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The André-Quillen spectral sequence for pre-logarithmic ring spectra

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The André-Quillen spectral sequence for pre-logarithmic ring spectra

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The André-Quillen spectral sequence for pre-logarithmic ring spectra

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We discuss an André-Quillen spectral sequence and an étale descent theorem for pre-logarithmic ring spectra.

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Chapter 1

Introduction

This dissertation constructs an André-Quillen spectral sequence for pre-logarithmic ring spectra. This theorem relate logarithmic topological André-Quillen homology to logarithmic topological Hochshild homology and provides a way of computing logarithmic topological Hochshild homology (which we explain below). The intended application of these results is to compute algebraic K-theory. Computations of algebraic K-theory are difficult in general, and one standard technique is to approximate algebraic K-theory with topological Hochschild homology. Computations of logarithmic topological Hochschild homology have already been used to compute algebraic K-theory in [8] and [2]. The theorems proven here provide additional tools for doing these computations.

We now explain the classical André-Quillen spectral sequence. Given a commutative unital ring R and a commutative R-algebra A, there is a spectral sequence [15], the André-Quillen spectral sequence,

$$E^2_{p,q} = H_p(\Lambda^q_A(\mathbb{L}_{A/R})) \Rightarrow HH^R_{p+q}(A).$$

where $\mathbb{L}_{A/R}$ is the cotangent complex of A over R, Λ^q_A is the q-th degree part of the exterior algebra, and $HH^R_*(A)$ is the Hochschild homology of A over R.

The cotangent complex $\mathbb{L}_{A/R}$ is a derived version of the A-module of Kähler differentials $\Omega^1_{A/R}$, the A-module that corepresents derivations: for an A-module X

$$Mod_A(\Omega^1_{A/R}, X) \cong Der(A, X).$$

The Hochschild homology groups are defined for associative R-algebras and can be thought of as a substitute for differential forms when A is non-commutative. The spectral sequence can thus be thought of as comparing the A as a commutative algebra to A as an associative algebra.

A spectral analogue of the André-Quillen spectral sequence was constructed in [14]. Given a ring spectrum R (as defined in [7] or [10] [23] [12]), and a commutative R-algebra A, there is an A-module $TAQ^R(A)$, originally defined in [3] and explained for symmetric spectra in [19], that serves as a spectral analogue of the cotangent complex $\mathbb{L}_{A/R}$. It can be defined as the corepresenting object for derivations: for an A-module X

$$Mod_A(TAQ^R(A), X) \simeq Der(A, X).$$

If A is an R-algebra, as defined in [23] [12], then [27] defines $THH^R(A)$, the spectral analogue of the Hochschild complex. The definition, which we will review in Chapter 2, uses the symmetric monoidal structure on the category of ring spectra to directly imitate the algebraic construction of Hochschild homology. Restricting to Eilenberg-Mac Lane spectra, one finds that

$$THH^{HR}_*(HA) \cong HH^R_*(A).$$

A discrete pre-logarithmic (or pre-log) ring (A, M, α) is a commutative unital ring A, a commutative unital monoid M, and a morphism of monoids $\alpha\colon M \to (A, \cdot)$ from M to the underlying multiplicative monoid of A, (the monoid obtained by forgetting the addition). Letting $A^\times$ denote the multiplicative units of A, a pre-log ring is a logarithmic ring (or log ring) if the map of monoids $\alpha^{-1}(A^\times) \to A^\times$ is an isomorphism. A log derivation is a pair $(d, d\log)\colon (A, M, \alpha) \to X$ consisting of a derivation $d\colon A \to X$ and a map of monoids $d\log\colon M \to X$ such that $d\alpha(m) = \alpha(m) d\log(m)$.

Log rings can be thought of as partial localizations. For example, if A is a discrete valuation ring, K its field of quotients, and $\alpha\colon \langle x \rangle \to (A, \cdot)$ the map from a free monoid on a single generator that assigns x to a generator of the unique maximal ideal of A, then

$$(A, A^\times) \to (A, A^\times \oplus \langle x \rangle) \to (K, K^\times)$$

can be considered a factorization of the map $A \to K$ that does not exist in rings.

Spectral analogues of pre-log rings were defined in [17] and [20], and logarithmic analogues of TAQ and THH were defined in [17] [19] [16]. A pre-log ring spectrum (Definition 2.1.10) is a triple (A, M, α), where A is a commutative unital symmetric ring spectrum, M is a commutative unital $\mathcal{J}$-space monoid, and $\alpha\colon M \to \Omega^{\mathcal{J}}(A)$ is a map of $\mathcal{J}$-space monoids where $\Omega^{\mathcal{J}}(A)$ plays the role of the underlying multiplicative monoid of A. The logarithmic topological André-Quillen homology $TAQ^{(R,P)}(A, M)$ is defined as the corepresenting object for logarithmic derivations:

$$Mod_A(TAQ^{(R,P)}(A,M),X) \simeq Der((A,M),X)$$

The logarithmic topological Hochschild homology $THH^{(R,P)}(A,M)$ has a definition formally similar to that of ordinary THH. We explain this in greater detail in Chapter 2.

We generalize the André-Quillen spectral sequence to pre-logarithmic ring spectra in Theorem 3.3.1. The precise statement is

Theorem (3.3.1). Given a spectrum E, a cofibrant object (R,P) of $\mathcal{P}$, and a cofibrant object (A,M) of $\mathcal{P}_{(R,P)}$, there is a spectral sequence with E^2-term

$$E^2_{p,q} = E_{p+q}((TAQ^{(R,P)}(A,M))^{\wedge q}/\Sigma_q)$$

converging to $E_{p+q}(THH^{(R,P)}(A,M))$. If E is multiplicative, then the spectral sequence is a spectral sequence of algebras.

Log structures were originally defined by Fontaine-Illusie and then expounded in [11]. Pre-logarithmic rings were introduced to homotopy theory in [8] where, in the course of computing the algebraic K-theory of K (a complete discrete valuation field of characteristic zero with valuation ring A and with perfect residue field k of characteristic $p > 2$) it was observed that there were cofiber sequences

$$THH(k) \to THH(A) \to THH(A|K)$$

and that the zeroth mod p homotopy group of $THH(A|K)$ has the structure of a pre-log ring.

The cofiber term $THH(A|K)$ is defined in terms of the category of A-modules. The construction formally resembles the cyclic bar construction used to define THH of a ring spectrum, but $THH(A|K)$ is not known to correspond to THH of any ring spectrum.

Over a series of papers [1] and [2] it was suggested that the approach of [8] might be extended to compute the mod (p, v_1) homotopy groups of $K(KU)$, the algebraic K-theory of (the p-completion of) topological K-theory. This was partly confirmed by the construction in [4] of general cofiber sequences for THH, and the specialization in [5] of these cofiber sequences to the case of KU:

$$THH(H\mathbb{Z}_{(p)}) \to THH(ku) \to THH(ku|KU).$$

It was conjectured in [17] that this latter computation would have a logarithmic interpretation. An André-Quillen spectral sequence and HKR theorem were also conjectured in [17, Remark 5.29] as part of an investigation of logarithmic structures in homotopy theory aimed at computing algebraic K-theory. It is shown in [17] that the mod p homotopy groups of $THH(A|K)$ are the same as those of a certain logarithmic THH. More recently, [16] showed that the cofiber term $THH(ku|KU)$ is stably equivalent to a logarithmic THH.

In Chapter 2, we review the necessary background material on prelogarithmic ring spectra, logarithmic topological André-Quillen homology and logarithmic topological Hochschild homology. In Chapter 3, we establish some results necessary to identify the E^2-term of the spectral sequence and we construct the spectral sequence. In Chapter 4, we prove an étale descent theorem.

Chapter 2

Pre-logarithmic ring spectra

In this chapter, we review the definitions of pre-logarithmic rings, logarithmic topological André-Quillen homology, and logarithmic topological Hochschild homology.

2.1 Pre-logarithmic ring spectra

In this section, we give a definition of pre-log ring spectra. The rings that we use are the commutative unital symmetric ring spectra. The monoids that we use are the commutative unital $\mathcal{J}$-space monoids. What makes $\mathcal{J}$-space monoids suitable for this purpose is that they form a symmetric monoidal category and they model the underlying multiplicative monoids of symmetric ring spectra.

One might consider using simplicial monoids, but the category of simplicial monoids fails to contain the underlying monoids of most symmetric ring spectra. For example, the category of connective ring spectra is Quillen equivalent to the category of ringlike E_∞ spaces. The underlying multiplicative monoid of a ringlike E_∞ space is a (multiplicative) E_∞ space. The category of simplicial monoids embeds in the category of E_∞ spaces, but is far from equivalent to it. For example, simplicial groups only reach Eilenberg-Mac Lane spaces.

The category of $\mathcal{J}$-space monoids on the other hand, is Quillen equivalent to E_∞ spaces over QS^0 [20, Theorem 4.11]. The space QS^0 has $\pi_0 QS^0 \cong \mathbb{Z}$, and the map to QS^0 lets us think of $\mathcal{J}$-space monoids as graded monoids. This makes it better suited to modeling the underlying multiplicative monoids of non-connective ring spectra

The category $\mathcal{J}$ has for objects pairs of finite sets (n, m) where we write n for $\{1, \ldots, n\}$. There is an object $(0, 0)$ where 0 denotes the empty set. A morphism $(\alpha, \beta, \phi)\colon (p, q) \to (r, s)$ in $\mathcal{J}$ consists of two injective maps $\alpha\colon p \to r$, and $\beta\colon q \to s$, and an isomorphism $\phi\colon r \setminus \alpha(p) \to s \setminus \beta(q)$ between the complements of the images of α and β. Letting $|p|$ denote the number of elements in p, the requirement that ϕ be an isomorphism implies that there are no morphisms from (p, q) to (r, s) unless $|r| - |p| = |s| - |q|$. The composite of morphisms $(\alpha_1, \beta_1, \sigma)\colon (p, q) \to (r, s)$ and $(\alpha_2, \beta_2, \tau)\colon (r, s) \to (t, u)$ is given by $(\alpha_2 \circ \alpha_1, \beta_2 \circ \beta_1, \phi)$ where ϕ is given by

$$\begin{cases} \tau & \text{on } t \setminus \alpha_2(r) \\ \beta_2 \circ \sigma \circ \alpha_2^{-1} & \text{on } \alpha_2(r \setminus \alpha_1(p)). \end{cases}$$

The category $\mathcal{J}$ has a tensor product $\sqcup\colon \mathcal{J} \times \mathcal{J} \to \mathcal{J}$ given by concatenation: $(p, q) \sqcup (r, s) = (p + r, q + s)$. The unit for the tensor product is $(0, 0)$. With this tensor product, $\mathcal{J}$ is a symmetric strict monoidal category.

Letting $\mathcal{S}$ denote the category of spaces (i.e. either simplicial sets or compactly generated weak Hausdorff spaces), *a $\mathcal{J}$-space* is a functor $\mathcal{J} \to \mathcal{S}$.

The category of $\mathcal{J}$-spaces is the functor category $\mathcal{S}^{\mathcal{J}}$.

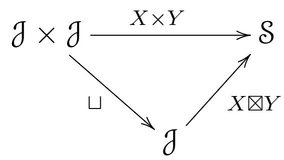
Example 2.1.1. Free $\mathcal{J}$-spaces

Defining functors $\mathcal{F}^{\mathcal{J}}_{p,q}\colon \mathcal{S} \to \mathcal{S}^{\mathcal{J}}$ by $\mathcal{F}^{\mathcal{J}}_{p,q}(X) = \mathcal{J}((p,q),-) \times X$ and $Ev_{p,q}\colon \mathcal{S}^{\mathcal{J}} \to \mathcal{S}$ by $Ev_{p,q}(A) = A_{p,q}$, there is an adjunction

$$\mathcal{S}^{\mathcal{J}}(\mathcal{F}^{\mathcal{J}}_{p,q}(X), A) \cong \mathcal{S}(X, Ev_{p,q}(A)).$$

We call a $\mathcal{J}$-space $\mathcal{F}^{\mathcal{J}}_{p,q}(X)$ a free $\mathcal{J}$-space.

Given two $\mathcal{J}$-spaces X, Y, there is a tensor product $X \boxtimes Y$ defined by left Kan extension of the cartesian product along the concatenation



In level j, this is given by the colimit

$$(X \boxtimes Y)_j = \operatorname*{colim}_{(p,q)\downarrow j} X_p \times X_q$$

A map out of the tensor product $X \boxtimes Y \to Z$ can be specified by a family of maps $X_p \times Y_q \to Z_{p \sqcup q}$. The unit for the tensor product is the $\mathcal{J}$-space $U = \mathcal{J}((0,0),-)$. There is an associator, left and right multiplications by the unit, and a symmetry making $\mathcal{S}^{\mathcal{J}}$ a symmetric monoidal category.

$\mathcal{S}^{\mathcal{J}}$ is also simplicially enriched and both tensored and cotensored over simplicial sets. Given a $\mathcal{J}$-space X, the tensor with a simplicial set T is taken levelwise so that

$$(X \times T)_j = X_j \times T$$

where on the right we use the cartesian product in simplicial sets or the cartesian product with the realization of T in topological spaces. Alternatively, this is the realization of the simplicial object given in simplicial level n by

$$(X \times T)_n = X \times \cdots \times X$$

with one factor of X for each element of T. The realization of a simplicial $\mathcal{J}$-space is the realization taken in each $\mathcal{J}$-space level so that

$$|X|_j = |X_j|$$

for a simplicial $\mathcal{J}$-space X. (For simplicial sets, realization means the diagonal of the bisimplicial set. For topological spaces it means geometric realization.) For details on the simplicial enrichment and the cotensor, see [20].

Proposition 2.1.2 ([20, Proposition 4.8])**.** There is a cofibrantly generated model structure on $\mathcal{S}^{\mathcal{J}}$, called the positive projective model structure, in which

- the generating cofibrations are the maps $\mathcal{F}^{\mathcal{J}}_{p,q}(\partial\Delta^n) \to \mathcal{F}^{\mathcal{J}}_{p,q}(\Delta^n)$ for $p \geq 1, n \geq 0$,
- the generating acyclic cofibrations are the maps $\mathcal{F}^{\mathcal{J}}_{p,q}(\Lambda^n_k) \to \mathcal{F}^{\mathcal{J}}_{p,q}(\Delta^n)$ for $p \geq 1, n \geq 0$,
- and the weak equivalences are the maps $f\colon X \to Y$ such that the induced maps on homotopy colimits $X_{h\mathcal{J}} \to Y_{h\mathcal{J}}$ is a weak equivalence

where the homotopy colimits $X_{h\mathcal{J}}$ are given by the bar construction i.e. the realization of the simplicial space

$$B_n(*, \mathcal{J}, X) = \coprod_{j_0 \to \cdots \to j_n} X_{j_0}.$$

With this model structure, $\mathcal{S}^{\mathcal{J}}$ is a simplicial model category, a monoidal model category satisfying the monoid axiom, and proper.

In addition to the positive projective model structure, $\mathcal{S}^{\mathcal{J}}$ has a flat model structure [20, 4.27], which has a strictly larger collection of cofibrant objects. The (p, q) latching space of a $\mathcal{J}$-space is the colimit

$$L_{(p,q)}X = \operatorname*{colim}_{\partial(\mathcal{J}\downarrow(p,q))} X$$

where $\partial(\mathcal{J} \downarrow (p, q))$ is the full subcategory of $\mathcal{J} \downarrow (p, q)$ excluding isomorphisms.

Definition 2.1.3. A $\mathcal{J}$-space X is *flat* if the map $L_{(p,q)}X \to X_{p,q}$ is a cofibration for all $(p, q) \in \mathcal{J}$.

We only mention this model structure for the following reason: If one of X or Y is flat, then there is a weak equivalence $X_{h\mathcal{J}} \times Y_{h\mathcal{J}} \to (X \boxtimes Y)_{h\mathcal{J}}$. This is given by the composite

$$X_{h\mathcal{J}} \times Y_{h\mathcal{J}} \cong (X \times Y)_{h\mathcal{J}\times h\mathcal{J}} \to (X \boxtimes Y \circ (- \sqcup -))_{h\mathcal{J}} \to (X \boxtimes Y)_{h\mathcal{J}}.$$

In addition, if X is flat, then by [20, Proposition 8.2] the functor $X \boxtimes (-)$ preserves $\mathcal{J}$-equivalences.

The commutative unital monoids in $\mathcal{S}^{\mathcal{J}}$, along with maps of such, form a category $\mathcal{CS}^{\mathcal{J}}$. This is the only category of $\mathcal{J}$-space monoids defined in this paper, so $\mathcal{J}$-spaces monoids are necessarily commutative and unital even if this is not explicitly mentioned.

A commutative unital $\mathcal{J}$-space monoid M can be defined by the data of a map

$$* \to M_{0,0}$$

and maps

$$M_p \times M_q \to M_{p \sqcup q}$$

for all $(p, q) \in \mathcal{J} \times \mathcal{J}$ such that

$$\begin{array}{ccc} M_p \times M_q & \longrightarrow & M_{p\sqcup q} \\ \downarrow & & \| \\ M_q \times M_p & \longrightarrow & M_{p\sqcup q} \end{array}$$

commutes.

Given a pair of morphisms $L \to N$ and $M \to N$ in $\mathcal{CS}^{\mathcal{J}}$, there is a morphism $L \boxtimes M \to N$ defined by the composite maps

$$L_p \times M_q \to N_p \times N_q \to N_{p\sqcup q}.$$

making $L \boxtimes M$ the coproduct of L and M in $\mathcal{CS}^{\mathcal{J}}$.

Example 2.1.4. Free $\mathcal{J}$-space monoids

Given a $\mathcal{J}$-space X, the free commutative unital $\mathcal{J}$-space monoid on X is given by

$$\mathbb{C}X = \coprod_{n\geq 0} X^{\boxtimes n}/\Sigma_n$$

Fixing an object P of $\mathcal{CS}^{\mathcal{J}}$, the comma category $\mathcal{CS}^{\mathcal{J}}_{P/}$ is a symmetric monoidal category. If M and N are two objects of $\mathcal{CS}^{\mathcal{J}}_{P/}$, the tensor product, written as $M \boxtimes_P N$, is given by the following pushout in $\mathcal{CS}^{\mathcal{J}}$

$$\begin{array}{ccc} P & \longrightarrow & N \\ \downarrow & & \downarrow \\ M & \longrightarrow & M \boxtimes_P N \end{array}$$

or equivalently, by the coproduct of M and N in $\mathcal{CS}^{\mathcal{J}}_{P/}$, or as the following coequalizer in $\mathcal{S}^{\mathcal{J}}$

$$M \boxtimes P \boxtimes N \rightrightarrows M \boxtimes N \to M \boxtimes_P N.$$

The unit for the tensor product is P. When $P = U$, there is no difference between $M \boxtimes_U N$ in $\mathcal{CS}^{\mathcal{J}}$ and $M \boxtimes N$ in $\mathcal{S}^{\mathcal{J}}$, and we omit the U subscript.

$\mathcal{CS}^{\mathcal{J}}_{P/}$ is a simplicial category, both tensored and cotensored over simplicial sets. Given a P-algebra M and a simplicial set T we define the P-algebra $M \otimes_P T$ as the realization of the simplicial P-algebra given in simplicial level n by

$$(M \otimes_P T)_n = M \boxtimes_P \cdots \boxtimes_P M$$

the coproduct of M with itself in $\mathcal{CS}^{\mathcal{J}}_{P/}$ with one summand for each element of T_n. The simplicial structure maps are induced from those of T. By forgetting the monoidal structure, we regard this as a simplicial object in $\mathcal{S}^{\mathcal{J}}$. The realization of a simplicial object in $\mathcal{CS}^{\mathcal{J}}$ is the realization of the underlying object in $\mathcal{S}^{\mathcal{J}}$. Sometimes we use $M \otimes_P T$ to refer to both the realization and the simplicial object. The context clarifies which is meant.

