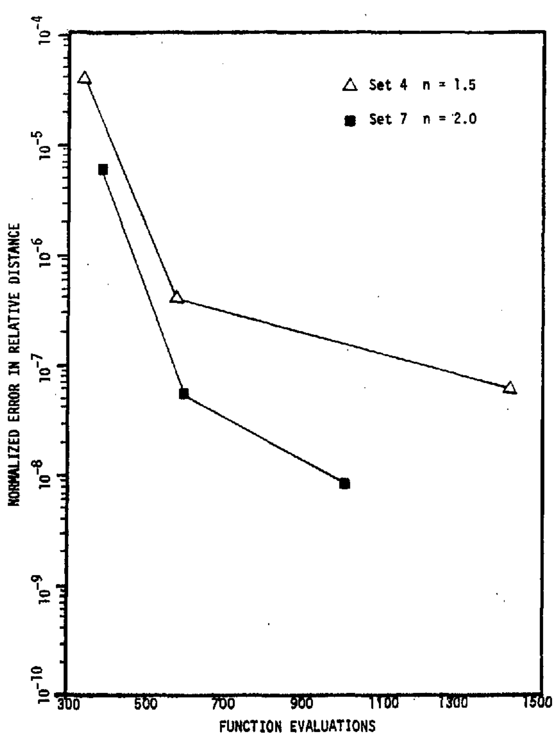


Figure 10.6c
Equation Set 4 vs. Equation Set 7

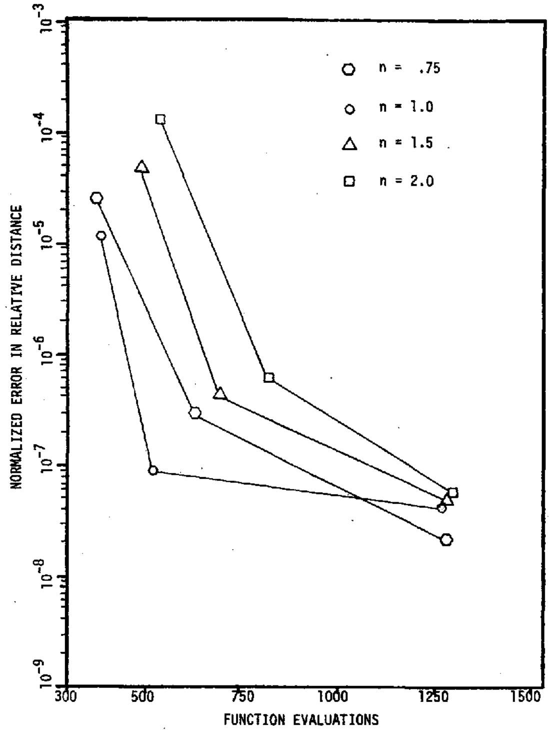


Figure 10.6d
Equation Set 5

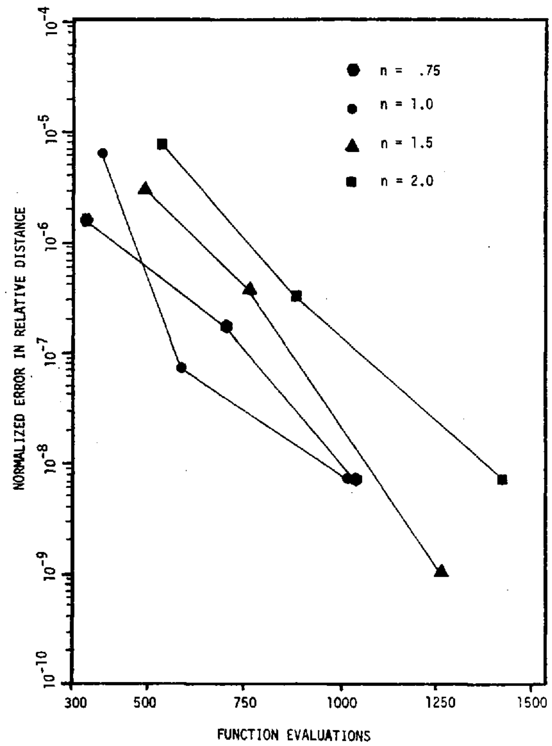


Figure 10.6e
Equation Set 8

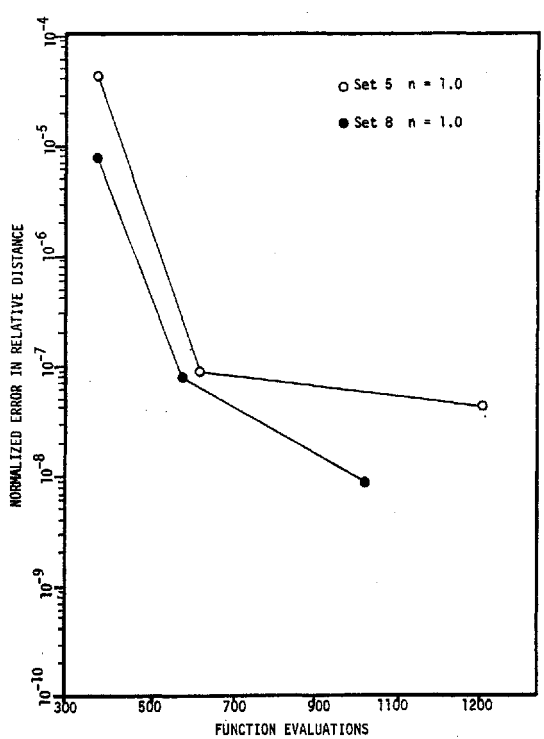


Figure 10.6f
Equation Set 5 vs. Equation Set 8

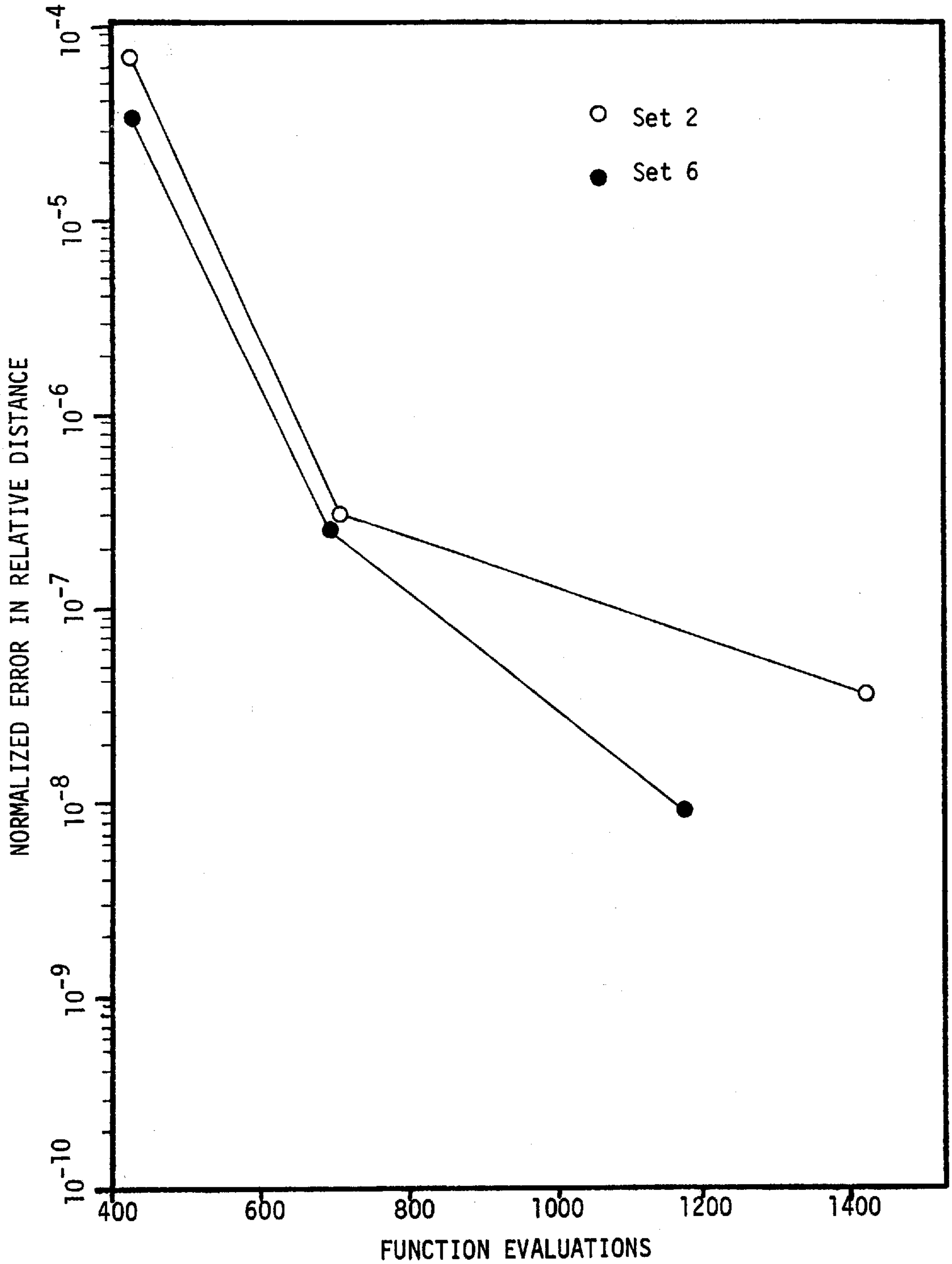


Figure 10.6g
Equation Set 2 vs. Equation Set 6

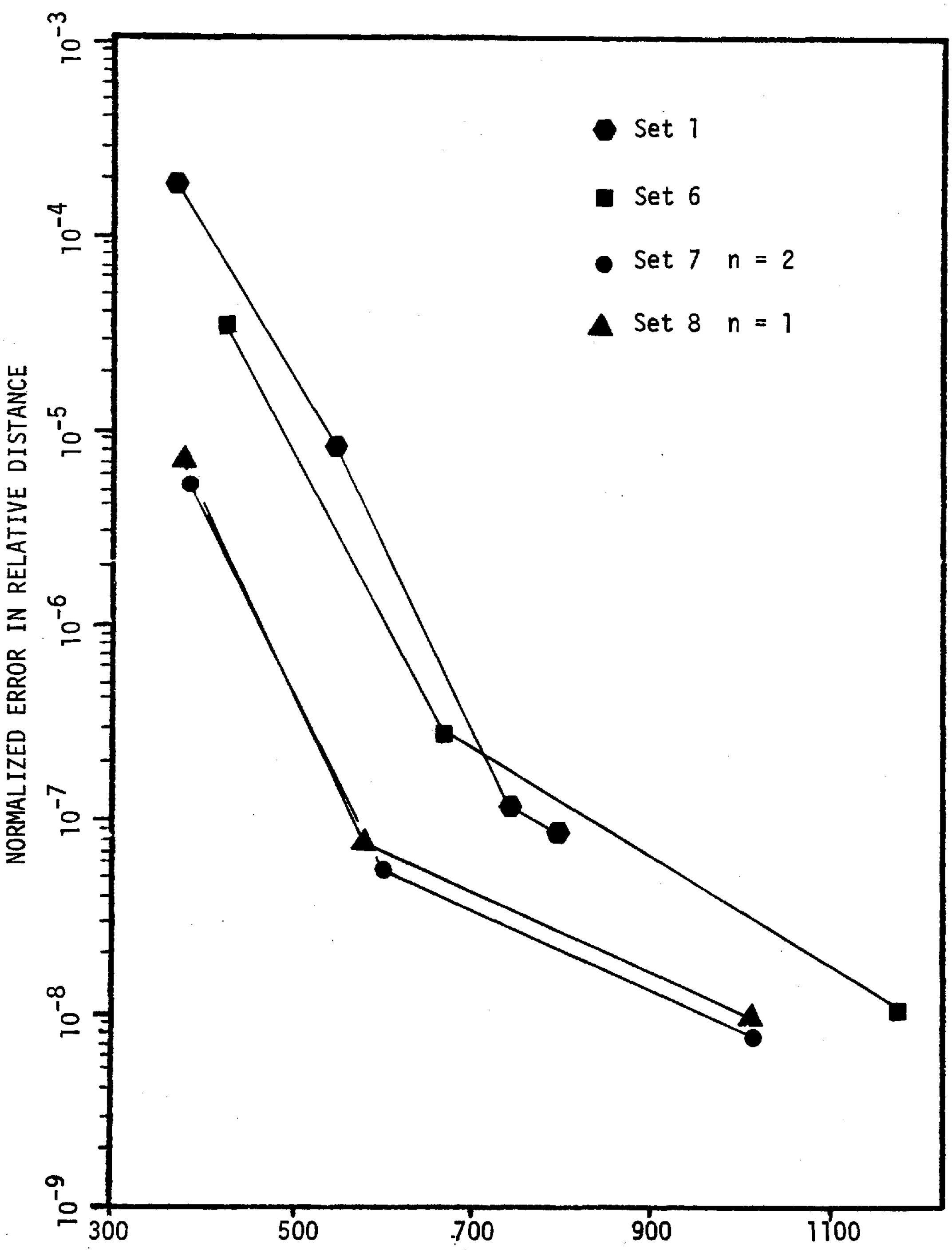
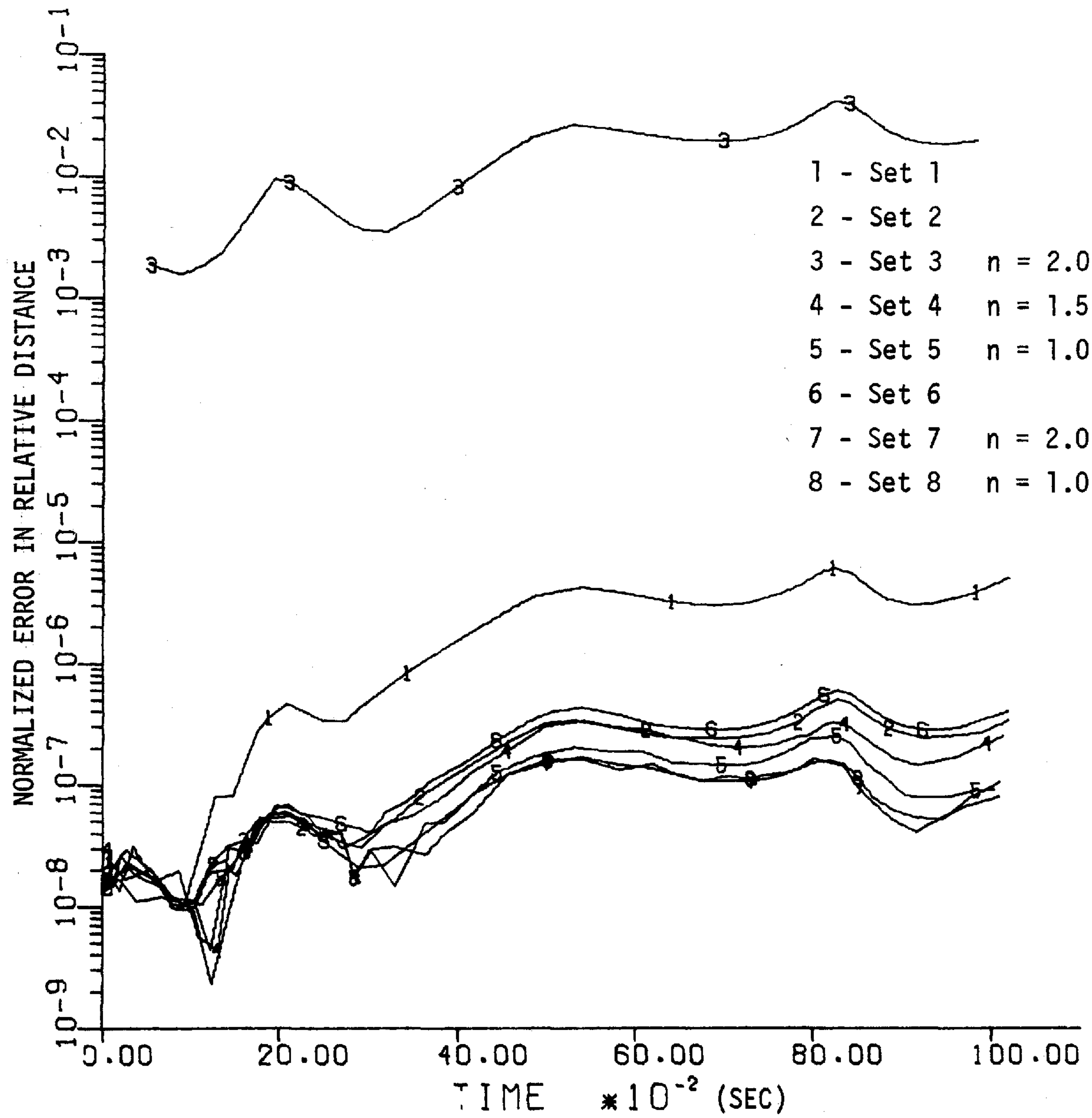


