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Analysis of a one-parameter family of triple close approaches occurring in stellar systems

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This paper is the first part of a trilogy dedicated to the role of triple encounters in the evolution of stellar systems. It shows how a symmetric triple collision may be perturbed to obtain a family of asymmetric triple close approaches with arbitrary high escape velocities and with the formation of binaries. The main result is that as the perturbation approaches zero, the product of the semimajor axis of the binary and the square of the escape velocity approach a value dependent only on the participating masses. The second part will describe results of the extensive numerical integrations, will offer detailed information on the one-parameter family of orbits generated, and will outline the intricate numerical techniques and controls used, without which no reliable numerical results may be obtained regarding the dynamic behavior of multicomponent stellar systems. The third and last part offers applications in stellar and galactic dynamics. In a parametric presentation it gives actual sizes of binaries formed and values of escape-velocities generated for triplets formed of Sun-like stars, of white dwarfs, of neutron stars, and of triple systems of galaxies.

INTRODUCTION

THE conjecture that sufficiently close simultaneous asymmetric approaches occurring in the problem of three bodies result in a binary formation and in an escaping third star has been recently supported by numerical evidence (Agekian 1967 and Szebehely 1967). This conjecture first mentioned by Birkhoff (1922, 1927) and later reformulated by Szebehely (1971, 1973) seems to be of fundamental importance in the global behavior of three gravitationally interacting stars. If orbits which are periodic even in the most general sense dominate the system, then the above conjecture is clearly an erroneous attempt to generalize the behavior of systems with low-angular momenta (**c**) to the global situation. In fact, the principal problem is the partition of the phase-space of the initial conditions. The regions of the phase space with bounded motions will probably be mixed with escape regions according to Hénon's (1974) recent conjecture of islands. Some of the large number of periodic orbits, recently discovered in the general problem of three bodies by Hénon (1974), Brouke (1974), Hadjidemetriou (1973), Standish (1970), and Szebehely (1970) show linear stability, some are unstable. These periodic orbits may be periodic only in a rotating coordinate system (relative periodicity), but from the point of view of boundedness, this kind of periodicity in a general sense is of great importance.

Since Sundman (1912) has shown that simultaneous close approaches occur only with small values of the total angular momentum, **c** (in fact for a triple collision $\mathbf{c}=0$ is a necessary condition), the study of systems with low values of **c** favors escape. Consider the equilateral Lagrangian solution which is a symmetric rotating configuration. If the mean motion is such that the virial coefficient is unity, the distances between the bodies is constant. If the velocity is reduced, the pulsating Lagrangian solution appears, and when the initial velocities of the participating bodies is zero (and consequently $\mathbf{c}=0$), a total collapse occurs. It is important to note that the symmetry is never destroyed in the Lagrangian solution, therefore, escape does not occur and all motions are periodic, even when **c** is small. It might be equally important to recall that in the general problem, when all three masses are of the same order of magnitude, the Lagrangian solutions are *all* unstable.

If asymmetric changes of the initial conditions are introduced and **c** is small, the equilateral configuration leads to escapes instead of periodic orbits. The existence of these motions are along the previously mentioned Birkhoff–Szebehely conjecture and this is the *raison d'être* of this paper. The instability of the unperturbed Lagrangian solutions, together with the escapes occurring in the associated perturbed system seem to offer an example, if not support, to the Laplacian instability as the dominant behavior.

THE MODEL

The unit masses occupy initially the apices of an equilateral triangle with unit sides. All initial velocities are parallel with one of the sides, say $\overline{m_1m_2}$. The velocities of m_1 and m_2 along the side $\overline{m_1m_2}$ are $v_0/2$ and the velocity of m_3 is v_0 in the opposite direction. Placing the center of mass of the system at the origin, it will stay there. The initial conditions are $x_{1,2}=\pm\frac{1}{2}$; $y_1=y_2=-y_3/2=-3^{-\frac{1}{2}}/2$; $\dot{x}_1=\dot{x}_2=-\dot{x}_3/2=v_0/2$; $\dot{y}_1=\dot{y}_2=\dot{y}_3=x_3=0$.

The complex vector

$$z_k=r_ke^{i\phi_k}$$

represents the location of the *k-th* body in the above system with $r_k(0)=3^{-\frac{1}{2}}$ and $\phi_k(0)=\pi/2-2\pi k/3$. Initially, therefore,

$$z_k(0)=3^{-\frac{1}{2}}ie^{-2\pi ik/3}.$$

The equations of motion are

$$\ddot{z}_k = G \sum_{\substack{i=1\\ i\neq k}}^{3} \frac{z_i - z_k}{|z_i - z_k|^3} m_i.$$