Example 2.1.5. $M \otimes_P S^0 = M \boxtimes_P M$. This is the realization of the constant simplicial object with

$$(M \otimes_P S^0)_n = M \boxtimes_P M.$$

Example 2.1.6. $M \otimes_P S^1$ is the realization of the simplicial object with $n+1$ summands of M in simplicial level n:

$$(M \otimes_P S^1)_n = \underbrace{M \boxtimes_P \cdots \boxtimes_P M}_{n+1}$$

The structure maps are induced from those of the circle $S^1 = \Delta^1/\partial\Delta^1$. If $d_i \colon S^1_n \to S^1_{n-1}$ is a face map, then there is an induced map

$$M \boxtimes_P S^1_n \to M \boxtimes_P S^1_{n-1}$$

defined on the summand indexed by $s \in S^1_n$ as the map

$$M \to \underbrace{M \boxtimes_P \cdots \boxtimes_P M}_{n}$$

corresponding to the inclusion of the $d_i(s)$ summand in $M \boxtimes_P S^1_{n-1}$. In the same way, the degeneracy maps of S^1 induce the degeneracy maps of $M \otimes_P S^1$.

The homotopy colimit $X_{h\mathcal{J}}$ of a commutative unital $\mathcal{J}$-space monoid is a monoid. It isn't typically commutative, but it is an E_∞-space. As a consequence, $\pi_0(X_{h\mathcal{J}})$ is a commutative unital monoid.

Proposition 2.1.7 ([20, Proposition 4.10])**.** The positive projective model structure on $\mathcal{S}^{\mathcal{J}}$ lifts to $\mathcal{CS}^{\mathcal{J}}$ via the inclusion and makes a $\mathcal{CS}^{\mathcal{J}}$ a cofibrantly generated model category.

- The generating cofibrations are given by applying $\mathbb{C}$ to the generating cofibrations of $\mathcal{S}^{\mathcal{J}}$,
- the generating acyclic cofibrations are given by applying $\mathbb{C}$ to the generating acyclic cofibrations of $\mathcal{S}^{\mathcal{J}}$,
- and the weak equivalences are the $\mathcal{J}$-equivalences.

The fibrations on $\mathcal{CS}^{\mathcal{J}}$ are the underlying fibrations of $\mathcal{S}^{\mathcal{J}}$. With this model structure, $\mathcal{CS}^{\mathcal{J}}$ is a simplicial model category, a monoidal model category, and proper. Furthermore, the underlying $\mathcal{J}$-spaces of cofibrant objects of $\mathcal{CS}^{\mathcal{J}}$ are flat. □

There is a Quillen adjunction

$$\mathbb{S}^{\mathcal{J}} \colon \mathcal{S}^{\mathcal{J}} \rightleftarrows Sp^{\Sigma} \colon \Omega^{\mathcal{J}}$$

relating $\mathcal{S}^{\mathcal{J}}$ with the positive projective model structure and Sp^{Σ} with the positive projective model structure. The functor $\mathbb{S}^{\mathcal{J}} \colon \mathcal{CS}^{\mathcal{J}} \to Sp^{\Sigma}$ is given by

$$\mathbb{S}^{\mathcal{J}}[X]_n = \bigvee_{k \geq 0} S^k \wedge_{\Sigma_k} X(n,k)_+$$

and the functor $\Omega^{\mathcal{J}} \colon Sp^{\Sigma} \to \mathcal{CS}^{\mathcal{J}}$ is given by

$$\Omega^{\mathcal{J}}(A)_{p,q} = \Omega^q(A_p).$$

This adjunction restricts to a Quillen adjunction

$$\mathbb{S}^{\mathcal{J}} \colon \mathcal{CS}^{\mathcal{J}} \rightleftarrows \{\text{commutative unital symmetric ring spectra}\} \colon \Omega^{\mathcal{J}}$$

For a commutative unital $\mathcal{J}$-space monoid M, the spectrum $\mathbb{S}^{\mathcal{J}}[M]$ is meant to be thought of as the algebra generated by M, while $\Omega^{\mathcal{J}}(A)$ is meant to be thought of as the underlying multiplicative monoid of the symmetric ring spectrum A.

One can define $\pi_0(\Omega^{\mathcal{J}}(A))$ by taking the colimit

$$\operatorname*{colim}_{(p,q)\in\mathcal{N}} \pi_0(\Omega^q(A_p))$$

over the wide subcategory $\mathcal{N}$ of $\mathcal{J}$ consisting only of standard inclusions. In this case, one has $\pi_0(\Omega^{\mathcal{J}}(A)) \cong \pi_*(A)$ where the homotopy groups being computed are the naive homotopy groups of A. When A is semistable, one gets the true homotopy groups.

Proposition 2.1.8. If A is a fibrant symmetric ring spectrum, then $\pi_0(\Omega^{\mathcal{J}}(A)_{h\mathcal{J}}) \cong \pi_*A/\{\pm 1\}$ as monoids.

Proof. This is [20, Proposition 4.24] and [20, Corollary 4.17]. □

Notation 2.1.9. Given a symmetric ring spectrum A let $\overline{\pi_*(A)}$ denote the quotient of $\pi_*(A)$ by $\{\pm 1\}$.

The quotient comes from the automorphisms of objects of $\mathcal{J}$. The automorphisms of (p, q) act on the loop coordinates of $\Omega^q(A_p)$ by permutations, and as a result, induce multiplication by ± 1 on $\pi_0(\Omega^q(A_p))$. This results in the quotient by ± 1 in the homotopy colimit. Such an occurrence is necessary because $\pi_* A$ is graded commutative (when the product is defined) rather than commutative while $\pi_0(\Omega^{\mathcal{J}}(A)_{h\mathcal{J}})$ must be commutative.

Definition 2.1.10. A *pre-logarithmic ring spectrum* or *pre-log ring* is a triple (A, M, α) where A is a commutative symmetric ring spectrum, M is a commutative unital J-space monoid, and $\alpha\colon \mathbb{S}^{\mathcal{J}}[M] \to A$ is a map of symmetric ring spectra.

The map α is frequently left out of the notation.

Definition 2.1.11. A *morphism of pre-log ring spectra* $(f, \phi)\colon (A, M, \alpha) \to (B, N, \beta)$ is a map of symmetric ring spectra $f\colon A \to B$ and a map of commutative unital J-space monoids $\phi\colon M \to N$ such that the diagram

$$\begin{array}{ccc} \mathbb{S}^{\mathcal{J}}[M] & \xrightarrow{\alpha} & A \\ \downarrow{\scriptstyle \mathbb{S}^{\mathcal{J}}[\phi]} & & \downarrow{\scriptstyle f} \\ \mathbb{S}^{\mathcal{J}}[N] & \xrightarrow{\beta} & B \end{array}$$

commutes.

Definition 2.1.12. A *canonical pre-log ring* is a pre-log ring $(\mathbb{S}^{\mathcal{J}}[M], M)$.

Let $\mathcal{P}$ denote the category of pre-log ring spectra. $\mathcal{P}$ is a symmetric monoidal category with the tensor product given by $\otimes = (\wedge, \boxtimes)$ (so that $(A, M) \otimes (B, N) = (A \wedge B, M \boxtimes N))$ and the unit given by (S, U). The tensor product is also a coproduct in $\mathcal{P}$.

Every pre-log ring is a pushout of ordinary and canonical pre-log rings in the sense that the following square is a pushout square

$$\begin{array}{ccc} (\mathbb{S}^{\mathcal{J}}[M], U) & \longrightarrow & (\mathbb{S}^{\mathcal{J}}[M], M) \\ \downarrow & & \downarrow \\ (A, U) & \longrightarrow & (A, M) \end{array}$$

The discussion after [16, Definition 4.5] shows the following

Proposition 2.1.13. The category $\mathcal{P}$ has a model structure in which a map $(f, \phi)\colon (A, M) \to (B, N)$ is a weak equivalence when f and ϕ are both weak equivalences, and a cofibration when ϕ, and the induced map $A \wedge_{\mathbb{S}^{\mathcal{J}}[M]} \mathbb{S}^{\mathcal{J}}[N] \to B$ are cofibrations.

With this model structure, the category of pre-log rings is a cofibrantly generated proper simplicial monoidal model category. The fibrations and weak equivalences are created by the forgetful functors to commutative symmetric ring spectra and commutative unital $\mathcal{J}$-space monoids. □

Given a pre-log ring (R, P), let $\mathcal{P}_{(R,P)}$ denote the comma category of pre-log rings under (R, P).

2.2 Logarithmic TAQ

This section reviews the definition of ordinary TAQ, then gives the definition of logarithmic TAQ.

Following the discussion in [3], we make the following definitions. Let A be a commutative unital symmetric ring spectrum. Let $\mathcal{M}_A$ be the category of A-modules, let $\mathcal{C}_A$ be the category of commutative A-algebras, let $\mathcal{C}_{A/A}$ be the category of commutative A-algebras augmented over A, and let $\mathcal{N}_A$ be the non-unital commutative A-algebras.

Define the functor $I\colon \mathcal{C}_{A/A} \to \mathcal{N}_A$ by the pullback in $\mathcal{M}_A$

$$\begin{array}{ccc} I(B) & \longrightarrow & B \\ \downarrow & & \downarrow \\ * & \longrightarrow & A \end{array}$$

Define a functor $K\colon \mathcal{N}_A \to \mathcal{C}_{A/A}$ by $K(X) = A \vee X$. When the ring A needs to be emphasized, we write K_A and I_A.

Proposition 2.2.1 ([3, Proposition 2.1])**.** There is an adjunction

$$\mathcal{C}_{A/A}(K(X), Y) \cong \mathcal{N}_A(X, I(Y)).$$

□

Define the functor $Q\colon \mathcal{N}_A \to \mathcal{M}_A$ by the pushout

$$\begin{array}{ccc} N^{\wedge 2} & \longrightarrow & N \\ \downarrow & & \downarrow \\ * & \longrightarrow & Q(N) \end{array}$$

in $\mathcal{M}_A$. Define a functor $Z\colon \mathcal{M}_A \to \mathcal{N}_A$ by $Z(X) = X$ with the zero product. When the ring A needs to be emphasized, we write Q_A and Z_A.

Proposition 2.2.2 ([3, Proposition 3.1])**.** There is an adjunction

$$\mathcal{M}_A(Q(X), Y) \cong \mathcal{N}_A(X, Z(Y)).$$

□

All four categories $\mathcal{M}_A$, $\mathcal{C}_A$, $\mathcal{C}_{A/A}$, and $\mathcal{N}_A$ inherit the positive projective model structure from Sp^Σ. This is proven for $\mathcal{M}_A$ and $\mathcal{C}_A$ in [12, Theorem 14.5 (i), (iii)] and [12, Theorem 15.2 (i)] respectively. For $\mathcal{C}_{A/A}$, we refer to [9, Theorem 7.6.5 (2)]. In the case of $\mathcal{N}_A$, the proof given in [3, Proposition 1.1] uses [7, VII Theorem 4.9] which is not quite appropriate for symmetric spectra, but which can be replaced by [12, Proposition 5.13].

Proposition 2.2.3 ([3, Proposition 2.2])**.** The (K, I) adjunction is a Quillen adjunction. □

Proposition 2.2.4. The (Q, Z) adjunction is a Quillen adjunction.

Proof. As noted in the discussion after [3, Proposition 3.1], Z preserves both weak equivalences and fibrations. □

With these definitions, [3] shows that there is an adjunction on homotopy categories

$$h\mathcal{M}_R(\mathbb{L}Q_R\mathbb{R}I_R(X), A) \cong h\mathcal{C}_{R/R}(X, \mathbb{L}K_R\mathbb{R}Z_R(A)).$$

In addition, it was shown that there is an isomorphism

$$h\mathcal{M}_A(\mathbb{L}Q_A\mathbb{R}I_A(B \wedge^{\mathbb{L}}_R A), X) \cong h\mathcal{C}_{R/A}(B, A \vee X).$$

Because $\mathcal{C}_{R/A}(A, A \vee X)$ can be thought of as the space of derivations of A into X, this isomorphism gives an interpretation of $\mathbb{L}Q_A\mathbb{R}I_A(A \wedge^{\mathbb{L}}_R A)$ as the corepresenting object for derivations. Consequently TAQ was defined as

$$TAQ^R(A) = \mathbb{L}Q_A\mathbb{R}I_A(A \wedge^{\mathbb{L}}_R A).$$

The logarithmic version of TAQ is defined analogously to the logarithmic Kähler differentials. For that we need to discuss group completion of $\mathcal{J}$-space monoids.

Definition 2.2.5 ([18, Definition 5.4 (iii)])**.** A map $f\colon X \to Y$ in $\mathcal{CS}^{\mathcal{J}}$ is a *weak equivalence after group completion* if the map $B(X_{h\mathcal{J}}) \to B(Y_{h\mathcal{J}})$ is a weak equivalence where B denotes the bar construction.

Definition 2.2.6. A $\mathcal{J}$-space monoid X is *grouplike* if $\pi_0 X_{h\mathcal{J}}$ is a group.

Proposition 2.2.7. The positive projective model structure on $\mathcal{CS}^{\mathcal{J}}$ has a left Bousfield localization in which

- The weak equivalences are the maps that are equivalences after group completion.
- The fibrant objects are the grouplike fibrant objects of the positive projective model structure.

□

We call this the *group completion model structure on* $\mathcal{CS}^{\mathcal{J}}$. Using the notation of [20], we denote this model category by $\mathcal{CS}^{\mathcal{J}}_{gp}$.

Definition 2.2.8 ([18, Definition 5.2 (ii)])**.** A *group completion* M^{gp} of an object M in $\mathcal{CS}^{\mathcal{J}}$ is a fibrant replacement in the group completion model structure.

Even with group completions of monoids, in order to mimic the definition of the logarithmic Kähler differentials it is necessary to have an interpretation of the group completion as a symmetric spectrum (in order to have all of the constituent objects in the same category). This is done by converting a monoid into a Γ-space.

Definition 2.2.9. A Γ-space X is *monoidlike* if for $S, T \in \Gamma^{op}$ the induced map $X(S \vee T) \to X(S) \times X(T)$ is a weak equivalence and $\pi_0 X(1^+)$ is a monoid.

Definition 2.2.10. A Γ-space X is *grouplike* if for $S, T \in \Gamma^{op}$ the induced map $X(S \vee T) \to X(S) \times X(T)$ is a weak equivalence and $\pi_0 X(1^+)$ is a group.

Remark 2.2.11. In the definition of monoidlike, it's actually redundant to require that $\pi_0 X(1^+)$ is a monoid. That follows from the first condition.

Proposition 2.2.12. Γ^{op}-$\mathcal{S}$ has a cofibrantly generated proper level model structure in which a map $X \to Y$ is a weak equivalence if and only if the map $X(S) \to Y(S)$ is a weak equivalence for every based set S, and a fibration if and only if $X(S) \to Y(S)$ is a fibration for every based set S. □

Proposition 2.2.13 ([18, Lemma 7.3])**.** The level model structure on Γ^{op}-$\mathcal{S}$ has a left Bousfield localization that is cofibrantly generated and proper. The fibrant objects in this model structure are levelwise fibrant and monoidlike. □

This is called the pre-stable model structure, and we'll denote it by $(\Gamma^{op}\text{-}\mathcal{S})_{pre}$.

Proposition 2.2.14 ([18, Lemma 7.14])**.** The pre-stable model structure on Γ^{op}-$\mathcal{S}$ has a left Bousfield localization that coincides with the stable Q-model structure on Γ-spaces. (Γ^{op}-$\mathcal{S}$ with the stable Q-model structure is Quillen equivalent to Sp^{Σ} with the positive projective model structure.) In this model structure, a map is a weak equivalence if the associated map of spectra is a weak equivalence. An object is fibrant if it is level fibrant and grouplike. □

We'll denote this model category by $(\Gamma^{op}\text{-}\mathcal{S})_{st}$.

Proposition 2.2.15. There is a functor $\gamma\colon \mathcal{CS}^{\mathcal{J}} \to \Gamma^{op}\text{-}\mathcal{S}$ that is equivalent to a composite of derived functors of Quillen equivalences. □

Definition 2.2.16. Let $b\mathcal{J} = \gamma(*)$ denote the image of the terminal object $* \in \mathcal{CS}^{\mathcal{J}}$ under γ.

Proposition 2.2.17. There are Quillen equivalences as follows:

1. [18, Corollary 7.12] The functor γ takes $\mathcal{CS}^{\mathcal{J}}$ with the positive projective model structure to the comma category $(\Gamma^{op}\text{-}\mathcal{S})_{pre} \downarrow b\mathcal{J}$ with the pre-stable model structure.

2. [18, Corollary 7.17] The functor γ takes $\mathcal{CS}^{\mathcal{J}}_{gp}$ with the group completion model structure to the comma category $(\Gamma^{op}\text{-}\mathcal{S})_{st} \downarrow b\mathcal{J}$ with the stable model structure.

□

A map $X \to Y$ in $\mathcal{CS}^{\mathcal{J}}_{gp}$ is a weak equivalence if and only if $\gamma(X) \to \gamma(Y)$ is a stable equivalence. By [18, Definition 3.11], given an object A of $\mathcal{CS}^{\mathcal{J}}$, $\gamma(A)(1^+) \cong A_{h\mathcal{J}}$, so γ takes $\mathcal{J}$-equivalences to level equivalences of Γ-spaces, and not merely pre-stable equivalences.

Because γ takes equivalences after group completion to stable equivalences, $\gamma(M)$ will play the role of the group completion of M in the definition of log TAQ.

Definition 2.2.18 ([19, Definition 5.20])**.** Let (A, M) be an object of $\mathcal{P}_{(R,P)}$. The *logarithmic topological André-Quillen homology spectrum* $TAQ^{(R,P)}(A, M)$ is a homotopy pushout in $\mathcal{M}_A$

$$TAQ^R(A) \coprod_{A \wedge_{\mathbb{S}^{\mathcal{J}}[M]} TAQ^{\mathbb{S}^{\mathcal{J}}[P]}(\mathbb{S}^{\mathcal{J}}[M])} A \wedge \gamma(M)/\gamma(P)$$

where $\gamma(M)/\gamma(P)$ is a cofibrant model of the symmetric spectrum associated to the cofiber of $\gamma(P) \to \gamma(M)$.

Log TAQ was constructed in [19] as the corepresenting object for logarithmic derivations. The map from $A \wedge_{\mathbb{S}^{\mathcal{J}}[M]} TAQ^{\mathbb{S}^{\mathcal{J}}[P]}(\mathbb{S}^{\mathcal{J}}[M])$ to $A \wedge \gamma(M)/\gamma(P)$ used to define this homotopy pushout was given implicitly by defining a natural transformation between the functors that they corepresent. We'll have more to say about this later. For the moment, we explain what logarithmic derivations are.

Let A be a commutative unital symmetric ring spectrum, and let X be a A-module. The spectrum $A \vee X$ has a product $(A \vee X) \wedge (A \vee X) \to A \vee X$

where X is given the zero product. i.e. there is an isomorphism

$$(A \vee X) \wedge (A \vee X) \cong (A \wedge A) \vee (A \wedge X) \vee (X \wedge A) \vee (X \wedge X)$$

and the product on $A \vee X$ comes from using this isomorphism, the product on A, the action of A on X, and the zero product $X \wedge X \to * \to X$.

Definition 2.2.19 ([19, Definition 5.2])**.** The *square-zero extension* $A \vee_f X$ is defined by the factorization $A \vee X \overset{\simeq}{\rightarrowtail} A \vee_f X \twoheadrightarrow A$ in commutative unital symmetric ring spectra.

Definition 2.2.20 ([18, Definition 2.5 (ii)])**.** Let M be a commutative unital $\mathcal{J}$-space monoid. The *units* $M^\times$ is the sub-$\mathcal{J}$-space monoid of M whose points correspond to units in $\pi_0(M_{h\mathcal{J}})$.

