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The combinatorics of reducible Dehn surgeries

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The combinatorics of reducible Dehn surgeries

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Dedicated to my wife Christa and the family we’re just beginning.

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The combinatorics of reducible Dehn surgeries

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We study reducible Dehn surgeries on nontrivial knots in S^3. The conjectured classification of such surgeries is known as the Cabling Conjecture, and partial progress toward the conjecture often comes in the form of a statement that an arbitrary reducible surgery resembles a cabled reducible surgery. One such resemblance is the Two Summands Conjecture: Dehn surgery on a knot in S^3 can only produce a manifold with at most two irreducible connected summands. In the event that a reducible surgery on a knot K in S^3 of slope r produces a manifold with more than two such summands, we show that $|r| \leq b$, where b denotes the bridge number of K. As a consequence, we rule out this possibility for knots with $b \leq 5$ and for positive braid closures.

We also study reducible Dehn surgeries without the assumption that the reducible manifold contains more than two connected summands. Specifically, if P is an essential planar surface in the exterior of a hyperbolic knot which completes to a reducing sphere in this surgery, then it is shown that the number of boundary components of P is at least ten.

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Chapter 1

Introduction

1.1 Dehn surgery on knots

Let K be a nontrivial knot in S^3, *i.e.* a smoothly embedded circle which does not bound a smoothly embedded disk. We begin by defining an operation on S^3, known as Dehn surgery, to obtain a different 3-manifold, and study the importance of this operation. To do so, construct the *exterior* of K, given by $X = \overline{S^3 \setminus N(K)}$. Then $H_1(\partial X) \cong \mathbb{Z} \times \mathbb{Z} = \langle [m], [l] \rangle$, where $m = \{pt\} \times \partial D^2$ in $N(K) \cong S^1 \times D^2$ and l is taken to be an essential simple closed curve in ∂X which is null-homologous in X. Then any essential simple closed curve on ∂X is homologous to $a[m] + b[l]$, for integers a and b. The set of unoriented isotopy classes of such curves are called *slopes*, and are in correspondence with $\mathbb{Q} \cup \{\frac{1}{0}\}$, where a curve homologous to $a[m] + b[l]$ is sent to $\frac{a}{b}$.

Definition 1.1.1. Let $r \in \mathbb{Q} \cup \{\frac{1}{0}\}$. Then r**-Dehn surgery** on K in S^3 is the operation which results in the manifold

$$S^3_r(K) = X \cup (S^1 \times D^2)/\sim,$$

where the identification is such that the r-slope on ∂X is glued to $\{pt\} \times \partial D^2$.

Intuitively, $S^3_r(K)$ is the manifold obtained by cutting out K and regluing it back in so that the r-slope now bounds a disk. In fact, given a knot in any closed, orientable, connected 3-manifold, one can analogously define Dehn surgery in that setting (though in general one does not have a canonical identification of slopes with elements of $\mathbb{Q} \cup \{\frac{1}{0}\}$). The famous Lickorish-Wallace Theorem [23], [43] states that any closed, orientable, connected 3-manifold can be obtained from any other by successive Dehn surgeries on knots. Said differently, every 3-manifold is a Dehn surgery on a *link* (a collection of disjoint knots) which can be taken to lie in the 3-sphere. Thus we see that Dehn surgery is an effective tool for studying 3-manifolds by deeply connecting it to the theory of knots.

A natural first task in attempting to use Dehn surgery to study 3-manifolds is determining the similarities between manifolds which are related by a single Dehn surgery. If we are considering beginning in S^3, this question becomes: how complicated can a manifold be if it is obtained as $S^3_r(K)$ for some K and r? To answer this question, we need to quantify the complexity of 3-manifolds. While there are many ways to do this, the method we will use is the prime decomposition of Kneser [22] and Milnor [28]. Recall that a 3-manifold M is *prime* if all connected sum decompositions are trivial: $M \cong M_1 \# M_2$ implies M_1 or M_2 is homeomorphic to S^3. That is to say, the 2-sphere defining this connected sum is *inessential, i.e.* it bounds a 3-ball. If M contains a sphere which does not bound a ball, we will say that M is *reducible*. It is a standard fact in 3-manifold topology that M is reducible if and only if M has a nontrivial connected sum decomposition or is homeomorphic to $S^1 \times S^2$.

The prime decomposition of M is a way of writing M as

$$M = \mathop{\#}_{i=1}^{n} M_i,$$

where each $M_i \ncong S^3$ and is prime, and this decomposition is unique up to rearranging the order of the summands.

To see that this is a good choice for complexity, consider the method by which a 3-manifold is classified. One must first cut the manifold along spheres into its unique set of prime connected summands [22],[28], then for each of these pieces one finds a collection of certain essential tori cutting the 3-manifold into Seifert fibered or atoroidal submanifolds [19, 20], [21] (here an orientable surface T of positive genus in M is *essential* if M contains no disks with boundary an essential curve on T, and T is not isotopic to a component of ∂M). After this is complete, only then can one begin to study the remaining irreducible, atoroidal 3-manifolds, which by Perelman's resolution of the Geometrization Conjecture [34], [35] have a very rigid geometric structure, and classification becomes feasible. Using this lens, understanding the prime decomposition of $S^3_r(K)$ is paramount.

Suppose $S^3_r(K)$ is reducible. Then it could be that K is unknotted and $r = 0$, producing $S^1 \times S^2$, but by the famous Property R theorem [4] this is the only such K and r giving a manifold with an $S^1 \times S^2$ connected summand. Thus we may assume that in this situation $S^3_r(K)$ will decompose as a connected sum and that $r \neq 0$. The standard example to consider is the

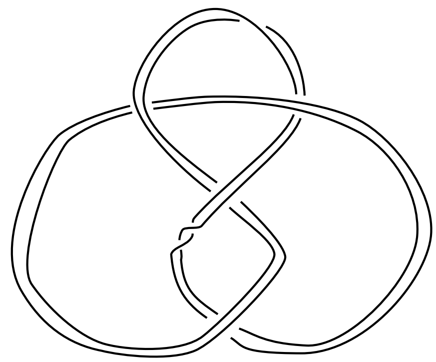


Figure 1.1: A cable knot

reducible surgery on a cable knot $C_{p,q}(K')$ with pattern knot K' [5]:

$$S^3_{\frac{pq}{1}}(C_{p,q}(K')) \cong L(q,p)\#S^3_{\frac{p}{q}}(K').$$

A (p,q)-cable knot K is constructed by taking a knot K' and replacing a regular neighborhood of K' by a solid torus containing a (p,q)-torus knot, and defining K to be the image of the torus knot. See Figure 1.1. The manifold $L(q,p)$ is the (q,p)-lens space, and must be nontrivial (*i.e.* not S^3 or $S^1 \times S^2$). In fact, the Cabling Conjecture asserts that this is the only possibility for K and r.

Conjecture 1.1.1 (Cabling Conjecture)**.** *If K is a nontrivial knot in S^3 with $S^3_r(K)$ reducible, then K is a cable knot and r is given by the cabling annulus.*

Here torus knots are considered to be cables of the unknot. The Cabling Conjecture is known to hold for many classes of knots, such as strongly invertible knots [3], symmetric knots [12], alternating knots [27], torus knots

[29], satellite knots [38], and arborescent knots [44], but in general it remains unsolved. One consequence of these results, along with Thurston's classification of knots [41], is that it suffices to assume K is hyperbolic.

Another direction of progress toward the Cabling Conjecture has been in showing that an arbitrary reducible surgery on a knot in S^3 coarsely resembles the cabled surgery: at least one summand is a nontrivial lens space [9], and the reducing slope r is an integer [8]. It follows that the $S^3_{\frac{p}{q}}(K)$ summand of the cable knot surgery is irreducible, so a cable knot's reducible surgery has two irreducible summands. This can be viewed as another approximation to the Cabling Conjecture, and remains unsolved in general.

Conjecture 1.1.2 (Two Summands Conjecture)**.** *If K is a nontrivial knot in S^3 with $S^3_r(K)$ reducible, then $S^3_r(K)$ consists of two irreducible connected summands.*

1.2 Statement of results

In this dissertation, we study reducible surgeries on knots in S^3 in both the general setting and this case of many summands. Suppose first that $S^3_r(K)$ is reducible and contains more than two prime summands. Then [42] shows that all but one of the summands are lens spaces, and [17] shows that all but two of the summands are integral homology spheres. Hence, $S^3_r(K) \cong L_1 \# L_2 \# Z$, where each L_i is a lens space and $H_1(Z;\mathbb{Z}) = 0$. Additionally, there is a history of providing bounds on the surgery coefficient in terms of the bridge number b of K. The bridge number is the minimal number of arcs in a bridge presentation of K, *i.e.* a knot, 2-sphere pair (K, S) such that the arcs of $K \setminus S$ are simultaneously isotopic into S on each side. To start, it is a consequence of the standard combinatorial techniques used that $|r| \leq b(b-1)$. In [36] this bound is improved to $|r| \leq (b-1)(b-2)$, and then to $|r| \leq \frac{1}{4}b(b+2)$ in [18]. Notice that these bounds are all quadratic in the bridge number. This is a consequence of the fact that they all arise from bounding $|\pi_1(L_i)|$ linearly in b, along with the fact that $|r| = |\pi_1(L_1)| \cdot |\pi_1(L_2)|$. Our first result is the establishing of a linear bound in this three summands case.

Theorem 1.2.1. *If Dehn surgery of slope r on a knot K in S^3 produces a manifold with more than two, and hence three connected summands, then $|r| \leq b$, where b is the bridge number of K. Consequently, the Two Summands Conjecture holds for knots with $b \leq 5$.*

As a consequence, we complete the proof of the Two Summands Conjecture for positive braid closures.

Corollary 1.2.2. *Let K be the closure of a positive braid in S^3. Then Dehn surgery on K produces at most two prime connected summands.*

The proof of Corollary 1.2.2 relies on the highly nontrivial fact [24] that hyperbolic positive knots (including the hyperbolic positive braid closures) could only have a single possible reducing slope of $2g-1$, where g denotes the genus of the knot, *i.e.* the minimal genus of a surface with boundary equal to the knot. As we will see, positive braid closures have a relationship between their genus and their bridge number that not all knots possess.

We remark that the statement regarding knots with $b \leq 5$ in Theorem 1.2.1 may already be known to hold in general: the Cabling Conjecture is shown to hold for knots with $b \leq 4$ in [14], and in [15] it is claimed that the $b = 5$ case has been additionally established in unpublished work. Another consequence of our work is a reproving of the Cabling Conjecture for knots with $b \leq 3$ (and in a technically quantifiable way, it is "almost" established for knots with $b \leq 5$). See Corollary 5.2.1 in Chapter 5 for a precise statement.

The second focus in this dissertation is on a well-studied complexity p of any reducing sphere $\widehat{P}$ in the general case of any reducible manifold arising from Dehn surgery on a knot in S^3. To define this complexity, recall that if K is a knot in S^3 and $Y = S^3_r(K)$, then Y contains a knot K' *dual* to K, *i.e.* $X = S^3 \setminus N(K) \cong Y \setminus N(K')$, so that there is some surgery on K' resulting in S^3 (K' is the image of $S^1 \times \{0\}$ contained in the filling solid torus in Y). Then after isotoping $\widehat{P}$ to be transverse to K', we define $p = |\widehat{P} \cap K'|$ to be minimized over all reducing spheres $\widehat{P}$. Since we may assume that $\widehat{P}$ is separating, and since X is irreducible, we see that p is a positive even integer. In fact, we see that $P = X \cap \widehat{P}$ is a planar surface in the exterior with p boundary components. If $p = 2$, then P is an essential annulus, and in fact it is shown in [7] that P is a cabling annulus and K is a cable knot. The Cabling Conjecture asserts that the remaining values of $p \geq 4$ should never occur. The case $p = 4$ is ruled out using the techniques of Gordon and Luecke [9] and has another proof in [26]. Our next result is the following:

Theorem 1.2.3. *Let K be a hyperbolic knot with a reducible surgery. If P is an essential planar surface which completes to a reducing sphere $\widehat{P}$ in the surgery, then the number p of boundary components of P is at least ten.*

The main techniques of this dissertation are the so-called graphs of intersection, introduced in [25] and [37] and made famous with the proof of the knot complement problem [9]. Our main point of departure is the primary technical Proposition of [9], which says that a certain combinatorial object called a *great web* must exist in the context of a reducible surgery on a knot in S^3. These great webs will be defined in Chapter 4 and discussed in detail throughout the dissertation, but for now it is sufficient to emphasize that a proof of the Cabling Conjecture would be complete if it is shown that the existence of great webs always give rise to contradictions. The proof of Theorem 1.2.3 is logically structured in this way: we prove that no great webs with $p < 10$ can exist. Additionally, we show (Corollary 4.1.3) that there is a nontrivial class of great webs that cannot exist, which are of arbitrarily large complexity in a way which we will discuss. This is a first result of its kind, and while this is a technical statement we emphasize that it suggests the validity of an attempt to positively resolve the Cabling Conjecture by studying these great webs.

We conclude this introduction by discussing the layout of the dissertation. In Chapter 2 we discuss the existence of surfaces Q and P in the knot exterior which intersect in a suitably essential way and complete to important closed surfaces in S^3 and $S^3_r(K)$, respectively. We use their intersections to construct graphs G_Q and G_P and discuss how these graphs contain a great deal of information about the topology of the manifolds S^3 and $S^3_r(K)$. In Chapter 3 we discuss a new index formula for these graphs, developed by Joshua Greene (and occurring here with his permission); the index allows for a strengthening of the standard arguments in this field of graphs of intersection (Theorem 3.0.6). In Chapter 4 we define great webs and use the index to discover much of their structure. We also define a *stratification of the Cabling Conjecture*, and use it to discuss how the remainder of the dissertation proves Theorems 1.2.1 and 1.2.3. The proofs of these Theorems occur in Chapters 5 and 7, respectively. We prove Corollary 1.2.2 in Chapter 6.

Chapter 2

Background

We begin by defining a pair of labeled graphs G_Q and G_P, and give some basic properties of these graphs. The construction that follows is part of a much more general field of study. For a survey of the possible situations in which such graphs of intersection can be defined, see [6].

Since our results apply to both the specific case of obtaining three summands by Dehn surgery on a knot in S^3, as well as the general case of obtaining *any* reducible manifold from a hyperbolic knot in S^3, we need to perform both setups, and discuss how they are related. We will refer to the former as the "three summands case", and the latter as the "general case".

2.1 Construction of the graphs G_Q and G_P in the three summands case

This construction first appeared in [18]. Suppose that $S^3_r(K) \cong L_1 \# L_2 \# Z$, where L_i is a lens space with $l_i = |\pi_1(L_i)|$, and Z is an integral homology sphere. For $i = 1, 2$, let P_i be a planar surface in the exterior of K, completing to a reducing sphere $\widehat{P}_i$ in $S^3_r(K)$ subject to the restrictions:

1. $\widehat{P}_i$ separates L_i from Z, in the sense that $S^3_r(K)$ cut along $\widehat{P}_i$ contains

two components, with a punctured L_i in one component and a punctured Z in the other, and

2. $p_i := |\partial P_i|$ is minimal among spheres with the above property.

