# TOPICS IN LAGRANGIAN AND HAMILTONIAN FLUID DYNAMICS: RELABELING SYMMETRY AND ION-ACOUSTIC WAVE STABILITY

APPROVED BY
DISSERTATION COMMITTEE:

Supervisor: P. J. Morrison

P. J. Morrison

R. D. Hazeltine

R. D. Hazeltine

W. Horton

W. Horton

Michael P. Marder

M. Marder

R de la Llave

R. de la Llave

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by

**NIKHIL SUBHASH PADHYE, B.Tech., M.A.**

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This dissertation is dedicated to
my mother Nalini,
my grandmother Kalindi Dravid,
and my wife Kathryn Olivia.

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Nikhil Padhye

Austin, Texas

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Nikhil Subhash Padhye, Ph.D.
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Relabeling symmetries of the Lagrangian action are found for the ideal, compressible fluid and magnetohydrodynamics (MHD). These give rise to conservation laws of potential vorticity (Ertel's theorem) and helicity in the ideal fluid, cross helicity in MHD, and a conservation law for an ideal fluid with three thermodynamic variables. The symmetry that gives rise to Ertel's theorem is generated by an infinite parameter group, and leads to a generalized Bianchi identity. The existence of a more general symmetry is also shown, with dependence on time and space derivatives of the fields, and corresponds to a family of conservation laws associated with the potential vorticity. In the Hamiltonian formalism, Casimir invariants of the noncanonical formulation are directly constructed from the symmetries of the reduction map from Lagrangian to Eulerian variables. Casimir invariants of MHD include a gauge-dependent family of invariants that incorporates magnetic helicity as a special case.

Novel examples of finite dimensional, noncanonical Hamiltonian dynamics are also presented: the equations for a magnetic field line flow with a symmetry direction, and Frenet formulas that describe a curve in 3-space.

In the study of Lyapunov stability of ion-acoustic waves, existence of negative energy perturbations is found at short wavelengths. The effect of adiabatic, ionic pressure on ion-acoustic waves is investigated, leading to explicit solitary and nonlinear periodic wave solutions for the adiabatic exponent $\gamma = 3$. In particular, solitary waves are found to exist at any wave speed above Mach number one, without an upper cutoff speed. Negative energy perturbations are found to exist despite the addition of pressure, which prevents the establishment of Lyapunov stability; however the stability of ion-acoustic waves is established in the KdV limit, in a manner far simpler than the proof of KdV soliton stability. It is also shown that the KdV free energy (Benjamin, 1972) is recovered upon evaluating (the negative of) the ion-acoustic free energy at the critical point, in the KdV approximation.

Numerical study of an ion-acoustic solitary wave with a negative energy perturbation shows transients with increased perturbation amplitude. The localized perturbation moves to the left in the wave-frame, leaving the solitary wave peak intact, thus indicating that the wave may be stable.

# Table of Contents

# Chapter 1

# Introduction

Lagrangian and Hamiltonian formulations of physical systems provide an effective framework for the systematic study of dynamics. Lagrangian action symmetries and the study of Lyapunov stability using Dirichlet's criterion (free energy principle) are two of the most striking tools offered by analytical dynamics. This dissertation is, in essence, the application of these two techniques to problems in fluid dynamics. The systems of interest are the ideal compressible fluid, magnetohydrodynamics (MHD), and the ion-acoustic fluid. Lagrangian action symmetries are found for the former two in Ch. 3, while the question of stability of wave solutions to the ion-acoustic fluid is addressed in Chs. 4 and 5.

Noncanonical Hamiltonian formalism is used throughout, and is briefly introduced in Ch. 2 and exemplified with the aid of two novel finite dimensional examples. The first example is that of the Frenet formulas that describe a curve in 3-space, while the second example gives the noncanonical formulation of the equations describing magnetic field line flow, when a symmetry direction exists. The ion-acoustic fluid, with and without ionic pressure, provides two examples of infinite-dimensional, noncanonical Hamiltonian structures (Morrison, 1982), which are briefly presented in Ch. 4. In Ch. 3, the noncanonical formulations of the ideal fluid and of MHD (Morrison and Greene, 1980) are utilized, but are not presented as parts of this dissertation.

Continuous symmetry transformations of the Lagrangian *labels* of fluid elements, or independent coordinates, which we refer to as *relabeling symmetries* following Salmon (1988), are presented in Ch. 3. These are mostly new results and give rise to the conservation laws of potential vorticity (Ertel's theorem) in the ideal fluid, and helicity, cross helicity in barotropic fluids and MHD respectively. The symmetry that gives rise to Ertel's theorem is especially interesting since it involves an infinite parameter continuous group, parametrized by arbitrary functions. For such symmetries, one has generalized Bianchi identities, or Noether's second theorem, in addition to Noether's first theorem. Both theorems are presented at the outset in Ch. 3; the main difference between the two theorems is that one requires that the equations of motion be satisfied, while the other does not.

A Lagrangian conservation law often, but not always, has a corresponding conservation law in Eulerian variables. The relation between these laws is clarified in Ch. 3. The Lagrangian conservation law that arises from the aforementioned symmetry of the infinite parameter group is an example of a conservation law that has no Eulerian equivalent; however upon integrating over the label domain, and utilizing the freedom offered by the arbitrary, parametrizing function, a simple Lagrangian conservation law is obtained, without any current. This transforms directly into the Eulerian advection law of potential vorticity, in addition to having a corresponding Eulerian conservation law.

Since potential vorticity is advected with the flow, any function of the potential vorticity is also advected. In fact, there exists a generalized version of the symmetry of the infinite parameter group, that has dependence on mixed time and label derivatives of the fields, and gives rise to the family of Eulerian advection laws associated with the potential vorticity.

The existence of the symmetry generated by the infinite parameter continuous group, for both barotropic and adiabatic flows, gives rise to three other interesting new results. Firstly, a specific choice of the arbitrary function leads to conservation laws which are recognized to be Weber's equations (Lamb, 1932). Secondly, the suspicion that a finite parameter symmetry may exist even after the introduction of one more advected thermodynamic variable, proves to be correct. This symmetry, and the corresponding conservation law, are also presented in Ch. 3. Thirdly, the potential energy functional obtained by expanding about a stationary equilibrium possesses a Bianchi identity and relates to spontaneous symmetry breaking, which gives rise to null eigenfunctions.

For barotropic fluid flows, Kelvin's circulation theorem is satisfied on any closed material, or Lagrangian, loop; when the flow is adiabatic, Kelvin's circulation theorem applies on isentropic material loops. Since the initial treatments (Newcomb, 1967; Bretherton, 1970) that related relabeling symmetries to conservation laws were concerned with Kelvin's circulation theorem, in Ch. 3 it is shown that the circulation theorem follows from Ertel's theorem.

For completeness, Galilean symmetries and the consequent conservation laws of energy, linear and angular momenta, and uniform motion of the center-of-mass are also included in Ch. 3. They also help to distinguish what is meant by relabeling symmetries; for example, space translation gives rise to linear momentum conservation, while label translation gives rise to Weber's equations in barotropic fluid flow.

Chapter 3 is concluded by deriving Casimir invariants of the noncanonical Hamiltonian formulations of the ideal fluid and MHD directly from the symmetries of the Euler-Lagrange map. Casimir invariants result as a consequence of the reduction from Lagrangian to Eulerian variables, which is referred to as the Euler-Lagrange map. The relabeling symmetry of the Lagrangian action, that gives rise to conservation of potential vorticity, is shown to be a symmetry of the Euler-Lagrange map and gives rise to a family of Casimir invariants associated with the potential vorticity. In the case of MHD, there exists a relabeling symmetry of the Euler-Lagrange map that leads to a family of Casimir invariants associated with the quantity $A \cdot B/\rho$ and includes magnetic helicity as a special case. It turns out that this is a gauge-dependent invariant; i.e. it is valid for a particular choice of the gauge for the vector potential. This invariant is an independent rediscovery of a known result (Gordin and Petviashvili, 1987), but the symmetry was previously unknown.

Lyapunov stability of solitary and nonlinear periodic waves of the ion-acoustic fluid is the subject of Chs. 4 and 5. It is shown that wave solutions lie on the extremal of a free energy functional of the ion-acoustic fluid, which is constructed from the Hamiltonian and Casimir invariants. In Ch. 4, it is established, by construction of examples, that the sign of the second variation of the free energy $\delta^2 F$ is indefinite. This indicates that the ion-acoustic waves may be unstable to short wavelength perturbations, which give rise to negative $\delta^2 F$. Such perturbations are henceforth referred to as negative energy perturbations.

One may suspect that the addition of ionic pressure may eliminate the existence of negative energy perturbations, hence the ion-acoustic equations are studied with the inclusion of ionic pressure, which is assumed to obey an adiabatic law. For the adiabatic exponent given by $\gamma = 3$, which corresponds to the exponent for a one degree-of-freedom system, it is found that an analogy to dynamics of a single particle can be set up, with a closed form expression for the potential. The parameter space, which consists of a pressure and a wave speed parameter, is explored and regimes are found in which solitary and nonlinear periodic waves exist. Interestingly, solitary waves exist for any wave speed greater than Mach number one; there is no upper cutoff on the wave speed, unlike the case without ionic pressure. Similarly, nonlinear periodic waves exist for any wave speed, without an upper cutoff.

Lyapunov stability can be established for a parameter range in which, unfortunately, there exist no wave solutions except (linearized) ones with vanishing amplitude. Negative energy perturbations thus exist for the solitary and nonlinear periodic waves, despite the addition of ionic pressure.

The time evolution of a negative energy perturbation of an ion-acoustic solitary wave (without ionic pressure) is studied numerically in Ch. 5. It is found that the localized negative energy perturbation grows rapidly in amplitude, and moves to the left in the frame of the wave moving to the right. The perturbation amplitude then decreases and the solitary wave peak passes through intact. This behavior is contrasted with the evolution of a localized positive energy perturbation which shows no transient growth of the perturbation amplitude. The time evolution of the perturbed uniform background at unit Mach number is also studied for the "negative energy perturbation" used earlier for the solitary wave. The uniform background, which is known to be Lyapunov stable, also shows transients with increased amplitude followed by decaying oscillation of the perturbation amplitude. These numerical results indicate that the ion-acoustic solitary waves may be stable.

The numerical algorithm used is an implicit finite difference scheme, with uniform time step and grid spacing. Newton's iterative method is used at each time step to solve the nonlinear equations. The boundary conditions require that the solution vanish at the edges of the grid.

A few words on the notation used in this dissertation are in order. Repeated indices are summed, except where mentioned otherwise. No special notation is used for vectors and matrices; they are to be understood by reference to the context.

# Chapter 2

# Noncanonical Hamiltonian Dynamics

Noncanonical variables are useful in a variety of physical systems, from the simple rigid rotating body, to the Eulerian description of fluid dynamics. This chapter is a brief introduction to noncanonical Hamiltonian formalism, and provides two unusual, finite-dimensional examples: the Frenet formulas that describe a curve in 3-space and the equations for magnetic field line flow with a symmetry direction. Examples of noncanonical Hamiltonian formulation in infinite dimensions are found in Ch. 4 for the ion-acoustic fluid, and the noncanonical formulations of the ideal fluid and MHD are encountered in Ch. 3. The interested reader may consult the review article by Morrison (1998) for a comprehensive treatment of noncanonical Hamiltonian dynamics.

The Hamiltonian formulation of a dynamical system requires the specification of a Hamiltonian and a Poisson bracket, effectively splitting the problem into kinematics—properties of the Poisson bracket—and dynamics—specified by the Hamiltonian. The Poisson bracket is defined in terms of a cosymplectic form $J^{ij}$:

$$\{f(z), g(z)\} := \frac{\partial f}{\partial z^i} J^{ij} \frac{\partial g}{\partial z^j}\,, \tag{2.1}$$

where $z^i$ $(i = 1, 2, \ldots, 2N)$ are the phase space coordinates, and $f$, $g$ are arbitrary functions. The phase space evolution is given by

$$\dot{z}^i = \{z^i, H(z)\} = J^{ij} \frac{\partial H}{\partial z^j}\,, \tag{2.2}$$

where $H(z)$ is the Hamiltonian.

In addition to the Leibniz property, the Poisson bracket must satisfy the conditions (Morrison, 1998, for example) of antisymmetry, linearity, and the Jacobi identity:

$$\{f,g\} = -\{g,f\}\,, \tag{2.3}$$

$$\{f,\alpha\, g + \beta\, h\} = \alpha\,\{f,g\} + \beta\,\{f,h\}\,, \tag{2.4}$$

$$\{\{f,g\},h\} + \{\{g,h\},f\} + \{\{h,f\},g\} = 0\,, \tag{2.5}$$

where $\alpha$, $\beta$ are constants. The Poisson bracket defined by Eq. (2.1) is evidently linear, while the antisymmetry and Jacobi property are assured if

$$J^{ij} = -J^{ji}\,, \tag{2.6}$$

$$J^{i\ell}\,\frac{\partial J^{jk}}{\partial z^\ell} + J^{j\ell}\,\frac{\partial J^{ki}}{\partial z^\ell} + J^{k\ell}\,\frac{\partial J^{ij}}{\partial z^\ell} = 0\,, \tag{2.7}$$

for all $i, j, k$. In particular, the Poisson bracket is *canonical* when the cosymplectic form is given by

$$J_c := \begin{bmatrix} 0_N & I_N \\ -I_N & 0_N \end{bmatrix}\,, \tag{2.8}$$

where $I_N$ and $0_N$ are $N \times N$ identity and null matrices respectively. The canonical Poisson bracket may be expressed as

$$\{f,g\}_c = \frac{\partial f}{\partial q^i}\,\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\,\frac{\partial g}{\partial q^i}\,, \tag{2.9}$$

where the first $N$ of the $z$'s have been denoted by $q$ and the rest by $p$, in order to obtain the familiar form of the canonical Poisson bracket, found in most textbooks.

Under a coordinate transformation $\bar{z}(z)$, the cosymplectic form transforms contravariantly to

$$J^{ij} = \frac{\partial \bar{z}^i}{\partial z^k}\, J_c^{k\ell}\, \frac{\partial \bar{z}^j}{\partial z^\ell}\,. \tag{2.10}$$

It is clear that $J^{ij}$ generally depends on the coordinates; when $J^{ij}$ satisfies Eqs. (2.6) and (2.7), but is not given by Eq. (2.8), it defines a *noncanonical* Poisson bracket. Examples of noncanonical Poisson brackets in finite dimensions are presented in Secs. 2.1 and 2.2.

An interesting feature of noncanonical Poisson brackets is the possibility of existence of *Casimir* invariants. A Casimir invariant $C(z)$ satisfies

$$\{C, f\} = 0 \quad \text{for any } f(z), \tag{2.11}$$

which implies that the gradient of $C$ is a null direction of the cosymplectic form:

$$J^{ij} \frac{\partial C}{\partial z^j} = 0 \quad \text{for all } i. \tag{2.12}$$

In particular, $\{C, H\}$ also vanishes, for *any* Hamiltonian $H(z)$. The Casimir invariant is thus a feature of the kinematics, and the dynamics is constrained to surfaces of constant Casimir. Another noteworthy feature of the noncanonical formulation is that conjugate pairs of coordinates are not necessary; odd dimensional systems are acceptable. It turns out that odd dimensionality implies the existence of Casimir invariants (Morrison, 1998).

The ideas expressed above can be extended to infinite dimensions by the use of functional calculus; however it is easier to work with the Poisson bracket directly than to introduce (the analogue of) a cosymplectic form. We briefly review examples of infinite dimensional noncanonical Hamiltonian formulations in Secs. 4.1.1 and 4.4.1; for more examples and an extensive treatment, see Morrison (1998).

## 2.1 Frenet Formulas for 3-D Curves

As a first example, we construct the noncanonical Hamiltonian formulation of the Frenet formulas, which describe a curve in three dimensions (Mathews and

Walker, 1970, for example), and are given by

$$\frac{d\alpha^i}{ds} = \kappa\,\beta^i\,, \tag{2.13}$$

$$\frac{d\beta^i}{ds} = -\kappa\,\alpha^i - \tau\,\gamma^i\,, \tag{2.14}$$

$$\frac{d\gamma^i}{ds} = \tau\,\beta^i\,, \tag{2.15}$$

where $s$ is a parameter that varies along the curve, $\kappa$ is the curvature and $\tau$ is the torsion. The vectors $\alpha$, $\beta$, and $\gamma$ are the tangent, principal normal, and binormal to the curve under consideration.

The Hamiltonian for the Frenet system is given by

$$H(z) = \frac{1}{2}\,(\alpha^2 + \beta^2 + \gamma^2)\,, \tag{2.16}$$

where the phase space coordinates are $z = (\alpha, \beta, \gamma)$. The noncanonical Poisson bracket is defined by Eq. (2.1), where the cosymplectic form is given by

$$J = \begin{bmatrix} 0_3 & \kappa\, I_3 & 0_3 \\ -\kappa\, I_3 & 0_3 & -\tau\, I_3 \\ 0_3 & \tau\, I_3 & 0_3 \end{bmatrix}\,. \tag{2.17}$$

The Casimir invariants are $C(\tau\,\alpha - \kappa\,\gamma)$, where $C$ is an arbitrary function of the argument $(\tau\,\alpha - \kappa\,\gamma)$. This example is somewhat unusual since the cosymplectic form and the Casimir invariants have explicit dependence on the parameter $s$, through $\kappa$ and $\tau$, which may depend on $s$; hence the Casimir invariants are not truly invariants, but satisfy $dC/ds = \partial C/\partial s$.

## 2.2 Magnetic Field Lines with Symmetry

One approach to the integration of magnetic field line flow equations involves locally gauging away one of the components of the vector potential (Cary and Littlejohn, 1983). The resulting action, $\int A \cdot d\ell$, is then easily expressed in a canonical Hamiltonian form, so that two coordinates are canonically conjugate, while the third coordinate serves as the parameter, replacing time in the usual sense. This requires the existence of a "third" coordinate which uniquely determines the other two.

When the magnetic field possesses a symmetry direction, one can formulate the magnetic field line flow problem as a Hamiltonian problem without the need for representation of the magnetic field in terms of potentials. In the formulation of Mezic and Wiggins (1994), this is accomplished by expressing two of the coordinates of the field lines canonically conjugate to each other, while the evolution of the third coordinate is given as a quadrature in terms of the other two. In addition to reviewing this formulation due to Mezic and Wiggins (1994), we also present the corresponding noncanonical Hamiltonian structure with three dynamical variables. The latter does involve the introduction of *one* scalar potential.

### 2.2.1 Canonical Formulation

Magnetic field line flow may be expressed as

$$\frac{dx^i}{ds} = B^i(x) \qquad i = 1, 2, 3. \tag{2.18}$$

In the above equation, $s$ is a parameter along the field line, $B$ is the magnetic field, and we will restrict our attention to the case where there is no time dependence. Evidently, the flow preserves volume since $\nabla \cdot B = 0$.

If the magnetic field possesses a volume preserving symmetry, that is if there exists a vector field $W$ such that $\nabla \cdot W = 0$, and the Lie bracket vanishes,

$$[B, W] := B \cdot \nabla W - W \cdot \nabla B = 0\,, \tag{2.19}$$

then there exists a transformation of coordinates $x^i(z^1, z^2, z^3)$, for which the system (2.18) may be written in the form (Mezic and Wiggins, 1994, theorem 2.2):

$$\frac{dz^1}{ds} = \frac{\partial}{\partial z^2} H(z^1, z^2)\,,$$

$$\frac{dz^2}{ds} = -\frac{\partial}{\partial z^1} H(z^1, z^2)\,,$$

$$\frac{dz^3}{ds} = k(z^1, z^2)\,. \tag{2.20}$$

The system is thus canonically Hamiltonian in $z^1$ and $z^2$, while $z^3$ is determined in terms of the canonical variables.

Consider, for example, the magnetic field equations in toroidal coordinates,

$$\frac{dr}{ds} = r(R + r\cos\phi)B^r\,,$$

$$\frac{d\phi}{ds} = (R + r\cos\phi)B^\phi\,,$$

$$\frac{d\psi}{ds} = rB^\psi\,, \tag{2.21}$$

where $R$ is the major radius, $r$ is the radial coordinate, and $\phi, \psi$ are the poloidal and toroidal angles respectively.

In simple tokamak models the magnetic field is often assumed to have the form (Gentle, 1996)

$$B^r = 0, \quad B^\phi = \frac{RB^0(r)}{(R + r\cos\phi)}\,, \quad B^\psi = B^\psi(r, \phi)\,.$$

(In fact, we allow a more general $\hat{\psi}$ component than most simple models require.) It is evident that the magnetic field has toroidal symmetry; more precisely, it may be verified that

$$W = (R + r\cos\phi)\,\hat{\psi}$$

commutes with the magnetic field and is divergenceless.

The two canonical coordinates, $z^1$ and $z^2$, must satisfy $W \cdot \nabla z^1 = 0 = W \cdot \nabla z^2$, while the third coordinate must satisfy $W \cdot \nabla z^3 = 1$ (Mezic and Wiggins, 1994). In our simple model, we have already chosen the appropriate coordinates so that $z^1 := r$, $z^2 := \phi$, $z^3 := \psi$. Furthermore the two canonical coordinates, $z^1$ and $z^2$, correspond to action and angle since the Hamiltonian is independent of $z^2$:

$$H(z^1) := -R \int^{z^1} B^0(x)\, dx\,, \tag{2.22}$$

while $k(z^1, z^2)$ is defined by

$$k(z^1, z^2) := z^1\, B^\psi(z^1, z^2)\,. \tag{2.23}$$

### 2.2.2 Noncanonical Formulation

A noncanonical Hamiltonian formulation can be found for the system represented by Eqs. (2.20) if there exists a function $\gamma(z^1, z^2)$, such that

$$\frac{\partial \gamma}{\partial z^1}\frac{\partial H}{\partial z^2} - \frac{\partial \gamma}{\partial z^2}\frac{\partial H}{\partial z^1} = k(z^1, z^2)\,. \tag{2.24}$$

It is easily shown that $\gamma$ exists when $z^1$ and $z^2$ are action-angle coordinates, and since action-angle coordinates can always be found for a one degree-of-freedom system, $\gamma$ always exists—at least as long as $H$ or $k$ do not have explicit dependence on the parameter $s$. The last of Eqs. (2.20) then takes the simple form, $d(z^3 - \gamma)/ds = 0$.

In the example that we have chosen, we find

$$\gamma(z^1, z^2) = -\frac{z^1}{RB^0(z^1)} \int^{z^2} B^\psi(z^1, x)\, dx\,, \tag{2.25}$$

to which an arbitrary function of $z^1$ may be added. The equations of motion may now be written as

$$\dot{z}^i = J_c^{ij}\, \frac{\partial H}{\partial z^j}\,, \qquad i = 1, 2, 3 \tag{2.26}$$

where the only non-vanishing components of $J$ are $J^{12} = 1 = -J^{21}$ and the new $z^3$ denotes $(z^3_{\text{old}} - \gamma(z^1, z^2))$. (The other two $z$'s are the same; a new symbol for $z^3$ has been resisted for the sake of simple notation.) If a coordinate change is now made, $y^i(z^1, z^2, z^3)$, the cosymplectic form changes to

$$J^{i\ell} = \frac{\partial y^i}{\partial z^1}\frac{\partial y^\ell}{\partial z^2} - \frac{\partial y^i}{\partial z^2}\frac{\partial y^\ell}{\partial z^1},$$

which is antisymmetric and obeys the Jacobi requirement, but may very well have dependence on the $y$'s. Observe that $z^3(y^1, y^2, y^3)$ may now be regarded as a Casimir invariant, since

$$J^{i\ell}\frac{\partial z^3}{\partial y^\ell} = \frac{\partial y^i}{\partial z^1}\frac{\partial y^\ell}{\partial z^2}\frac{\partial z^3}{\partial y^\ell} - \frac{\partial y^i}{\partial z^2}\frac{\partial y^\ell}{\partial z^1}\frac{\partial z^3}{\partial y^\ell} = 0 \quad \text{for all } i\,. \tag{2.27}$$

# Chapter 3

# Symmetries in Hydrodynamics and Magnetohydrodynamics

In this chapter, we begin with a tabulation of symmetries of the Lagrangian actions of the ideal compressible fluid and MHD, and list the corresponding conservation laws in Lagrangian variables, that arise from Noether's (first) theorem. One of the symmetries is found to be generated by an *infinite* parameter continuous group, i.e. parametrized by *arbitrary functions* (Padhye and Morrison, 1996b). For such symmetries one has generalized Bianchi identities, or Noether's second theorem, in addition to the usual statement of Noether's first theorem; both theorems are presented in Sec. 3.1. The Lagrangian formulation of the ideal fluid and of MHD is introduced in Sec. 3.2.

Conservation laws in Lagrangian, or material, variables often have corresponding conservation laws in Eulerian variables. This correspondence is established in Sec. 3.3. In the two sections that follow it, Eulerian conservation laws are stated along with the Lagrangian conservation laws, wherever possible. In Sec. 3.4, usual symmetries of the ten-parameter Galilei group—infinitesimal time and space translations, space rotations, and Galilean boosts—are presented. These lead to the familiar conservation laws of energy, linear and angular momenta, and uniform motion of the center-of-mass.

*Relabeling symmetries* are of special interest in this chapter, and are presented in Sec. 3.5. These are continuous symmetry transformations of the

Lagrangian *labels* of fluid elements, or independent coordinates, which we refer to as *relabeling symmetries*, following the nomenclature introduced earlier (Salmon, 1988). A symmetry is found, generated by an infinite parameter continuous group, for the ideal fluid Lagrangian, giving rise to Ertel's theorem of conservation of potential vorticity (Pedlosky, 1979); the corresponding generalized Bianchi identity is also presented. Other finite dimensional relabeling symmetries are found, which give rise to helicity conservation in the fluid, and cross helicity conservation in barotropic MHD. A relabeling symmetry and the corresponding conservation law for a fluid which has one extra *label* or advected quantity, in addition to the entropy, is also presented.

The relabeling symmetry which gives rise to the conservation of helicity is not a point transformation, but has dependence on the gradient of the velocity. It also turns out that the symmetry of the infinite parameter group, that gives rise to Ertel's theorem, can be generalized (Padhye and Morrison, 1996a) and has a complicated dependence on mixed space and time derivatives of the fields. The generalized symmetry changes the velocity field, except *on orbit*, i.e. the velocity is unchanged only for solutions to the variational problem. The family of conservation laws that arise from this symmetry correspond to the fact that if the potential vorticity is advected, so is any function of the potential vorticity.

The first discussion of relabeling symmetries seems to have been made by Newcomb (1962), where they were called exchange symmetries. Since Lagrangian symmetries have been related to Kelvin's circulation theorem (Newcomb, 1967; Bretherton, 1970), it is pointed out, in Sec. 3.5.1, that the circulation theorem can be derived from Ertel's theorem. It is also pointed out that Weber's equations (Lamb, 1932) arise from a special choice of the relabeling symmetry which gives rise to Ertel's theorem. More recently, Ertel's theorem has been connected to fluid element relabeling (Ripa, 1981; Salmon, 1982), however the treatment here is more general than Ripa (1981) and differs from Salmon (1982). Ertel's theorem and the relabeling symmetry that gives rise to it has also been recently discussed in the context of oceanic flows (Muller, 1995). Conservation of cross helicity in barotropic MHD flows has previously been linked to Lagrangian symmetries (Calkin, 1963), but of a different Lagrangian and unrelated to relabeling.

In concluding Sec. 3.5, it is shown that the potential energy functional obtained by expanding about a stationary equilibrium possesses a Bianchi identity and relates to spontaneous symmetry breaking, which gives rise to null eigenfunctions.

We are concerned with the Hamiltonian framework in Sec. 3.6 and show that the map from Lagrangian variables to Eulerian variables for a fluid, possesses one of the relabeling symmetries seen in earlier sections. The noncanonical Poisson bracket for the fluid in Eulerian form (Morrison and Greene, 1980) has Casimir invariants, which we construct directly from the relabeling symmetry. This rounds out the usual picture of reduction from Lagrangian to Eulerian variables (Morrison, 1998; Marsden and Ratiu, 1994, for example). In the case of MHD, symmetries of the reduction from material to Eulerian variables give rise to Casimir invariants too, including a family of invariants which incorporates magnetic helicity as a special case.

## 3.1 Noether's Theorems

The action for a classical field theory may be written as

$$S[q] := \int_{\mathcal{D}} \mathcal{L}(q, \partial q, x)\, d^n x\,, \tag{3.1}$$

where $\mathcal{L}$ is the Lagrangian density and $q(x) = (q^1, q^2, \cdots, q^m)$ are the fields which depend on the variables $x = (x^0, x^1, \cdots, x^{n-1})$ and $\partial q$ denotes the derivatives of the fields with respect to the variables. The variable $x^0$ may be regarded as time and the others as space, and they belong to the domain $\mathcal{D}$.

Under point transformations (Utiyama, 1959; Sundermeyer, 1982, for example),

$$\hat{x}^i = \hat{x}^i(x)\,, \qquad \hat{q}^j = \hat{q}^j(q,x)\,, \tag{3.2}$$

the action transforms to

$$\hat{S}[\hat{q}] := \int_{\hat{\mathcal{D}}} \hat{\mathcal{L}}(\hat{q}, \hat{\partial}\hat{q}, \hat{x})\, d^n\hat{x} = S[q]\,, \tag{3.3}$$

where the second equality expresses covariance of the action and implies that the Lagrangian density must transform as

$$\hat{\mathcal{L}}(\hat{q}, \hat{\partial}\hat{q}, \hat{x}) = \frac{\partial(x)}{\partial(\hat{x})}\, \mathcal{L}(q, \partial q, x)\,, \tag{3.4}$$

where $\partial(x)/\partial(\hat{x})$ denotes the Jacobian of the transformation.

Furthermore we seek transformations that leave the form of the Euler-Lagrange equations invariant, i.e. we seek (a subset of) symmetry transformations. Evidently, for such transformations $\hat{S}[\hat{q}] = S[\hat{q}]$, which implies

$$\hat{\mathcal{L}}(\hat{q}, \hat{\partial}\hat{q}, \hat{x}) - \mathcal{L}(\hat{q}, \hat{\partial}\hat{q}, \hat{x}) = \partial_i \Lambda^i\,, \tag{3.5}$$

where $\Lambda$ is a vector with zero flux across the boundary of $\hat{\mathcal{D}}$. Such transformations, for which the Lagrangian density differs at most by a divergence, are called invariant transformations. In particular, if $\partial_0\Lambda^0 = 0$, i.e. the divergence is only spatial, the Lagrangian (which is the space integral of the Lagrangian density) is invariant and if the divergence term is altogether absent, the Lagrangian *density* is invariant.

We now consider invariant point transformations that have the following infinitesimal form:

$$\hat{x}^i = x^i + \delta x^i(x)\,, \qquad \hat{q}^j(\hat{x}) = q^j(x) + \Delta q^j(q,x)\,. \tag{3.6}$$

Derivatives of the fields change accordingly:

$$\Delta(\partial_j q^i) = \partial_j(\Delta q^i) - (\partial_j \, \delta x^k)(\partial_k q^i) \, , \tag{3.7}$$

where $\Delta(\partial_j q^i)$ is defined to be the first order piece of $\hat{\partial}_j \hat{q}^i - \partial_j q^i$. Finite transformations can be constructed by iteration of such infinitesimal ones.

Up to first order, $\delta x^j$ and $\Delta q^i$ may be considered functions of either the new or the old variables and the Jacobian may be written as

$$\frac{\partial(\hat{x})}{\partial(x)} = 1 + \partial_i \, \delta x^i \, . \tag{3.8}$$

The differential form of Eq. (3.5) is thus

$$\bar{\delta}\mathcal{L} + \mathcal{L} \, \partial_i \, \delta x^i + \partial_i \, \delta\Lambda^i = 0 \, , \tag{3.9}$$

where $\bar{\delta}\mathcal{L}$ is defined to be the first order piece of $\mathcal{L}(\hat{q}, \hat{\partial}\hat{q}, \hat{x}) - \mathcal{L}(q, \partial q, x)$ and $\Lambda^i$ is written as $\delta\Lambda^i$ to indicate that it is also of first order.

For convenience we define, to first order,

$$\delta q^i(x) := \hat{q}^i(x) - q^i(x) = \Delta q^i - \delta x^j \, \partial_j q^i \, . \tag{3.10}$$

Thus while $\delta q^i$ is the change in the field at a fixed point, $\Delta q^i$ is the change relative to a transformed point. Equation (3.9) may now be written as

$$\partial_i \, \delta J^i + S_i \, \delta q^i = 0 \, , \tag{3.11}$$

where $S_i$'s denote functional derivatives of the action with respect to $q^i$'s, that is,

$$S_i := \frac{\delta S}{\delta q^i} = \frac{\partial \mathcal{L}}{\partial q^i} - \partial_j \frac{\partial \mathcal{L}}{\partial(\partial_j q^i)} \tag{3.12}$$

and the current $\delta J^i$ is defined by,

$$\delta J^i := \mathcal{L} \, \delta x^i + \frac{\partial \mathcal{L}}{\partial(\partial_i q^j)} \, \delta q^j + \delta\Lambda^i \, . \tag{3.13}$$

### 3.1.1 First Theorem

We now note that when the equations of motion are satisfied, i.e. $S_i \equiv 0$ in Eq. (3.11), we are left with

$$\partial_i \, \delta J^i = 0 \,. \tag{3.14}$$

The conservation law expressed by Eq. (3.14) may be recognized as the usual expression of Noether's (first) theorem (Noether, 1918; Tavel, 1971, for an English translation).

### 3.1.2 Second Theorem

Another possibility (Noether, 1918) is to integrate Eq. (3.11) to get

$$\int_{\mathcal{D}} S_i \, \delta q^i \, d^n x = 0 \,. \tag{3.15}$$

Consider, for example, invariant transformations which have the form (Anderson and Bergmann, 1951)

$$\delta x^i = \varepsilon(x) \, \chi^i(x) \,, \quad \Delta q^i = \varepsilon(x) \, \phi^i(x) + \partial_j \varepsilon(x) \, \psi^{ij}(q, x) \,, \tag{3.16}$$

where $\varepsilon(x)$ is an infinitesimal, arbitrary function of $x$. (In general there may exist a set of independent symmetries, in which case one adds a subscript to the $\varepsilon$'s.)

For such transformations,

$$\int_{\mathcal{D}} S_i \, \delta q^i \, d^n x = \int_{\mathcal{D}} \varepsilon(x) \left[ S_i \phi^i - S_i (\partial_j q^i) \chi^j - \partial_j (S_i \psi^{ij}) \right] \, d^n x = 0 \,, \tag{3.17}$$

where we have used Eq. (3.10) to express $\delta q^i$ and integrated by parts to get rid of the derivative on $\varepsilon$, assuming that the boundary term vanishes. The arbitrariness of $\varepsilon$ allows us to choose it so that the boundary terms disappear.

And since the integral in Eq. (3.17) vanishes for arbitrary $\varepsilon(x)$, the Dubois-Reymond lemma implies

$$S_i\left[\phi^i - (\partial_j q^i)\chi^j\right] - \partial_j(S_i\psi^{ij}) = 0\,. \tag{3.18}$$

Note that when the equations of motion are satisfied the terms $S_i\left[\phi^i - (\partial_j q^i)\chi^j\right]$ and $\partial_j(S_i\psi^{ij})$ vanish separately (and trivially); this is replaced by the weaker condition, Eq. (3.18), when the equations of motion are not necessarily satisfied.

Equation (3.18), which depends crucially on $\varepsilon(x)$ being an arbitrary function of $x$ rather than a constant parameter, is an example of the identity of Noether's second theorem, also referred to as a generalized Bianchi identity. It is particularly interesting since it is satisfied independently of the equations of motion and its existence indicates that not all Euler-Lagrange equations of motion are independent. For this reason it is also called a *strong* conservation law as opposed to the *weak* conservation law expressed by Eq. (3.14) which requires the equations of motion.

It is also noteworthy that for such transformations, with an arbitrary $\varepsilon(x)$ as in Eq. (3.16), the weak conservation law itself splits into more than one statement. This follows from $\varepsilon(x)$ and its derivatives being independent, hence terms multiplying them must vanish independently.

## 3.2 Lagrangian Formulations of Hydrodynamics and Magnetohydrodynamics

Here, the Lagrangians of the ideal compressible fluid and MHD, and the resulting equations of motion in Lagrangian, as well as Eulerian, variables are introduced. In Sec. 3.3, the relation between Lagrangian and Eulerian conservation laws is clarified, paving the way for application of Noether's theorems to the ideal fluid and MHD in Secs. 3.4 and 3.5.

The variable $x^0$ of the previous section is replaced explicitly by time $\tau$, and three other components of $x$ are to be interpreted as the labels $a$ of the Lagrangian fluid elements; e.g. these could be the initial positions of the fluid elements, $q(\tau = 0)$. The variables $r = q(a;\tau)$ keep track of the position of the fluid element labeled $a$. At any time the mapping $q(a;\tau)$ between $r$ and $a$ is an invertible mapping of a domain $D$, and to simplify matters, $D$ is assumed time independent, although the fluid is compressible.

The fluid Lagrangian density $\mathcal{L}$ may be written as (Eckart, 1938; Herivel, 1955; Eckart, 1960; Newcomb, 1962):

$$\mathcal{L}(q, \dot{q}, \mathcal{J}, a) = \rho_0 \left[ \frac{1}{2}\dot{q}^2 - U\left(\frac{\rho_0}{\mathcal{J}}, s\right) - \Phi(q) \right], \tag{3.19}$$

where $\rho_0(a)$ is the initial density distribution and $\dot{q}$ denotes the time derivative of $q$, keeping the label fixed. The internal or potential energy per unit mass is denoted by $U$ and is assumed to be a function of two thermodynamic quantities, viz. the density $\rho_0/\mathcal{J}$ and the entropy $s$. Additional conservative forces on the fluid can be accounted for by including a potential $\Phi(q)$. We assume adiabaticity, so that $s = s(a)$ only. The entropy of a fluid element does not change in time; it is simply advected with the flow.

The volume element transforms as

$$d^3r = \mathcal{J}\, d^3a\,, \tag{3.20}$$

where $\mathcal{J}$ denotes the Jacobian $\partial(q)/\partial(a)$ of the transformation $q : a \to r$. Conservation of mass implies

$$\rho\, d^3r = \rho_0\, d^3a \quad \Rightarrow \quad \rho(r,t) = \rho_0(a)\, \mathcal{J}^{-1}(a;\tau)\Big|_{a=q^{-1}(r;t)}, \tag{3.21}$$

where $\rho(r,t)$ denotes the fluid density in the transformed coordinates. Mass conservation is thus built into the Lagrangian. Note that we have introduced the symbol $t \equiv \tau$ for later convenience in distinguishing between the spatial and label coordinates, i.e. $r, t$ and $a, \tau$ domains.

Note that $\rho_0$ can be eliminated by transforming the label space $a$ to $\bar{a}$ with Jacobian,

$$\frac{\partial(\bar{a})}{\partial(a)} = \rho_0(a)\,, \tag{3.22}$$

so that the volume measure changes to compensate for any change in the mass of the fluid element, thus maintaining unit density throughout the fluid volume. In these coordinates

$$\rho(r,t) = \mathcal{J}^{-1}(a;\tau)\Big|_{a=q^{-1}(r;t)}\,, \tag{3.23}$$

and the Lagrangian density is

$$\mathcal{L}(q,\dot{q},\mathcal{J},a) = \frac{1}{2}\dot{q}^2 - U(\mathcal{J}^{-1},s) - \Phi(q)\,, \tag{3.24}$$

where we have dropped the bar over $a$. In the rest of this chapter we assume that the label space has been chosen appropriately, so that Eq. (3.24) defines the fluid Lagrangian density.

The Lagrangian equations of motion are given by

$$S_i = -\ddot{q}_i - A_i^{\;j}\,\partial_j p - \frac{\partial\Phi}{\partial q^i} = 0\,, \tag{3.25}$$

where the pressure $p$ is defined by

$$p(\rho,s) = \rho^2\,\frac{\partial U}{\partial\rho}(\rho,s)\,, \tag{3.26}$$

and we have made use of

$$\frac{\partial\mathcal{L}}{\partial(\partial_i q^j)} = \rho^2\,\frac{\partial U}{\partial\rho}\,A_j^{\;i} = p\,A_j^{\;i}\,. \tag{3.27}$$

The Eulerian form of the equations of motion is obtained from Eq. (3.25) upon noting

$$\frac{\partial}{\partial\tau} = \frac{\partial}{\partial t} + \dot{q}^i\,\frac{\partial}{\partial r^i} = \frac{\partial}{\partial t} + v^i\,\frac{\partial}{\partial r^i}\,, \tag{3.28}$$

and, as noted in Eq. (A.4),

$$A_i^{\ j}\,\frac{\partial}{\partial a^j}=\mathcal{J}\,\frac{\partial}{\partial r^i}\,, \tag{3.29}$$

whereby we get the equations,

$$\frac{\partial v_i}{\partial t}+v^j\,\frac{\partial v_i}{\partial r^j}=-\frac{1}{\rho}\,\frac{\partial p}{\partial r^i}-\frac{\partial\Phi}{\partial r^i}\,. \tag{3.30}$$

In the equations above, the velocity $v(r,t)$ is defined by

$$v^i(r,t):=\dot{q}^i(a;\tau)\Big|_{a=q^{-1}(r;t)}\,, \tag{3.31}$$

where the right hand side is evaluated at $a=q^{-1}(r;t)$.

The Lagrangian density for MHD (Newcomb, 1962) has one additional term,

$$\mathcal{L}_{\mathrm{M}}=\frac{1}{2}\dot{q}^2-U(\mathcal{J}^{-1},s)-\Phi(q)-\frac{1}{2}\,\mathcal{J}^{-1}\,B_0^j\,\partial_j q^i\,B_0^k\,\partial_k q_i\,, \tag{3.32}$$

where $B_0^i(a)$ are components of the magnetic field as a function of the labels, e.g. the initial magnetic field. The magnetic field transforms as

$$B^i(r,t)=\mathcal{J}^{-1}\,B_0^j\,\partial_j q^i\Big|_{a=q^{-1}(r;t)}\,, \tag{3.33}$$

which has the inverse,

$$B_0^i=B^j\,A_j^{\ i}\Big|_{r=q(a;\tau)}\,. \tag{3.34}$$

Equation (3.33) ensures that magnetic flux density is conserved on the fluid elements. Note that surface elements transform as

$$d\sigma_i=A_i^{\ j}\,d\sigma_{0j} \tag{3.35}$$

where $d\sigma_{0j}$ denotes the area element in the $j$th direction in label space while $d\sigma_i$ denotes the area element in the $i$th direction in configuration space. The magnetic flux density thus transforms as

$$B^i\,d\sigma_i=\mathcal{J}^{-1}\,B_0^j\,\partial_j q^i\,A_i^{\ k}\,d\sigma_{0k}=B_0^k\,d\sigma_{0k}\,, \tag{3.36}$$

where we have used Eq. (A.6).