Definition 2.2.21 ([18, Definition 2.11)])**.** Let A be a commutative unital symmetric ring spectrum. The *units* $\mathrm{GL}_1^{\mathcal{J}}(A)$ is $\Omega^{\mathcal{J}}(A)^\times$.

Definition 2.2.22 ([19, Definition 5.3])**.** The $\mathcal{J}$-space monoid $(1+X)^{\mathcal{J}}$ is the cofibrant replacement of the pullback of the following diagram

$$\begin{array}{ccc} & & \mathrm{GL}_1^{\mathcal{J}}(A \vee_f X) \\ & & \downarrow \\ U & \longrightarrow & \mathrm{GL}_1^{\mathcal{J}}(A) \end{array}$$

(recall that U is the initial object in $\mathcal{CS}^{\mathcal{J}}$).

Definition 2.2.23 ([19, Construction 5.6])**.** Given a pre-log ring (A, M), the monoid $(M+X)^{\mathcal{J}}$ is defined by the following factorization in $\mathcal{CS}^{\mathcal{J}}$

$$M \boxtimes (1+X)^{\mathcal{J}} \overset{\simeq}{\rightarrowtail} (M+X)^{\mathcal{J}} \twoheadrightarrow M$$

in the positive projective model structure.

Definition 2.2.24. The pre-log ring $(A \vee_f X, (M+X)^{\mathcal{J}})$ is the *square-zero extension* of (A, M).

Definition 2.2.25 ([19, Definition 5.14])**.** The *space of logarithmic derivations* is the space

$$\mathcal{P}_{(R,P)/(A,M)}((A, M), (A \vee_f X, (M+X)^{\mathcal{J}})).$$

Proposition 2.2.26 ([19, Proposition 5.21])**.** There is a weak equivalence

$$\mathcal{M}_A(TAQ^{(R,P)}(A, M), X) \simeq \mathcal{P}_{(R,P)/(A,M)}((A, M), (A \vee_f X, (M+X)^{\mathcal{J}})).$$

□

2.3 Logarithmic THH

In this section, we give the definition of logarithmic THH.

Definition 2.3.1 ([16, Definition 3.17])**.** The *repletion* N^{rep} of a commutative unital $\mathcal{J}$-space monoid over M, e.g. $N \to M$, is a factorization

$$N \overset{\simeq}{\rightarrowtail} N^{rep} \twoheadrightarrow M$$

in the group completion model structure.

Definition 2.3.2. A commutative unital $\mathcal{J}$-space monoid N over M, e.g. $N \to M$ is *replete* if the map $N \to M$ is a fibration in the group completion model structure.

Definition 2.3.3 ([16, Definition 3.19]). A map $N \to M$ in $\mathcal{CS}^{\mathcal{J}}$ is *virtually surjective* if $\pi_0(N^{gp}_{h\mathcal{J}}) \to \pi_0(M^{gp}_{h\mathcal{J}})$ is surjective.

Example 2.3.4. Given $P \longrightarrow M$ in $\mathcal{CS}^{\mathcal{J}}$, the composite

$$M \cong M \boxtimes_P P \to M \boxtimes_P M \to M$$

is the identity. Taking group completions, and then homotopy colimits, one has

$$M^{gp}_{h\mathcal{J}} \to (M \boxtimes_P M)^{gp}_{h\mathcal{J}} \to M^{gp}_{h\mathcal{J}}$$

which is also the identity. So $\pi_0(M \boxtimes_P M)^{gp}_{h\mathcal{J}} \to \pi_0 M^{gp}_{h\mathcal{J}}$ is surjective. Thus $M \boxtimes_P M \to M$ is virtually surjective.

Remark 2.3.5. The same argument applies to show that $M \otimes_P T \to M$ is virtually surjective for any simplicial set T. In particular, $M \otimes_P S^1$ is virtually surjective.

Proposition 2.3.6 ([16, Corollary 3.21]). Let $\epsilon\colon N \to M$ be a virtually surjective map in $\mathcal{CS}^{\mathcal{J}}$. Then the repletion N^{rep} is $\mathcal{J}$-equivalent to the homotopy pullback of $M \longrightarrow M^{gp} \longleftarrow N^{gp}$ in the positive projective model structure. □

Letting $B^{cy}(M)$ denote $(M \otimes_P S^1)$, and letting $B^{rep}(M)$ denote $(M \otimes_P S^1)^{rep}$, we imitate the definition of logarithmic THH given in [16, Definition 4.6]:

Definition 2.3.7. Let (A, M) be a cofibrant object of $\mathcal{P}_{(R,P)}$. The *logarithmic topological Hochschild homology spectrum* $THH^{(R,P)}(A, M)$ is given by

$$THH^R(A) \bigwedge_{\mathbb{S}^{\mathcal{J}}[B^{cy}(M)]} \mathbb{S}^{\mathcal{J}}[B^{rep}(M)].$$

When (A, M) is not cofibrant, take a factorization in $\mathcal{P}$

$$(R, P) \rightarrowtail (A, M)^{cof} \xrightarrow{\simeq} \!\!\!\!\rightarrow (A, M)$$

and define $THH^{(R,P)}(A, M) = THH^{(R,P)}((A, M)^{cof})$.

Chapter 3

The André-Quillen spectral sequence

In this chapter we give two constructions of André-Quillen spectral sequences. The first is a straightforward translation of the spectral sequence of [14] into logarithmic terms. The second, Theorem 3.3.1 is an adaptation of the technique to produce a multiplicative spectral sequence.

3.1 Identifying the E^2-term

In this section, we prove some results necessary to identify the E^2-term of the spectral sequence. Because pre-log rings are pushouts of ordinary rings and canonical pre-log rings, we address the ordinary and canonical pre-log parts separately, and then explain what happens on the pushouts.

Lemma 3.1.1. Let $\mathcal{D}$ be a model category. Let $B \to B'$ be a map in $\mathcal{D}$. There is a Quillen adjunction

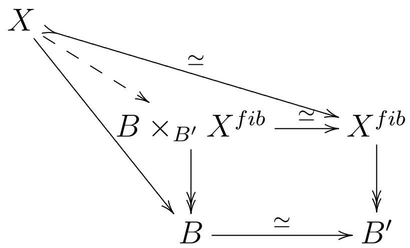
$$\mathcal{D}_{/B'}(X, A) \cong \mathcal{D}_{/B}(X, B \times_{B'} A)$$

where the left adjoint is given by postcomposition with $B \to B'$ and the right adjoint is induced by pullback.

Proof. Cofibrations in $\mathcal{D}_{/B'}$ are maps over B' that are cofibrations in $\mathcal{D}$, and likewise for cofibrations in $\mathcal{D}_{/B}$. Similarly, weak equivalences in $\mathcal{D}_{/B'}$ are maps over B' that are weak equivalences, and likewise for $\mathcal{D}_{/B}$. Therefore, the left adjoint, postcomposition with $B \to B'$, preserves cofibrations and weak equivalences, and is thus a left Quillen functor. □

Lemma 3.1.2. Let $\mathcal{D}$ be a right proper model category. Suppose that $B \xrightarrow{\simeq} B'$ is a weak equivalence in $\mathcal{D}$. Then the adjunction of Lemma 3.1.1 is a Quillen equivalence.

Proof. Given a cofibrant object X, we need to show that $X \to B \times_{B'} X^{fib}$ is a weak equivalence where $X \overset{\simeq}{\rightarrowtail} X^{fib} \twoheadrightarrow B'$ is a factorization in $\mathcal{D}$. The dotted arrow in the next diagram is the counit of the adjunction.



It follows from 2-of-3 that $X \to B \times_{B'} X^{fib}$ is a weak equivalence.

Given a fibrant object $A \in \mathcal{D}_{/B'}$, we need to show that $(B \times_{B'} A)^{cof} \to A$ is a weak equivalence where $(B \times_{B'} A)^{cof} \twoheadrightarrow B \times_{B'} A$ is a cofibrant replacement in $\mathcal{D}$. Because $\mathcal{D}$ is right proper, the map $B \times_{B'} A \to A$ is a weak equivalence and we have the following diagram

$$\begin{array}{ccccc} (B \times_{B'} A)^{cof} & & & & \\ & \searrow^{\simeq} & & & \\ & & B \times_{B'} A & \xrightarrow{\simeq} & A \\ & & \downarrow & & \downarrow \\ & & B & \xrightarrow{\simeq} & B'. \end{array}$$

Consequently, $(B \times_{B'} A)^{cof} \to A$ is a weak equivalence.

This shows that this is a Quillen equivalence. □

Define $F\colon \mathcal{CS}^{\mathcal{J}}_{M/\Omega^{\mathcal{J}}(T)} \to \mathcal{C}_{\mathbb{S}^{\mathcal{J}}[M]/T}$ by sending an object N in $\mathcal{CS}^{\mathcal{J}}_{M/\Omega^{\mathcal{J}}(T)}$ to $\mathbb{S}^{\mathcal{J}}[N]$ with augmentation $\mathbb{S}^{\mathcal{J}}[N] \to \mathbb{S}^{\mathcal{J}}[\Omega^{\mathcal{J}}(T)] \to T$, the composite of $\mathbb{S}^{\mathcal{J}}$ applied to $N \to \Omega^{\mathcal{J}}(T)$ and the counit of the $(\mathbb{S}^{\mathcal{J}}, \Omega^{\mathcal{J}})$ adjunction.

Define $G\colon \mathcal{C}_{\mathbb{S}^{\mathcal{J}}[M]/T} \to \mathcal{CS}^{\mathcal{J}}_{M/\Omega^{\mathcal{J}}(T)}$ by sending Y in $\mathcal{C}_{\mathbb{S}^{\mathcal{J}}[M]/T}$ to $\Omega^{\mathcal{J}}(Y)$ with a map $M \to \Omega^{\mathcal{J}}(\mathbb{S}^{\mathcal{J}}[M]) \to \Omega^{\mathcal{J}}(Y)$, the composite of $\Omega^{\mathcal{J}}$ applied to $\mathbb{S}^{\mathcal{J}}[M] \to Y$ and the unit of the $(\mathbb{S}^{\mathcal{J}}, \Omega^{\mathcal{J}})$ adjunction.

Lemma 3.1.3. There is a Quillen adjunction

$$\mathcal{C}_{\mathbb{S}^{\mathcal{J}}[M]/T}(F(N), Y) \cong \mathcal{CS}^{\mathcal{J}}_{M/\Omega^{\mathcal{J}}(T)}(N, G(Y))$$

between $\mathcal{C}_{\mathbb{S}^{\mathcal{J}}[M]/T}$ with the positive projective model structure and $\mathcal{CS}^{\mathcal{J}}_{M/\Omega^{\mathcal{J}}(T)}$ with the positive projective model structure.

Proof. First, we show that there is an adjunction. A map $F(N) \to Y$ in

$\mathcal{C}_{\mathbb{S}^{\mathcal{J}}[M]/T}$ corresponds to a commutative diagram

$$
\begin{array}{ccccc}
 & & \mathbb{S}^{\mathcal{J}}[M] & & \\
 & \swarrow & & \searrow^{a} & \\
\mathbb{S}^{\mathcal{J}}[N] & & \xrightarrow{\quad b \quad} & & Y \\
 & \searrow_{c} & & \swarrow & \\
 & & T & &
\end{array}
$$

Taking the adjoints of a, b, and c coming from the $(\mathbb{S}^{\mathcal{J}}, \Omega^{\mathcal{J}})$ adjunction, this corresponds to a commutative diagram in $\mathcal{C}\mathcal{S}^{\mathcal{J}}$

$$
\begin{array}{ccccc}
 & & M & & \\
 & \swarrow & & \searrow^{a^*} & \\
N & & \xrightarrow{\quad b^* \quad} & & \Omega^{\mathcal{J}}(Y) \\
 & \searrow_{c^*} & & \swarrow & \\
 & & \Omega^{\mathcal{J}}(T) & &
\end{array}
$$

which encodes a map $N \to G(Y)$. If we instead start with the bottom diagram, taking adjoints will produce the top one.

Now we show that the adjunction is a Quillen adjunction. A map in $\mathcal{C}\mathcal{S}^{\mathcal{J}}_{M/\Omega^{\mathcal{J}}(\mathbb{S}^{\mathcal{J}}[M])}$, for example

$$
\begin{array}{ccccc}
 & & M & & \\
 & \swarrow & & \searrow & \\
L & & \xrightarrow{\quad f \quad} & & N \\
 & \searrow & & \swarrow & \\
 & & \Omega^{\mathcal{J}}(T) & &
\end{array}
$$

is a cofibration if f is a cofibration in $\mathcal{C}\mathcal{S}^{\mathcal{J}}$. Applying F, the resulting map is a cofibration if $\mathbb{S}^{\mathcal{J}}[f]$ is a cofibration. Because $\mathbb{S}^{\mathcal{J}}$ is a left Quillen functor, this is the case, so F preserves cofibrations. The same argument shows that F preserves acyclic cofibrations. $\square$

Notation 3.1.4. Let $\times^h$ denote homotopy pullback. The category is to be understood from context.

Lemma 3.1.5. Suppose that M is cofibrant in $\mathcal{CS}^{\mathcal{J}}$, and that N is an object in $\mathcal{CS}^{\mathcal{J}}_{M/M}$ with a virtually surjective augmentation. Then N^{rep}, the repletion of N over M, and $M \boxtimes (U \times^h_{M^{gp}} N^{gp})$ are $\mathcal{J}$-equivalent in $\mathcal{CS}^{\mathcal{J}}_{M/M}$.

Proof. Let $N^{gp} \to M^{gp}$ be the fibrant replacement of $N \to M$ in $\mathcal{CS}^{\mathcal{J}}_{gp}$, and let $N^{gp} \rightarrowtail^{\simeq} T \twoheadrightarrow M^{gp}$ be a factorization in $\mathcal{CS}^{\mathcal{J}}_{pos}$. We will produce the following zig-zag of $\mathcal{J}$-equivalences in $\mathcal{CS}^{\mathcal{J}}_{M/M}$

$$
\begin{array}{ccccc}
N^{rep} & \xrightarrow{\simeq} & M \times_{M^{gp}} T & \xleftarrow{\simeq} & M \boxtimes (U \times_{M^{gp}} T) \\
 & & & \swarrow_{\simeq} & \\
 & & M \boxtimes (U \times^h_{M^{gp}} T) & \xleftarrow{\simeq} & M \boxtimes (U \times^h_{M^{gp}} N^{gp}).
\end{array}
$$

Because $N \to M$ is virtually surjective, it follows from [16, Corollary 3.21] that N^{rep} is $\mathcal{J}$-equivalent to $M \times_{M^{gp}} T$. Examining the proof of that corollary shows that the two are in fact $\mathcal{J}$-equivalent under N and over M i.e. there is a commutative diagram

$$
\begin{array}{ccc}
 & N & \\
\swarrow & & \searrow \\
N^{rep} & \xrightarrow{\simeq} & M \times_{M^{gp}} T \\
\searrow & & \swarrow \\
 & M &
\end{array}
$$

By precomposing with the map $M \rightarrowtail N$, we see that they are $\mathcal{J}$-equivalent in $\mathcal{CS}^{\mathcal{J}}_{M/M}$.

Consider the map $M \boxtimes (U \times_{M^{gp}} T) \to M \times_{M^{gp}} T$ induced from the map $M \to M \times_{M^{gp}} T$ just discussed and the map $U \times_{M^{gp}} T \to M \times_{M^{gp}} T$ induced on pullbacks. This also fits into a commutative diagram in $\mathcal{CS}^{\mathcal{J}}_{M/M}$

$$\begin{array}{ccccc} & & M & & \\ & \swarrow & & \searrow & \\ M \boxtimes (U \times_{M^{gp}} T) & & \longrightarrow & & M \times_{M^{gp}} T \\ & \searrow & & \swarrow & \\ & & M & & \end{array}$$

Taking homotopy colimits, we have $(M \boxtimes (U \times_{M^{gp}} T))_{h\mathcal{J}} \to (M \times_{M^{gp}} T)_{h\mathcal{J}}$ and because M is cofibrant in $\mathcal{CS}^{\mathcal{J}}$, [21, Lemma 2.25] implies that there is a weak equivalence $M_{h\mathcal{J}} \times (U \times_{M^{gp}} T)_{h\mathcal{J}} \xrightarrow{\simeq} (M \boxtimes (U \times_{M^{gp}} T))_{h\mathcal{J}}$. The composite

$$M_{h\mathcal{J}} \times (U \times_{M^{gp}} T)_{h\mathcal{J}} \longrightarrow (M \times_{M^{gp}} T)_{h\mathcal{J}}$$

is, by [20, Corollary 11.4], weakly equivalent to

$$M_{h\mathcal{J}} \times (U_{h\mathcal{J}} \times^h_{M^{gp}_{h\mathcal{J}}} T_{h\mathcal{J}}) \longrightarrow M_{h\mathcal{J}} \times^h_{M^{gp}_{h\mathcal{J}}} T_{h\mathcal{J}}.$$

Because $U_{h\mathcal{J}}$ is contractible, there is a weak equivalence

$$M_{h\mathcal{J}} \times (\{1\} \times^h_{M^{gp}_{h\mathcal{J}}} T_{h\mathcal{J}}) \xrightarrow{\simeq} M_{h\mathcal{J}} \times (U_{h\mathcal{J}} \times^h_{M^{gp}_{h\mathcal{J}}} T_{h\mathcal{J}})\ .$$

In the commutative diagram

$$\begin{array}{ccc} M_{h\mathcal{J}} \times (\{1\} \times^h_{M^{gp}_{h\mathcal{J}}} T_{h\mathcal{J}}) & \xrightarrow{\simeq} & M_{h\mathcal{J}} \times (U_{h\mathcal{J}} \times^h_{M^{gp}_{h\mathcal{J}}} T_{h\mathcal{J}}) \\ & \searrow \quad \swarrow & \\ & M_{h\mathcal{J}} \times^h_{M^{gp}_{h\mathcal{J}}} T_{h\mathcal{J}} & \end{array}$$

The left diagonal map can be modeled using the cobar construction. In level n, this is a map

$$M_{h\mathcal{J}} \times \{1\} \times (M^{gp}_{h\mathcal{J}})^{\times 2n+1} \times T_{h\mathcal{J}} \longrightarrow M_{h\mathcal{J}} \times (M^{gp}_{h\mathcal{J}})^{\times 2n+1} \times T_{h\mathcal{J}}$$

$$(m, 1, m_1, \ldots, m_{2n+1}, t) \longmapsto (m, mm_1, \ldots, mm_{2n+1}, mt)$$

where we have suppressed maps $M_{h\mathcal{J}} \to M^{gp}_{h\mathcal{J}}$ and $M_{h\mathcal{J}} \to T_{h\mathcal{J}}$ and product maps to keep the notation uncluttered. Because $M^{gp}_{h\mathcal{J}}$ and $T_{h\mathcal{J}}$ are grouplike, multiplication by m is an equivalence, so this is levelwise a weak equivalence. The induced map on the totalizations, and thus the homotopy limits, is a weak equivalence. It follows from 2-of-3 that

$$M_{h\mathcal{J}} \times (U_{h\mathcal{J}} \times^h_{M^{gp}_{h\mathcal{J}}} T_{h\mathcal{J}}) \longrightarrow M_{h\mathcal{J}} \times^h_{M^{gp}_{h\mathcal{J}}} T_{h\mathcal{J}}$$

is a weak equivalence, so

$$M_{h\mathcal{J}} \times (U \times_{M^{gp}} T)_{h\mathcal{J}} \longrightarrow (M \times_{M^{gp}} T)_{h\mathcal{J}}$$

is a weak equivalence. It also follows from 2-of-3 that

$$(M \boxtimes (U \times_{M^{gp}} T))_{h\mathcal{J}} \longrightarrow (M \times_{M^{gp}} T)_{h\mathcal{J}}$$

a weak equivalence.