Here "punctured" means the interiors of some finite number of disjoint 3-balls have been removed. Let $p = p_1 + p_2$. It can be shown using standard techniques that P_1 can be chosen to be disjoint from P_2. Let $P = P_1 \cup P_2$. By [4], we may find a meridional planar surface Q in the exterior of K, completing to a sphere $\widehat{Q}$ in S^3, such that no arc component of $Q \cap P$ is boundary-parallel in Q or in P, and so that $\partial Q \cap \partial P$ is minimized. This allows us to construct a graph G_Q on $\widehat{Q}$ whose set of (fat) vertices are the disk components of $\widehat{Q} \setminus \mathrm{Int}(Q)$ (*i.e.* meridian disks of the filling solid torus) and whose edges are the arc components of $Q \cap P$. Similarly, we construct graphs G_P^i on $\widehat{P}_i$, and define $G_P = G_P^1 \amalg G_P^2$. The *faces*, or complementary regions, of G_Q and G_P have boundaries consisting of arcs alternatingly lying on the vertices and the edges of G_Q and G_P; these will be called *corners* and *edges*, respectively. A face will thus be referred to as an n-gon if n is the number of edges in its boundary. Then the choice of Q above translates to the fact that no face of G_Q or G_P is a monogon.

Label the boundary components of P as $u_1, \ldots, u_p$ so that they occur in order along the boundary torus. Notice some of these components lie on P_1 and the rest lie on P_2. This allows us to assign a pair of labels to each edge of G_Q, corresponding to the two vertices of G_P to which the edge is incident. We will call an oriented edge a λ-*edge* if it has a label λ at its tail, and we will call an edge with labels λ_1 and λ_2 a (λ_1, λ_2)-*edge*. There will often be a reason for the ordering of the tuple (λ_1, λ_2) to assist the reader, but technically we consider this to be unordered. The notation $\langle \lambda, \lambda+1 \rangle$ will be used to designate the labeling of corners. A λ-*cycle* is a cycle of G_Q which can be oriented to be a collection of λ-edges. Since r is an integer, each boundary component of P intersects each boundary component of Q exactly once. Thus at a single vertex of G_Q, one sees these labels occur once in either clockwise or counterclockwise order based on the orientations of K and Q; call those vertices *negative* and *positive*, respectively. Since $\widehat{Q}$ is separating, precisely $\frac{q}{2}$ vertices of G_Q are positive. Letting $q = |\partial Q|$, label the boundary components of Q as $v_1, \ldots, v_q$, and proceed similarly. Edges can be given a sign as well: an edge of G_Q (respectively, G_P) is *positive* if it connects two vertices of G_Q (respectively, G_P) of the same sign; otherwise, the edge is *negative*. By considering the orientability of all relevant submanifolds, we have the *parity rule*: an arc component of $Q \cap P$ is a positive edge of G_Q if and only if it is a negative edge of G_P. A λ-cycle is a *great* λ-*cycle* if it bounds a disk in $\widehat{Q}$ containing only vertices of the same sign. Note that the construction of Q forces $\frac{q}{2} \leq b$ (this is nontrivial, and comes from the use of *thin position* for knots in its construction; see [4] for more information). We may add elements of $\mathbb{Z}/p\mathbb{Z}$ to labels of ∂P in the obvious way: given a label $\lambda \in \{1, \ldots, p\}$ the label $\lambda + 1$ is either defined or is taken to be 1; similarly for ∂Q.

The technique of graphs of intersection proceeds by deriving combinatorial restrictions on the graphs, which in turn give rise to topological restrictions on the manifolds. For example, a disk face of a graph whose corners are all $\langle \lambda, \lambda+1 \rangle$ for some label λ and whose vertices are all of the same sign is called a *Scharlemann cycle*, and it is well known (see for example [6]) that a Scharlemann cycle face of G_Q gives rise to a lens space summand of $S^3_r(K)$ (in the context we are in, where $\widehat{P}$ is a union of spheres). Clearly then no Scharlemann cycle can exist in G_P, else there would be a lens space summand in S^3. The highlighted face in Figure 5.1a is a Scharlemann cycle.

In the three summands case, however, there are two lens space summands of different orders, and hence two different types of Scharlemann cycles may occur on G_Q. To clarify, let G_i be the subgraph of G_Q consisting of all vertices and with edge set given by $G_Q \cap G_P^i$, so that G_i contains only those edges of G_Q which lie on G_P^i. Then we have the following properties, which essentially say that the graph G_Q is well-behaved.

Proposition 2.1.1 ([18]). *Suppose σ_i is a Scharlemann cycle of G_i, so that its labels are x and y in G_Q. Then*

1. *$y = x + 1$, i.e. there are no edges of G_{3-i} in σ_i, so that it is a genuine Scharlemann cycle of G_Q,*

2. *σ_i contains exactly l_i edges in its boundary for some (fixed) integer $l_i \geq 2$, $i = 1, 2$, and*

3. *if σ_i' is another Scharlemann cycle of G_i, then its corners are also labeled $(x, x+1)$.*

Define the *length* of a Scharlemann cycle to be the number of edges in its boundary. As we will see (Lemma 4.0.3), G_Q will contain at least one Scharlemann cycle of each length in the three summands case. Relabel the boundary components of P so that the corners of any length l_1 Scharlemann cycle are labeled (1,2). Let $(x, x+1)$ denote the corners of any length l_2 Scharlemann cycle. Let

$$L = \{3, \ldots, x-1, x+2, \ldots, p\},$$

and call L the set of *regular* labels. The set of *Scharlemann* labels is thus $\{1, 2, x, x+1\}$.

2.2 Construction of G_P and G_Q in the general case

In the general case, one begins by defining P to be a planar surface completing to a reducing sphere as before, but now subject to global minimality: $p = |\partial P|$ is minimal among all planar surfaces completing to reducing spheres in $S^3_r(K)$. Then Q is defined analogously, as are G_Q, G_P, the labels and notions of positivity and negativity of vertices and edges, the parity rule, etc. In the general case, there will be at least one Scharlemann cycle (Lemma 4.0.2), and there is exactly one pair of labels (say, 1 and 2) and one length (say, l) for all the Scharlemann cycles of G_Q [11], so the regular labels will be defined to be $L = \{3, 4, \ldots, p\}$. Since $\widehat{P}$ separates $S^3_r(K)$ and K', the faces of

G_Q may be denoted as *odd* or *even* based on whether they contain corners of the form $\langle 2k-1, 2k\rangle$ or $\langle 2k, 2k+1\rangle$, respectively. In fact, it is clear that in the three summands case, if without loss of generality $p_1 \leq p_2$, then removing P_2 from the discussion and proceeding analogously brings us to the general case.

2.3 Additional definitions and considerations

One notation that will be used throughout the dissertation is a cyclic ordering of the labels around a fat vertex. Note that given a pair of labels α and β, the statement $\alpha < \beta$ is not well-defined. However, given triples of labels, one finds that statements of the form $\alpha < \beta < \gamma$ are well-defined; they mean that on a positive (respectively negative) vertex, beginning at the label α one can travel counterclockwise (respectively clockwise) to the label β, then to γ, before returning to α. Another notation that we will use is with regards to faces adjacent to Scharlemann cycles. If f is a face of G_Q, let $m(f)$ denote the number of Scharlemann cycles to which f is adjacent (*i.e.* they share an edge). This quantity will prove to have a profound effect on the face f.

Additionally, any subsurface E of $\widehat{Q}$ or $\widehat{P}$ will be assumed to be a union of faces and vertices of G_Q or G_P, respectively, and we will not distinguish between such a surface and the graph contained on it (except to be slightly careful in Definition 4.0.2). This will imply that $G_E := G_Q \cap E$ is a well-defined object in the realm of intersection graphs. We will say that G_E is *closed* if G_Q contains no Scharlemann cycles adjacent to (*i.e.* sharing an edge with) ∂E, independent of whether the Scharlemann cycle is contained in E or its complement. In practice, should such a condition fail for E, then the Scharlemann cycle and all faces adjacent to it should be added to E to create a disk E' containing E with closed graph $G_{E'}$ (this process may need to be repeated with new Scharlemann cycles). To avoid confusion we will say E is *graphically closed* to indicate that G_E is closed. We will say that E is *positive* if all of the vertices in G_E are positive.

Finally, in [14], a subgraph of G_Q called a *strict* (or sometimes *new*) *great cycle* is defined. In the general case, a strict great cycle is a great (λ-)cycle which is not a Scharlemann cycle. It is shown that strict great cycles contain important subgraphs called closed clusters. A *closed cluster* is a graphically closed disk consisting of Scharlemann cycles and faces called *two-cornered faces*, which only consist of the corners $\langle 2, 3\rangle$ and $\langle p, 1\rangle$. (This is an important distinction: two-cornered faces use those and only those specific corners and not just any two types of corners.) Two-cornered faces are *exterior* if they are adjacent to a single Scharlemann cycle, otherwise they are *interior*. Hoffman then goes on to show that, in the general case, closed clusters always give rise to contradictions.

Theorem 2.3.1. *[Hoffman] Suppose that G_Q is a graph in the general case.*

1. *(Lemma 3.0.3 of [14]) G_Q does not contain a strict (or new) great cycle.*

2. *G_Q does not contain a closed cluster.*

We remark that the second condition is proved in the process of proving the first. We broke down Theorem 2.3.1 into two parts because we will discuss new ways of finding closed clusters in Chapter 3, and so these new methods still give rise to contradictions. The second part is proven by showing that the sphere $\widehat{P}$ was not minimal to begin with, by finding a *seemly pair* of two-cornered faces. This pair satisfies some technical conditions that translate into the ability to construct two new spheres, $\widehat{P}'$ and $\widehat{P}''$, both with fewer intersections with K', and which have the property that both spheres arise from $\widehat{P}$ by an exchange of a disk or an isotopy. Thus if one of the spheres, $\widehat{P}'$ say, is inessential, then it bounds a 3-ball, and this 3-ball gives an isotopy of $\widehat{P}$ to $\widehat{P}''$. If this happens, or if $\widehat{P}'$ was essential, then $\widehat{P}$ does not realize the minimum value of p, giving the desired contradiction.

We gave a rough outline of this proof for multiple reasons. The first reason is that we will certainly be interested in existence of closed clusters, as many of our arguments will give rise to them. The second reason is that there is one place in Chapter 7 that we in fact need to mimic this argument, as we will find a pair of disks which have no reason to actually satisfy the technical conditions of being a seemly pair, but they nonetheless may be used to give the same contradiction.

Chapter 3

A combinatorial index

In this chapter, we define a index that we may assign to the faces of G_Q (or G_P) in the general case. This index is due to Joshua Greene, and occurs here with his permission. We note that this index is definable in a more general setting (of graphs of intersection); we will restrict its development to the general case of the Cabling Conjecture. As we discussed in Chapter 2, the existence of a closed cluster in the graph G_Q gives a contradiction to Theorem 2.3.1. We will use the index to give rise to closed clusters in G_Q; in fact, the constructions here generalize the notions of strict great cycles, but we will not use this. We will also assume $p > 2$, which we may by [7].

Let γ be an oriented simple arc in G_Q which:

1. is incident to only positive vertices,
2. doesn't intersect the interior of any fat vertex,
3. begins and ends at edge endpoints, and
4. only travels counterclockwise around each fat vertex it encounters.

Equivalently, γ consists of a sequence of corners and edges of G_Q (possibly multiple corners at any vertex) which can be taken to be a positive subpath of the boundary of some faces of G_Q, oriented clockwise. The paths γ, γ', and $\overline{\gamma}'$ of Figure 3.1 are examples.

Let V be an abstract positive vertex of G_Q. Define $f_\gamma : \gamma \to \partial V$ such that $f_\gamma|v$ travels counterclockwise along ∂V exactly as it does at the fat vertex v, and $f_\gamma|e$ travels clockwise along ∂V from the label of the tail of e to the label of its head. Let $q : \partial V \to V'$ be the quotient of ∂V by the edge endpoints, so that V' is a bouquet of p circles, which are in correspondence with the set of corners of V. Let $g : H_1(V') \to \mathbb{Z}$ be defined so that (the quotient of) any corner of V, oriented from λ to $\lambda + 1$, maps to $-1 \in \mathbb{Z}$. We define the *partial degree* of γ, $\deg(\gamma)$, to be the element of $\mathbb{Z}[\frac{1}{p}]$ given by

$$\deg(\gamma) = \frac{1}{p}\, g(q_*(f(\gamma))).$$

The quantity $\deg(\gamma)$ will be an integer if and only if γ begins and ends at some edge endpoints with the same labels (*e.g.* if γ is the boundary of a union of faces of G_Q). We define $\deg(\gamma + \gamma') = \deg(\gamma) + \deg(\gamma')$ for paths γ and γ'.

Definition 3.0.1. Let F be a positive subsurface of $\widehat{Q}$. Then the **index** of F is defined by $I(F) = \deg(\partial F) - \chi(F)$.

It is clear from this definition that Scharlemann cycles are the only faces of G_Q of negative index, since the degree of their boundary is zero, and every other face has boundary of positive degree. The reason for working with

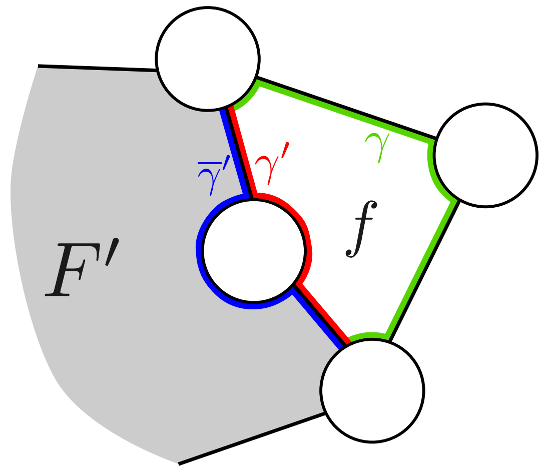


Figure 3.1: Adding the face f to F'

the index of a face, rather than the degree of its boundary, is that the index is additive over subfaces.

Lemma 3.0.2. *Let F be a positive subsurface of $\widehat{Q}$. Then* $I(F) = \sum_{\substack{f \subseteq F \\ f \text{ a face of } G_Q}} I(f)$.