The Lagrangian equations of motion are given by

$$S_i = -\ddot{q}_i - A_i{}^j\,\partial_j p - \frac{\partial\Phi}{\partial q^i} + \partial_j\left(B_0^j\,\mathcal{J}^{-1}\,B_0^\ell\,\partial_\ell q_i\right)$$

$$- A_i{}^j\,\partial_j\left(\frac{1}{2}\,\mathcal{J}^{-2}\,B_0^k\,\partial_k q^m\,B_0^\ell\,\partial_\ell q_m\right) = 0\,, \quad (3.37)$$

where we have made use of

$$\frac{\partial\mathcal{L}_{\mathrm{M}}}{\partial(\partial_i q^j)} = p\,A_j{}^i + \frac{1}{2}\,\mathcal{J}^{-2}\,A_j{}^i\,B_0^k\,\partial_k q^m\,B_0^\ell\,\partial_\ell q_m - B_0^i\,\mathcal{J}^{-1}\,B_0^\ell\,\partial_\ell q_j\,. \quad (3.38)$$

Following the same procedure as in the case of hydrodynamics, the Eulerian form of the equations of motion is obtained from Eq. (3.37), yielding

$$\frac{\partial v_i}{\partial t} + v^j\,\frac{\partial v_i}{\partial r^j} = -\frac{1}{\rho}\,\frac{\partial p}{\partial r^i} - \frac{\partial\Phi}{\partial r^i} + \frac{1}{\rho}\,B^j\,\frac{\partial B_i}{\partial r^j} - \frac{1}{2\,\rho}\,\frac{\partial B^2}{\partial r^i}\,, \quad (3.39)$$

where the reader will recognize that the last two terms are the $i$th components of the $J \times B$ force, for $J = \nabla \times B$.

## 3.3 Relation of Eulerian to Lagrangian Conservation Laws

Consider any conservation law in Lagrangian coordinates, represented by

$$\frac{\partial I^0}{\partial\tau} + \frac{\partial I^i}{\partial a^i} = 0\,. \quad (3.40)$$

It is evident that under the transformation from the material labels $a, \tau$ to the spatial coordinates $r = q(a;\tau), t = \tau$, Eq. (3.40) does not *directly* transform to a conservation law. It can, however, be shown to correspond to an conservation law in $r, t$ coordinates:

$$\frac{\partial}{\partial t}(\rho\,\tilde{I}^0) + \frac{\partial}{\partial r^i}\left(\rho\,v^i\,\tilde{I}^0 + [A^{-1}]_j{}^i\,\tilde{I}^j\right) = 0\,, \quad (3.41)$$

where $\rho$ denotes $\mathcal{J}^{-1}$, $\tilde{I}^0(r,t) := I^0(a,\tau)$, and $\tilde{I}^j(r,t) := I^j(a,\tau)$, and $a = q^{-1}(r;t)$.

Note that we have carefully avoided using the term *Eulerian conservation law* to describe Eq. (3.41), since we reserve the term *Eulerian variables* for observable physical quantities, in the absence of the knowledge of $q(a;\tau)$. (Only local properties of $q(a;\tau)$, such as $v = \dot{q}$ and $\rho = \partial(q)/\partial(a)$, are observable, and part of the Eulerian description. Furthermore, these are known as functions of $r, t$, and not $a, \tau$.) For instance, an arbitrary function of the labels can be transformed to $r, t$ coordinates, but is not considered Eulerian. On the other hand, the function $s(a)$ is observable, hence $\tilde{s}(r,t) = s(a)$ is part of the Eulerian description. In Eq. (3.41), there is no guarantee that $\tilde{I}^0$ and $[A^{-1}]_j{}^i\,\tilde{I}^j$ have Eulerian representation, viz. in terms of $r$, $t$, $v(r,t)$, $s(r,t)$, $\rho(r,t)$, and their derivatives.

In the rest of this section, we prove that Eq. (3.40) leads to Eq. (3.41). Integration of Eq. (3.40) over the label domain leads to

$$\frac{d}{d\tau}\int_D I^0\, d^3a = 0\,, \tag{3.42}$$

assuming that the perpendicular component of the current $I$ vanishes on the boundary of the domain. The transformation from $a$ to $r$, with Jacobian $\mathcal{J} = \partial(q)/\partial(a)$, leads to a new representation for the invariant:

$$\frac{d}{dt}\int_D \rho\,\tilde{I}^0\, d^3r = 0\,, \tag{3.43}$$

where $\rho$ denotes $\mathcal{J}^{-1}$, consistent with Eq. (3.23). From Eq. (3.43), it follows that there exists a current $\tilde{J}(r,t)$ which satisfies

$$\frac{\partial}{\partial t}(\rho\,\tilde{I}^0) + \tilde{\nabla}\cdot\tilde{J} = 0\,, \tag{3.44}$$

and we assume that there is no net flux of the current on the boundary of the spatial domain. Our next task is to relate $\tilde{J}$ to $I^0$ and $I$, which necessitates proving several steps involving determinants and cofactors.

Under the transformation of coordinates from $a$ to $r$, Eq. (3.40) changes to

$$\frac{\partial \tilde{I}^0}{\partial t} + v^i \frac{\partial \tilde{I}^0}{\partial r^i} + \frac{\partial q^i}{\partial a^j} \frac{\partial \tilde{I}^j}{\partial r^i} = 0 \,, \tag{3.45}$$

where $\partial q^i / \partial a^j$ is evaluated at $a = q^{-1}(r;t)$. Straightforward application of the Leibniz rule to the first term in Eq. (3.44) yields,

$$\begin{aligned} \frac{\partial}{\partial t}(\rho \tilde{I}^0) &= \tilde{I}^0 \frac{\partial \rho}{\partial t} + \rho \frac{\partial \tilde{I}^0}{\partial t} \\ &= \tilde{I}^0 \frac{\partial \rho}{\partial t} - \rho v^i \frac{\partial \tilde{I}^0}{\partial r^i} - \rho \frac{\partial q^i}{\partial a^j} \frac{\partial \tilde{I}^j}{\partial r^i} \,, \end{aligned} \tag{3.46}$$

where we have made use of Eq. (3.45) to express $\partial \tilde{I}^0 / \partial t$ in terms of spatial derivatives.

The third term on the right hand side of Eq. (3.46), may now be rewritten as a divergence:

$$-\rho \frac{\partial q^i}{\partial a^j} \frac{\partial \tilde{I}^j}{\partial r^i} = -[A^{-1}]_j{}^i \frac{\partial \tilde{I}^j}{\partial r^i} = -\frac{\partial}{\partial r^i}\left([A^{-1}]_j{}^i \tilde{I}^j\right), \tag{3.47}$$

where we have used Eqs. (A.3) and (A.5) in the form applicable to the inverse transformation.

Next, we prove that

$$\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial r^i}(\rho v^i) \,, \tag{3.48}$$

which is the continuity equation in Eulerian variables, but is simply an identity from the Lagrangian point of view. We rewrite Eq. (3.48) as

$$\frac{\partial \rho}{\partial t} + v^i \frac{\partial \rho}{\partial r^i} = -\rho \frac{\partial v^i}{\partial r^i} \,, \tag{3.49}$$

so that the left hand side is recognized to be $\partial \mathcal{J}^{-1} / \partial \tau$. Using the chain rule on the right hand side yields,

$$\frac{\partial \mathcal{J}^{-1}}{\partial \tau} = -\mathcal{J}^{-1} \frac{\partial [q^{-1}]^j}{\partial r^i} \frac{\partial \dot{q}^i}{\partial a^j} \,, \tag{3.50}$$

which is equivalent to

$$\frac{\partial \mathcal{J}}{\partial \tau} = \mathcal{J}\, \frac{\partial [q^{-1}]^j}{\partial r^i} \frac{\partial \dot{q}^i}{\partial a^j} = A_j{}^i \, \frac{\partial \dot{q}^j}{\partial a^i}\,, \tag{3.51}$$

where Eq. (A.5) has been used in obtaining the right hand side. Equation (3.51) is identical to Eq. (A.8), which is derived in App. A, thus proving that Eq. (3.48) is an identity in Lagrangian coordinates. Collecting the results of Eqs. (3.47) and (3.48), Eq. (3.46) can be expressed in the form of a conservation law, reproducing Eq. (3.41).

## 3.4 Galilean Symmetries

The Lagrangian actions of the ideal fluid and MHD are invariant under the action of the ten-parameter Galilei group consisting of time and space translations, space rotations, and Galilean boosts. These symmetries give rise to the Lagrangian, as well as Eulerian, conservation laws of energy, linear and angular momenta, and uniform motion of the center-of-mass.

### 3.4.1 Energy Conservation

The transformation,

$$\begin{aligned} \delta \tau &:= \hat{\tau} - \tau = \varepsilon\,, \\ \delta a &:= \hat{a} - a = 0\,, \\ \Delta q &:= \hat{q}(\hat{a};\hat{\tau}) - q(a;\tau) = 0\,, \end{aligned} \tag{3.52}$$

implies the change in derivatives,

$$\Delta \dot{q} = 0\,, \quad \text{and} \quad \Delta \partial_j q^i = 0\,. \tag{3.53}$$

The change in $q$ at a fixed point is given by

$$\delta q^i = -\varepsilon\, \dot{q}^i\,, \tag{3.54}$$

and the transformed Lagrangian density satisfies,

$$\hat{\mathcal{L}}(\hat{q}, \hat{\partial}\hat{q}, \hat{a}) - \mathcal{L}(\hat{q}, \hat{\partial}\hat{q}, \hat{a}) = 0\,, \tag{3.55}$$

which shows that the transformation given by Eq. (3.52) leaves the Lagrangian density invariant. The conservation law, in Lagrangian variables, is given by

$$\varepsilon \frac{\partial}{\partial \tau}\left(\dot{q}^2 - \mathcal{L}\right) - \varepsilon \frac{\partial}{\partial a^i}\left(p\, \dot{q}^j\, A_j{}^i\right) = 0\,, \tag{3.56}$$

where we have used Noether's first theorem. Using the result of Sec. 3.3 we obtain the Eulerian conservation law,

$$\frac{\partial}{\partial t}\left[\rho\left(\frac{v^2}{2} + U(\rho, s) + \Phi(r)\right)\right] + \frac{\partial}{\partial r^i}\left[\rho\left(\frac{v^2}{2} + U(\rho, s) + \Phi(r) - p\right) v^i\right] = 0\,, \tag{3.57}$$

which is recognized to be the law of energy conservation.

The corresponding result for MHD is

$$\frac{\partial}{\partial \tau}\left(\dot{q}^2 - \mathcal{L}_{\mathrm{M}}\right) - \frac{\partial}{\partial a^i}\left[\left(p\, A_j{}^i - B_0^i\, \mathcal{J}^{-1}\, B_0^\ell\, \partial_\ell q_j + \frac{1}{2}\mathcal{J}^{-2}\, A_j{}^i\, B_0^k\, \partial_k q^m\, B_0^\ell\, \partial_\ell q_m\right) \dot{q}^j\right] = 0\,, \tag{3.58}$$

which has the Eulerian equivalent,

$$\frac{\partial}{\partial t}\left[\rho\left(\frac{v^2}{2} + U(\rho, s) + \Phi(r)\right) + \frac{B^2}{2}\right] + \tilde{\nabla}\cdot\left[\rho\left(\frac{v^2}{2} + U(\rho, s) + \Phi(r) - p\right) v + B\,(v \cdot B)\right] = 0\,. \tag{3.59}$$

### 3.4.2 Momentum Conservation

The transformation,

$$\delta\tau := \hat{\tau} - \tau = 0\,,$$

$$\delta a := \hat{a} - a = 0\,,$$

$$\Delta q^i := \hat{q}^i(\hat{a};\hat{\tau}) - q^i(a;\tau) = \varepsilon^i\,, \tag{3.60}$$

implies the change in derivatives,

$$\Delta \dot{q}^i = 0\,, \qquad \Delta(\partial_j q^i) = 0\,, \tag{3.61}$$

and the change in $q$ at a fixed point,

$$\delta q^i = \varepsilon^i\,. \tag{3.62}$$

The Lagrangian density transforms to $\hat{\mathcal{L}}(\hat{q}, \hat{\partial}\hat{q}, \hat{a}) = \mathcal{L}(q, \partial q, a)$, thus yielding

$$\hat{\mathcal{L}}(\hat{q}, \hat{\partial}\hat{q}, \hat{a}) - \mathcal{L}(\hat{q}, \hat{\partial}\hat{q}, \hat{a}) = \varepsilon^i\,\frac{\partial\phi}{\partial q^i}\,. \tag{3.63}$$

The right hand side vanishes if $\partial\phi/\partial q^i = 0$ for all $i$; if it is only true for one or two values of $i$, we set the $\varepsilon^i$ corresponding to the nonzero direction(s) to zero. Assuming this to be true, the transformation of Eq. (3.60) leaves the Lagrangian density invariant, and Noether's first theorem leads to the conservation law in Lagrangian variables,

$$\varepsilon^j\,\frac{\partial}{\partial\tau}\,\dot{q}_j + \varepsilon^j\,\frac{\partial}{\partial a^i}\,(p\,A_j{}^i) = 0\,. \tag{3.64}$$

Following Eq. (3.41), we are able to write the corresponding Eulerian conservation law,

$$\frac{\partial}{\partial t}\,(\rho\,v_j) + \frac{\partial}{\partial r^i}\,(\rho\,v^i\,v_j + p\,\delta_j{}^i) = 0\,, \tag{3.65}$$

which is recognized to be the law of conservation of momentum.

The corresponding result for MHD is

$$\frac{\partial}{\partial\tau}\,\dot{q}_j + \frac{\partial}{\partial a^i}\left(p\,A_j{}^i + \frac{1}{2}\,\mathcal{J}^{-2}\,A_j{}^i\,B_0^k\,\partial_k q^m\,B_0^\ell\,\partial_\ell q_m\right.$$

$$\left. - \, B_0^i\,\mathcal{J}^{-1}\,B_0^\ell\,\partial_\ell q_j\right) = 0\,, \quad (3.66)$$

which has the Eulerian equivalent,

$$\frac{\partial}{\partial t}\,(\rho\,v_j) + \frac{\partial}{\partial r^i}\left(\rho\,v^i\,v_j + (p + \frac{B^2}{2})\,\delta_j{}^i - B^i\,B_j\right) = 0\,. \quad (3.67)$$

### 3.4.3 Angular Momentum Conservation

The transformation,

$$\delta\tau := \hat{\tau} - \tau = 0\,,$$

$$\delta a := \hat{a} - a = 0\,,$$

$$\Delta q^i := \hat{q}^i(\hat{a};\hat{\tau}) - q^i(a;\tau) = \epsilon^{ijk}\,\varepsilon_j\,q_k\,, \quad (3.68)$$

which implies the change in derivatives,

$$\Delta\dot{q}^i = \epsilon^{ijk}\,\varepsilon_j\,\dot{q}_k\,, \qquad \Delta(\partial_\ell q^i) = \epsilon^{ijk}\,\varepsilon_j\,\partial_\ell q_k\,, \quad (3.69)$$

and the change in $q$ at a fixed point,

$$\delta q^i = \epsilon^{ijk}\,\varepsilon_j\,q_k\,. \quad (3.70)$$

The Lagrangian density transforms to $\hat{\mathcal{L}}(\hat{q},\hat{\partial}\hat{q},\hat{a}) = \mathcal{L}(q,\partial q,a)$, thus yielding

$$\hat{\mathcal{L}}(\hat{q},\hat{\partial}\hat{q},\hat{a}) - \mathcal{L}(\hat{q},\hat{\partial}\hat{q},\hat{a}) = \epsilon^{ijk}\,\varepsilon_j\left(-p\,A_i{}^\ell\,\partial_\ell q_k + q_k\,\frac{\partial\Phi}{\partial q^i}\right)\,. \quad (3.71)$$

The first term on the right hand side may be rewritten using $A_i{}^\ell\,\partial_\ell q_k = \mathcal{J}\,\delta_{ik}$, which, together with the antisymmetry of $\epsilon^{ijk}$, is seen to vanish. The second term vanishes if $\partial\Phi/\partial q^i = 0$ for all $i$, or if $\Phi = \Phi(q^2)$. Assuming this to be true, the transformation given by Eqs. (3.68) leaves the Lagrangian density invariant, and Noether's first theorem leads to the conservation law in Lagrangian variables,

$$\varepsilon_i \frac{\partial}{\partial \tau} (\epsilon^{ijk} q_j \dot{q}_k) + \varepsilon_i \frac{\partial}{\partial a^\ell} (p \, \epsilon^{ijk} q_j \, \delta_k{}^m A_m{}^\ell) = 0 \, . \tag{3.72}$$

Following Eq. (3.41), we are able to write the corresponding Eulerian conservation law,

$$\frac{\partial}{\partial t} (\rho \, \epsilon^{ijk} r_j v_k) + \frac{\partial}{\partial r^\ell} \left( \epsilon^{ijk} r_j (\rho \, v_k \, v^\ell + p \, \delta_k{}^\ell) \right) = 0 \, , \tag{3.73}$$

which is recognized to be the law of conservation of angular momentum.

The MHD counterpart to Eq. (3.71) has the following additional terms on the right hand side:

$$+\epsilon^{ijk} \varepsilon_j \left( \frac{1}{2} \mathcal{J}^{-2} B_0^m \partial_m q^s B_0^n \partial_n q_s A_i{}^\ell \partial_\ell q_k - \mathcal{J}^{-1} B_0^m \partial_m q_j B_0^\ell \partial_\ell q_k \right) , \tag{3.74}$$

which vanish due to the antisymmetry of $\epsilon^{ijk}$, and we get the following Lagrangian conservation law from Noether's first theorem:

$$\frac{\partial}{\partial \tau} (\epsilon^{ijk} q_j \dot{q}_k) + \frac{\partial}{\partial a^\ell} \left[ \epsilon^{ijk} q_j \left( p \, \delta_k{}^m A_m{}^\ell \right. \right.$$

$$\left. \left. + \frac{1}{2} \mathcal{J}^{-2} B_0^u \partial_u q^s B_0^n \partial_n q_s \, \delta_k{}^m A_m{}^\ell - \mathcal{J}^{-1} B_0^\ell B_0^n \partial_n q_k \right) \right] = 0 \, . \tag{3.75}$$

The corresponding Eulerian conservation law is given by

$$\frac{\partial}{\partial t} (\rho \, \epsilon^{ijk} r_j v_k) + \frac{\partial}{\partial r^\ell} \left[ \epsilon^{ijk} r_j \left( \rho \, v_k \, v^\ell + p \, \delta_k{}^\ell + \frac{B^2}{2} \delta_k{}^\ell - B_k B^\ell \right) \right] = 0 \, . \tag{3.76}$$

### 3.4.4 Galilean Boosts

The transformation,

$$\delta \tau := \hat{\tau} - \tau = 0 \, ,$$

$$\delta a := \hat{a} - a = 0 \, ,$$

$$\Delta q^i := \hat{q}^i(\hat{a}; \hat{\tau}) - q^i(a; \tau) = \varepsilon^i \, t \, , \tag{3.77}$$

implies the change in the derivatives,

$$\Delta\dot{q}^i = \varepsilon^i\,, \qquad \Delta(\partial_j q^i) = 0\,, \tag{3.78}$$

and the change in $q$ at a fixed point,

$$\delta q^i = \varepsilon^i\, t\,. \tag{3.79}$$

The Lagrangian density transforms to $\hat{\mathcal{L}}(\hat{q}, \hat{\partial}\hat{q}, \hat{a}) = \mathcal{L}(q, \partial q, a)$, thus yielding

$$\hat{\mathcal{L}}(\hat{q}, \hat{\partial}\hat{q}, \hat{a}) - \mathcal{L}(\hat{q}, \hat{\partial}\hat{q}, \hat{a}) = -\frac{\partial}{\partial\tau}\,(\varepsilon^i\, q_i) + t\,\varepsilon^i\,\frac{\partial\Phi}{\partial q^i}\,. \tag{3.80}$$

The right hand side is a divergence if $\partial\Phi/\partial q^i = 0$ for all $i$; if it is only true for one or two values of $i$, we set the $\varepsilon^i$ corresponding to the nonzero direction(s) to zero. Assuming this to be true, the transformation given by Eqs. (3.77) is an invariant transformation, and Noether's first theorem, with $\delta\Lambda^0 = -\varepsilon^i\, q_i$, leads to the conservation law in Lagrangian variables,

$$\varepsilon^j\,\frac{\partial}{\partial\tau}\,(q_j - \dot{q}_j\,\tau) - \varepsilon^j\,\frac{\partial}{\partial a^i}\,(p\,\tau\,A_j{}^i) = 0\,. \tag{3.81}$$

Following Eq. (3.41), we are able to write the corresponding Eulerian conservation law,

$$\frac{\partial}{\partial t}\,\rho\,(r_j - v_j\,t) + \frac{\partial}{\partial r^i}\left(\rho\,r_j\,v^i - t\,(\rho\,v^i\,v_j + p\,\delta_j{}^i)\right) = 0\,, \tag{3.82}$$

which is recognized to be the conservation law expressing uniform motion of the center of mass.

In the case of MHD, the Lagrangian conservation law is given by

$$\frac{\partial}{\partial\tau}\,(q_j - \dot{q}_j\,\tau) - \frac{\partial}{\partial a^i}\,\tau\left[A_j{}^i\left(p + \frac{1}{2}\,\mathcal{J}^{-2}\,B_0^u\,\partial_u q^s\,B_0^n\,\partial_n q_s\right)\right.$$
$$\left. - \,\mathcal{J}^{-1}\,B_0^i\,B_0^n\,\partial_n q_i\right] = 0\,, \tag{3.83}$$

which corresponds to the Eulerian conservation law,

$$\frac{\partial}{\partial t}\,\rho\,(r_j - v_j\,t) + \frac{\partial}{\partial r^i}\left[\rho\,r_j\,v^i\right.$$
$$\left. - \,t\,(\rho\,v^i\,v_j + p\,\delta_j{}^i + \frac{B^2}{2}\,\delta_j{}^i - B^i\,B_j)\right] = 0\,. \tag{3.84}$$

## 3.5 Relabeling Symmetries

Invariant transformations of the form,

$$\delta\tau := \hat{\tau} - \tau = 0\,,$$

$$\delta a := \hat{a} - a \neq 0\,,$$

$$\Delta q := \hat{q}(\hat{a};\hat{\tau}) - q(a;\tau) = 0\,, \tag{3.85}$$

are explored in this section. The functional dependence of $\delta a$ is not specified; it may depend on $a$, $q$, and derivatives of $q$. In the case of symmetries giving rise to the conservation of potential vorticity in fluids, conservation of cross helicity in MHD, and the conservation law for a fluid with an additional advected thermodynamic quantity, $\delta a$ depends on $a$ alone, while in the case of the generalized symmetry associated with potential vorticity, as well as the symmetry corresponding to helicity conservation in a fluid, $\delta a$ has a complicated dependence on first and mixed second derivatives of $q$ in time and label space.

Note that only the label coordinates $a$ are transformed in Eq. (3.85); each component of $q$ transforms as a scalar, hence the name *relabeling symmetry* arises. A relabeling transformation induces the change,

$$\Delta\dot{q}^i = -\delta\dot{a}^k\,\partial_k q^i\,, \qquad \Delta(\partial_j q^i) = -\partial_j\delta a^k\,\partial_k q^i\,, \tag{3.86}$$

in the derivatives of $q$, while the first order change in $q$ at a fixed point is given by

$$\delta q^i = -\delta a^k\,\partial_k q^i\,. \tag{3.87}$$

The transformed fluid Lagrangian density can be expressed using Eq. (3.4) and, up to first order, leads to

$$\hat{\mathcal{L}}(\hat{q},\hat{\partial}\hat{q},\hat{a}) - \mathcal{L}(\hat{q},\hat{\partial}\hat{q},\hat{a}) = -\left[\frac{1}{2}\dot{q}^2 - U - \Phi - \frac{1}{\mathcal{J}}\frac{\partial U}{\partial\rho}\right]\nabla\cdot\delta a$$
$$+\frac{\partial U}{\partial s}\delta a\cdot\nabla s + \dot{q}_i\,\delta\dot{a}\cdot\nabla q^i\,. \tag{3.88}$$

The MHD counterpart to Eq. (3.88) has the following additional terms on the right hand side:

$$\frac{1}{\mathcal{J}} \partial_j q^i \, \partial_k q_i \Big( B_0^j \, B_0^k \, \partial_\ell \delta a^\ell + B_0^j \, \delta a^\ell \, \partial_\ell B_0^k - B_0^j \, B_0^\ell \, \partial_\ell \delta a^k \Big) \, .$$

### 3.5.1 Potential Vorticity Conservation in the Ideal Fluid

From Eq. (3.88), it follows immediately that $\delta a$ which satisfies,

$$\nabla \cdot \delta a = 0 \, , \quad \delta a \cdot \nabla s = 0 \, , \quad \text{and} \quad \delta \dot{a} = 0 \, , \tag{3.89}$$

leaves the Lagrangian density invariant. These requirements assure that the relabeling does not alter the mass, lies within isentropic surfaces, and does not change the velocity field. They are met by

$$\delta a = \nabla s \times \nabla \varepsilon \, , \tag{3.90}$$

where $\varepsilon = \varepsilon(a)$ is an infinitesimal, arbitrary function of the label alone and hence is advected.

For this symmetry, Noether's first theorem, stated in Eqs. (3.13) and (3.14), yields the Lagrangian conservation law,

$$\frac{\partial}{\partial \tau} \left( \dot{q}_j \, \nabla q^j \cdot \nabla s \times \nabla \varepsilon \right) + \frac{\partial}{\partial a^i} \Big( (p - \rho \, \mathcal{L}) \, A_j{}^i \, \nabla s \times \nabla \varepsilon \cdot \nabla q^j \Big) = 0 \, . \tag{3.91}$$

The corresponding conservation law in $r, t$ coordinates is given by

$$\frac{\partial}{\partial t} \Big( v \cdot \tilde{\nabla} \tilde{s} \times \tilde{\nabla} \tilde{w} \Big) + \tilde{\nabla} \cdot \Big[ v \, (v \cdot \tilde{\nabla} \tilde{s} \times \tilde{\nabla} \tilde{w}) - \Big( \rho \frac{v^2}{2} - \rho U - \rho \Phi - p \Big) \tilde{\nabla} \tilde{s} \times \tilde{\nabla} \tilde{w} \Big] = 0 \, . \tag{3.92}$$

where the infinitesimal function $\varepsilon(a)$ has been replaced by a finite function $w(a) =: \tilde{w}(r, t)$, which is an arbitrary function of the label. Since it involves the arbitrary advected quantity $\tilde{w}(r, t)$, the conservation law expressed above is not Eulerian. Only physically meaningful quantities that can be measured are Eulerian; thus $\tilde{s}(r,t) = s(a)$ is Eulerian, but $\tilde{w}(r,t)$ is not. An example of a physically meaningful choice for $\tilde{w}(r,t)$ is dye concentration, assuming that the dye does not affect the dynamics in any way, and is simply advected with the flow.

The arbitrariness of $\varepsilon(a)$, which prevents Eq. (3.91) from having an equivalent Eulerian conservation law, proves to be beneficial in another way. Integration of Eq. (3.91) over the label space leads to vanishing of the divergence term, and integration by parts allows us to isolate $\varepsilon(a)$, giving

$$\frac{d}{dt}\int_D \varepsilon(a)\,\nabla\dot{q}_j\cdot\nabla q^j\times\nabla s\,d^3a = 0\,. \tag{3.93}$$

The arbitrariness of $\varepsilon(a)$ then leads us to the simple material conservation law,

$$\frac{\partial}{\partial\tau}Q_s = 0\,, \tag{3.94}$$

where $Q_s$ is the potential vorticity associated with the advected quantity $s$, and is defined, in terms of Lagrangian variables, by

$$Q_s := \nabla\dot{q}_j\cdot\nabla q^j\times\nabla s\,. \tag{3.95}$$

The Eulerian expression for the potential vorticity is obtained by using the chain rule to transform the derivatives in Eq. (3.95), yielding,

$$\begin{aligned}\tilde{Q}_s &= \epsilon^{\ell mn}\,\frac{\partial v_j}{\partial r^i}\,\frac{\partial q^i}{\partial a^\ell}\,\frac{\partial q^j}{\partial a^m}\,\frac{\partial q^k}{\partial a^n}\,\frac{\partial\tilde{s}}{\partial r^k}\\ &= \epsilon^{ijk}\,\mathcal{J}\,\frac{\partial v_j}{\partial r^i}\,\frac{\partial\tilde{s}}{\partial r^k}\\ &= \rho^{-1}\,\tilde{\nabla}\tilde{s}\cdot\tilde{\nabla}\times v\,,\end{aligned} \tag{3.96}$$

where we have used Eq. (A.7). The Lagrangian conservation law expressed by Eq. (3.94) is thus transformed to the Eulerian advection law,

$$\frac{\partial}{\partial t}\tilde{Q}_s + v\cdot\tilde{\nabla}\tilde{Q}_s = 0\,, \tag{3.97}$$

where Eq. (3.28) has been used to convert the partial time derivatives. Equation (3.97) is called Ertel's theorem of conservation of potential vorticity (Pedlosky, 1979), despite the fact that it is an advection law. It is, of course, closely related to the conservation law,

$$\frac{\partial}{\partial t}\rho\tilde{Q}_s + \tilde{\nabla}\cdot\rho v\tilde{Q}_s = 0\,, \tag{3.98}$$

which corresponds to Eq. (3.94).

Previously, the conservation of potential vorticity was derived from a (different) Lagrangian symmetry for incompressible stratified flows (Ripa, 1981), and from a constrained variational principle (Salmon, 1982). Relabeling symmetry has also been related to Kelvin's circulation theorem (Newcomb, 1967; Bretherton, 1970). One treatment (Bretherton, 1970) expresses the symmetry in terms of $q$ rather than $a$; we can easily make correspondence by using the one-to-one mapping between $q$ and $a$, which implies

$$\delta q = -\delta a\cdot\nabla q = \frac{\tilde{\nabla}\varepsilon\times\tilde{\nabla}s}{\rho}\,. \tag{3.99}$$

The use of relabeling symmetry seems to have been made first by Newcomb (1967) where a relabeling symmetry is found for an incompressible, ideal fluid without internal energy, $U$.

*Generalized Bianchi Identity*

The lack of arbitrariness in time of $\varepsilon(a)$, which arose due to the last condition of Eqs. (3.89) and can be traced to the kinetic energy term, prevents us from using Eq. (3.15) directly. (In essence this is because Hamilton's principle for particles is not parameterization invariant.) Instead we integrate the equivalent of Eq. (3.11) over the label space (not time) to get the generalized Bianchi identity

$$\frac{\partial}{\partial\tau}\left(\nabla\dot{q}_i\cdot\nabla q^i\times\nabla s\right) + \nabla S_i\cdot\nabla q^i\times\nabla s = 0\,, \tag{3.100}$$

It can be verified that the above equation is satisfied for *any* $q(a;\tau)$ by using the explicit form for $S_i$ given in Eq. (3.25), which leads to

$$\nabla p \times \nabla s \cdot \nabla \mathcal{J} + \nabla \cdot [\nabla \times (\Phi \nabla s)] = 0\,. \tag{3.101}$$

Each of the terms in the above equation vanish independently and identically; the first, because $p$ may be written as a function of $s$ and $\mathcal{J}$, and the second requires no explanation. When the equations of motion are satisfied $S_i \equiv 0$ and Eq. (3.100) reduces to Eq. (3.94), as might be expected.

As mentioned in Sec. 3.1, Eq. (3.100) shows that the equations of motion are not independent. Another interpretation is that there exist several other "equations of motion" for which the potential vorticity $Q_s$ is independent of time. An example is the case,

$$S_i = \sigma_i(s, q^i)\,, \qquad \text{(no sum on } i) \tag{3.102}$$

for which it is clear that Eq. (3.100) yields $\partial Q_s/\partial\tau = 0$. When the equations of motion are satisfied $S_i \equiv 0$ and Eq. (3.100) reduces to Eq. (3.94), as might be expected.

*Weber's equations*

Consider the case of barotropic flow, i. e. the Lagrangian density is defined as in Eq. (3.24), but the internal energy $U$ has no dependence on the entropy $s$. The only requirement of the relabeling symmetry in this case is that it be divergenceless, and is given by

$$\delta a = \nabla \times \varepsilon\,. \tag{3.103}$$

Specifically, pick the divergence-free, translation symmetry,

$$\delta a^i = \varepsilon^i\,, \tag{3.104}$$

where $\varepsilon^i$'s are constants. For this symmetry, $\delta q^i = -\varepsilon^j \, \partial_j q^i$, and Noether's first theorem yields,

$$\frac{\partial}{\partial \tau} \left( \dot{q}_j \, \partial_i q^j \right) + \frac{\partial}{\partial a^i} \left( p \, \mathcal{J} - \mathcal{L} \right) = 0 \,, \tag{3.105}$$

where we have used Eq. (A.6) to express $A_j{}^i \, \partial_k q^j = \mathcal{J} \, \delta_k{}^i$, and the independence of the $\varepsilon^i$'s has allowed us to separate the components. The time integral of Eqs. (3.105) are referred to as Weber's equations (Lamb, 1932, Sec. 15).

In the presence of the second thermodynamic variable, entropy $s$, the specific choice,

$$\delta a = \nabla s \times \varepsilon = \nabla \times (\varepsilon \, s) \,, \tag{3.106}$$

where $\varepsilon$ is a constant vector, satisfies Eqs. (3.89), thereby qualifying it to be an invariant transformation, and leads to the result

$$\left\{ \frac{\partial}{\partial \tau} \left( \dot{q}_j \, \nabla q^j \right) + \nabla (p \, \mathcal{J} - \mathcal{L}) \right\} \times \nabla s = 0 \,, \tag{3.107}$$

where we have used Noether's first theorem. Equations (3.107) express the fact that Weber's equations are satisfied on surfaces of constant entropy $s$, as may be expected.

*Generalized symmetry*

Since the potential vorticity $Q_s$ is advected, any function of $Q_s$ is also advected; therefore we expect a family of Lagrangian conservation laws associated with potential vorticity. Indeed there exists a symmetry more general than that expressed in Eq. (3.90), which gives rise to these conservation laws. The transformation,

$$\delta a(a, \partial q, \partial \dot{q}, \partial^2 q, \partial^2 \dot{q}) = \nabla s \times \nabla \varepsilon \,, \tag{3.108}$$

where $\varepsilon = \varepsilon(Q_s, a)$ is an infinitesimal, arbitrary function of its arguments and $Q_s$ is defined by Eq. (3.95), induces the change,

$$\Delta \dot{q}^i = -\nabla s \times \nabla \dot{\varepsilon} \cdot \nabla q^i \,, \quad \Delta(\partial_j q^i) = -\partial_j \delta a^k \, \partial_k q^i \,, \tag{3.109}$$

in the derivatives of $q$, and the change in $q$ at a fixed point is given by

$$\delta q^i = -\nabla s \times \nabla \varepsilon \cdot \nabla q^i \,. \tag{3.110}$$

The transformation of the Lagrangian density is expressed by

$$\hat{\mathcal{L}}(\hat{q}, \hat{\partial}\hat{q}, \hat{a}) - \mathcal{L}(\hat{q}, \hat{\partial}\hat{q}, \hat{a}) = \dot{q}_i \, \nabla s \times \nabla \dot{\varepsilon} \cdot \nabla q^i \,. \tag{3.111}$$

In order to show that the right hand side is a four-divergence, we write,

$$\dot{q}_i \, \nabla s \times \nabla \dot{\varepsilon} \cdot \nabla q^i = \nabla \cdot \left( \dot{\varepsilon} \, \dot{q}_i \, \nabla q^i \times \nabla s \right) - \dot{\varepsilon} \, \nabla \dot{q}_i \cdot \nabla q^i \times \nabla s \,, \tag{3.112}$$

where the second term is recognized to be $-\dot{\varepsilon}\, Q_s$. Next we observe that the definition of $\bar{\varepsilon}$,

$$\frac{\partial \bar{\varepsilon}}{\partial Q_s}(Q_s, a) := Q_s \, \frac{\partial \varepsilon}{\partial Q_s}(Q_s, a) \,, \tag{3.113}$$

enables us to express $\partial \bar{\varepsilon}/\partial \tau = \dot{\varepsilon}\, Q_s$, and we have the result,

$$\hat{\mathcal{L}}(\hat{q}, \hat{\partial}\hat{q}, \hat{a}) - \mathcal{L}(\hat{q}, \hat{\partial}\hat{q}, \hat{a}) = -\frac{\partial \bar{\varepsilon}}{\partial \tau} + \nabla \cdot \left( \dot{\varepsilon} \, \dot{q}_i \, \nabla q^i \times \nabla s \right) , \tag{3.114}$$

which shows that Eq. (3.108) defines an invariant transformation. The resulting conservation law in Lagrangian variables is given by

$$\begin{aligned} &\frac{\partial}{\partial \tau} \left( \bar{\varepsilon} + \dot{q}_i \, \nabla s \times \nabla \varepsilon \cdot \nabla q^i \right) \\ &\qquad + \nabla \cdot \left( (p\, \mathcal{J} - \mathcal{L}) \, \nabla s \times \nabla \varepsilon - \dot{\varepsilon} \, \dot{q}_i \, \nabla q^i \times \nabla s \right) = 0 \,, \end{aligned} \tag{3.115}$$

which results in the Eulerian advection law,

$$\frac{\partial}{\partial t} f(\tilde{Q}_s) + v \cdot \tilde{\nabla} f(\tilde{Q}_s) = 0 \,, \tag{3.116}$$

where $f(\tilde{Q}_s)$ is an arbitrary function of the potential vorticity.

*Kelvin's circulation theorem*

We now proceed to show the connection of Ertel's theorem to Kelvin's circulation theorem. Integrating Eq. (3.94) over a volume, $V$, fixed in label space and contained in the domain, $D$, and using Gauss' divergence theorem gives

$$\frac{d}{dt}\oint_{\Sigma} s\,\nabla q^i \times \nabla \dot{q}_i \cdot d\sigma_0 = 0\,, \tag{3.117}$$

where $\Sigma$ is the surface enclosing $V$ and $d\sigma_0$ is the infinitesimal surface element. Now if $V$ is chosen to be any volume sandwiched between parts of two surfaces of constant entropy separated by a small value, $\delta s$, the contribution to the integral from the sides is small and one has

$$\delta s\,\frac{d}{dt}\int_S \nabla q^i \times \nabla \dot{q}_i \cdot d\sigma_0 = 0\,, \tag{3.118}$$

where $S$ is a part of any surface of constant $s$. Using the second of Eqs. (3.20) we can thus write for non-zero $\delta s$

$$\frac{d}{dt}\int_{\tilde{S}} \tilde{\nabla} \times v \cdot d\sigma = 0\,. \tag{3.119}$$

In the above equation $d\sigma$ is an infinitesimal surface element in $r$ space and the isentropic surface $S$ which was fixed in label space is now considered to be an isentropic surface $\tilde{S}$ in $r$ space which evolves in time but is made up of the same fluid elements. Equation (3.119) is Kelvin's circulation theorem and is true on surfaces of constant entropy.

For a homentropic fluid, or equivalently for barotropic flows, instead of Eq. (3.91) we simply get

$$\frac{\partial}{\partial t}(\nabla \dot{q}_i \times \nabla q^i) = 0\,, \tag{3.120}$$

which implies

$$\frac{d\tilde{Q}_w}{dt} := \frac{d}{dt}\left(\frac{1}{\rho}\,\tilde{\nabla}\tilde{w} \cdot \tilde{\nabla} \times v\right) = 0 \tag{3.121}$$

for *any* advected quantity, $\tilde{w}(r,t) := w(a)$. It is thus quite clear that Kelvin's circulation theorem holds on *any* material surface for barotropic flows.

### 3.5.2 Helicity Conservation in the Barotropic Ideal Fluid

For barotropic flows, i.e. $U = U(\rho)$ only, the relabeling transformation,

$$\delta a = \varepsilon \, \nabla q^i \times \nabla \dot{q}_i \,, \tag{3.122}$$

where $\varepsilon$ is an infinitesimal parameter, leads to the transformation of the Lagrangian density:

$$\hat{\mathcal{L}}(\hat{q}, \hat{\partial}\hat{q}, \hat{a}) - \mathcal{L}(\hat{q}, \hat{\partial}\hat{q}, \hat{a}) = \varepsilon \, \dot{q}_k \, \nabla q^i \times \nabla \ddot{q}_i \cdot \nabla q^k \,. \tag{3.123}$$

After some manipulation, it can be shown that the right hand side in the above equation may be written as a four-divergence:

$$\begin{aligned} \varepsilon \, \dot{q}_k \, \nabla q^i \times \nabla \ddot{q}_i \cdot \nabla q^k = {} & \frac{\partial}{\partial t}\left(\frac{\varepsilon}{2} \dot{q}_k \nabla q^i \times \nabla \dot{q}_i \cdot \nabla q^k\right) \\ & + \nabla \cdot \left(\frac{\varepsilon}{2} \, q^k \, \nabla q^i \times \nabla(\dot{q}_k \, \ddot{q}_i) - \frac{\varepsilon}{2} \frac{\dot{q}^2}{2} \nabla q^i \times \nabla \dot{q}_i\right) . \end{aligned} \tag{3.124}$$

Note that the transformation has velocity dependence; it is not a point transformation.

The Lagrangian conservation law is seen to be

$$\begin{aligned} & \frac{\partial}{\partial \tau}(\dot{q}_k \, \nabla q^i \times \nabla \dot{q}_i \cdot \nabla q^k) \\ & - \nabla \cdot \left[\left(\frac{\dot{q}^2}{2} - 2 \frac{\partial}{\partial \rho} \rho U - 2\Phi\right) \nabla q^i \times \nabla \dot{q}_i + q^k \, \nabla q^i \times \nabla(\ddot{q}_i \, \dot{q}_k)\right] = 0 \,, \end{aligned} \tag{3.125}$$

which has the corresponding Eulerian conservation law,

$$\begin{aligned} & \frac{\partial}{\partial t}\left(v \cdot \tilde{\nabla} \times v\right) + \tilde{\nabla} \cdot \Big[v \, (v \cdot \tilde{\nabla} \times v) \\ & \qquad - \left(\frac{v^2}{2} - 2 \frac{\partial}{\partial \rho} \rho U - 2 \, \Phi(r)\right) \tilde{\nabla} \times v - r^k \, \tilde{\nabla} \times (\alpha \, v_k)\Big)\Big] = 0 \,, \end{aligned} \tag{3.126}$$

where

$$\alpha(r, t) := \frac{\partial v}{\partial t} + v \cdot \tilde{\nabla} v \,.$$

Integration over the spatial domain, assuming that the surface terms vanish, gives the helicity invariant:

$$\frac{d}{dt}\int_D v \cdot \tilde{\nabla} \times v \, d^3r = 0 \, . \tag{3.127}$$

### 3.5.3 Cross Helicity Conservation in Barotropic MHD

At the beginning of Sec. 3.5, it was noted that the MHD counterpart to Eq. (3.88) has the additional terms,

$$\frac{1}{\mathcal{J}}\partial_j q^i \, \partial_k q_i \Big( B_0^j \, B_0^k \, \partial_\ell \delta a^\ell + B_0^j \, \delta a^\ell \, \partial_\ell B_0^k - B_0^j \, B_0^\ell \, \partial_\ell \delta a^k \Big) \, ,$$

which vanish if $\delta a$ is any function of the labels multiplying $B_0$. But we also require that the conditions obtained previously, Eqs. (3.89), be satisfied; this leads to overspecification and consequently there exists no relabeling symmetry $\delta a$ that satisfies all the requirements. (It is for this reason that the potential energy functional for MHD does not exhibit spontaneous symmetry breaking, unlike the fluid case discussed in Sec. 3.5.5, and is thus reminiscent of the Higgs mechanism in quantum field theory.)