Now using the factorization given earlier

$$\begin{array}{ccc} N^{gp} & \xrightarrow{\simeq} & T \\ & \searrow \quad \swarrow & \\ & M^{gp} & \end{array}$$

take homotopy pullbacks along $U \rightarrowtail M^{gp}$, to obtain a commutative diagram

$$\begin{array}{ccccc} & & U & & \\ & \swarrow & \downarrow & \searrow & \\ U \times^h_{M^{gp}} N^{gp} & \xrightarrow{\simeq} & U \times^h_{M^{gp}} T & \xleftarrow{\simeq} & U \times_{M^{gp}} T \\ & \searrow & \downarrow & \swarrow & \\ & & U & & \end{array}$$

The pullback $U \times_{M^{gp}} T$ is a homotopy pullback because $T \to M^{gp}$ is a fibration. The induced map $U \times^h_{M^{gp}} T \leftarrow U \times_{M^{gp}} T$ is thus a weak equivalence.

Because M is cofibrant, $M \boxtimes (-)$ preserves $\mathcal{J}$-equivalences and we obtain a commutative diagram

$$\begin{array}{ccccc} & & M & & \\ & \swarrow & \downarrow & \searrow & \\ M \boxtimes (U \times^h_{M^{gp}} N^{gp}) & \xrightarrow{\simeq} & M \boxtimes (U \times^h_{M^{gp}} T) & \xleftarrow{\simeq} & M \boxtimes (U \times_{M^{gp}} T) \\ & \searrow & \downarrow & \swarrow & \\ & & M & & \end{array}$$

□

Lemma 3.1.6. Given a map $M \to N$, there is an adjunction

$$\mathcal{CS}^{\mathcal{J}}_{M/N}(M \boxtimes X, Y) \cong \mathcal{CS}^{\mathcal{J}}_{/N}(X, Y)$$

between $M \boxtimes (-) \colon \mathcal{CS}^{\mathcal{J}}_{/N} \to \mathcal{CS}^{\mathcal{J}}_{M/N}$ and the forgetful functor. If M is a cofibrant commutative unital $\mathcal{J}$-space monoid, then there is an induced adjunction on homotopy categories.

Proof. This is an adjunction because $\boxtimes$ is a coproduct in $\mathcal{CS}^{\mathcal{J}}$. When M is cofibrant, $M \boxtimes (-)$ preserves both cofibrations and $\mathcal{J}$-equivalences. $\square$

Lemma 3.1.7. There is an adjunction

$$\mathcal{CS}^{\mathcal{J}}_{/N}(X, Y) \cong \mathcal{CS}^{\mathcal{J}}_{/U}(X, U \times_N Y).$$

between the forgetful functor (the functor that postcomposes with $U \to N$), and $U \times_N (-)\colon \mathcal{CS}^{\mathcal{J}}_{/N} \to \mathcal{CS}^{\mathcal{J}}_{/U}$. This adjunction induces an adjunction on homotopy categories.

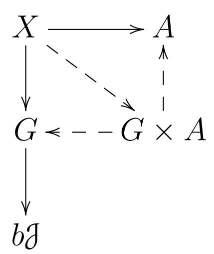
Proof. This is a special case of 3.1.1. $\square$

Lemma 3.1.8. Let G be an object of $\Gamma^{op}\text{-}\mathcal{S} \downarrow b\mathcal{J}$. Let $U\colon (\Gamma^{op}\text{-}\mathcal{S} \downarrow b\mathcal{J})_{/G} \to \Gamma^{op}\text{-}\mathcal{S}$ be the forgetful functor, and let $P\colon \Gamma^{op}\text{-}\mathcal{S} \to (\Gamma^{op}\text{-}\mathcal{S} \downarrow b\mathcal{J})_{/G}$ be the functor $P(A) = G \times A$. There is a Quillen adjunction

$$(\Gamma^{op}\text{-}\mathcal{S})(UX, A) \cong (\Gamma^{op}\text{-}\mathcal{S} \downarrow b\mathcal{J})_{/G}(X, P(A))$$

where $(\Gamma^{op}\text{-}\mathcal{S} \downarrow b\mathcal{J})_{/G}$ has the induced model structure of a comma category.

Proof. If X is an object of $(\Gamma^{op}\text{-}\mathcal{S} \downarrow b\mathcal{J})_{/G}$, and A is an object of $\Gamma^{op}\text{-}\mathcal{S}$, then a map $X \to A$ determines a unique map $X \to G \times A$ into the product.



Similarly, a map $X \to G \times A$ determines a map $X \to A$ by projecting onto the second coordinate. These are inverse isomorphisms by construction, and naturality follows from the universal property of the product.

A map $X \to A$ in $(\Gamma^{op}\text{-}\mathcal{S} \downarrow b\mathcal{J})_{/G}$ is a cofibration if and only if it is a map in $(\Gamma^{op}\text{-}\mathcal{S} \downarrow b\mathcal{J})_{/G}$ and a cofibration in $\Gamma^{op}\text{-}\mathcal{S}$, so the forgetful functor preserves cofibrations. Similarly, the forgetful functor preserves acyclic cofibrations. □

Let $\mathbb{P}_A^+ \colon \mathcal{N}_A \to \mathcal{M}_A$ be the functor that produces the free non-unital A-algebra on a given A-module, which is the left adjoint in a free-forgetful adjunction. Explicitly,

$$\mathbb{P}_A^+(X) = \bigvee_{k>0} X^{\wedge k}/\Sigma_k$$

with the smash products being taken over A.

Let $\mathbb{P}_A \colon \mathcal{M}_A \to \mathcal{C}_A$ be the functor that produces the free commutative unital A-algebra on a given A-module, which is the left adjoint in a free-forgetful adjunction. Explicitly,

$$\mathbb{P}_A(X) = \bigvee_{k\geq 0} X^{\wedge k}/\Sigma_k$$

with the smash products being taken over A.

Proposition 3.1.9. Given a cofibrant object M in $\mathcal{CS}^{\mathcal{J}}$, and a cofibration $M \rightarrowtail N$ in $\mathcal{CS}^{\mathcal{J}}$, there is an isomorphism in $h\mathcal{M}_{\mathbb{S}^{\mathcal{J}}[M]}$

$$\mathbb{L}Q_{\mathbb{S}^{\mathcal{J}}[M]}\mathbb{R}I_{\mathbb{S}^{\mathcal{J}}[M]}(\mathbb{S}^{\mathcal{J}}[N^{rep}]) \cong \mathbb{S}^{\mathcal{J}}[M] \wedge \gamma(N)/\gamma(M)$$

Proof. Let Y be a fibrant object in $\mathcal{M}_{\mathbb{S}^\partial[M]}$. Using the adjunction of Proposition 2.2.4, there is an isomorphism

$$h\mathcal{M}_{\mathbb{S}^\partial[M]}(\mathbb{L}Q_{\mathbb{S}^\partial[M]}\mathbb{R}I_{\mathbb{S}^\partial[M]}(\mathbb{S}^\partial[N^{rep}]),Y) \cong h\mathcal{N}_{\mathbb{S}^\partial[M]}(\mathbb{R}I_{\mathbb{S}^\partial[M]}(\mathbb{S}^\partial[N^{rep}]),Y)$$

Using the adjunction of Proposition 2.2.3, there is an isomorphism

$$h\mathcal{N}_{\mathbb{S}^\partial[M]}(\mathbb{R}I_{\mathbb{S}^\partial[M]}(\mathbb{S}^\partial[N^{rep}]),Y) \cong h\mathcal{C}_{\mathbb{S}^\partial[M]/\mathbb{S}^\partial[M]}(\mathbb{S}^\partial[N^{rep}],\mathbb{S}^\partial[M]\vee Y)$$

Using the equivalence of Lemma 3.1.2, there is an isomorphism

$$h\mathcal{C}_{\mathbb{S}^\partial[M]/\mathbb{S}^\partial[M]}(\mathbb{S}^\partial[N^{rep}],\mathbb{S}^\partial[M]\vee Y) \cong h\mathcal{C}_{\mathbb{S}^\partial[M]/\mathbb{S}^\partial[M]^{fib}}(\mathbb{S}^\partial[N^{rep}],\mathbb{S}^\partial[M]\vee Y)$$

There is a stable equivalence

$$Y \xrightarrow{\simeq} \mathbb{S}^\partial[M]^{fib} \wedge_{\mathbb{S}^\partial[M]} Y$$

given by smashing Y with $\mathbb{S}^\partial[M] \rightarrowtail^{\simeq} \mathbb{S}^\partial[M]^{fib}$. Taking a fibrant replacement in $\mathcal{M}_{\mathbb{S}^\partial[M]^{fib}}$,

$$\mathbb{S}^\partial[M]^{fib} \wedge_{\mathbb{S}^\partial[M]} Y \xrightarrow{\simeq} Z$$

we have an isomorphism

$$h\mathcal{C}_{\mathbb{S}^\partial[M]/\mathbb{S}^\partial[M]^{fib}}(\mathbb{S}^\partial[N^{rep}],\mathbb{S}^\partial[M]\vee Y) \cong h\mathcal{C}_{\mathbb{S}^\partial[M]/\mathbb{S}^\partial[M]^{fib}}(\mathbb{S}^\partial[N^{rep}],\mathbb{S}^\partial[M]^{fib}\vee Z).$$

We take a fibrant replacement in $\mathcal{C}_{\mathbb{S}^\partial[M]/\mathbb{S}^\partial[M]^{fib}}$

$$\mathbb{S}^\partial[M]^{fib}\vee Z \xrightarrow{\simeq} \mathbb{S}^\partial[M]^{fib} \vee_f Z \twoheadrightarrow \mathbb{S}^\partial[M]^{fib}$$

and obtain the isomorphism

$$h\mathcal{C}_{\mathbb{S}^{\mathcal{J}}[M]/\mathbb{S}^{\mathcal{J}}[M]^{fib}}(\mathbb{S}^{\mathcal{J}}[N^{rep}], \mathbb{S}^{\mathcal{J}}[M]^{fib} \vee Z) \cong h\mathcal{C}_{\mathbb{S}^{\mathcal{J}}[M]/\mathbb{S}^{\mathcal{J}}[M]^{fib}}(\mathbb{S}^{\mathcal{J}}[N^{rep}], \mathbb{S}^{\mathcal{J}}[M]^{fib} \vee_f Z).$$

Using Lemma 3.1.3, there is an isomorphism

$$h\mathcal{C}_{\mathbb{S}^{\mathcal{J}}[M]/\mathbb{S}^{\mathcal{J}}[M]^{fib}}(\mathbb{S}^{\mathcal{J}}[N^{rep}], \mathbb{S}^{\mathcal{J}}[M]^{fib} \vee_f Z) \cong h\mathcal{C}\mathcal{S}^{\mathcal{J}}_{M/\Omega^{\mathcal{J}}(\mathbb{S}^{\mathcal{J}}[M]^{fib})}(N^{rep}, \Omega^{\mathcal{J}}(\mathbb{S}^{\mathcal{J}}[M]^{fib} \vee_f Z)).$$

Using Lemma 3.1.5, Lemma 3.1.6, and Lemma 3.1.7, we have isomorphisms

$$\begin{aligned} &h\mathcal{C}\mathcal{S}^{\mathcal{J}}_{M/\Omega^{\mathcal{J}}(\mathbb{S}^{\mathcal{J}}[M]^{fib})}(N^{rep}, \Omega^{\mathcal{J}}(\mathbb{S}^{\mathcal{J}}[M]^{fib} \vee_f Z)) \\ &\quad\cong h\mathcal{C}\mathcal{S}^{\mathcal{J}}_{M/\Omega^{\mathcal{J}}(\mathbb{S}^{\mathcal{J}}[M]^{fib})}(M \boxtimes (U \times^h_{M^{gp}} N^{gp}), \Omega^{\mathcal{J}}(\mathbb{S}^{\mathcal{J}}[M]^{fib} \vee_f Z)) \\ &\quad\cong h\mathcal{C}\mathcal{S}^{\mathcal{J}}_{/\Omega^{\mathcal{J}}(\mathbb{S}^{\mathcal{J}}[M]^{fib})}(U \times^h_{M^{gp}} N^{gp}, \Omega^{\mathcal{J}}(\mathbb{S}^{\mathcal{J}}[M]^{fib} \vee_f Z)) \\ &\quad\cong h\mathcal{C}\mathcal{S}^{\mathcal{J}}_{/U}(U \times^h_{M^{gp}} N^{gp}, (1+Z)^{\mathcal{J}}). \end{aligned}$$

Applying γ, using the fact that $\gamma(U)$ is contractible, and Lemma 3.1.8, we have isomorphisms

$$\begin{aligned} &h\mathcal{C}\mathcal{S}^{\mathcal{J}}_{/U}(U \times^h_{M^{gp}} N^{gp}, (1+Z)^{\mathcal{J}}) \\ &\quad\cong h(\Gamma^{op}\text{-}\mathcal{S} \downarrow b\mathcal{J})_{pre/\gamma(U)}(\gamma(U \times^h_{M^{gp}} N^{gp}), \gamma((1+Z)^{\mathcal{J}})) \\ &\quad\cong h(\Gamma^{op}\text{-}\mathcal{S})_{pre}(\gamma(U \times^h_{M^{gp}} N^{gp}), \gamma((1+Z))^{\mathcal{J}})) \end{aligned}$$

Because the stable model structure is a left Bousfield localization of the pre-stable model structure, and $\gamma((1+Z)^{\mathcal{J}})$ is grouplike, there is an isomorphism

$$h(\Gamma^{op}\text{-}\mathcal{S})_{pre}(\gamma(U \times^h_{M^{gp}} N^{gp}), \gamma((1+Z)^{\mathcal{I}})) \cong h(\Gamma^{op}\text{-}\mathcal{S})_{st}(\gamma(U \times^h_{M^{gp}} N^{gp}), \gamma((1+Z)^{\mathcal{I}})).$$

Because γ preserves homotopy pullbacks, and takes equivalences after group completion to stable equivalences, we know that

$$\gamma(U \times^h_{M^{gp}} N^{gp}) \simeq \gamma(U) \times^h_{\gamma(M)} \gamma(N).$$

Applying γ to $M \rightarrowtail N \longrightarrow M$ we see that

$$\gamma(N) \simeq \gamma(M) \vee (\gamma(N)/\gamma(M)).$$

Consequently, $\gamma(U \times^h_{M^{gp}} N^{gp})$ is stably equivalent to $\gamma(U) \times^h_{\gamma(M)} (\gamma(M) \vee \gamma(N)/\gamma(M))$. Now, $\gamma(U)$ is contractible, so the homotopy pullback is the homotopy fiber of the map $\gamma(M) \vee (\gamma(N)/\gamma(M)) \to \gamma(M)$ and thus stably equivalent to $\gamma(N)/\gamma(M)$.

This establishes the stable equivalence

$$\gamma(U \times^h_{M^{gp}} N^{gp}) \simeq \gamma(N)/\gamma(M)$$

so there is an isomorphism

$$h(\Gamma^{op}\text{-}\mathcal{S})_{st}(\gamma(U \times^h_{M^{gp}} N^{gp}), \gamma((1+Z)^{\mathcal{I}})) \cong h(\Gamma^{op}\text{-}\mathcal{S})_{st}(\gamma(N)/\gamma(M), \gamma((1+Z)^{\mathcal{I}})).$$

Because Γ^{op}-$\mathcal{S}$ is Quillen equivalent to connective spectra, there is an isomorphism

$$h(\Gamma^{op}\text{-}\mathcal{S})_{st}(\gamma(N)/\gamma(M), \gamma((1+Z)^{\mathcal{I}})) \cong h\mathcal{M}_S(\gamma(N)/\gamma(M), \gamma((1+Z)^{\mathcal{I}}))$$

and because $\gamma(N)/\gamma(M)$ is connective and $(1+Z)^\partial$ is a connective cover, there is an isomorphism

$$h\mathcal{M}_S(\gamma(N)/\gamma(M), \gamma((1+Z)^\partial)) \cong h\mathcal{M}_S(\gamma(N)/\gamma(M), Z).$$

Finally, there is a sequence of isomorphisms

$$\begin{aligned} h\mathcal{M}_S(\gamma(N)/\gamma(M), Z) &\cong h\mathcal{M}_S(\gamma(N)/\gamma(M), Y) \\ &\cong h\mathcal{M}_{\mathbb{S}^\partial[M]}(\mathbb{S}^\partial[M] \wedge \gamma(N)/\gamma(M), Y) \end{aligned}$$

The first comes from the stable equivalence

$$Y \xrightarrow{\simeq} \mathbb{S}^\partial[M]^{fib} \wedge_{\mathbb{S}^\partial[M]} Y \xrightarrow{\simeq} Z.$$

The second is the extension of scalars to $\mathbb{S}^\partial[M]$. The desired result now follows from the Yoneda lemma. □

Corollary 3.1.10. Let P be a cofibrant object in $\mathcal{CS}^\partial$, and let $P \rightarrowtail M$ be a cofibration in $\mathcal{CS}^\partial$. Then there is an isomorphism in $h\mathcal{M}_{\mathbb{S}^\partial[M]}$

$$\mathbb{L}Q_{\mathbb{S}^\partial[M]}\mathbb{R}I_{\mathbb{S}^\partial[M]}(\mathbb{S}^\partial[(M \otimes_P S^0)^{rep}]) \cong \mathbb{S}^\partial[M] \wedge \gamma(M)/\gamma(P)$$

Proof. This is the case $N = M \otimes_P S^0 = M \boxtimes_P M$, so

$$\mathbb{L}Q_{\mathbb{S}^\partial[M]}\mathbb{R}I_{\mathbb{S}^\partial[M]}(\mathbb{S}^\partial[(M \boxtimes_P M)^{rep}]) \cong \mathbb{S}^\partial[M] \wedge \gamma(M \boxtimes_P M)/\gamma(M)$$

and we have to identify the cofiber $\gamma(M \boxtimes_P M)/\gamma(M)$. Take a factorization in $\mathcal{M}_{\mathbb{S}^\partial[M]}$

$$\gamma(P) \rightarrowtail \gamma(M)^{cof} \xrightarrow{\simeq} \gamma(M).$$

Because γ preserves homotopy pushouts, $\gamma(M \boxtimes_P M)$ is weakly equivalent to a pushout

$$\gamma(M \boxtimes_P M) \simeq \gamma(M) \coprod_{\gamma(P)} \gamma(M)^{cof},$$

so the cofiber of $\gamma(M) \to \gamma(M \boxtimes_P M)$ is stably equivalent to the iterated pushout in the following diagram

$$\begin{array}{ccc} \gamma(P) & \rightarrowtail & \gamma(M)^{cof} \\ \downarrow & & \downarrow \\ \gamma(M) & \rightarrowtail & \gamma(M) \coprod_{\gamma(P)} \gamma(M)^{cof} \\ \downarrow & & \downarrow \\ * & \rightarrowtail & \gamma(M)/\gamma(P) \end{array}$$

Thus $\gamma(M \boxtimes_P M)/\gamma(M) \simeq \gamma(M)/\gamma(P)$. □

Corollary 3.1.11. Let P be a cofibrant object in $\mathcal{CS}^{\mathcal{J}}$, and let $P \rightarrowtail M$ be a cofibration in $\mathcal{CS}^{\mathcal{J}}$. Then there is an isomorphism in $h\mathcal{M}_{\mathbb{S}^{\mathcal{J}}[M]}$

$$\mathbb{L}Q_{\mathbb{S}^{\mathcal{J}}[M]} \mathbb{R}I_{\mathbb{S}^{\mathcal{J}}[M]}(\mathbb{S}^{\mathcal{J}}[(M \otimes_P S^1)^{rep}]) \cong \Sigma \mathbb{S}^{\mathcal{J}}[M] \wedge \gamma(M)/\gamma(P)$$

Proof. Here, we take $N = M \otimes_P S^1$ and identify the cofiber $\gamma(M \otimes_P S^1)/\gamma(M)$. In the following, let $M \otimes_P S^1$ denote the simplicial object and let $|M \otimes_P S^1|$ denote the realization. (Recall that the realization of a simplicial object in $\mathcal{CS}^{\mathcal{J}}$ is the realization as an object of $\mathcal{S}^{\mathcal{J}}$ i.e. the $\mathcal{J}$-levelwise realization.)