Proof. We will show that

$$\deg(\partial F) = V_0(F) - E_0(F) + \sum_{\substack{f \subseteq F \\ f \text{ a face of } G_Q}} \deg(\partial f), \tag{3.1}$$

where $V_0(F)$ and $E_0(F)$ are the numbers of interior vertices and edges of F, respectively. With this, we see that

$$\begin{aligned}
I(F) &= \deg(\partial F) - \chi(F) \\
&= \left(V_0(F) - E_0(F) + \sum_{f \subseteq F} \deg(\partial f) \right) - \left(V(F) - E(F) + \sum_{f \subseteq F} \chi(f) \right) \\
&= -\chi(\partial F) + \sum_{f \subseteq F} I(f) \\
&= \sum_{f \subseteq F} I(f).
\end{aligned}$$

We prove Equation 3.1 by induction on the number n of faces in F. It is a tautology when $n = 1$, so suppose for all subsurfaces with fewer than n faces, Equation 3.1 holds. Let f be a face of F, and define $F' = F \setminus f$ as well as two curves making up ∂f:

$$\gamma' = \partial f \cap F'$$

and

$$\gamma = \partial f \setminus \gamma'.$$

Let $\overline{\gamma'}$ be the path consisting of the same edges of γ', but oppositely oriented, so that it travels in the other way around each vertex they encounter. See Figure 3.1. Then in determining how $\deg(\partial F)$ differs from $\deg(\partial F')$, one deletes the path $\overline{\gamma}'$ from $\partial F'$ and adds the path γ. Thus we see that

$$\begin{aligned}
\deg(\partial F) &= \deg(\partial F') - \deg(\overline{\gamma}') + \deg(\gamma) \\
&= \deg(\partial F') + (\deg(\gamma') + E(\gamma') - V_0(\gamma')) + \deg(\gamma) \\
&= \deg(\partial F') + E(\gamma') - V_0(\gamma') + \deg(\partial f),
\end{aligned}$$

because the path $\overline{\gamma}'$ followed by γ' has degree $E(\gamma') - V_0(\gamma')$, for $E(\gamma')$ the number of edges in γ', and $V_0(\gamma')$ the number of interior vertices of γ'. Then the induction hypothesis gives that Equation 3.1 holds for F', and so we see that

$$\begin{aligned}
\deg(\partial F) &= \deg(\partial F') + E(\gamma') - V_0(\gamma') + \deg(\partial f), \\
&= \left(V_0(F') - E_0(F') + \sum_{\substack{f \subseteq F \\ f \text{ a face of } G_Q}} \deg(\partial f) \right) + E(\gamma') - V_0(\gamma') + \deg(\partial f) \\
&= V_0(F) - E_0(F) + \sum_{\substack{f \subseteq F \\ f \text{ a face of } G_Q}} \deg(\partial f),
\end{aligned}$$

because $E_0(F) = E_0(F') + E(\gamma')$ and $V_0(F) = V_0(F') + V_0(\gamma')$. This is the desired statement. □

Thus if F is a positive subsurface of $\widehat{Q}$, then $I(F) < 0$ implies that F contains a Scharlemann cycle. We will see that something much stronger holds. As a first attempt to show that the index captures a great deal of the combinatorial structure, we find that the index can detect faces being two-cornered.

Lemma 3.0.3. *Let f be a positive face of G_Q, adjacent to $m(f)$ Scharlemann cycles. Then*

$$I(f) \geq m(f) - \chi(f). \tag{3.2}$$

If $m(f) > 0$, then equality holds if and only if f is a two-cornered face.

Proof. Decompose ∂f as

$$\partial f = \bigcup_{i=1}^{m} \gamma_i \cup \bigcup_{i=1}^{m} \delta_i,$$

where each γ_i consists of a $(1,2)$-edge of f, as well as the two corners adjacent to this edge, and δ_i is the subpath of ∂f between γ_i and γ_{i+1} (here we mean $i+1 \pmod m$). Note $m(f) \leq m$. Then $\deg(\gamma_i) = \frac{p-3}{p}$. Now $\deg(\delta_i)$ is nonnegative, since each corner it travels across contributes $-\frac{1}{p}$ but gives rise to another edge of δ_i, contributing at least $\frac{1}{p}$ to its degree. Since δ_i is a path beginning at a label 3, and ending at p, and is nonnegative, we have that

$$\deg(\partial f) = \sum_{i=1}^{m} \frac{p-3}{p} + \sum_{i=1}^{m} \deg(\delta_i) \geq m\left(\frac{p-3}{p} + \frac{3}{p}\right) = m \geq m(f),$$

giving the desired result. □

We will need the following Lemma from elementary graph theory, we include a proof for the reader's convenience.

Lemma 3.0.4. *Let T be a nonempty finite tree with a bipartition $V(T) = \mathcal{B} \amalg \mathcal{W}$. Let $w = |\mathcal{W}|$ and $b = |\mathcal{B}|$. Suppose that the valence of each $v \in \mathcal{B}$ is $k \geq 2$ (so no $v \in \mathcal{B}$ is a leaf). Then $(k-1)b + 1 = w$.*

Proof. We will refer to the vertices of $\mathcal{B}$ and $\mathcal{W}$ as "black" and "white", respectively. We induct on b. If $b = 0$ then T is just a single white vertex, and the statement holds; if $b = 1$ then T is a star with one black vertex and k white vertices, and the statement holds. Suppose the statement is true for all trees satisfying the hypotheses with fewer black vertices than T. Let v be a black vertex of T. Consider the forest T' obtained by deleting v from T (and removing all edges which were incident to it). The number of components of T' is k, since T is a tree and the valence of v is k. Label these components $T_1, T_2, \ldots, T_k$ and define b_i and w_i accordingly. Clearly $b = 1 + \sum b_i$ and $w = \sum w_i$. Each black vertex of T_i still has valence k, and so the inductive hypothesis applies to say that $b_i = \dfrac{w_i - 1}{k-1}$, $1 \leq i \leq k$. Thus

$$\begin{aligned} b &= 1 + \sum_{i=1}^{k} b_i, \\ &= 1 + \sum_{i=1}^{k} \frac{w_i - 1}{k-1}, \\ &= 1 + \frac{1}{k-1}\left(-k + \sum_{i=1}^{k} w_i\right), \\ &= \frac{(k-1) + (-k+w)}{k-1}, \\ &= \frac{w-1}{k-1}, \end{aligned}$$

as desired. $\square$

The following lemma says that the index can identify a closed cluster.

Lemma 3.0.5. *Suppose σ is a Scharlemann cycle of G_Q inside of a positive, graphically closed disk in $\widehat{Q}$. Let Δ be the smallest graphically closed disk containing σ. Then*

$$I(\Delta) = -1 + \sum_{f \in \mathcal{F}} (I(f) + \chi(f) - m(f)).$$

Thus Δ is a closed cluster if and only if $I(\Delta) = -1$.

Intuitively, the index of a disk Δ satisfying the hypotheses of Lemma 3.0.5 captures each time a face in Δ fails to have equality hold in Inequality 3.2; $I(\Delta)$ is -1 plus each of these failures. If the inequality never fails to be equality, then every non-Scharlemann-cycle face is two-cornered and Δ is a closed cluster. Another important example is when $I(\Delta) = 0$. Then only one (disk) face f in Δ fails to have equality hold in Inequality 3.2, and this face has $I(f) = m(f)$. In this situation, we call Δ an *almost closed cluster* with unique *high-index* face f. See Figure 3.2 for an example. Almost closed clusters will play a vital role in the proof of Theorem 1.2.3. All other non-Scharlemann-cycle faces in the cluster will be called *low-index*.

Proof of Lemma 3.0.5. Note first that any face $f \in \mathfrak{F}$ is a disk, because if it were not, either $\widehat{Q}$ would have nontrivial topology or Δ would not be a disk. Note also that as Δ is graphically closed and lies in a positive, graphically closed disk, Δ is positive. We thus have by Lemma 3.0.3 that each non-Scharlemann-cycle face f of Δ has $I(f) \geq m(f) - 1$ and equality holds if and only if f is a two-cornered face. Suppose Δ contains m Scharlemann cycles and n non-Scharlemann-cycle faces. Let Δ^* be the dual graph to Δ. That is, each face of Δ gives rise to a vertex of Δ^*, and two such vertices are adjacent via an edge in Δ^* if and only if the corresponding faces were adjacent in Δ. Then if we color the Scharlemann cycles black and the non-Scharlemann-cycle faces white (and Δ^* accordingly), Lemma 3.0.4 applies to Δ^* to say that $n = m(|\sigma| - 1) + 1$. Note additionally that we may count the number $E = E_0(\Delta)$ of interior edges of Δ in two ways: using the Scharlemann cycles to say

$$E = |\sigma| \cdot m,$$

while using the non-Scharlemann-cycle faces to say

$$E = \sum_{f \in \mathcal{F}} m(f).$$

Hence we find that

$$\begin{aligned}
I(\Delta) &= \sum_{f \subseteq \Delta} I(f) \\
&= -m + \sum_{f \in \mathcal{F}} I(f) \\
&= -m + \left(m \cdot |\sigma| - \sum_{f \in \mathcal{F}} m(f) \right) + \sum_{f \in \mathcal{F}} I(f) \\
&= m(|\sigma| - 1) + \sum_{f \in \mathcal{F}} (I(f) + m(f)) \\
&= (-1 + n) + \sum_{f \in \mathcal{F}} (I(f) + m(f)) \\
&= -1 + \sum_{f \in \mathcal{F}} (I(f) + \chi(f) - m(f)),
\end{aligned}$$

as desired. The final equality holds because each $f \in \mathcal{F}$ is a disk. The final sum is zero if and only if Δ is a closed cluster, since each f realizes the minimum value $I(f) = m(f) - 1$ and hence is a closed cluster by Lemma 3.0.3. □

Perhaps the most valuable ability of the index is in finding closed clusters and characterizing when they do not exist in non-positive index, positive, graphically closed disks.

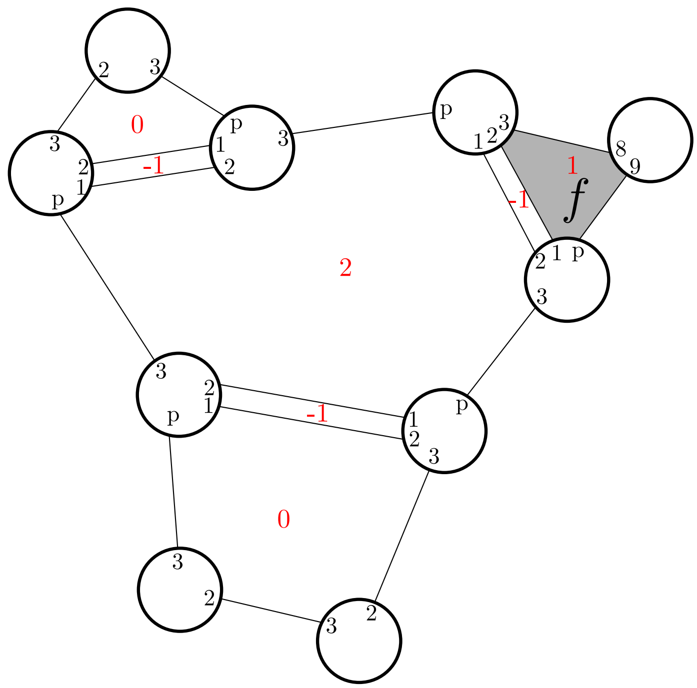


Figure 3.2: An almost closed cluster. Here the red numbers indicate the index of a face, and the face f is the unique high-index face.

Theorem 3.0.6. *Let E be a positive, graphically closed subsurface of $\widehat{Q}$. If $I(E) < 0$, then E contains a closed cluster. If $I(E) = 0$ and E does not contain a closed cluster, then for every face f in E exactly one of the following holds:*

1. *$I(f) = m(f) = 0$,*

2. *$I(f) = -1$, and thus f is a Scharlemann cycle in an almost closed cluster, or*

3. *f is a non-Scharlemann-cycle face in an almost closed cluster D, and either*

(a) $I(f) = m(f) - 1$*, making* f *a low-index two-cornered face, or*

(b) $m(f) = I(f) > 0$*, making* f *the unique high-index face of* D*.*

Proof. Let Δ be the smallest graphically closed surface containing all Scharlemann cycles of E (if E does not contain any Scharlemann cycles, then let $\Delta = \emptyset$; if there are many Scharlemann cycles, Δ may be disconnected). Then each face f of $E \setminus \Delta$ is not a Scharlemann cycle, nor is it adjacent to one. Hence $I(f) \geq 0$ for all such faces f. Clearly, then,

$$I(F \setminus \Delta) = I(F) - I(\Delta) = \sum_{f \subseteq F \setminus \Delta} I(f) \geq 0.$$

Hence $I(\Delta) \leq I(F) \leq 0$ (the index of the empty set is defined to be zero). If $f \in F \setminus \Delta$ had positive index, or if any component of Δ had positive index, then there would be another component of Δ with negative index, which by Lemma 3.0.5 would be a closed cluster. This gives all the desired statements. $\square$

Note that we are working in the general case (of the Cabling Conjecture), so a closed cluster cannot arise in $\widehat{Q}$. As we will see, there is a natural choice of positive, index zero, graphically closed disk in which we make all of our arguments. To conclude this Chapter, we will determine some constraints on the labeling of all non-Scharlemann-cycle faces such a disk. Generically, faces of positive, index 0, graphically closed disks are index 0 themselves. When they are not, we obtain a great deal of information about their labeling. We will say a *two-cornered path* in the boundary of a face f consists only of the corners $\langle p, 1\rangle$ and $\langle 2, 3\rangle$.

Lemma 3.0.7. *Let f be a face of a positive, index 0, graphically closed disk. Orient ∂f clockwise. If there is a subpath δ of ∂f with* $\deg(\delta) \geq 1$ *not beginning at the labels 1 or 2 and not ending at the labels 1, 2, or p, and δ contains no $(1,2)$ edges, then $I(f) = 0$. If $I(f) > 0$ then $\partial f = \gamma \cup \delta$ where δ is a path from 3 to p adjacent to no Scharlemann cycles, and γ is a two-cornered path from p to 3 adjacent to all $m(f)$ Scharlemann cycles. Further, we have one of the following.*

1. *f is a low-index two-cornered face in an almost closed cluster, and* $\deg(\delta) = \frac{3}{p}$ *and* $\deg(\gamma) = m(f) - \frac{3}{p}$*, or*

2. *f is the unique high-index face of an almost closed cluster and one of:*

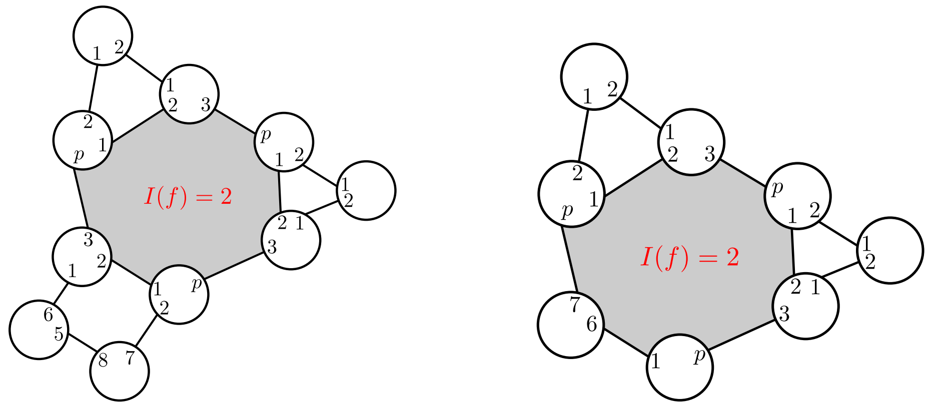
   (a) $\deg(\delta) = \frac{3}{p}$ *and* $\deg(\gamma) = 1 + m(f) - \frac{3}{p}$*, so that one $(1,2)$-edge of f is not incident to a Scharlemann cycle, making f a high-index two-cornered face, or*

   (b) $\deg(\delta) = 1 + \frac{3}{p}$ *and* $\deg(\gamma) = m(f) - \frac{3}{p}$*, so that all $(1,2)$-edges of f are incident to Scharlemann cycles, and f is not two-cornered.*

Examples of each of the two types of high-index faces are shown in Figure 3.3.

Proof. Consider a decomposition of ∂f as

$$\partial f = \bigcup_{i=1}^{k} \gamma_i \cup \bigcup_{i=1}^{k} \delta_i,$$



(a) A high-index two-cornered face (type (a)).

(b) A high-index, non-two-cornered face (type (b)).