A solution can, however, be found for a barotropic flow, i.e. a flow for which $U$, and hence $p$, depend only on the density $\rho$. (A solution can also be found without imposing the restriction of barotropicity in the case where the entropy $s$ is a flux label, i.e. $B_0 \cdot \nabla s = 0$, but that is a very restrictive initial condition.) The second of Eqs. (3.89) is eliminated for barotropic MHD, which leads to the symmetry,

$$\delta a = \varepsilon(x_0, y_0) \, B_0 \, , \tag{3.128}$$

where $x_0(a)$ and $y_0(a)$ are flux labels. In other words, the initial magnetic field is expressible as $\nabla x_0 \times \nabla y_0$. However the existence of flux labels $x_0(a)$ and $y_0(a)$ is not crucial; if they do not exist one simply thinks of $\varepsilon$ as an infinitesimal constant parameter.

For this symmetry, Noether's (first) theorem yields the Lagrangian conservation law,

$$\frac{\partial}{\partial \tau}\left(\dot{q}_j\, B_0 \cdot \nabla q^j\right) + \nabla \cdot \left[B_0 \left(\frac{\dot{q}^2}{2} - \frac{d}{d\rho}\rho U - \Phi\right)\right] = 0\,, \qquad (3.129)$$

which has the corresponding Eulerian conservation law,

$$\frac{\partial}{\partial t}(v \cdot B) + \tilde{\nabla} \cdot \left[v\,(v \cdot B) + B\left(\frac{v^2}{2} - \frac{d}{d\rho}\rho U - \Phi(r)\right)\right] = 0\,. \qquad (3.130)$$

Integration over the spatial domain yields the invariant,

$$\frac{d}{dt}\int_D v \cdot B\, d^3 r = 0\,, \qquad (3.131)$$

which is commonly referred to as the cross helicity invariant. Prior to this work conservation of cross helicity was derived from a Lagrangian symmetry involving Clebsch potentials and the polarization (Calkin, 1963). See also (Lundgren, 1963).

### 3.5.4 Ideal Fluid with an Additional Label

Having seen in Sec. 3.5.1 that the relabeling symmetry is of an infinite parameter group, we might expect that a symmetry may exist even after the introduction of one more thermodynamic variable. Here, we introduce the advected quantity $c(a)$ as an additional thermodynamic quantity, which, for instance, may correspond to the concentration of a dissolved solute. The fluid Lagrangian density is given by

$$\mathcal{L}_c(q, \dot{q}, \mathcal{J}, a) = \frac{1}{2}\dot{q}^2 - U(\mathcal{J}^{-1}, s, c) - \Phi(q)\,, \qquad (3.132)$$

for which the change induced by a general relabeling transformation is expressed by the right hand side of Eq. (3.88), with the additional term $-(\partial U/\partial c)\,\delta a \cdot \nabla c$.

The relabeling must therefore be done on surfaces of constant $c$, as well as constant $s$, in order to ensure that the Lagrangian density is invariant, and is given by

$$\delta a = \varepsilon\, \nabla s \times \nabla c\,, \tag{3.133}$$

where $\varepsilon$ is a constant parameter. The symmetry is thus of a one parameter group; symmetry of the infinite parameter group no longer exists upon introduction of $c(a)$.

The Lagrangian conservation law that corresponds to the above symmetry is given by

$$\frac{\partial}{\partial \tau}\left(\dot{q}_j\, \nabla s \times \nabla c \cdot \nabla q^j\right) + \frac{\partial}{\partial a^i}\left((p - \rho\, \mathcal{L}_c)\, A_j{}^i\, \nabla s \times \nabla c \cdot \nabla q^j\right) = 0\,, \tag{3.134}$$

which has the Eulerian equivalent,

$$\frac{\partial}{\partial t}\left(v \cdot \tilde{\nabla}\tilde{s} \times \tilde{\nabla}\tilde{c}\right) + \tilde{\nabla} \cdot \Big[v\,(v \cdot \tilde{\nabla}\tilde{s} \times \tilde{\nabla}\tilde{c}) - \left(\rho\frac{v^2}{2} - \rho U - \rho\,\Phi - p\right) \tilde{\nabla}\tilde{s} \times \tilde{\nabla}\tilde{c}\Big] = 0\,. \tag{3.135}$$

On comparing Eq. (3.135) to Eq. (3.92), it may seem, at first glance, that they are identical, with $\tilde{c}$ replaced by $\tilde{g}$, however note that $U$ has dependence on $\tilde{c}$ in Eq. (3.135), while there is no dependence of $U$ on $\tilde{g}$ in Eq. (3.92).

### 3.5.5 Spontaneous Symmetry Breaking

In the stability analysis of stationary fluid equilibria (in particular MHD) one often considers the second variation of energy functionals. As an example consider the potential energy functional:

$$W := \int_D \Big[U(\rho, s) + \Phi(q)\Big]\, d^3a\,. \tag{3.136}$$

The equilibrium $q_e$ is considered to be an extremal point of $W$ and the second variation is checked for definiteness at the equilibrium. Noting that $W$ possesses the same symmetry as expressed earlier by Eq. (3.90) (but without any restriction on the time dependence since, here, the integral is only over space), leads to a generalized Bianchi identity:

$$\nabla\left(\frac{\delta W}{\delta q^i}\right)\times\nabla q^i\cdot\nabla s=0\,. \tag{3.137}$$

The functional derivatives of $W$, which are set to zero to obtain the extremal point, are thus not all independent of each other.

The existence of the symmetry also relates to spontaneous symmetry breaking and Goldstone's theorem (Abers and Lee, 1973, for example), which are concepts of field theory (Morrison and Eliezer, 1986, in the context of noncanonical Hamiltonian theory). We describe this for static equilibria, but a more general development exists. For the potential energy functional the analogue of (3.15) is

$$\delta_* W=\int_D\frac{\delta W}{\delta q^i}\delta_* q^i\,d^3a\equiv 0\,, \tag{3.138}$$

where $\delta_* q$ is given by Eq. (3.99). Taking a second variation of (3.138) yields

$$\delta_*^2 W=\int_D\left(\delta_* q^i\frac{\delta^2 W[q]}{\delta q^j\delta q^i}\cdot\delta q^j-\frac{\delta W}{\delta q^i}\frac{\partial\delta q^i}{\partial a^j}\delta_* a^j\right)d^3a\equiv 0\,, \tag{3.139}$$

where this second $\delta q$ is arbitrary and the dot indicates that the operator on the left acts on the quantity to the right. Evaluating (3.139) on an equilibrium point $q_e$ yields

$$\delta_*^2 W_e=\int_D\delta q^j\frac{\delta^2 W[q_e]}{\delta q^i\delta q^j}\cdot\delta_* q_e^i\,d^3a\equiv 0\,. \tag{3.140}$$

Since (3.140) vanishes for arbitrary $\delta q$, it follows that

$$\frac{\delta^2 W[q_e]}{\delta q^i\delta q^j}\cdot\delta_* q_e^i\equiv 0\,. \tag{3.141}$$

There are two ways to solve (3.141): either (i) $\delta_* q_e^i=-(\partial q_e^i/\partial a^j)\delta_* a^j=0$, which implies that the equilibrium point has the same relabeling symmetry as $W$ [a notably trivial case since $q_e(s_0(a))$], and no symmetry is broken, or

(ii) $\delta_* q_e^i \neq 0$, which implies that $\delta^2 W[q_e]/\delta q^i \delta q^j$ has $\delta_* q_e$ as a null eigenvector, and symmetry is "spontaneously broken." Observe that $\delta_* q_e$ is a zero frequency eigenfunction of the linearized equations of motion written in Lagrangian variables.

Since relabeling is a symmetry group, it is obvious that one can make a finite displacement from the equilibrium point and remain on the same level set of $W$. This can be seen by iterating the above variational procedure. For example, the next variation of (3.139) gives

$$\begin{aligned}\delta_*^3 W &= \int_D \left( \delta_* q^i \left[ \frac{\delta^3 W[q]}{\delta q^k \delta q^j \delta q^i} \cdot \delta q^j \right] \cdot \delta q^k - 2 \frac{\partial \delta q^i}{\partial a^j} \delta_* a^j \frac{\delta^2 W[q]}{\delta q^k \delta q^i} \cdot \delta q^k \right) d^3 a \\ &\equiv 0\,, \end{aligned} \tag{3.142}$$

which when evaluated on $q_e$ yields

$$\delta_*^3 W_e = \int_D \delta_* q^i \left[ \frac{\delta^3 W[q_e]}{\delta q^k \delta q^j \delta q^i} \cdot \delta q^j \right] \cdot \delta q^k \, d^3 a \equiv 0\,. \tag{3.143}$$

This procedure is analogous to Taylor expanding a potential energy function about an equilibrium of a finite system that lies in a trough. This was worked out explicitly to all orders for the special case of toroidal geometry in (Ilgisonis and Pastukhov, 1995). Although in terms of Lagrangian variables the equilibria that are connected by the relabeling transformation are distinct, it is evident by the definition of relabeling that in the Eulerian description these equilibria are identical.

## 3.6 Symmetries of the Euler-Lagrange Map

In this section, we directly construct the Casimir invariants of the noncanonical Hamiltonian formulations of hydrodynamics and MHD (Morrison and Greene, 1980), from the symmetries of the reduction map from Lagrangian to Eulerian variables.

### 3.6.1 Ideal Fluid Casimir Invariants

The Hamiltonian density for the ideal, compressible fluid, in terms of Lagrangian variables, is given by

$$
\begin{aligned}
H[\pi, q] &:= \int_D \mathcal{H}(\pi, q, \partial q, a)\, d^3a \\
&:= \int_D \left[\frac{\pi^2}{2} + U(\mathcal{J}, s) + \Phi(q)\right] d^3a\,, \qquad (3.144)
\end{aligned}
$$

which together with the canonical Poisson bracket,

$$
[F, G] = \int_D \left[\frac{\delta F}{\delta q}\cdot\frac{\delta G}{\delta \pi} - \frac{\delta G}{\delta q}\cdot\frac{\delta F}{\delta \pi}\right] d^3a\,, \qquad (3.145)
$$

produces the ideal fluid equations of motion.

The transformation,

$$
\begin{aligned}
\hat{a} &= a + \delta a(a)\,, \\
\Delta q &:= \hat{q}(\hat{a};\tau) - q(a;\tau) \equiv 0\,, \\
\Delta \pi &:= \hat{\pi}(\hat{a};\tau) - \pi(a;\tau) \equiv 0\,, \qquad (3.146)
\end{aligned}
$$

leads to the change in derivatives of $q$ given by $\Delta(\partial_j q^i) = -\partial_j \delta a^k\, \partial_k q^i$, and the first order change in $q$ and $\pi$ at a fixed point,

$$
\delta q^i = -\delta a^j\, \partial_j q^i \qquad \text{and} \qquad \delta \pi_i = -\delta a^j\, \partial_j \pi_i\,. \qquad (3.147)
$$

The first order change in the Hamiltonian density is expressed by

$$
\begin{aligned}
\hat{\mathcal{H}}(\hat{\pi}, \hat{q}, \partial\hat{q}, \hat{a}) - \mathcal{H}(\hat{\pi}, \hat{q}, \partial\hat{q}, \hat{a}) &:= \frac{\partial(a)}{\partial(\hat{a})}\mathcal{H}(\pi, q, \partial q, a) - \mathcal{H}(\hat{\pi}, \hat{q}, \partial\hat{q}, \hat{a}) \\
&= -(\mathcal{H} + p\,\mathcal{J})\,\nabla\cdot\delta a - \frac{\partial U}{\partial s}\,\delta a\cdot\nabla s\,. \qquad (3.148)
\end{aligned}
$$

It is clear that the Hamiltonian is invariant for the relabeling symmetry, viz. that given by Eq. (3.90). Thus for the same Poisson bracket in terms of the new variables, the form of the equations of motion is left unaltered under such a relabeling.

The existence of this symmetry of the Hamiltonian density indicates that one may be able to obtain an alternative formulation of the dynamics in terms of variables which inherently possess this symmetry. This is indeed the case for the reduction (Morrison, 1998; Marsden and Ratiu, 1994, and references therein) to Eulerian variables, which is conveniently represented by the following:

$$\begin{aligned}\rho(r,t) &:= \int_D \delta(r - q(a;\tau))\, d^3a\,,\\ \sigma(r,t) &:= \int_D s(a)\, \delta(r - q(a;\tau))\, d^3a\,,\\ M(r,t) &:= \int_D \pi(a,\tau)\, \delta(r - q(a;\tau))\, d^3a\,.\end{aligned} \tag{3.149}$$

When one considers variations of the Eulerian variables $\rho$, $\sigma$, and $M$, that are *induced by relabeling*, we see that

$$\begin{aligned}\delta\rho &= \int_D \nabla\cdot\delta a\, \delta(r - q(a;\tau))\, d^3a\,,\\ \delta\sigma &= \int_D \left[s\,\nabla\cdot\delta a + \delta a\cdot\nabla s\right]\, \delta(r - q(a;\tau))\, d^3a\,,\\ \delta M &= \int_D \pi\,\nabla\cdot\delta a\, \delta(r - q(a;\tau))\, d^3a\,.\end{aligned} \tag{3.150}$$

The conditions for vanishing of these variations, together with the constraint, $\pi = \dot{q}$, are the same as those of Eqs. (3.89). Thus the relabeling given by Eq. (3.90) is also a symmetry of the map from Lagrangian to Eulerian variables.

In the framework resulting from the reduction to Eulerian variables, we are naturally interested in functionals $F[q,\pi]$ which can be expressed in terms of the Eulerian variables $\tilde{F}[\rho,\sigma,M]$. Evidently, this is not possible for all $F[q,\pi]$; but note that $\tilde{F}[\rho,\sigma,M]$ has the relabeling symmetry mentioned above since

$\rho$, $\sigma$, and $M$ have it. Therefore, at the very least, one demands that $F[q,\pi]$ display the same symmetry.

This consideration gives rise to a scheme for obtaining Casimir invariants, special invariants that arise in the Eulerian framework, from knowledge of the symmetry. Since the variation of $F$ must vanish when the variations $\delta_* q$ and $\delta_* \pi$ arise from the relabeling symmetry, $\delta_* a$, we demand

$$\delta_* F = \int_D \left[\frac{\delta F}{\delta q} \cdot \delta_* q \; + \frac{\delta F}{\delta \pi} \cdot \delta_* \pi\right] d^3a = 0\,. \tag{3.151}$$

It is clear that if there exists a functional $C_1[q,\pi]$, such that

$$\delta_* q = -\frac{\delta C_1}{\delta \pi} \qquad \text{and} \qquad \delta_* \pi = \frac{\delta C_1}{\delta q}\,, \tag{3.152}$$

its Poisson bracket with any $F$ belonging to the class of functionals satisfying Eq. (3.151), vanishes. This will be the case when the Poisson bracket is expressed in terms of Eulerian, noncanonical variables (Morrison and Greene, 1980) and therefore, by definition, $C_1$ is a Casimir invariant. Obviously, Casimir invariants are constants of motion for any dynamics with a Hamiltonian that can be expressed in terms of Eulerian variables.

As might be expected from Eq. (3.93), and easily checked, the functional $C_1$ defined by

$$C_1[q,\pi] := \int_D \varepsilon(a)\, \nabla \pi_i \cdot \nabla q^i \times \nabla s \, d^3a\,, \tag{3.153}$$

is the generator of the symmetry, i.e. it satisfies

$$[C_1, q^i] = -\frac{\delta C_1}{\delta \pi_i} = -\delta_* a \cdot \nabla q^i =: \delta_* q^i \tag{3.154}$$

and

$$[C_1, \pi_i] = \frac{\delta C_1}{\delta q^i} = \Delta_* \pi_i - \delta_* a \cdot \nabla \pi_i =: \delta_* \pi_i\,. \tag{3.155}$$

The Eulerian expression for the Casimir invariants $C_1$ yields

$$\tilde{C}_1[\rho, s, v] = \int_D \rho\, \mathcal{C}_1(\tilde{Q}_s)\, d^3r\,, \tag{3.156}$$

where $\mathcal{C}_1(\tilde{Q}_s)$ is arbitrary and $s(r,t) := \sigma(r,t)/\rho(r,t) = s(a = q^{-1}(r;t))$.

Evidently, the Poisson bracket of a functional $C_2[q,\pi]$, with *any* $F$ also vanishes if

$$\frac{\delta C_2}{\delta q} = 0 = \frac{\delta C_2}{\delta \pi}\,. \tag{3.157}$$

This is true when the integrand of $C_2$ is an arbitrary function of the labels and independent of $q$ and $\pi$. There exists no Eulerian representation for most such functionals; however

$$C_2[s] := \int_D \mathcal{C}_2(s)\, d^3a \tag{3.158}$$

does have an Eulerian representation, where $\mathcal{C}_2(s)$ is arbitrary.

Combining Eqs. (3.156) and (3.158), we arrive at a general expression for the Casimir invariants in Eulerian form:

$$\tilde{C}[\rho, s, v] := \int_D \rho\, \mathcal{C}(s, \tilde{Q}_s)\, d^3r\,, \tag{3.159}$$

where $\mathcal{C}$ is an arbitrary function of either argument.

In the noncanonical Hamiltonian formulation of the fluid, a Casimir has to satisfy the conditions:

$$\tilde{\nabla}\cdot\left(\rho\frac{\delta \tilde{C}}{\delta M}\right) = 0\,, \quad \frac{\delta \tilde{C}}{\delta M}\cdot\tilde{\nabla}\left(\frac{\sigma}{\rho}\right) = 0 \quad \text{and} \tag{3.160}$$

$$M_j\,\tilde{\nabla}\frac{\delta \tilde{C}}{\delta M_j} + \frac{\delta \tilde{C}}{\delta M}\cdot\tilde{\nabla}\left(\frac{M}{\rho}\right) + \rho\,\tilde{\nabla}\frac{\delta \tilde{C}}{\delta \rho} + \sigma\,\tilde{\nabla}\frac{\delta \tilde{C}}{\delta \sigma} = 0\,. \tag{3.161}$$

The equivalence of these conditions to the symmetry conditions, Eqs. (3.89), is seen when one notes that if $\tilde{C}$ can be expressed as a functional of $\rho$, $\sigma$, and $M$, then

$$\frac{\delta C}{\delta \pi} = \frac{\delta \tilde{C}}{\delta M} \quad \text{and} \quad \frac{\delta C}{\delta q} = \mathcal{J}\left[\rho\,\tilde{\nabla}\frac{\delta \tilde{C}}{\delta \rho} + \sigma\,\tilde{\nabla}\frac{\delta \tilde{C}}{\delta \sigma} + M_i\,\tilde{\nabla}\frac{\delta \tilde{C}}{\delta M_i}\right]. \tag{3.162}$$

The use of Eqs. (3.154) and (3.155) then leads to Eqs. (3.160) and (3.161) when $\delta a$ satisfies Eqs. (3.89) and vice versa. Note that for Casimirs satisfying Eq. (3.157), the conditions reduce to a simpler form,

$$\frac{\delta \tilde{C}_2}{\delta M} = 0 = \rho \tilde{\nabla} \frac{\delta \tilde{C}_2}{\delta \rho} + \sigma \tilde{\nabla} \frac{\delta \tilde{C}_2}{\delta \sigma} . \tag{3.163}$$

For barotropic flow, the potential vorticity corresponding to any advected quantity $w(a)$ is also advected, as expressed in Eq. (3.121). Therefore one can use $Q_w$ to generate yet another advected quantity, $Q_{Q_w}$ and so on; from one advected quantity we can generate an infinite family of advected quantities. Thus the Casimir has the form

$$C[\rho, w, v] = \int_D \rho\, \mathcal{C}(w, \tilde{Q}_w, \tilde{Q}_{\tilde{Q}_w}, \cdots)\, d^3 r\,, \tag{3.164}$$

where $\mathcal{C}(w, \tilde{Q}_w, \tilde{Q}_{\tilde{Q}_w}, \cdots)$ is an arbitrary function of its arguments.

### 3.6.2 MHD Casimir Invariants

The discussion in the previous section leads us to expect the existence of Casimirs, in the Hamiltonian formulation, which satisfy Eq. (3.157) and which may be expressible in terms of $\rho$, $s$, $v$, and $B$. It is easily verified that $B \cdot \tilde{\nabla} \tilde{w}/\rho = B_0 \cdot \nabla w$, where $\tilde{w}(r,t) := w(a)$ is an arbitrary advected quantity, and leads to the Eulerian expression:

$$C[\rho, s, B] := \int_D \rho\, \mathcal{C}_1 \left[ s, \frac{B}{\rho} \cdot \tilde{\nabla}(s), \frac{B}{\rho} \cdot \tilde{\nabla}\left( \frac{B}{\rho} \cdot \tilde{\nabla}(s) \right), \cdots \right] d^3 r\,, \tag{3.165}$$

where $\mathcal{C}_1$ is an arbitrary function of its arguments. This form for the Casimirs is given in (Henyey, 1982); we obtain a more general expression next.

The Lagrange-Euler map for the magnetic field,

$$B^i(r,t) := \int_D B_0^j(a)\, \frac{\partial q^i}{\partial a^j}\, \delta(r - q(a;\tau))\, d^3 a\,, \tag{3.166}$$

and its corresponding vector potential representation,

$$A_i(r, t; A_0, q) = \int_D A_{0j}(a) \frac{\partial a^j}{\partial q^i} \delta(r - q(a; \tau)) \, \mathcal{J} \, d^3 a \,, \tag{3.167}$$

lead to the conclusion that $A \cdot B/\rho = A_0 \cdot B_0$, within a gauge restriction. We note that in Eq. (3.167), one may add to $A_0(a)$, the gradient of a gauge, $\phi_0(a, t)$, which leads to a corresponding gauge choice, $\phi(r, t) := \phi_0(q^{-1}(r; t), t)$, for $A(r, t)$. But for the validity of $A \cdot B/\rho = A_0 \cdot B_0$, we must restrict the gauge to be advected, $\phi(r, t) := \phi_0(q^{-1}(r; t))$, which is equivalent to demanding that all explicit time dependence be removed from $A_0$. With this choice it can be seen that the vector potential in Eulerian coordinates satisfies the equation

$$\frac{\partial A}{\partial t} = v \times B - \tilde{\nabla}(A \cdot v) \,. \tag{3.168}$$

This gauge choice and the corresponding invariant has been discussed previously (Gordin and Petviashvili, 1987), but not arrived at from an argument about the symmetry of the Euler-Lagrange map.

Thus, more generally, the Casimir invariants are expressed by

$$C[\rho, s, A] := \int_D \rho \, \mathcal{C}\Big[ s, \frac{A \cdot B}{\rho}, \frac{B}{\rho} \cdot \tilde{\nabla}(s), \frac{B}{\rho} \cdot \tilde{\nabla}\Big(\frac{A \cdot B}{\rho}\Big), \frac{B}{\rho} \cdot \tilde{\nabla}\Big(\frac{B}{\rho} \cdot \tilde{\nabla}(s)\Big), \cdots \Big] \, d^3 r \,, \tag{3.169}$$

where $B$ is understood to be an abbreviation for $\tilde{\nabla} \times A$. Operating within the restricted choice of gauges mentioned earlier, we note that the addition of a gauge, $A \to A + \tilde{\nabla}\phi$, changes $A \cdot B/\rho$ by the term $B \cdot \tilde{\nabla}\phi/\rho$, which is also advected. The numerical value of $C[\rho, s, A]$ thus depends on the gauge, but after the initial choice of the gauge has been made, it nevertheless is a constant of the motion. It is clear that magnetic helicity $\int A \cdot B \, d^3 a$ is a special case of this family of invariants.

For the barotropic case, the Casimir is written most generally as

$$C[\rho, v, A] = \int_D \rho \left[ \frac{v \cdot B}{\rho} + \mathcal{C}\Big( \frac{A \cdot B}{\rho}, \frac{B \cdot \tilde{\nabla}}{\rho} \Big( \frac{A \cdot B}{\rho} \Big), \cdots \Big) \right] d^3 r \,, \tag{3.170}$$

where $\mathcal{C}$ is an arbitrary function of its arguments. In the case where flux labels exist globally, the Casimir is given by

$$C[v, x, y] = \int_D f(x, y)\, v \cdot \nabla x \times \nabla y \, d^3 r \,, \tag{3.171}$$

where $f$ is an arbitrary function of the flux labels, $x(r,t) := x_0(q^{-1}(r;t))$ and $y(r,t) := y_0(q^{-1}(r;t))$.

# Chapter 4

# Lyapunov Stability of Ion-Acoustic Waves

The existence of ion-acoustic solitary and periodic wave solutions has been known for some time, and has also been observed in laboratory plasmas. Ion-acoustic equations, which are introduced in the next section, have also been known to reduce to the KdV equation in the limit of small amplitude and unit Mach number. The Lyapunov stability of KdV solitons has attracted much attention and has been established (Benjamin, 1972; Bona, 1975), however not much attention has been paid to the Lyapunov stability of wave solutions of the ion-acoustic fluid, which is the focus of attention in this chapter.

The noncanonical Hamiltonian formulation of ion-acoustic waves is presented in the next section, and Galilean invariance is shown. Solitary and periodic wave solutions are presented in Sec. 4.2, which are shown to correspond to the critical point of a free energy functional $F$ in Sec. 4.3. The sign of $\delta^2 F$ is proved to be indefinite by explicit construction of perturbations, thereby preventing the establishment of Lyapunov stability. Perturbations with short wavelength, or strong curvature, give rise to negative $\delta^2 F$. The numerical study of the evolution of such negative energy perturbations is the subject of Ch. 5, where it is found that the perturbation is initially amplified, but saturates, and the solitary wave peak passes through, without being disrupted.

The effect of inclusion of ionic pressure, which obeys an adiabatic law, is discussed in the latter half of this chapter. The noncanonical Hamiltonian formulation is presented in Sec. 4.4, and it is found in Sec. 4.5 that the system can be explicitly solved for solitary and periodic wave solutions, when the adiabatic exponent $\gamma$ equals three, which corresponds to a one degree-of-freedom system. Interestingly, solitary wave solutions exist at all wave speeds above one, and do not have an upper cutoff, unlike the case without ionic pressure. In the study of Lyapunov stability in Sec. 4.6, we find that negative energy perturbations still exist, except for solutions with vanishing amplitude.

Lyapunov stability of ion-acoustic solitary waves is proved in the KdV limit in Sec. 4.7 without having to deal with the harder problem of proving KdV soliton stability.

## 4.1 Ion-Acoustic Equations

Ion-acoustic equations are applicable to plasmas with a cold ion component surrounded by hot electrons. The equations are composed of a continuity equation, a momentum equation, and Poisson's equation for the electric potential:

$$\begin{aligned} \frac{\partial n}{\partial t} + \frac{\partial}{\partial x}(nv) &= 0\,, \\ \frac{\partial v}{\partial t} + \frac{\partial}{\partial x}\left(\frac{v^2}{2} + \frac{e\,\phi}{m}\right) &= 0\,, \\ \frac{1}{4\pi e}\frac{\partial^2 \phi}{\partial x^2} &= n_0 \exp\left(\frac{e\,\phi}{T_e}\right) - n\,, \end{aligned} \tag{4.1}$$

where $n\,, v$, and $m$ are ion number density, velocity, and mass respectively. Electronic thermal energy is denoted by $T_e$, and $n_0$ is a constant corresponding to the electron and ion number densities in regions of zero electric field. The electric potential is denoted $\phi$.

In Eqs. (4.1), note that the Poisson equation assumes electron density $n_e := n_0 \exp(e\,\phi/T_e)$. This expression for the electron density arises from the electron momentum equation,

$$\frac{\partial v_e}{\partial t} + \frac{\partial}{\partial x}\left(\frac{v_e^2}{2} - \frac{e\,\phi}{m_e}\right) = -\frac{1}{m_e\,n_e}\frac{\partial p_e}{\partial x}\,, \tag{4.2}$$

where $p_e$ is the electron pressure. Since the electron mass $m_e$ is small, we neglect the change of electron momentum, which leaves us with a simple balance of electric force and pressure,

$$e\,\frac{\partial \phi}{\partial x} = \frac{1}{n_e}\frac{\partial p_e}{\partial x}\,. \tag{4.3}$$

Assuming that the equation of state is given by $p_e = n_e\,T_e$, and the electrons are isothermal, we integrate Eq. (4.3) to get $n_e = n_0\,\exp(e\,\phi/T_e)$.

Equations (4.1) can be rewritten in a tidier fashion by defining dimensionless dependent and independent variables:

$$\bar{x} := k_D\,x\,, \quad \bar{t} := \omega_P\,t\,, \quad \bar{n} := \frac{n}{n_0}\,, \quad \bar{v} := \frac{v}{c_0}\,, \quad \bar{\phi} := \frac{e\,\phi}{T_e}\,. \tag{4.4}$$

The natural scales of the ion-acoustic system appear in the above equations – $k_D$ is the inverse Debye length, $\omega_P$ is the ionic plasma frequency, and $c_0$ is the ion-acoustic speed:

$$k_D^2 := \frac{4\,\pi\,e^2\,n_0}{T_e}\,, \quad \omega_P^2 := \frac{4\,\pi\,e^2\,n_0}{m}\,, \quad c_0^2 := \frac{T_e}{m}\,. \tag{4.5}$$

Dropping the bars over the newly defined dimensionless variables, Eqs. (4.1) are recast:

$$\begin{aligned} \frac{\partial n}{\partial t} + \frac{\partial}{\partial x}(nv) &= 0\,, \\ \frac{\partial v}{\partial t} + \frac{\partial}{\partial x}\left(\frac{v^2}{2} + \phi\right) &= 0\,, \\ \frac{\partial^2 \phi}{\partial x^2} &= \mathrm{e}^{\phi} - n\,. \end{aligned} \tag{4.6}$$

Henceforth the term *ion-acoustic equations* will refer to Eqs. (4.6).

The name *ion-acoustic* has its origin in the dispersion relation that arises on linearizing the equations around a uniform background (Chen, 1984; Nicholson, 1983):

$$\omega^2 = \frac{k^2}{1+k^2}\,. \tag{4.7}$$

For small $k$ the dispersion relation is approximately given by $\omega^2 \approx k^2$, which is identical to the dispersion relation for sound waves in a neutral gas. Although the rapidly moving electrons shield the ions, charge neutrality is not perfect. This leads to compression and rarefaction of ion density caused by quasi-electrostatic interaction between ion bunches. Furthermore, comparision of the ion-acoustic speed $c_0^2 = T_e/m$ to the sound speed $c_s^2 = \gamma p/\rho$ shows that inertia is provided by the ion mass, while electrons provide the restoring pressure. Experimental verification of ion-acoustic waves was first accomplished with the aid of a Q-machine (Wong et al., 1964).

### 4.1.1 Hamiltonian Formulation

The Hamiltonian for the ion-acoustic system is given by

$$H[n,v] := \int_D \left( n\frac{v^2}{2} + \frac{\phi_x^2}{2} + (\phi - 1)\exp\phi \right) dx\,, \tag{4.8}$$

where $\phi_x$ denotes $\partial\phi/\partial x$. The noncanonical Hamiltonian formulation (see Ch. 2 for definition) is completed by defining a Poisson bracket, which for the ion-acoustic system is given by (Morrison, 1982),

$$\{\mathcal{F},\mathcal{G}\} := \int_D \left( \frac{\delta\mathcal{G}}{\delta n}\frac{\partial}{\partial x}\frac{\delta\mathcal{F}}{\delta v} - \frac{\delta\mathcal{F}}{\delta n}\frac{\partial}{\partial x}\frac{\delta\mathcal{G}}{\delta v} \right) dx\,. \tag{4.9}$$

Note that $\phi$ is not independently varied, but is related to $n$ by the Poisson equation in Eqs. (4.6), so that the first order variation of $H[n,v]$ induced by

$\delta n$ is given by

$$\delta H[\delta n;\delta\phi] = \int_D \left(\frac{v^2}{2}\,\delta n + \phi_x\,\delta\phi_x + \phi\,\mathrm{e}^{\phi}\,\delta\phi\right)dx$$
$$= \int_D \left(\frac{v^2}{2}\,\delta n + \phi\left(-\delta\phi_{xx} + \mathrm{e}^{\phi}\,\delta\phi\right)\right)dx\,, \tag{4.10}$$

where we have used integration by parts and assumed that $\delta\phi_x$ vanishes on the boundary of $D$. From the first order variation of the Poisson equation in Eqs. (4.6), recognizing that

$$\delta n = \mathrm{e}^{\phi}\,\delta\phi - \delta\phi_{xx}\,, \tag{4.11}$$

we get the result,

$$\delta H[\delta n;\delta\phi] = \int_D \left(\frac{v^2}{2} + \phi\right)\delta n\,dx \quad \Rightarrow \quad \frac{\delta H}{\delta n} = \frac{v^2}{2} + \phi\,. \tag{4.12}$$

It is then readily verified that time evolutions of $n$ and $v$ given by

$$n_t = \{n, H\} \quad \text{and} \quad v_t = \{v, H\} \tag{4.13}$$

reproduce the first two of Eqs. (4.6).

### 4.1.2 Invariants

We now proceed to enumerate known invariants associated with the ion-acoustic equations. The first two ion-acoustic equations are already in the form of conservation laws, which immediately leads to the invariants,

$$N := \int_D n\,dx \quad \text{and} \quad U := \int_D v\,dx\,, \tag{4.14}$$

which correspond to the total ion number and velocity respectively. The boundary terms,

$$n\,v|_{\partial D} \quad \text{and} \quad \left.\frac{v^2}{2} + \phi\right|_{\partial D}\,,$$

have been assumed to vanish in arriving at the invariants. On infinite domains with non-zero asymptotic values of velocity and density, the invariants must be defined carefully in order to be finite. For example, if the ion number density and velocity tend to values $n_\infty$ and $v_\infty$ at $\pm\infty$, the above definitions of $N$ and $U$ must be renormalized to

$$N := \int_{-\infty}^{+\infty} (n - n_\infty)\, dx\,, \quad \text{and} \quad U := \int_{-\infty}^{+\infty} (v - v_\infty)\, dx\,. \tag{4.15}$$

Evolution of the momentum density $n\,v$, as determined by the ion-acoustic equations, is expressed by the equation,

$$\frac{\partial}{\partial t}(n\,v) + \frac{\partial}{\partial x}(n\,v^2) + n\,\frac{\partial \phi}{\partial x} = 0\,, \tag{4.16}$$

which is seen to be a conservation law upon noting that the Poisson equation in Eqs. (4.6) allows us to express

$$n\,\frac{\partial \phi}{\partial x} = \frac{\partial}{\partial x}\left(\mathrm{e}^{\phi} - \frac{\phi_x^2}{2}\right)\,. \tag{4.17}$$

The corresponding invariant is the total momentum,

$$P := \int_D n\,v\,dx\,, \tag{4.18}$$

where, again, we have assumed that surface terms vanish.

There is in fact another invariant, corresponding to a Galilean boost, in which time appears explicitly,

$$I := \int_D (x - v\,t)\,n\,dx\,, \tag{4.19}$$

and the conservation law corresponding to it is

$$\frac{\partial}{\partial t}\Big(n\,(x - v\,t)\Big) + \frac{\partial}{\partial x}\left(n\,v\,(x - v\,t) - t\,\mathrm{e}^{\phi} + t\,\frac{\phi_x^2}{2}\right) = 0\,. \tag{4.20}$$

It is interesting to note that the current in the above conservation law has terms linear in time; for $I$ to be invariant in time, the current must vanish at the (spatial) boundaries.

Finally, we note that the antisymmetry of the bracket given by Eq. (4.9) and the absence of explicit time dependence of the Hamiltonian given by Eq. (4.8), ensure that it is invariant. Specifically, the conservation law is

$$\frac{\partial}{\partial t}\left(n\frac{v^2}{2}+\frac{\phi_x^2}{2}+(\phi-1)\mathrm{e}^{\phi}\right)+\frac{\partial}{\partial x}\left(n\,v\left(\frac{v^2}{2}+\phi\right)-\phi\,\phi_{x\,t}\right)=0\,. \tag{4.21}$$

*Casimir Invariants*

It is noteworthy that in the Hamiltonian formulation, the invariants $N$ and $U$, given by Eq. (4.14), commute with any functional, i. e. $\{N,\mathcal{G}\}$ and $\{V,\mathcal{G}\}$ vanish for *any* $\mathcal{G}$, and are hence Casimir invariants. (See Ch. 2 for the relation between noncanonical Hamiltonian formulation and existence of Casimir invariants.)

### 4.1.3 Galilean Invariance

Anticipating that we will switch perspectives between inertial frames of reference, we note that upon making the transformation, $v = u + c$, where $c$ is a constant, the ion-acoustic equations change to

$$\frac{\partial n}{\partial t}+\frac{\partial}{\partial x}(nu)+c\frac{\partial n}{\partial x}=0 \quad \text{and} \quad \frac{\partial u}{\partial t}+\frac{\partial}{\partial x}\left(\frac{u^2}{2}+\phi\right)+c\frac{\partial u}{\partial x}=0\,, \tag{4.22}$$

while the Poisson equation is unchanged. Transformation of the independent variables to $(\xi,\tau) := (x - c\,t, t)$, leads to transformed partial derivatives:

$$\frac{\partial}{\partial t}=\frac{\partial}{\partial \tau}-c\frac{\partial}{\partial \xi} \qquad \text{and} \qquad \frac{\partial}{\partial x}=\frac{\partial}{\partial \xi}\,.$$

It is thus clear that the extra term which appears in each of Eqs. (4.22) is cancelled by the transformation of the independent variables, and we are left with

$$\frac{\partial n}{\partial \tau}+\frac{\partial}{\partial \xi}(nu)=0 \quad \text{and} \quad \frac{\partial u}{\partial \tau}+\frac{\partial}{\partial \xi}\left(\frac{u^2}{2}+\phi\right)=0\,, \tag{4.23}$$

where $n(\xi, \tau)$ and $u(\xi, \tau)$ transform as scalars. The ion-acoustic equations are thus Galilean invariant.

On the other hand, the boundary conditions are not Galilean invariant. For example, in the case of solitary wave solutions that we will discuss shortly, the "laboratory frame" is the one in which the ion velocity is zero at infinity, however in the frame moving at the solitary wave speed $c$, the ion velocity is $-c$ at infinity.

## 4.2 Solitary and Periodic Wave Solutions

As might be expected from their name, ion-acoustic equations indeed possess travelling wave solutions, which we review in this section (Davidson, 1972; Krall and Trivelpiece, 1973, for example). The assumption that all quantities depend only on $x - ct$, or equivalently, depend only on $x$ in the frame moving with the wave speed $c$, leads to simple integration of the continuity and momentum equations, giving algebraic equations. The Poisson equation transforms to a nonlinear ordinary differential equation:

$$n\,u + \alpha = 0\,, \qquad u^2 = 2(\beta - \phi)\,, \qquad \phi'' = \mathrm{e}^{\phi} - n\,, \tag{4.24}$$

where the prime denotes differentiation with respect to $x$, and $\alpha\,, \beta$ are arbitrary constants. Note that we have also transformed to the inertial frame moving with the wave, where the velocity of ions $u(x,t)$ is related to their velocity in the lab frame $v(x + ct, t)$, by $v(x + ct, t) = u(x,t) + c$. This is the natural frame for our analysis.

It is evident from the second of Eqs. (4.24) that $\phi$ is restricted to $\phi \leq \beta$ for $u$ to be real. Furthermore, the direction of $u$ cannot be determined from Eqs. (4.24) alone. Without loss of generality, we pick positive sign for $\alpha$, which implies that in order for the ion number density, $n = -\alpha/u$, to be non-negative, it is necessary that $u$ be negative. In other words, ion movement is unidirectional in the frame of the wave.

Choice of the sign of $\alpha$ having been made, we express $n$ in terms of $\phi$ and rewrite the Poisson equation:

$$\phi'' = \mathrm{e}^{\phi} - \frac{\alpha}{\sqrt{2(\beta - \phi)}} =: -\frac{dV}{d\phi}(\phi)\,. \tag{4.25}$$

Equation (4.25) is recognized to be analogous to the case of single particle dynamics in a time independent potential. The electric potential $\phi$ is analogous to the spatial coordinate of the particle and the spatial variable $x$ is analogous to the role of time in this analogy. The potential $V(\phi)$ can be expressed explicitly:

$$V(\phi) = -\mathrm{e}^{\phi} - \alpha\sqrt{2(\beta - \phi)}\,. \tag{4.26}$$

We are interested in real valued $V(\phi)$, therefore it is well defined only for $\phi \leq \beta$. This analogy to particle dynamics was first made for magnetosonic waves and later for ion-acoustic waves (Davis et al., 1958; Sagdeev, 1966).

Analogous to the conservation of the sum of kinetic and potential energies for particle dynamics in a potential, here we have the result,

$$E := \frac{1}{2}(\phi')^2 + V(\phi) = \text{constant}\,. \tag{4.27}$$

The above equation arises from Eq. (4.25) after multiplication by $\phi'$ and integration over the $x$ domain.

The physical picture we have in mind is an undisturbed background of uniform density. Therefore we restrict our attention to wave solutions satisfying $u = -c$ and $n = 1$ for all values of $x$ at which $\phi = 0$. Note that the requirement $u = -c$ ensures that the velocity of ions $v$ in the laboratory frame, vanishes at infinity. This specification fixes the values of $\alpha$ and $\beta$:

$$\alpha = c \qquad \text{and} \qquad \beta = \frac{c^2}{2}\,. \tag{4.28}$$

In what follows, we restrict our attention to solutions with the above values of $\alpha$ and $\beta$.