Figure 10.6h
Comparison of Equations for Satellite Pair F



TIME HISTORY OF ERROR PAIR F

Figure 10.6i

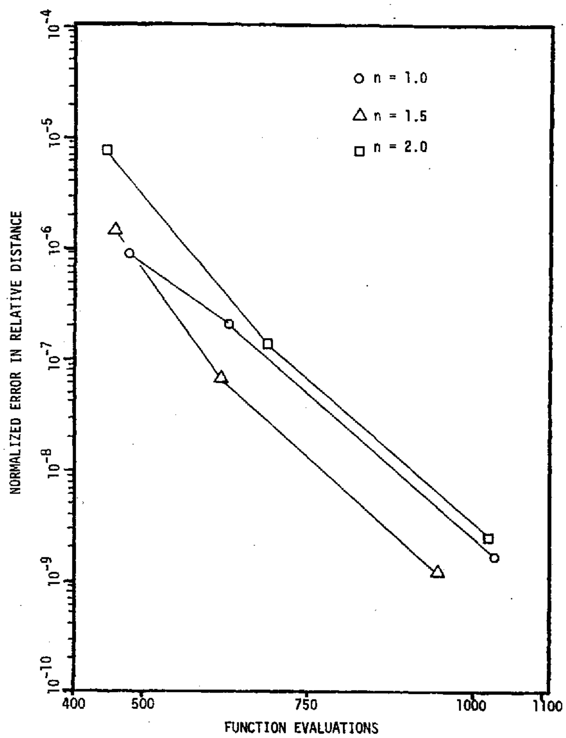


Figure 10.7a
Equation Set 4

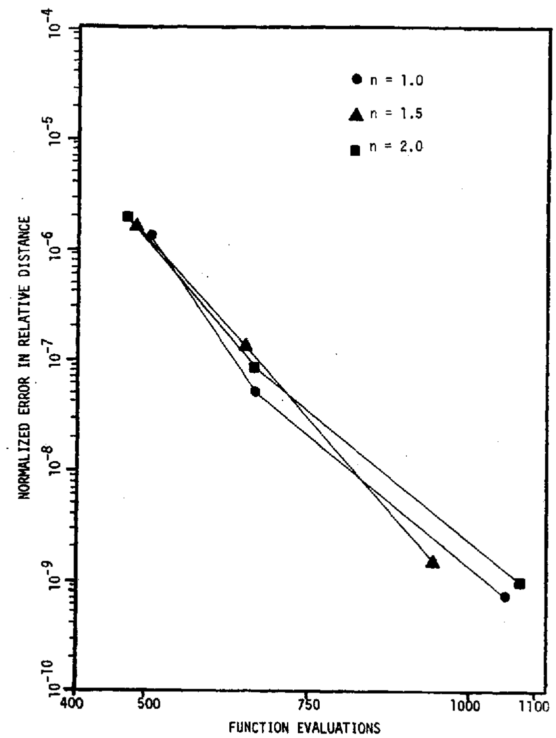


Figure 10.7b
Equation Set 7

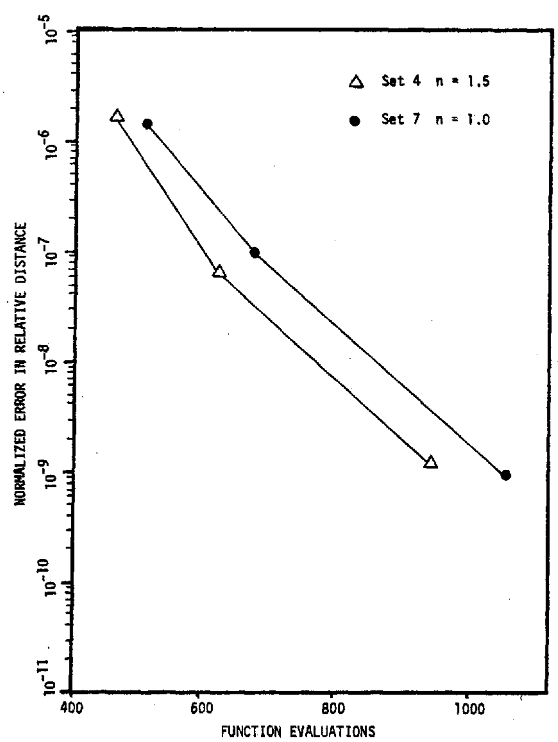


Figure 10.7c
Equation Set 4 vs. Equation Set 7

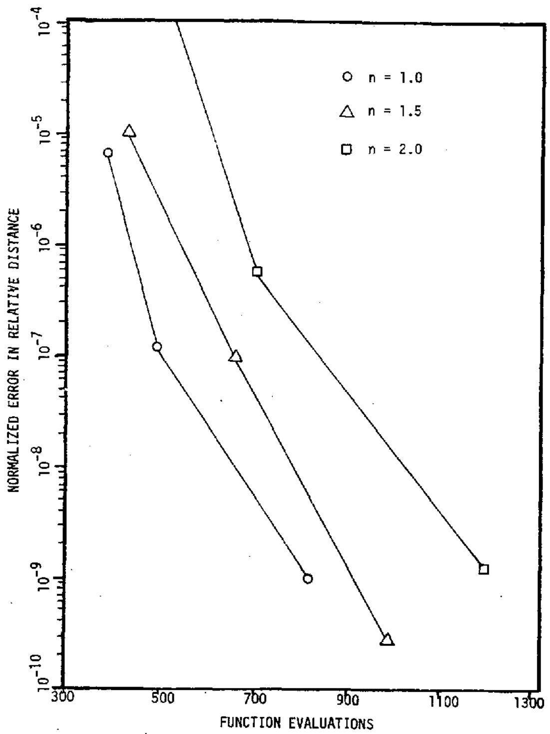


Figure 10.8d
Equation Set 5

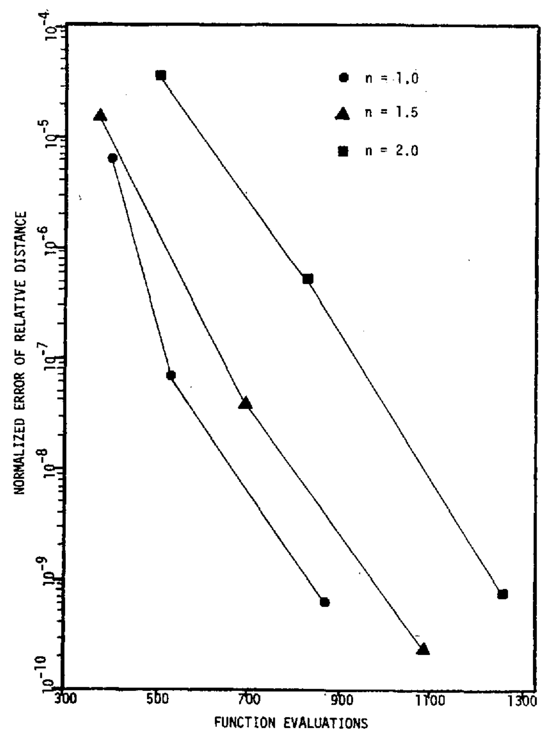


Figure 10.8e
Equation Set 8

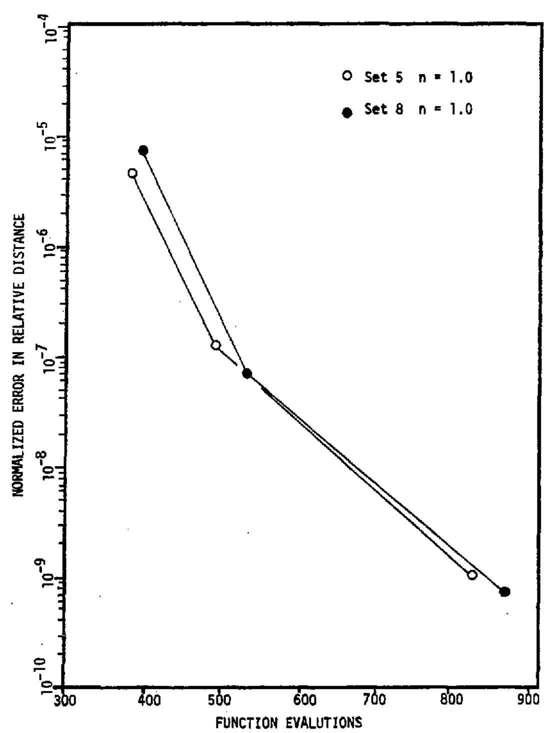


Figure 10.8f
Equation Set 5 vs. Equations Set 8

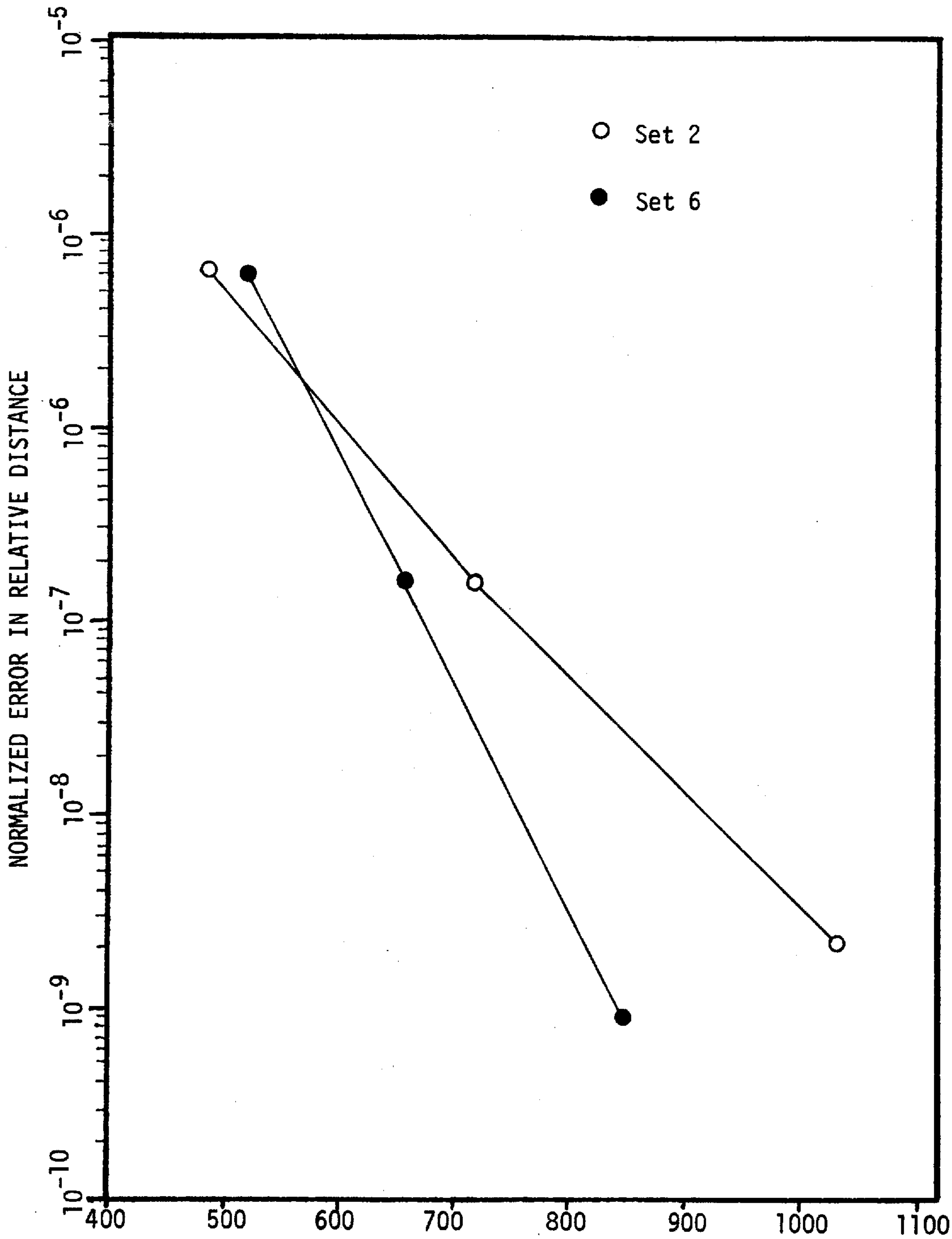


Figure 10.7g
Equation Set 2 vs. Equation Set 6

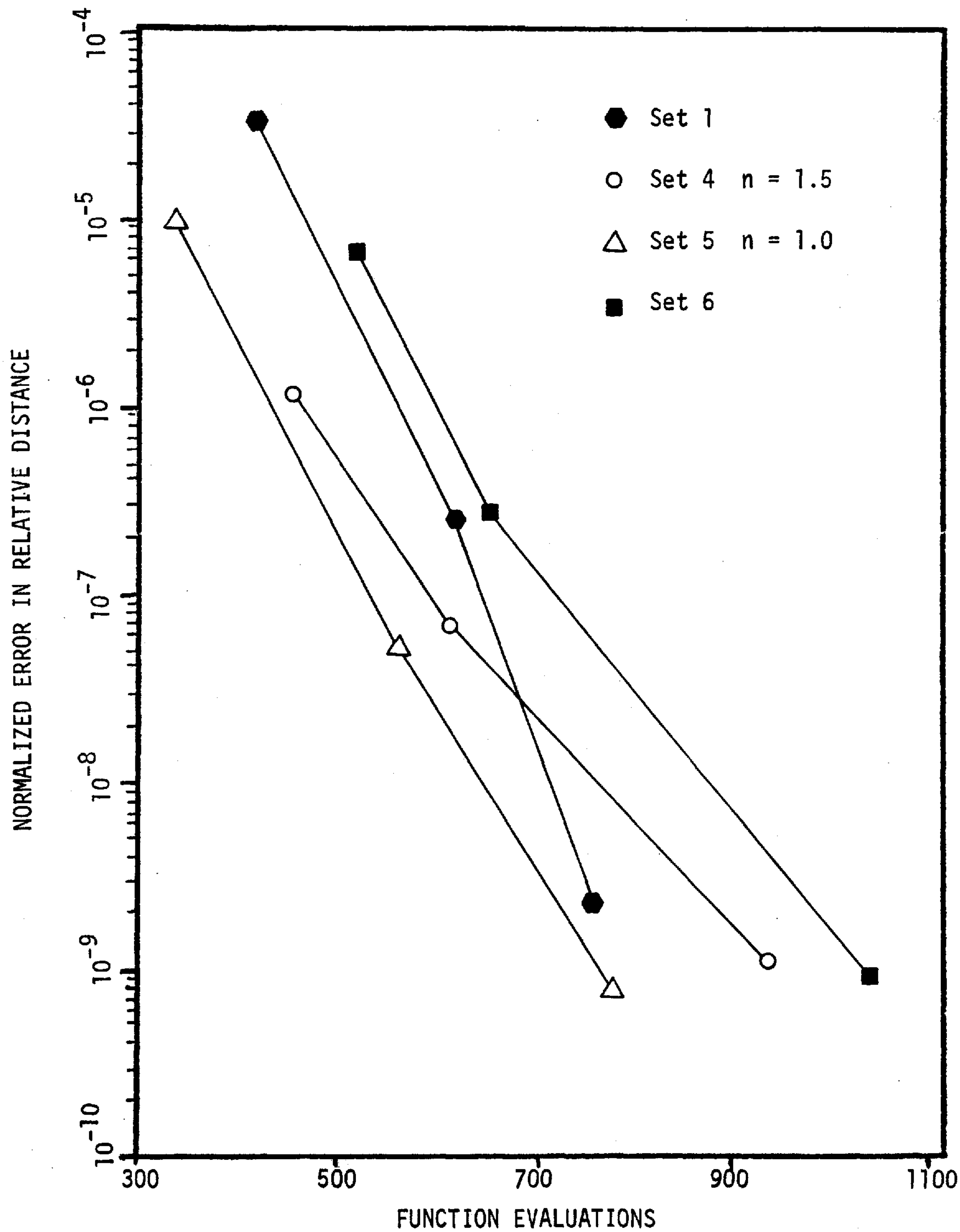
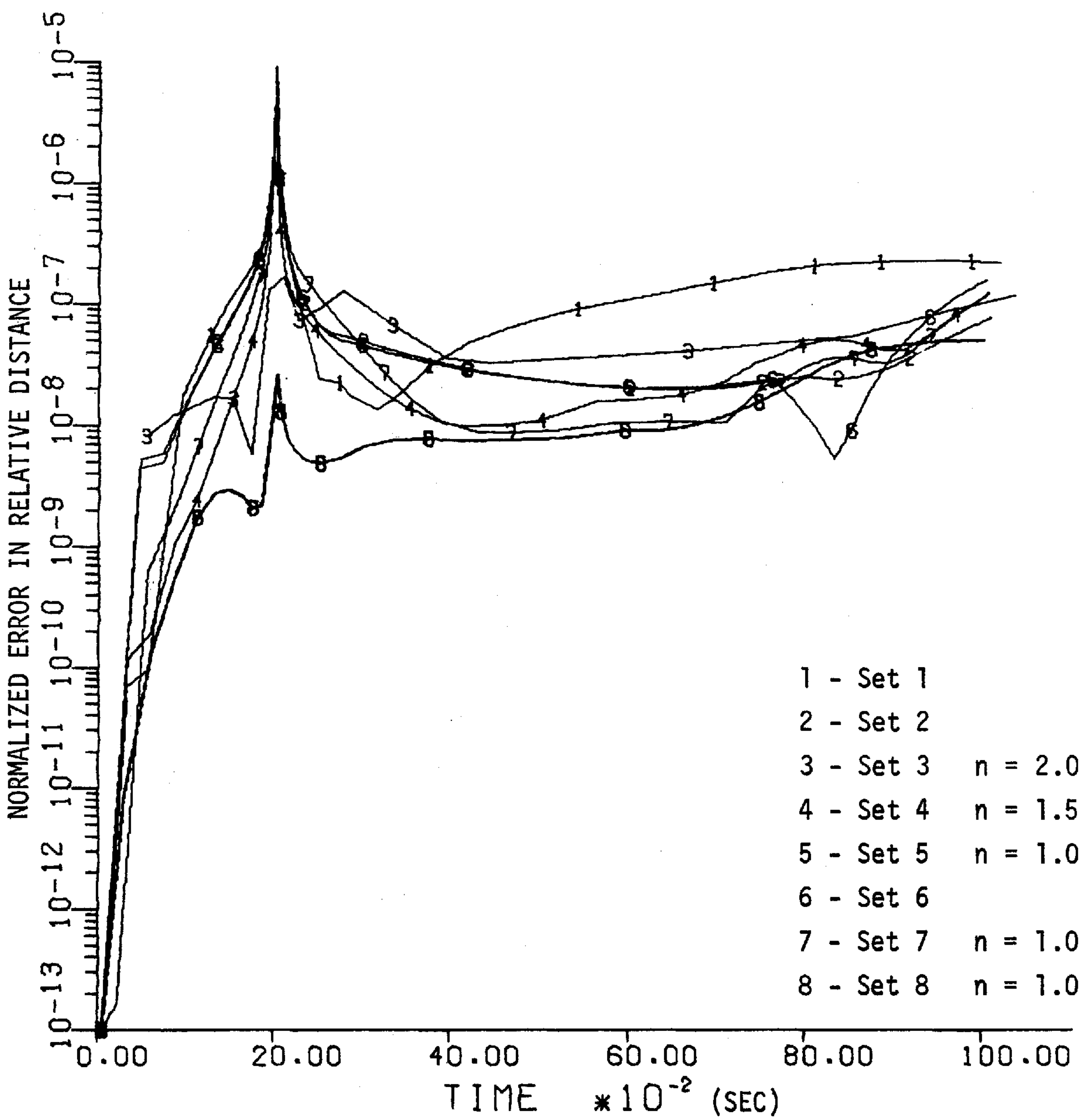


Figure 10.7h
Comparison of Equations for Satellite Pair G



TIME HISTORY OF ERROR PAIR G

Figure 10.7i

a decided advantage for equation sets 2 and 6, (Figure 10.7g). The integration of the relative vector (Figure 10.7h) does not help the efficiency of the Cowell methods, while the time smoothing does seem to offer some advantage, especially for set 5. Again, a look at the magnitude of the relative position at the final time Figure 7b shows that Equation set 1 was not hampered by having to do a subtraction of two nearly equal numbers at the final term, and yet equation set 5 which must calculate relative accelerations by subtracting two nearly equal numbers (when satellites are at perigee) is still more efficient. This would tend to substantiate the feeling that the smoothing process in itself helps the efficiency of the methods.

The importance of the time chosen for the comparison alluded to above can be easily seen from Figure 10.7i. The relative accuracies of the equation sets varies widely, and a comparison at another time might show quite different results.

4.8 Satellite Pair H

From Figure 8b it can be seen that this pair model also approaches the point of minimum separation abruptly, but not as sharply as Pair G. The effects of this on the use of the N-S method are to make them more efficient for sets 4 vs. 7 (Figure 10.8c) and very competitive for sets 5 vs. 8 (Figure 10.8f). On the Modified Cowell sets (set 2 and 6), the N-S equations become more competative as the accuracy requested increases. But it appears that set 2 is more efficient at low accuracies. Figure 10.8h shows the integration of the relative vector in the Cowell methods to be less efficient, while the use of time smoothing appears to increase the efficiency, especially for set 8.