Figure 3.3: The two types of high-index faces. Both of these examples have $I(f) = m(f) = 2$.

where each γ_i consists of the maximal two-cornered path containing a $(1, 2)$-edge of f, and each δ_i does not contain any $(1, 2)$-edges. This is similar to the decomposition of ∂f as in Lemma 3.0.3, but now $k \leq m(f) + 1$ by Theorem 3.0.6. If f is a two-cornered face, then $\gamma_1 = \partial f$, and the result holds. If $\delta_1 = \partial f$, then $I(f) = 0$. Assume then that f is the high-index, non-two-cornered face of an almost closed cluster. Note that by the maximality of each γ_i, each δ_i has at least one corner which is not $\langle p, 1\rangle$ or $\langle 2, 3\rangle$, and so $\deg(\delta_i) \geq 1 + \frac{3}{p}$. If $k = 1$, then the proof is complete, so assume $k \geq 2$. Then

$$\sum_{i=1}^{k} \deg(\gamma_i) = \left(m(f) - \frac{3k}{p}\right) \text{ or } \left(1 + m(f) - \frac{3k}{p}\right),$$

depending on if there is a $(1, 2)$-edge of f not incident to a Scharlemann cycle.

Thus

$$\begin{aligned}
\deg(\partial f) &\geq \sum_{i=1}^{k} \deg(\gamma_i) + \sum_{i=1}^{k} \deg(\delta_i) \\
&\geq \left(m(f) - \frac{3k}{p} \right) + k \left(1 + \frac{3}{p} \right) \\
&\geq m(f) + k,
\end{aligned}$$

hence $I(F) > m(f)$, contradicting Theorem 3.0.6. Hence $k = 1$, giving the desired conclusion. $\square$

Chapter 4

The existence and structure of great webs

In the resolution of the knot complement problem [9] a subgraph called a great web is defined, and it is shown that sufficiently large great webs contain Scharlemann cycles. This is integral to the proof, as in that context the two surfaces are both spheres in distinct copies of S^3, so the existence of a lens space summand in either is impossible.

Definition 4.0.2. A **great k-web** is a subgraph Λ of G_Q satisfying the following properties:

1. The graph Λ lies in a disk D_Λ of $\widehat{Q}$ such that every vertex in D_Λ is the same sign and is a vertex of Λ, and
2. With precisely k total exceptions, every edge endpoint incident to any vertex of Λ is an endpoint of an edge of Λ. These exceptional edges are called **λ-ghosts**, where λ is the label of the edge at the vertex in Λ.

See Figures 4.1 and 5.1a for examples. As previously mentioned, we will not distinguish Λ from D_Λ. Let us say that two properly embedded, compact surfaces with boundary R and S in a knot exterior *intersect essentially* if one can construct graphs G_R and G_S analogous to the setup of Chapter 2 which satisfy all the non-triviality requirements described, see [6] for the most general setup. The following is the main combinatorial result of [9], where a notion of *representing a type* is defined and used to arrive at the existence of great webs.

Theorem 4.0.1 (Gordon-Luecke, [9])**.** *Suppose Q is a connected, properly embedded planar surface, P is a (possibly disconnected) properly embedded surface with no closed components, and Q and P intersect essentially. Let $p = |\partial P|$, and assume that $\Delta > 1 - \frac{\chi(\widehat{P})}{p}$, where Δ is the geometric intersection number between the slopes given by ∂Q and ∂P. Then either G_P represents all types or G_Q contains a great $(p - \chi(\widehat{P}))$-web.*

It should be noted that the statement of this result in [9] does not *a priori* allow P to be disconnected. This must be carefully checked; we refer the interested reader to Appendix 1 for further discussion. It follows from [33] and [9] that G_P representing all types implies that the manifold containing $\widehat{Q}$ contains a summand with nontrivial torsion in its first homology. In our context, this manifold is S^3, so we find that G_Q must contain a great $(p-2)$-web in the general case, or a great $(p-4)$-web in the three summands case. Throughout the remainder of the dissertation, all great webs will be assumed to be one of these two options, depending on the relevant case, and we will suppress the integer k from the notation.

A very well-known corollary to Theorem 4.0.1 is the existence of a lens space summand in the surgered manifold. This is a consequence of the fact

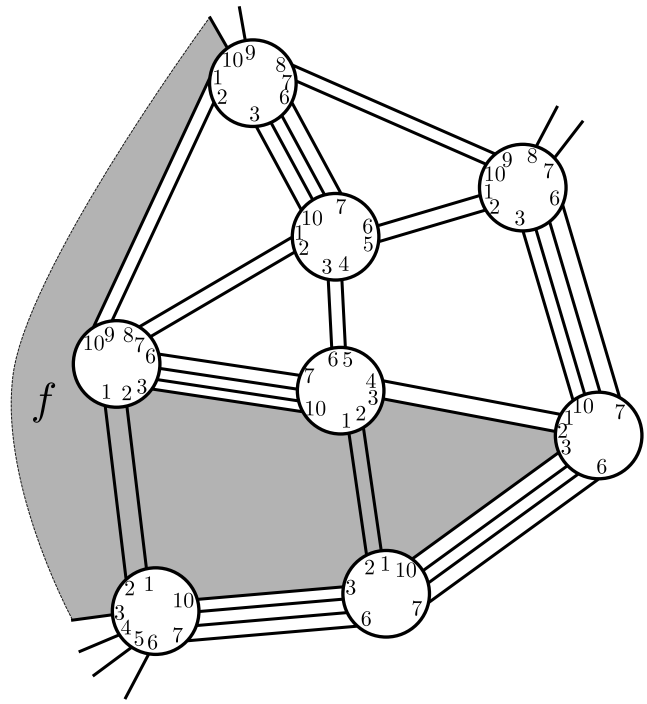


Figure 4.1: A great 8-web, for $p = 10$. The shaded region is the (formal) closed cluster $\overline{C}$.

that a great web must contain a Scharlemann cycle. In fact, much more can be said about the structure of a great web.

Lemma 4.0.2. *Let Λ be a great web with positive vertices in the general case.*

1. *There is exactly one λ-ghost for each regular label $\lambda \in L$. The ghosts of Λ occur in counterclockwise order around $\partial\Lambda$, in the sense that between the λ- and $(\lambda+1)$-ghosts $\partial\Lambda$ consists of edges labeled $(\lambda+1, \lambda)$ (possibly zero such edges, in which case the $(\lambda+1)$-ghost immediately follows the λ-ghost).*

2. *Construct a formal disk face f by connecting the 3-ghost to the p-ghost. Then $\overline{\Lambda} = \Lambda \cup f$ contains a closed cluster $\overline{C}$.*

See Figure 4.1. The edges labeled $(\lambda+1, \lambda)$ that occur between the λ- and $(\lambda+1)$-ghosts are called *nice* boundary edges. Note that the second condition completely describes the boundary of Λ between the 3- and the p-ghost. Since Λ cannot contain a closed cluster the face f is a two-cornered face in $\overline{C}$, and in fact all $(1,2)$-edges of $\partial\Lambda$ are edges of Scharlemann cycles of $\overline{C}$, each of which is graphically closed into Λ; additionally, there is at least one such Scharlemann cycle. Before we prove Lemma 4.0.2, we state a weaker version for the three summands case. Note that something equally as strong as Lemma 4.0.2 could be proven for the three summands case, but we will not need it.

Lemma 4.0.3. *Let Λ be a great web in the three summands case.*

1. *There is exactly one λ-ghost for each $\lambda \in L$.*

2. *Λ contains a length l_1 Scharlemann cycle σ_1 and a length l_2 Scharlemann cycle σ_2.*

The proof of Lemma 4.0.3 is a subset of the proof of Lemma 4.0.2, so we will not include it.

Proof of Lemma 4.0.2. Suppose there is no λ-ghost, for some label λ. Beginning at some vertex of Λ, construct a λ-edged path, *i.e.* a path which may be oriented such that the tail of each edge is labeled λ. Since there is no λ-ghost, this path cannot leave Λ. Hence it encloses a disk which is a great λ-cycle. It is clear that great λ-cycles have index -1, since every corner and every edge counts λ in the degree, which must hence be zero. Thus this λ-cycle must in fact be a Scharlemann cycle, as otherwise it is graphically closed and contains a closed cluster. Thus if $\lambda \in L$, there must be a λ-ghost, and hence there must be exactly one of each.

Suppose λ and $\lambda + 1$ are regular labels, and that their ghosts do not occur at the same vertex of Λ. Beginning at the vertex v containing the λ-ghost, consider the boundary edges in the counterclockwise direction. The first boundary edge has the label $\lambda + 1$ at v. Let γ be the maximal path along $\partial\Lambda$ beginning at v such that γ consists of $(\lambda + 1, \lambda)$-edges. If γ continues all the way to the $(\lambda + 1)$-ghost, then condition (1) is satisfied. Otherwise, there is some edge in $\partial\Lambda$, immediately beyond γ, which is a $(\lambda + 1, \alpha)$-edge e for $\alpha \neq \lambda$. Let v' be the vertex where e has a label α. Since there is precisely one λ-ghost, there is a λ-edged path in Λ from v' to v. This path cuts off a disk $D \subseteq \Lambda$ with the property that every vertex of D has its $(\lambda+1)$-edge contained in D. Thus D contains a $(\lambda + 1)$-cycle, a contradiction. Hence γ must extend all the way to the $(\lambda + 1)$-ghost as desired.

Construct f and $\overline{\Lambda}$ as stated. Then clearly $\overline{\Lambda}$ is positive and graphically closed, as the edges along its boundary are labeled entirely with regular labels. Condition (2) is hence proven by showing that $\deg(\partial\overline{\Lambda}) = 0$ and appealing to Theorem 3.0.6. This is seen by considering the regular value $\frac{3}{2}$. Decompose the degree count into vertex-edge pairs; each vertex counts $\frac{3}{2}$ once negatively, and then the subsequent edge counts $\frac{3}{2}$ positively. $\square$

Throughout the remainder of the dissertation, suppose Λ is a great web in G_Q for the knot K, and (after possibly reversing the orientation of $\widehat{Q}$) the vertices of Λ are positive.

4.1 Placements of great webs in the general case

Throughout this subsection, suppose we are in the general case. Note then that the closed cluster $\overline{C}$ of Lemma 4.0.2 has $I(\overline{C}) = -1$, so if Λ' is defined by

$$\Lambda' = \Lambda \setminus \overline{C},$$

then $I(\Lambda') = 0$ and Λ' is clearly positive, so Theorem 3.0.6 applies to this disk. Λ' will be a main focus of our arguments.

Let Γ be the subgraph of G_P consisting of those edges which in G_Q belong to Λ. Let $D_1, \ldots, D_m$ be the complementary regions of the subgraph of Γ consisting of all edges between u_1 and u_2. These regions will be called the *Scharlemann subregions* of Γ; given a Scharlemann cycle σ of Λ, there is a way to glue these n regions together into the $l = |\sigma|$ complementary regions of σ, called the *Scharlemann regions* of σ. See Figures 5.1a and 5.1b for examples of Λ and Γ, respectively (disregard the captions for now).

In order to prove Theorems 1.2.1 and 1.2.3 as well as related results we will be concerned about the way the regular vertices $u_3, \ldots, u_p$ lie in these

Scharlemann subregions. To formalize this, reindex the D_i such that the first n Scharlemann subregions contain a regular vertex (and the rest are bigons in Γ). Define the *labeled placement* of Λ to be the set of sets

$$\{v(D_i) : 1 \leq i \leq n\},$$

and the *placement* of Λ will be the multiset

$$\{|v(D_i)| : 1 \leq i \leq n\}.$$

If p is small, one may use placements of webs to divide the problem into a small number of cases. For example, the proof of the $p = 4$ case in [26] divides the $p = 4$ problem into the two possible placements, namely $\{1, 1\}$ and $\{2\}$. Likewise, if $p = 6$ one possible placement would be $\{2, 2\}$. This gives a *stratification of the Cabling Conjecture*, namely into cases of the form ($p = p_0$, placement of the $p_0 - 2$ regular vertices). This may appear to give a great deal of cases, even for a value for p as small as $p = 6$ or 8, but we have the following restrictions on placements.

Lemma 4.1.1. *Let D_i be a Scharlemann subregion of Γ, for $1 \leq i \leq m$.*

1. *$v(D_i)$ consists of the same number of positive and negative vertices. Thus $|v(D_i)|$ is even.*

2. *$|v(D_i)| \leq \frac{p}{2} - 1$.*

The first condition of Lemma 4.1.1 will be proven in Chapter 5; the discussion from now until that point will be independent of this condition.

The second condition is a very common argument in the field of graphs of intersection, see for example [1]. Note that Lemma 4.1.1 provides another proof of the $p = 4$ case, because the placement $\{1, 1\}$ violates condition (1) while the placement $\{2\}$ violates condition (2). In a similar spirit, we see that Lemma 4.1.1 reduces the $p = 6$ problem to the placement $\{2, 2\}$, and the $p = 8$ problem to the placement $\{2, 2, 2\}$. To complete the proof of Theorem 1.2.3, we show the following.

Proposition 4.1.2. *The placement* $\{2, 2, \ldots, 2\}$ *cannot exist.*

The proof decomposes into two cases: $p = 6$, and $p > 6$; both of these proofs will be in Chapter 7. Note that one may use Proposition 4.1.2 to rule out the existence of webs of simultaneously arbitrarily high p and q, (related to valence and number of vertices, respectively). This is due to the fact that the proof of Proposition 4.1.2 only requires the statement that the regular vertices of G_P are only connected in pairs by edges of Γ. We state this as a Corollary.

Corollary 4.1.3. *No web gives rise to any placement which is a subgraph of a placement* $\{2, 2, \ldots, 2\}$. *There are infinite families of webs which cannot exist because of this, containing webs with simultaneously arbitrarily large* p *and* q.

Proof. We give an example of a family Λ_m^p of such webs for $m \geq 1$, $p \geq 6$ of arbitrarily large p and v, and recall $v = |V(\Lambda_m^p)| \leq \frac{q}{2}$. Fix p and m, and define $n = \frac{1}{2}(m(p-4) + p - 6)$. Let Λ_m^p be the web defined in Figure 4.2. There are vertices t, t_i for $1 \leq i \leq m$, and s_j for $1 \leq j \leq n$ which do not have all

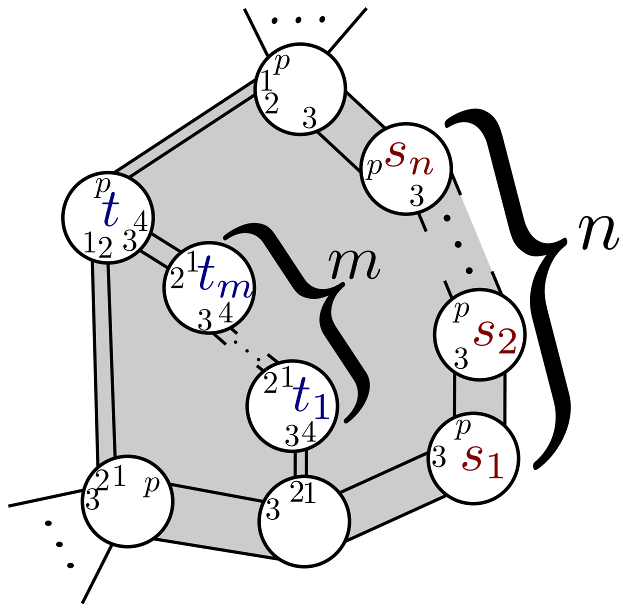


Figure 4.2: The webs Λ_m^p.

their edges drawn in Figure 4.2. These edges are added to Λ_m^p by the rule that no s_j has any of its 1- or $2-$ edges connecting to any other $s_{j'}$, $j \neq j'$. This specifies a unique web with $v = m + n + 4$ and of valence p, and the labeling of all of its edges is a subset of the placement $\{2, 2, \ldots, 2\}$, and so it is ruled out by Proposition 4.1.2. $\square$

There is nothing unique about the webs Λ_m^p, there are many families of webs which are ruled out in this way. To our knowledge, this is the first nontrivial result of its kind: [7] and [26] consider low values of p, and [14] considers low values of q. We believe that this result, while technical in nature, suggests that great webs tend to contain enough information to arrive at a contradiction. In the next section, we prove another result which provides more evidence of this fact. In fact, the author has yet to find a great web in either case which does not give rise to *some* contradiction.