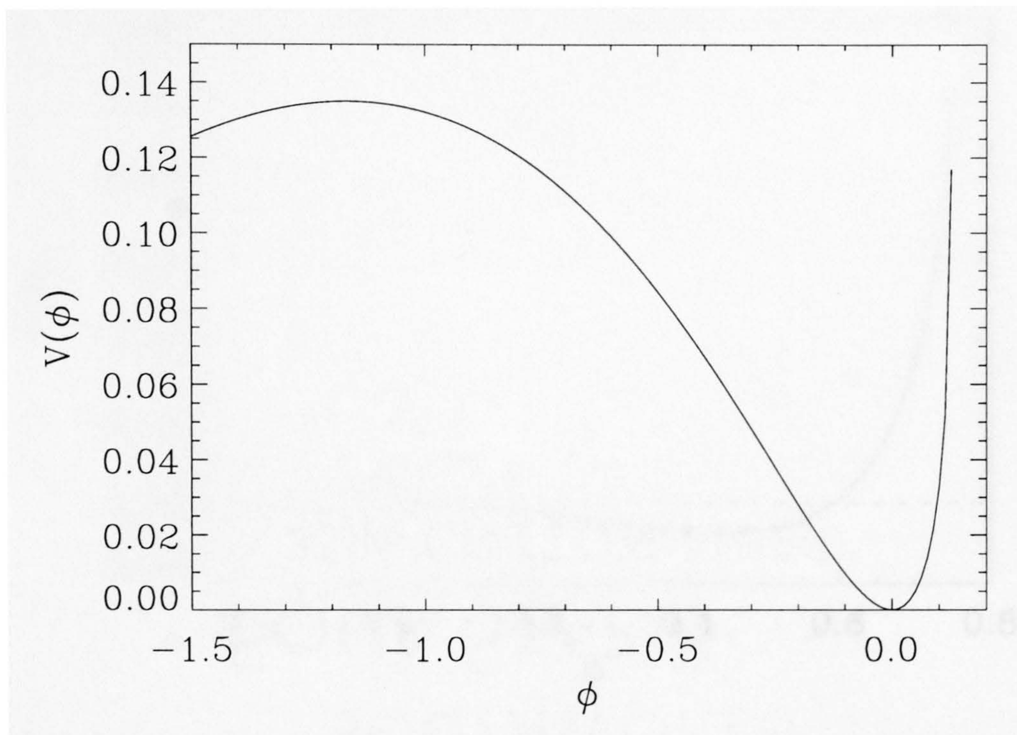


Figure 4.1: $V(\phi)$ for $c = 0.5$ has a minimum at the origin, showing the existence of periodic waves.

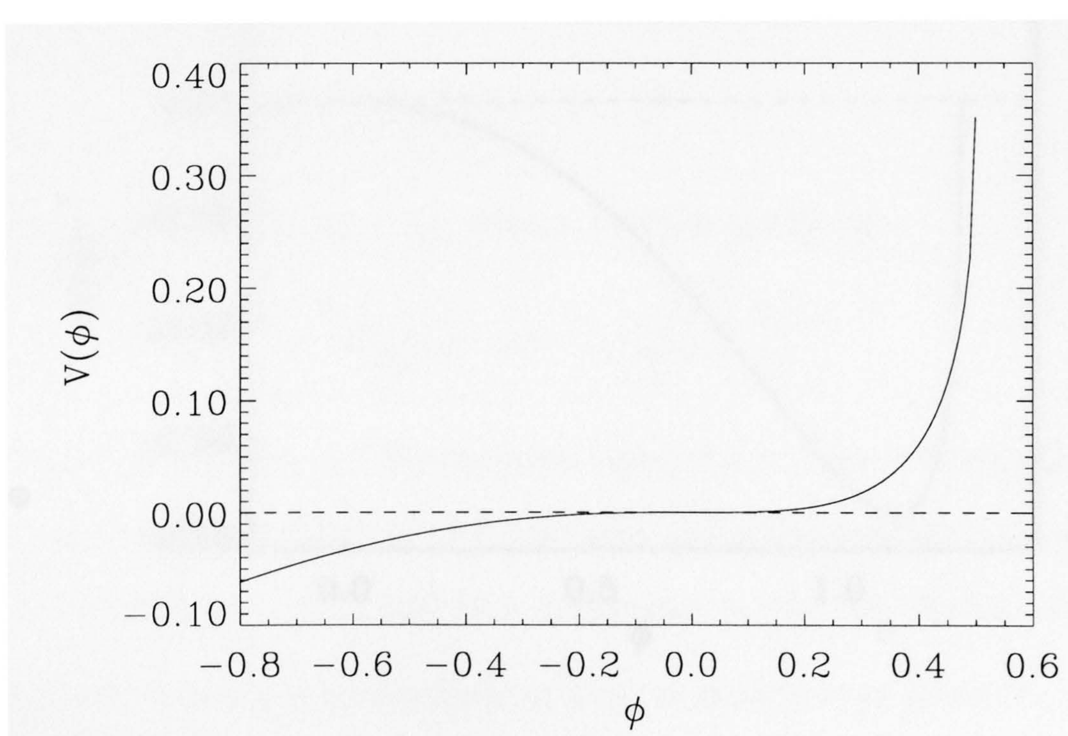


Figure 4.2: $V(\phi)$ for $c = 1$. The origin is an inflexion point.

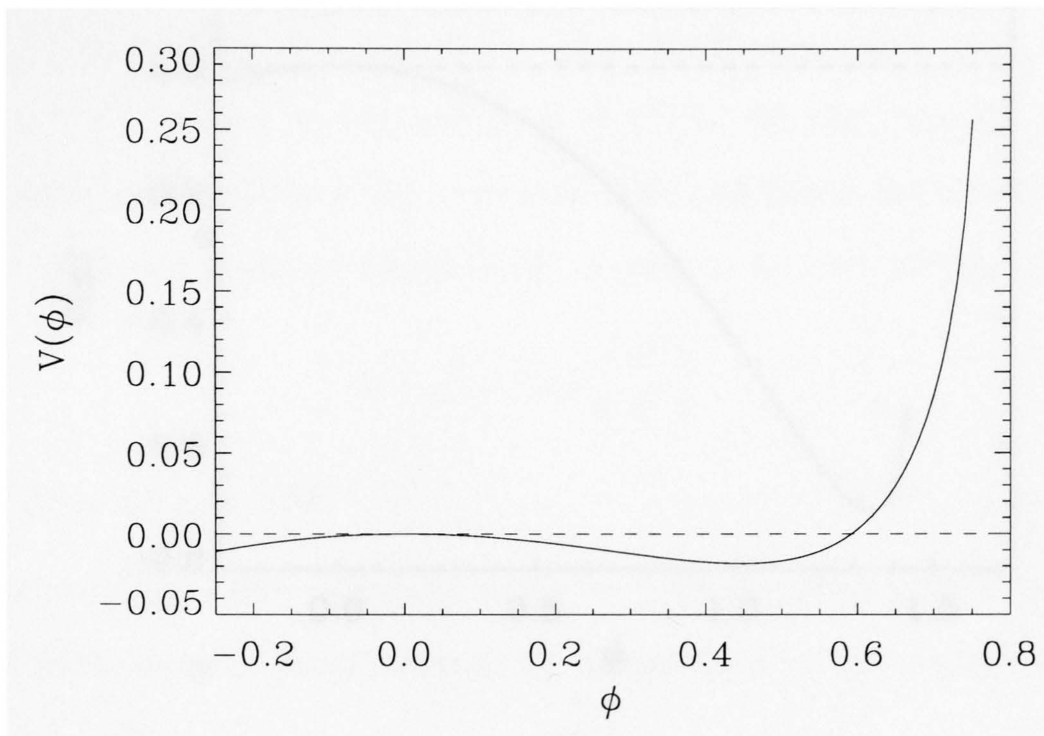


Figure 4.3: $V(\phi)$ for $c = 1.23$, showing the existence of a solitary wave with amplitude $\approx 0.6$. The origin is a maximum point of $V(\phi)$.

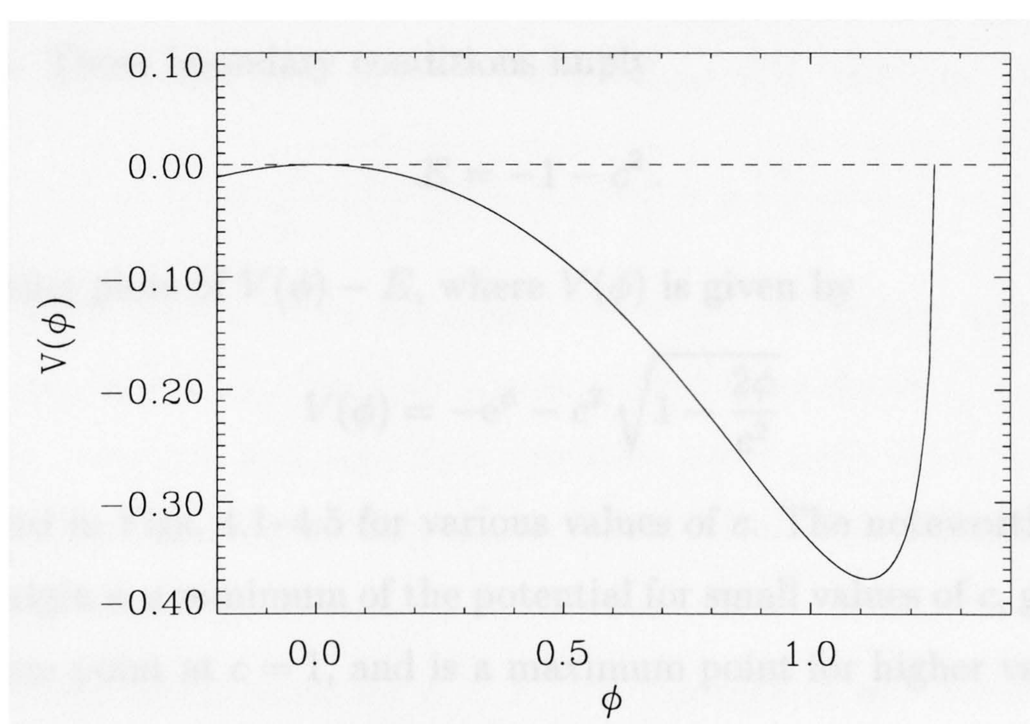


Figure 4.4: $V(\phi)$ for $c = 1.58$, showing the existence of a solitary wave with amplitude $\approx 1.2$. This is the limiting value of $c$, where the potential ceases to be real exactly at the turning point.

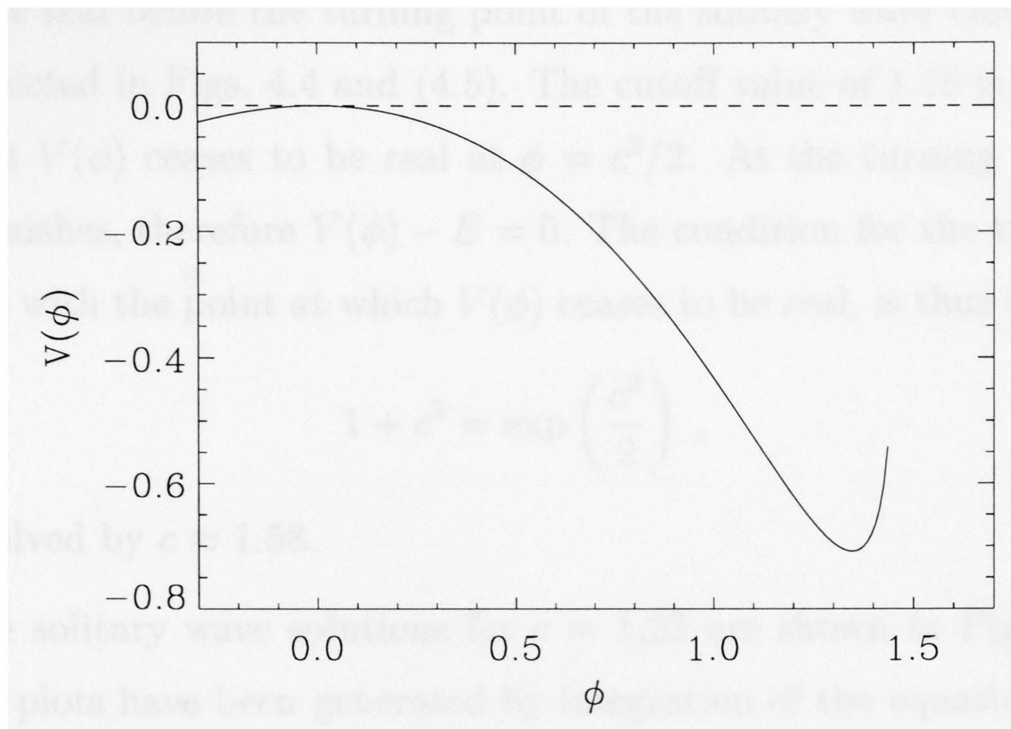


Figure 4.5: $V(\phi)$ for $c = 1.7$. The potential ceases to be real before the turning point is reached, hence there is no solitary wave.

Solitary wave solutions are obtained on seeking solutions which, at $x \to \pm\infty$, satisfy $\phi' = 0$ in addition to the above requirements of $\phi = 0$, $u = -c$, and $n = 1$. These boundary conditions imply

$$E = -1 - c^2 \, .$$

The resulting plots of $V(\phi) - E$, where $V(\phi)$ is given by

$$V(\phi) = -\mathrm{e}^{\phi} - c^2 \sqrt{1 - \frac{2\phi}{c^2}} \tag{4.29}$$

are depicted in Figs. 4.1–4.5 for various values of $c$. The noteworthy feature is that the origin is a minimum of the potential for small values of $c$, goes through an inflection point at $c = 1$, and is a maximum point for higher values of $c$.

Solitary waves are homoclinic orbits, as can be seen from the phase space picture shown in Fig. 4.6. The phase space coordinates for the ion-acoustic fluid are $\phi$ and $\phi_x$, analogous to $q$ and $\dot{q}$ for particle dynamics. Solitary wave solutions exist for $c$ ranging between 1 and 1.58; at $c = 1.58$ the potential ceases to be real before the turning point of the solitary wave can be reached. This is depicted in Figs. 4.4 and (4.5). The cutoff value of 1.58 is obtained by noting that $V(\phi)$ ceases to be real at $\phi = c^2/2$. At the turning point of the orbit $\phi'$ vanishes, therefore $V(\phi) - E = 0$. The condition for the turning point to coincide with the point at which $V(\phi)$ ceases to be real, is thus expressed by

$$1 + c^2 = \exp\left(\frac{c^2}{2}\right),$$

which is solved by $c \approx 1.58$.

The solitary wave solutions for $c = 1.23$ are shown in Figs. (4.7) and (4.8). The plots have been generated by integration of the equation,

$$\phi'' = -\frac{dV}{d\phi}(\phi), \tag{4.30}$$

where $V(\phi)$ is defined by Eq. (4.29). Numerical integration has been carried out using Mathematica, after breaking up the second order equation into two first order ones. Since we are looking for a homoclinic orbit, the boundary conditions are critical; note that for small values of $\phi$, the slope $\phi'$ is given by

$$\phi' = \sqrt{-2\Big(V(\phi) + 1 + c^2\Big)} \approx \phi\,\frac{\sqrt{c^2-1}}{c}, \tag{4.31}$$

where we have used a series expansion for $V(\phi)$. Boundary conditions with smaller slopes give rise to periodic waves, while higher slopes give rise to divergent orbits. Machine and algorithm precision necessarily limit the range over which we can simulate a solitary wave; it either becomes periodic or diverges beyond that range.

Periodic waves exist for $c < 1$. An example of the periodic potential $V(\phi)$ for $c = 0.5$ is shown in Fig. 4.1. The nonlinear periodic wave solution corresponding to $c = 0.5$ and $E = -1-c^2+0.05 = -1.2$, obtained by numerical integration, is depicted in Figs. 4.9 and 4.10. Periodic wave solutions also exist for $c > 1$, as is evident from Figs. 4.3–4.5; however the oscillations are about a

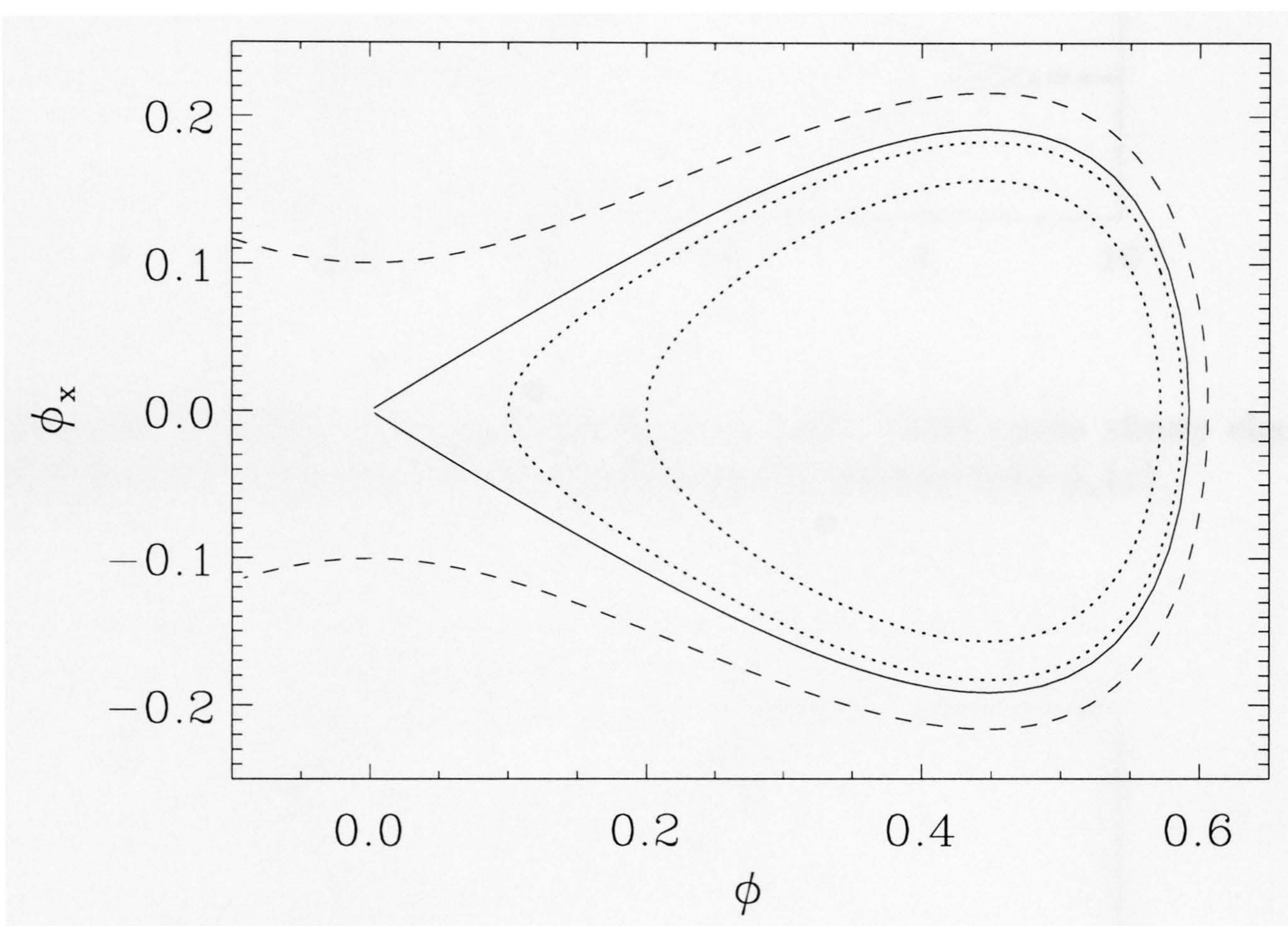


Figure 4.6: Orbits in phase space for $c = 1.23$. The solitary wave corresponds to the homoclinic orbit (solid curve), which separates the inner region with periodic orbits (dashed, closed curves) from the outer region with divergent orbits (dashed curve). The family of orbits is parametrized by $E$.

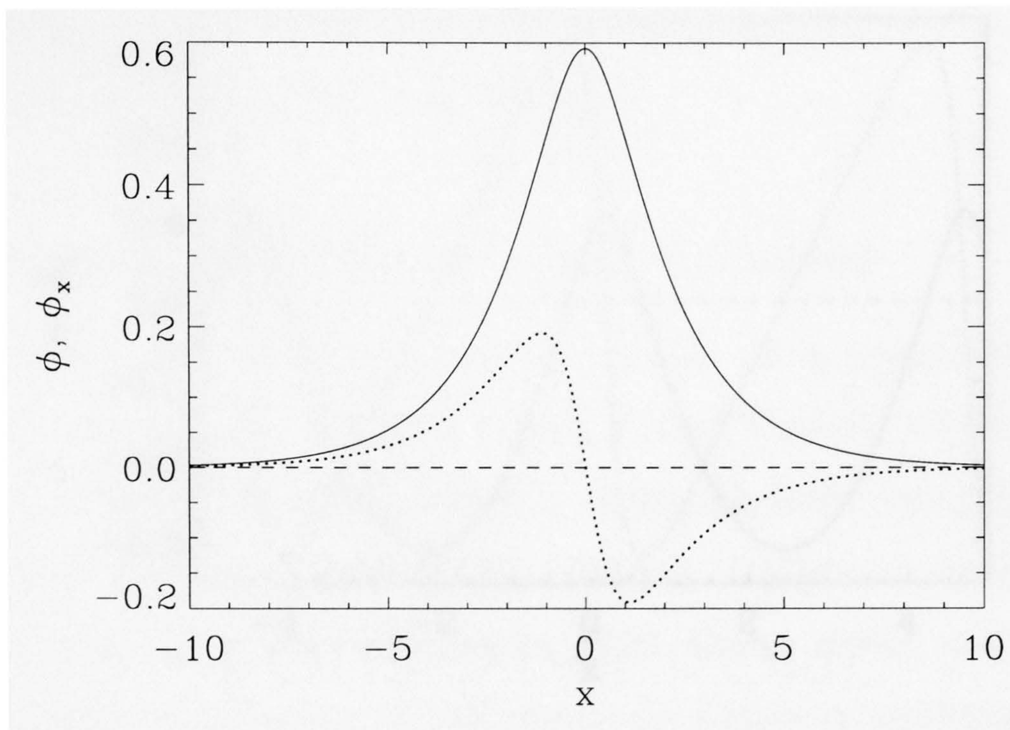


Figure 4.7: Solitary wave solutions for $c = 1.23$. Solid curve shows electric potential $\phi(x)$, and dotted curve shows negative electric field $\phi_x(x)$.

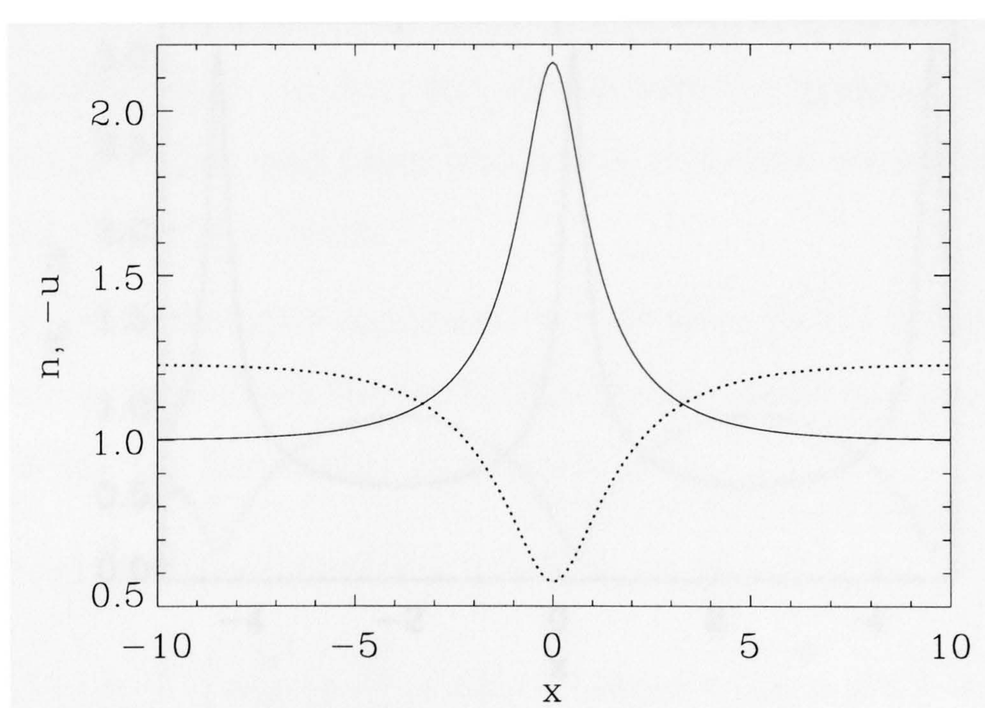


Figure 4.8: Solitary wave solutions for $c = 1.23$. Solid curve shows ion number density $n(x)$, and dotted curve shows negative ion velocity $-u(x)$.

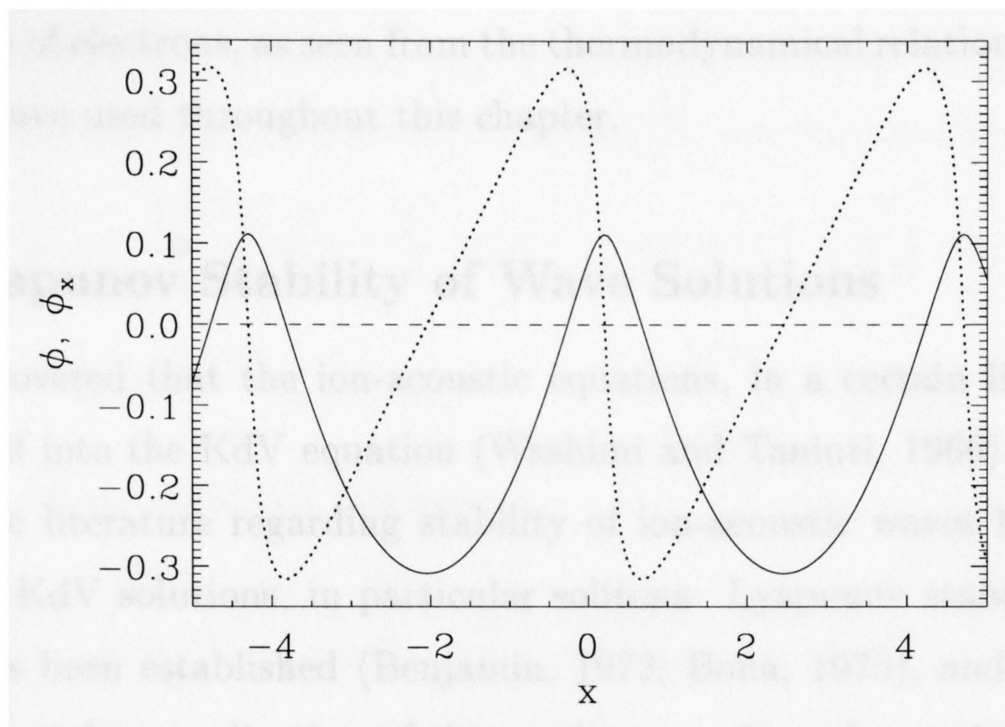


Figure 4.9: Nonlinear periodic wave solutions for $c = 0.5$, $E = -1.2$. Solid curve shows electric potential $\phi(x)$, and dotted curve shows negative electric field $\phi_x(x)$.

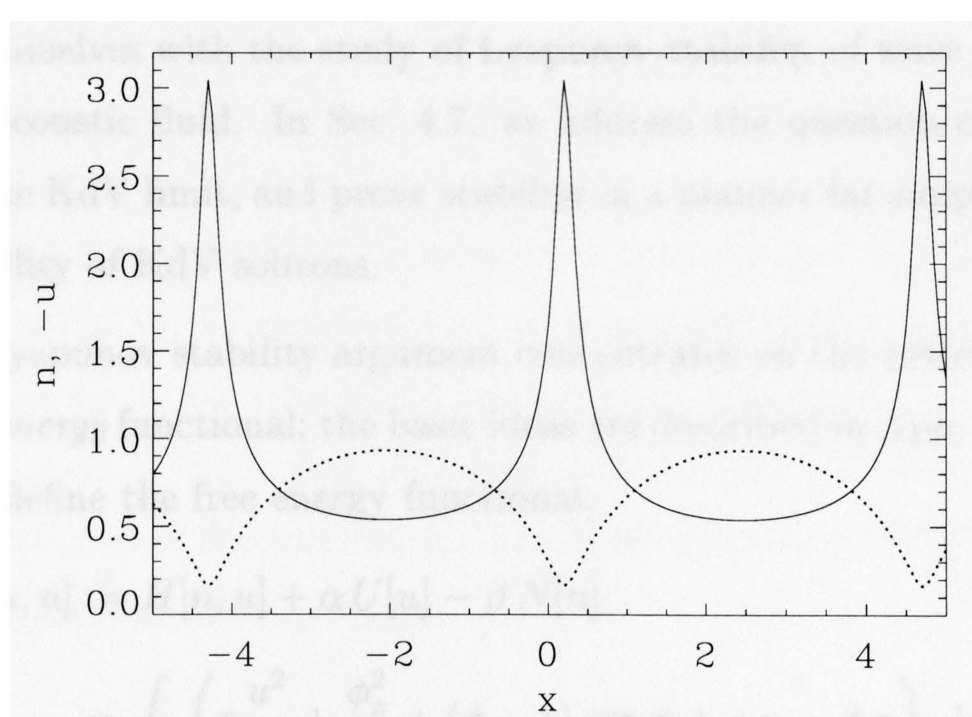


Figure 4.10: Nonlinear periodic wave solutions for $c = 0.5$, $E = -1.2$. Solid curve shows ion number density $n(x)$, and dotted curve shows negative ion velocity $-u(x)$.

positive value of $\phi$, which is no longer a quasineutral situation, but corresponds to an excess of electrons, as seen from the thermodynamical relation $n_e = \exp\phi$, which we have used throughout this chapter.

## 4.3 Lyapunov Stability of Wave Solutions

It was discovered that the ion-acoustic equations, in a certain limit, can be transformed into the KdV equation (Washimi and Taniuti, 1966). Since then most of the literature regarding stability of ion-acoustic waves has been restricted to KdV solutions, in particular solitons. Lyapunov stability of KdV solitons has been established (Benjamin, 1972; Bona, 1975), and further improvements and generalization of the results to a class of equations, of which KdV is but one, have been made (Bona and Soyeur, 1994, and references therein). There has also been work done on the stability of ion-acoustic waves when various dissipative effects, such as electron thermal conductivity, electron viscosity etc., are taken into account (Kaw, 1973, and references therein). Here we concern ourselves with the study of Lyapunov stability of wave solutions to the full ion-acoustic fluid. In Sec. 4.7, we address the question of Lyapunov stability in the KdV limit, and prove stability in a manner far simpler than the proof of stability of KdV solitons.

The Lyapunov stability argument concentrates on the extremal properties of a *free energy* functional; the basic ideas are described in App. B. Towards this end, we define the free energy functional,

$$\begin{aligned} F[n,u] &:= H[n,u] + \alpha\, U[u] - \beta\, N[n] \qquad (4.32) \\ &= \int_D \left( n\frac{u^2}{2} + \frac{\phi_x^2}{2} + (\phi-1)\exp\phi + \alpha\, u - \beta\, n \right) dx\,, \end{aligned}$$

which is constructed from the invariants of motion defined in Sec. 4.1.2. Travelling wave solutions satisfying Eqs. (4.24) lie on the extremal of $F$, as can be seen from setting $\delta F/\delta n = 0$ and $\delta F/\delta u = 0$.

For studying stability of solitary wave solutions, a slight modification of the above definition of the free energy is required in order to be meaningful on an infinite domain. The invariants, $H$, $U$, and $N$, are renormalized,

$$H := \int_{-\infty}^{+\infty} \left( n\frac{u^2}{2} + \frac{\phi_x^2}{2} + (\phi - 1)\exp\phi + 1 - \frac{c^2}{2} \right) dx \,,$$

$$U := \int_{-\infty}^{+\infty} (u + c)\, dx \,, \qquad N := \int_{-\infty}^{+\infty} (n - 1)\, dx \,, \tag{4.33}$$

which leads to the redefined free energy,

$$F[n,u] := \int_{-\infty}^{+\infty} \left( n\,\frac{u^2}{2} + \frac{\phi_x^2}{2} + (\phi - 1)\exp\phi + c\,u - n\,\frac{c^2}{2} + 1 + c^2 \right) . \tag{4.34}$$

Solitary waves lie on the extremal of the above free energy, as can be seen from the conditions $\delta F/\delta n = 0$ and $\delta F/\delta u = 0$, for an extremal point.

Stability of a solution which corresponds to an extremal point of the free energy is assured if it is a minimum point. It is therefore crucial to study the second variation of the free energy:

$$\delta^2 F[\delta n, \delta u] = \frac{1}{2} \int_D \left[ n\,\delta u^2 + 2u\,\delta n\,\delta u + \delta\phi_x^2 + \delta\phi^2\,(1+\phi)\exp\phi \right] dx \,, \tag{4.35}$$

where $n, u$, and $\phi$, are equilibrium quantities lying on the extremal of $F$, so that the first variation $\delta F$ vanishes for any perturbation $\delta n$, $\delta u$. Perturbation of the potential $\delta\phi$ is not considered independent, but related to $\delta n$ through the Poisson equation. Note that the terms $\alpha\, U$ and $\beta\, N$ do not contribute to the second variation of the free energy, hence the integrand of $\delta^2 F$ is identical for both, periodic and solitary waves.

*Stability of the Uniform Background*

Note that the uniform background $\phi = 0$, $n = 1$, $u = -c$ is a solution of the ion-acoustic equations for all wave speeds $c$. The Lyapunov stability of this uniform background can be proved upon shifting to the laboratory frame of the ions by adding momentum to the free energy. (This procedure is not useful for the wave solutions since they "pick out" a natural wave speed.) The free energy for this case may be written as

$$F_{\text{bg}}[n,u] := H[n,u] + \frac{c^2}{2}\, N[n] + c\, P[n,u]\,, \tag{4.36}$$

which has the second variation,

$$\delta^2 F_{\text{bg}} = \frac{1}{2}\int_{-\infty}^{+\infty} (\delta u^2 + \delta\phi^2 + \delta\phi_x^2)\, dx\,. \tag{4.37}$$

Clearly, the second variation of the free energy is positive definite and the norm to measure the perturbation (see App. B) is also evident, thus proving Lyapunov stability of the uniform background.

### 4.3.1 Stability of Solitary Waves

In this section we refer to the "second variation of the free energy" simply as "energy", and our goal is to find the existence of negative energy perturbations of ion-acoustic solitary waves, if any. The existence of negative energy perturbations would imply that Lyapunov stability cannot be established. In this regard, note that the non-existence of negative energy perturbations is necessary, but not sufficient to prove Lyapunov stability. For the proof of Lyapunov stability, there is the further requirement of existence of a norm; see App. B. As in Eq. (4.35), the energy is given by,

$$\begin{aligned}\delta^2 F[\delta n, \delta u] = \frac{1}{2}\int_{-\infty}^{+\infty} \Big[ &\frac{1}{n}(n\,\delta u + u\,\delta n)^2 \\ &+ \delta\phi_x^2 + \delta\phi^2\,(1+\phi)\mathrm{e}^{\phi} - \frac{u^2}{n}\delta n^2 \Big]\, dx\,, \end{aligned} \tag{4.38}$$

where we have completed the square, $(n\,\delta u + u\,\delta n)^2$, in order to eliminate the cross term $2\,u\,\delta n\,\delta u$, in favor of a sum of squares.

We now pick the worst case variation of the velocity, i. e. $\delta u$ which leads to the smallest possible energy. Evidently, the worst case variation $\hat{\delta u}$ satisfies

$$n\,\hat{\delta u} + u\,\delta n = 0\,, \tag{4.39}$$

which leads to the elimination of $\delta u$ and leaves us with the energy,

$$\begin{aligned}\delta^2 \hat{F}[\delta n] &:= \delta^2 F[\delta n, \hat{\delta u}] \\ &= \frac{1}{2}\int_{-\infty}^{+\infty} \left[\delta\phi_x^2 + \delta\phi^2(1+\phi)\mathrm{e}^{\phi} - \psi^2\delta n^2\right] dx\,. \end{aligned} \tag{4.40}$$

In the above equation, we have defined

$$\psi^2(x) := \frac{u^2}{n}(x)\,, \tag{4.41}$$

which will be found to be of convenience in what follows. In the rest of this subsection, we concentrate on showing that the sign of the energy $\delta^2\hat{F}[\delta n]$ is indefinite.

The principal type of integral inequality that we use throughout this section is quite simple:

$$\int_D g_1(x)\,dx \;\le\; \int_D g_2(x)\,, \qquad \text{for} \quad g_1(x) \le g_2(x) \quad \forall\, x \in D\,. \tag{4.42}$$

The difficulty lies mainly in establishing $g_1(x) \le g_2(x)$. A technique that we use repeatedly in this section, is to express $g_1(x)$ as a sum of squares, and modify the coefficients of the positive definite terms to obtain a greater function $g_2(x)$. As an example, consider

$$g_1(x) = p_1(x)\,q_1^2(x) - p_2(x)\,q_2^2(x)\,, \tag{4.43}$$

and

$$\sup_{x\in D}\Big(p_1(x)\Big) =: p_{1s}\,, \qquad \inf_{x\in D}\Big(p_2(x)\Big) =: p_{2i}\,. \tag{4.44}$$

It follows that

$$g_1(x) \;\leq\; g_2(x) := p_{1s}\, q_1^2(x) - p_{2i}\, q_2^2(x)\,. \tag{4.45}$$

inequality (4.45) of the above example is of much use in the discussion that follows.

Linearized variation of the Poisson equation relates $\delta n$ and $\delta\phi$:

$$\delta n = \mathrm{e}^{\phi}\, \delta\phi - \delta\phi_{xx}\,, \tag{4.46}$$

which, when squared, leads to

$$\delta n^2 = \mathrm{e}^{2\phi}\, \delta\phi^2 - 2\,\mathrm{e}^{\phi}\delta\phi\, \delta\phi_{xx} + \delta\phi_{xx}^2\,. \tag{4.47}$$

The integral of $\delta n^2$ cannot directly be expressed as an integral of a sum of squares of $\delta\phi$ and its derivatives, hence we multiply Eq. (4.47) by $\exp\,(-\phi)$ and integrate over the infinite domain to get

$$I_{\delta n} := \int_{-\infty}^{+\infty} \mathrm{e}^{-\phi}\delta n^2\, dx = \int_{-\infty}^{+\infty} \left(\mathrm{e}^{\phi}\, \delta\phi^2 + 2\,\delta\phi_x^2 + \mathrm{e}^{-\phi}\, \delta\phi_{xx}^2\right) dx\,, \tag{4.48}$$

where we have integrated by parts, assuming $\delta\phi\, \delta\phi_x$ vanishes at $\pm\infty$, to get the $\delta\phi_x^2$ term.

The objective of these manipulations is to find bounds on the integral of $\delta n^2$ in terms of integrals of a sum of squares of $\delta\phi$ and its derivatives, which we are now in a position to accomplish:

$$I_{\delta n} \;\leq\; \int_{-\infty}^{+\infty} \delta n^2\, dx \;\leq\; \mathrm{e}^{\phi_s}\, I_{\delta n}\,. \tag{4.49}$$

In inequality (4.49), the equalities are true only when $\phi(x)$ vanishes identically, and we have made use of the fact that $\phi(x)$ is non-negative for solitary wave solutions. The symbol $\phi_s$ denotes the maximum value of $\phi(x)$:

$$\phi_s := \sup_{x\in R}|\phi(x)|\,. \tag{4.50}$$

The upper bound in inequality (4.49) is seen to be true upon noting

$$\mathrm{e}^{\phi_s}\, I_{\delta n} = \int_{-\infty}^{+\infty} \mathrm{e}^{(\phi_s-\phi)}\, \delta n^2\, dx\,.$$

Clearly, $\exp\,(\phi_s - \phi)$ always exceeds or equals one, and $\delta n^2$ being always positive, it follows that $\exp(\phi_s)\, I_{\delta n}$ is the upper bound. Similarly, the lower bound in inequality (4.49) is proved by noting that the coefficient $\exp(-\phi)$, multiplying the positive definite quantity $\delta n^2$, is always less than or equal to one.

We now proceed to find an upper bound on the energy. Evidently,

$$\delta^2 \hat{F}[\delta n] \;\leq\; \delta^2 \hat{F}_1[\delta n]\,, \tag{4.51}$$

where

$$\delta^2 \hat{F}_1[\delta n] := \frac{1}{2}\int_{-\infty}^{+\infty} \left[\delta\phi_x^2 + \delta\phi^2\,(1+\phi)\mathrm{e}^{\phi}\right] dx - \psi_i^2 \int_{-\infty}^{+\infty} \delta n^2\, dx\,, \tag{4.52}$$

and

$$\psi_i^2 := \inf_{x\in R}|\psi^2(x)|\,. \tag{4.53}$$

In arriving at inequality (4.51), we have again made use of the property that a smaller coefficient multiplying a positive definite quantity leads to a smaller quantity.

Furthermore, replacement of $\int \delta n^2$ in Eq. (4.52) by the lower bound given in inequality (4.49), leads to yet another upper bound on $\delta^2 \hat{F}[\delta n]$. The advantage of this upper bound is the elimination of $\delta n^2$, leading to a sum of terms involving $\delta\phi^2$, $\delta\phi_x^2$, and $\delta\phi_{xx}^2$:

$$\delta^2 \hat{F}[\delta n] \;\leq\; \delta^2 \hat{F}_1[\delta n] \;\leq\; \delta^2 \hat{F}_2[\delta\phi]\,, \tag{4.54}$$

$$\text{where}\quad \delta^2 \hat{F}_2[\delta\phi] := \frac{1}{2}\int_{-\infty}^{+\infty} \{\delta\phi^2(1+\phi-\psi_i^2)\mathrm{e}^{\phi} + \delta\phi_x^2(1-2\,\psi_i^2) - \delta\phi_{xx}^2\,\psi_i^2\,\mathrm{e}^{-\phi}\}\, dx\,. \tag{4.55}$$

It is noteworthy that in Eq. (4.55), the highest order derivative term, $\delta\phi_{xx}^2$, has a negative coefficient, indicating that perturbations may be found with a negative upper bound, which would guarantee the existence of negative energy perturbations.

An argument similar to the above, replacing $\psi_i^2$ in inequality (4.51) by $\psi_s^2 := \sup_{x\in R}|\psi^2(x)| = c^2$, leads to a lower bound on the energy,

$$\delta^2\hat{F}_2[\delta n] \geq \int_{-\infty}^{+\infty}\Big\{\delta\phi^2(1 - c^2\,e^{2\phi_s}) + \delta\phi_x^2(1 - 2\,c^2\,e^{\phi_s}) - \delta\phi_{xx}^2\,c^2\,e^{\phi_s}\Big\}dx\,, \tag{4.56}$$

however we shall not have much use for this lower bound. Observe that the lower bound is negative for $c \geq 1$, which is the solitary wave regime.