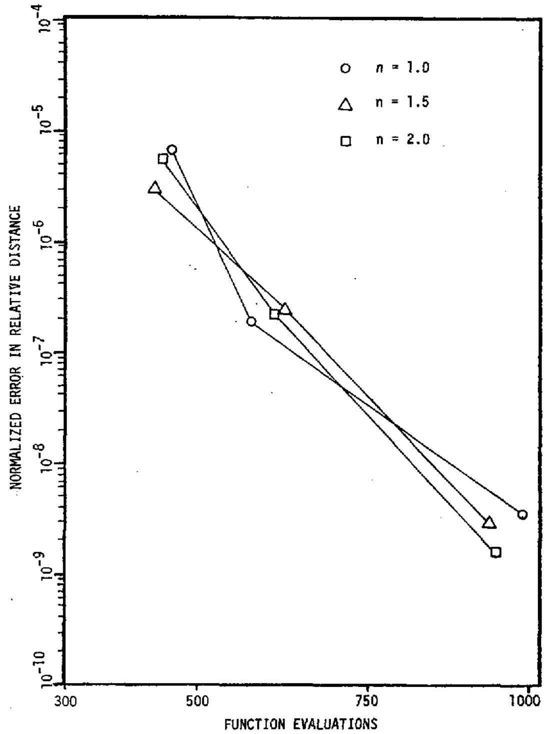


Figure 10.8a
Equation Set 4

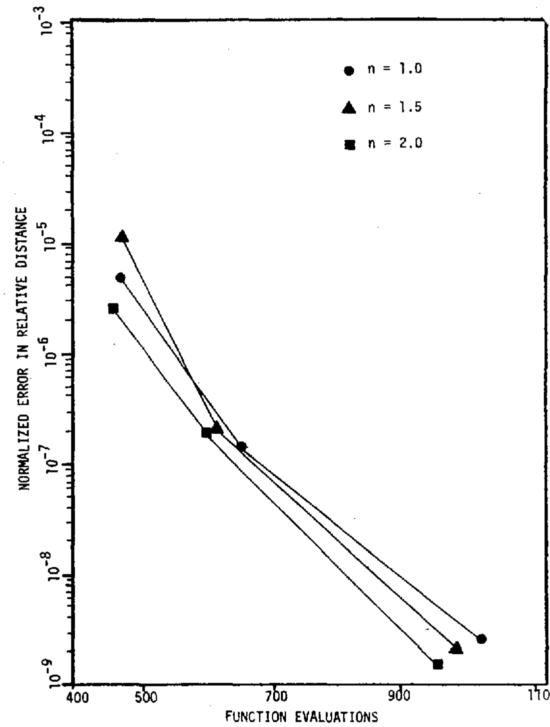


Figure 10.8b
Equation Set 8

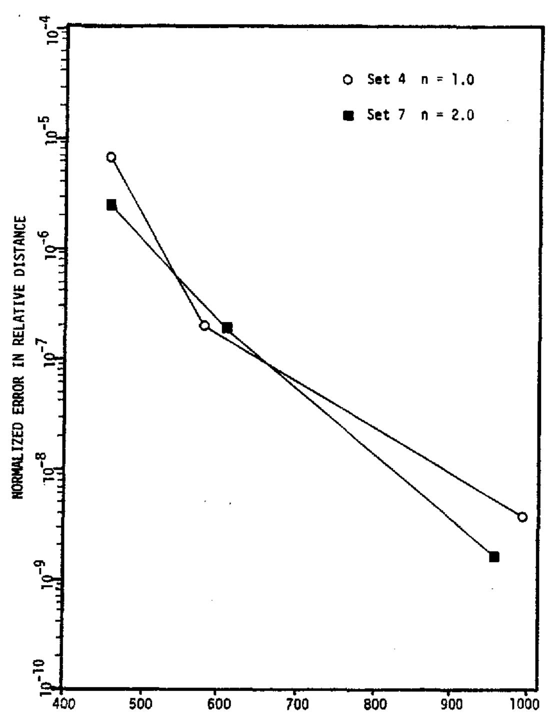


Figure 10.8c
Equation Set 4 vs. Equation Set 7

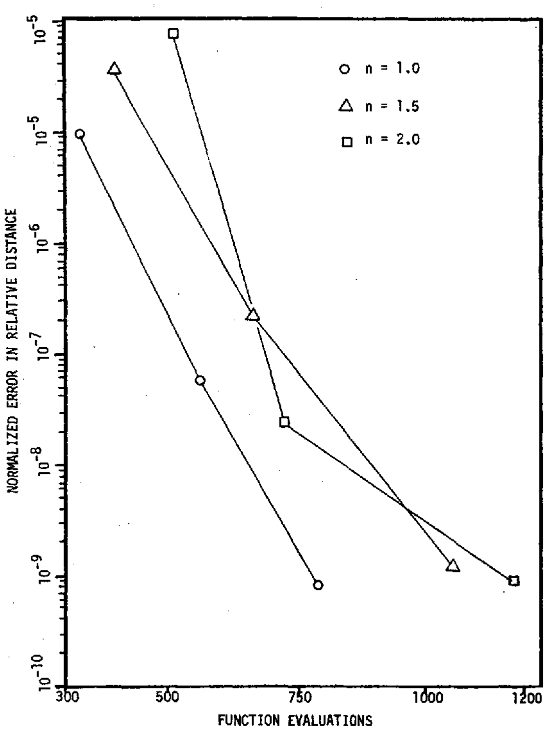


Figure 10.7d
Equation Set 5