4.2 Great webs are large

We next show that, roughly, great webs tend to contain a great deal of topological information, because they cannot have a large number of parallel edges.

Definition 4.2.1. Let Λ be a great web in the either the general or the three summands case. Then Λ is **small** if there exists a collection of $\frac{p}{2}$ parallel edges in Λ containing no (length two) Scharlemann cycles, otherwise Λ is **large**. Such a collection of parallel edges is called a **full quota**.

The following Proposition uses a pair of techniques which are very different from each other. The $p \equiv 0 \pmod 4$ case is an adaptation of an argument of Howie [18, Lemma 2.4]. The $p \equiv 2 \pmod 4$ case is due to Cameron Gordon and John Luecke; it is an adaptation of an argument of Thompson [40]. The author is indebted to Professors Gordon and Luecke for their generosity in sharing their knowledge, which occurs here with their permission.

Proposition 4.2.1. *Let Λ be a great web in either case. Then Λ is large.*

Proof. It is an immediate consequence of [18, Lemma 2.4] that Λ must be large in the three summands case. In fact, something stronger holds: the bigons given by a full quota not need be located entirely in a single parallelism class of edges. We will need this rigidity in the general case.

Suppose first that Λ is a great web in the general case and $p \equiv 0 \pmod 4$. Define the 1-handle H_i to be the component of the filling solid torus contained between the fat vertices u_i and u_{i+1} which contains no other u_j. Let $z \in P$ and define g_i to be an element of $\pi_1(S^3_r(K), z)$ represented by a curve which travels from z through $\widehat{P}$ to the vertex u_i, over ∂H_i running parallel to the arcs of $Q \cap H_i$ to u_{i+1}, then back to z through $\widehat{P}$. Note that $\mu = g_1 g_2 \cdots g_p$ is the image of a meridian of K in $\pi_1(S^3_r(K))$. Thus $\pi_1(S^3_r(K)) / \langle\langle \mu \rangle\rangle$ is trivial. However, if Λ is small, then Λ contains a full quota, which in turn contains bigons B_i which give rise to relations $g_{a+i} g_{a-i} = 1$, for $i = 1, \dots, \frac{p-2}{2}$. Thus we see that $\mu = g_a w g_b w^{-1}$, for $b = a + \frac{p}{2}$.

Note that g_a and g_b are elements of a factor of $\pi_1(S^3_r(K)) = \mathbb{Z}/l\mathbb{Z} * G$. Since $p \equiv 0 \pmod 4$, these are elements of the same factor, and hence the normal closure $\langle\langle g_a w g_b w^{-1} \rangle\rangle$ has nontrivial index in $\pi_1(S^3_r(K))$. This is a contradiction.

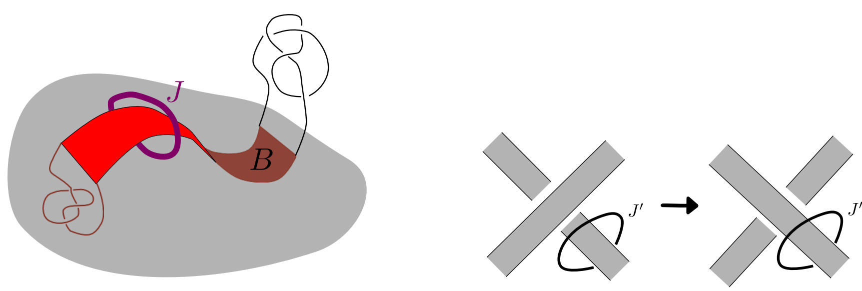
Hence we may suppose that we are in the general case with $p \equiv 2 \pmod 4$. Let K' denote the dual knot of K in $S^3_r(K)$. Then if Λ is small, there is a disk B on Q giving the parallelism between the outermost edges of the full quota. This B is a band demonstrating K' as a band sum in $S^3_r(K)$, see Figure 4.3a. Note that the assumption $p \equiv 2 \pmod 4$ is important, because the tangles (corresponding to cores of H_a and H_b) must lie on opposite sides of $\widehat{P}$ in order for us to say that they do not link. Let J be a *meridian* of the band B, *i.e.* perform the following construction: fix an identification $N(B) \cong B \times [-1,1]$ so that $B_0 = B \times \{0\}$ is a slightly wider band than B, let α be a properly embedded arc in B_0 connecting the two components of $B_0 \cap \partial E(K)$ where $E(K)$ denotes the exterior of K (the 'long' edges of the band B_0), and define

$$J = (\alpha \times \{\pm 1\}) \cup (\partial\alpha \times [-1,1]) .$$

Let Y be the result of performing 0-framed Dehn surgery on J. Then since J was an unknot in $S^3_r(K)$ (for example, it bounded the disk $\alpha \times [-1,1]$), Y contains an $S^1 \times S^2$ connected summand. In fact, since J', the dual knot to J in Y, bounds a disk disjoint from K' (thought of here as living in Y), we may use this disk to isotope the band B and K' in Y to see that it is an unknotted band which doesn't link either summand of the band sum. See Figure 4.3b. That is, in Y, K' is a connected sum, and additionally lies entirely within in a connected summand distinct from the $S^1 \times S^2$ connected summand. Perform the dual surgery on K' in $S^3_r(K)$ to obtain a knot in S^3 (that is, J, thought of here as lying in S^3) with a surgery containing an $S^1 \times S^2$ connected summand (that is, the manifold obtained from Y by performing the dual surgery on K'). To summarize, we have the following commutative diagram of manifolds and surgeries on these manifolds:

$$\begin{tikzcd}
S^3 \arrow[r, bend left, "K"] \arrow[d, "J"'] & S^3_r(K) \arrow[l, bend left, "K'"] \arrow[d, "J"] \\
S^3_0(J) \arrow[r, bend left, "K"] & Y \arrow[l, bend left, "K'"]
\end{tikzcd}$$

Then by the resolution of the Poenaru Conjecture [4], J is an unknot in S^3. Then for any disk D in S^3 with $J = \partial D$, we must have $K \cap D \neq \emptyset$, as otherwise we could perform in $S^3_r(K)$ the same debanding as we did in Y to see that K' is a nontrivial connected sum, and this would contradict the fact that K is hyperbolic. Thus no disk bounded by J is disjoint from K, so K lies in the solid torus $V = \overline{S^3 \setminus N(J)}$, and does not lie inside a ball in V. Further, since J was an unknot in $S^3_r(K)$ by construction, the manifold $V_r(K)$ resulting from surgery on K in V is reducible. Hence K is a cable by [38], contradicting the assumption that K is hyperbolic. $\square$



(a) K' is a band sum in $S^3_r(K)$, and J is a meridian of the band B.

(b) Passing pieces of the band and the knot K' over the disk bounded by J' (not pictured).

Figure 4.3

The group elements g_i for $i = 1, \ldots, g_p$ will be of great importance for the proof of Theorem 1.2.3, so we conclude this chapter with a definition to codify this. Given a face f of G_Q, we will say f *is of the form* $g_{i_1} g_{i_2} \ldots g_{i_k}$ if the corners of f are $\langle i_1, i_1+1 \rangle$, followed by $\langle i_2, i_2+1 \rangle$, $\ldots$, followed by $\langle i_k, i_k+1 \rangle$. This depends on the starting place for reading off the corners of f, so the form of f is well-defined up to cyclically permuting the group elements. In addition, we will say f is of the form $g_{i_1}g_{i_2}\ldots g_{i_k}*$ to indicate that f is of the form $g_{i_1}g_{i_2}\ldots g_{i_k}g_{i_{k+1}}\ldots g_{i_n}$ for some as-of-yet determined g_{i_j}'s, $k+1 \leq j \leq n$.

Chapter 5

Divisibility of great webs

5.1 The Divisibility Proposition and Theorem 1.2.1

Our main result of this Chapter is the following. Let $v = |V(\Lambda)|$, for Λ a great web in either case. Recall that in the general case l is the order of the fundamental group of a lens space summand; similarly for l_1 and l_2 in the three summands case.

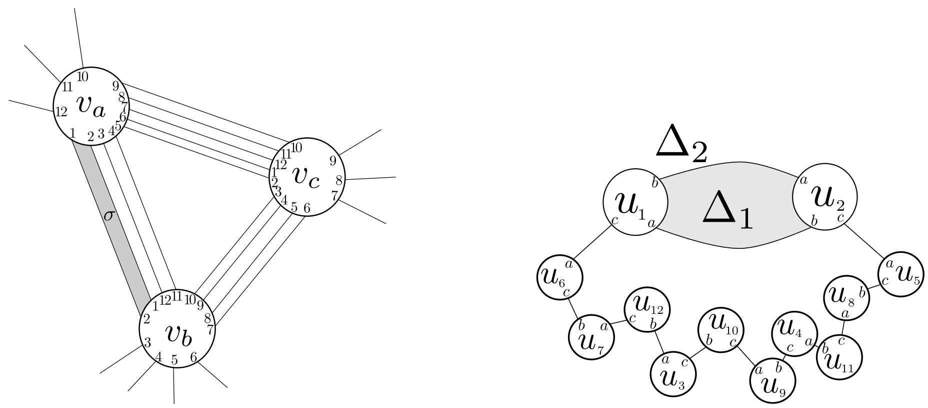
Proposition 5.1.1. *In the general case, l divides v; in the three summands case, l_i divides v for $i = 1, 2$.*

Recall the definition of Γ from Chapter 4: Γ is the subgraph of G_P consisting of all the vertices of G_P along with those edges of G_P which in G_Q are the edges of Λ:

$$V(\Gamma) = V(G_P), \text{ and}$$

$$E(\Gamma) = E(\Lambda).$$

Note that $E(\Gamma)$ does not contain (those edges of G_P corresponding to) ghost edges of Λ. Let the *regular* and *Scharlemann* vertices of Γ be those whose label corresponds to regular and Scharlemann labels, respectively. Then by Lemma 4.0.2 or 4.0.3 the valence of a regular vertex of Γ is $v - 1$, and the



(a) A great 10-web Λ on G_Q for $p = 12$ in the general case

(b) The graph Γ on G_P for the great web Λ pictured in Figure 5.1a. Here, $n_1(\Delta_1) = 0$, while $n_1(\Delta_2) = 1$, because of the location of the c-edge at u_1.

Figure 5.1: The graphs Λ and Γ. The fact that all the regular vertices of Γ are in a single Scharlemann region (namely Δ_2) is a phenomenon of p being quite small, and does not occur in general.

valence of a Scharlemann vertex of Γ is v. We will refer to the edges in $E(\Gamma)$ as Γ-*edges*. It should be noted that no Γ-edges are adjacent to one another at any vertex when considered as edges of G_P; in particular, both labels of each Γ-edge correspond to vertices of Λ, and are hence the same sign, so because $\widehat{Q}$ is separating there must at the very least be an edge with a label corresponding to a vertex of opposite sign between them. In fact, there are an odd number of edges of G_P between any pair of Γ-edges. Hence when we say that any two Γ-edges are consecutive, we are referring to the subgraph Γ.

Proof of Proposition 5.1.1. Let E denote the Γ-edges which are incident to two Scharlemann vertices. Thus the edges of E are incident to the vertices

u_1 and u_2 in the general case, and one of the pairs u_1 and u_2 or u_x and u_{x+1} in the three summands case. For clarity we will restrict our attention to the edges connecting the vertices u_1 and u_2, and show that l or l_1 divides v; the proof of l_2 dividing v is the same. Additionally, for clarity we suppress the difference between the notation in the two cases, namely σ and σ_1, l and l_1, and P and P_1 and just use the notation from the general case. Recall that we defined the union of the *Scharlemann subregions* of Γ to be

$$\widehat{P} \ \setminus \ (u_1 \cup u_2 \cup \bigcup_{e \in E} e),$$

consisting of disks $D_1, \ldots, D_m$. Each disk D_j has a collection of Γ-edges interior to D_j and incident to u_1, call the number of such edges $n_1(D_j)$. Similarly define $n_2(D_j)$ using u_2. We first claim that $n_1(D_j) = n_2(D_j)$ for each j. To see this, note that all Γ-edges are negative edges of G_P, so Γ is a bipartite graph, and hence we may count its edges interior to D_j by counting edges incident to positive vertices or by counting edges incident to negative vertices. Since u_1 and u_2 are of opposite sign, this gives:

$$n_1(D_j) \equiv \left|E\big(\Gamma \cap \mathrm{Int}(D_j)\big)\right| \pmod{v-1}, \text{ and}$$
$$n_2(D_j) \equiv \left|E\big(\Gamma \cap \mathrm{Int}(D_j)\big)\right| \pmod{v-1}.$$

Hence we have

$$n_1(D_j) \equiv n_2(D_j) \pmod{v-1}.$$

Since there is at least one Scharlemann cycle (σ, say) in Λ on the labels $\{1, 2\}$, we know that $0 \leq n_i(D_j) < v - 1$ for $i = 1,\ 2$. Hence

$$n_1(D_j) = n_2(D_j)$$

for each $j \in \{1, \ldots, m\}$.

Now consider just those edges of E pertaining to σ; these edges cut $\widehat{P}$ into *Scharlemann regions* $\Delta_1, \ldots, \Delta_l$. Each Δ_k is a union of some number of subregions of Γ along some Γ-edges between u_1 and u_2. If $n_1(\Delta_k)$ and $n_2(\Delta_k)$ are defined analogously, we see that $n_1(\Delta_k) = n_2(\Delta_k)$ for each k: if Δ_k contains m_k subregions $D_1^k, D_2^k \ldots, D_{m_k}^k$, then

$$\begin{aligned} n_1(\Delta_k) &= (m_k - 1) + \sum_{j=1}^{m_k} n_1(D_j^k) \\ &= (m_k - 1) + \sum_{j=1}^{m_k} n_2(D_j^k) \\ &= n_2(\Delta_k). \end{aligned}$$

Note that we could have proven $n_1(\Delta_k) = n_2(\Delta_k)$ for Scharlemann regions from first principles as we did $n_1(D_j) = n_2(D_j)$ for subregions of Γ. We provided the proof in this manner because it is slightly stronger, and it will make the language of placements more clear.