The simplicial object $M \otimes_P S^1$ is Reedy cofibrant (see [9, Chapter 15]). To see this, we have to check that

$$(sk_{n-1} M \otimes_P S^1)_n \to (M \otimes_P S^1)_n$$

is a cofibration for $n \geq 0$, where sk_n is the n-skeleton. When $n = 0$ this is $U \rightarrowtail M$ and when $n = 1$ this is $M \rightarrowtail M \boxtimes_P M$ both of which are cofibrations. When $n > 1$, note that

$$\begin{aligned}
(sk_{n-1} M \otimes_P S^1)_n &\cong \operatorname*{colim}_{k \in \Delta^{op}_{n-1} \downarrow n} (M \otimes_P S^1)_k \\
&\cong \operatorname*{colim}_{k \in \Delta^{op}_{n-1} \downarrow n} |M \otimes_P S^1_k| \\
&\cong |M \otimes_P \operatorname*{colim}_{k \in \Delta^{op}_{n-1} \downarrow n} S^1_k| \\
&\cong |M \otimes_P (sk_{n-1} S^1)_n| \\
&\cong |M \otimes_P S^1_n| \\
&\cong (M \otimes_P S^1)_n
\end{aligned}$$

The result is that the map $(sk_{n-1} M \otimes_P S^1)_n \to (M \otimes_P S^1)_n$ is the identity. Thus $M \otimes_P S^1$ is Reedy cofibrant.

By [9, Proposition 18.7.4], this implies that the map $\operatorname{hocolim}_{\Delta^{op}} M \otimes_P S^1 \to |M \otimes_P S^1|$ is a $\mathcal{J}$-equivalence. Applying γ produces a stable equivalence

$$\gamma(\operatorname*{hocolim}_{\Delta^{op}} M \otimes_P S^1) \to \gamma|M \otimes_P S^1|,$$

and because γ preserves homotopy colimits, there is a stable equivalence

$$\operatorname*{hocolim}_{\Delta^{op}} \gamma(M \otimes_P S^1) \simeq \gamma|M \otimes_P S^1|$$

Consequently, the cofiber $\gamma(|M \otimes_P S^1|)/\gamma(M)$ is stably equivalent to the cofiber of

$$\operatorname*{hocolim}_{\Delta^{op}} \gamma(M) \to \operatorname*{hocolim}_{\Delta^{op}} \gamma(M \otimes_P S^1)$$

which is in turn stably equivalent to

$$\operatorname*{hocolim}_{\Delta^{op}} \gamma(M \otimes_P S^1)/\gamma(M)$$

where $\gamma(M \otimes_P S^1)/\gamma(M)$ is a cofiber of simplicial gamma spaces.

The cofiber can be computed levelwise. In simplicial level n, $\gamma(M \otimes_P S^1)$ is stably equivalent to

$$\gamma(M) \amalg_{\gamma(P)} \underbrace{\gamma(M)^{cof} \amalg_{\gamma(P)} \cdots \amalg_{\gamma(P)} \gamma(M)^{cof}}_{n}$$

with one $\gamma(M)$ summand and n summands equal to $\gamma(M)^{cof}$. The desired cofiber is the cofiber of the insertion into the first summand

$$\gamma(M) \to \gamma(M) \amalg_{\gamma(P)} \gamma(M)^{cof} \amalg_{\gamma(P)} \cdots \amalg_{\gamma(P)} \gamma(M)^{cof}.$$

The cofiber is thus

$$\gamma(M)/\gamma(P) \vee \cdots \vee \gamma(M)/\gamma(P)$$

a wedge of n copies of $\gamma(M)/\gamma(P)$.

Having the cofiber, we need to understand the hocolim. The first degeneracy $d_1 \colon \gamma(M \otimes_P S^1)_1 \to \gamma(M \otimes_P S^1)_2$ from level 1 to level 2 is the map

$$\gamma(M) \amalg_{\gamma(P)} \gamma(M)^{cof} \to \gamma(M) \amalg_{\gamma(P)} \gamma(M)^{cof} \amalg_{\gamma(P)} \gamma(M)^{cof}$$

which skips the last $\gamma(M)^{cof}$ summand, and the induced maps on cofibers is the inclusion

$$\gamma(M)/\gamma(P) \to \gamma(M)/\gamma(P) \vee \gamma(M)/\gamma(P)$$

omitting the last $\gamma(M)/\gamma(P)$ summand. Similarly, the zeroth degeneracy is the inclusion omitting the first summand. Therefore level 2 is degenerate. In higher levels, each degeneracy omits a summand, so it follows by induction that the simplicial object is entirely degenerate above level 1. This allows us to conclude that the resulting simplicial object is Reedy cofibrant. To see this we check that the maps

$$(sk_{n-1})_n \to \underbrace{\gamma(M)/\gamma(P) \vee \cdots \vee \gamma(M)/\gamma(P)}_{n}$$

are cofibrations. When $n = 0$, this is the identity map $* \to *$, and when $n = 1$, this is the inclusion $* \to \gamma(M)/\gamma(P)$ both of which are cofibrations. When $n > 1$, we've just seen that this simplicial object is degenerate so this map is the identity map. So the simplicial object is Reedy cofibrant.

Again, [9, Proposition 18.7.4] implies that the homotopy colimit over Δ^{op} is stably equivalent to the realization. The realization is $\Sigma\gamma(M)/\gamma(P)$. $\square$

Now, we would like to assert that there is an isomorphism in $h\mathcal{M}_A$ between $TAQ^{(R,P)}(A, M)$ and the following homotopy pushout of A-modules

$$\mathbb{L}Q\mathbb{R}I(A \wedge_R A) \coprod_{A \wedge_{\mathbb{S}^{\mathcal{J}}[M]} \mathbb{L}Q\mathbb{R}I(\mathbb{S}^{\mathcal{J}}[M \boxtimes_P M])} A \wedge_{\mathbb{S}^{\mathcal{J}}[M]} \mathbb{L}Q\mathbb{R}I(\mathbb{S}^{\mathcal{J}}[(M \boxtimes_P M)^{rep}]).$$

where $\mathbb{L}Q\mathbb{R}I$ stands for the composite $Q \circ (-)^{cof} \circ I \circ (-)^{fib} \colon \mathcal{C}_{A/A} \to \mathcal{M}_A$ (or $\mathcal{C}_{\mathbb{S}^{\mathcal{J}}[M]/\mathbb{S}^{\mathcal{J}}[M]} \to \mathcal{M}_{\mathbb{S}^{\mathcal{J}}[M]}$ as appropriate). The maps come from applying this composite of functors to $\mathbb{S}^{\mathcal{J}}[M \boxtimes_P M] \to A \wedge_R A$ and $\mathbb{S}^{\mathcal{J}}[M \boxtimes_P M] \to \mathbb{S}^{\mathcal{J}}[(M \boxtimes_P M)^{rep}]$. The definition of non-logarithmic TAQ and the weak equivalence of

Corollary 3.1.10 shown above almost make this true. It remains to be shown that the map

$$\mathbb{L}Q\mathbb{R}I(\mathbb{S}^{\mathcal{J}}[M \boxtimes_P M]) \to \mathbb{L}Q\mathbb{R}I(\mathbb{S}^{\mathcal{J}}[(M \boxtimes_P M)^{rep}])$$

is weakly equivalent to the map

$$TAQ^{\mathbb{S}^{\mathcal{J}}[P]}(\mathbb{S}^{\mathcal{J}}[M]) \to \mathbb{S}^{\mathcal{J}}[M] \wedge \gamma(M)/\gamma(P)$$

used to define log TAQ.

In the definition of TAQ given in [19, Definition 5.20], this map is given implicitly by showing that there is a natural transformation of functors

$$\mathcal{M}_{\mathbb{S}^{\mathcal{J}}[M]}(TAQ^{\mathbb{S}^{\mathcal{J}}[P]}(\mathbb{S}^{\mathcal{J}}[M]), X) \longleftarrow \mathcal{M}_{\mathbb{S}^{\mathcal{J}}[M]}(\mathbb{S}^{\mathcal{J}}[M] \wedge \gamma(M)/\gamma(P), X).$$

Therefore, we can show that the two maps are weakly equivalent by showing that the following square commutes.

$$\begin{array}{ccc} h\mathcal{M}_{\mathbb{S}^{\mathcal{J}}[M]}(\mathbb{L}Q\mathbb{R}I(\mathbb{S}^{\mathcal{J}}[M \boxtimes_P M]), X) & \longleftarrow & h\mathcal{M}_{\mathbb{S}^{\mathcal{J}}[M]}(\mathbb{L}Q\mathbb{R}I(\mathbb{S}^{\mathcal{J}}[(M \boxtimes_P M)^{rep}]), X) \\ \Big\| & & \Big\updownarrow \cong \\ h\mathcal{M}_{\mathbb{S}^{\mathcal{J}}[M]}(TAQ^{\mathbb{S}^{\mathcal{J}}[P]}(\mathbb{S}^{\mathcal{J}}[M]), X) & \longleftarrow & h\mathcal{M}_{\mathbb{S}^{\mathcal{J}}[M]}(\mathbb{S}^{\mathcal{J}}[M] \wedge \gamma(M)/\gamma(P), X). \end{array}$$

The vertical isomorphism on the left comes from the definition of TAQ. The vertical isomorphism on the right has just been shown. Before doing this, we'll need the following results.

Lemma 3.1.12. There is an adjunction

$$\mathcal{CS}^{\mathcal{J}}_{M/M}(M \boxtimes X, Y) \cong \mathcal{CS}^{\mathcal{J}}_{/U}(X, U \times_M Y)$$

When M is cofibrant, it induces an adjunction on homotopy categories.

Proof. This is the composite adjunction of Lemma 3.1.6 and Lemma 3.1.7. □

Lemma 3.1.13. Let X be a cofibrant object of $\mathcal{CS}^{\mathcal{J}}_{/U}$. The unit of the derived adjunction

$$X \to U \times_M (M \boxtimes X)^{fib}$$

is a $\mathcal{J}$-equivalence.

Proof. $(M \boxtimes X)^{fib}$ is a fibrant replacement in $\mathcal{CS}^{\mathcal{J}}_{M/M}$ i.e. there is a factorization

$$(M \boxtimes X) \xrightarrow{\simeq} (M \boxtimes X)^{fib} \twoheadrightarrow M.$$

So the pullback $U \times_M (M \boxtimes X)^{fib}$ is a homotopy pullback, and by [20, Corollary 11.4]

$$(U \times_M (M \boxtimes X)^{fib})_{h\mathcal{J}} \simeq U_{h\mathcal{J}} \times^h_{M_{h\mathcal{J}}} (M \boxtimes X)_{h\mathcal{J}}.$$

Because X is cofibrant,

$$U_{h\mathcal{J}} \times^h_{M_{h\mathcal{J}}} (M \boxtimes X)_{h\mathcal{J}} \simeq U_{h\mathcal{J}} \times^h_{M_{h\mathcal{J}}} M_{h\mathcal{J}} \times X_{h\mathcal{J}}.$$

And because $U_{h\mathcal{J}}$ is contractible, this is weakly equivalent to $X_{h\mathcal{J}}$. □

Lemma 3.1.14. If Y has the form $(M \boxtimes X)^{fib}$, then the counit of the derived adjunction is a weak equivalence.

$$M \boxtimes (U \times_M (M \boxtimes X)^{fib})^{cof} \to (M \boxtimes X)^{fib}$$

Proof. The cofibrant replacement is the result of a factorization

$$U \rightarrowtail (U \times_M (M \boxtimes X)^{fib})^{cof} \twoheadrightarrow U \times_M (M \boxtimes X)^{fib}.$$

So

$$(M \boxtimes (U \times_M (M \boxtimes X)^{fib})^{cof})_{h\mathcal{J}} \simeq M_{h\mathcal{J}} \times (U \times_M (M \boxtimes X)^{fib})_{h\mathcal{J}}.$$

The right-hand side is in turn weakly equivalent to

$$M_{h\mathcal{J}} \times (X_{h\mathcal{J}}).$$

□

Lemma 3.1.15. There is an isomorphism

$$h\mathcal{CS}^{\mathcal{J}}_{/U}(X, X') \to h\mathcal{CS}^{\mathcal{J}}_{M/M}(M \boxtimes X^{cof}, M \boxtimes (X')^{cof})$$

Proof. This follows from the previous two lemmas. □

Proposition 3.1.16. The following square commutes.

$$\begin{array}{ccc} h\mathcal{M}_{\mathbb{S}^{\mathcal{J}}[M]}(\mathbb{L}Q\mathbb{R}I(\mathbb{S}^{\mathcal{J}}[M \boxtimes_P M]), X) & \longleftarrow & h\mathcal{M}_{\mathbb{S}^{\mathcal{J}}[M]}(\mathbb{L}Q\mathbb{R}I(\mathbb{S}^{\mathcal{J}}[(M \boxtimes_P M)^{rep}]), X) \\ \| & & \updownarrow \cong \\ h\mathcal{M}_{\mathbb{S}^{\mathcal{J}}[M]}(TAQ^{\mathbb{S}^{\mathcal{J}}[P]}(\mathbb{S}^{\mathcal{J}}[M]), X) & \longleftarrow & h\mathcal{M}_{\mathbb{S}^{\mathcal{J}}[M]}(\mathbb{S}^{\mathcal{J}}[M] \wedge \gamma(M)/\gamma(P), X). \end{array}$$

Proof. Let $\mathcal{O}$ denote $\Omega^{\mathcal{J}}(\mathbb{S}^{\mathcal{J}}[M] \vee_f X)$. First, use the $(\mathbb{L}Q\mathbb{R}I, \mathbb{L}K\mathbb{R}Z)$ adjunction and the $(\mathbb{S}^{\mathcal{J}}, \Omega^{\mathcal{J}})$ adjunction to transform the top row of the square as follows

$$\begin{array}{ccc} h\mathcal{CS}^{\mathcal{J}}_{M/\Omega^{\mathcal{J}}(\mathbb{S}^{\mathcal{J}}[M])}(M \boxtimes_P M, \mathcal{O}) & \longleftarrow & h\mathcal{CS}^{\mathcal{J}}_{M/\Omega^{\mathcal{J}}(\mathbb{S}^{\mathcal{J}}[M])}((M \boxtimes_P M)^{rep}, \mathcal{O}) \\ \updownarrow \cong & & \updownarrow \cong \\ h\mathcal{M}_{\mathbb{S}^{\mathcal{J}}[M]}(\mathbb{L}Q\mathbb{R}I(\mathbb{S}^{\mathcal{J}}[M \boxtimes_P M]), X) & \longleftarrow & h\mathcal{M}_{\mathbb{S}^{\mathcal{J}}[M]}(\mathbb{L}Q\mathbb{R}I(\mathbb{S}^{\mathcal{J}}[(M \boxtimes_P M)^{rep}]), X) \end{array}$$

We use the natural transformations of functors used to define TAQ in [19] to transform the bottom row:

$$\begin{array}{ccc} h\mathcal{M}_{\mathbb{S}^{\mathcal{J}}[M]}(TAQ^{\mathbb{S}^{\mathcal{J}}[P]}(\mathbb{S}^{\mathcal{J}}[M]),X) & \longleftarrow & h\mathcal{M}_{\mathbb{S}^{\mathcal{J}}[M]}(\mathbb{S}^{\mathcal{J}}[M]\wedge\gamma(M)/\gamma(P),X) \\ \updownarrow\cong & & \updownarrow\cong \\ h\mathcal{C}_{\mathbb{S}^{\mathcal{J}}[P]/\mathbb{S}^{\mathcal{J}}[M]}(\mathbb{S}^{\mathcal{J}}[M],\mathbb{S}^{\mathcal{J}}[M]\vee_f X) & \longleftarrow & h\mathcal{CS}^{\mathcal{J}}_{P/M}(M,(M+X)^{\mathcal{J}}) \end{array}$$

and then the isomorphism

$$h\mathcal{C}_{\mathbb{S}^{\mathcal{J}}[P]/\mathbb{S}^{\mathcal{J}}[M]}(\mathbb{S}^{\mathcal{J}}[M],\mathbb{S}^{\mathcal{J}}[M]\vee_f X)\cong h\mathcal{CS}^{\mathcal{J}}_{P/M}(M,\Omega^{\mathcal{J}}(\mathbb{S}^{\mathcal{J}}[M]\vee_f X))$$

coming from the $(\mathbb{S}^{\mathcal{J}},\Omega^{\mathcal{J}})$ adjunction in order to transform the lower left corner.

As a result, it needs to be shown that the following square commutes

$$\begin{array}{ccc} h\mathcal{CS}^{\mathcal{J}}_{P/\Omega^{\mathcal{J}}(\mathbb{S}^{\mathcal{J}}[M])}(M,\mathcal{O}) & \longleftarrow & h\mathcal{CS}^{\mathcal{J}}_{M/\Omega^{\mathcal{J}}(\mathbb{S}^{\mathcal{J}}[M])}((M\boxtimes_P M)^{rep},\mathcal{O}) \\ \| & & \updownarrow\cong \\ h\mathcal{CS}^{\mathcal{J}}_{P/\Omega^{\mathcal{J}}(\mathbb{S}^{\mathcal{J}}[M])}(M,\mathcal{O}) & \longleftarrow & h\mathcal{CS}^{\mathcal{J}}_{P/M}(M,(M+X)^{\mathcal{J}}). \end{array}$$

Using extension of scalars, we prefer to work with the following square

$$\begin{array}{ccc} h\mathcal{CS}^{\mathcal{J}}_{M/\Omega^{\mathcal{J}}(\mathbb{S}^{\mathcal{J}}[M])}(M\boxtimes_P M,\mathcal{O}) & \longleftarrow & h\mathcal{CS}^{\mathcal{J}}_{M/\Omega^{\mathcal{J}}(\mathbb{S}^{\mathcal{J}}[M])}((M\boxtimes_P M)^{rep},\mathcal{O}) \\ \| & & \updownarrow\cong \\ h\mathcal{CS}^{\mathcal{J}}_{M/\Omega^{\mathcal{J}}(\mathbb{S}^{\mathcal{J}}[M])}(M\boxtimes_P M,\mathcal{O}) & \longleftarrow & h\mathcal{CS}^{\mathcal{J}}_{M/M}(M\boxtimes_P M,(M+X)^{\mathcal{J}}). \end{array}$$

The upper horizontal map comes from precomposition with the repletion map $M\boxtimes_P M\to(M\boxtimes_P M)^{rep}$. The lower horizontal map comes from postcomposition with $(M+X)^{\mathcal{J}}\to\Omega^{\mathcal{J}}(\mathbb{S}^{\mathcal{J}}[M]\vee_f X)$. The vertical map on the right is implicit in the arguments that we gave above in Proposition 3.1.9 and Corollary 3.1.10 which show that

$$h\mathcal{CS}^{\partial}_{M/\Omega^{\partial}(\mathbb{S}^{\partial}[M])}((M \boxtimes_P M)^{rep}, \Omega^{\partial}(\mathbb{S}^{\partial}[M] \vee_f X)) \cong h\mathcal{M}_{\mathbb{S}^{\partial}[M]}(\mathbb{S}^{\partial}[M] \wedge \gamma(M)/\gamma(P), X)$$

and a similar argument given in [19, Proposition 5.19] that shows that

$$h\mathcal{CS}^{\partial}_{P/M}(M, (M+X)^{\partial}) \cong h\mathcal{M}_{\mathbb{S}^{\partial}[M]}(\mathbb{S}^{\partial}[M] \wedge \gamma(M)/\gamma(P), X).$$

We make this more explicit here. Using Lemma 3.1.5, Lemma 3.1.6, and Lemma 3.1.7, we have isomorphisms

$$\begin{aligned} &h\mathcal{CS}^{\partial}_{M/\Omega^{\partial}(\mathbb{S}^{\partial}[M])}((M \boxtimes_P M)^{rep}, \Omega^{\partial}(\mathbb{S}^{\partial}[M] \vee_f X)) \\ &\quad \cong h\mathcal{CS}^{\partial}_{M/\Omega^{\partial}(\mathbb{S}^{\partial}[M])}(M \boxtimes U \times^h_{M^{gp}} (M \boxtimes_P M)^{gp}, \Omega^{\partial}(\mathbb{S}^{\partial}[M] \vee_f X)) \\ &\quad \cong h\mathcal{CS}^{\partial}_{/\Omega^{\partial}(\mathbb{S}^{\partial}[M])}(U \times^h_{M^{gp}} (M \boxtimes_P M)^{gp}, \Omega^{\partial}(\mathbb{S}^{\partial}[M] \vee_f X)) \\ &\quad \cong h\mathcal{CS}^{\partial}_{/U}(U \times^h_{M^{gp}} (M \boxtimes_P M)^{gp}, (1+X)^{\partial}) \end{aligned}$$

Now, using Lemma 3.1.12, there are isomorphisms

$$\begin{aligned} &h\mathcal{CS}^{\partial}_{/U}(U \times^h_{M^{gp}} (M \boxtimes_P M)^{gp}, (1+X)^{\partial}) \\ &\quad \cong h\mathcal{CS}^{\partial}_{M/M}(M \boxtimes (U \times^h_{M^{gp}} (M \boxtimes_P M)^{gp}), M \boxtimes (1+X)^{\partial}) \\ &\quad \cong h\mathcal{CS}^{\partial}_{M/M}((M \boxtimes_P M)^{rep}, (M+X)^{\partial}) \end{aligned}$$

If it could be shown that $(M+X)^{\partial}$ is replete over M, (which given its construction seems rather likely), then the adjunction between the positive projective and the group completion model structures would immediately imply that precomposing with the repletion $M \boxtimes_P M \to (M \boxtimes_P M)^{rep}$ is an isomorphism. It is however quite difficult to identify fibrations in the group completion model structure, so we resort to the following departure through gamma spaces.