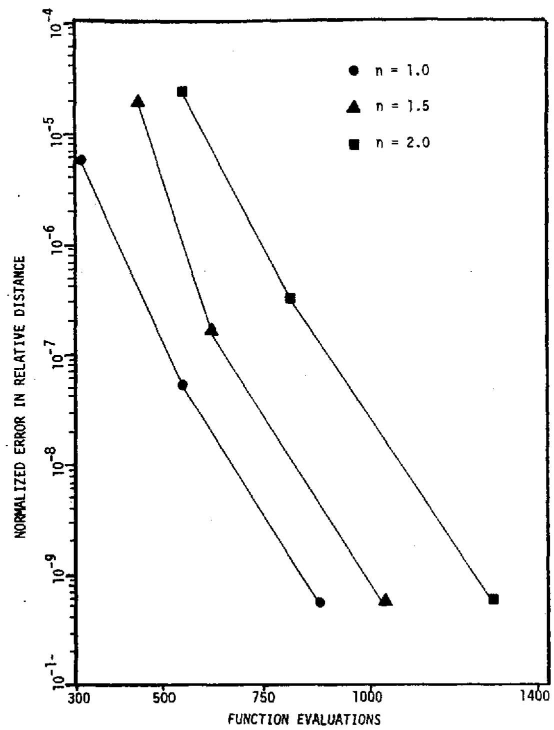


Figure 10.6e
Equation Set 8

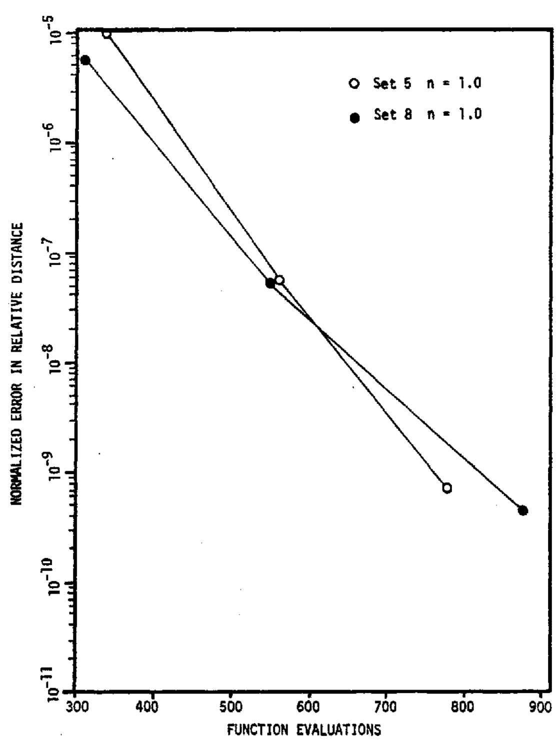


Figure 10.7f
Equation Set 5 vs. Equation Set 8

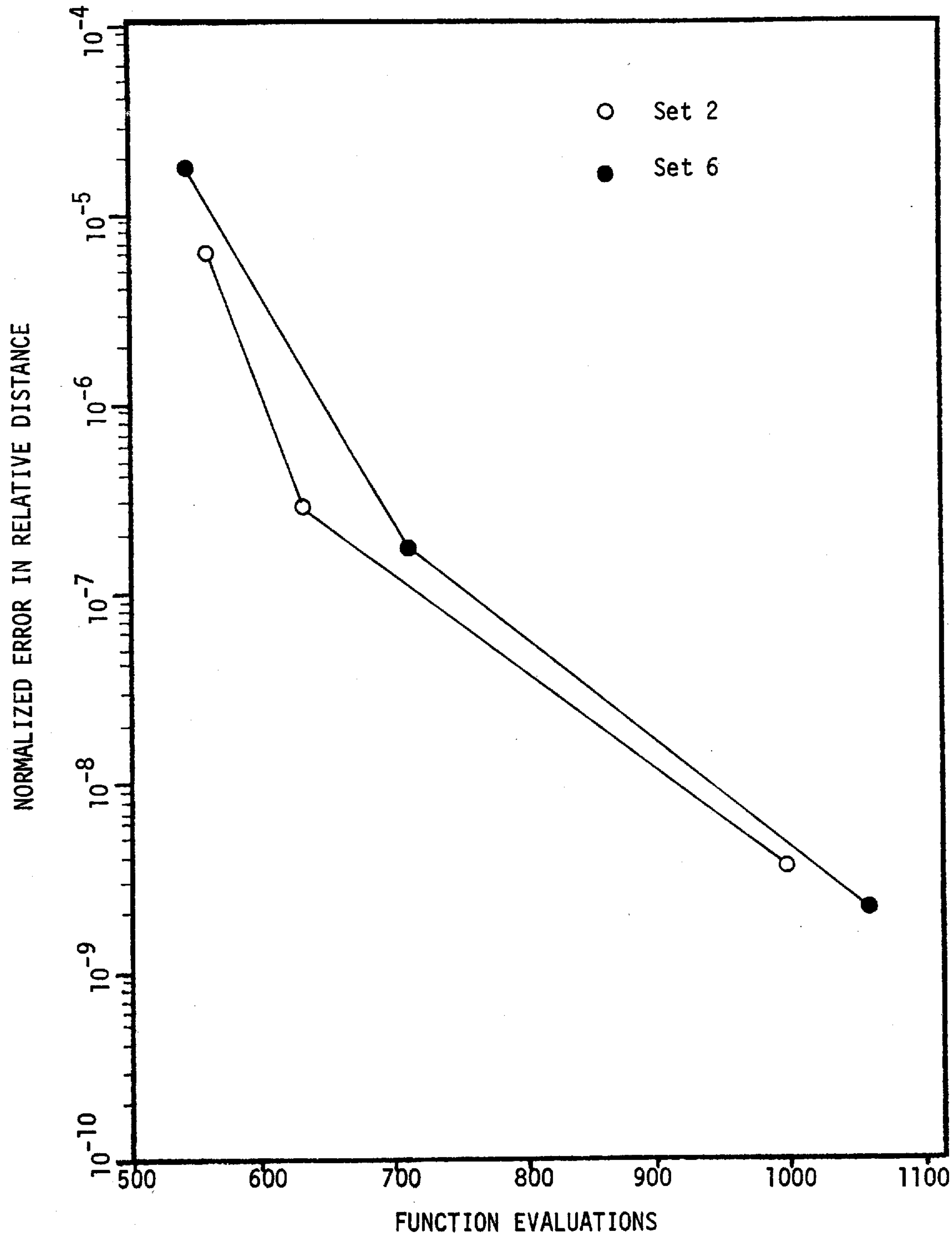


Figure 10.8g
Equation Set 2 vs. Equation Set 6

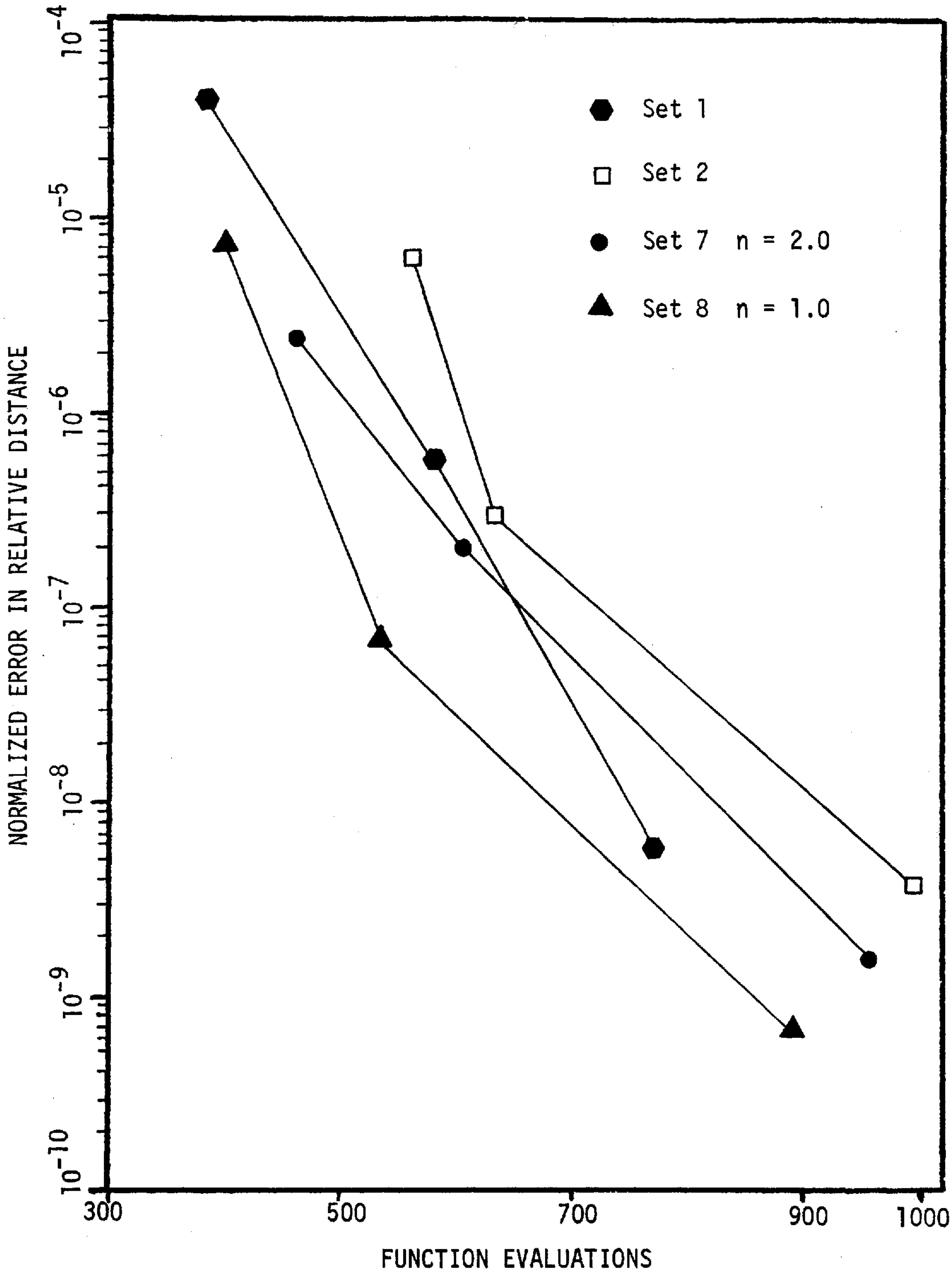
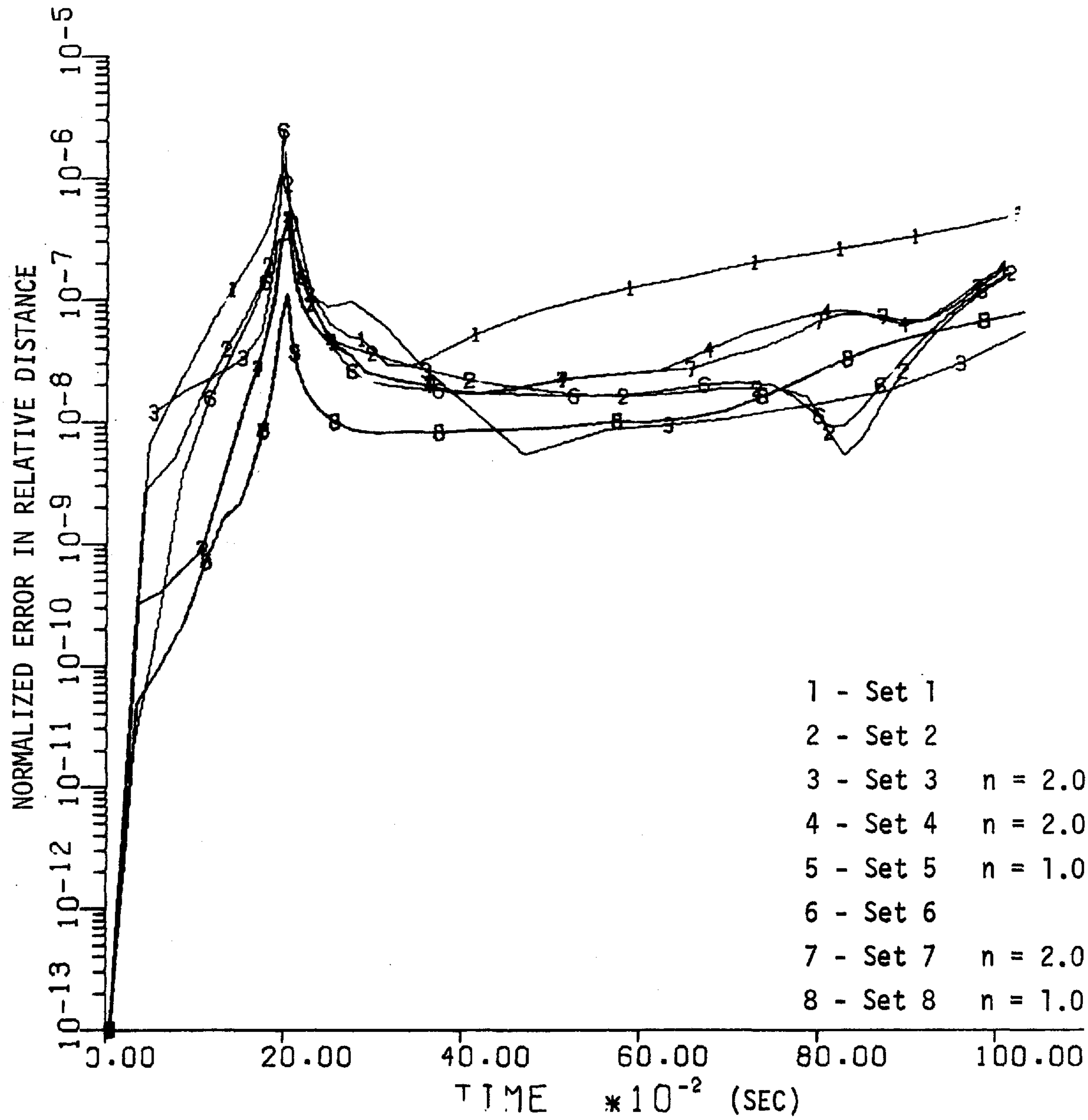