Consider the torus T_{Sch} obtained by tubing $\widehat{P}$ along the boundary of the 1-handle H_1, where (recall) H_1 is the portion of the filling solid torus between u_1 and u_2 not containing fat vertices of G_P in its interior. On T_{Sch}, $\partial\sigma$ is an $\frac{l}{s}$-curve, corresponding to obtaining the lens space summand $L(l, s)$. The Δ_k may be ordered such that the interior Γ-edges of Δ_k at u_1 run over H_1 to the interior Γ-edges of Δ_{k+1} at u_2, where by $k+1$ we mean $k+1 \pmod{l}$. This follows from the fact that an interior Γ-edge with a label λ at u_1 corresponds to an edge labeled 1 at a vertex v_λ in Λ; since there are no 1- or 2-ghosts, the edge labeled 2 at v_λ is an edge of Λ, and hence the λ-edge at u_2 is a Γ-edge. Since the converse is also true, we see that $n_1(\Delta_k) = n_2(\Delta_{k+1})$, and thus $n_i(\Delta_k) = n$ is the same for all k and for $i = 1, 2$; this implies that $(1+n)\cdot l = v$ by counting the valence of u_1 in Γ, and thus l divides v. □

Proof of Theorem 1.2.1. Recall that in the three summands case, l_1 and l_2 are relatively prime and that $|r| = l_1 \cdot l_2$. Both of these statements are due to the fact that $H_1(S^3_r(K)) \cong \mathbb{Z}/r\mathbb{Z}$. We thus see that r divides v (possibly trivially, *i.e.* $r = \pm v$). Since the vertices of Λ are all positive, $v \leq \frac{q}{2} \leq b$, and the result follows. This finishes the Two Summands Conjecture for knots with $b \leq 5$ because one has $2 \leq l_1 < l_2$ (without loss of generality). □

5.2 Corollaries to the Divisibility Proposition

To conclude this chapter, we give three corollaries with a more technical flavor to them that additionally result from Proposition 5.1.1. First, we reprove the Cabling Conjecture for 2- and 3-bridge knots. This was originally done in [14]. This is accomplished by seeing that the number of vertices in a great web cannot be prime, something which is forced by those low bridge numbers.

Corollary 5.2.1. *In the general case, l cannot equal v, so v cannot be prime. Hence the Cabling Conjecture holds for knots with $b \leq 3$, and modulo the case $l = 2$, $v = 4$, for knots with $b \leq 5$.*

Proof. Suppose Λ is a great web in the general case with $l = v$ (if v is prime this is forced by Proposition 5.1.1), so that Λ consists of a Scharlemann cycle

σ on the labels $(1, 2)$ and a collection of edges which connect two vertices in σ. Note then that there is only one Scharlemann cycle in Λ. Suppose every edge of Λ is parallel to an edge of σ. Pick any vertex v_0 of σ not containing the λ-ghost of Λ, for $\lambda = \frac{p+2}{2}$. Then the λ-edge of v_0 is parallel to one of the two edges of σ incident to v_0, and either way gives rise to a full quota. Thus suppose not all edges of Λ are parallel to edges of σ, and let e be such an edge. Then there is an arc γ which is a subarc of $\partial\sigma$ with $\partial\gamma = \partial e$, such that $\gamma \cup e$ bounds a disk D in Λ. By the definition of e, γ contains an interior vertex. By considering edges e' interior to D which are not parallel to edges of γ, one finds that there must be some interior vertex v_0 of γ which is incident only to the two vertices adjacent to it along γ. Such a v_0 necessarily gives rise to a full quota.

In either situation, we see that Λ is small, contradicting Proposition 4.2.1. $\square$

By slightly strengthening the proof of Proposition 5.1.1 we are able to prove the first part of Lemma 4.1.1.

Proof of Lemma 4.1.1, part 1. Returning to the proof of Proposition 5.1.1 (and its notation), when proving $n_1(D_j) = n_2(D_j)$ for a subregion D_j of Γ we only cared about counting $\left|E\big(\Gamma \cap \mathrm{Int}(D_j)\big)\right| \pmod{v-1}$. If we are more careful in our counting we get the following (suppose without loss of generality u_1 is positive):

$$\left|E\big(\Gamma \cap \mathrm{Int}(D_j)\big)\right| = (v-1)\cdot|\{\text{positive regular vertices in } D_j\}| + n_1(D_j), \text{ and}$$
$$\left|E\big(\Gamma \cap \mathrm{Int}(D_j)\big)\right| = (v-1)\cdot|\{\text{negative regular vertices in } D_j\}| + n_2(D_j).$$

By first removing $n_1(D_j) = n_2(D_j)$ from the resulting equation, we see that in fact

$$|\{\text{positive regular vertices in } D_j\}| = |\{\text{negative regular vertices in } D_j\}|.$$

□

Finally, in [18], a subgraph called a *sandwiched disk* is defined and used to establish the bound $|r| \leq \frac{1}{4}b(b+2)$ in the three summands case. A sandwiched disk in G_Q has boundary the union of two arcs, each of which is a subarc of a Scharlemann cycle, with the additional property that there are no vertices in the interior of the disk. It is shown that a sandwiched disk gives rise to a contradiction through a careful analysis of its interior structure. Since Λ needs to contain a Scharlemann cycle of each length l_i and all its vertices are the same sign, it appears to be a good place to search for sandwiched disks. The following result says that when we restrict our attention to the great web, a disk whose boundary is sandwiched cannot exist, independent of the number of vertices in its interior.

Corollary 5.2.2. *In the three summands case, no Scharlemann cycle of Λ intersects another in more than one vertex.*

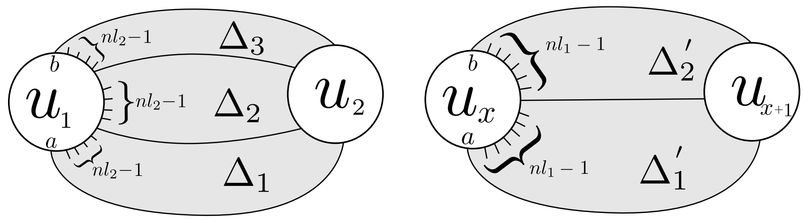


Figure 5.2: The Γ-edges between a and b are counted for Corollary 5.2.2. In the Figure, $k_1 = 2$ and $k_2 = 1$.

Proof. Suppose that $\sigma_1 \cap \sigma_2$ contains at least $v_a \cup v_b$, for vertices v_a and v_b of Λ. Note that we know that the two Scharlemann cycles are different lengths, by Lemma 2.1.1, part 3. By Proposition 5.1.1, we see that $v = nl_1l_2$. Then we see that the number n_1 of Γ-edges incident to u_1 lying interior to a single Scharlemann region of G_P^1 corresponding to σ_1 is the same for each region. Thus $n_1 = \frac{nl_1l_2-l_1}{l_1} = nl_2 - 1$. Likewise, the number n_x of Γ-edges incident to u_x lying interior to a single Scharlemann region of G_P^2 corresponding to σ_2 has $n_x = nl_1 - 1$. See Figure 5.2. Now as labels, $|a - b|$ is fixed, and this quantity can be counted by using the number of edge endpoints between the labels at u_1, and also at u_x, and we may set these two counts equal to one another. However, we are not interested in $|a - b|$; we will instead count just those labels of Γ-edges lying between a and b. We see that, if between a and b on G_P^i we have k_i edges of σ_i, then

$$k_1 + (k_1 + 1) \cdot (nl_2 - 1) = k_2 + (k_2 + 1) \cdot (nl_1 - 1),$$

$$k_1 + nl_2 + k_1nl_2 - k_1 - 1 = k_2 + nl_1 + k_2nl_1 - k_2 - 1,$$

$$k_1l_2 = k_2l_1.$$

Since l_1 and l_2 are relatively prime, we see that l_2 divides k_2. But k_2 is some number of Scharlemann cycle edges of σ_2 lying in G_P^2 between a particular pair of such Scharlemann cycle edges, and hence $k_2 \leq l_2 - 2$ (*i.e.* k_2 certainly doesn't count the a-edge nor the b-edge of u_x). Hence l_2 could not possibly divide k_2, a contradiction. Thus a pair of Scharlemann cycles in Λ can share at most one vertex. $\square$

Chapter 6

Positive braids satisfy the Two Summands Conjecture

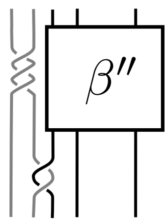
In this section we complete the proof that closures of positive braids satisfy the Two Summands Conjecture. As a result of Theorem 1.2.1, $|r| \leq b$. As discussed in the introduction, [24] established the strong condition that if a hyperbolic positive knot has a reducible Dehn surgery of slope r, then $r = 2g - 1$. Thus we must relate the bridge number of K to the genus of K, a task which may be accomplished if one restricts from positive knots to the closure of positive braids. Recall that a *braid* on n strands is built from the braid letters $\{\sigma_i^{\pm 1}\}_{i=1}^{n-1}$, where σ_i positively exchanges the ith and $(i+1)$st strands. A *positive* braid only uses the letters σ_i, and none of the ${\sigma_i}^{-1}$.

Proof of Corollary 1.2.2. . Let K be the closure of a positive braid. Then K is a positive knot, so by [24] if K has a reducible Dehn surgery of slope r, then $r = 2g - 1$, where g denotes the genus of K. Hence it suffices to show that if K has a reducible surgery with three summands, then $r \neq 2g - 1$. We may assume by Theorem 1.2.1 that the bridge number b of K is at least 6, and that $|r| \leq b$. Let β be a positive braid representing K subject to the constraint that the strand number n of β is minimized among positive braids representing K. By [39], the surface obtained from $\hat{\beta}$ by taking n disks and adding positive bands for each braid letter σ_i is fibered, and hence minimal genus. Thus if $e = e(\beta)$ denotes the total exponent sum, we have $2g-1 = e-n$. Let $e_i = e_i(\beta)$ denote the exponent sum of σ_i in β. Now if any $e_i = 1$, then there exists a sphere passing only through this crossing demonstrating $\hat{\beta}$ as a connected sum. Since K is hyperbolic, one of the two knots is the unknot, and this allows us to destabilize β and obtain a positive braid on fewer strands representing K. Thus every σ_i occurs at least twice in β, giving $e \geq 2(n-1)$. In fact, define $e = 2n - 2 + s$ for s a non-negative integer. Noting that $\hat{\beta}$ is a bridge presentation for K, we see that

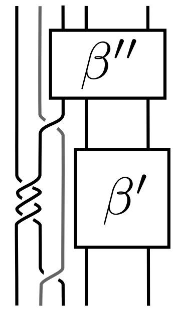
$$\begin{aligned}
2g - 1 &= e - n \\
&= 2n - 2 + s - n \\
&= n + (s - 2) \\
&\geq b + (s - 2) \\
&\geq |r| + (s - 2),
\end{aligned}$$

so if $s \geq 3$ then the proof is complete. We proceed with an analysis of possible braid words. Since $b \geq 6$, we have that σ_1, σ_2, σ_3, σ_{n-2}, and σ_{n-1} all exist and are distinct. Denote braid closure equivalence by $\sim$, that is, two braids $\beta_1 \sim \beta_2$ if and only if they are related by a sequence of braid group relations, stabilizations, and conjugations (cyclic permutations of the braid word).

Our first claim is that at least one of $e_1 \geq 3$ or $e_2 \geq 4$ must hold. Suppose $e_1 = 2$ and $e_2 \leq 3$, and note that $\beta \not\sim {\sigma_1}^2\beta'$, as then it is clear that



(a) The closure of ${\sigma_1}^3\beta''{\sigma_2}^2$ is a link.



(b) The closure of $\beta''\sigma_2{\sigma_1}^3\beta'\sigma_2$ is a link.

Figure 6.1

K is a link of at least two components. Thus it must be that

$$\beta \sim \sigma_1\beta'\sigma_1\beta'',$$

where $e_2(\beta')$ and $e_2(\beta'')$ are nonzero, as otherwise ${\sigma_1}^2$ could be chosen to occur in β. Note $e_1(\beta') = 0$ and $e_1(\beta'') = 0$. Since $e_2 \leq 3$, at least one of β' and β'' would have exactly one occurrence of σ_2. This means that we could change β so that

$$\begin{aligned} \beta &\sim \sigma_1\sigma_2\sigma_1\beta''' \\ &\sim \sigma_2\sigma_1\sigma_2\beta''' \\ &\sim \sigma_1\sigma_2\beta'''\sigma_2, \\ &\sim \sigma_2\beta'''\sigma_2, \end{aligned} \tag{6.1}$$

and $e_1(\sigma_2\beta'''\sigma_2) = 0$, contradicting the minimality of n. Thus one of $e_1 \geq 3$ or $e_2 \geq 4$ must hold, and similarly must one of $e_{n-1} \geq 3$ or $e_{n-2} \geq 4$. If $s \leq 2$, this forces $e_1 = 3$, $e_{n-1} = 3$, and $e_i = 2$ for $i \in \{2, 3, \ldots, n-2\}$.

Next we see that $\beta \not\sim {\sigma_2}^2\beta'$ $(\sim {\sigma_1}^3\beta''{\sigma_2}^2)$, since $e_2 = 2$ and hence K is a link of at least two components, see Figure 6.1a. Thus we see that

$$\beta \sim \sigma_2 \beta' \sigma_2 \beta'',$$

where β' and β'' each contain a σ_1 or σ_3. Without loss of generality, we have two cases: $e_1(\beta') = 2$ or 3. If $e_1(\beta') = 2$, then we use the relation $\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$ twice to remove σ_1 from β as in the process described in 6.1, a contradiction. Thus as a last case, assume $e_1(\beta') = 3$. Then either $e_3(\beta') = 0$ or 1, since $e_3(\beta'') \geq 1$. If $e_3(\beta') = 1$, we proceed by using β'' to reduce $e_2(\beta)$ to 1 and again see $\hat{\beta}$ as a connected sum, to remove the first three or last $n-4$ strands of β. Finally, if $e_3(\beta') = 0$, then Figure 6.1b shows that K is in fact a link of at least two components, a contradiction.

Thus $s \geq 3$, and K could not produce three connected summands by Dehn surgery. $\square$

While the proof of Corollary 1.2.2 is certainly sufficient, it is perhaps a first approximation to the phenomenon that minimal positive braid words representing a nontrivial knot with sufficiently large bridge number seem to have large exponent sum relative to their strand number (*i.e.* $e > 2n$).

Question 6.0.3. *For which classes of knots is there a relationship between the bridge number of a knot and the difference $e - n$ of a minimal-strand braid representing the knot?*

Note that by the resolution of the Jones Conjecture [2], $e-n$ is a knot invariant, where one is careful in defining e to be the *algebraic* exponent sum. For positive braids, the total exponent sum and the algebraic exponent sum are the same. The statement $e - n > b$ is false in general, even among positive braids: ${\sigma_1}^3$ is a minimal braid for the trefoil.

It would be interesting to apply a similar argument as in the proof of Corollary 1.2.2, if possible, to a class of knots called *L-space knots*. An L-space knot is a knot in S^3 with a positive surgery resulting in an L-space, a manifold whose Heegaard Floer homology is minimal in a suitable sense. Lens spaces, elliptic manifolds, and connected sums thereof are all L-spaces, and there are many others. See [31] for the definition of Heegaard Floer homology, and [32] for the definitions and basic facts about L-spaces and L-space knots.