Apply γ to get

$$\begin{aligned} &h\mathcal{CS}^{\mathcal{J}}_{M/M}((M \boxtimes_P M)^{rep}, (M+X)^{\mathcal{J}}) \\ &\qquad\qquad \cong h(\Gamma^{op}\text{-}\mathcal{S} \downarrow b\mathcal{J})_{pre_{\gamma(M)/\gamma(M)}}(\gamma((M \boxtimes_P M)^{rep}), \gamma((M+X)^{\mathcal{J}})). \end{aligned}$$

From the definition of $(M+X)^{\mathcal{J}}$, we know that there is a weak equivalence

$$M_{h\mathcal{J}} \times (1+X)^{\mathcal{J}}_{h\mathcal{J}} \to (M+X)^{\mathcal{J}}_{h\mathcal{J}}$$

and consequently a level equivalence

$$\gamma(M) \times \gamma((1+X)^{\mathcal{J}}) \to \gamma((M+X)^{\mathcal{J}}).$$

Thus, there is an isomorphism

$$\begin{aligned} &h(\Gamma^{op}\text{-}\mathcal{S} \downarrow b\mathcal{J})_{pre_{\gamma(M)/\gamma(M)}}(\gamma((M \boxtimes_P M)^{rep}), \gamma((M+X)^{\mathcal{J}})) \\ &\qquad \cong h(\Gamma^{op}\text{-}\mathcal{S} \downarrow b\mathcal{J})_{pre_{\gamma(M)/\gamma(M)}}(\gamma((M \boxtimes_P M)^{rep}), \gamma(M) \times \gamma((1+X)^{\mathcal{J}})) \end{aligned}$$

Because $\gamma((1+X)^{\mathcal{J}})$ is grouplike,

$$\begin{aligned} &h(\Gamma^{op}\text{-}\mathcal{S} \downarrow b\mathcal{J})_{pre_{\gamma(M)/\gamma(M)}}(\gamma((M \boxtimes_P M)^{rep}), \gamma(M) \times \gamma((1+X)^{\mathcal{J}})) \\ &\qquad \cong h(\Gamma^{op}\text{-}\mathcal{S} \downarrow b\mathcal{J})_{st_{\gamma(M)/\gamma(M)}}(\gamma((M \boxtimes_P M)^{rep}), \gamma(M) \times \gamma((1+X)^{\mathcal{J}})) \end{aligned}$$

The map $\gamma(M \boxtimes_P M) \to \gamma((M \boxtimes_P M)^{rep})$ is a stable equivalence because $M \boxtimes_P M \to (M \boxtimes_P M)^{rep}$ is an equivalence after group completion, so precomposition yields an isomorphism

$$h(\Gamma^{op}\text{-}\mathcal{S} \downarrow b\mathcal{J})_{st_{\gamma(M)/\gamma(M)}}(\gamma((M \boxtimes_P M)^{rep}), \gamma(M) \times \gamma((1+X)^{\mathcal{J}})) \\ \cong h(\Gamma^{op}\text{-}\mathcal{S} \downarrow b\mathcal{J})_{st_{\gamma(M)/\gamma(M)}}(\gamma(M \boxtimes_P M), \gamma(M) \times \gamma((1+X)^{\mathcal{J}})).$$

The remaining isomorphisms only reverse those that came before the last.

$$h(\Gamma^{op}\text{-}\mathcal{S} \downarrow b\mathcal{J})_{st_{\gamma(M)/\gamma(M)}}(\gamma(M \boxtimes_P M), \gamma(M) \times \gamma((1+X)^{\mathcal{J}})) \\ \cong h(\Gamma^{op}\text{-}\mathcal{S} \downarrow b\mathcal{J})_{pre_{\gamma(M)/\gamma(M)}}(\gamma(M \boxtimes_P M), \gamma(M) \times \gamma((1+X)^{\mathcal{J}})) \\ \cong h(\Gamma^{op}\text{-}\mathcal{S} \downarrow b\mathcal{J})_{pre_{\gamma(M)/\gamma(M)}}(\gamma(M \boxtimes_P M), \gamma((M+X)^{\mathcal{J}})) \\ \cong h\mathcal{C}\mathcal{S}^{\mathcal{J}}_{M/M}(M \boxtimes_P M, (M+X)^{\mathcal{J}})$$

So altogether, we have an isomorphism

$$h\mathcal{C}\mathcal{S}^{\mathcal{J}}_{M/\Omega^{\mathcal{J}}(\mathbb{S}^{\mathcal{J}}[M])}((M \boxtimes_P M)^{rep}, \Omega^{\mathcal{J}}(\mathbb{S}^{\mathcal{J}}[M] \vee_f X)) \cong h\mathcal{C}\mathcal{S}^{\mathcal{J}}_{M/M}(M \boxtimes_P M, (M+X)^{\mathcal{J}})$$

that factors maps from $(M \boxtimes_P M)^{rep}$ into $\Omega^{\mathcal{J}}(\mathbb{S}^{\mathcal{J}}[M] \vee_f X)$ through $(M+X)^{\mathcal{J}}$, and then restricts through the repletion $M \boxtimes_P M \to (M \boxtimes_P M)^{rep}$. □

The proofs of the following two lemmas carry over from S-modules to symmetric spectra without change.

Lemma 3.1.17 ([3, Lemma 4.4])**.** Let A be a cofibrant commutative symmetric ring spectrum, B a cofibrant commutative A-algebra, and C a fibrant and cofibrant A-algebra over A. In the homotopy category $h\mathcal{N}_B$,

$$\mathbb{R}I_A(C) \wedge^{\mathbb{L}}_A B \cong \mathbb{R}I_B(C \wedge^{\mathbb{L}}_A B)$$

Lemma 3.1.18 ([3, Lemma 4.5])**.** Let A be a cofibrant commutative symmetric ring spectrum, B a cofibrant commutative A-algebra, and N a cofibrant A-NUCA. In the homotopy category $h\mathcal{M}_B$,

$$\mathbb{L}Q_A(N) \wedge^{\mathbb{L}}_A B \cong \mathbb{L}Q_B(N \wedge^{\mathbb{L}}_A B)$$

We will apply these under slightly different hypotheses.

Lemma 3.1.19. Let M be a cofibrant object of $\mathcal{CS}^{\mathcal{J}}$, let A be a cofibrant object of $\mathcal{C}_{\mathbb{S}^{\mathcal{J}}[M]}$, and let T be a cofibrant object of $\mathcal{C}_{\mathbb{S}^{\mathcal{J}}[M]/\mathbb{S}^{\mathcal{J}}[M]}$. Then there is an isomorphism in $h\mathcal{M}_A$

$$\mathbb{L}Q_A\mathbb{R}I_A(A \wedge^{\mathbb{L}}_{\mathbb{S}^{\mathcal{J}}[M]} T) \cong A \wedge^{\mathbb{L}}_{\mathbb{S}^{\mathcal{J}}[M]} \mathbb{L}Q_{\mathbb{S}^{\mathcal{J}}[M]}\mathbb{R}I_{\mathbb{S}^{\mathcal{J}}[M]}(T).$$

Proof. Let the superscript $fcaa$ denote a fibrant replacement in $\mathcal{C}_{A/A}$. By definition,

$$\mathbb{L}Q_A\mathbb{R}I_A(A \wedge^{\mathbb{L}}_{\mathbb{S}^{\mathcal{J}}[M]} T) = Q_A((I_A((A \wedge_{\mathbb{S}^{\mathcal{J}}[M]} T)^{fcaa}))^{cna}).$$

Let $T \xrightarrow{\simeq} T^{fcsmsm}$ be a fibrant replacement in $\mathcal{C}_{\mathbb{S}^{\mathcal{J}}[M]/\mathbb{S}^{\mathcal{J}}[M]}$. This is a stable equivalence in $\mathbb{S}^{\mathcal{J}}[M]$-modules, so there is a stable equivalence of A-modules

$$Q_A((I_A((A \wedge_{\mathbb{S}^{\mathcal{J}}[M]} T)^{fcaa}))^{cna}) \xrightarrow{\simeq} Q_A((I_A((A \wedge_{\mathbb{S}^{\mathcal{J}}[M]} T^{fcsmsm})^{fcaa}))^{cna}).$$

By Lemma 3.1.17, there is a stable equivalence in $\mathcal{N}_A$

$$I_A((A \wedge_{\mathbb{S}^{\mathcal{J}}[M]} T^{fcsmsm})^{fcaa}) \xleftarrow{\simeq} A \wedge_{\mathbb{S}^{\mathcal{J}}[M]} I_{\mathbb{S}^{\mathcal{J}}[M]}(T^{fcsmsm}).$$

This implies a stable equivalence in $\mathcal{M}_A$

$$Q_A((I_A((A \wedge_{\mathbb{S}^{\mathcal{J}}[M]} T^{fcsmsm})^{fcaa}))^{cna}) \xleftarrow{\simeq} Q_A((A \wedge_{\mathbb{S}^{\mathcal{J}}[M]} I_{\mathbb{S}^{\mathcal{J}}[M]}(T^{fcsmsm}))^{cna}).$$

Taking a cofibrant replacement

$$I_{\mathbb{S}^{\mathcal{J}}[M]}(T^{fcsmsm}) \overset{\simeq}{\twoheadleftarrow} (I_{\mathbb{S}^{\mathcal{J}}[M]}(T^{fcsmsm}))^{cnsm}$$

in $\mathcal{N}_{\mathbb{S}^{\mathcal{J}}[M]}$, there is a stable equivalence

$$Q_A((A \wedge_{\mathbb{S}^{\mathcal{J}}[M]} I_{\mathbb{S}^{\mathcal{J}}[M]}(T^{fcsmsm}))^{cna}) \simeq Q_A((A \wedge_{\mathbb{S}^{\mathcal{J}}[M]} (I_{\mathbb{S}^{\mathcal{J}}[M]}(T^{fcsmsm}))^{cnsm})^{cna})$$

and applying Lemma 3.1.18 to the right-hand side yields a stable equivalence

$$\begin{aligned} Q_A((A \wedge_{\mathbb{S}^{\mathcal{J}}[M]} (I_{\mathbb{S}^{\mathcal{J}}[M]}(T^{fcsmsm}))^{cnsm})^{cna}) \\ &\simeq A \wedge_{\mathbb{S}^{\mathcal{J}}[M]} Q_{\mathbb{S}^{\mathcal{J}}[M]}((I_{\mathbb{S}^{\mathcal{J}}[M]}(T^{fcsmsm}))^{cnsm}) \\ &= A \wedge^{\mathbb{L}}_{\mathbb{S}^{\mathcal{J}}[M]} \mathbb{L}Q_{\mathbb{S}^{\mathcal{J}}[M]}\mathbb{R}I_{\mathbb{S}^{\mathcal{J}}[M]}(T)). \end{aligned}$$

□

Lemma 3.1.20. Let B be a cofibrant object in $\mathcal{C}_{A/A}$, and let $B \rightarrowtail X$ and $B \rightarrowtail Y$ be cofibrations in $\mathcal{C}_{A/A}$. Then there is an isomorphism in $h\mathcal{M}_A$

$$\mathbb{L}Q\mathbb{R}I(X \wedge^{\mathbb{L}}_B Y) \cong \mathbb{L}Q\mathbb{R}I(X) \coprod_{\mathbb{L}Q\mathbb{R}I(B)} \mathbb{L}Q\mathbb{R}I(Y)$$

where the right hand side denotes the homotopy pushout of

$$Q((I(X^{fcaa}))^{cna} \longleftarrow Q((I(B^{fcaa}))^{cna} \longrightarrow Q((I(Y^{fcaa}))^{cna}$$

Proof. Let $X \rightarrowtail X^{fcaa}$, and $B \rightarrowtail X^{fcaa}$, and $Y \rightarrowtail X^{fcaa}$ be fibrant replacements in $\mathcal{C}_{A/A}$. Take a factorization $B^{fcaa} \rightarrowtail (X^{fcaa})^{cof} \overset{\simeq}{\twoheadrightarrow} X^{fcaa}$ into a cofibration and an acyclic fibration, and note that there is a lift as in the following diagram

$$\begin{array}{ccc}
B & \xrightarrow{\simeq} & B^{fcaa} \\
\big\downarrow & & \big\downarrow \\
 & & (X^{fcaa})^{cof} \\
 & \nearrow & \big\downarrow \simeq \\
X & \xrightarrow{\simeq} & X^{fcaa}
\end{array}$$

Doing the same with Y, there is a stable equivalence

$$X \wedge_B Y \xrightarrow{\simeq} (X^{fcaa})^{cof} \wedge_{B^{fcaa}} (Y^{fcaa})^{cof}.$$

The unit of the (K, I) adjunction gives us stable equivalences

$$(X^{fcaa})^{cof} \xleftarrow{\simeq} K((I(X^{fcaa})^{cof})^{cna})$$

which, using the factorization

$$(I(X^{fcaa})^{cof})^{cna} \xleftarrow{\simeq} I(X^{fcaa})^{cof})^{cna_2} \longleftarrow (I(B^{fcaa}))^{cna}$$

we extend to

$$(X^{fcaa})^{cof} \xleftarrow{\simeq} K(I((X^{fcaa})^{cof})^{cna}) \xleftarrow{\simeq} K(I((X^{fcaa})^{cof})^{cna_2}).$$

This allows us to write a stable equivalence

$$\begin{aligned}
&(X^{fcaa})^{cof} \wedge_{B^{fcaa}} (Y^{fcaa})^{cof} \\
&\qquad \simeq K(I((X^{fcaa})^{cof})^{cna_2}) \wedge_{K((I(B^{fcaa}))^{cna})} K(I((Y^{fcaa})^{cof})^{cna_2}).
\end{aligned}$$

Because K is a left adjoint, the right-hand side here is isomorphic to a pushout

$$K(I((X^{fcaa})^{cof})^{cna_2}) \wedge_{K((I(B^{fcaa}))^{cna})} K(I((Y^{fcaa})^{cof})^{cna_2}) \\ \cong K\left(I((X^{fcaa})^{cof})^{cna_2} \coprod_{(I(B^{fcaa}))^{cna}} I((Y^{fcaa})^{cof})^{cna_2} \right)$$

Applying $Q((I((-)^{fcaa})^{cna}))$, there are stable equivalences

$$\begin{aligned}
&\mathbb{L}Q\mathbb{R}I(X \wedge_B^{\mathbb{L}} Y) = Q((I((X \wedge_B Y)^{fcaa})^{cna})) \\
&\simeq Q((I((K\left(I((X^{fcaa})^{cof})^{cna_2} \coprod_{(I(B^{fcaa}))^{cna}} I((Y^{fcaa})^{cof})^{cna_2} \right))^{fcaa})^{cna})) \\
&\simeq Q\left(I((X^{fcaa})^{cof})^{cna_2} \coprod_{(I(B^{fcaa}))^{cna}} I((Y^{fcaa})^{cof})^{cna_2} \right) \\
&\simeq Q(I((X^{fcaa})^{cof})^{cna_2}) \coprod_{Q((I(B^{fcaa}))^{cna})} Q(I((Y^{fcaa})^{cof})^{cna_2})
\end{aligned}$$

The next to last reduction comes from the unit of the (K, I) adjunction and the last because Q is a left adjoint and thus commutes with pushouts.

Reversing the factorizations done earlier, we have stable equivalences

$$\begin{aligned}
&Q(I((X^{fcaa})^{cof})^{cna_2}) \coprod_{Q((I(B^{fcaa}))^{cna})} Q(I((Y^{fcaa})^{cof})^{cna_2}) \\
&\simeq Q(I((X^{fcaa})^{cof})^{cna}) \coprod_{Q((I(B^{fcaa}))^{cna})} Q(I((Y^{fcaa})^{cof})^{cna}) \\
&\simeq Q((I(X^{fcaa}))^{cna}) \coprod_{Q((I(B^{fcaa}))^{cna})} Q((I(Y^{fcaa}))^{cna}) \\
&= \mathbb{L}Q\mathbb{R}I(X) \coprod_{\mathbb{L}Q\mathbb{R}I(B)} \mathbb{L}Q\mathbb{R}I(Y)
\end{aligned}$$

where on the right-hand side of each line the coproduct now denotes a homotopy pushout rather than a pushout. □

$TAQ^R(A)$ was defined in [3] as $\mathbb{L}Q_A\mathbb{R}I_A(A \wedge_R^{\mathbb{L}} A)$, and [3, Proposition 3.2] shows that it is the corepresenting object for derivations. Logarithmic TAQ was defined in [19] as the corepresenting object for logarithmic derivations. It turns out that log TAQ is also $\mathbb{L}Q\mathbb{R}I$ of a certain ring.