The case of triple collision corresponds to zero initial velocities or to $\phi_k(t)=\phi_k(0)=\text{constant}$, and $r_1=r_2=r_3=r(t)$. These conditions give, with $G=1$,

$$\ddot{r} = \frac{d^2r}{dt^2} = -3^{-\frac{1}{2}}r^{-2},$$

the integral of which is

$$(\dot{r})^2/2 = 3^{-\frac{1}{2}}r^{-1} - 1.$$

Note that the (regularizable) triple collision is described by the solution of the above two-body equations:

$$r = 2^{-1}3^{-\frac{1}{2}}(1+\cos\sqrt{2}\tau),$$

where $r d\tau = dt$, or $t = 2^{-1}3^{-\frac{3}{4}}(\tau + 2^{-\frac{1}{2}}\sin 2^{\frac{1}{2}}\tau)$, which latter is a simple version of Kepler's equation. The time to collision is $\tau_c = \pi 2^{-\frac{1}{2}}$ or $t_c = (24)^{-\frac{1}{4}}\pi$.

The nonzero initial velocities introduce an asymmetry. The three distances are not equal any more, in fact up to order two, at the beginning of the motion $(t=0)$ we have $r_{12}=1-at^2$, $r_{23}=1+bt+ct^2$ and $r_{31}=1-bt+ct^2$, where $a=\frac{3}{2}$, $b=3v_0/4$ and $c=\frac{3}{2}(\frac{9}{16}v_0^2-1)$. The coefficients a, b, and c follow from the series expansions for the position vectors z_k. For instance, $x_3=v_0t+\cdots$, $y_3=y_3(0)-(\sqrt{3}/2)t^2+\cdots$, etc. Note that for small t, the inequality stands: $r_{31}\leq r_{12}\leq r_{23}$, therefore, at the beginning of the motion an asymmetry is established which is controlled by the initial velocity through the coefficient b, and which results in the escape of m_2.

The following dynamical parameters may be obtained by using their conventional definitions [see for instance Szebehely (1973)]. Initially, the moment of inertia of the system is $I(0)=\sum m_i r_i^2(0)=1$, with $\dot{I}(0)=0$, and $\ddot{I}(0)=3v_0^2-6$. At $t=0$ the kinetic energy is $T(0)=\frac{1}{2}\sum m_i \dot{\mathbf{r}}_i^2(0)=3v_0^2/4$ and the potential energy is

$$V = -F = -\sum_{1\leq i<j\leq 3} \frac{Gm_im_j}{r_{ij}} = -3.$$

The angular momentum is $|\mathbf{c}|=\sqrt{3}v_0/2$ and the virial coefficient at $t=0$ is $\alpha=v_0^2/2$. The total energy is given by $E_t=-3(1-v_0^2/4)$.

ANALYSIS

(1) The motion begins with a contraction as long as $\ddot{I}(0)<0$ or $v_0\leq\sqrt{2}$. In this study the initial condition $(v_0=0)$ is slightly disturbed, so v_0 is arbitrarily small $(v_0\ll 1)$, $\ddot{I}(0)<0$, and $E_t<0$. *The initial collapse results in a minimum value of the moment of inertia,*

$$I_{\min} \geq v_0^4/64,$$

which bound is obtained as follows.

As originally shown by Sundman (1912), the function

$$L = \frac{1}{(I)^{\frac{1}{2}}}(\dot{I}^2 + 4\mathbf{c}^2) - 8E_t(I)^{\frac{1}{2}}$$

has the property that if I increases, L does not decrease, and if I decreases, L does not increase. Variations of this function play important roles in the works of Birkhoff (1927), Siegel (1956), and Szebehely (1973).

Estimating the value of $I_{\min}$ occurring at the first collapse we have $I_1=I_0$, $\dot{I}_1=0$, $I_2=I_{\min}$, $\dot{I}_2=0$, and $L_1\geq L_2$ since $I_1>I_2$. Consequently,

$$L_1 = \frac{4\mathbf{c}^2}{(I)_0^{\frac{1}{2}}} + 8|E_t|(I)^{\frac{1}{2}} \geq \frac{4\mathbf{c}^2}{(I)_{\min}^{\frac{1}{2}}} + 8|E_t|(I)_{\min}^{\frac{1}{2}} = L_2,$$

and substituting the appropriate values, we have the desired result.

(2) The asymmetric triple close approach results in a binary and in an escaper. The (asymptotic) hyperbolic escape velocity relative to the center of mass of the three bodies, v_∞ and the asymptotic value of of the semimajor axis of the binary a_∞, depend on the perturbation measured by v_0. *As* $v_0\rightarrow 0^+$, *the product* $a_\infty v_\infty^2 \rightarrow \frac{2}{3}$.

To show this, first, attention is directed to the double limit-process involved in the above result. After the binary is formed, the distance between the escaper and the binary, $r\rightarrow\infty$, the velocity of the escaper, $v\rightarrow v_\infty$, and the semimajor axis of the binary a approaches asymptotically the value a_∞. In this limit process the original three-body problem approaches its partition into two two-body problems. The escaper and the center of mass of the binary form a hyperbolic two-body problem and the members of the binary form an elliptic two-body problem.