Figure 10.8h
Comparison of Equations for Satellite Pair H



TIME HISTORY OF ERROR PAIR H

Figure 10.8i

The reader's attention is now directed to Figure 10.8i, the time history plot for Pair H . It should be observed that once again the relative accuracies of the methods vary drastically with time, and for this reason any conclusion drawn from the efficiencies of the sets at the comparison may not be valid for other times.

4.9 Data Comparisons

Since the needs of each individual user of this data will be different, it would be impossible to anticipate all required comparisons. and make them in this report. However, since all the data are presented, the user can make further comparisons as needed for any purpose. Selected comparisions will now be presented which illustrate the information available in the data.

The next step is to determine how Equation set 5 (time smoothing applied to one satellite and the relative vector based upon the magnitude of both satellites position vector) reacts to different situations. For the case where the closeness of approach is varied (Pair A vs. Pair B) it can be seen from Figures 10.1d and 10.2d that the efficiency of set 5 is very sensitive to changes in n for both satellite pairs, and $n = 1.0$ is most efficient by far for both. A comparison of Figures 10.1f and 10.2f shows how set 5 relates to set 8, which is plotted in Figures 10.1h and 10.2h. This comparison shows that while for Pair A, set 5 is very competitive with set 4, and more efficient than either the Standard or Modified Cowell procedures, it is clearly less efficient than set 4 for Pair B and has moved much closer to the standard Cowell procedure. From this it would appear that the introduction of the magnitude of the second satellite into the smoothing equation has decreased the gains realized by the integration of the relative vector.

The reaction of set 5 to a change in semi-major axis is next examined through a comparison of Pairs C and E. Figures 10.3d and 10.5d show set 5 to be sensitive to changes in n, but the most efficient choice of n in both cases is still $n = 1.0$. When the plot for set 5 is compared to set 8 in Figure 10.5f, and its location extrapolated relative to set 8 in Figure 10.5h, a comparison with Figure 10.3h shows that the change in semi-major axis has not had a noticeable effect upon set 5, relative to sets 1, 2, or 4. The effect of the change in eccentricities between Pairs E and F is first noticed in Figures 10.5d and 10.6d. Set 5 is still sensitive to changes in n, but $n = 1$ is most efficient for both pairs. Again, in order to compare set 5 to sets 1, 2, and 4 it is first necessary to look at Figures 10.5f and 10.6f to note its position relative to set 8. Once this is done, it can be seen that although in both cases the efficiency is better than that attained by set 2, it is again hampered in the high eccentricity case (as was the other method involving integration of the relative vector). Set 5 and set 4 are very close to one another although it appears that in the case of the low eccentricity (and therefore less abrupt approach) set 5 was more efficient while for Pair F, set 4 was more efficient.

To further study the effect of the abrupt approach on set 5, Pairs D, G, and H are now compared. Figures 10.4h, 10.7h and 10.8h all exhibit the same trend. Over the entire range, set 5 is more efficient than sets 4 and 2, and for Pairs G and H, set 5 is much better than the Standard Cowell, but seems to be approaching it as the accuracies get lower. For

Pair D, set 5 is not quite as efficient as the standard Cowell. The overall conclusions are in general that set 5 has better efficiency than the other methods involving the integration of the relative vector. However, it suffers from the limitations imposed upon the methods by the abruptness of approach of the two satellites.

In order to observe how the integration of the relative vector is affected by the variation in the closeness of approach between Pairs A and B it should first be noted from Figures 10.1g and 10.2g that the N-S method does not affect the set 2 equations drastically. A comparison of 10.1h and 10.2h shows, however, that the Modified Cowell method with the N-S implementation (set 6) is more efficient than the Standard Cowell formulation (set 1) in both cases. Note that for Pair B, with its much closer approach, the relative gain in efficiency is far greater.

The way that a change in the semi-major axis of the two satellites affects the relative efficiency of the Modified Cowell equation can be investigated by comparing the results of Pairs C and E. Figures 10.3g and 10.5g show a larger change in efficiency for set 2 with the implementation of the N-S technique. Figure 10.3h and 10.5h both show the Modified Cowell with N-S technique to be competitive with the Standard Cowell method for low accuracy (10^{-6} and above) regions, while in high accuracy range (below 10^{-6}) the Modified Cowell procedure is much more efficient. On both pairs, almost one order of magnitude gain in accuracy can be achieved with the same number of function evaluations. From this comparison, it appears that the size of the orbit does not significantly affect the relative efficiencies of the Standard Cowell and Modified Cowell formulations.

The next effect studied is how changing the eccentricities of both satellites affects the efficiency of the Modified Cowell equations relative to the Standard Cowell set. Pairs E and F are almost identical except that the eccentricities of both satellites are increased, while this will change the size of the orbits also) from above we have seen that this effect is slight). The first thing to note is that in Figures 10.5g and 10.6g, the application of the N-S method has increased the efficiency of set 6 by an appreciable amount. Now, from comparision Figures 10.5h and 10.6h, it can be seen that for both Pairs E and F, the Modified Cowell are again competitive with the Standard Cowell for low accuracies. For high accuracies, in Pair E the modified Cowell set is much more efficient than the Standard Cowell, while for Pair F the reverse is true. This comparison would tend to substantiate the observation that as the orbits become more eccentric, the advantages gained by integration of the relative vector become smaller. A comparison of the time history of the magnitudes of the relative positions of the two satellite pairs (Figures 5b and 6b), suggests a possible reason for this. From the figures it can be seen that for the more eccentric pair, Pair F, the satellites approach one another more abruptly at perigee ($t \cong 22$ seconds and $t \cong 85$ seconds) than the near circular orbits (Pair E). The separation of the two satellites for these two pairs is approximately the same, while the eccentric satellites are nearer the central body.

For the Standard Cowell Set, this abrupt approach is no problem as the integrator will just integrate the orbits independently and not know of the change. Figures 5b and 6b also show that at the final time where the errors are calculated, the relative distance is quite large, so the calculation of

$\overline{r} = \overline{r}_2 - \overline{r}_1$ results in the least loss in significance for set 1, and is a very optimistic situation for the set.

To further observe the effects of this type of abrupt approach, the reader's attention is directed to Pairs D, G, and H. From Figure 10.4h it can be seen that, as expected, the integration of the Modified Cowell set is not as the Standard Cowell set. From an examination of Figure 4b, the abruptness of the approach can again be seen. Figure 10.7g shows that the implementation of the N-S technique has greatly improved the efficiency of the Modified Cowell set, but even so, from Figure 10.7h, set 6 is much less efficient than the Standard Cowell set. Again, the time history of relative position for Pair E, (Figure 6b) shows a very abrupt approach. Finally, Figure 10.8h shows this same trend; the integration of the relative vector in set 2 has drastically reduced the efficiency of the Cowell equations. Figure 8b shows the abrupt approach which hampers the relative equations.

The integration of the relative vector in general increases the efficiencies of the Cowell type of equations when there are no abrupt approaches, and the benefits of this Modified set appear to fall off as the approaches become more abrupt. The trade-off point for efficiency between the Modified Cowell and the Standard Cowell sets is dependent upon how rapidly the relative vector changes near the close approaches.

The behavior of equation set 4 (time smoothing applied to one satellite and the relative vector based upon the magnitude of the satellite position vector) will now be discussed. The first effect studied is how the accuracy of the equation is affected by a change in the closeness of approach between Pairs A and B. As pointed out previously, the value of n has a more dramatic impact upon Pair B for set 4 than for Pair A. For Pair A, the largest change in efficiency is between $n=1$ and $n=2$ and is approximately 30%. The other major difference between these curves is that for Pair A, $n=1$ is most efficient for most accuracies, while for Pair B $n=1.5$ is better over most of the range. From Figure 10.1c, it can be seen that the inclusion of the N-S technique does not improve accuracy for Pair A a great deal. From Figure 10.1h it can be seen that set 7 is more efficient for Pair A than the Standard Cowell method. From a comparison of Figures 10.1c and h, it can be seen that for Pair A, set 4 is more efficient than the Standard Cowell set as well as the Modified Cowell set at high accuracies, and competitive with them for lower accuracies.