*Positive Energy Perturbations*

Positive energy perturbations are easily found. Setting either $\delta u$, or $\delta n$ (and hence $\delta\phi$) identically to zero in the expression for the energy given by Eq. (4.38), we see that the energy is positive. In particular, picking the non-trivial perturbation to be an odd function of $x$ conserves the invariants, $P$, $N$, and $U$, given in Section 4.1.2, since the equilibrium quantities $n$ and $u$ are even functions of $x$.

*Negative Energy Perturbations*

We establish the existence of negative energy perturbations in the solitary wave regime $c \in (1, 1.58)$ by construction of an example. The example is constructed in a manner which also preserves the invariants $P$, $N$, and $U$, of Section 4.1.2, to ensure that the perturbation giving rise to negative energies is accessible on the invariant surfaces. In particular, $N$ and $U$ surfaces are Casimir surfaces.

We pick $\delta\phi$ to be the sum of two equal and opposite Gaussian distribu-

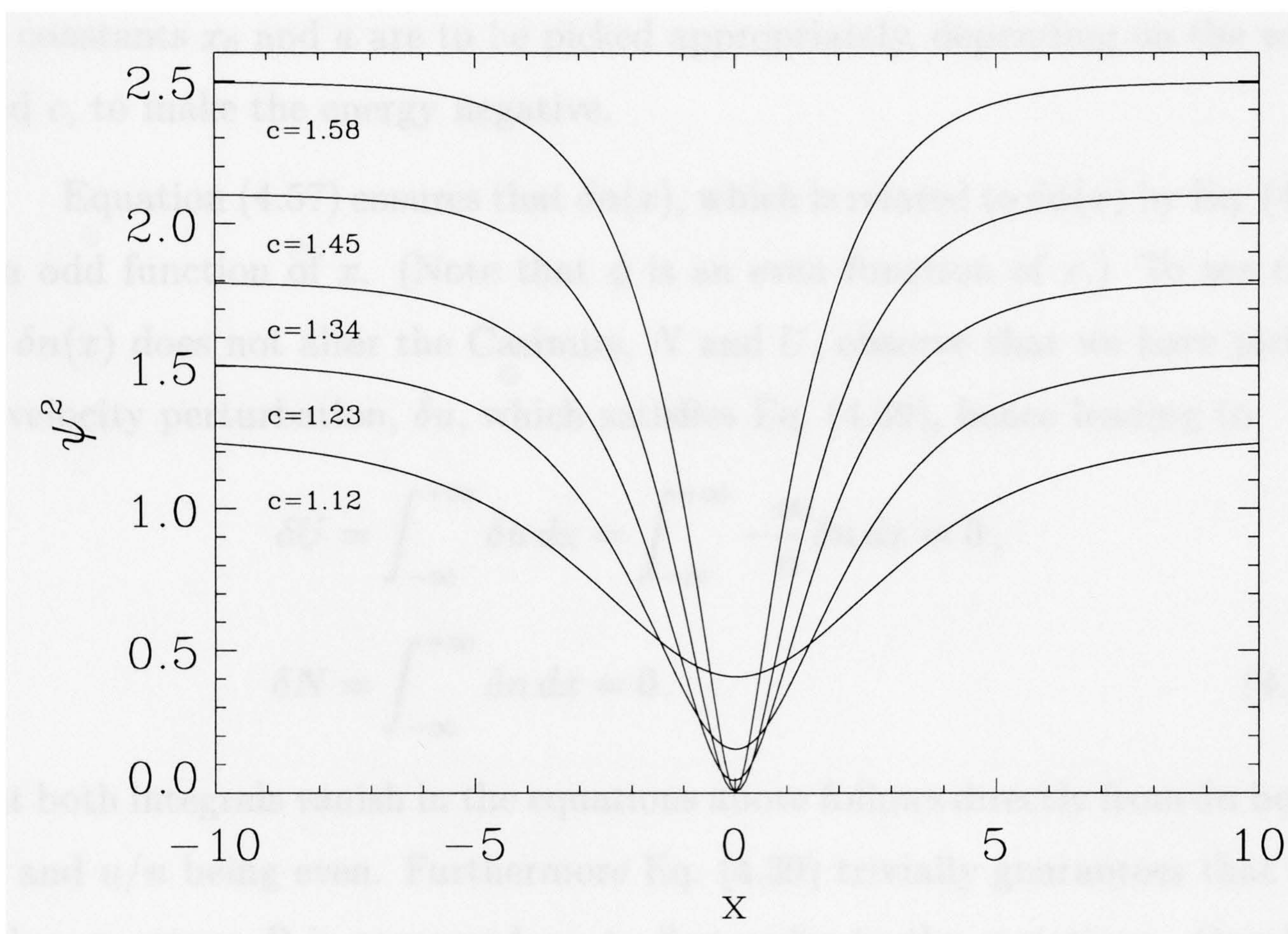


Figure 4.11: $\psi^2(x)$ corresponding to solitary waves of various speeds $c$. Note that $\psi^2(x)$ sharply falls to zero at the center and is close to $c^2$ for the most part of the real line.

tions, placed equidistantly on either side of the origin:

$$\delta\phi(x) = G_+(x) - G_-(x)\,, \tag{4.57}$$

where

$$G_+(x) := g\ \exp\left[-\frac{(x-x_0)^2}{a^2}\right], \quad G_-(x) := g\ \exp\left[-\frac{(x+x_0)^2}{a^2}\right]. \tag{4.58}$$

The constants $x_0$ and $a$ are to be picked appropriately, depending on the wave speed $c$, to make the energy negative.

Equation (4.57) ensures that $\delta n(x)$, which is related to $\delta\phi(x)$ by Eq. (4.46), is an odd function of $x$. (Note that $\phi$ is an even function of $x$.) To see that odd $\delta n(x)$ does not alter the Casimirs, $N$ and $U$, observe that we have picked the velocity perturbation, $\hat{\delta u}$, which satisfies Eq. (4.39), hence leading to

$$\delta U = \int_{-\infty}^{+\infty} \hat{\delta u}\, dx = \int_{-\infty}^{+\infty} -\frac{u}{n}\,\delta n\, dx = 0\,,$$

$$\delta N = \int_{-\infty}^{+\infty} \delta n\, dx = 0\,. \tag{4.59}$$

That both integrals vanish in the equations above follows directly from $\delta n$ being odd and $u/n$ being even. Furthermore Eq. (4.39) trivially guarantees that the total momentum $P$ is conserved up to first order in the variations. One last property to be noted is that the highly localized structure of the Gaussians in Eq. (4.58) ensures that the perturbation has finite energy, where we use the word "energy" to mean perturbation of the Hamiltonian; not the second variation of the free energy.

Having established that the perturbations under consideration are accessible on surfaces of constant Casimirs, as well as the surface of constant momentum, we now proceed with our aim to find $x_0$ and $a$ as functions of the wave speed $c$, which lead to negative values of the energy expressed by

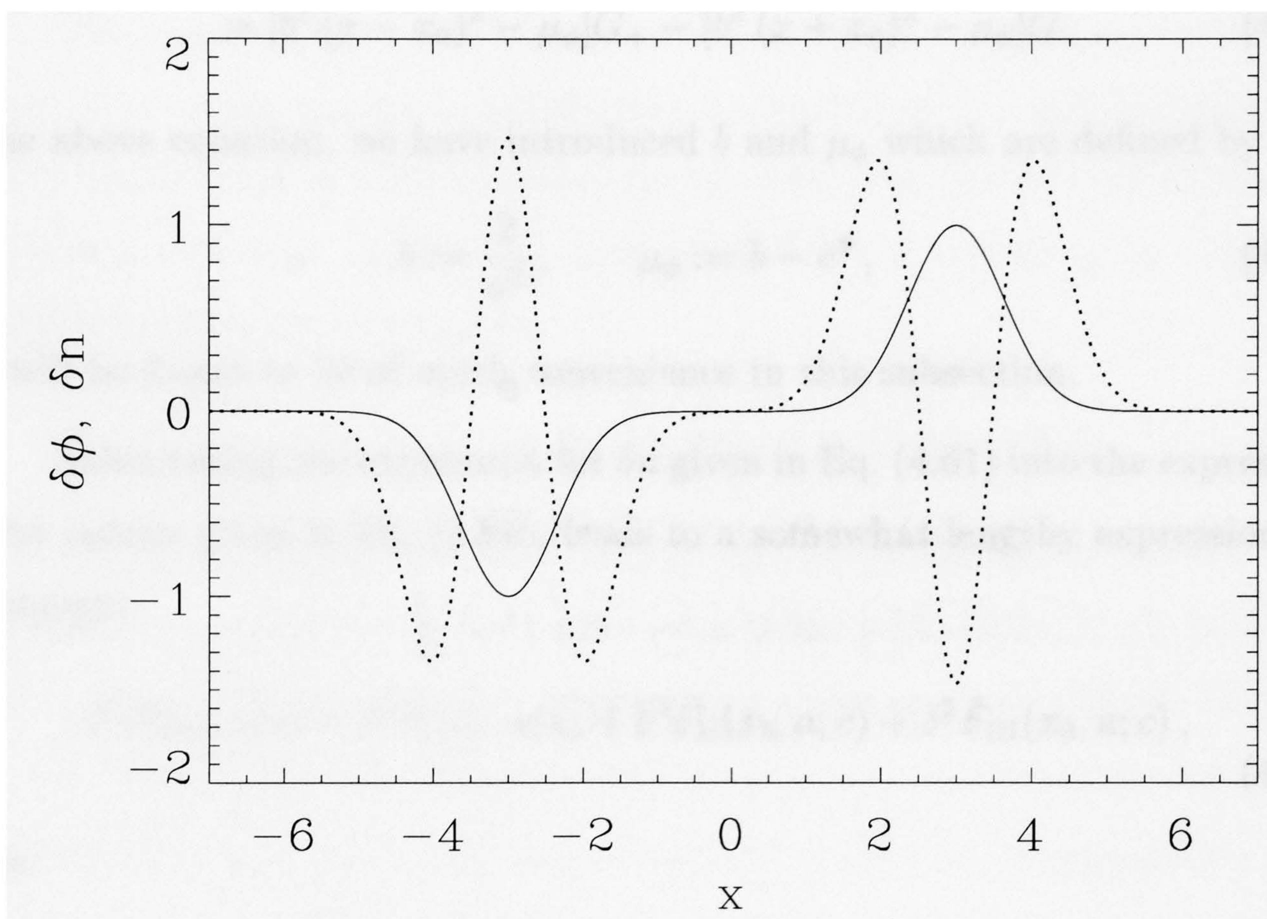


Figure 4.12: Solid curve shows the perturbation $\delta\phi$, comprising two equal and opposite Gaussians, assumed to be of unit amplitude for this illustration. Dotted curve shows the corresponding $\delta n$, which has been computed assuming $e^{\phi} \approx 1$.

Eq. (4.40). An alternative way to write the energy is,

$$\delta^2 \hat{F}[\delta n] = \int_{-\infty}^{+\infty} \left(\delta n\, \delta\phi + \phi\, \mathrm{e}^{\phi}\, \delta\phi^2 - \psi^2\, \delta n^2\right) dx\,, \tag{4.60}$$

and $\delta n$ is given by

$$\begin{aligned}\delta n &= \mathrm{e}^{\phi}\, \delta\phi - \delta\phi_{xx} \\ &= [b^2\,(x - x_0)^2 - \mu_\phi]G_+ - [b^2\,(x + x_0)^2 - \mu_\phi]G_-\,.\end{aligned} \tag{4.61}$$

In the above equation, we have introduced $b$ and $\mu_\phi$ which are defined by

$$b := \frac{2}{a^2}\,, \qquad \mu_\phi := b - \mathrm{e}^{\phi}\,, \tag{4.62}$$

and will be found to be of much convenience in this subsection.

Substituting the expression for $\delta n$ given in Eq. (4.61) into the expression for the energy given in Eq. (4.60), leads to a somewhat lengthy expression for the energy:

$$\delta^2 \hat{F}(x_0, a; c) = \delta^2 \hat{F}_{\mathrm{I}}(x_0, a; c) + \delta^2 \hat{F}_{\mathrm{II}}(x_0, a; c) + \delta^2 \hat{F}_{\mathrm{III}}(x_0, a; c)\,, \tag{4.63}$$

where

$$\delta^2 \hat{F}_{\mathrm{I}}(x_0, a; c) := \int_{-\infty}^{+\infty} (\phi\, \mathrm{e}^{\phi} - \mu_\phi - \mu_\phi^2\, \psi^2)\,(G_+^2 + G_-^2 - 2\,G_0^2)\, dx\,, \tag{4.64}$$

$$\begin{aligned}\delta^2 \hat{F}_{\mathrm{II}}(x_0, a; c) := \int_{-\infty}^{+\infty} & b^2\,(1 + 2\,\mu_\phi\, \psi^2)\, \big[(x - x_0)^2\, G_+^2 \\ & + (x + x_0)^2\, G_-^2 - 2\,(x^2 + x_0^2)\, G_0^2\big]\, dx\,,\end{aligned} \tag{4.65}$$

$$\begin{aligned}\delta^2 \hat{F}_{\mathrm{III}}(x_0, a; c) := -\int_{-\infty}^{+\infty} & \psi^2\, b^4\, \big[(x - x_0)^4\, G_+^2 + (x + x_0)^4\, G_-^2 \\ & - 2\,(x^2 - x_0^2)^2\, G_0^2\big]\, dx\,.\end{aligned} \tag{4.66}$$

In the above equations, we have introduced another convenient symbol $G_0^2$, which is defined to be

$$G_0^2(x) := G_+(x)\, G_-(x) = g^2\, \mathrm{e}^{-b\, x_0^2}\, \mathrm{e}^{-b\, x^2} \,. \tag{4.67}$$

It is now our intention to find upper bounds on each of $\delta^2 \hat{F}_{\mathrm{I}}$, $\delta^2 \hat{F}_{\mathrm{II}}$, and $\delta^2 \hat{F}_{\mathrm{III}}$, so that their sum provides us with an upper bound on $\delta^2 \hat{F}$.

The upper bounds on $\delta^2 \hat{F}_{\mathrm{II}}$ and $\delta^2 \hat{F}_{\mathrm{III}}$ are both positive, as we will see shortly. The crucial negative contribution comes from $\delta^2 \hat{F}_{\mathrm{I}}$, and its magnitude must be made large enough to overcome the other two positive contributions. In order to find such an upper bound on $\delta^2 \hat{F}_{\mathrm{I}}$, it is necessary to take into account the following two regions:

$$D_0 = [-x_1, x_1] \,, \qquad D_1 = (-\infty, -x_1) \,\cup\, (x_1, +\infty) \,, \tag{4.68}$$

that is, $D_0$ is the closed interval $[-x_1, x_1]$, and $D_1$ is the rest of the real line. The point $x_1$ is chosen so that the maximum value of $\phi(x)$ for $x \in D_1$ is $\kappa\, c^2/2$, where $\kappa$ is a positive constant. Solitary wave solutions are defined only for $\phi \leq c^2/2$, hence $\kappa$ must be less than one.

Furthermore, we note that $\phi(x)$ and $\psi^2(x)$ are even functions of $x$, and are monotonic in each semi-infinite region. The monotonicity and even-ness imply that the maximum value of $\phi(x)$ occurs at $\pm x_1$. These considerations are expressed in symbolic form:

$$\sup\nolimits_{x \in D_1} |\phi(x)| = \phi(\pm x_1) = \kappa\, \frac{c^2}{2} =: \phi_{s(1)} \,, \tag{4.69}$$

$$\inf\nolimits_{x \in D_1} |\phi(x)| = \phi(\pm\infty) = 0 \,, \tag{4.70}$$

$$\sup\nolimits_{x \in D_1} |\psi^2(x)| = \psi^2(\pm\infty) = c^2 \,, \tag{4.71}$$

$$\inf\nolimits_{x \in D_1} |\psi^2(x)| = \psi^2(\pm x_1) = c^2 \sqrt{1-\kappa} =: \psi^2_{i(1)} \,. \tag{4.72}$$

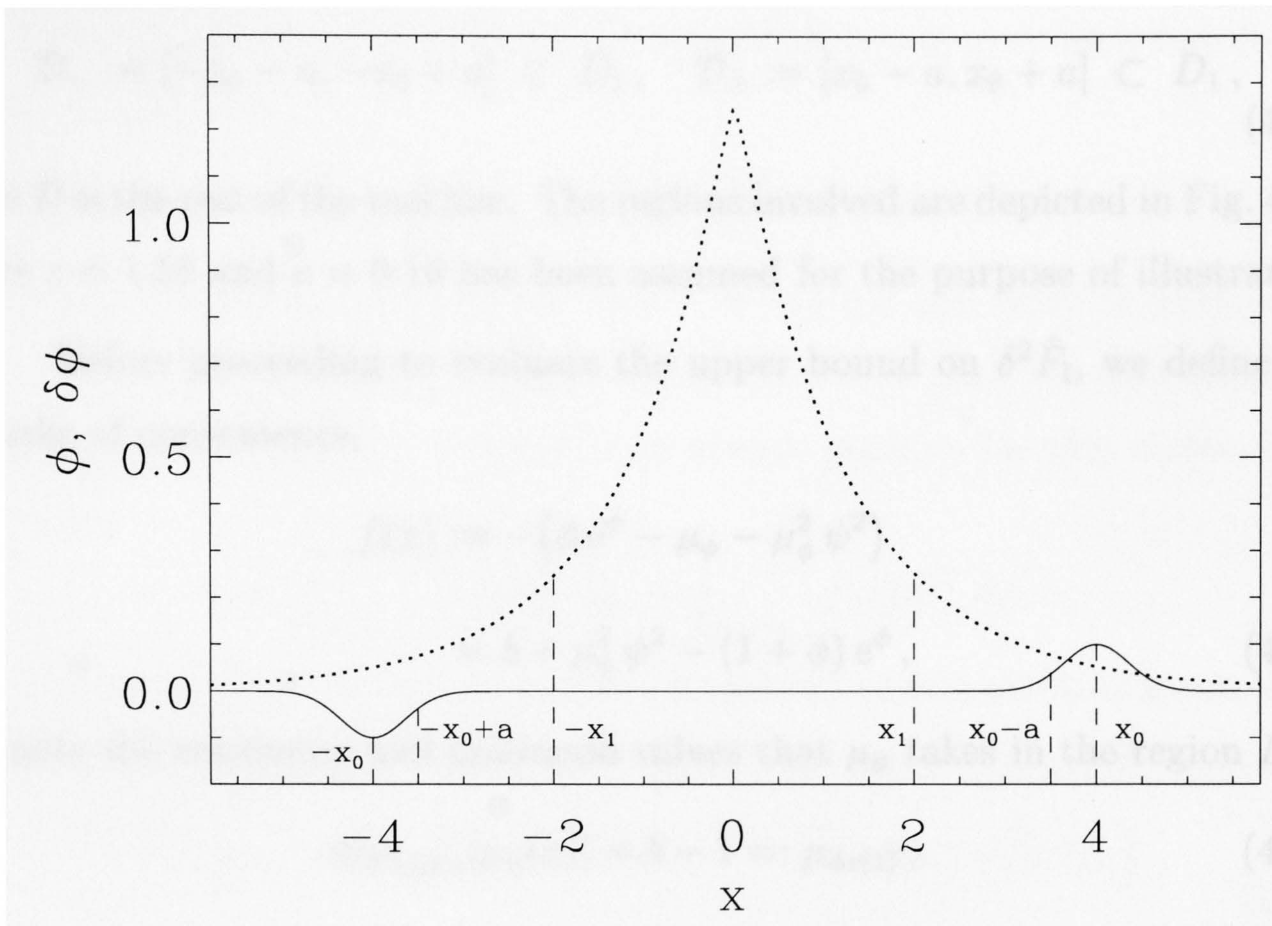


Figure 4.13: Dotted curve depicts the solitary wave solution $\phi(x)$ for $c = 1.58$, while the solid curve shows the perturbation $\delta\phi(x)$. Note that the peaks of the Gaussians lie away from the solitary wave peak; specifically, $x_0 - a > x_1$ is satisfied, so that $\mathcal{D}_-$ and $\mathcal{D}_+$ lie in the domain $D_1$. The region to the left of $-x_1$ and right of $x_1$ makes up the domain $D_1$. The half width of the Gaussians is $a = 0.5$; $x_1 = 2.0$ has been chosen to correspond to $\kappa = 0.19$.

The need to consider two separate regions is thus clear. Negative contribution to the energy comes from the regions in which $\psi^2$ is at its peak value of $c^2$, as is evident from Eq. (4.40). If the perturbation is constructed to lie mainly in the domain $D_1$, where $\psi_{i(1)}^2$ is large (hence $\kappa$ and $\phi_{s(1)}$ are small), the energy can be made negative.

We now require that the intervals $[-x_0 - a, -x_0 + a]$ and $[x_0 - a, x_0 + a]$ lie in the domain $D_1$, so that the peaks of the perturbation are in the region where $\psi^2$ is large. Thus we split the domain of integration into subdomains,

$\mathcal{D}_-$, $\mathcal{D}_+$, and $\mathcal{D}$:

$$\mathcal{D}_- := [-x_0 - a, -x_0 + a] \subset D_1 , \quad \mathcal{D}_+ := [x_0 - a, x_0 + a] \subset D_1 , \tag{4.73}$$

while $\mathcal{D}$ is the rest of the real line. The regions involved are depicted in Fig. 4.13, where $c = 1.58$ and $\kappa = 0.19$ has been assumed for the purpose of illustration.

Before proceeding to evaluate the upper bound on $\delta^2 \hat{F}_{\rm I}$, we define, for the sake of convenience,

$$\begin{aligned} f(x) &:= -\left(\phi\, e^{\phi} - \mu_\phi - \mu_\phi^2\, \psi^2\right) \\ &= b + \mu_\phi^2\, \psi^2 - (1 + \phi)\, e^{\phi} , \end{aligned} \tag{4.74}$$

and note the maximum and minimum values that $\mu_\phi$ takes in the region $D_1$:

$$\sup_{x \in D_1} |\mu_\phi(x)| = b - 1 =: \mu_{\phi s(1)} , \tag{4.75}$$

$$\inf_{x \in D_1} |\mu_\phi(x)| = b - e^{\phi_s} =: \mu_{\phi i(1)} . \tag{4.76}$$

In turn, this determines the upper and lower bounds on $f(x)$,

$$f_{i(1)} \leq f(x) \leq f_{s(1)} , \quad \text{for} \quad x \in D_1 , \tag{4.77}$$

$$f_{s(1)} := b + \mu_{\phi s(1)}^2\, c^2 - 1 , \tag{4.78}$$

$$f_{i(1)} := b + \mu_{\phi i(1)}^2\, \psi_{i(1)}^2 - (1 + \phi_{s(1)})\, e^{\phi_s(1)} . \tag{4.79}$$

The preceding considerations now enable us to calculate an upper bound on the energy:

$$\begin{aligned} \delta^2 \hat{F}_{\rm I}(x_0, a; c) &= 2 \int_{-\infty}^{+\infty} f(x)\, G_0^2\, dx - \int_{-\infty}^{+\infty} f(x)\, G_+^2\, dx - \int_{-\infty}^{+\infty} f(x)\, G_-^2\, dx \\ &\leq 2\, f_{s(1)} \int_{-\infty}^{+\infty} G_0^2\, dx - \int_{\mathcal{D} \cup \mathcal{D}_-} f(x)\, G_+^2\, dx - \int_{\mathcal{D} \cup \mathcal{D}_+} f(x)\, G_-^2\, dx \\ &\quad - f_{i(1)} \int_{\mathcal{D}_+} G_+^2\, dx - f_{i(1)} \int_{\mathcal{D}_-} G_-^2\, dx , \end{aligned} \tag{4.80}$$

where we have once again used the technique exemplified by Eq. (4.45). Furthermore, we pick $b$, and hence the Gaussian half width $a$, to be a function of the wave speed,

$$b = \frac{2}{a^2} = \left(1 + \frac{c^2}{2}\right) \mathrm{e}^{c^2/2}\,, \tag{4.81}$$

which assures us that $f(x) \geq 0$ for all $x \in R$ and for any value of wave speed $c$, as can be seen from Eq. (4.74) and the knowledge that the equilibrium is defined only for $\phi \leq c^2/2$. Since $f(x)$ is always positive for this choice for $b$, the terms,

$$-\int_{\mathcal{D}\,\cup\,\mathcal{D}_-} f(x)\, G_+^2\, dx \;-\; \int_{\mathcal{D}\,\cup\,\mathcal{D}_+} f(x)\, G_-^2\, dx\,,$$

may be dropped from the upper bound in inequality (4.80), yielding a somewhat greater upper bound. The remaining integrals are easily evaluated, leading to

$$\delta^2 \hat{F}_{\mathrm{I}}(x_0, a; c) \;\leq\; g^2 \left(2\,\gamma\, \mathrm{e}^{-b\,x_0^2}\, f_{s(1)} - \sqrt{2\,\pi}\, a\, \mathrm{Erf}(\sqrt{2})\, f_{i(1)}\right)\,, \tag{4.82}$$

where

$$\gamma := \sqrt{\frac{\pi\, a}{\sqrt{2}}}\,, \tag{4.83}$$

and

$$\mathrm{Erf}(z) := \frac{2}{\sqrt{\pi}} \int_0^z \mathrm{e}^{-t^2}\, dt\,, \quad \Rightarrow \quad \mathrm{Erf}(\sqrt{2}) \;\approx\; 0.95\,. \tag{4.84}$$

Thanks to the strong damping provided by $\exp(-b\,x_0^2)$, the positive first term in inequality (4.82) can be made as small as we wish, simply by increasing $x_0$, i. e. pushing the Gaussians further apart, thus allowing the negative term to dominate.

The upper bounds on $\delta^2 \hat{F}_{\mathrm{II}}$ and $\delta^2 \hat{F}_{\mathrm{III}}$, defined by Eqs. (4.65) and (4.66), are relatively easily obtained:

$$\delta^2 \hat{F}_{\mathrm{II}}(x_0, c) \;\leq\; b^2\,(1 + 2\,\mu_{\phi s}\, c^2) \int_{-\infty}^{+\infty} \left[(x - x_0)^2\, G_+^2 + (x + x_0)^2\, G_-^2\right] dx$$

$$- \,2 \int_{-\infty}^{+\infty} \left[(x^2 + x_0^2)\, G_0^2\right] dx\,, \tag{4.85}$$

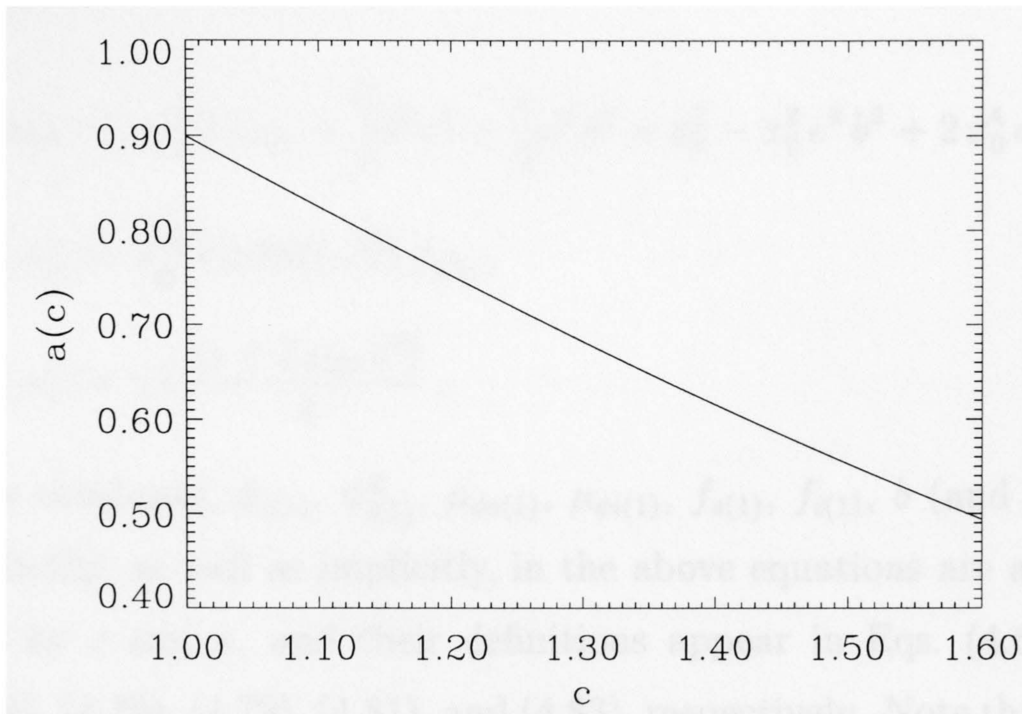


Figure 4.14: Plot shows half-width of the Gaussians $G_+$ and $G_-$, which form the perturbation, as a function of the wave speed.

$$\delta^2 \hat{F}_{\rm III}(x_0, c) \quad \leq \quad 2\, g^2\, c^2\, b^4 \int_{-\infty}^{+\infty} (x^2 \quad - \quad x_0^2)^2\, G_0^2\, dx\,. \qquad (4.86)$$

The integrals are straightforward to evaluate, resulting in

$$\delta^2 \hat{F}_{\rm II}(x_0, c) \;\leq\; \gamma\, g^2 \left[ b\, \frac{(1 + 2\, \mu_{\phi s}\, c^2)}{4} - 2\, b^2\, \mathrm{e}^{-b\, x_0^2} \left( \frac{a^2}{8} + x_0^2 \right) \right], \qquad (4.87)$$

$$\delta^2 \hat{F}_{\rm III}(x_0, c) \;\leq\; \gamma\, g^2\, \mathrm{e}^{-b\, x_0^2} \left( \frac{5}{8} b^2\, c^2 - x_0^2\, c^2\, b^3 + 2\, x_0^4\, c^2\, b^4 \right). \qquad (4.88)$$

Adding the upper bounds on $\delta^2 \hat{F}_{\rm I}$, $\delta^2 \hat{F}_{\rm II}$, and $\delta^2 \hat{F}_{\rm III}$, as given by Eqs. (4.82), (4.87) and (4.88), finally yields an upper bound on the energy that is suitable to our purpose of proving the existence of negative energy perturbations:

$$\delta^2 \hat{F} \leq g^2 \left( \mathcal{C}_1(c, \kappa) - \mathcal{C}_2(c, \kappa) + \mathrm{e}^{-b\, x_0^2}\, \mathcal{C}_3(c, \kappa, x_0) \right), \qquad (4.89)$$

where

$$\mathcal{C}_3(c,\kappa,x_0) := \gamma\left(2\,f_{s(1)} + \frac{5}{8}b^2\,c^2 - \frac{1}{4}a^2\,b^2 + x_0^2 - x_0^2\,c^2\,b^3 + 2\,x_0^4\,c^2\,b^4\right),$$

$$\mathcal{C}_2(c,\kappa) := \sqrt{2\,\pi}\,a\,\mathrm{Erf}(\sqrt{2})\,f_{i(1)}\,,$$

$$\mathcal{C}_1(c,\kappa) := \gamma\,b\,\frac{(1+2\,\mu_{\phi s}\,c^2)}{4}\,. \tag{4.90}$$

The various constants, $\phi_{s(1)}$, $\psi^2_{i(1)}$, $\mu_{\phi s(1)}$, $\mu_{\phi i(1)}$, $f_{s(1)}$, $f_{i(1)}$, $b$ (and $a$), $\gamma$, that appear explicitly, as well as implicitly, in the above equations are all uniquely determined by $c$ and $\kappa$, and their definitions appear in Eqs. (4.69), (4.72), (4.75), (4.76), (4.78), (4.79), (4.81), and (4.83), respectively. Note that choosing a value for $\kappa$ is equivalent to picking the point, $x_1$, at which the domain is split. It is evident from inequality (4.89) that for $\mathcal{C}_1 - \mathcal{C}_2 < 0$, the upper bound is negative for a large enough choice of $x_0$. Although $\mathcal{C}_3$ is a polynomial of fourth order in $x_0$, it is rendered inconsequential by the strong damping term multiplying it.

To conclude this subsection, we compute the energy upper bound for $\kappa = 0.19$ and $x_0 = 1.5$, to show that it is negative for all values of $c$ in the solitary wave regime. The result is depicted in Fig. 4.15. For $x_0 > 1.5$, i. e. as the Gaussians are pushed outwards, the energy upper bound becomes progressively more negative.

For consistency with the assumptions made in our argument, it is required further that

$$x_0 \;\geq\; x_1 + a\,, \tag{4.91}$$

so that the interval $[x_0 - a, x_0 + a]$ lies to the right of $x_1$. The value of $x_1$ is found from the relation, $\phi(x_1) = \kappa\,c^2/2$. The equilibrium solution, $\phi(x)$, is required for this purpose and is found by numerical integration of Eq. (4.30). As an example, for $c = 1.58$, $x_1 \approx 2.0$ and from Fig. 4.14, we see that $a \approx 0.5$,

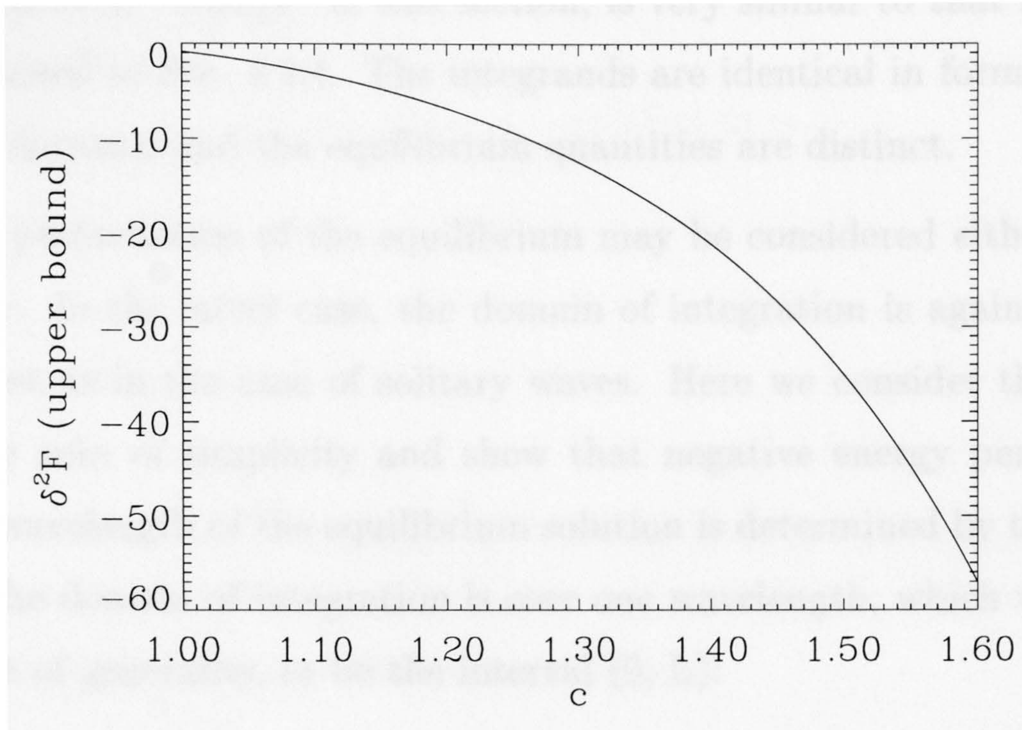


Figure 4.15: Upper bound on $\delta^2 F$ is seen to be negative for all values of $c$ in the solitary wave regime. ($x_0 = 1.5$)

therefore $x_0$ must exceed 2.5. Recall that this example has been portrayed in Fig. 4.13.

The solitary wave width tends to infinity as the amplitude tends to zero, which corresponds to $c \to 1+$. One might be tempted to believe that $x_0$ increases indefinitely as the wave speed approaches unity, however note that for low values of $c$, the wave amplitude never reaches $\kappa c^2/2$ and $\psi_i^2$ is not close to zero. There is no need to break the domain into three parts, i. e. the result of Eq. (4.55) is sufficient, and the restriction of inequality (4.91) is removed. Any value of $x_0$ may be chosen that makes the right hand side of inequality (4.89) negative; from Fig. 4.15 it is evident that $x_0 = 1.5$ is sufficient for this purpose.

### 4.3.2 Stability of Periodic Waves

Nonlinear periodic wave solutions exist for wave speed $c < 1$, as has been described in Sec. 4.2. The expression for the second variation of the free energy, referred to as "energy" in this section, is very similar to that for solitary waves, discussed in Sec. 4.3.1. The integrands are identical in form, while the integration domains and the equilibrium quantities are distinct.

The perturbation of the equilibrium may be considered either periodic or aperiodic. In the latter case, the domain of integration is again the entire real line, just as in the case of solitary waves. Here we consider the periodic case for the sake of simplicity and show that negative energy perturbations exist. The wavelength of the equilibrium solution is determined by the value of $c$ and $E$. The domain of integration is over one wavelength, which we assume, without loss of generality, to be the interval (0, L):

$$\delta^2 F[\delta n, \delta u] = \frac{1}{2}\int_0^L \left[\frac{1}{n}(n\,\delta u + u\,\delta n)^2 \right.$$
$$\left. + \,\delta\phi_x^2 + \delta\phi^2\,(1+\phi)\mathrm{e}^{\phi} - \frac{u^2}{n}\delta n^2\right] dx\,. \quad (4.92)$$

Following the same argument as in Sec. 4.3.1 in going from Eq. (4.38) to Eq. (4.40), we are led to

$$\delta^2\hat{F}[\delta n] := \delta^2 F[\delta n, \hat{\delta u}] = \frac{1}{2}\int_0^L \left[\delta\phi_x^2 + \delta\phi^2(1+\phi)\mathrm{e}^{\phi} - \psi^2\delta n^2\right] dx\,. \quad (4.93)$$

We now show that the sign of $\delta^2 F$ is indefinite.

The existence of positive energy perturbations is seen trivially, as in Sec. 4.3.1, simply by setting one of the perturbations, $\delta u$ or $\delta n$, to zero in Eq. (4.92). In order to establish the existence of negative energy perturbations, consider the perturbation,

$$\delta\phi(x) = \delta\phi_0\,\sin(k\,x)\,, \quad k := \frac{2\,\pi\,N}{L}\,, \quad (4.94)$$

where $N$ is an integer. Evidently, $\delta\phi$ vanishes at the boundary points, 0 and $L$. The ion number density perturbation is given by

$$\delta n = \mathrm{e}^{\phi}\,\delta\phi - \delta\phi_{xx} = \left(\mathrm{e}^{\phi} + k^2\right)\,\delta\phi\,. \quad (4.95)$$

Observing that

$$\int_0^L \cos^2(kx)\, dx = \int_0^L \sin^2(kx)\, dx = \frac{L}{2}\,, \tag{4.96}$$

Eq. (4.93) yields,

$$\begin{aligned}\delta^2 \hat{F}(k) = \frac{1}{2}\,\delta\phi_0^2 \Bigg\{ k^2\,\frac{L}{2} \\ - \int_0^L \sin^2(kx) \left[\psi^2\,\left(\mathrm{e}^{\phi} + k^2\right)^2 - (1+\phi)\,\mathrm{e}^{\phi}\right] dx \Bigg\}\,,\end{aligned} \tag{4.97}$$

for the specific perturbation that we have chosen.

We denote the upper bound on $\phi(x)$, for $x \in [0, L]$, by $\phi_s$ and the lower bound on $\psi^2(x)$, by $\psi_i^2$, so that

$$\begin{aligned}\delta^2 \hat{F} &\leq \frac{1}{2}\,\delta\phi_0^2 \left\{ k^2\,\frac{L}{2} - \int_0^L \sin^2(kx)\,\left[\psi_i^2\,\left(1+k^2\right)^2 - (1+\phi_s)\,\mathrm{e}^{\phi_s}\right] dx \right\} \\ &\leq \delta\phi_0^2\,\frac{L}{4} \left[k^2 + (1+\phi_s)\,\mathrm{e}^{\phi_s} - \psi_i^2\,\left(1+k^2\right)^2\right]\,.\end{aligned} \tag{4.98}$$

We are thus left with the simple task of determining the sign of the quadratic form,

$$\delta^2 \hat{F}(y) = A\,y^2 + B\,y + C\,, \tag{4.99}$$

where $y := k^2$ and the constants, $A$, $B$, and $C$, are defined by

$$\begin{aligned}A &:= -\psi_i^2\,, \\ B &:= 1 - 2\,\psi_i^2\,, \\ C &:= (1+\phi_s)\,\mathrm{e}^{\phi_s} - \psi_i^2\,.\end{aligned} \tag{4.100}$$

That the above quadratic form, and hence the energy, is negative for large values of $k$ (or $y$) is evident since the coefficient $A$ of the quadratic term is negative. Evidently, this method for proving negative energy perturbations fails when $\psi_i^2 = 0$, in which case it becomes necessary to consider separately the regions in which $\psi_i^2$ is zero and non-zero, similar to the argument in Sec. 4.3.1. The perturbation would then be constructed so that it is non-zero in the regions where $\psi_i^2$ is non-zero and close to zero elsewhere; however it is not our intention to explicitly construct such an example here.

For the sake of precision, we note that the quadratic form is negative when $y$ exceeds the highest root of $A\,y^2 + B\,y + C = 0$. Both roots are real in this case, since the discriminant $(B^2 - 4\,A\,C)$ is positive definite, thus it follows that for

$$k^2 = y > \frac{(1 - 2\,\psi_i^2) + \sqrt{1 + 4\,\psi_i^2\,\phi_s\,\mathrm{e}^{\phi_s} + 4\,\psi_i^2\,(\mathrm{e}^{\phi_s} - 1)}}{2\,\psi_i^2}\,, \tag{4.101}$$

the upper bound on the energy is negative.

## 4.4 Ion-Acoustic Equations with Pressure

In our treatment of the stability of the ion-acoustic equations in Sec. 4.3, we noted that small wavelength perturbations are a source of negative energies, making it impossible to prove Lyapunov stability. The source of the negative energy lies in the term $-\psi^2\,\delta n^2$ in Eq. (4.38). Naturally, one wonders if consideration of ionic pressure, which was neglected in the ion-acoustic equations, may prevent the negative energy perturbations from arising. It is this question that we seek to answer next.

In this section we present the ion-acoustic equations modified to include ionic pressure, their Hamiltonian structure, and invariants. Travelling wave solutions are explored in the next section, and the question of stability is discussed in Sec. 4.6.

The ion-acoustic equations with ionic pressure are

$$
\begin{aligned}
\frac{\partial n}{\partial t} + \frac{\partial}{\partial x}(nv) &= 0\,, \\
\frac{\partial v}{\partial t} + \frac{\partial}{\partial x}\left(\frac{v^2}{2} + \frac{e\phi}{m} + \frac{h(n)}{m}\right) &= 0\,, \\
\frac{1}{4\pi e}\frac{\partial^2 \phi}{\partial x^2} &= n_0 \exp\left(\frac{e\phi}{T_e}\right) - n\,,
\end{aligned}
\tag{4.102}
$$

where the only modification to Eqs. (4.6) is the addition of the gradient of the enthalpy $h(n)$ to the momentum equation. The enthalpy is assumed to depend only on the ion number density $n$, and is related to the pressure $p$ by

$$
\frac{dh}{dn} = \frac{1}{n}\frac{dp}{dn}\,. \tag{4.103}
$$

Later, we specifically pick the pressure to obey the adiabatic law for one dimension.