The argument above required positivity of braids. There are several generalizations of positivity, such as *homogeneity* [39], which still gives a fibered knot. L-space knots need not have a positive (or homogeneous) braid representative, but they have a *strongly quasipositive* braid representative [13], which is a different generalization of positivity, and are fibered [30]. Additionally, in [16] Jennifer Hom, Tye Lidman, and the author show that L-space knots have the same property that the only possible reducing slope among hyperbolic representatives is $2g - 1$. Thus they are another good candidate class of knots for which to attempt to resolve the Two Summands Conjecture using Theorem 1.2.1 and the techniques of Corollary 1.2.2.

Chapter 7

Ruling out small values of p

In this Chapter, we prove Proposition 4.1.2, which as we discussed in Chapter 4 completes the proof of Theorem 1.2.3. To begin, recall we have a great web Λ in the general case, which contains a positive, index 0, graphically closed disk Λ' obtained from Λ by removing the (formal) closed cluster $\overline{C}$ (see the discussion immediately following Lemma 4.0.2). Thus Theorem 3.0.6 applies to greatly control the structure of the faces of Λ'. We defined Γ to be the subgraph of G_P consisting of the edges of Λ (thought of as lying in G_P). Additionally, in Section 4.1 we defined the placement of Λ (or Γ). Finally, we defined the *form* of a face of G_Q at the end of Chapter 4.

7.1 $S^3_r(K)$ contains an $\mathbb{R}P^3$ summand

In this section, we argue that if the placement is $\{2, 2, \ldots, 2\}$, then the lens space summand of $S^3_r(K)$ is in fact a lens space of order two, and is hence homeomorphic to $\mathbb{R}P^3$. To begin, we will need to use Lemma 3.0.7 to travel into Λ from $\partial\Lambda$ as much as possible. Order the nice boundary edges of Λ in a counterclockwise fashion, starting with the first (nice) boundary edge beginning after the 3-ghost (*i.e.* emanating from the vertex containing the

3-ghost).

Lemma 7.1.1. *Let $(\alpha+1,\alpha)$ be the labels of a nice boundary edge e of Λ. Then there are edges parallel to e into Λ' with the labels*

$$(\alpha+1,\alpha),(\alpha+2,\alpha-1),\dots,(\alpha+n+1,\alpha-n),$$

where either

1. *$\alpha-n=3$ or $\alpha+n+1=p$, or*

2. *there is some $m\le n+1$ such that $\{\alpha+n+2,\alpha-n,\alpha+m+1,\alpha-m\}$ is a subset of an element of the labeled placement.*

Proof. Let v_a and v_b be the vertices at which e is labeled $\alpha+1$ and α, respectively. Let f be the face of Λ' incident to e. The next edge e' into Λ' at v_a is labeled $\alpha+2$. If one of $\alpha+2=1$ or $\alpha-1=2$, then the first condition is satisfied. Thus assume both are regular labels. If $\alpha+2$ is in the same element of the labeled placement as $\alpha+1$, then the second condition is satisfied. Otherwise, e' does not have labels $(\alpha+2,\alpha+1)$, and so is labeled $(\alpha+2,\lambda)$ for $\lambda\le\alpha-1<\alpha+2$. Thus the path $\delta=e\cup e'$ has $\deg_f(\delta)\ge 1$ and is adjacent to no Scharlemann cycles, hence Lemma 3.0.7 gives that $I(f)=0$. Thus $\lambda=\alpha-1$ and f is of the form $g_{\alpha+1}g_{\alpha-1}^k$. If $k>1$, then the second condition is satisfied, as there is an $(\alpha,\alpha-1)$-edge. Otherwise, f is a bigon.

Suppose we have found edges with labels

$$(\alpha+2,\alpha-1),\dots,(\alpha+n,\alpha-(n-1))$$

parallel to e into Λ'. Then $\alpha+n+1=1$ implies that $\alpha+n=p$, and $\alpha-n=2$ implies that $\alpha-(n-1)=3$, and the first condition is satisfied. Thus assume the labels $\alpha+n+1$ and $\alpha-n$ are regular. Let f_n be the face containing these corners $\langle \alpha+n, \alpha+n+1\rangle$ at v_a and $\langle \alpha-n, \alpha-(n-1)\rangle$ at v_b, and let e_n be the edge of ∂f_n between these corners. If the second condition is not satisfied, the $(\alpha+n+1)$-edge e'_n leaving v_a is labeled $(\alpha+n+1, \lambda)$, for some $\lambda \leq \alpha-n < \alpha+n+1$. See Figure 7.1. Thus again the path $\delta = e'_n \cup e_n$ again gives by Lemma 3.0.7 that $I(f)=0$, and so f is a bigon, or the second condition is satisfied.

We may continue this process until one of the two conditions is satisfied, by induction. $\square$

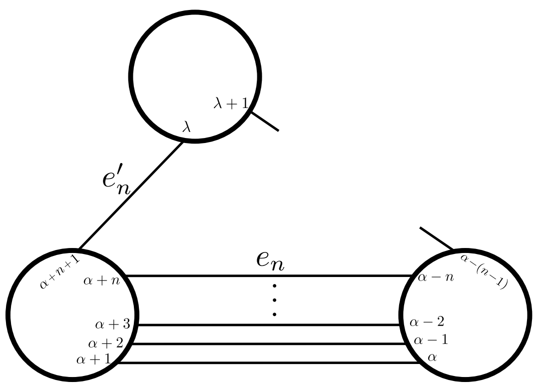


Figure 7.1

As a consequence, we see that if the placement is $\{2,2,\ldots,2\}$, then the labeled placement is

$$\left\{\{p,3\},\{p-1,4\},\ldots,\left\{\frac{p}{2}+2,\frac{p}{2}+1\right\}\right\}.$$

This is due to the fact that Λ contains at least one two-cornered face, and all two-cornered faces contain at least one $(p, 3)$-edge. Thus $\alpha - n = 3$ or $\alpha + n + 1 = p$, and the vertices u_3 and u_p are adjacent in Γ only to each other and u_1 and u_2. If indeed the final edge was a $(2, 3)$- or $(p, 1)$-edge, then the collection of edges parallel to the some boundary $(\alpha + 1, \alpha)$-edge would form a full quota, a contradiction. Hence the final edge in the parallelism class is a $(p, 3)$-edge, and so gives the desired labeled placement.

Lemma 7.1.2. *If the placement is* $\{2, 2, \ldots, 2\}$*, then* $|\sigma| = 2$*, so that the surgered manifold contains an* $\mathbb{R}P^3$ *connected summand.*

Proof. Suppose that the placement is $\{2, 2, \ldots, 2\}$ and $|\sigma| > 2$. We first claim that no two-cornered face is a bigon. To see this, note that the $(p, 3)$-edge of any two-cornered face of G_Q is parallel in G_P to any other $(p, 3)$-edge, since there are no other vertices in the Scharlemann sub-region containing u_3 and u_p. Let D be any collection of bigons defining a parallelism between a $\left(\frac{p}{2} + 2, \frac{p}{2} + 1\right)$-edge and a $(p, 3)$-edge (*e.g.* parallel to a nice boundary $\left(\frac{p}{2} + 2, \frac{p}{2} + 1\right)$-edge). Then if f is a two-cornered face bigon, and E is the parallelism in G_P between the $(p, 3)$-edges of f and D, then the band $B = f \cup E \cup D$ shows that K is a satellite knot, and hence couldn't be a counterexample to the Cabling Conjecture, using the methods of Proposition 4.2.1. Thus no two-cornered face is a bigon.

Suppose for some label $\lambda \in L$ satisfying $3 \leq \lambda \leq \frac{p}{2} + 1$, there is a $(\lambda, 2)$-edge in Λ' with an odd face $f_\lambda \subseteq \Lambda'$ adjacent to it. Then the number of edges in Γ between u_λ and $u_{p+3-\lambda}$ is not equal to $v = |V(\Lambda)|$, its maximum possible value. Hence there is a $(1, p+3-\lambda)$-edge in Λ', and the odd face $f_{p+2-\lambda}$ adjacent to this edge lies in Λ'. Since these faces are odd and not Scharlemann cycles, they are index zero and hence of the form

$$g_\lambda {g_1}^{k_\lambda} \qquad \text{and} \qquad g_{p+2-\lambda} {g_1}^{k_{p+2-\lambda}}$$

for integers k_λ and $k_{p+2-\lambda}$. We next claim that $k_\lambda + k_{p+2-\lambda} = |\sigma|$. To see this, construct a torus T_{Sch} by taking $\widehat{P}$ and tubing by ∂H_1. Fix the basis for $H_1(T_{\text{Sch}})$ given by $\mathcal{B} = \{g_1, [\partial u_1]\}$. This torus has a $(|\sigma|, k)$-curve on it given by any Scharlemann cycle of Λ. The vertices u_λ and $u_{\lambda+1}$ lie somewhere on T_{Sch}, and one may decompose ∂f_λ into an arc between these vertices on ∂H_λ and an arc α_λ on T_{Sch}. Similarly define $\alpha_{p+2-\lambda}$. Then since u_λ and $u_{p+3-\lambda}$ lie in the same Scharlemann region, as do $u_{\lambda+1}$ and $u_{p+2-\lambda}$, there is a simple closed curve γ consisting of arcs connecting these pairs of vertices in the complement of $\mathcal{B}$ together with α_λ and $\alpha_{p+2-\lambda}$. By construction, γ is a curve which is a $(k_\lambda + k_{p+2-\lambda}, s)$-curve on T_{Sch}, yet it is essential and disjoint from the boundary of a Scharlemann cycle. Thus $k_\lambda + k_{p+2-\lambda} = |\sigma|$. Note that the existence of $(1, p+3-\lambda)$-edges also give rise to $(\lambda, 2)$-edges in an analogous way. We will refer to any face of the form $g_\lambda {g_1}^{k_\lambda}$ as an *almost Scharlemann cycle* (ASC) of type λ. As we will see, we may define k_λ for each odd $\lambda \in \{3, \ldots, \widehat{\frac{p}{2}+1}, \ldots, p-1\}$ (if $\frac{p}{2}+1$ is not odd, then the set is all odd $\lambda \neq 1$) . The assumption $|\sigma| > 2$ implies that each pair k_λ and $k_{p+2-\lambda}$ has at least one of the integers greater than 1.

We now define a path in the dual graph of Λ' to complete the argument. Throughout the proof, we encourage the reader to refer to the example in Figure 7.2. To begin the definition, we add a simplifying assumption:

$$\begin{aligned} &\text{not all of } k_\lambda = 1 \text{ for } 3 \leq \lambda < \frac{p}{2} + 1, \text{ and} \\ &\text{not all of } k_\lambda = 1 \text{ for } \frac{p}{2} + 1 < \lambda \leq p - 1. \end{aligned} \tag{7.1}$$

We will see that this is the generic case, and once we see how to make the argument here it will be clear how to extend it. Since every two-cornered face is not a bigon, and $C = \overline{C} \cap \Lambda$ contains an exterior two-cornered face, we may find an odd face f_1 of Λ' incident to a $(3,2)$- or $(1,p)$-edge. The face f_1 is an ASC of type 3 or $p-1$. If f_1 is not a bigon (*i.e.* if $k_3 > 1$ or $k_{p-1} > 1$), then it contains a $(1,2)$-edge in its boundary, and so is adjacent to a face h_1 across this edge. If f_1 is a bigon, then it is adjacent through its $(4,1)$- or $(p-1,2)$-edge to a face f_1^1 of the form $g_p g_4 *$. By Lemma 3.0.7, f_1^1 is a face of the form $g_4 g'$, where g' is two-cornered. We will call any face of the form $g_\lambda g'$ with g' two-cornered an *almost two-cornered face* (A2CF) of type λ. Note that such an A2CF contains two edges with one of the labels λ and $\lambda + 1$ in each, one which we just entered via an ASC of type $\lambda \pm 1$, and the other which is adjacent to an ASC of type $\lambda \mp 1$, which we define to be f_1^2. This process repeats until we discover a face h_1 adjacent to an ASC, by Assumption 7.1. See Figure 7.2.

Note that h_1 is a two-cornered face which lies in Λ' (it can't lie outside of Λ by construction, and it cannot lie in C because we entered it from an ASC). If $I(h_1) = 0$, this is a clear computation, and otherwise it follows from

Lemma 3.0.7. We need to find an index 0 two-cornered face h_1'. $I(h_1) = 0$, define $h_1' = h_1$ (this is like $h_1 = h_1'$ and $h_2 = h_2'$ in Figure 7.2). If $I(h_1) = m(h_1) - 1 > 0$, then h_1 is the unique high-index two-cornered face of an almost closed cluster D. This is due to the fact that the face from which we entered h_1 is an ASC, and not an actual Scharlemann cycle. Travel through D to reach an (index 0) exterior two-cornered face h_1' (this is like h_3 and h_3' in Figure 7.2). In either case, we have found an index 0 two-cornered face h_1' which must contain a $(3, 2)$- or $(1, p)$-edge which we have not previously visited. This edge is incident to an ASC f_2 of type 3 or $p - 1$, and the process starts over.

To summarize, we have defined a path in the dual graph of G_Q which does not leave Λ', because it cannot enter C and it does not use edges of $\partial\Lambda$. Also, whenever the path travels through an almost closed cluster, if enters the cluster from the unique high-index face. We have recorded the conditions of how each type of face is entered via this dual path in Table 7.1.

Since Λ' is compact, this dual path must return to itself. Table 7.1 shows that the only way this can occur is by entering a high-index two-cornered face h_i a second time: all other options either use the unique edge with a certain label pair for that face, or occur inside of an almost closed cluster, so we first rejoined the path at the unique high-index face of the cluster. However, we only enter high-index two-cornered faces via ASCs, and so we have shown that h_i is adjacent to at least two ASCs. This implies that the index of h_i is at least one more than $m(h_i)$, giving rise to a closed cluster in Λ', a contradiction.

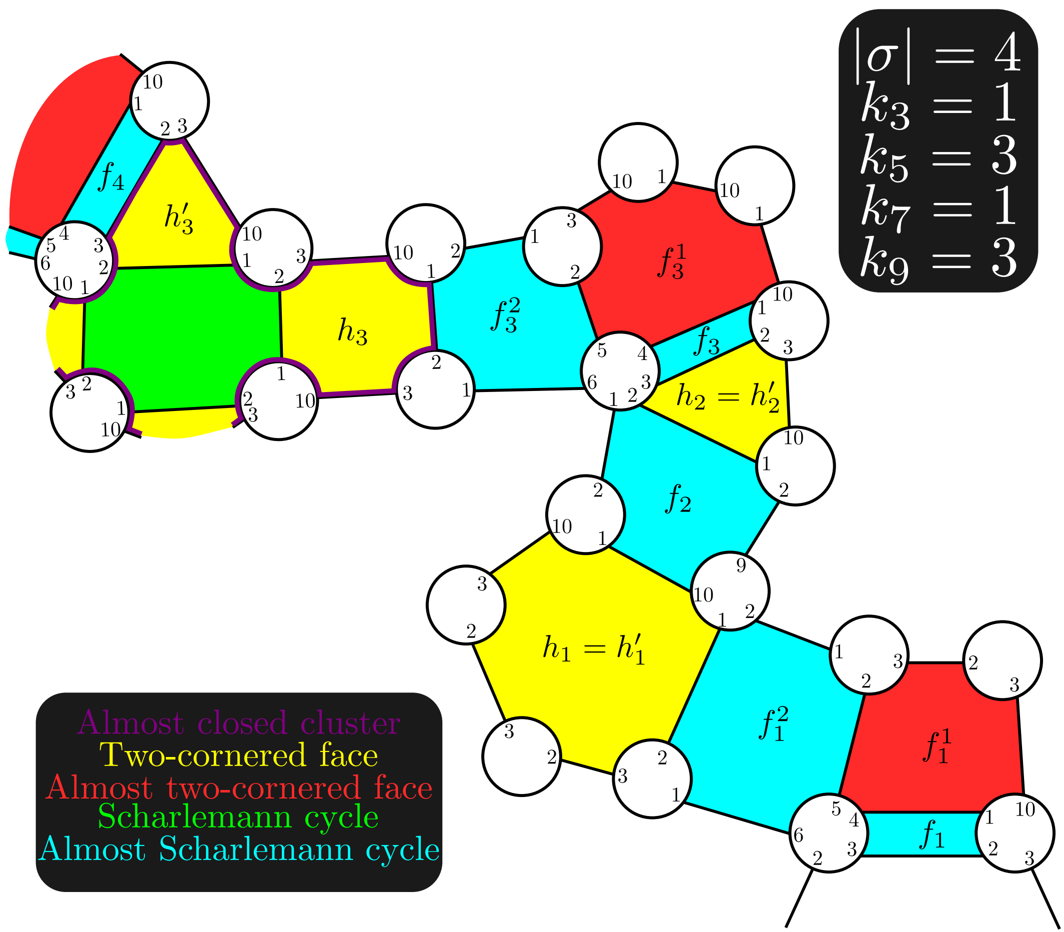


Figure 7.2: Finding the dual path. Here $p = 10$, and the colors correspond to the type of face (see legend).