If the graph of Equation set 4 vs. set 7 (Figure 10.2c) is now compared with Figure 10.2h, it can be seen that set 4 for Pair B is much more efficient than the Standard Cowell set, and the Modified Cowell set. The relative improvement in set 4 between Pairs A and B is dramatic. This improvement is even greater than that realized by integration of the relative vector rather than the Standard Cowell set. Thus, the time smoothing has apparently greatly improved the efficiency of the integration, and this is not due to the integration of the relative vector alone.

To see how the semi-major axis size affects the efficiencies of set 4, examination of Pairs C and E is useful. First, from Figures 10.3a and 10.4a it is clear that this change in size has not changed the relative effects of n on the set. If the reader looks at Figure 10.3c to see how set 4 looks relative to set 7, then looks at Figure 10.3h to approximately locate set 4 relative to set 7, and then follows the same procedure for Figure 10.5c and h, it can be seen that the efficiency of set 4 relative to set 1 for Pair E has not been appreciably effected. Thus, the efficiency gain realized by the time smoothing does not appear to be significantly effected by the orbit size.

The effect of the eccentricity on set 4 can be observed in its efficiency relative to the Standard Cowell equations for Pairs E and F. Figures 10.3a and 10.6a (difference is eccentricity) show that set 4 is not extremely sensitive to the value of n for either Pair. Again, in order to compare the relative efficiency of set 4 relative to the Cowell and Modified Cowell equations it is first necessary to observe the curves of Set 4 relative to set 7 for Pairs E and F.

Next, compare set 7 to sets 1 and 2 in Figures 10.5c and 10.6c. Once this is done, it can be seen that for both pairs, the time smoothing has increased the efficiency beyond that for set 1 for both pairs in the low accuracy region. However, for Pair E in high accuracy regions, time smoothing results in a more efficient procedure than set 1 while it is almost the same as set 1 for Pair F. It should be observed that this is the same trend as was observed for the Modified Cowell set, and indeed this set also integrates the relative vector.

If set 4 is compared to set 2, it appears that set 4 is always more efficient than set 2, due to the time smoothing, while set 4 reacts in the same manner as set 2 in the case of an abrupt approach. To substantiate this, a comparison of Pairs D, G, and H is in order. Comparison of Figures 10.4h, 10.7h, and 10.8h tend to support this hypothesis. From the figures the smoothing does tend to increase the efficiency relative to set 2, but the fact that the relative vector is being integrated hurts the efficiency of the method relative to set 1.

In summary, it appears that the time smoothing enhances the efficiency of the integration of set 4 a great deal. However, the integration of the relative vector introduces the same form of problem for this set (set 4 as was noted for set 2, the Modified Cowell set, when abrupt approaches are integrated. Again it is necessary to remember that due to the final time, the efficiencies given for set 1 are very optimistic, and would likely be much lower if output were desired at a point of minimum separation.

The Nacozy-Szebehely method applied to sets 2, 4 and 5 (Sets 6, 7, and 8 respectively) had very consistant effects. If the differences between each of these sets is looked at for Pairs A and B, it can be seen that the N-S technique increased the efficiencies of each of the methods for higher accuracies for Pair A, and increased their efficiencies even more for the closer approach in Pair B.

The semi-major axis difference between Pairs C and E has an interesting effect upon the efficiencies of sets 2, 4, and 5 and their N-S counterparts. For Pair C, there is little increase in efficiency for the N-S versions at high accuracies, while for Pair E there is a dramatic increase for each of the methods at high accuracies. For the comparison of Pairs E and F, the implementation of the N-S technique behaves similarly for both However, it is even more efficient for the moderately eccentric case than for the near circular case.

The increases in efficiency due to the N-S procedure in the above cases are quite large, in many cases allowing an accuracy increase of almost two orders of magnitude for the same number of function evaluations. A comparison of Pairs D, G, and H shows a little more erratic behavior in the advantages of the N-S method. For Pairs D and G, the method has little effect on sets 4 and 8 (Figures 10.4c and 10.7c) but tends to decrease the efficiency, while for Pair H, the efficiency is increased slightly. Equation sets 5 and 8, on the other hand, are slightly aided by the N-S technique in Pair B, while it is slightly hampered for Pairs G and H. For Pairs D, set 2 is only slightly effected by the N-S method, while for Pair G there is a dramatic increase in its efficiency, and finally for Pair H there is only a small decrease in efficiency. The N-S technique appears to help the methods when there is a sustained close approach, but when the approach is rapid and short, the method will in general make little difference.

While it is hoped that this chapter has presented most of the major comparisons, the graphs included will allow the reader to make any other comparisons which he may desire. The conclusions reached will now be grouped in a more compact form in the next chapter.

CHAPTER V

Conclusions and Recommendations

It is concluded that for the following cases the Standard Cowell set will either be most efficient, or very competative with the other sets:

1) The desired accuracy on the relative vector is low (10^{-3} and above), or
2) The satellites have an abrupt approach.

For the time smoothed equations, the first factor which it was necessary to determine was the fact value of n. As stated by Nacozy[9], the most advantageous choice for n, when studying one vehicles motion is dependent upon many factors, such as

1) location in the two body orbit,
2) the order of the integration method,
3) forces included in the perturbation model, and
4) the eccentricity of the two body reference orbit.

Clearly, the most efficient choice for n for each equation set cannot be determined for all cases, but will only be discussed for the cases run here.

The proper choice for n, the time smoothing exponent for equation set 4 varies, from pair to pair, and it is interesting to note that the introduction of the N-S method to perform the subtraction can alter the most efficient choice of n (i.e., set 7). The proper choice of n seems to be very problem dependent, and will probably have to be determined for the specific model being used. The value $n=2$ was most commonly best for sets 4 and 7.

Equation set 5 was very consistent in that $n=1.0$ was most efficient. This behavior was repeated in its N-S counterpart, set 8. It is interesting to note that in the time transformation function, for sets 4 and 7

$$f = 1/(r_2)^n \tag{29}$$

and the transformation function for set 5 and 8

$$f = 1/(r_1 r_2)^n \tag{30}$$

the choice of $n=2$ for Eq. (29), and $n=1$ for Eq. (30) produce values which are very close to one another if $r_1 \approxeq r_2$.

The N-S method appears to enhance the efficiencies of the methods in almost all cases. This is most evident when the satellites are in close proximity for a reasonable length of time, i.e., the orbits are similar.

In conclusion it is recommended that when the orbits are dissimilar, that is, they approach the point of close separation very abruptly, the standard Cowell equations will probably be either most efficient, or very competative with the other methods described here. When the two orbits are similar, the modified Cowell equations, incorporating the N-S technique will probably offer great advantages at high tolerance. A very important point to note here is the fact that the machine used (CDC 6600) carries approximately 14 digits, as the close approaches in which 5 digits were lost will probably not show the advantages of the N-S technique as clearly as on a machine of lesser word length. Another point to be considered is that the N-S method requires extra time to do the subtractions. In cases where the force model is quite simple, the time savings of this method might be reduced, but in cases when a more sophisticated force model is used, its advantages will be increased. The time smoothing equations can be very economical to use, but the amount of trajectory-time information required by the user and frequency with which this information is required can reduce its overall economy as far as computing time is concerned. If only position information is required, then no transformations between t and s are required after the initialization. If velocity and acceleration data is required, however, the transformations might become too expensive if the force model is simple. If the force model is more complex, the amount of time spent in the transformations may be small compared to the savings realized from fewer function evaluations. Again, it must be pointed out that these conclusions rest on the basis that the trends in relative efficiencies of the methods desired using a two-body model carried over when more sophisticated force models are used. Since the two body accelerations are dominant in most instances, it is believed that while the optimal value for n will probably change for each model, the overall trends will extrapolate the more complex force models.

Equation set 3 suffers from the effects of the method used for time matching. If a much more efficient method can be developed for determining information for both satellites at the same time, such as an interpolation procedure, it is felt that this set might be competative with the other time smoothed sets, and for the case of dissimilar orbits might be more efficient, (as the Standard Cowell is more efficient than modified Cowell for these orbits).

This paper scratches the surface of a class of problems which deserve greater study. Other topics of study which could be investigated are:

1) An investigation of the effects of the word size of the computer used on the efficiencies realized in the implementation of the N-S technique.
2) More efficient means of obtaining information about both satellites from equation set 3.
3) The possibility of choosing the best value for the time smoothing exponent at each step, analogous to the method proposed by Nacozy for integration of a single body.

Appendix

As stated in Chapter 2, the time smoothed equations are all based on a transformation of the independent variable of integration, from t to a new independent variable of integration s. This transformation usually adds a new differential equation to the system, relating t and s. The common form of this relationship is

$$ds = f\,dt \tag{1}$$

where f is a function of magnitudes of the position vectors of satellite 1 and/or satellite 2.

The first step in obtaining the time smoothed equations is to determine the relationship between the acceleration of a vector $\bar{\xi}$ with respect to t, $\ddot{\bar{\xi}}$, and with respect to s, $\bar{\xi}''$. The second derivative of $\bar{\xi}$ with respect to time can be written

$$\ddot{\bar{\xi}} = \frac{d^2\bar{\xi}}{dt^2} = \frac{d}{dt}\left[\frac{d\bar{\xi}}{dt}\right]$$

$$= \frac{d}{dt}\left[\frac{d\bar{\xi}}{ds}\;\frac{ds}{dt}\right] \tag{2}$$

If $\frac{ds}{dt}$ is replaced by f as indicated by equation (1) and the differentiation is carried out, the equation can be written as:

$$\frac{d}{dt}\left[\frac{d\bar{\xi}}{ds}\right] f + \frac{d\bar{\xi}}{ds}\frac{d}{dt}\left[f\right]$$

$$= \frac{ds}{dt}\frac{dt}{ds}\frac{d}{dt}\left[\frac{d\bar{\xi}}{ds}\right] f + \frac{d\bar{\xi}}{ds}\;\frac{d}{dt}\left[f\right]$$

$$= f^2\left[\frac{d^2}{ds^2}\bar{\xi}\right] + \frac{d\bar{\xi}}{ds}\;\frac{d}{dt}\left[f\right] \tag{3}$$

Letting $\dot{\bar{\eta}}$ denote differentiation of a vector $\bar{\eta}$ with respect to time, and $\bar{\eta}'$ denote differentiation with respect to S, equation (3) becomes

$$\ddot{\bar{\xi}} = \bar{\xi}'' f^2 + \bar{\xi} \; \dot{f} \tag{4}$$

This equation relates accelerations in the time space to accelerations in the time smoothed space. The corresponding relationship between t and s can be obtained from equation (1) by a separation of variables and integration. The result is

$$\int_{t_o}^{t_f} dt = \int_{s_o}^{s_f} \frac{ds}{f}$$

or

$$t = \int_{s_o}^{s_f} \frac{ds}{f} \tag{5}$$

In the first time smoothed equation set, each satellite is independently smoothed with respect to the magnitude of its position from the central body. Letting

$$f_1 = \frac{ds_1}{dt_1} = \frac{1}{a_1 r_1^{\,n_1}} \tag{6}$$

and

$$f_2 = \frac{ds_2}{dt_2} = \frac{1}{a_1 r_1^{\,n_1}} \tag{7}$$

it can be noted that f_1 and f_2 are both of the form $f = \frac{1}{ar^n}$.