Introducing the same dimensionless dependent and independent variables as in Eqs. (4.4) and (4.5), with the further definition,

$$
\bar{h}(\bar{n}) := \frac{h(n)}{m\,c_0^2}\,,
$$

and dropping the bars over the dimensionless variables, we arrive at the equations,

$$
\begin{aligned}
\frac{\partial n}{\partial t} + \frac{\partial}{\partial x}(nv) &= 0\,, \\
\frac{\partial v}{\partial t} + \frac{\partial}{\partial x}\left(\frac{v^2}{2} + \phi + h(n)\right) &= 0\,, \\
\frac{\partial^2 \phi}{\partial x^2} &= \mathrm{e}^{\phi} - n\,.
\end{aligned}
\tag{4.104}
$$

Equations (4.104) represent the ion-acoustic system with pressure that will be our focus of attention in this and the next two sections to follow.

### 4.4.1 Hamiltonian Formulation

The Hamiltonian formulation is very similar to that seen in Sec. 4.1.1. The Poisson bracket is identical to the one defined in Eq. (4.9), while the new Hamiltonian is given by

$$H[n, v] := \int_D \left( n\frac{v^2}{2} + \frac{\phi_x^2}{2} + (\phi - 1)\exp\phi + n\,U(n) \right)\, dx\,. \tag{4.105}$$

The potential energy function $U(n)$ is related to the previously defined pressure and enthalpy by,

$$p(n) = n^2\,\frac{dU}{dn} \qquad \text{and} \qquad h(n) = \frac{d}{dn}(n\,U)\,. \tag{4.106}$$

In particular, if the pressure obeys an adiabatic law,

$$p(n) = p_0\, n^{\gamma}\,, \tag{4.107}$$

the functions $U(n)$ and $h(n)$ are given by

$$U(n) = \frac{p_0\, n^{\gamma-1}}{(\gamma - 1)} \quad \text{and} \quad h(n) = \frac{p_0\, \gamma\, n^{\gamma-1}}{(\gamma - 1)}\,. \tag{4.108}$$

The reader may verify quite easily that the time evolutions of $n$ and $v$ given by $n_t = \{n, H\}$ and $v_t = \{v, H\}$ reproduce the first and second of Eqs. (4.104).

### 4.4.2 Invariants

In Eqs. (4.104), the continuity and velocity equations are already in the form of conservation laws, hence it follows immediately that

$$N := \int_D n\, dx \quad \text{and} \quad U := \int_D v\, dx\,, \tag{4.109}$$

are invariants. In fact, they are more than that: analogous to the case without ionic pressure seen in Sec. 4.1.2, here too $N$ and $U$ are Casimirs of the Hamiltonian formulation.

The other invariants seen in Sec. 4.1.2 are also invariants of the ion-acoustic system with pressure, with minor changes to the conservation laws:

$$\frac{\partial}{\partial t}(n\,v) \quad + \quad \frac{\partial}{\partial x}\left(n\,v^2 + \mathrm{e}^{\phi} - \frac{\phi_x^2}{2} + p\right) \quad = \quad 0\,, \quad (4.110)$$

$$\frac{\partial}{\partial t}\left(n\frac{v^2}{2} + \frac{\phi_x^2}{2} + (\phi - 1)\,\mathrm{e}^{\phi} + n\,U(n)\right)$$

$$+ \frac{\partial}{\partial x}\left(n\,v\left(\frac{v^2}{2} + \phi + h(n)\right) - \phi\,\phi_{xt}\right) = 0\,, \quad (4.111)$$

$$\frac{\partial}{\partial t}\Big(n\,(x \; - \; v\,t)\Big) \; + \; \frac{\partial}{\partial x}\left(n\,v\,(x - v\,t) - t\,\mathrm{e}^{\phi} + t\,\frac{\phi_x^2}{2} - t\,p\right) \; = \; 0\,. \quad (4.112)$$

The invariance of momentum $P$, energy or Hamiltonian $H$, and the invariant $I$ with explicit dependence on time, follow from the above equations.

## 4.5 Solitary and Periodic Wave Solutions

Ion-acoustic equations with pressure are Galilean invariant, just as in the case without pressure seen in Sec. 4.1.3. Our treatment, henceforth, will be carried out in the inertial frame moving at the wave speed $c$, where the velocity is denoted $u$, and all equilibrium quantities have dependence only on the spatial coordinate $x$. The elimination of explicit time dependence leads to simple integration of Eqs. (4.104) yielding

$$n\,u + \alpha = 0\,, \qquad u^2 = 2\,[\beta - \phi - h(n)]\,, \qquad \phi'' = \exp\phi - n\,, \quad (4.113)$$

where $\alpha$, $\beta$ are arbitrary constants.

We now assume that the pressure obeys an adiabatic law, so that $h(n) = p_0\,\gamma\,n^{\gamma-1}/(\gamma-1)$. If we are to find solutions in a manner identical to that in Sec. 4.2, i. e. analogous to single particle dynamics, it is necessary to express

$n$ in terms of $\phi$. This can be accomplished in terms of elementary functions for the case $\gamma = 3$, which is the adiabatic exponent corresponding to systems with one degree of freedom.

For the choice $\gamma = 3$, the enthalpy is $h(n) = \frac{3}{2}\, p_0\, n^2$, and it is seen from Eqs. (4.113) that $n^2$ satisfies the quadratic equation,

$$\frac{1}{2}\, n^4 + \frac{(\phi - \beta)}{3\, p_0}\, n^2 + \frac{\alpha^2}{6\, p_0} = 0\,. \tag{4.114}$$

The ion number density can thus lie on either of two branches satisfying

$$n_{(\pm)}(\phi) = \sqrt{\eta(\phi) \pm \sqrt{\eta^2(\phi) - \bar{\alpha}^2}}\,, \tag{4.115}$$

where

$$\eta(\phi) := \frac{(\beta - \phi)}{3\, p_0} \qquad \text{and} \qquad \bar{\alpha}^2 := \frac{\alpha^2}{3\, p_0}\,. \tag{4.116}$$

The third of Eqs. (4.113) can thus be rewritten $\phi'' = \mathrm{e}^{\phi} - n_{(\pm)}(\phi)$, which may be cast in the form,

$$\phi'' = -\frac{dV_{(\pm)}}{d\phi}(\phi)\,, \tag{4.117}$$

where

$$V_{(\pm)}(\phi) := -\, \mathrm{e}^{\phi} - p_0\, n^3_{(\pm)}(\phi) - \frac{\alpha^2}{n_{(\pm)}(\phi)}\,. \tag{4.118}$$

We are thus left with the problem of finding solutions for the motion of a single particle in time independent potentials $V_{(+)}(\phi)$ and $V_{(-)}(\phi)$.

Before proceeding to specify the solutions, we investigate the limit of $V_{(+)}(\phi)$ and $V_{(-)}(\phi)$ as $p_0 \to 0$. One expects to recover the potential $V(\phi)$, defined in Eq. (4.26), from one of the two branches $V_{(+)}(\phi)$ and $V_{(-)}(\phi)$, in the limit as $p_0$ vanishes. Taylor series expansion yields,

$$
\begin{aligned}
n_{(\pm)}(\phi) &= \left[\eta \pm \eta \left(1 - \frac{\overline{\alpha}^2}{\eta^2}\right)^{1/2}\right]^{1/2} \\
&= \left[\eta \pm \eta \left(1 - \frac{1}{2}\frac{\overline{\alpha}^2}{\eta^2} - \frac{1}{8}\frac{\overline{\alpha}^4}{\eta^4} - \cdots\right)\right]^{1/2} \qquad (4.119) \\
&= \left(\frac{(\beta-\phi)}{3\,p_0} \pm \frac{(\beta-\phi)}{3\,p_0} \mp \frac{1}{2}\frac{\alpha^2}{(\beta-\phi)} \mp \frac{3}{8}\frac{p_0\,\alpha^4}{(\beta-\phi)^3} \mp \mathcal{O}(p_0^2)\right)^{1/2}.
\end{aligned}
$$

It is clear from the above equation that only the "minus branch", $n_{(-)}$, is finite in the limit $p_0 \to 0$; the limiting value being given by

$$
\lim_{p_0\to 0}\left[n_{(-)}(\phi)\right] = \frac{\alpha}{\sqrt{2(\beta-\phi)}}, \qquad (4.120)
$$

which is identical to $n(\phi)$ in Sec. 4.2. Making use of Eq. (4.120) in Eq. (4.118), notice that the term $p_0\,n_{(-)}^3$ vanishes in the limit $p_0 \to 0$, and we are left with

$$
\lim_{p_0\to 0}\left[V_{(-)}(\phi)\right] = -\mathrm{e}^{\phi} - \alpha\,\sqrt{2\,(\beta-\phi)}\,, \qquad (4.121)
$$

which is identical to $V(\phi)$ defined in Eq. (4.26). For small $p_0$, we therefore expect the wave solutions to lie on the "minus branch".

For an equilibrium wave solution to exist, it is necessary that $V_{(\pm)}(\phi)$ have at least one minimum. Note that this is required for a solitary wave as well; although the homoclinic orbit starts and ends at an X point, there must exist a turning point, which is possible only if the potential has at least one minimum in between the X point and the turning point, so that it may "climb back up". In order to have a minimum, it is necessary that $d^2V_{(\pm)}/d\phi^2$ have regions in which it is positive.

$$
\begin{aligned}
\frac{dV_{(\pm)}}{d\phi} &= -\,\mathrm{e}^{\phi} + n_{(\pm)}(\phi) \\
\Rightarrow \quad \frac{d^2V_{(\pm)}}{d\phi^2} &= -\,\mathrm{e}^{\phi} \mp \frac{n_{(\pm)}}{6\,p_0\,\sqrt{\eta^2 - \overline{\alpha}^2}}\,. \qquad (4.122)
\end{aligned}
$$

It is evident from the equation above that, $n_{(\pm)}$ being positive, $d^2V_{(+)}/d\phi^2$ is always negative, hence no solution exists on the "plus branch". We only consider the "minus branch" henceforth.

The constants $\alpha$ and $\beta$ are no longer arbitrary if we demand $n_{(-)} = 1$ and $u = -c$ when $\phi = 0$. The relation $u = -\alpha/n_{(-)}$ together with Eq. (4.115) imply,

$$\alpha = c \quad \text{and} \quad \beta = \frac{c^2}{2} + \frac{3\,p_0}{2}\,. \tag{4.123}$$

In obtaining the expression for $\beta$ we have used the squaring operation twice, thus losing some sign information. Plugging the value for $\beta$ back into Eq. (4.115) and checking for consistency gives the equation,

$$\frac{c^2}{6\,p_0} + \frac{1}{2} - \sqrt{\left(\frac{c^2}{6\,p_0} - \frac{1}{2}\right)^2} = 1\,, \tag{4.124}$$

which is an identity only if

$$\frac{c^2}{6\,p_0} - \frac{1}{2} \geq 0 \quad \Rightarrow \quad p_0 \leq \frac{c^2}{3}\,. \tag{4.125}$$

For $p_0$ greater than $c^2/3$, the square root term in Eq. (4.124) is of the opposite sign, and consistency is achieved only on the "plus branch", which is of little interest to us. The result expressed by inequality (4.125) is important; we are restricted to finding solutions in the parameter regime $p_0 \leq c^2/3$. It turns out, as we will see in Sec. 4.6, that $p_0 \geq c^2/3$ assures Lyapunov stability.

The condition $n_{(-)} = 1$ at $\phi = 0$ assures us that $\phi = 0$ is a critical point of $V_{(-)}(\phi)$, as may be seen from the first of Eqs. (4.122). We now study $d^2V_{(-)}/d\phi^2$ at $\phi = 0$ to find the parameter regimes in which the origin corresponds to O and X points. From the second of Eqs. (4.122) it may be seen that

$$\left.\frac{d^2V_{(-)}}{d\phi^2}\right|_{\phi=0} = \left(c^2 - 3\,p_0\right)^{-1} - 1\,, \tag{4.126}$$

which is positive if $p_0 > (c^2-1)/3$. Combining this result with inequality (4.125) leads to

$$\frac{c^2}{3} - \frac{1}{3} < p_0 \leq \frac{c^2}{3}, \tag{4.127}$$

which is the parameter regime in which the origin is a minimum of $V_{(-)}(\phi)$, indicating existence of periodic waves.

Solitary waves are homoclinic orbits which start and end at the origin, and their existence requires the origin to be a maximum of $V_{(-)}(\phi)$. It is clear from the preceding paragraph that the origin is a maximum for

$$p_0 < \frac{c^2}{3} - \frac{1}{3}. \tag{4.128}$$

Since $p_0$ is a non-negative constant, it is evident from inequality (4.128) that solitary wave solutions may exist only for $c \geq 1$.

Inequality (4.128) is necessary but not sufficient for the existence of solitary waves. To extract the parameter regime of existence of solitary waves more accurately, we compare $V_{(-)}(0)$ to $V_{(-)}(\phi_c)$, where $\phi_c$ is the cutoff value of $\phi$ beyond which $V_{(-)}(\phi)$ is not defined. Solitary waves exist for parameter values for which $V_{(-)}(\phi_c) \geq V_{(-)}(0)$. The cutoff value $\phi_c$ is obtained by noting the value of $\phi$ at which the square root term in Eq. (4.115) vanishes, and is seen to be

$$\phi_c = \frac{c^2}{2} + \frac{3\,p_0}{2} - c\sqrt{3\,p_0} = \frac{1}{2}\left(c - \sqrt{3\,p_0}\right)^2. \tag{4.129}$$

The idea behind comparision of $V_{(-)}(0)$ and $V_{(-)}(\phi_c)$ is quite simple: we would like to check if $V_{(-)}(\phi)$ "climbs back up" to provide a turning point for the homoclinic orbit.

Observing that $n_{(-)}(0) = 1$ and $n_{(-)}(\phi_c) = \sqrt{c/\sqrt{3\,p_0}}$ we get

$$V_{(-)}(0) = -1 - p_0 - c^2, \tag{4.130}$$

$$V_{(-)}(\phi_c) = -e^{\phi_c} - \frac{4}{3}(3\,p_0)^{1/4}\,c^{3/2}. \tag{4.131}$$

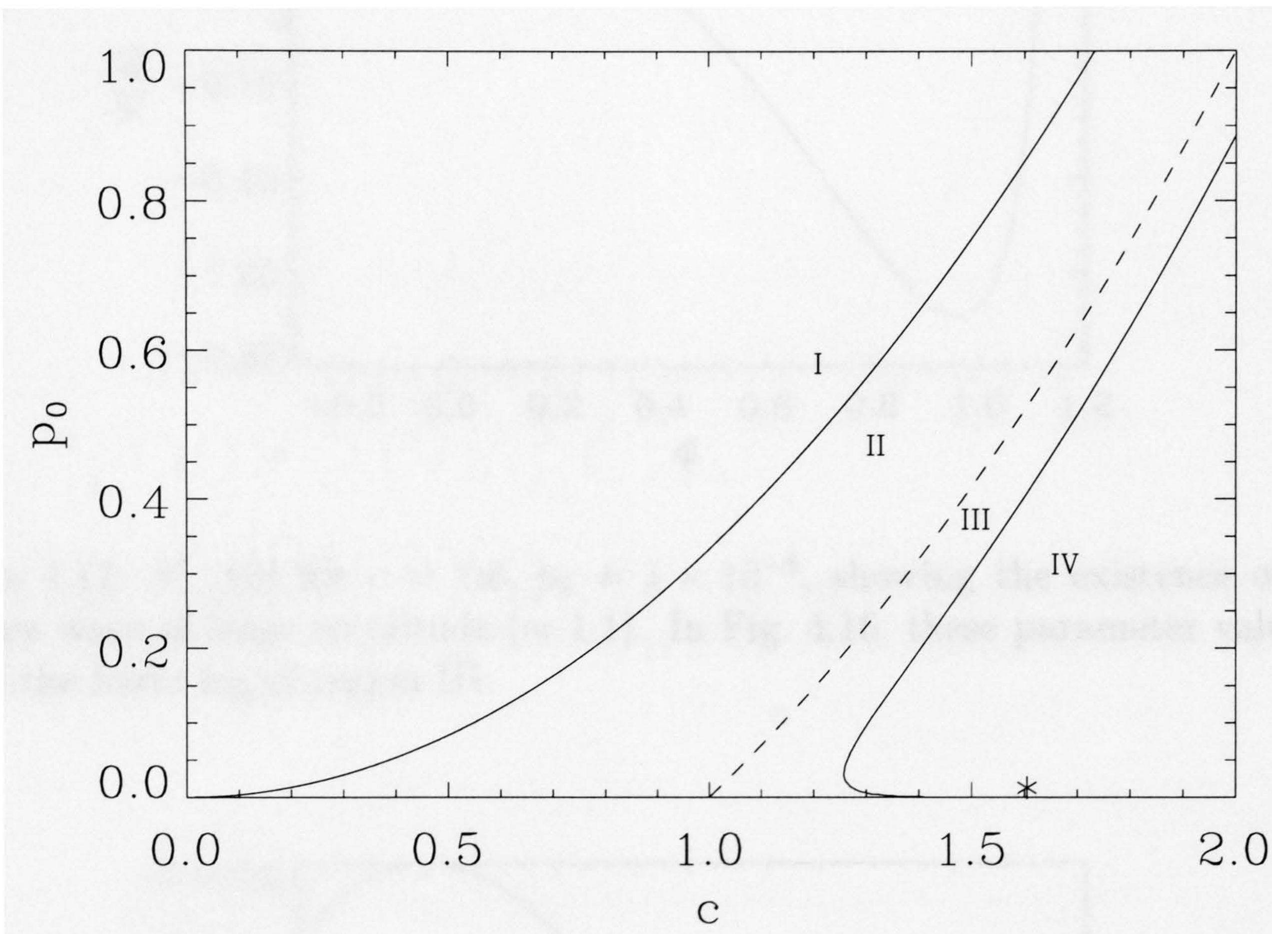


Figure 4.16: Parameter regimes of wave solutions. For parameter values in region II, the waves are periodic, while solitary waves are found in region III. The dashed curve is the locus of inflexion points and is part of the parabola, $(c^2 - 1)/3$, while the curve separating regions I and II is part of the parabola, $c^2/3$. The asterisk on the $c$ axis marks the end of the extremely narrow lower leg of the solitary wave regime. The upper leg of region III continues indefinitely, becoming narrower for higher values of $c$. The amplitude of the periodic waves vanishes on both of the bounding curves of region II, while the amplitude of the solitary waves vanishes at the upper bounding curve (dashed) of region III. There are no wave solutions in regions I and IV.

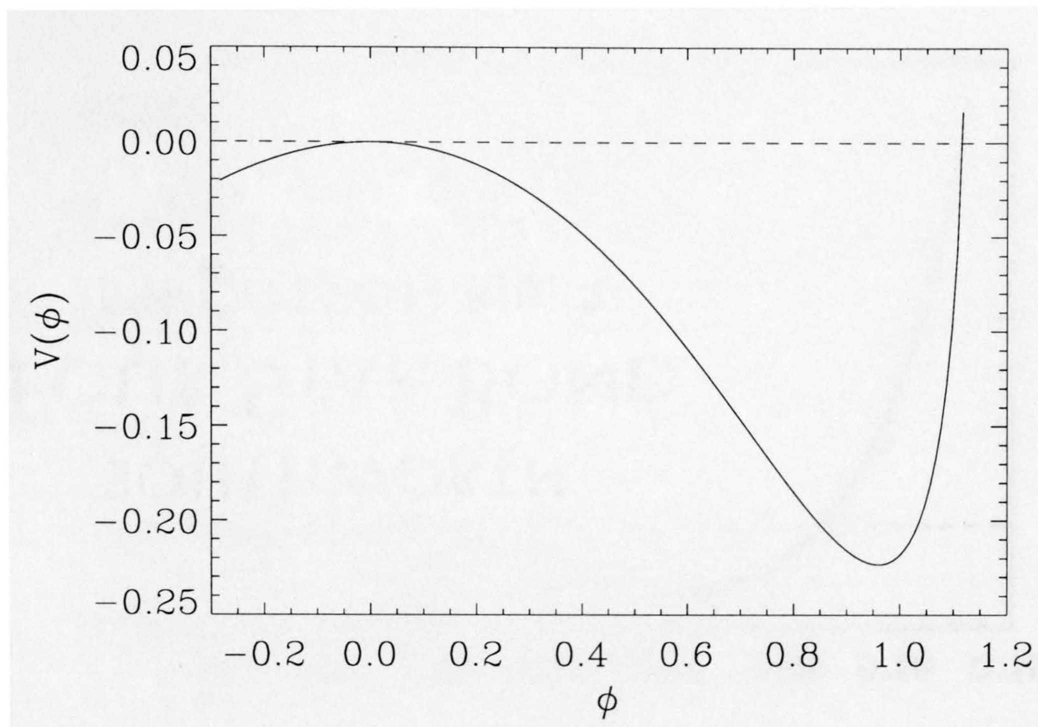


Figure 4.17: $V_{(-)}(\phi)$ for $c = 1.5$, $p_0 = 1 \times 10^{-5}$, showing the existence of a solitary wave of large amplitude ($\approx 1.1$). In Fig. 4.16, these parameter values lie on the lower leg of region III.

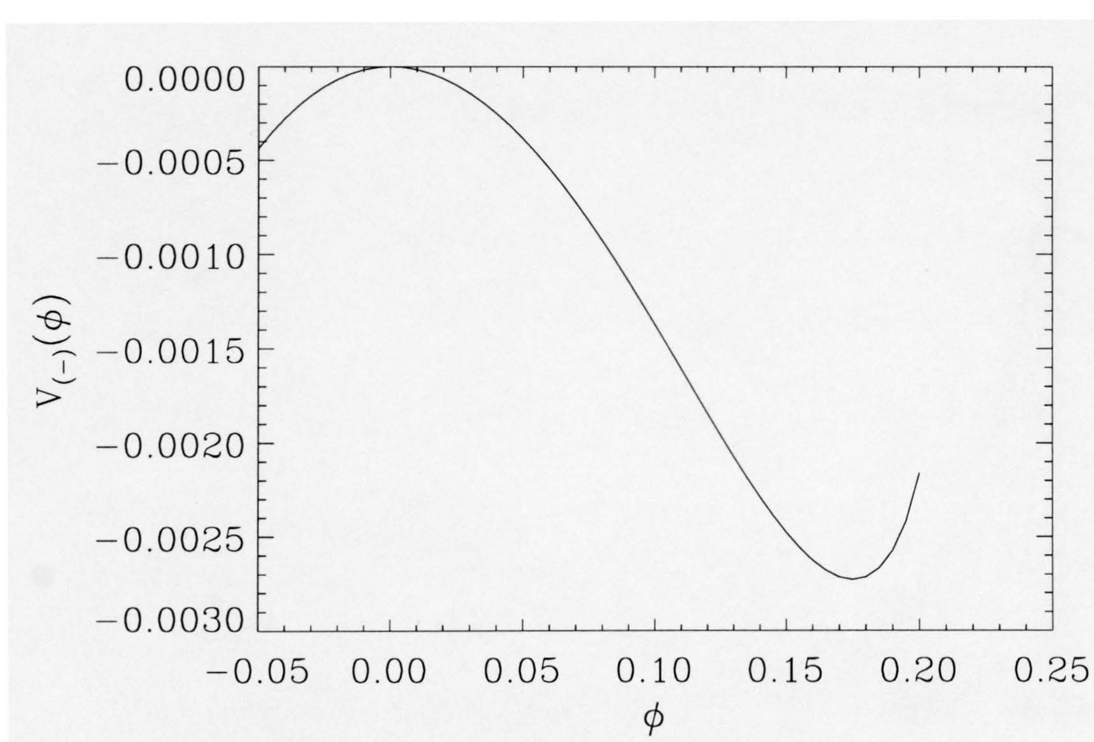


Figure 4.18: $V_{(-)}(\phi)$ for $c = 1.5$, $p_0 = 0.25$. The potential ceases to be real before the turning point can be reached, hence there is no solitary wave. In Fig. 4.16, these parameter values lie in region IV.

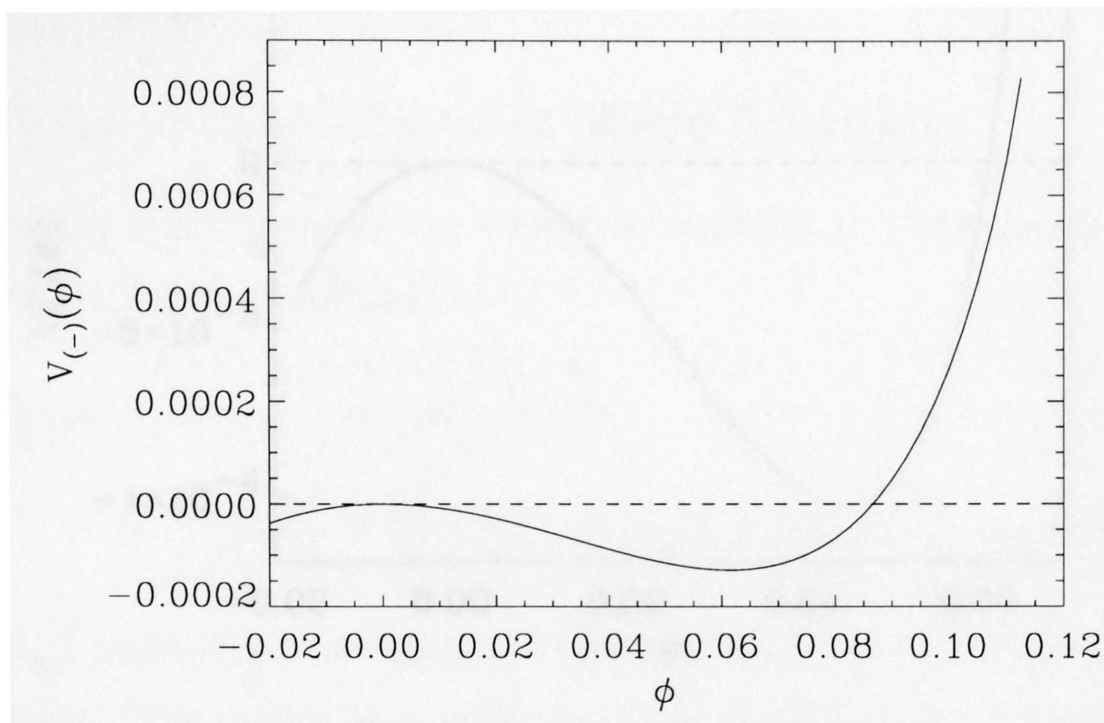


Figure 4.19: $V_{(-)}(\phi)$ for $c = 1.5$, $p_0 = 0.35$, showing the existence of a solitary wave of small amplitude ($\approx 0.09$). In Fig. 4.16, these parameter values lie on the upper leg of region III.

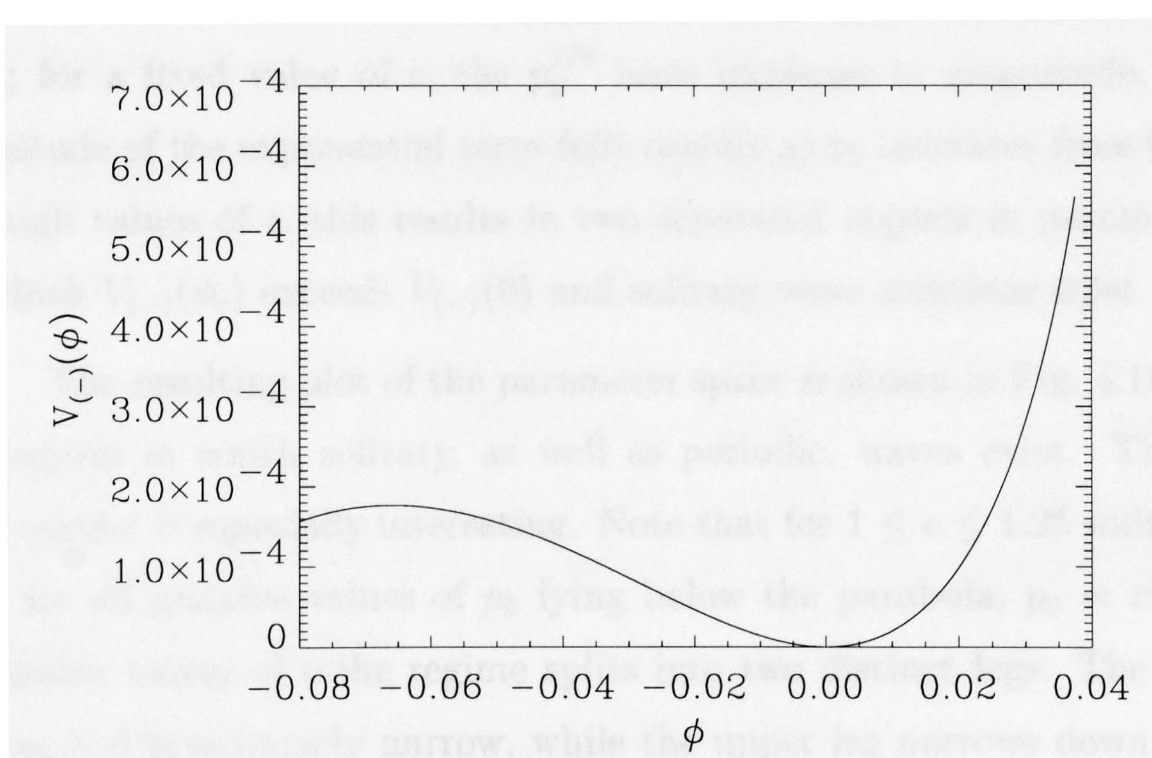


Figure 4.20: $V_{(-)}(\phi)$ for $c = 1.5$, $p_0 = 0.5$, showing the existence of periodic waves. In Fig. 4.16, these parameter values lie in region II.

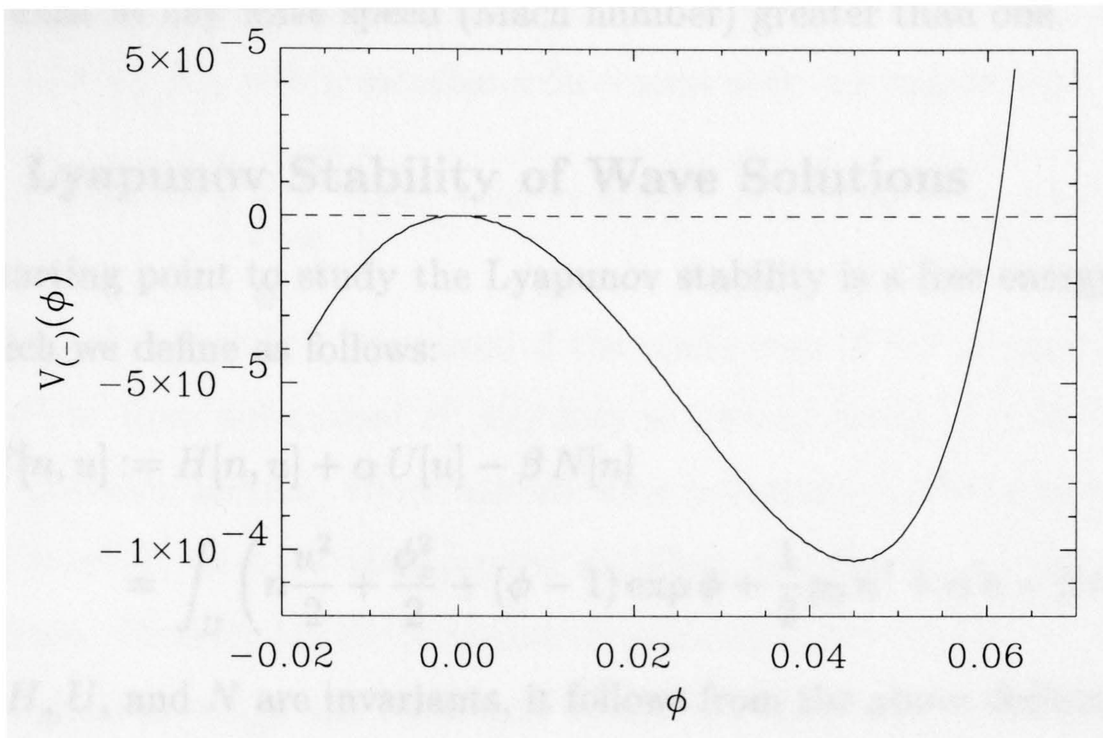


Figure 4.21: $V_{(-)}(\phi)$ for $c = 2$, $p_0 = 0.9$, showing the existence of a solitary wave of small amplitude ($\approx 0.06$). In Fig. 4.16, these parameter values lie on the upper leg of region III. Note that the wave speed $c = 2$ is higher than the upper cutoff of 1.58 for ion-acoustic equations without ionic pressure.

$V_{(-)}(0)$ is linear in $p_0$, while $V_{(-)}(\phi_c)$ is a somewhat more involved function of $p_0$; for a fixed value of $c$, the $p_0^{1/4}$ term increases in magnitude, while the magnitude of the exponential term falls rapidly as $p_0$ increases from 0 to $c/\sqrt{3}$. For high values of $c$, this results in two separated regions in parameter space for which $V_{(-)}(\phi_c)$ exceeds $V_{(-)}(0)$ and solitary wave solutions exist.

The resulting plot of the parameter space is shown in Fig. 4.16, showing the regions in which solitary, as well as periodic, waves exist. The solitary wave regime is especially interesting. Note that for $1 \leq c \leq 1.25$ solitary waves exist for all positive values of $p_0$ lying below the parabola, $p_0 = c^2/3 - 1/3$. For higher values of $c$ the regime splits into two distinct legs. The lower leg, near $p_0 = 0$ is extremely narrow, while the upper leg narrows down and stays close to $p_0 = c^2/3 - 1/3$. The lower leg is finite in length and no longer exists beyond $c = 1.58$, which is the usual cutoff speed for ion-acoustic solitary waves without the pressure term. The upper leg continues indefinitely, thus solitary waves exist at any wave speed (Mach number) greater than one.

## 4.6 Lyapunov Stability of Wave Solutions

The starting point to study the Lyapunov stability is a free energy functional $F$, which we define as follows:

$$
\begin{aligned}
F[n,u] &:= H[n,u] + \alpha\, U[u] - \beta\, N[n] \qquad (4.132)\\
&= \int_D \left( n\frac{u^2}{2} + \frac{\phi_x^2}{2} + (\phi - 1)\exp\phi + \frac{1}{2} p_0\, n^3 + \alpha\, u - \beta\, n \right) dx\,.
\end{aligned}
$$

Since $H$, $U$, and $N$ are invariants, it follows from the above definintion that $F$ is invariant. The reader may verify that the conditions for a critical point of $F$, given by $\delta F/\delta n = 0$ and $\delta F/\delta u = 0$, reproduce Eqs. (4.113). $F$ satisfies the criteria required of a free energy functional.

The free energy defined by Eqs. (4.132) is very similar to the free energy defined by Eqs. (4.32) for the case without pressure, and the second variation of the free energy,

$$
\delta^2 F[\delta n, \delta u] = \frac{1}{2}\int_D \Big[ n\,\delta u^2 + 2u\,\delta n\,\delta u + 3\,p_0\, n\,\delta n^2 + \delta\phi_x^2 + \delta\phi^2\,(1+\phi)\mathrm{e}^{\phi} \Big]\, dx\,, \qquad (4.133)
$$

differs from Eq. (4.35) by a single term, $3\,p_0\, n\,\delta n^2/2$. This positive definite term is capable of suppressing the negative energy perturbations, if it is large enough. To see this more easily, we isolate the terms involving $\delta u$ by completing the square, and rewrite Eq. (4.133):

$$
\delta^2 F[\delta n, \delta u] = \frac{1}{2}\int_{-\infty}^{+\infty} \left[ \frac{1}{n}(n\,\delta u + u\,\delta n)^2 + \delta\phi_x^2 + \delta\phi^2\,(1+\phi)\,\mathrm{e}^{\phi} + n\left(3\,p_0 - \frac{u^2}{n^2}\right)\delta n^2 \right] dx\,. \qquad (4.134)
$$

Following the same procedure as in Sec. 4.3.1, we pick the worst case variation of the velocity $\hat{\delta u}$, which satisfies $n\,\hat{\delta u} + u\,\delta n = 0$, leaving us with the energy,

$$\delta^2 \hat{F}[\delta n] = \frac{1}{2}\int_{-\infty}^{+\infty}\left[\delta\phi_x^2 + \delta\phi^2(1+\phi)\mathrm{e}^{\phi} + n\left(3\,p_0 - \frac{u^2}{n^2}\right)\delta n^2\right] dx\,. \tag{4.135}$$

It is clear that stability is assured if the coefficient of $\delta n^2$ is positive for all $x$. Since $u^2/n^2$ does not exceed $c^2$, stability is assured for $p_0 \geq c^2/3$, but as noted in the previous section, there are no wave solutions in the region $p_0 > c^2/3$. For $p_0 = c^2/3$, we have thus proved stability, but the waves are of vanishing amplitude. Thus for the finite amplitude solutions that interest us, there always exist regions in which the coefficient of $\delta n^2$ is negative.

We will not rigorously prove the existence of negative energy perturbations in this section, but offer instead an argument based on the findings of the somewhat lengthy treatment in Sec. 4.3.1. The negative energy perturbations that arose in the case without ionic pressure were due to the negative coefficient of $\delta n^2$. Recall that $\delta n = \exp\phi\,\delta\phi - \phi_{xx}$, hence a negative coefficient of $\delta n^2$ implies the same negative coefficient of the highest order term, $\delta\phi_{xx}^2$. A simple way to see that highest order derivative terms with negative coefficients imply the existence of negative energy perturbations is to note that in Fourier space $\delta\phi_{xx}^2$ corresponds to $k^4\,\delta\phi^2$, where we have assumed a single mode for simplicity. It is then clear that for high enough $k$, the $\delta\phi_{xx}^2$ term with the negative coefficient outweighs the other positive terms.

Yet another way to treat the matter in the case of solitary waves is to replace the $\psi_i^2$ term in the treatment in Sec. 4.3.1 by the minimum value of $u^2/n - 3\,n\,p_0$ in a region far enough away from the center of the solitary wave so that $u^2/n - 3\,n\,p_0$ is positive. Then the same Gaussian perturbation constructed in Sec. 4.3.1 may be used to prove the existence of negative energy perturbations. (The Gaussians would have to be placed farther away, and have smaller half widths than in the case of solitary waves without ionic pressure.)

For periodic waves, the perturbation may be chosen concentrated in the region where $3\,n\,p_0 - u^2/n$ is negative and close to zero elsewhere, to get negative energy perturbations. The effect of the pressure is to push the negative energy perturbations to shorter wavelengths, but is not enough to suppress them.

## 4.7 KdV Limit of Ion-Acoustic Equations

The KdV equation can be derived from the ion-acoustic equations (without ionic pressure) by a transformation (Washimi and Taniuti, 1966; Horton and Ichikawa, 1996, Sec. 5.4, for a concise treatment), which is an example of *reductive perturbation theory* (Taniuti, 1974) formulated later. The spatial variable is transformed to one moving at unit velocity and $n$, $v$, $\phi$, are expressed as perturbation series in a small parameter $\epsilon$. The most striking feature of the transformation is the scaling of time and space variables by powers of $\epsilon$:

$$\xi := \epsilon^{1/2}(x - t)\,, \qquad \tau := \epsilon^{3/2} t\,, \tag{4.136}$$

whereupon the derivatives transform to $\partial/\partial t = \epsilon^{3/2}\,\partial/\partial\tau - \epsilon^{1/2}\,\partial/\partial\xi$ and $\partial/\partial x = \epsilon^{1/2}\,\partial/\partial\xi$, and Eqs. (4.6) are transformed to

$$\begin{aligned} \epsilon\frac{\partial n}{\partial\tau} + \frac{\partial}{\partial\xi}(nv - n) &= 0\,, \\ \epsilon\frac{\partial v}{\partial\tau} + \frac{\partial}{\partial\xi}\left(\frac{v^2}{2} + \phi - v\right) &= 0\,, \\ \epsilon\frac{\partial^2\phi}{\partial\xi^2} &= \exp\phi - n\,. \end{aligned} \tag{4.137}$$

In Eqs. (4.137), $n$, $v$, and $\phi$ are transformed as scalars, and we now express them as series expansions:

$$\begin{aligned} n &= 1 + \epsilon\, n_1 + \epsilon^2\, n_2 + \cdots\,, \\ v &= \epsilon\, v_1 + \epsilon^2\, v_2 + \cdots\,, \\ \phi &= \epsilon\,\phi_1 + \epsilon^2\,\phi_2 + \cdots\,. \end{aligned} \tag{4.138}$$

It is evident that to zeroth order in $\epsilon$ we have assumed unit number-density, vanishing velocity and electric potential.

Using the series expressions for $n$, $v$, and $\phi$ in Eqs. (4.137) gives, to first order in $\epsilon$,

$$n_1 = v_1 = \phi_1 \,. \tag{4.139}$$

Unlike ordinary perturbation theory, the evolution of the first order quantities is determined at the next order. The second order quantities satisfy the equations,

$$\begin{aligned}
\frac{\partial n_1}{\partial \tau} + \frac{\partial}{\partial \xi}(n_1 v_1 + v_2 - n_2) &= 0\,, \\
\frac{\partial v_1}{\partial \tau} + \frac{\partial}{\partial \xi}\left(\frac{v_1^2}{2} + \phi_2 - v_2\right) &= 0\,, \\
\frac{\partial^2 \phi_1}{\partial \xi^2} - \frac{\phi_1^2}{2} &= \phi_2 - n_2 \,.
\end{aligned} \tag{4.140}$$

We can now eliminate all second order quantities by adding the first two of Eqs. (4.140) and using the third equation to express the quantity $(\phi_2 - n_2)$ in terms of first order quantities. Furthermore it is clear from Eqs. (4.139) that we can eliminate both, $v_1$ and $\phi_1$. The result is the KdV equation in terms of $n_1$ alone:

$$\frac{\partial n_1}{\partial \tau} + n_1 \frac{\partial n_1}{\partial \xi} + \frac{1}{2}\frac{\partial^3 n_1}{\partial \xi^3} = 0\,. \tag{4.141}$$

Note that the standard form of the KdV equation, without the factor 1/2 multiplying the final term in the equation above, is obtained by scaling $n_1$ and $\tau$ by 1/2 and 2 respectively.