The second limit process refers to the behavior of the members of the one-parameter (v_0) family as the perturbing initial velocity v_0 approaches zero.

After the binary was formed, the total energy, E_t, may be written as

$$E_t = E_e + E_b + E_{eb},$$

where E_e is the escape energy, E_b is the energy stored in the binary, and E_{eb} is the correction due to three-body effects. Equations for E_e and E_b may be written from two-body consideration as follows:

$$E_e = \frac{Mm_2}{2(m_1+m_3)}v^2 - \frac{Gm_2(m_1+m_3)}{r},$$

$$E_b = -\frac{Gm_1m_3}{2a},$$

where m_1 and m_3 form the binary and M is the total mass.

Several escape conditions exist in the literature. For instance, Standish (1971) has shown that it is sufficient for escape that

$$E_e \geq \frac{Gm_1m_2m_3}{m_1+m_3}\frac{d^2}{r^2(r-d)}+\frac{Mm_2}{2(m_1+m_3)}v^2\sin^2\alpha,$$

and that

$$r>d=\frac{G}{|E_t|}\sum_{1\leq i<j\leq 3} m_im_j,$$

where α is the angle between $\mathbf{v}$ and $\mathbf{r}$.

These conditions are satisfied and escape does occur for sufficiently small perturbations. This follows from the fact that as $v_0 \to 0^+$, $I_{\min} \to 0^+$. The asymmetric triple close approach, after reaching $I_{\min}$ generates sufficiently large values of I and $\dot{I}$ for escape. For an indication of this process see Birkhoff (1927) and for its detailed analysis, Szebehely (1973). Note that high-precision numerical integrations also verify the analytical expectations. In fact, escape of m_2 occurs in the region of $0<v_0\leq 0.186$.

In the first limit process E_t is fixed,

$$E_{eb}\to 0,\quad E_b\to -\frac{Gm_1m_3}{2a_\infty},$$

and

$$E_e\to\frac{Mm_2v_\infty^2}{2(m_1+m_3)},$$

or

$$E_t=\frac{Mm_2}{2(m_1+m_3)}v_\infty^2-\frac{Gm_1m_3}{2a_\infty}.$$

In the second limit process we consider the general form of the total energy

$$E_t=\tfrac{3}{4}mv_0^2-\frac{3Gm^2}{l},$$

where $m=m_1=m_2=m_3$ and l is the length of the side of the equilateral triangle at the beginning of the motion. Equating the two forms of the total energy we have

$$a_\infty v_\infty^2=G\frac{m_1m_3}{m_2}\left(\frac{m_1+m_3}{M}\right)+\tfrac{3}{2}a_\infty\frac{m}{m_2}\left(\frac{m_1+m_3}{M}\right)\left(v_0^2-4\frac{Gm}{l}\right).$$

As $v_0\to 0^+$, the following limiting values are obtained: $I_{\min}\to 0^+$, $\mathbf{c}\to 0^+$, $E_t\to -3Gm^2l^{-1}$, $a_\infty\to 0^+$, $v_\infty\to\infty$, and consequently

$$a_\infty v_\infty^2\to G\frac{m_1m_3}{m_2}\left(\frac{m_1+m_3}{M}\right),$$

which limit for $m_1=m_2=m_3=1$ and $G=1$ become $\frac{2}{3}$ as stated.

Note that if the above simplifying substitution is made prior to the second limit process, we have $a_\infty(v_\infty^2-v_0^2+4)=\frac{2}{3}$, allowing an orderly presentation of numerically established members of the family.

ASTRONOMICAL CONSEQUENCES AND REMARKS

The not unpleasing result, $a_\infty v_\infty^2\to\frac{2}{3}$ may be put in other forms, utilizing the minimum of inertia ($I_{\min}v_\infty^4\geq 4/9$), or other parameters of the family.

It is essential to point out that in the second limit process, representing the members of the family of close approaches, the angular momentum approaches zero and the total energy approaches -3. The process allows the generation of arbitrary high-escape velocities with arbitrary close binaries.

The sizes of the participating bodies determine in physical and astronomical problems the meaningful minimum values of a_∞ and, consequently, the maximum obtainable value of the escape velocity. The details of such computations as well as the results of the extensive numerical integrations will be presented in the sequel. An example, however, might clarify the process. Consider three white dwarfs of solar masses $M=M_\odot$ and of radii $R=0.005R_\odot$ located initially at $l=1$ pc distances. If the perturbing velocity is such that the semimajor axis of the binary formed is $a_\infty=100R$, then the escape velocity is 524 km/sec. If the perturbation is reduced and a closer binary is formed ($a_\infty=10R$), then the escape velocity increases to 1440 km/sec. It may be shown that high-density neutron stars produce escapers with relativistic velocities.

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