For this form

$$f = \frac{d}{dt} (ar^n)^{-1}$$

$$= -(ar^n)^{-2} \; na \; r^{n-1} \; \frac{dr}{dt}$$

$$= \frac{-n}{ar^{n+1}} \frac{dr}{ds} \frac{ds}{dt} = \frac{-n}{ar^{n+1}} \frac{dr}{ds} \; f$$

or finally,

$$\dot{f} = \frac{-n}{a^2 r^{2n+1}} \frac{dr}{ds} \tag{8}$$

Therefore,

$$\dot{f}_1 = \frac{-n_1}{a_1^2 r_1^{2n_1+1}} \frac{dr_1}{ds_1}, \quad \dot{f}_2 = \frac{-n_2}{a_1^2 r^{2n_2+1}} \frac{dr_2}{ds_2}$$

$$f_1^2 = \frac{1}{a_1^2 r_1^{2n_2}} \qquad f_2^2 = \frac{1}{a_2 r_2^{2n_2}} \tag{9}$$

Substitution of $\bar{r}_1$ and $\bar{r}_2$ individually into equation (4), and the application of equations (9) to the result yields

$$\ddot{\bar{r}}_1 = \frac{\bar{r}_1''}{a_1^2 r_1^{2n_1+1}} - \frac{n_2}{a_1^2 r_1^{2n_1+1}} \frac{dr_1}{ds_1} \frac{d\bar{r}_1}{ds_1}$$

and

$$\ddot{\bar{r}}_1 = \frac{\bar{r}_2''}{a_2^2 r_2^{2n_2+1}} - \frac{n_2}{a_2^2 r_2^{2n_2+1}} \frac{dr_2}{ds_s} \frac{d\bar{r}_2}{ds_2} \tag{10}$$

Letting the accelerations in time space be expressed as:

$$\ddot{\bar{r}}_1 = \frac{-\mu(\bar{r}_1)}{r_1^3} + P_1(\bar{r}_1, \dot{\bar{r}}_1) \tag{11}$$

and

$$\ddot{\bar{r}}_2 = \frac{-\mu(\bar{r}_2)}{r_2^3} + P_2(\bar{r}_2, \dot{\bar{r}}_2)$$

the accelerations of $\bar{r}_1$ and $\bar{r}_2$ can be introduced into equations (10). Solving the resulting equations for $\bar{r}''$ to obtain the equations of motion for fime smoothed space the equations become

$$\bar{r}_1'' = \frac{n_1}{r_1}\frac{dr_1}{ds_1}\frac{d\bar{r}_1}{ds_1} - \mu\, a_1^2\, r_1^{2n_1-3}\,\bar{r}_1 + \bar{P}\,(\bar{r}_1,\dot{\bar{r}}_1)\, a_1^2\, r_1^{2n}$$

and (12)

$$\bar{r}_2'' = \frac{n_2}{r_2}\frac{dr_2}{ds_2}\frac{d\bar{r}_2}{ds_2} - \mu\, a_2^2\, r_2^{2n_2-3}\,\bar{r}_2 + \bar{P}(\bar{r}_2,\dot{\bar{r}}_2)\, a_2^2\, r_1^{2n_2}$$

At this point it is noted that since each satellite has been independently smoothed, the relations between t and s are different for each satellite. If f_1 is substituted into Equation (5), the relationship between t and s for the first satellite is found to be

$$t_1 = \int_{s_{10}}^{s_{1t}} a_1\, r_1^{n_1}\, ds_1 \tag{13}$$

While for the second satellite the relationship is

$$t_2 = \int_{s_{20}}^{s_{2t}} a_2 r_2^{n_2}\, ds_2 \tag{14}$$

Thus for this equation set, two new first order differential equations have been introduced. A major disadvantage of this formulation is that since relative position is our primary interest, it will be necessary to match t_1 and t_2 at each desired output point.

Similar logic as was used in developing the Cowell equations for one satellite and the relative vector may be used to develop a second set of time smoothed equations. For this set, (Eq. set 2) one satellite and the relative vector are smoothed, both with respect to the magnitude of the position of the second satellite, (can be 1 or 2). The relation between t and s is, as before,

$$f = \frac{ds}{dt}\ \frac{1}{a_2 r_2^{n_2}} \tag{15}$$

The expressions for f^2 and $\dot{f}$ derived previously and stated in equations (9) are still valid and need only be applied to $\ddot{\bar{r}}$ and $\ddot{\bar{r}}_2$. The equation for $\ddot{\bar{r}}_2$ is unaltered and is restated below

$$\bar{r}_2{}'' = \frac{n^2}{r_2} \frac{dr_2}{ds_s} \frac{d\bar{r}_2}{ds_2} - \mu\, a_2^2\, r_2^{2n_2-3}\, \bar{r}_2 + P(\bar{r}_2,\dot{\bar{r}}_2) a_2^2\, r_2^{2n_2} \qquad (16)$$

The first step in developing the expression for $\bar{r}''$ is substitution into equation (3).

$$\ddot{\bar{r}} = \bar{r}''\, f^2 + \bar{r}'\, \dot{f} \qquad (17)$$

If the equations for f^2 and $\dot{f}$ are substituted into equation (17), and $\ddot{\bar{r}}$ is replaced by

$$\ddot{\bar{r}} = \ddot{\bar{r}}_1 - \ddot{\bar{r}}_2 = -\mu \left(\frac{r_1}{r_1^3} - \frac{\bar{r}_2}{r_2^3}\right) + \bar{P}(\bar{r}_1,\dot{\bar{r}}_1) - \bar{P}(\bar{r}_2,\dot{\bar{r}}_2) \qquad (18)$$

the following equation is obtained:

$$\bar{r}'' = \frac{n_2}{r_2} r_2{}^{1}\bar{r}^{1} + a_2^2\, r_2{}^{2n} \left[\bar{P}(\bar{r}_1,\dot{\bar{r}}_1) - P(\bar{r}_2,\dot{\bar{r}}_2) + \mu\left(\frac{\bar{r}_2}{r_2^3} - \frac{\bar{r}_1}{r_1^3}\right)\right] \qquad (19)$$

It should be noted that one direct consequence of this coupling is to eliminate the need for two relations between t and s. The time can be directly calculated from equation (5) as

$$t = \int_{so}^{s_t} \frac{ds}{f} \qquad (20)$$

Now the time matching required for the first set of time smoothing equations is not necessary.

A third set of smoothed equations, suggested by Dr. V. G. Szebehely can be obtained by smoothing one satellite and the relative vector with respect to the product of the magnitudes of the positions of both satellites.

Here f has the form

$$f = \frac{ds}{dt} = (r_1 r_2)^{-n} \tag{21}$$

and $\dot{f}$ can be developed as follows

$$\dot{f} = -n(r_1 r_2)^{-n-1} \left(r_1 \frac{dr_2}{dt} + r_2 \frac{dr_1}{dt} \right)$$

$$= -n\ (r_1 r_2)^{-(n+1)} \left(r_1 \frac{dr_2}{ds}\frac{ds}{dr} + r_2 \frac{dr_2}{ds}\frac{ds}{dt} \right)$$

$$= -n\ (r_1 r_2)^{-(n+1)} \cdot f \cdot (r_1 r_2' + r_2 r_1^{1})$$

$$= -n\ (r_1 r_2)^{-(2n+1)} (r_1 r_2' + r_2 r_1') \tag{22}$$

As before, the expressions for f and $\dot{f}$ can be applied to equation (4) for $\bar{r}_2$, resulting in

$$\ddot{\bar{r}}_2 \quad \frac{1}{(r_1 r_2)^{2n}} \bar{r}'' - \frac{n}{(r_1 r_2)^{2n+1}} (r_1 r_2' + r_2 r_1') \tag{23}$$

Solving for $\bar{r}_2''$ one obtains

$$\bar{r}_2'' = \ddot{\bar{r}}_2 (r_1 r_2)^{2n} + \frac{n}{r_1 r_2} (r_1 r_2' + r_2 r_1')\ \bar{r}_2' \tag{24}$$

When the acceleration of $\bar{r}_2$ with respect to time from equations (11) is introduced into equation (24), the smoothed equation for $\bar{r}_2$ emerges as

$$\bar{r}_2'' = \frac{n}{r_1 r_2} (r_1 r_2' + r_2 r_1')\bar{r}_2' - (r_1 r_2)^{2n} \frac{\mu r_2}{r_2^3} + P(\bar{r}_2, \dot{\bar{r}}_2)(r_1 r_2)^{2n} \tag{25}$$

The relative acceleration vector in the transformed system now needs to be developed. Recalling the acceleration from equation (18),

$$\ddot{\bar{r}} = \ddot{\bar{r}}_1 - \ddot{\bar{r}}_2 = -\mu \left[\frac{r_1}{r_1^{\,3}} - \frac{r_2}{r_2^{\,3}}\right] + \left[P_1(\bar{r}_1,\dot{\bar{r}}_1) - P(\bar{r}_2,\dot{\bar{r}}_2)\right] \tag{18}$$

and substituting f, $\dot{f}$, and $\ddot{\bar{r}}$ into equation (17), we arrive at

$$-\mu \left[\frac{r_1}{r_1^3} - \frac{r_2}{r_2^3}\right] + \left[\bar{P}(\bar{r}_1\dot{\bar{r}}_1) - \bar{P}(\bar{r}_2,\dot{\bar{r}}_2)\right] = \bar{r}''(r_1r_2)^{-2n} - \frac{n}{r_1r_2}(r_1r_2'+r_2r_1')$$

Rewriting the equation to isolate $\bar{r}''$, produces

$$\bar{r}'' = \frac{n}{r_1r_2}(r_1r_2'+r_2r_1')\bar{r}' - \mu \frac{\left[\frac{\bar{r}_1}{r_1^{\,3}} - \frac{\bar{r}_2}{r_2^{\,2}}\right]}{(r_1r_2)^{-2n}} + \frac{\bar{P}(\bar{r}_1\dot{\bar{r}}_1) - \bar{P}(\bar{r}_2,\dot{\bar{r}}_2)}{(r_1r_2)^{-2n}} \tag{26}$$

Once again the relationship between time and s may be obtained from equation (5) as

$$t = \int_{s_o}^{s_f} (r_1r_2)^n \, ds$$

Again there is no need for the time matching required by the first set of smoothed equations.

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