### 4.7.1 KdV Free Energy

Seeking travelling wave solutions of Eq. (4.141), i.e. with dependence on $\zeta := \xi - C\tau$, leads to the second order nonlinear ordinary differential equation,

$$n_1'' + n_1^2 - 2Cn_1 = 0\,, \tag{4.142}$$

where the prime denotes differentiation with respect to $\zeta$, and we have integrated once, setting the constant of integration to zero. Eq. (4.142) may also be obtained as a condition for extremizing a free energy $F_{\mathrm{KdV}}$, which is a linear combination of two invariants $M$ and $V$, of the KdV equation:

$$\begin{aligned} F_{\mathrm{KdV}}[n_1] &:= M[n_1] + C\,V[n_1] \\ &= \int_{-\infty}^{+\infty} d\zeta \left[\frac{1}{2}(n_1')^2 - \frac{n_1^3}{3}\right] + C \int_{-\infty}^{+\infty} d\zeta\, n_1^2\,. \end{aligned} \tag{4.143}$$

It is easily checked that the first variation, $\delta F_{\mathrm{KdV}}/\delta n_1$, set to zero, gives Eq. (4.142), which has well known soliton solutions parametrized by $C$. The positivity of $\delta^2 F_{\mathrm{KdV}}$ with respect to a certain norm, and hence the Lyapunov stability of KdV solitons has been established (Benjamin, 1972; Bona, 1975). See also (Bona and Soyeur, 1994) and references therein.

### 4.7.2 Mechanism of Removal of Ion-Acoustic Negative Energy Perturbations in the KdV Limit

In order to make a transition from the free energy for ion-acoustic waves to that for the KdV equation, we seek to define the free energy for solitary wave solutions of ion-acoustic equations in the form of Eqs. (4.137) rather than Eqs. (4.6). The reason for this lies in the time dependence of the coordinate transformation given by Eq. (4.136). The free energy, on the other hand, is a spatial integral; hence Eq. (4.34) cannot serve as a starting point for our argument.

Assuming wave solutions that depend on $\zeta = \xi - C\,\tau$, the first two of Eqs. (4.137) are easily integrated, yielding

$$n\,v - n\,(1 + \epsilon C) = \alpha \quad \text{and} \quad \frac{v^2}{2} + \phi - v\,(1 + \epsilon C) = \beta\,. \tag{4.144}$$

The constants $\alpha$ and $\beta$ are easily determined for solitary wave solutions, assuming $n = 1$ and $v = 0$ asymptotically. The result is $\alpha = -1 - \epsilon C$ and $\beta = 0$.

The Hamiltonian for the system represented by Eqs. (4.137) is

$$H_{\text{ia}}[n,v;\epsilon] := \frac{1}{\epsilon}\int_{-\infty}^{+\infty}\left(n\,\frac{v^2}{2} - n\,v + \frac{\epsilon}{2}\,\phi_\xi^2 + (\phi-1)\,\mathrm{e}^{\phi} + 1\right)d\xi\,, \tag{4.145}$$

while the definition of the Poisson bracket of Eq. (4.9) is retained. The constant, 1, has been added to the Hamiltonian so that the integrand vanishes asymptotically. The invariants $N$, $U$, and $P$, of the ion-acoustic system of Eqs. (4.6) are also invariants of Eqs. (4.137), as may be verified by the reader. Thus we define the free energy,

$$\begin{aligned} F_{\text{ia}}[n,v;\epsilon] &:= \epsilon\,H_{\text{ia}} - \epsilon\,C\,P + (1+\epsilon\,C)\,U \\ &= \int_{-\infty}^{+\infty}\Bigg(n\,\frac{v^2}{2} + \frac{\epsilon}{2}\,\phi_\zeta^2 + (\phi-1)\,\mathrm{e}^{\phi} + 1 \\ &\qquad\qquad + (1+\epsilon\,C)\,(v - n\,v)\Bigg)d\zeta\,, \end{aligned} \tag{4.146}$$

The reader may verify that Eqs. (4.144) are the conditions for a critical point of the free energy $F_{\text{ia}}$.

Substituting the series expansions for $n$, $v$, and $\phi$ given by Eqs. (4.138) in the expression for the free energy, we get the following series expression:

$$\begin{aligned} F_{\text{ia}}[n_1,v_1,n_2,v_2,\cdots;\epsilon] = \int_{-\infty}^{+\infty} d\zeta \Bigg\{ &\epsilon^2\left[\frac{v_1^2}{2} - n_1\,v_1 + \frac{\phi_1^2}{2}\right] \\ &+\epsilon^3\Bigg[v_2\,(v_1-n_1) + (\phi_1\,\phi_2 - n_2\,v_1) + \left(n_1\frac{v_1^2}{2} + \frac{\phi_1^3}{3}\right) \\ &+\frac{1}{2}\left(\frac{\partial\phi_1}{\partial\zeta}\right)^2 - C\,n_1\,v_1\Bigg] + \mathcal{O}(\epsilon^4)\Bigg\}. \end{aligned} \tag{4.147}$$

Note that $\phi_1\,,\phi_2\,,\cdots$ are not independent; they are related to $n_1\,,n_2\,,\cdots$ via the Poisson equation, given by the third of Eqs. (4.137). Equating terms of order $\epsilon$ and $\epsilon^2$ in the Poisson equation, gives the relations

$$\phi_1 = n_1 \quad \text{and} \quad \frac{\partial^2\phi_1}{\partial\xi^2} - \frac{\phi_1^2}{2} = \phi_2 - n_2\,, \tag{4.148}$$

which we have already encountered as parts of Eqs. (4.139) and (4.140). Equations (4.148) play an important role in what follows; note that they are entirely a consequence of the $\epsilon$ ordering.

We isolate the lowest order term in $\epsilon$ from Eq. (4.147),

$$\begin{aligned} F_{\rm ia}^{(0)}[n_1, v_1] &:= \int_{-\infty}^{+\infty} \left( \frac{v_1^2}{2} - n_1\, v_1 + \frac{n_1^2}{2} \right) d\zeta \\ &= \frac{1}{2} \int_{-\infty}^{+\infty} \left(v_1 - n_1\right)^2 d\zeta \,, \end{aligned} \tag{4.149}$$

where we have eliminated $\phi_1$ using $\phi_1 = n_1$. The critical point of $F_{\rm ia}^{(0)}[n_1, v_1]$ is given by the conditions $\delta F_{\rm ia}^{(0)}/\delta n_1 = 0$ and $\delta F_{\rm ia}^{(0)}/\delta v_1 = 0$, or simply by observation in this case, yielding

$$n_1 = v_1 \,, \tag{4.150}$$

at the critical point. We have thus recovered Eq. (4.139). Furthermore, it is evident from

$$\delta^2 F_{\rm ia}^{(0)}[\delta n_1, \delta v_1] = \frac{1}{2} \int_{-\infty}^{+\infty} \left(\delta v_1 - \delta n_1\right)^2 d\zeta \,, \tag{4.151}$$

(or simply by observation) that the critical point is a minimum and $\delta^2 F_{\rm ia}^{(0)}$ is positive definite. Observe that perturbations for which $\delta n_1 = \delta v_1$ are exceptions to the positive definiteness of $\delta^2 F_{\rm ia}^{(0)}$. These are perturbations that connect neighbouring equilibrium states, which is related to spontaneous symmetry breaking and existence of null eigenfunctions. With the exception of such perturbations, we have succeeded in proving Lyapunov stability in the KdV ordering, far more easily than proving stability of KdV solitons.

The main reason for the removal of negative energy perturbations, which are present in the ion-acoustic fluid, is the $\epsilon$ ordering involved in taking the KdV limit, which leads to

$$n = \mathrm{e}^{\phi} - \epsilon\, \phi'' \,, \tag{4.152}$$

rather than

$$n = \mathrm{e}^{\phi} - \phi'' , \tag{4.153}$$

which is true in the absence of any assumption about ordering. As we have already noted in Secs. (4.3.1) and (4.3.2), the negative energy perturbations are at short wavelengths. Comparision of Eqs. (4.152) and (4.153) shows that the dispersion is assumed to be small in the KdV limit, effectively restricting the ion-acoustic equations to long wavelengths, and avoiding the problematic short wavelengths. The latter is not unexpected; it is well known that the KdV equation models small amplitude, long wavelength modes in fluids (Su and Gardner, 1969, for example).

One final point to note is that the KdV free energy, defined by Eq. (4.143), is seen to be *negative* of the free energy $F_{\mathrm{ia}}$ defined by Eq. (4.147), up to leading order in $\epsilon$, when evaluated at the critical point, $n_1 = v_1$. It is evident that the term of order $\epsilon^2$ in Eq. (4.147) vanishes at the critical point. Among terms of order $\epsilon^3$, $(\phi_1\,\phi_2 - n_2\,v_1)$ may be written as $n_1\,(\phi_2 - n_2)$, which can then be expressed in terms of $n_1$ alone, using Eqs. (4.148). Collecting all the terms written in terms of $n_1$, and after integrating by parts once, we get

$$\begin{aligned} \mathcal{F}_{\mathrm{ia}} &:= \frac{1}{\epsilon^3} F_{\mathrm{ia}}\bigg|_{n_1 = v_1} \\ &= - \int_{-\infty}^{+\infty} \left\{ \left( \frac{1}{2} \left( \frac{\partial n_1}{\partial \zeta} \right)^2 - \frac{n_1^3}{3} \right) + C\, n_1^2 + \mathcal{O}(\epsilon) \right\} d\zeta \, . \end{aligned} \tag{4.154}$$

It is clear, upon comparision of Eq. (4.154) to Eq. (4.143), that neglecting terms of order $\epsilon$, the free energy $\mathcal{F}_{\mathrm{ia}}$ is identical to $F_{\mathrm{KdV}}$, except for a minus sign.

# Chapter 5

# Numerical Study of Stability of Ion-Acoustic Waves

As seen in Sec. 4.3, Lyapunov stability of ion-acoustic waves cannot be established due to the existence of negative energy perturbations, i.e. there exist perturbations for which the second variation of the free energy is negative. Is the ion-acoustic solitary wave unstable with respect to these negative energy perturbations? In order to address this question, we construct an implicit finite difference scheme to explore the time evolution of the initial value problem. We bear in mind that numerical algorithms cannot definitively prove instability, since the concept of stability is an infinitesimal one, however we can address the question of instability with respect to small, finite amplitude perturbations.

The implicit finite difference scheme (Candy, 1998; Press et al., 1992, Ch. 19) that is used for the numerical computation is explained in Sec. 5.1. The results of three numerical simulations are contained in Sec. 5.2. The first simulation shows the evolution of a solitary wave with an initial negative energy perturbation. The perturbation amplitude is seen to grow rapidly and drops after the solitary wave peak moves to the right, leaving the perturbation behind it.

In the second numerical simulation, this behavior is contrasted with the evolution of a solitary wave with an initial positive energy perturbation, which exhibits no growth of the perturbation size. In the figures shown here, which are all of the density $n(x,t)$, the initial profiles of the first and second numerical simulations are identical. In this regard, note that the difference between the positive and negative energy perturbations that are used here, lies in the velocity perturbation, as explained later in Sec. 5.2.

The third numerical simulation shows the evolution of the uniform background of unit Mach number with an initial perturbation which is identical to the negative energy perturbation of the solitary wave seen earlier. Again, transients with increased amplitude are observed which decay in time as the perturbation moves to the left. The Lyapunov stability of the uniform background has been established at all speeds in Sec. 4.3.

## 5.1 Finite Difference Scheme

The equations of interest are given by Eqs. (4.6), but in a frame moving with the solitary wave, so that $v$ may be replaced by $u$, as seen in Sec. 4.1.3. Since finite difference schemes are easier to solve when the solutions vanish at the edges of the simulation box, we carry out the transformation,

$$n \to 1+n \qquad \text{and} \qquad u \to -c+u\,, \tag{5.1}$$

which leads to the equations,

$$\begin{aligned}
\frac{\partial n}{\partial t} + \frac{\partial}{\partial x}(n\,u - n\,c + u) &= 0\,,\\
\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(\frac{u^2}{2} - c\,u + \phi\right) &= 0\,,\\
\frac{\partial^2 \phi}{\partial x^2} &= \mathrm{e}^{\phi} - 1 - n\,.
\end{aligned} \tag{5.2}$$

Equations (5.2) are the nonlinear model equations that we wish to solve numerically.

We use the notation $\hat{z}$ to denote the value of the function $z$ at the next time step, a time $\Delta t$ later. (The symbol $z$ denotes one of $n$, $u$, or $\phi$.)

The $x$-grid points are each separated by $\Delta x$, and numbered from 1 to $N$; the subscript $i$ is used to denote the value of a function at the $i$th grid point. In addition to propagating $z_i$ forward in time, we intend to introduce one more tier of iteration, explained shortly hereafter; for this purpose we introduce the superscript $(j+1)$. The discretized equations may thus be written as:

$$\frac{1}{\Delta t}\left(\hat{n}_i^{(j+1)} - n_i\right) + \frac{1}{2\,\Delta x}\left(\hat{n}_{i+1}^{(j+1)}\,\hat{u}_{i+1}^{(j+1)} - \hat{n}_{i-1}^{(j+1)}\,\hat{u}_{i-1}^{(j+1)}\right.$$
$$\left. - \,c\,\hat{n}_{i+1}^{(j+1)} + c\,\hat{n}_{i-1}^{(j+1)} + \hat{u}_{i+1}^{(j+1)} - \hat{u}_{i-1}^{(j+1)}\right) = 0\,, \quad (5.3)$$

$$\frac{1}{\Delta t}\left(\hat{u}_i^{(j+1)} - u_i\right) + \frac{1}{2\,\Delta x}\left(\frac{1}{2}\left(\hat{u}_{i+1}^{(j+1)}\right)^2 - \frac{1}{2}\left(\hat{u}_{i-1}^{(j+1)}\right)^2\right.$$
$$\left. - \,c\,\hat{u}_{i+1}^{(j+1)} + c\,\hat{u}_{i-1}^{(j+1)} + \hat{\phi}_{i+1}^{(j+1)} - \hat{\phi}_{i-1}^{(j+1)}\right) = 0\,, \quad (5.4)$$

$$\frac{1}{\Delta x^2}\left(\hat{\phi}_{i+1}^{(j+1)} - 2\,\hat{\phi}_i^{(j+1)} + \hat{\phi}_{i-1}^{(j+1)}\right) + 1 - \exp\hat{\phi}_i^{(j+1)} + \hat{n}_i^{(j+1)} = 0\,. \quad (5.5)$$

As may be expected, our intention is to linearize the above equations and solve for the small differences $\hat{z}_i - z_i$. Since the equations are nonlinear, we use Newton's method to find the solution at each time step; this is the second tier of iteration for which we introduced the superscript $(j+1)$ above. The $(j+1)$th approximation to the true solution $\hat{z}$ is thus expressed by

$$\hat{z}_i^{(j+1)} = \hat{z}_i^{(j)} + \delta z_i^{(j)}\,, \qquad j = 0, 1, 2, \cdots, J\,. \quad (5.6)$$

Discarding terms that are not linear in $\delta n$, $\delta u$, and $\delta\phi$, we obtain the equations,

$$-\frac{1}{\Delta t}\,\delta n_i^{(j)} - \frac{1}{2\,\Delta x}\left[\delta n_{i+1}^{(j)}\left(\hat{u}_{i+1}^{(j)} - c\right) + \delta u_{i+1}^{(j)}\left(\hat{n}_{i+1}^{(j)} + 1\right)\right.$$
$$\left. - \,\delta n_{i-1}^{(j)}\left(\hat{u}_{i-1}^{(j)} - c\right) - \delta u_{i-1}^{(j)}\left(\hat{n}_{i-1}^{(j)} + 1\right)\right]$$
$$= \frac{1}{\Delta t}\left(\hat{n}_i^{(j)} - n_i\right) + \frac{1}{2\,\Delta x}\left(\hat{n}_{i+1}^{(j)}\,\hat{u}_{i+1}^{(j)} - \hat{n}_{i-1}^{(j)}\,\hat{u}_{i-1}^{(j)}\right.$$
$$\left. - \,c\,\hat{n}_{i+1}^{(j)} + c\,\hat{n}_{i-1}^{(j)} + \hat{u}_{i+1}^{(j)} - \hat{u}_{i-1}^{(j)}\right)\,, \quad (5.7)$$

$$
\begin{aligned}
&- \frac{1}{\Delta t}\,\delta u_i^{(j)} - \frac{1}{2\,\Delta x}\left[\delta u_{i+1}^{(j)}\left(\hat{u}_{i+1}^{(j)} - c\right) - \delta u_{i-1}^{(j)}\left(\hat{u}_{i-1}^{(j)} - c\right)\right. \\
&\qquad\qquad\qquad\qquad \left. + \, \delta\phi_{i+1}^{(j)} - \delta\phi_{i-1}^{(j)}\right] \\
&\qquad = \frac{1}{\Delta t}\left(\hat{u}_i^{(j)} - u_i\right) + \frac{1}{2\,\Delta x}\left(\frac{1}{2}\left(\hat{u}_{i+1}^{(j)}\right)^2 - \frac{1}{2}\left(\hat{u}_{i-1}^{(j)}\right)^2\right. \\
&\qquad\qquad\qquad\qquad \left. - \, c\,\hat{u}_{i+1}^{(j)} + c\,\hat{u}_{i-1}^{(j)} + \hat{\phi}_{i+1}^{(j)} - \hat{\phi}_{i-1}^{(j)}\right) , \qquad (5.8)
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{1}{\Delta x^2}\left(\delta\phi_{i+1}^{(j)} - 2\,\delta\phi_i^{(j)} + \delta\phi_{i-1}^{(j)}\right) + \delta\phi_i^{(j)}\,\exp\hat{\phi}_i^{(j)} - \delta n_i^{(j)} \\
&\qquad = \frac{1}{\Delta x^2}\left(\hat{\phi}_{i+1}^{(j)} - 2\,\hat{\phi}_i^{(j)} + \hat{\phi}_{i-1}^{(j)}\right) + 1 - \exp\hat{\phi}_i^{(j)} + \hat{n}_i^{(j)} . \qquad (5.9)
\end{aligned}
$$

Observe that when $\delta n$, $\delta u$, and $\delta\phi$ vanish, Eqs. (5.7)–(5.9) are identical to the nonlinear discretized equations given by Eqs. (5.3)–(5.5).

The algorithm successively solves for $\delta z_i^{(0)}, \delta z_i^{(1)}, \cdots$ until it vanishes, or falls below a very small cutoff value; this cutoff value determines the maximum number of iterations, $J+1$. At the final Newton-Raphson iteration, we obtain the approximate solution,

$$
\hat{z}_i := \hat{z}_i^{(J)} . \qquad (5.10)
$$

Note that for $j = 0$, $\hat{z}_i^{(0)}$ is set equal to the value at the previous time step, $z_i$.

In matrix form, Eqs. (5.7)–(5.9) may be written as:

$$
A^{(j)} \cdot \delta z^{(j)} = B^{(j)} , \qquad \text{(no sum on } j) \qquad (5.11)
$$

where $z := \left(n_1, u_1, \phi_1, n_2, u_2, \phi_2, \cdots, n_N, u_N, \phi_N\right)$. The matrices $A^{(j)}$ and $B^{(j)}$ are functions of $z$ and the step sizes. The dimension of the matrix $A^{(j)}$ is $3\,N \times 3\,N$, but the non-zero values are contained only in a narrow diagonal band, with three lower diagonals and four upper diagonals.

One way to represent the matrix $A^{(j)}$ is in terms of blocks of $3 \times 3$ matrices:

$$A^{(j)} = \begin{bmatrix} D_1^{(j)} & U_2^{(j)} & & \cdots & & & \\ L_1^{(j)} & D_2^{(j)} & U_3^{(j)} & \cdots & & & \\ & L_2^{(j)} & D_3^{(j)} & U_4^{(j)} \cdots & & & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ & & & \cdots & L_{N-3}^{(j)} & D_{N-2}^{(j)} & U_{N-1}^{(j)} \\ & & & \cdots & & L_{N-2}^{(j)} & D_{N-1}^{(j)} \; U_N^{(j)} \\ & & & \cdots & & & L_{N-1}^{(j)} \; D_N^{(j)} \end{bmatrix}, \tag{5.12}$$

where all unspecified elements are zero. The blocks on the diagonal are given by

$$D_i^{(j)} = \begin{bmatrix} 2\,\Delta x & 0 & 0 \\ 0 & 2\,\Delta x & 0 \\ -\Delta x^2 & 0 & 2 + \Delta x^2 \exp \hat{\phi}_i^{(j)} \end{bmatrix}. \tag{5.13}$$

The blocks above the diagonal are given by

$$U_i^{(j)} = \begin{bmatrix} \Delta t\left(\hat{u}_i^{(j)} - c\right) & \Delta t\left(1 + \hat{n}_i^{(j)}\right) & 0 \\ 0 & \Delta t\left(\hat{u}_i^{(j)} - c\right) & \Delta t \\ 0 & 0 & -1 \end{bmatrix}, \tag{5.14}$$

while the blocks below the diagonal are given by

$$L_i^{(j)} = \begin{bmatrix} -\Delta t\left(\hat{u}_i^{(j)} - c\right) & -\Delta t\left(1 + \hat{n}_i^{(j)}\right) & 0 \\ 0 & -\Delta t\left(\hat{u}_i^{(j)} - c\right) & -\Delta t \\ 0 & 0 & -1 \end{bmatrix}. \tag{5.15}$$

The matrix elements of $B^{(j)}$ are given by

$$\begin{aligned} B_{3i-2}^{(j)} = -2\Delta x\big(\hat{n}_i^{(j)} - n_i\big) - \Delta t\big(&\hat{n}_{i+1}^{(j)}\,\hat{u}_{i+1}^{(j)} - \hat{n}_{i-1}^{(j)}\,\hat{u}_{i-1}^{(j)} \\ &- c\,\hat{n}_{i+1}^{(j)} + c\,\hat{n}_{i-1}^{(j)} + \hat{u}_{i+1}^{(j)} - \hat{u}_{i-1}^{(j)}\big), \end{aligned} \tag{5.16}$$

$$\begin{aligned} B_{3i-1}^{(j)} = -2\Delta x\big(\hat{u}_i^{(j)} - u_i\big) - \Delta t\bigg(&\frac{1}{2}\left(\hat{u}_{i+1}^{(j)}\right)^2 - \frac{1}{2}\left(\hat{u}_{i-1}^{(j)}\right)^2 \\ &- c\,\hat{u}_{i+1}^{(j)} + c\,\hat{u}_{i-1}^{(j)} + \hat{\phi}_{i+1}^{(j)} - \hat{\phi}_{i-1}^{(j)}\bigg), \end{aligned} \tag{5.17}$$

$$B_{3i}^{(j)} \;=\; \hat{\phi}_{i+1}^{(j)} \;-\; 2\,\hat{\phi}_i^{(j)} \;+\; \hat{\phi}_{i-1}^{(j)} \;+\; \Delta x^2 \big(1 \;+\; \hat{n}_i^{(j)} \;-\; \exp \hat{\phi}_i^{(j)}\big)\,, \tag{5.18}$$

where $i = 1, 2, \ldots, N$. Note that for $i = 1$ and $i = N$, some of terms in the definition of the elements of $B^{(j)}$ do not exist; these are taken to be zero. For instance, the definition of $\left[B^{(j)}\right]_1$ involves $\hat{n}_0^{(j)}$ and $\hat{u}_0^{(j)}$, which are assumed to vanish. It is for this reason that we require the solutions to be zero at the edges of the simulation box; when the solution is non-zero near the edges, the numerical results are no longer trustworthy.

The numerical code takes the initial data, and propagates it forward in time at each $x$ grid point. At each time step, the matrices $A^{(j)}$ and $B^{(j)}$ are redefined, and an algorithm DGBTRF, developed by the Numerical Algorithms Group (NAG), factorizes the matrix $A^{(j)}$ into upper and lower triangular matrices for faster inversion. Another NAG algorithm DGBTRS, then inverts the matrix equation, yielding solutions to $\delta z_i^{(j)}$. The Newton-Raphson method iterations continue till the sum over $i$ of absolute values of all the $\delta z_i^{(j)}$'s falls below a cutoff value, which has been set at $10^{-12}$ for the calculations presented here. (For this cutoff value, it was observed that three iterations were performed at almost every time step.) The $z$ values are then reset, and the procedure is repeated at every time step.

## 5.2 Numerical Results

The first numerical simulation computes the time evolution of a solitary wave ($c = 1.23$) with a negative energy perturbation, which is defined by

$$\delta u = -\frac{u - c}{1 + n}\,\delta n\,, \qquad \delta n = \mathrm{e}^{\phi}\,\delta\phi - \delta\phi_{xx}\,, \tag{5.19}$$

where $\delta\phi$ is the sum of two Gaussians, as in Sec. 4.3.1. The Gaussian half-width is set to $1/2$, which is shorter than necessary, as may be seen from Fig. 4.14. The Gaussians are centered five Debye lengths to either side of the solitary

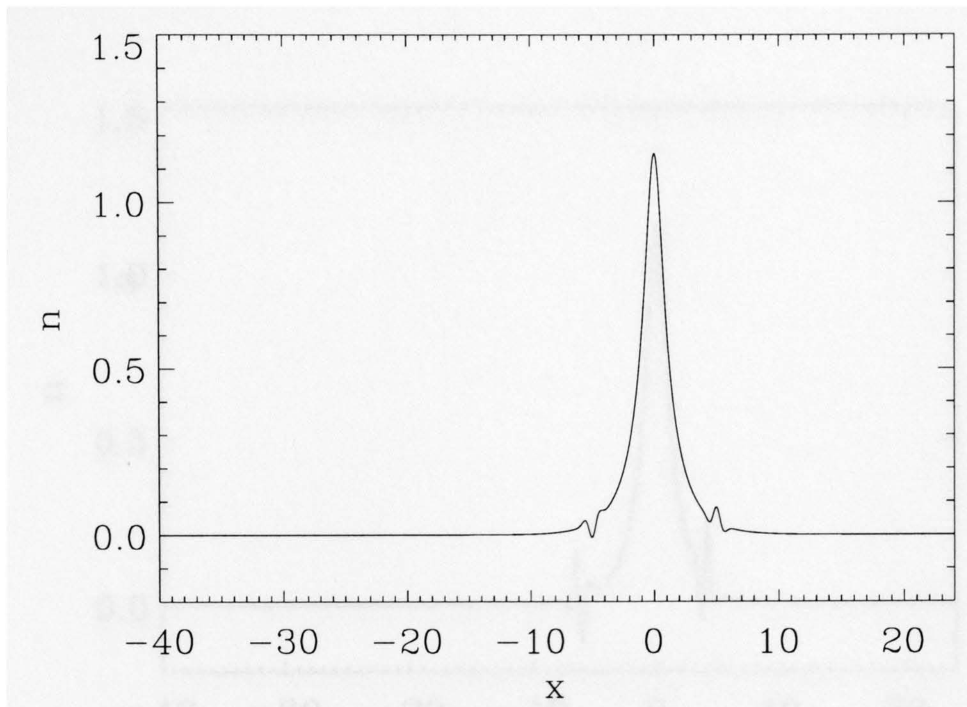


Figure 5.1: Solitary wave profile with a negative energy perturbation (see text) at $t = 0$.

wave peak. These settings generate a perturbation which gives rise to negative $\delta^2 F$.

The initial solitary wave profile and perturbations were generated using Mathematica. The perturbation amplitude is set to $g = 0.05$. Runs were also made for amplitude $g = 0.1$, with no qualitative difference in the results. The time step is $\Delta t = 0.005$, and the $x$ grid has 6401 points separated by $\Delta x = 0.01$. Runs were made with time steps twice and ten times 0.005, with minor quantitative differences.

The results are shown in Figs. 5.1–5.8. Observe that the perturbation quickly grows in amplitude initially and then diminishes. The solitary wave peak moves through the perturbation, coming out intact on the right. The fine structures, seen in Figs. 5.4 and (5.5), are not subgrid phenomena; there are approximately 100 grid points spanning each structure. Figures 5.9 and 5.10 show that the fine structures are well resolved.

The second numerical simulation that is presented here, computes the

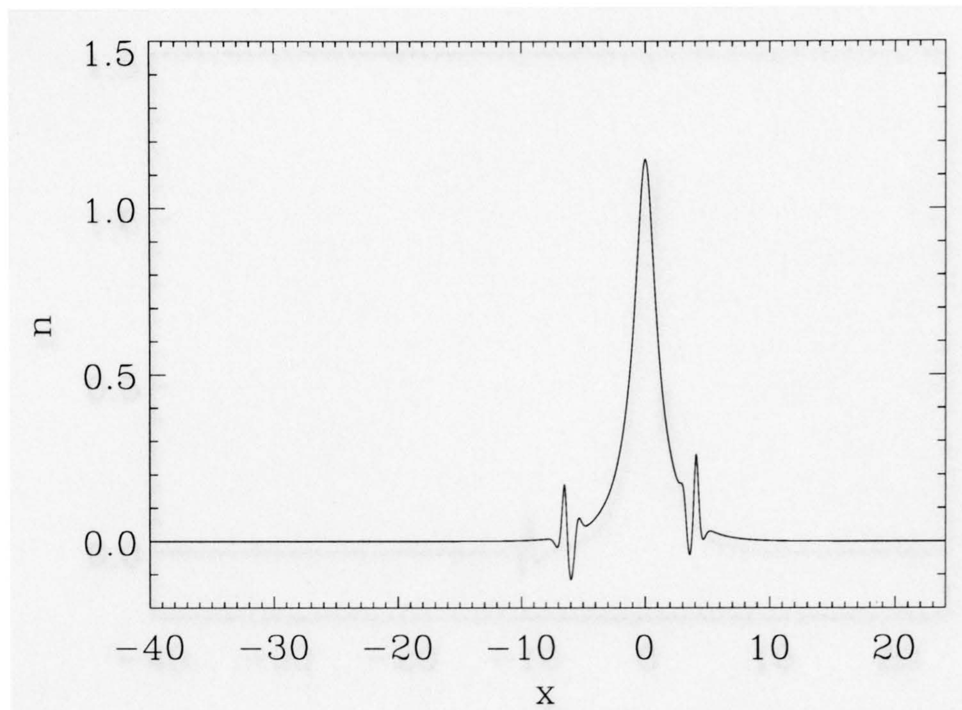


Figure 5.2: Evolution of the perturbed solitary wave at $t = 1$. Note the increased amplitude of the initial perturbation.

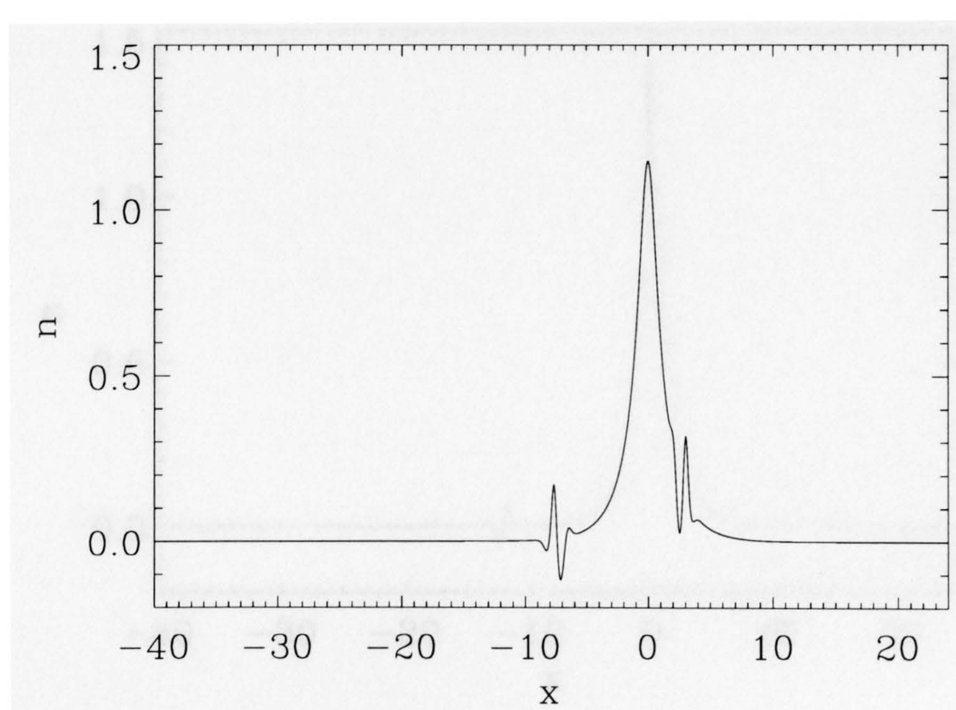


Figure 5.3: Evolution of the perturbed solitary wave at $t = 2$.

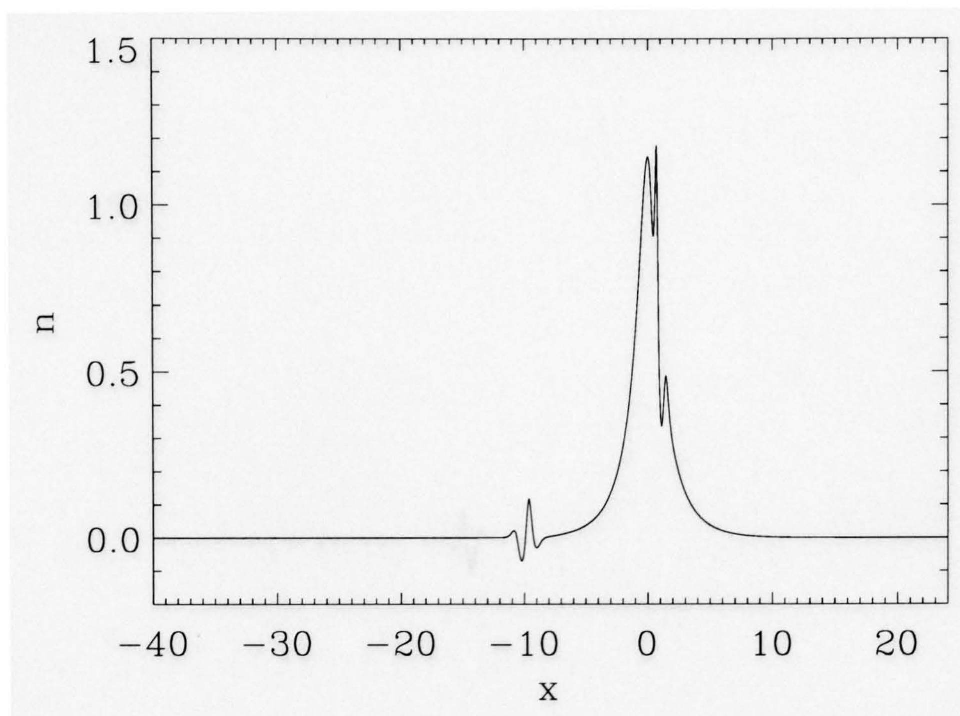


Figure 5.4: Evolution of the perturbed solitary wave at $t = 4$.

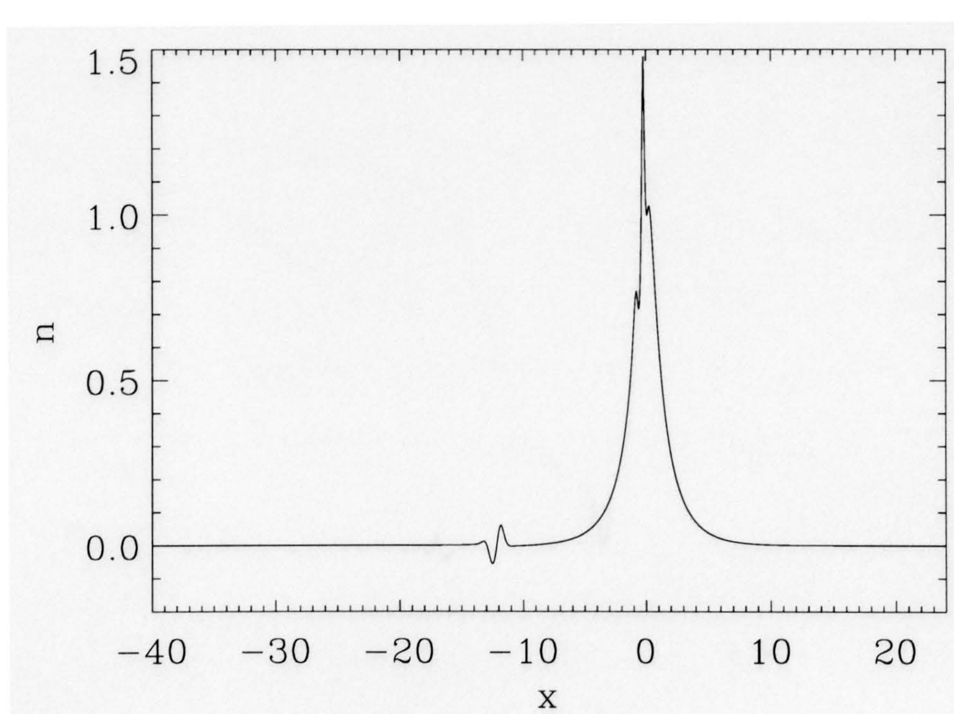


Figure 5.5: Evolution of the perturbed solitary wave at $t = 6$.

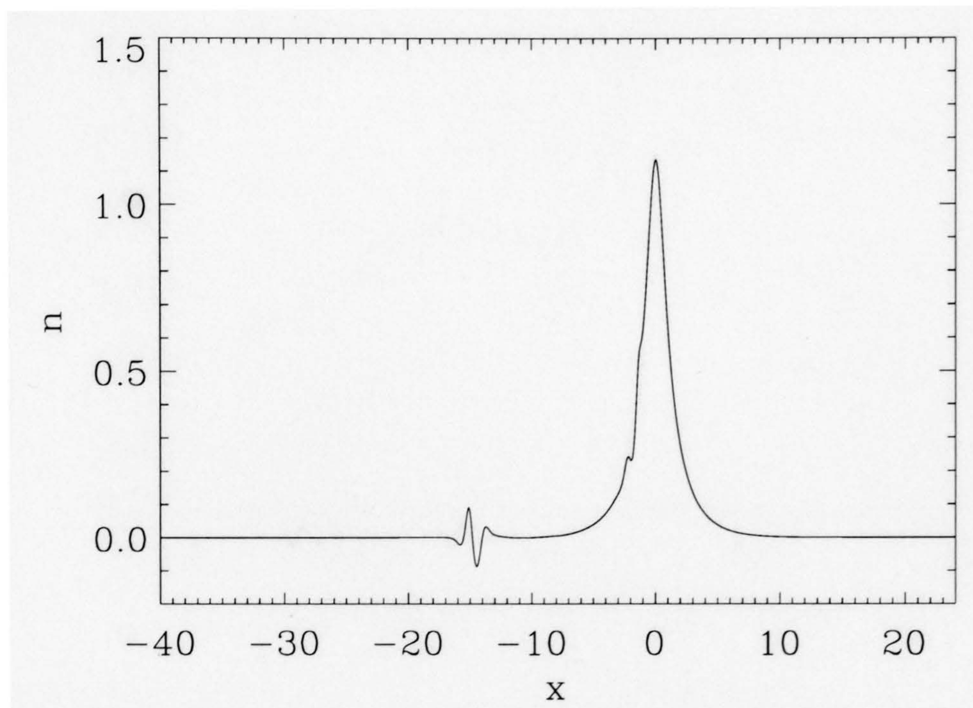


Figure 5.6: Evolution of the perturbed solitary wave at $t = 8$.

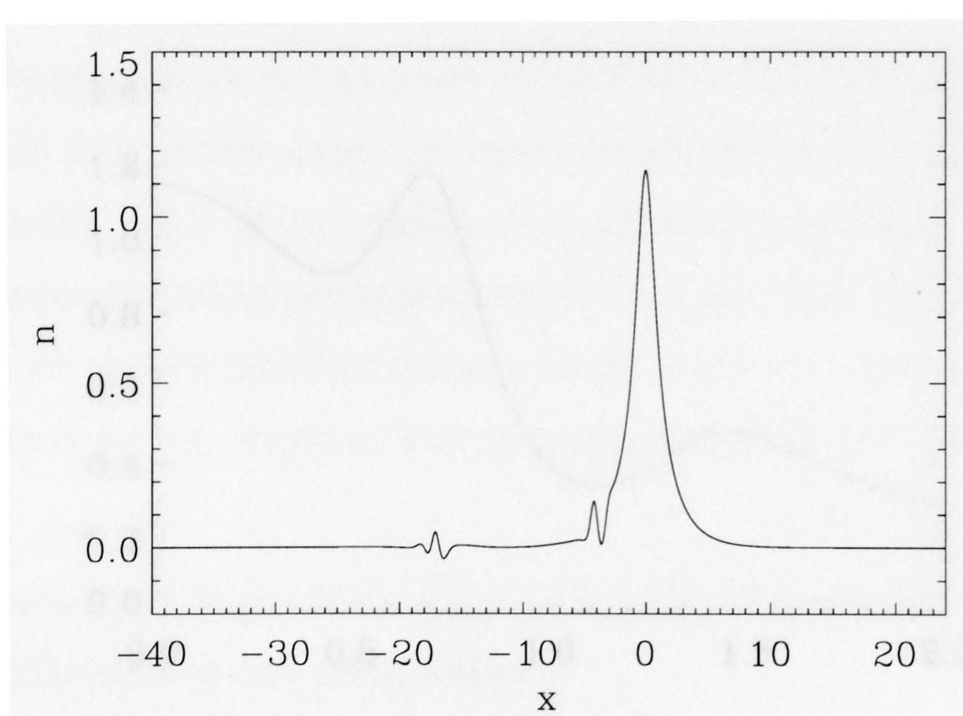


Figure 5.7: Evolution of the perturbed solitary wave at $t = 10$.

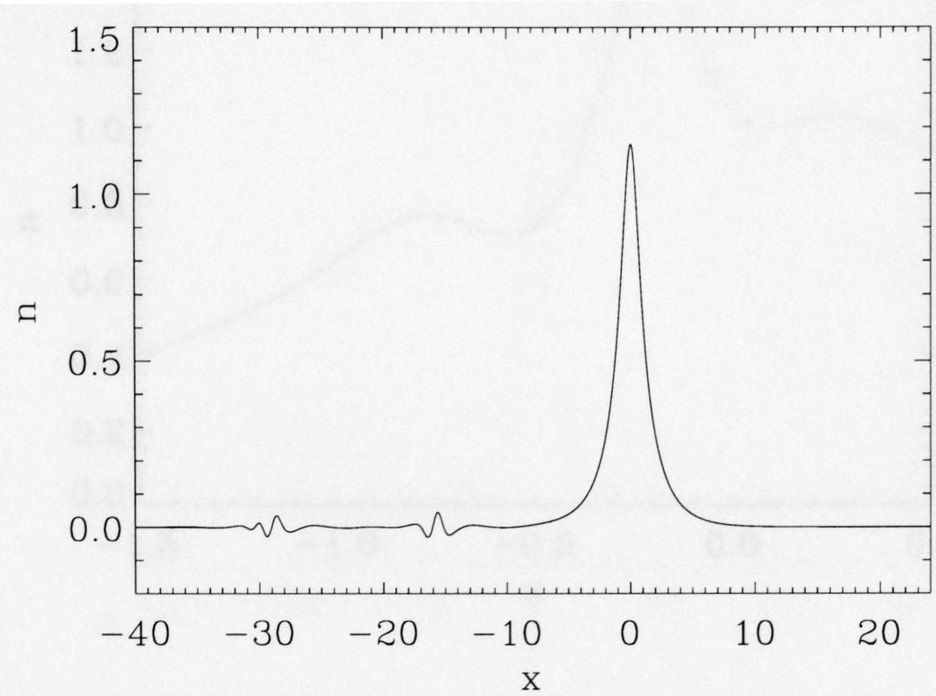


Figure 5.8: Evolution of the perturbed solitary wave at $t = 20$.