Proposition 3.1.21. Let (R, P) be a cofibrant pre-log ring spectrum, and let (A, M) be a cofibrant object of $\mathcal{P}_{(R,P)}$. Then there is an isomorphism in $h\mathcal{M}_A$

$$\mathbb{L}Q_A\mathbb{R}I_A(A \wedge_R A \wedge_{A \wedge_{\mathbb{S}^{\mathcal{J}}[M]} \mathbb{S}^{\mathcal{J}}[M \boxtimes_P M]} A \wedge_{\mathbb{S}^{\mathcal{J}}[M]} \mathbb{S}^{\mathcal{J}}[(M \boxtimes_P M)^{rep}]) \cong TAQ^{(R,P)}(A, M)$$

Proof. The cofibrancy of the hypotheses ensures that all of the tensor products are derived. We apply Lemma 3.1.20, Lemma 3.1.19, the definition of ordinary TAQ, and Corollary 3.1.10. □

Proposition 3.1.22. Let (R, P) be a cofibrant pre-log ring spectrum, and let (A, M) be a cofibrant object of $\mathcal{P}_{(R,P)}$. Then there is an isomorphism in $h\mathcal{M}_A$

$$\mathbb{L}Q_A\mathbb{R}I_A(THH^{(R,P)}(A, M)) \cong \Sigma TAQ^{(R,P)}(A, M)$$

Proof. Base change the definition of $THH^{(R,P)}(A, M)$ to get

$$THH^{(R,P)}(A, M) \simeq THH^R(A) \bigwedge_{A \wedge_{\mathbb{S}^{\mathcal{J}}[M]} \mathbb{S}^{\mathcal{J}}[B^{cy}(M)]} A \wedge_{\mathbb{S}^{\mathcal{J}}[M]} \mathbb{S}^{\mathcal{J}}[B^{rep}(M)]$$

Applying $\mathbb{L}Q_A\mathbb{R}I_A$ to this homotopy colimit, we get the homotopy colimit

$$\mathbb{L}Q_A\mathbb{R}I_A(THH^R(A)) \coprod_{\mathbb{L}Q_A\mathbb{R}I_A(A \wedge_{\mathbb{S}^{\mathcal{J}}[M]} \mathbb{S}^{\mathcal{J}}[B^{cy}(M)])} \mathbb{L}Q_A\mathbb{R}I_A(A \wedge_{\mathbb{S}^{\mathcal{J}}[M]} \mathbb{S}^{\mathcal{J}}[B^{rep}(M)])$$

where we have in mind a point-set diagram as in Lemma 3.1.20.

Because A is a cofibrant $\mathbb{S}^{\beth}[M]$-algebra, the tensor products with A over $\mathbb{S}^{\beth}[M]$ are derived. Using Lemma 3.1.19, we have

$$\mathbb{L}Q_A\mathbb{R}I_A(A \wedge_{\mathbb{S}^{\beth}[M]} \mathbb{S}^{\beth}[B^{cy}(M)]) \cong A \wedge_{\mathbb{S}^{\beth}[M]} \mathbb{L}Q_{\mathbb{S}^{\beth}[M]}\mathbb{R}I_{\mathbb{S}^{\beth}[M]}(\mathbb{S}^{\beth}[B^{cy}(M)])$$

and

$$\mathbb{L}Q_A\mathbb{R}I_A(A \wedge_{\mathbb{S}^{\beth}[M]} \mathbb{S}^{\beth}[B^{rep}(M)]) \cong A \wedge_{\mathbb{S}^{\beth}[M]} \mathbb{L}Q_{\mathbb{S}^{\beth}[M]}\mathbb{R}I_{\mathbb{S}^{\beth}[M]}(\mathbb{S}^{\beth}[B^{rep}(M)])$$

so the homotopy colimit above is isomorphic to

$$\Sigma TAQ^R(A) \coprod_{A \wedge_{\mathbb{S}^{\beth}[M]} \Sigma TAQ^{\mathbb{S}^{\beth}[P]}(\mathbb{S}^{\beth}[M])} A \wedge_{\mathbb{S}^{\beth}[M]} \Sigma TAQ^{(\mathbb{S}^{\beth}[P],P)}(\mathbb{S}^{\beth}[M], M).$$

This is stably equivalent to

$$\begin{aligned}
&\Sigma TAQ^R(A) \coprod_{\Sigma A \wedge_{\mathbb{S}^{\beth}[M]} TAQ^{\mathbb{S}^{\beth}[P]}(\mathbb{S}^{\beth}[M])} \Sigma A \wedge_{\mathbb{S}^{\beth}[M]} TAQ^{(\mathbb{S}^{\beth}[P],P)}(\mathbb{S}^{\beth}[M], M) \\
&\cong \sum \left(TAQ^R(A) \coprod_{A \wedge_{\mathbb{S}^{\beth}[M]} TAQ^{\mathbb{S}^{\beth}[P]}(\mathbb{S}^{\beth}[M])} A \wedge_{\mathbb{S}^{\beth}[M]} TAQ^{(\mathbb{S}^{\beth}[P],P)}(\mathbb{S}^{\beth}[M], M) \right) \\
&\cong \Sigma TAQ^{(R,P)}(A, M).
\end{aligned}$$

□

3.2 The spectral sequence

In this section, we translate the André-Quillen spectral sequence constructed in [14] from S-modules into symmetric spectra using the Quillen equivalence of [25]. This is necessary because the proof given there uses features of S-modules that are not available in symmetric spectra.

From [25], there is a monoidal Quillen equivalence

$$\Lambda\colon \mathcal{M}_A^\Sigma \leftrightarrows \mathcal{M}_{\Lambda(A)} \colon \Phi$$

between $\mathcal{M}_A^\Sigma$, the category of A-modules where A is flat (i.e. S-cofibrant) as a symmetric spectrum, and $\mathcal{M}_{\Lambda(A)}$, the category of $\Lambda(A)$-modules where $\Lambda(A)$ is an EKMM S-algebra. This restricts to give a monoidal adjunction between non-unital commutative symmetric A-algebras and non-unital commutative $\Lambda(A)$-algebras

$$\Lambda\colon \mathcal{N}_A^\Sigma \rightleftarrows \mathcal{N}_{\Lambda(A)} \colon \Phi.$$

The unit and counit are maps of non-unital commutative algebras. Thus, the triangle identities hold because they hold for modules. Similarly, Φ preserves fibrations and acyclic fibrations because it preserves both of these classes of maps for modules. The left Quillen functor Λ can be seen to be a Quillen equivalence because it factors as a composite of Quillen equivalences

$$\mathcal{N}_A^\Sigma \xrightarrow{K} \mathcal{C}_{A/A}^\Sigma \xrightarrow{\Lambda} \mathcal{C}_{\Lambda(A)/\Lambda(A)} \xrightarrow{I} \mathcal{N}_A.$$

The adjunction is a monoidal adjunction with Λ strong monoidal and Φ lax monoidal. Because Λ is strong monoidal, to see that the adjunction is a monoidal Quillen adjunction it is only necessary to check that for a cofibrant replacement $A^{\text{cof}} \xrightarrow{\simeq} A$ in $\mathcal{N}_A^\Sigma$, the image $\Lambda(A^{\text{cof}}) \longrightarrow \Lambda(A)$ is a weak equivalence. The map $A^{\mathrm{cof}} \xrightarrow{\simeq} A$ is a weak equivalence of A-modules and the result has already been established for $\mathcal{M}_A^\Sigma$. Summarizing, the adjunction

$$\Lambda\colon \mathcal{N}_A^\Sigma \rightleftarrows \mathcal{N}_{\Lambda(A)} \colon \Phi.$$

is a monoidal Quillen equivalence.

Proposition 3.2.1. Let N be the fiber of the map $THH^{(S,U)}(A,M) \to A$ (which is necessarily in $\mathcal{N}_A^\Sigma$) and assume that N is 0-connected. There is a spectral sequence with E^1-term

$$E^1_{s,t} = \pi_{t-s}\left[\bigwedge^{s-1} \Sigma TAQ^{(S,U)}(A,M)\right]_{h\Sigma_{s-1}}$$

converging to $\pi_{t-s}N$.

Proof. We construct the logarithmic André-Quillen spectral sequence by applying [14, Corollary 2.7] which shows that given a 0-connected N' in $h\mathcal{N}_A$ there is a spectral sequence with

$$E^1_{s,t} = \pi_{t-s}\left[\bigwedge^{s-1} \mathbb{L}Q(N')\right]_{h\Sigma_{s-1}}$$

and converging to $\pi_{t-s}N'$.

If $N' \cong \mathbb{L}\Lambda(N)$, then because Λ is monoidal and a left adjoint, we can conclude that

$$\begin{aligned}\pi_{t-s}\left[\bigwedge^{s-1} \mathbb{L}Q(N')\right]_{h\Sigma_{s-1}} &\cong \pi_{t-s}\left[\bigwedge^{s-1} \mathbb{L}Q\mathbb{L}\Lambda(N)\right]_{h\Sigma_{s-1}} \\ &\cong \pi_{t-s}\mathbb{L}\Lambda\left[\bigwedge^{s-1} \mathbb{L}Q(N)\right]_{h\Sigma_{s-1}}.\end{aligned}$$

Because $X \to \mathbb{R}\Phi\mathbb{L}\Lambda(X)$ is an isomorphism, we know that $\pi_*(X) \cong \pi_*\mathbb{R}\Phi\mathbb{L}\Lambda(X)$, and [25] showed that Φ preserves homotopy groups, from which we conclude that $\pi_*\mathbb{R}\Phi\mathbb{L}\Lambda(X) \cong \pi_*\mathbb{L}\Lambda(X)$. So we have $\pi_*(X) \cong \pi_*\mathbb{L}\Lambda(X)$. Setting $X = \left[\bigwedge^{s-1}\mathbb{L}Q(N')\right]_{h\Sigma_{s-1}}$, we conclude that

$$\pi_{t-s}\left[\bigwedge^{s-1}\mathbb{L}Q(N)\right]_{h\Sigma_{s-1}} \cong \pi_{t-s}\mathbb{L}\Lambda\left[\bigwedge^{s-1}\mathbb{L}Q(N)\right]_{h\Sigma_{s-1}}.$$

and setting $X = N$, we conclude that

$$\pi_{t-s}N \cong \pi_{t-s}N'.$$

This gives us the spectral sequence in symmetric spectra that we desired.

When N is the augmentation ideal of $THH^{(S,U)}(A, M) \to A$, we've seen that there is an isomorphism in $h\mathcal{M}_A^\Sigma$

$$\mathbb{L}Q(N) = \mathbb{L}Q\mathbb{R}I(THH^{(S,U)}(A, M)) \simeq \Sigma TAQ^{(S,U)}(A, M)$$

so we can write the E^1-term of this spectral sequence as

$$\pi_{t-s}\left[\bigwedge^{s-1}\Sigma TAQ^{(S,U)}(A, M)\right]_{h\Sigma_{s-1}}.$$

□

3.3 Multiplicative structure in the spectral sequence

In this section, we modify the construction of the André-Quillen spectral sequence given in [14] to produce a multiplicative spectral sequence relating log TAQ and log THH. The main idea is to filter THH by powers of the augmentation ideal. The André-Quillen spectral sequence is the resulting spectral sequence. Because we don't have actual powers of ideals to work with, we use the bar construction to expose the multiplication in the augmentation ideal and simulate powers of the ideal.

Define the functor $Q_n \colon \mathcal{N}_A \to \mathcal{M}_A$ for $n \geq 2$ by the pushout in $\mathcal{M}_A$

$$\begin{array}{ccc} N^{\wedge_A n} & \longrightarrow & N \\ \downarrow & & \downarrow \\ * & \longrightarrow & Q_n(N), \end{array}$$

and define the functor $I^n \colon \mathcal{N}_A \to \mathcal{N}_A$ for $n \geq 2$ by the pullback in $\mathcal{M}_A$

$$\begin{array}{ccc} I^n(N) & \longrightarrow & N \\ \downarrow & & \downarrow \\ * & \longrightarrow & Q_n(N). \end{array}$$

Set $I^1(N) = N$.

Theorem 3.3.1. Given a spectrum E, a cofibrant object (R, P) of $\mathcal{P}$, and a cofibrant object (A, M) of $\mathcal{P}_{(R,P)}$, there is a spectral sequence with E^2-term

$$E^2_{p,q} = E_{p+q}((TAQ^{(R,P)}(A, M))^{\wedge q}/\Sigma_q)$$

converging to $E_{p+q}(THH^{(R,P)}(A, M))$. If E is multiplicative, then the spectral sequence is a spectral sequence of algebras.

Proof. First construct a filtration of $THH^{(R,P)}(A, M)$, then show that there is a product compatible with the filtration. The filtration is analogous to taking the powers of the augmentation ideal.

Take a fibrant replacement

$$THH^{(R,P)}(A,M) \xrightarrow{\simeq} THH^{(R,P)}(A,M)^{fib} \twoheadrightarrow A$$

and consider $N^c = I(THH^{(R,P)}(A,M)^{fib})^{cof}$, the cofibrant replacement of the augmentation ideal.

Let $(\mathbb{P}^+_A \colon \mathcal{M}_A \to \mathcal{M}_A, \mu\colon (\mathbb{P}^+_A)^2 \to \mathbb{P}^+_A, \eta\colon 1 \to \mathbb{P}^+_A)$ be the monad associated to the free-forgetful adjunction $(\mathbb{P}^+_A, U, \eta, \xi)$. So $\mu\colon (\mathbb{P}^+_A)^2 N \to \mathbb{P}^+_A N$ is induced by the maps

$$N^{i_1} \wedge_A \cdots \wedge_A N^{i_k} \to N^{i_1+\cdots+i_k}$$

and $\eta(N)$ is equal to the inclusion of the first summand.

As in [14], let $B_*N^c = B_*(\mathbb{P}^+_A, (\mathbb{P}^+_A)^n, N^c)$ be the simplicial bar construction with

$$B_n(\mathbb{P}^+_A, (\mathbb{P}^+_A)^n, N^c) = (\mathbb{P}^+_A)^{n+1} N^c$$

in simplicial level n and structure maps

$$\begin{aligned} d_i &= (\mathbb{P}^+_A)^i \mu_{(\mathbb{P}^+_A)^{n-i-1}} \quad \text{for } 0 \le i < n \\ d_n &= \mathbb{P}^+_A \xi \\ s_i &= (\mathbb{P}^+_A)^{i+1} \eta_{(\mathbb{P}^+_A)^{n-i}} \quad \text{for } 0 \le i \le n \end{aligned}$$

There is a homotopy equivalence (of simplicial non-unital commutative A-algebras) $B_*N^c \to N^c$. Using the functor I^n, we obtain a sequence

$$\cdots \longrightarrow I^{n+1}(B_*N^c) \longrightarrow I^n(B_*N^c) \longrightarrow \cdots \longrightarrow B_*N^c.$$

The map $I^{n+1}(B_*N^c) \to I^n(B_*N^c)$ is given in simplicial level k by the following inclusion of spectra

$$((\mathbb{P}^k(N^c))^{\wedge n+1}/\Sigma_{n+1} \vee (\mathbb{P}^k(N^c))^{\wedge n+2}/\Sigma_{n+2} \vee \cdots) \longrightarrow (\mathbb{P}^k(N^c))^{\wedge n}/\Sigma_n \vee (\mathbb{P}^k(N^c))^{\wedge n+1}/\Sigma_{n+1} \vee \cdots) \quad .$$

The filtration quotients are simplicial A-modules with $\mathbb{P}^k(N^c)^{\wedge n}/\Sigma_n$ in simplicial level k. Looking at the definitions, one sees that this is $Q_2(B_*N^c)^{\wedge n}/\Sigma_n$.

Realization gives a sequence of inclusions of spectra. We extend this to the right via

$$|B_*N^c| \longrightarrow A \vee |B_*N^c|$$

Thus there is a filtration

$$\cdots \longrightarrow |I^{n+1}(B_*N^c)| \longrightarrow |I^n(B_*N^c)| \longrightarrow \cdots \longrightarrow |B_*N^c| \longrightarrow A \vee |B_*N^c| \simeq THH^{(R,P)}(A, M)$$

Taking E-homology groups, this gives rise to a spectral sequence with E^2-term

$$E_{p+q}(|Q_2(B_*N^c)^{\wedge q}/\Sigma_q|)$$

that converges to $E_{p+q}(THH^{(R,P)}(A, M))$. The homotopy equivalence $|B_*N^c| \to N^c$ induces weak equivalences

$$|Q_2(B_*N^c)^{\wedge q}/\Sigma_q| \simeq Q_2(N^c)^{\wedge q}/\Sigma_q$$

so the E^2 term is isomorphic to

$$E^2_{p,q} = E_{p+q}((TAQ^{(R,P)}(A,M))^{\wedge q}/\Sigma_q).$$

There are products $|I^r(B_*N^c)| \wedge_A |I^s(B_*N^c)| \to |I^{r+s}(B_*N^c)|$ induced from simplicial maps given in simplicial level k by

$$\left(\bigvee_{i\geq r} \mathbb{P}^k(N^c)^{\wedge r}/\Sigma_r\right) \wedge_A \left(\bigvee_{i\geq s} \mathbb{P}^k(N^c)^{\wedge s}/\Sigma_s\right) \to \bigvee_{i\geq r+s} \mathbb{P}^k(N^c)^{\wedge r+s}/\Sigma_{r+s}$$

when $r, s \neq 0$. In addition, there are products $(A \vee |B_*N^c|) \wedge_A |I^n(B_*N^c)| \to |I^n(B_*N^c)|$ and a product $(A \vee |B_*N^c|) \wedge_A (A \vee |B_*N^c|) \to (A \vee |B_*N^c|)$ using the products just defined and the A-action.

When E is multiplicative, then the product on the filtration induces a product on the spectral sequence. □

One consequence of the spectral sequence is a kind of étale descent theorem.

Definition 3.3.2 ([19, Definition 5.22])**.** A morphism $(R,P) \to (A,M)$ in $\mathcal{P}$ is *formally log étale* if $TAQ^{(R,P)}(A,M)$ is contractible. We will write just log étale for formally log étale.

Suppose now that $(A,M) \to (B,N)$ is a log étale extension, and consider the map

$$B \wedge_A THH^{(R,P)}(A,M) \to THH^{(R,P)}(B,N).$$

Both the domain and the codomain can be filtered to obtain the following map of filtrations

$$\begin{array}{ccccccc}
\cdots \longrightarrow & B \wedge_A I_A^2 & \longrightarrow & B \wedge_A I_A & \longrightarrow & B \wedge_A THH^{(R,P)}(A, M) \\
& \downarrow & & \downarrow & & \downarrow \\
\cdots \longrightarrow & I_B^2 & \longrightarrow & I_B & \longrightarrow & THH^{(R,P)}(B, N)
\end{array}$$

where we abbreviate $I_A^q = |I^q(B_* N^c)|$ for $N = I(THH^{(R,P)}(A, M)^{fib})$, and similarly for I_B^q. The filtration quotients of the top row are

$$B \wedge_A I_A^q / I_A^{q+1} \simeq B \wedge_A (\Sigma TAQ^{(R,P)}(A, M))^{\wedge q}/\Sigma_q$$

while the filtration quotients of the lower row are

$$I_B^q / I_B^{q+1} \simeq (\Sigma TAQ^{(R,P)}(B, N))^{\wedge q}/\Sigma_q$$

Because the extension is étale, these are weakly equivalent.

So we have a map of spectral sequences with isomorphic E^2-terms, and as a result, $E_*(B \wedge_A THH^{(R,P)}(A, M))$ and $E_*(THH^{(R,P)}(B, N))$ have the same associated graded groups. Thus the André-Quillen spectral sequence gives a form of étale descent for log THH.

Example 3.3.3. The map

$$ku \wedge_\ell THH(\ell | L) \to THH(ku | KU),$$

conjectured to be a weak equivalence in [1], induces an isomorphism in $V(1)$-homology. Weak equivalences

$$THH^{(S,U)}(\ell, i_* \mathrm{GL}_1^{\mathcal{J}}(L)) \simeq THH(\ell | L)$$

and

$$THH^{(S,U)}(ku, i_*\mathrm{GL}_1^{\mathcal{J}}(KU)) \simeq THH(ku|KU)$$

were demonstrated in [16, Theorem 6.13], and in [19, Theorem 6.1] the map $(\ell, i_*\mathrm{GL}_1^{\mathcal{J}}(L)) \to (ku, i_*\mathrm{GL}_1^{\mathcal{J}}(KU))$ was shown to be an étale extension. Because $V(1)_*\ell \cong \mathbb{F}_p$, all $V(1)$-homology groups are $\mathbb{F}_p$-modules, and there are no extension issues in the spectral sequence, so there is an isomorphism in $V(1)$-homology:

$$V(1)_*(ku \wedge_\ell THH(\ell|L)) \cong V(1)_*THH(ku|KU).$$

Chapter 4

The Étale descent theorem

In this chapter, we prove a logarithmic analogue of an étale descent theorem. (Theorem 4.1.2). We define the terms log thh-étale, prove the étale descent theorem, and show how the notion of log étale defined in Definition 3.3.2 relates to the standard notion in logarithmic algebraic geometry.