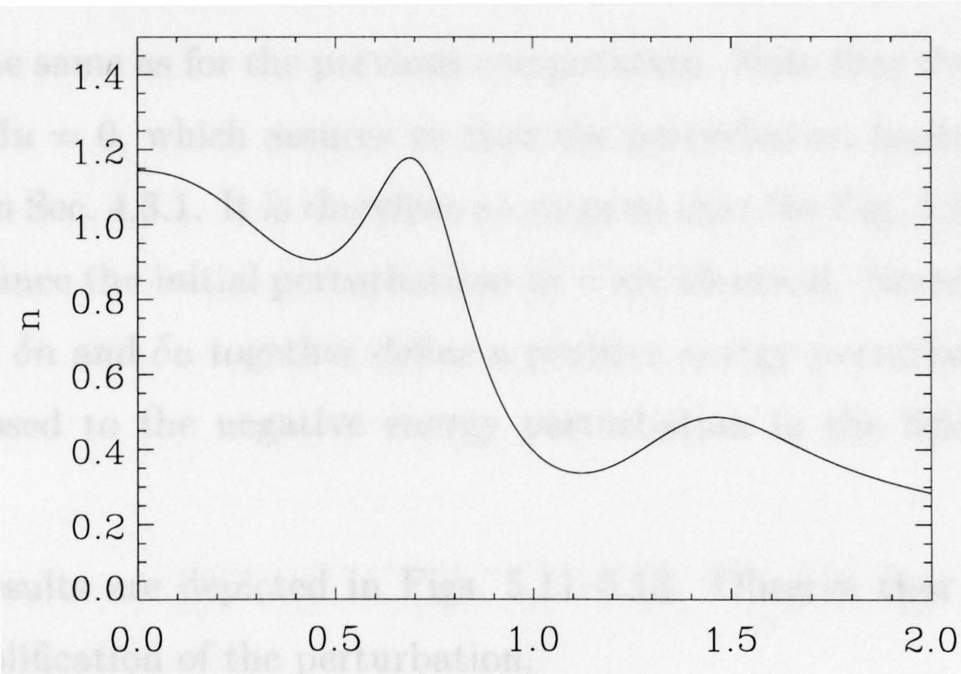


Figure 5.9: Enlarged view of the fine structure in Fig. 5.4.

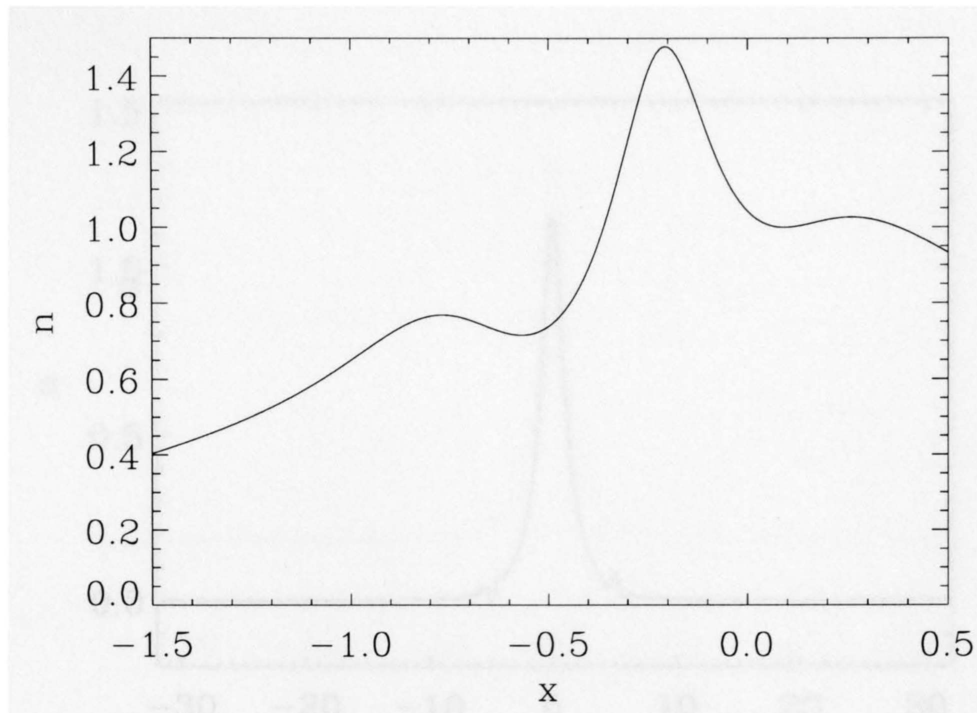


Figure 5.10: Enlarged view of the fine structure in Fig. 5.5.

time evolution of a solitary wave ($c = 1.23$) with a positive energy perturbation, which is defined by

$$\delta u = 0\,, \qquad \delta n = \mathrm{e}^{\phi}\,\delta\phi - \delta\phi_{xx}\,, \tag{5.20}$$

where $\delta\phi$ is the same as for the previous computation. Note that the difference, here, is that $\delta u = 0$, which assures us that the perturbation leads to positive $\delta^2 F$, as seen in Sec. 4.3.1. It is therefore no surprise that the Fig. 5.1 is identical to Fig. 5.11, since the initial perturbations in $n$ are identical. Nevertheless, the perturbations $\delta n$ and $\delta u$ together define a positive energy perturbation in this case, as opposed to the negative energy perturbation in the first numerical simulation.

The results are depicted in Figs. 5.11–5.13. Observe that there is no transient amplification of the perturbation.

The final numerical simulation that is presented here, computes the time evolution of the stable uniform background with an initial perturbation identical to the negative energy perturbation of the solitary wave seen earlier.

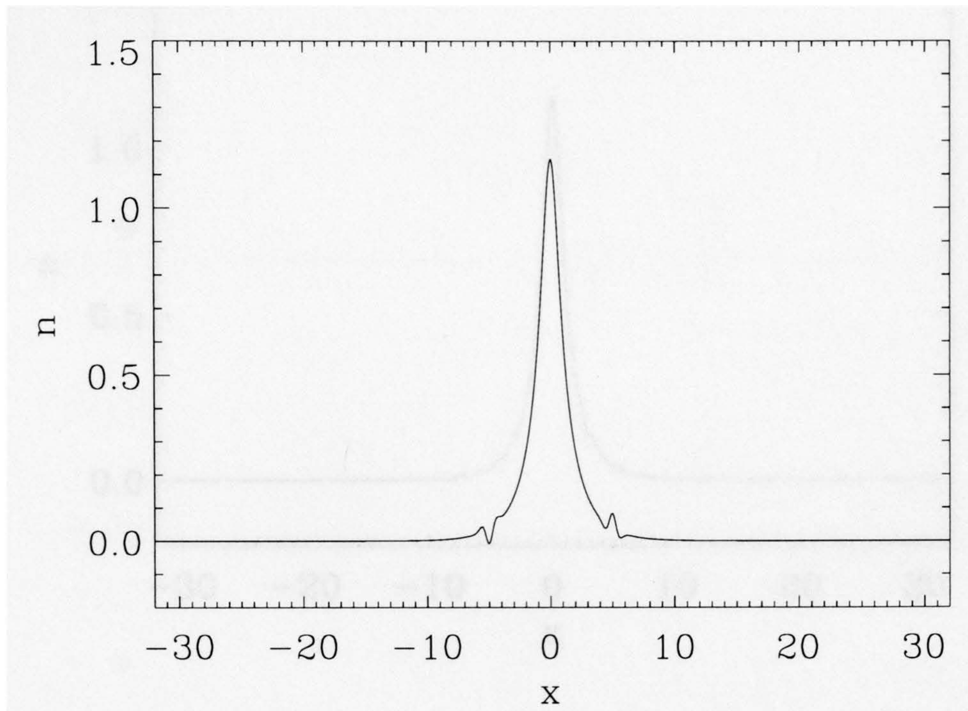


Figure 5.11: Solitary wave profile with a positive energy perturbation (see text) at $t = 0$.

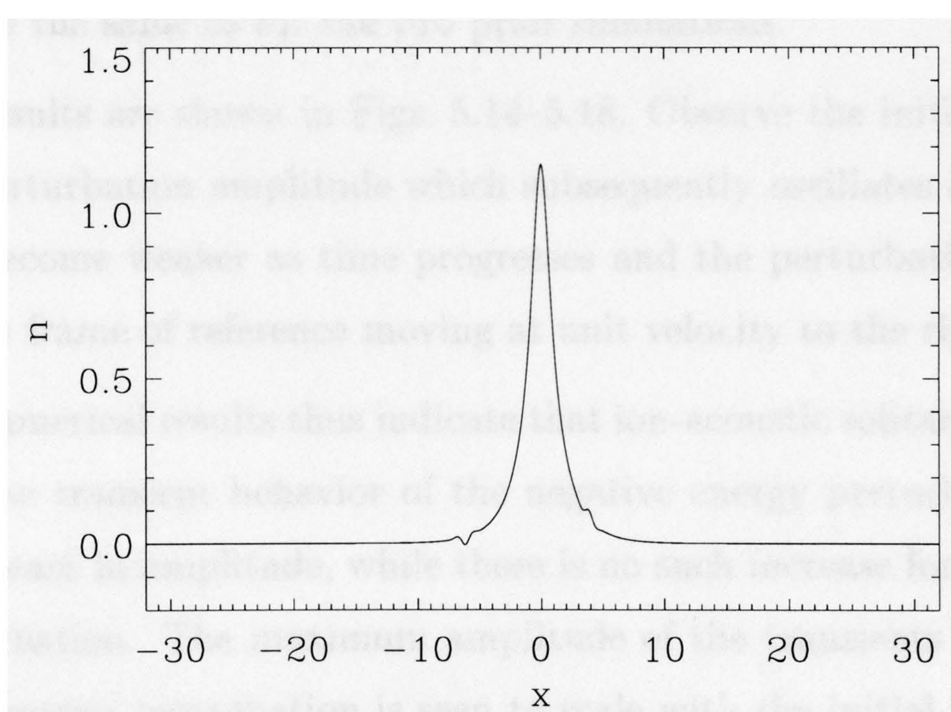


Figure 5.12: Evolution of the perturbed solitary wave at $t = 1$; compare Fig. 5.2.

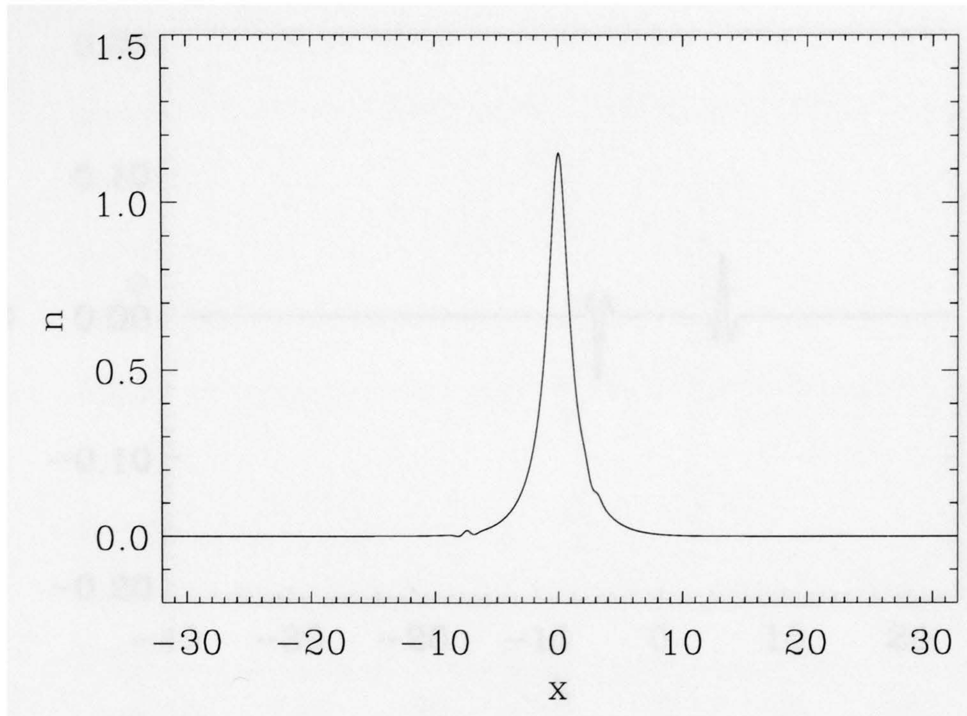


Figure 5.13: Evolution of the perturbed solitary wave at $t = 2$; compare Fig. 5.3.

The uniform background is the solution $n = u = 0$, corresponding to $c = 1$, and the perturbation is defined by Eq. (5.19). The Gaussian half-width and separation are the same as for the two prior simulations.

The results are shown in Figs. 5.14–5.18. Observe the initial amplification of the perturbation amplitude which subsequently oscillates in time. The oscillations become weaker as time progresses and the perturbation moves to the left in the frame of reference moving at unit velocity to the right.

The numerical results thus indicate that ion-acoustic solitary waves may be stable. The transient behavior of the negative energy perturbation shows an initial increase in amplitude, while there is no such increase for the positive energy perturbation. The maximum amplitude of the transients arising from the negative energy perturbation is seen to scale with the initial perturbation amplitude as shown in Fig. 5.19. Since the nonlinear saturation level of an instability often scales with the initial growth rate (and not initial perturbation amplitude), this offers some support to the claim of stability with respect to

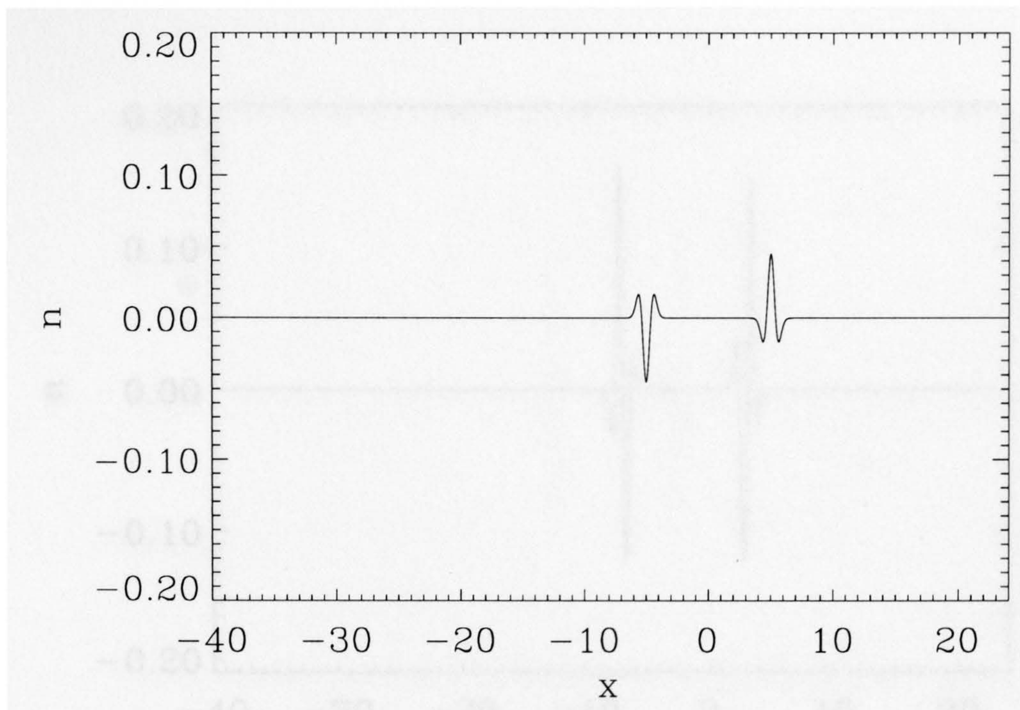


Figure 5.14: Perturbed uniform background at $t = 0$. The perturbation is identical to the negative energy perturbation of the solitary wave seen earlier; the uniform background has been established as being Lyapunov stable in Ch. 4.

the supremum norm.

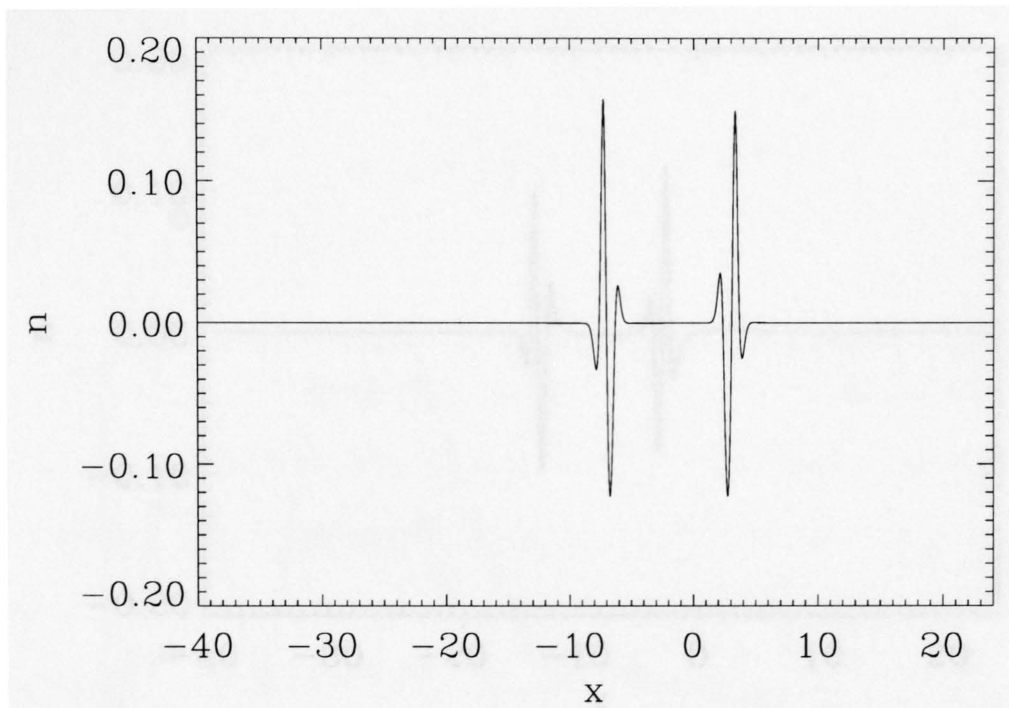


Figure 5.15: Evolution of the perturbed uniform background at $t = 2$. Note the increased amplitude of the perturbation.

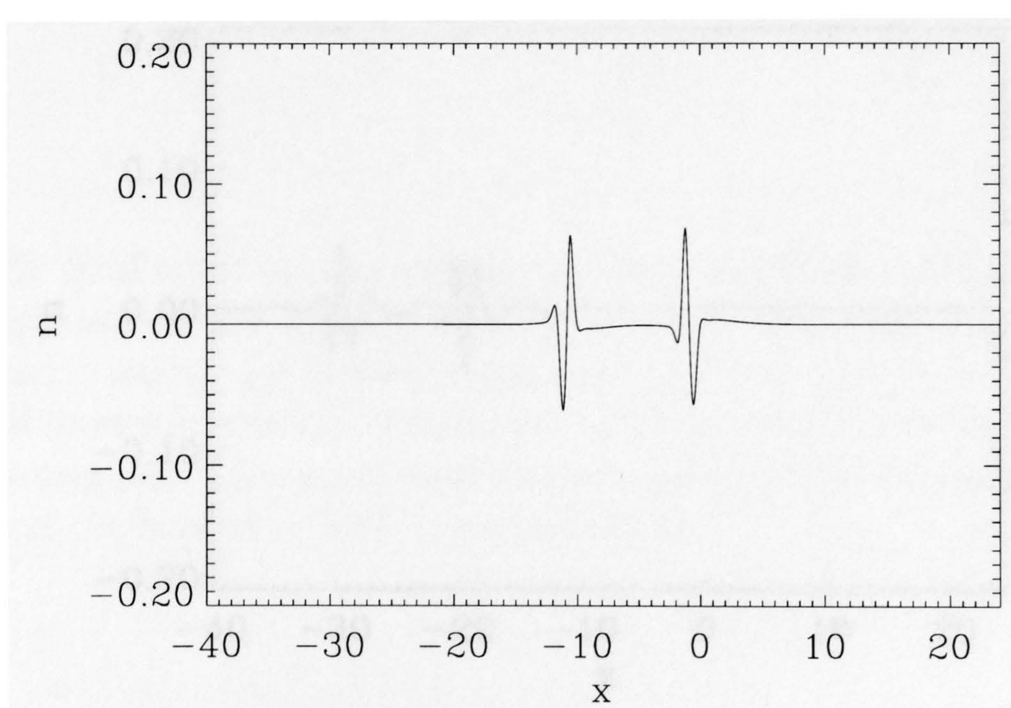


Figure 5.16: Evolution of the perturbed uniform background at $t = 6$.

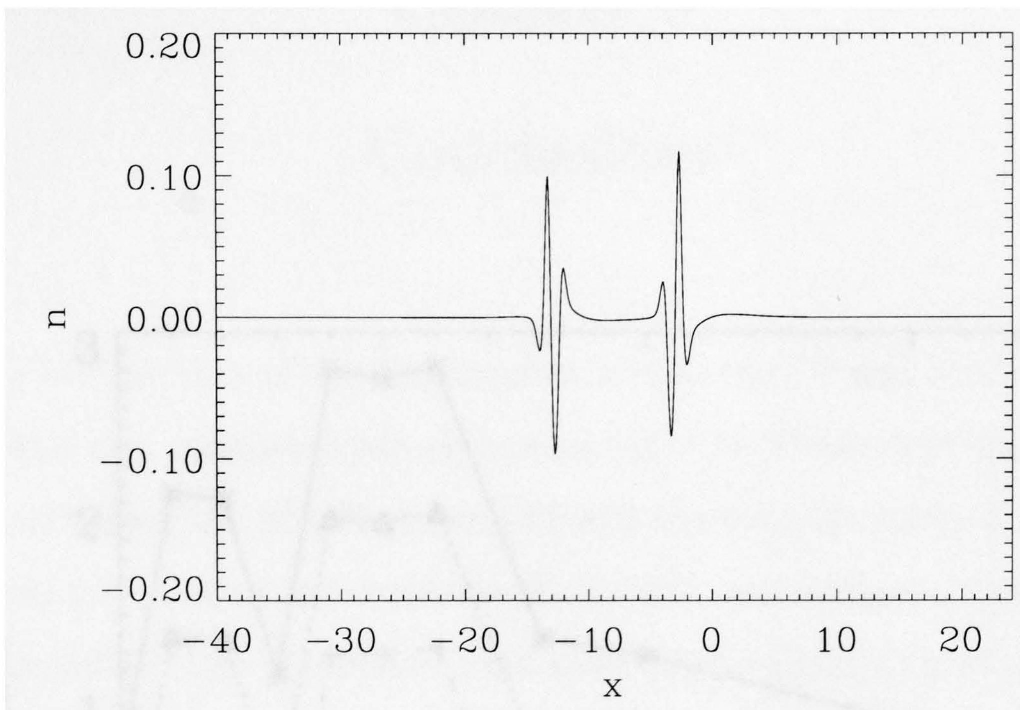


Figure 5.17: Evolution of the perturbed uniform background at $t = 8$.

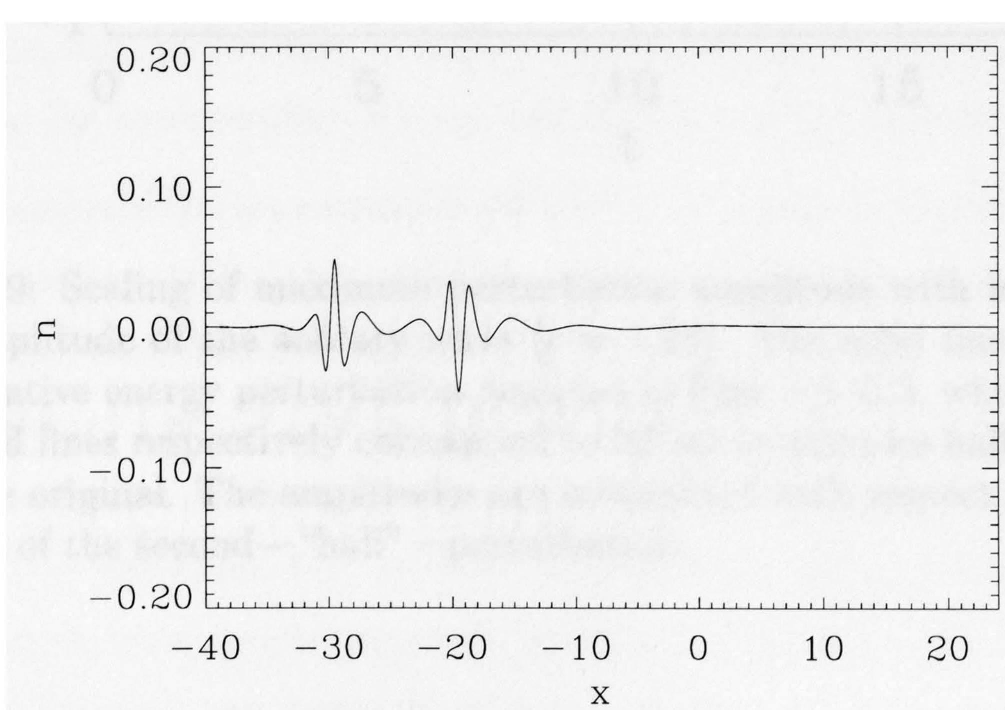


Figure 5.18: Evolution of the perturbed uniform background at $t = 25$.

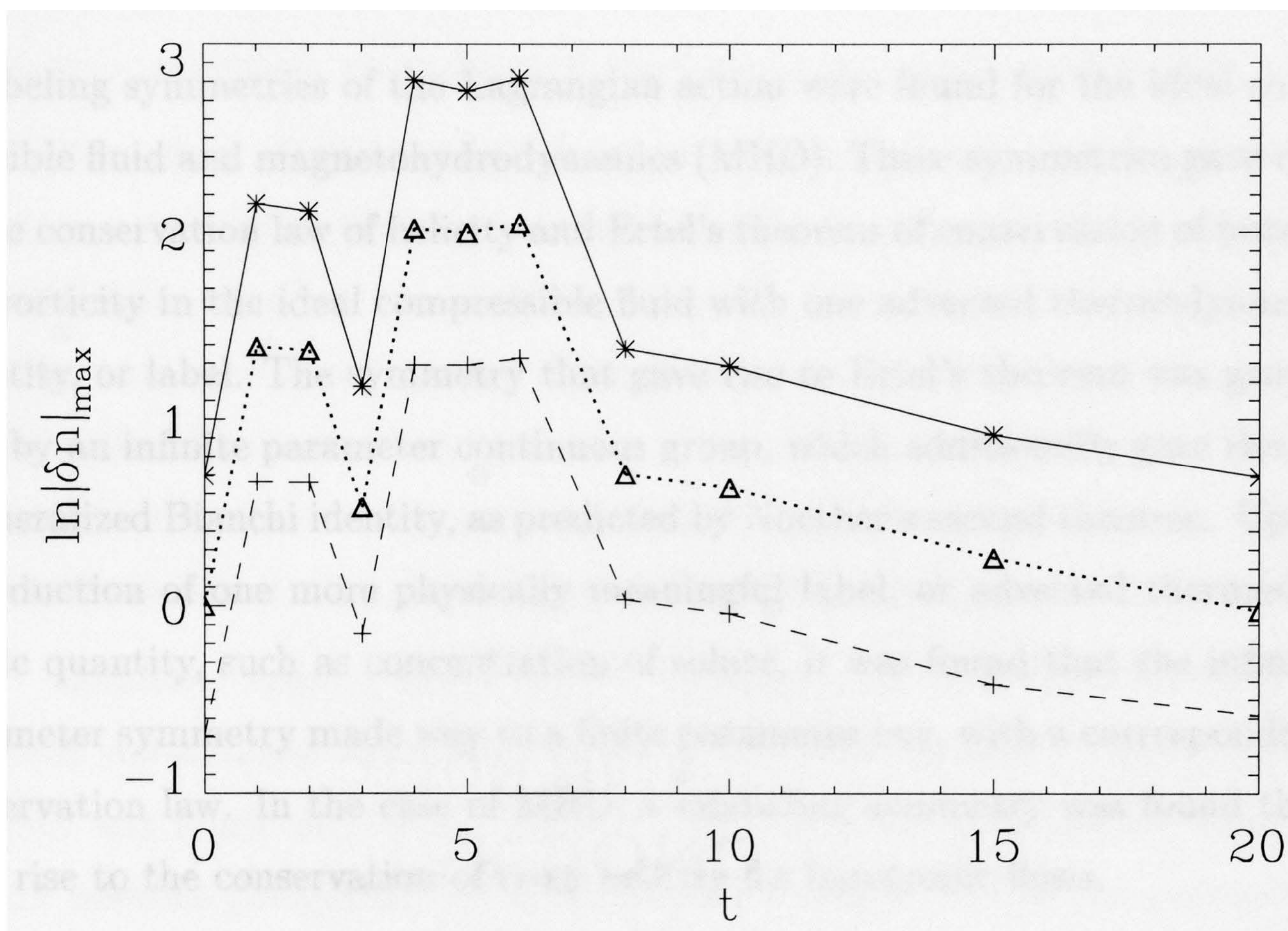


Figure 5.19: Scaling of maximum perturbation amplitude with initial perturbation amplitude of the solitary wave ($c = 1.23$). The solid line corresponds to the negative energy perturbation depicted in Figs. 5.1–5.8, while the dotted and dashed lines respectively correspond to initial amplitudes half and quarter that of the original. The amplitudes are normalized with respect to the initial amplitude of the second—"half"—perturbation.

# Chapter 6

# Conclusions

Relabeling symmetries of the Lagrangian action were found for the ideal compressible fluid and magnetohydrodynamics (MHD). These symmetries gave rise to the conservation law of helicity and Ertel's theorem of conservation of potential vorticity in the ideal compressible fluid with one advected thermodynamic quantity, or label. The symmetry that gave rise to Ertel's theorem was generated by an infinite parameter continuous group, which additionally gave rise to a generalized Bianchi identity, as predicted by Noether's second theorem. Upon introduction of one more physically meaningful label, or advected thermodynamic quantity, such as concentration of solute, it was found that the infinite parameter symmetry made way to a finite parameter one, with a corresponding conservation law. In the case of MHD, a relabeling symmetry was found that gave rise to the conservation of cross helicity for barotropic flows.

Casimirs of the noncanonical Hamiltonian formulations of the ideal compressible fluid and MHD, that result from the reduction of Lagrangian to Eulerian variables, were found directly from relabeling symmetries of the reduction map. Families of Casimirs associated with potential vorticity and entropy were found this way for the fluid. For MHD, a relabeling symmetry of the Euler-Lagrange map was found that gave rise to a family of Casimirs associated with $A \cdot B/\rho$, which incorporates magnetic helicity as a special case. It was shown that such invariants are gauge-dependent, i.e. they are invariants for certain choices of the gauge of the vector potential.

In the study of Lyapunov stability of ion-acoustic solitary and nonlinear periodic waves, it was shown that there exists a free energy functional, constructed from the Hamiltonian and Casimir invariants, for which the wave solutions were seen to lie on a critical point. The second variation of the free energy was shown to be of indefinite sign, by explicit construction of perturbations. Negative energy perturbations arose at short wavelengths, and were shown to be accessible on surfaces of constant Casimirs and momentum.

The effect of addition of adiabatic, ionic pressure to the ion-acoustic equations was studied. Solitary and nonlinear periodic waves were found when the adiabatic exponent was given by $\gamma = 3$. In particular, solitary waves exist for any wave speed greater than Mach number one, without an upper cutoff, unlike the case without pressure. The ionic pressure was not sufficient to suppress negative energy perturbations in the parameter regime in which the nonlinear waves exist, hence Lyapunov stability could not be established.

Lyapunov stability of solitary wave solutions was established in the limit in which ion-acoustic equations reduce to the KdV equation. The proof of stability was much shorter and simpler than the proof of Lyapunov stability of KdV solitons (Benjamin, 1972). Lyapunov stability could be established in the KdV limit since the limiting process effectively eliminated the problematic short wavelengths at which negative energy perturbations were found for the ion-acoustic equations. It was also shown that the KdV free energy is recovered upon evaluating (the negative of) the ion-acoustic free energy at the critical point in the KdV limit of small amplitude and unit wave speed.

The time evolution of an ion-acoustic solitary wave with an initial (small, finite amplitude) negative energy perturbation was simulated numerically. Transients with increased perturbation amplitude were observed, and the solitary wave peak moved to the right, leaving the perturbation behind it. For a positive energy perturbation of the same amplitude, no transient increase in perturbation amplitude was observed. The same "negative energy perturbation" of the stable uniform background, which is a trivial solution corresponding to unit wave speed, also showed transients with increased amplitude followed by decaying oscillation of the perturbation amplitude. These numerical results suggest that the ion-acoustic solitary waves may be stable.

Other independent results presented in this dissertation include the non-canonical Hamiltonian structures of the Frenet formulas that describe a curve in 3-space and of the equations describing magnetic field line flow with a symmetry direction. The relationship between Lagrangian and Eulerian conservation laws in fluid dynamics was also clarified.

# Appendix A

# Determinant Identities

Determinant identities are of substantial use (Truesdell, 1954; Newcomb, 1967, for example) in the Lagrangian formulation of compressible fluid theories. Here we start with two basic definitions and derive a few useful identities. The cofactor of the matrix element $\partial q^i/\partial a^j$ is denoted by $A_i{}^j$, and is defined by,

$$A_i{}^j := \frac{1}{2}\,\epsilon_{ik\ell}\,\epsilon^{jmn}\,\frac{\partial q^k}{\partial a^m}\,\frac{\partial q^\ell}{\partial a^n}\,. \tag{A.1}$$

The determinant $\mathcal{J}$ may be defined in terms of $A_i{}^j$ :

$$\mathcal{J}\,\delta_i{}^k := A_i{}^j\,\frac{\partial q^k}{\partial a^j}\,. \tag{A.2}$$

Next we proceed to derive results that follow from Eqs. (A.1) and (A.2).

Differentiating Eq. (A.1) with respect to $a^j$, we see that $A_i{}^j$ commutes with $\partial/\partial a^j$ :

$$\frac{\partial}{\partial a^j}\left(A_i{}^j\,\cdot\right) = A_i{}^j\,\frac{\partial}{\partial a^j}\,, \tag{A.3}$$

where we have used the antisymmetry of $\epsilon^{jmn}$. It follows that

$$A_i{}^j\,\frac{\partial}{\partial a^j} = A_i{}^j\,\frac{\partial q^k}{\partial a^j}\,\frac{\partial}{\partial r^k} = \mathcal{J}\,\frac{\partial}{\partial r^i}\,, \tag{A.4}$$

where we have used Eq. (A.2) to obtain the right hand side.

On contracting Eq. (A.2) with $\partial[q^{-1}]^\ell/\partial r^k$, we get

$$\mathcal{J}\,\delta_i{}^k\,\frac{\partial}{\partial r^k}\,[q^{-1}]^\ell = A_i{}^j\,\frac{\partial q^k}{\partial a^j}\,\frac{\partial}{\partial r^k}\,[q^{-1}]^\ell = A_i{}^j\,\delta_j{}^\ell\,,$$

from which it follows that

$$\mathcal{J}\,\frac{\partial}{\partial r^i}\,[q^{-1}]^\ell = A_i{}^\ell\,. \tag{A.5}$$

Contracting Eq. (A.5) with $\partial q^i/\partial a^m$ leads to

$$\mathcal{J}\,\delta_m{}^\ell = A_i{}^\ell\,\frac{\partial q^i}{\partial a^m}\,. \tag{A.6}$$

Observe from Eqs. (A.2) and (A.6) that contraction of $A_i{}^j$ with respect to either index leads to identical results.

Another useful identity is obtained by contracting Eq. (A.2) with $\epsilon^{i\ell m}$:

$$\begin{aligned}
\mathcal{J}\,\epsilon^{k\ell m} &= \frac{1}{2}\,\epsilon^{i\ell m}\,\epsilon_{iuv}\,\epsilon^{jst}\,\frac{\partial q^u}{\partial a^s}\,\frac{\partial q^v}{\partial a^t}\,\frac{\partial q^k}{\partial a^j} \\
&= \frac{1}{2}\,(\delta_u{}^\ell\,\delta_v{}^m - \delta_v{}^\ell\,\delta_u{}^m)\epsilon^{jst}\,\frac{\partial q^k}{\partial a^j}\,\frac{\partial q^u}{\partial a^s}\,\frac{\partial q^v}{\partial a^t} \\
&= \epsilon^{jst}\,\frac{\partial q^k}{\partial a^j}\,\frac{\partial q^\ell}{\partial a^s}\,\frac{\partial q^m}{\partial a^t}\,.
\end{aligned} \tag{A.7}$$

All the identities above are also valid for the inverse transformation $r \to a$, with the obvious replacements: $q \to q^{-1}$, $\partial/\partial a^j \to \partial/\partial r^j$, $A \to A^{-1}$, $\mathcal{J} \to \mathcal{J}^{-1}$.

In order to obtain an expression for the time derivative of $\mathcal{J}$, we differentiate Eq. (A.7) to get

$$\begin{aligned}
\frac{\partial \mathcal{J}}{\partial \tau} &= \epsilon^{ijk}\,\frac{\partial \dot{q}^1}{\partial a^i}\,\frac{\partial q^2}{\partial a^j}\,\frac{\partial q^3}{\partial a^k} + \epsilon^{jki}\,\frac{\partial \dot{q}^2}{\partial a^j}\,\frac{\partial q^3}{\partial a^k}\,\frac{\partial q^1}{\partial a^i} + \epsilon^{kij}\,\frac{\partial \dot{q}^3}{\partial a^k}\,\frac{\partial q^1}{\partial a^i}\,\frac{\partial q^2}{\partial a^j} \\
&= A_1{}^i\,\frac{\partial \dot{q}^1}{\partial a^i} + A_2{}^j\,\frac{\partial \dot{q}^2}{\partial a^j} + A_3{}^k\,\frac{\partial \dot{q}^3}{\partial a^k} \\
&= A_j{}^i\,\frac{\partial \dot{q}^j}{\partial a^i}\,.
\end{aligned} \tag{A.8}$$

For the conversion from Lagrangian to Eulerian conservation laws, the relation $[A^{-1}]_j{}^i\, A_k{}^j = \delta_k{}^i$ is of much use, the proof of which is as follows:

$$\begin{aligned}
[A^{-1}]_j{}^i\, A_k{}^j &= \frac{1}{4}\,\epsilon^{iuv}\,\epsilon_{jst}\,\epsilon^{jmn}\,\epsilon_{kxy}\,\frac{\partial}{\partial r^u}\,[q^{-1}]^s\,\frac{\partial}{\partial r^v}\,[q^{-1}]^t\,\frac{\partial q^x}{\partial a^m}\,\frac{\partial q^y}{\partial a^n} \\
&= \frac{1}{4}\,\epsilon^{iuv}\,\epsilon_{kxy}\left(\frac{\partial}{\partial r^u}\,[q^{-1}]^m\,\frac{\partial}{\partial r^v}\,[q^{-1}]^n\right. \\
&\qquad \left. -\frac{\partial}{\partial r^u}\,[q^{-1}]^n\,\frac{\partial}{\partial r^v}\,[q^{-1}]^m\right)\frac{\partial q^x}{\partial a^m}\,\frac{\partial q^y}{\partial a^n} \\
&= \frac{1}{4}\,\epsilon^{iuv}\,\epsilon_{kxy}\left(\delta_u{}^x\,\delta_v{}^y - \delta_u{}^y\,\delta_v{}^x\right) = \frac{1}{2}\,\epsilon^{iuv}\,\epsilon_{kuv} \\
&= \delta_k{}^i\,. \qquad \text{(A.9)}
\end{aligned}$$

# Appendix B

# Free Energy Principle of Stability

Here we briefly discuss some basic ideas of stability (Hazeltine and Meiss, 1992, for example) pertinent to our treatment of stability of ion-acoustic waves in Ch. 3. We are interested in the stability of an equilibrium solution to the set of equations,

$$\frac{\partial z}{\partial t} + \mathcal{O} \cdot z = 0 \,, \tag{B.1}$$

where the vector $z$ represents the dynamical variables, while $\mathcal{O}$, for our purpose, is some partial differential operator. The equilibrium solution $z_0$ satisfies $\mathcal{O} \cdot z_0 = 0$.

Perhaps the most fundamental and intuitive definition of stability is that of Lyapunov.

Lyapunov Stability: An equilibrium $z_0$ is stable if, for any neighborhood $U$ of $z_0$, all initial conditions in some smaller neighborhood $V$, contained in $U$, stay in $U$ for all time.

The most effective way to show Lyapunov stability for a set of partial differential equations involves a *free energy functional.* It is required that such a free energy functional $F$, if it exists, must be an invariant and have a critical point at the equilibrium $z_0$.

Free energy functionals can generally be found for equilibria of Hamiltonian systems. Existence of $F$ allows the use of the following property:

Dirichlet's Criterion (or Free Energy Principle): If the second variation of $F$ at $z_0$ is positive definite,

$$\delta^2 F[\zeta; z_0] \,\geq\, C\,\|\zeta\|^2 \quad \text{for any } \zeta\,, \tag{B.2}$$

then $z_0$ is Lyapunov stable.

*Proof.* Eq. (B.2) implies that the surfaces of constant $F$ are ellipsoids near $z_0$, hence the invariance of $F$ maintains the motion on one of these ellipsoids. This is the simplest way to understand Dirichlet's criterion.

A more rigorous way to understand Dirichlet's criterion is to note that since $F$ has a critical point at $z_0$, the first variation vanishes at $z_0$, and we are left with

$$F[z_0 + \zeta] - F[z_0] = \delta^2 F[\zeta; z_0] + \epsilon(\|\zeta\|)\,\|\zeta\|^2 \tag{B.3}$$

where $\epsilon(\|\zeta\|) \to 0$ as $\|\zeta\| \to 0$. Choosing $\|\zeta\|$ small enough so that $|\epsilon| < C/2$, it is established that

$$F[z_0 + \zeta] - F[z_0] \,>\, \tfrac{1}{2} C\,\|\zeta\|^2\,. \tag{B.4}$$

The left hand side of inequality (B.4) is constant in time since $F$ is an invariant; therefore $\|\zeta\|^2$ is bounded for all time. ∎

Note that the idea of stability is associated with a norm, which was left unspecified in inequality (B.2). Stability is generally norm-dependent; a system may be stable with respect to a certain norm, but not others.

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