4.1 The Étale descent theorem

This section concerns itself with proving a logarithmic version of the étale descent theorem. The following definition of (thh-)étale is adapted from [13].

Definition 4.1.1. A pre-log ring extension $(R, P) \to (A, M)$ is *thh-étale* if $A \simeq THH^{(R,P)}(A, M)$.

Applying $\mathbb{L}Q_A\mathbb{R}I_A$ to an equivalence $A \xrightarrow{\simeq} THH^{(R,P)}(A, M)$ gives $* \simeq \Sigma TAQ^{(R,P)}(A, M)$ showing that thh-étale extensions are étale. The example [13, Example 3.5] shows that the converse is not generally true, although it is true in some cases.

Theorem 4.1.2 (Étale descent). Let (R, P) be a cofibrant object of $\mathcal{P}$. Let $(R, P) \rightarrowtail (A, M) \rightarrowtail (B, N)$ be a pair of cofibrations in $\mathcal{P}$. If $(A, M) \rightarrowtail (B, N)$

is a thh-étale extension, then

$$B \wedge_A THH^{(R,P)}(A, M) \simeq THH^{(R,P)}(B, N)$$

Proof. Thh-étale means that

$$B \simeq THH^{(A,M)}(B, N) \cong B \otimes_A S^1 \wedge_{\mathbb{S}^{\mathcal{J}}[N \otimes_M S^1]} \mathbb{S}^{\mathcal{J}}[(N \otimes_M S^1)^{rep}].$$

Additionally, one has

$$A \cong A \wedge_R (R \otimes_R S^1) \wedge_{\mathbb{S}^{\mathcal{J}}[M \boxtimes_P (P \otimes_P S^1)]} \mathbb{S}^{\mathcal{J}}[(M \boxtimes_P (P \otimes_P S^1))^{rep}]$$

and by definition

$$THH^{(R,P)}(A, M) = A \otimes_R S^1 \wedge_{\mathbb{S}^{\mathcal{J}}[M \otimes_P S^1]} \mathbb{S}^{\mathcal{J}}[(M \otimes_P S^1)^{rep}].$$

Therefore $B \wedge_A THH^{(R,P)}(A, M)$ is an iterated colimit

$$\begin{array}{ccccc}
B \otimes_A S^1 & \longleftarrow & \mathbb{S}^{\mathcal{J}}[N \otimes_M S^1] & \rightarrowtail & \mathbb{S}^{\mathcal{J}}[(N \otimes_M S^1)^{rep}] \\
\uparrow & & \uparrow & & \uparrow \\
A \wedge_R (R \otimes_R S^1) & \longleftarrow & \mathbb{S}^{\mathcal{J}}[M \boxtimes_P (P \otimes_P S^1)] & \rightarrowtail & \mathbb{S}^{\mathcal{J}}[(M \boxtimes_P (P \otimes_P S^1))^{rep}] \\
\downarrow & & \downarrow & & \downarrow \\
A \otimes_R S^1 & \longleftarrow & \mathbb{S}^{\mathcal{J}}[M \otimes_P S^1] & \rightarrowtail & \mathbb{S}^{\mathcal{J}}[(M \otimes_P S^1)^{rep}]
\end{array}$$

obtained by taking rows first. Taking the columns first gives $THH^{(R,P)}(B, N)$.

In the case of the first column, we have the pushout

$$B \otimes_A S^1 \longleftarrow A \wedge_R (R \otimes_R S^1) \rightarrowtail A \otimes_R S^1.$$

This is the realization of a simplicial object given in simplicial level n as a pushout

$$B \wedge_A \cdots \wedge_A B \longleftarrow A \wedge_R R \wedge_R \cdots \wedge_R R \rightarrowtail A \wedge_R \cdots \wedge_R A.$$

This is isomorphic to

$$B \wedge_A (B \wedge_R A) \wedge_A \cdots \wedge_A (B \wedge_R A)$$

which is in turn isomorphic to

$$B \wedge_R \cdots \wedge_R B.$$

The second column and third columns can be understood in the same way. For the third column, it is necessary to know that the repletion commutes with the pushout. This is shown in [16, Lemma 4.26]. □

The preceding was proved in [13] in the case of ordinary (non-logarithmic) rings with the additional hypothesis that A and B be connective, but it doesn't seem to be necessary.

4.2 Log étaleness

This section concerns itself with the relationship between the usual notion of log étale in algebraic geometry and the one defined in Definition 3.3.2.

Until now, we have not needed the notion of log ring spectra because the functors TAQ and THH aren't sensitive to the difference between a pre-log ring and its logification. We've taken advantage of this to ignore log rings and logification, but an alternative choice would be to use this fact to restrict attention to log rings. For example, the notion of log smoothness is defined for log schemes, and not for pre-log schemes.

We review the basic notions of logarithmic structures from [11], restricting to the affine case, and then show how the notion of log étale morphism for discrete log rings relates to the notion of log étale introduced in Definition 3.3.2.

Definition 4.2.1. A *log structure* (M, α) on a ring A is a pre-log structure such that

$$\alpha^{-1}(A^\times) \xrightarrow{\cong} A^\times$$

is an isomorphism. The triple (A, M, α) is called a log ring.

Here $A^\times$ denotes the multiplicative units of A. We typically leave α out of the notation and write (A, M) for a log ring instead of (A, M, α).

Given a pre-log ring (A, M), we can produce a *logification* (A, M^a) via the following pushout

$$\begin{array}{ccc} \alpha^{-1}(A^\times) & \longrightarrow & M \\ \downarrow & & \downarrow \\ A^\times & \longrightarrow & M^a \end{array}$$

The log ring (A, M^a) is initial among log rings under (A, M).

Definition 4.2.2. A monoid is *coherent* if it is finitely generated.

Definition 4.2.3. A monoid is *integral* if $ac = bc$ implies that $a = b$.

Definition 4.2.4. A log ring (A, M) is *fine* if M is integral and coherent.

Definition 4.2.5. A *chart* for a fine log ring (A, M) is a map $P \to M$ from a fine monoid P to M such that $P^a \cong M$.

Definition 4.2.6. A *chart* for a morphism $(A, M) \to (B, N)$ of fine log rings is a triple $(P \to M, Q \to N, P \to Q)$ consisting of a chart for (A, M), a chart for (B, N), and a morphism of monoids $P \to Q$.

Given a pre-log ring (A, M), there is a pre-log ring spectrum (HA, FM) where HA is an Eilenberg-Mac Lane spectrum and FM is notation for $\mathcal{F}^{\mathcal{J}}_{0,0}M$ using the functor $\mathcal{F}^{\mathcal{J}}_{0,0}$ described in Example 2.1.1.

Lemma 4.2.7. If (A, P) is a discrete pre-log ring with logification (A, M), then (HA, FP) and (HA, FM) have the same logification.

Proof. The logification of (HA, FM, β) is by definition, the pair $(HA, (FM)^a)$, where $(FM)^a$ is given by factoring the map $\beta^{-1}(\mathrm{GL}_1^{\mathcal{J}}(HA)) \to \mathrm{GL}_1^{\mathcal{J}}(HA)$ in $\mathcal{CS}^{\mathcal{J}}$ and taking the pushout

$$\begin{array}{ccc}
\beta^{-1}(\mathrm{GL}_1^{\mathcal{J}}(HA)) & \longrightarrow & FM \\
\downarrow & & \downarrow \\
G & \longrightarrow & (FM)^a \\
\simeq \downarrow & & \\
\mathrm{GL}_1^{\mathcal{J}}(HA). & &
\end{array}$$

The monoid $\beta^{-1}(\mathrm{GL}_1^{\mathcal{J}}(HA))$ is the pullback of the following diagram

$$\begin{array}{ccc} & & FM \\ & & \downarrow \\ \mathrm{GL}_1^{\mathcal{J}}(HA) & \longrightarrow & \Omega^{\mathcal{J}}(HA). \end{array}$$

Because $\mathrm{GL}_1^{\mathcal{J}}(HA) \to \Omega^{\mathcal{J}}(HA)$ is an inclusion, the pullback is $F(A^{\times})$. Therefore, the pushout defining the logification of FM is

$$\begin{array}{ccc} F(A^{\times}) & \longrightarrow & FM \\ \downarrow & & \downarrow \\ G & \longrightarrow & (FM)^{\beta} \end{array}$$

Because M is the logification of P,

$$FM = F(A^{\times} \oplus_{\alpha^{-1}(A^{\times})} P) \cong F(A^{\times}) \boxtimes_{F(\alpha^{-1}(A^{\times}))} FP$$

Therefore the logification of FM is given by the iterated pushout

$$\begin{array}{ccc} F(\alpha^{-1}(A^{\times})) & \longrightarrow & FP \\ \downarrow & & \downarrow \\ F(A^{\times}) & \longrightarrow & FM \\ \downarrow & & \downarrow \\ G & \longrightarrow & (FM)^{a} \end{array}$$

Because FM is $\mathcal{J}$-equivalent to the homotopy pushout of the top square, FM^{a} is $\mathcal{J}$-equivalent to the homotopy pushout $B(G, F(\alpha^{-1}(A^{\times})), FP)$ of the larger square. So FP and FM have $\mathcal{J}$-equivalent logifications. $\square$

Consider the following commutative square of log rings

$$\begin{array}{ccc} (T', L') & \longleftarrow & (B, N) \\ \uparrow & & \uparrow \\ (T, L) & \longleftarrow & (A, M) \end{array}$$

where $T' \cong T/I$ for a square-zero ideal I, and L' is the logification of L in T'.

Definition 4.2.8 ([11, Section 3.3])**.** A morphism $f\colon (A, M) \to (B, N)$ of fine log rings is *étale* if for any diagram as above, a unique map $(B, N) \to (T, L)$ exists making the diagram commute.

Proposition 4.2.9 ([11, Theorem 3.5])**.** Let $f\colon (A, M) \to (B, N)$ be a morphism of fine log rings, and let $P \to M$ be a chart. Then f is étale if and only if étale-locally on B there is a chart $(P \to M, Q \to N, P \to Q)$ for f such that

- the kernel and cokernel of the map $P^{gp} \to Q^{gp}$ are finite with order invertible in B
- the morphism $A \otimes_{\mathbb{Z}[P]} \mathbb{Z}[Q] \to B$ is étale

The notion of formally étale introduced earlier generalizes the notion of étale for fine discrete log rings in the following sense.

Proposition 4.2.10. If $(A, M) \to (B, N)$ is an étale discrete log ring extension, then $(HA, FM) \to (HB, FN)$ is a formally étale log ring spectrum extension.

Proof. Let $\{B \to B_i\}$ be an étale cover, and let $(B, N) \to (B_i, N_i)$ be the induced maps of pre-log rings where N_i is the logification of N in B_i. By Lemma 4.2.7 and [16, Theorem 4.24], there is a stable equivalence

$$THH^{(HA,FM)}(HB_i, FN_i) \simeq THH^{(HA,FP)}(HB_i, FQ).$$

Using the given factorization

$$A \to A \otimes_{\mathbb{Z}[P]} \mathbb{Z}[Q] \to B_i$$

we can produce a factorization

$$(HA, FP) \to (H(A \otimes_{\mathbb{Z}[P]} \mathbb{Z}[Q]), FQ) \to (HB_i, FQ)$$

where the second map is étale.

In order to apply THH, we take cofibrant replacements

$$\begin{array}{ccccc} (HA, FP) & \rightarrowtail & (H(A \otimes_{\mathbb{Z}[P]} \mathbb{Z}[Q])^{cof}, FQ^{cof}) & \rightarrowtail & (HB_i^{cof}, FQ^{cof}) \\ \| & & \downarrow \simeq & & \downarrow \simeq \\ (HA, FP) & \longrightarrow & (H(A \otimes_{\mathbb{Z}[P]} \mathbb{Z}[Q]), FQ) & \longrightarrow & (HB_i, FQ). \end{array}$$

Because the vertical maps are weak equivalences, the map

$$(H(A \otimes_{\mathbb{Z}[P]} \mathbb{Z}[Q])^{cof}, FQ^{cof}) \to (HB_i^{cof}, FQ^{cof})$$

is étale.

For notational simplicity, in the discussion that follows we will assume that these cofibrant replacements have been done, and do not write the superscripts.

Because the second map is étale, there is a stable equivalence

$$TAQ^{(HA,FP)}(HB_i, FQ)$$
$$\simeq HB_i \wedge_{H(A\otimes_{\mathbb{Z}[P]}\mathbb{Z}[Q])} TAQ^{(HA,FP)}(H(A \otimes_{\mathbb{Z}[P]} \mathbb{Z}[Q]), FQ).$$

Writing A_{PQ} for $H(A \otimes_{\mathbb{Z}[P]} \mathbb{Z}[Q])$ and using the definition, this is stably equivalent to

$$HB_i \wedge_{A_{PQ}} TAQ^{HA}(A_{PQ}) \amalg_{HB_i \wedge_{\mathbb{S}^{\mathcal{J}}[FQ]} TAQ^{\mathbb{S}^{\mathcal{J}}[FP]}(\mathbb{S}^{\mathcal{J}}[FQ])} HB_i \wedge \gamma(FQ)/\gamma(FP).$$

Now, $\gamma(FP)(1^+) \simeq P$, so $\gamma(FP)$ is stably equivalent to the prespectrum whose spaces are $\{P, BP, B^2P, \ldots\}$. It follows that $\gamma(FP)$ is stably equivalent to the Eilenberg-Mac Lane spectrum HP^{gp}, and thus that the cofiber $\gamma(FQ)/\gamma(FP)$ is stably equivalent to HQ^{gp}/P^{gp}. The smash product

$$HB_i \wedge \gamma(FQ)/\gamma(FP)$$

is, by the hypothesis on Q^{gp}/P^{gp}, contractible, so $TAQ^{(HA,FP)}(HB_i, FQ)$ is stably equivalent to

$$HB_i \wedge_{H(A\otimes_{\mathbb{Z}[P]}\mathbb{Z}[Q]} TAQ^{HA}(H(A \otimes_{\mathbb{Z}[P]} \mathbb{Z}[Q])) \amalg_{HB_i \wedge_{\mathbb{S}^{\mathcal{J}}[FQ]} TAQ^{\mathbb{S}^{\mathcal{J}}[FP]}(\mathbb{S}^{\mathcal{J}}[FQ])} *.$$

i.e. the cofiber of

$$HB_i \wedge_{\mathbb{S}^{\mathcal{J}}[FQ]} \mathbb{L}Q\mathbb{R}I(\mathbb{S}^{\mathcal{J}}[Q] \wedge_{\mathbb{S}^{\mathcal{J}}[FP]} \mathbb{S}^{\mathcal{J}}[FQ]) \to HB_i \wedge_{A_{PQ}} \mathbb{L}Q\mathbb{R}I(A_{PQ} \wedge_{HA} A_{PQ})$$

or by the discussion following Lemma 3.1.17 and Lemma 3.1.18

$$\mathbb{L}Q_{HB_i}\mathbb{R}I_{HB_i}(HB_i \wedge_{\mathbb{S}^{\mathcal{J}}[FP]} \mathbb{S}^{\mathcal{J}}[FQ]) \to \mathbb{L}Q_{HB_i}\mathbb{R}I_{HB_i}(HB_i \wedge_{HA} H(A \otimes_{\mathbb{Z}[P]} \mathbb{Z}[Q])).$$

We can produce this cofiber by applying $\mathbb{L}Q_{HB_i}\mathbb{R}I_{HB_i}$ to the pushout of

$$\begin{array}{ccc} HB_i \wedge_{\mathbb{S}^{\mathcal{J}}[FP]} \mathbb{S}^{\mathcal{J}}[FQ] & \longrightarrow & HB_i \wedge_{HA} H(A \otimes_{\mathbb{Z}[P]} \mathbb{Z}[Q]) \\ \downarrow & & \\ HB_i \wedge_{\mathbb{S}^{\mathcal{J}}[FQ]} \mathbb{S}^{\mathcal{J}}[FQ]. & & \end{array}$$

This pushout is an iterated pushout which, taking the pushouts in the other order, can be written as

$$HB_i \wedge_{HA \wedge_{\mathbb{S}^{\mathcal{J}}[FP]} \mathbb{S}^{\mathcal{J}}[FQ]} H(A \otimes_{\mathbb{Z}[P]} \mathbb{Z}[Q]).$$

Applying $\mathbb{L}Q_{HB_i}\mathbb{R}I_{HB_i}$ produces

$$HB_i \wedge_{H(A \otimes_{\mathbb{Z}[P]} \mathbb{Z}[Q])} TAQ^{HA \wedge_{\mathbb{S}^{\mathcal{J}}[FP]} \mathbb{S}^{\mathcal{J}}[FQ]}(H(A \otimes_{\mathbb{Z}[P]} \mathbb{Z}[Q]).$$

Now, $HA \wedge_{\mathbb{S}^{\mathcal{J}}[FP]} \mathbb{S}^{\mathcal{J}}[FQ]$ is an HA-module, so there are maps

$$H\pi_0(HA \wedge_{\mathbb{S}^{\mathcal{J}}[FP]} \mathbb{S}^{\mathcal{J}}[FQ]) \to HA \wedge_{\mathbb{S}^{\mathcal{J}}[FP]} \mathbb{S}^{\mathcal{J}}[FQ]$$

and

$$HA \wedge_{\mathbb{S}^{\mathcal{J}}[FP]} \mathbb{S}^{\mathcal{J}}[FQ] \to H\pi_0(HA \wedge_{\mathbb{S}^{\mathcal{J}}[FP]} \mathbb{S}^{\mathcal{J}}[FQ])$$

which compose to

$$H(A \otimes_{\mathbb{Z}[P]} \mathbb{Z}[Q]) \to HA \wedge_{\mathbb{S}^{\mathcal{J}}[FP]} \mathbb{S}^{\mathcal{J}}[FQ] \to H(A \otimes_{\mathbb{Z}[P]} \mathbb{Z}[Q]).$$

Applying $TAQ^{HA\wedge_{\mathbb{S}^{\mathcal{J}}[FP]}\mathbb{S}^{\mathcal{J}}[FQ]}(-)$, produces an equivalence

$$\begin{aligned} & TAQ^{HA\wedge_{\mathbb{S}^{\mathcal{J}}[FP]}\mathbb{S}^{\mathcal{J}}[FQ]}(H(A\otimes_{\mathbb{Z}[P]}\mathbb{Z}[Q])) \\ \to\ & TAQ^{HA\wedge_{\mathbb{S}^{\mathcal{J}}[FP]}\mathbb{S}^{\mathcal{J}}[FQ]}(HA\wedge_{\mathbb{S}^{\mathcal{J}}[FP]}\mathbb{S}^{\mathcal{J}}[FQ]) \\ \to\ & TAQ^{HA\wedge_{\mathbb{S}^{\mathcal{J}}[FP]}\mathbb{S}^{\mathcal{J}}[FQ]}(H(A\otimes_{\mathbb{Z}[P]}\mathbb{Z}[Q])) \end{aligned}$$

but the middle term is contractible, so $TAQ^{HA\wedge_{\mathbb{S}^{\mathcal{J}}[FP]}\mathbb{S}^{\mathcal{J}}[FQ]}(H(A\otimes_{\mathbb{Z}[P]}\mathbb{Z}[Q]))$ is contractible. □

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