Table 7.1: Conditions of the dual path

We only enter	from	via
ASCs	two-cornered faces or A2CFs	$(\lambda, 2)$-edges for $3 \leq \lambda \leq \frac{p}{2}$, or $(1, \lambda)$-edges for $\frac{p}{2}+2 \leq \lambda \leq p$.
A2CFs	ASCs	$(\lambda+1, 1)$-edges for $3 \leq \lambda \leq \frac{p}{2}$, or $(2, \lambda-1)$-edges for $\frac{p}{2}+2 \leq \lambda \leq p-1$.
two-cornered faces (high-index)	ASCs	$(1, 2)$-edges.
Scharlemann cycles	two-cornered faces in almost closed clusters	$(1, 2)$-edges.
two-cornered faces (low-index)	Scharlemann cycles in almost closed clusters	$(1, 2)$-edges.

The proof is hence completed by describing the modification to the argument in the case when the Assumption 7.1 is not true. The only difference occurs when $k_\lambda = 1$ for either all of the ASCs of type λ with $3 \leq \lambda < \frac{p}{2}+1$ or all of the ASCs of type λ with $\frac{p}{2}+1 < \lambda \leq p-1$. In this situation, if $p \equiv 0 \pmod 4$ the path could travel all the way through to a $(1, \frac{p}{2}+2)$- or a $(\frac{p}{2}+1, 2)$-edge. The odd face f incident to this edge is of the form ${g_{\frac{p}{2}}}^k {g_1}^{|\sigma|}$, for the same reason that $k_\lambda + k_{p+2-\lambda} = |\sigma|$. This face serves the same purpose as an ASC of type $\frac{p}{2}+1$, so we could extend the argument in this case. Thus we may assume that $p \equiv 2 \pmod 4$, and without loss of generality $1 = k_3 = k_5 = \ldots = k_{\frac{p}{2}}$. If the dual path is defined such that we arrive at a $(\frac{p}{2}+1, 1)$-edge, then the next face f is of the form $g_p g_{\frac{p}{2}+1} *$. While it is possible that the $\left\langle \frac{p}{2}+1, \frac{p}{2}+2 \right\rangle$-

corner repeats, so that f is not an A2CF, we see by Lemma 3.0.7 that f is of the form ${g_{\frac{p}{2}+1}}^k g'$, for g' two-cornered. This face has a unique $(\frac{p}{2}+2, 2)$-edge; define f' to be the face adjacent to f along this edge. Lemma 3.0.7 gives that f' is either an ASC of type $\frac{p}{2}+2$ or a face of type $g_{\frac{p}{2}+2} g_{\frac{p}{2}} {g_1}^{|\sigma|}$. Either way, the face f' serves the same purpose as an ASC and has a $(1,2)$-edge which we may travel through, as desired. □

7.2 Proof of Proposition 4.1.2

We are now prepared to prove Proposition 4.1.2. The proof will divide into two cases: $p = 6$ and $p > 6$, which are of a very different flavor from one another. In particular, the argument to rule out the $p = 6$ case is quite subtle, requiring first that we determine precisely what $\partial\Lambda$ looks like, and then show that Γ could not be constructed subject to these constraints. It is interesting to note that some webs with $p = 6$ and with a very small number of vertices are the hardest to rule out, since no argument seems to apply to them *except* for this labeling-type argument. In most other cases, there are many distinct arguments that give rise to a contradiction (for example, a labeling-type argument seems to always work).

Proof of Proposition 4.1.2. Recall that if we have a placement of $\{2, 2, \ldots, 2\}$ then the 3- through $(\frac{p}{2}+1)$-ghosts are all at the same vertex of Λ. If the $(\frac{p}{2}+2)$-through p-ghosts do not all lie on the same vertex, then there is a nice boundary $(\lambda+1, \lambda)$-edge for some $\lambda \in L$ satisfying $(\frac{p}{2}+2) \leq \lambda \leq p-1$. However, since we know the labeled placement is $\left\{\{p,3\},\{p-1,4\},\ldots,\left\{\frac{p}{2}+2,\frac{p}{2}+1\right\}\right\}$, no such edge may exist. Thus the ghosts are evenly divided between two vertices, and between these vertices along $\partial\Lambda$ are parallelism classes of edges labeled $(p,3)$ through $(\frac{p}{2}+2,\frac{p}{2}+1)$. See Figure 7.3. Additionally, in the language of Lemma 7.1.2, all the integers $k_\lambda = 1$ for each odd $\lambda \in L$ (except for possibly $\frac{p}{2}+1$).

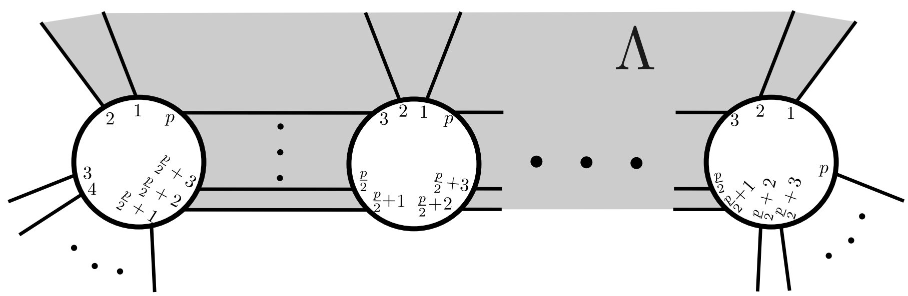


Figure 7.3: The boundary of Λ for the placement $\{2,2,\ldots,2\}$

We will first show that we may find a vertex v satisfying:

$$v \in \partial\Lambda \text{ between the } \left(\frac{p}{2}+1\right)\text{- and } \left(\frac{p}{2}+2\right)\text{-ghosts whose 2-edge is labeled } (3,2). \tag{7.2}$$

If there is a nice boundary $(3,2)$-edge, then the vertex containing the 3- through $(\frac{p}{2}+1)$-ghosts will suffice. Suppose then that there is no nice boundary $(3,2)$-edge, and begin at the Scharlemann cycle incident to the vertex containing the 3-ghost. Travel dually into the two-cornered face f_1. See Figure 7.4. Note that f_1 is not a bigon, else it is parallel to $\partial\Lambda$, and Λ contains a full quota. If the edge labeled e in Figure 7.4 is a $(1,2)$-edge, then dually travel through it, then through the Scharlemann cycle, and repeat with the next two-cornered face f_2 (recall that C is graphically closed into Λ, so we know we may proceed in this way). We cannot travel all the way through to the $(\frac{p}{2}+2)$-ghost, because then there are many vertices whose regular labels do not have paths leading to their respective ghosts. Thus we find a $(3,2)$-edge, giving rise to the desired vertex.

This edge is incident to a bigon of type g_3g_1 (because it is an ASC of type 3). Travel dually though this bigon to a face F_1 of the form $g_2g_pg_4*$. By Lemma 3.0.7, F_1 is in fact of the form $g_pg_4{g_2}^k$. If $k \neq 1$, then the vertex containing the last $\langle 2,3\rangle$-corner of F_1 is another vertex with Property 7.2. Dually travel through the bigon of type g_3g_1 at this new vertex to a face F_2, also of the form $g_pg_4{g_2}^k$. We know that we will not reach the $(\frac{p}{2}+2)$-ghost in this way, because each time we find a bigon of type g_3g_1, and a $(1,4)$-edge could not be a boundary edge (since $p > 4$). If F_2 has more than one $\langle 2,3\rangle$ corner, we may repeat this process to find F_3, F_4, *etc.* This process must terminate, and so we eventually arrive at a face G_1 which is of the form $g_pg_4g_2$. In the same spirit, find a vertex w satisfying an analogous condition to Property 7.2, but for $(1,p)$-edges. We perform an analogous search, arriving at a face G_2 of the form $g_pg_{p-2}g_2$.

Case 1: $p > 6$. In this case, there is a bigon B of the form $g_{p-2}g_4$. Construct the sub-3-manifold X of $S^3_r(K)$ given by

$$X = N(\widehat{P} \cup H_2 \cup H_4 \cup H_{p-2} \cup H_p \cup G_1 \cup B \cup G_2).$$

Then $H_1(X)$ contains 2-torsion, since the disks G_1, G_2, and B could be banded

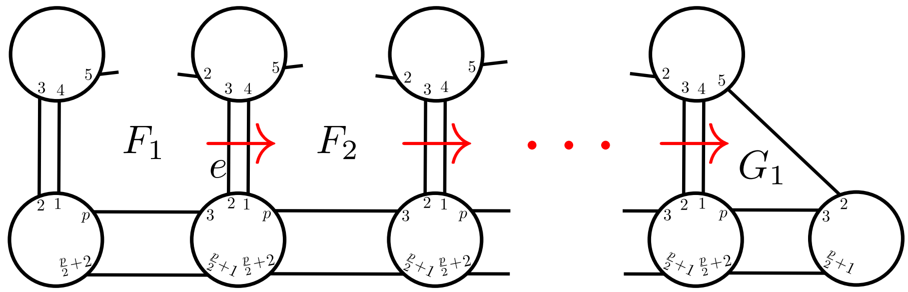


Figure 7.4: Finding 2-torsion on the even side of P

together to give rise to a disk D of the form $(g_2 g_p)^2$. Now X lies in the even side M_2 of $\widehat{P}$, and has boundary $S \amalg T$, where S is isotopic to $\widehat{P}$ and T is a torus. In fact, we may obtain D from $G_1 \cup B \cup G_2$ via handleslides of the H_i in order to notice that the image of $i_* : H_1(T) \to H_1(X)$ does not contain this $\mathbb{Z}/2\mathbb{Z}$ subgroup. Hence we find that $H_1(M_2)$ contains 2-torsion. However, we know that the odd side contains (in fact, is exactly) an $\mathbb{R}P^3$ connected summand, and so also contains 2-torsion. This contradicts the fact that $H_1(S^3_r(K)) = \mathbb{Z}/2\mathbb{Z} \oplus H_1(M_2)$ is cyclic.

Case 2: $p = 6$**.** In this case, $p - 2 = 4$, and so the disks G_1 and G_2 are both of the form $g_6 g_4 g_2$, hence the above argument will not suffice (indeed, they may even be the same disk). We will show that the graph Γ could not be constructed because the labeling of the edges incident to its vertices will be inconsistent.

Let σ be a Scharlemann cycle of Λ. Let v_a and v_b be the vertices of σ. We claim that at least one of the 3-edges of v_a and v_b is not a $(3,2)$-edge. Suppose this is not the case. Note that the number of $(6,1)$-edges of

Γ is equal to the number of $(3, 2)$-edges of Γ. Let n be the number of labels $\mu \in \{1, 2, \ldots, q\}$ satisfying $a \leq \mu \leq b$ such that v_μ is a vertex of Γ. Said differently, n is the number of edge endpoints between the labels a and b on the vertices u_k of Γ, unless the k-ghost occurs at one of these vertices v_μ, in which case this number is $n - 1$. However, it is clear from Figure 7.5 that

$$n - 1 \leq |\{(3, 2)\text{-edges of } \Gamma\}| = |\{(6, 1)\text{-edges of } \Gamma\}| \leq n - 2.$$

Here the first inequality is by definition of n, and the third uses the labeling of edges at u_1. Similarly, we see that at least one of the 6-edges of v_a and v_b is not a $(6, 1)$-edge. Note that these edges could be ghosts or $(6, 3)$-edges of Γ.

Let D' be the Scharlemann subregion containing the vertices u_4 and u_5, and define $D = \widehat{P} \setminus \mathrm{Int}(D')$. Then D contains all edges of Scharlemann cycles and two-cornered faces of Λ, and satisfies $|D \cap K'| = 4$, where K' is the dual knot to K. Suppose F_1 and F_2 are two-cornered faces of Λ satisfying the condition that their boundaries do not cobound an annulus on the surface $\Sigma = \partial N(D \cup H_2 \cup H_p)$. Then the argument of Hoffman discussed in Chapter 2 (the sketch of the proof of Theorem 2.3.1) gives a reduction of the sphere $\widehat{P}$; essentially, the disks F_1 and F_2 will serve the role of a seemly pair. Hence the boundaries of any pair of two-cornered faces of Λ cobound an annulus on Σ, hence are freely homotopic in the knot exterior. This implies that the disks are of the same form. We conclude that all two-cornered faces of Λ are exterior and of the same form (and in fact parallel on Σ, which we will use).

We next need to discover exactly what $\partial\Lambda$ looks like. Suppose that

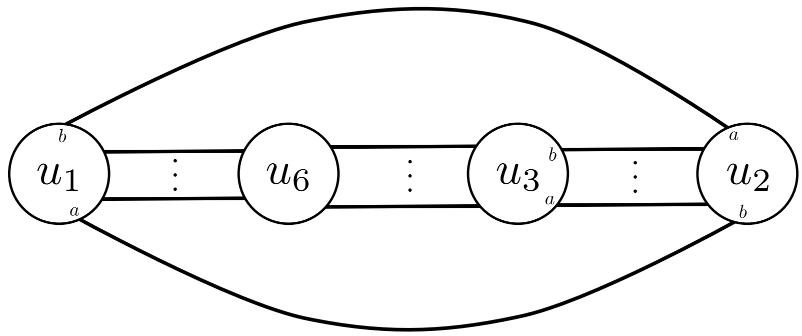


Figure 7.5: Without loss of generality, this is a subgraph of Γ if the Scharlemann cycle on the vertices v_a and v_b has both of its 3-edges are $(3,2)$-edges.

neither set of ghosts occur at the vertices of any Scharlemann cycle, so that the first and last Scharlemann cycles along $\partial\Lambda$ are adjacent to nice boundary $(1,6)$- and $(3,2)$-edges, respectively. Hence the two-cornered faces adjacent to them are of the form $g_2^k g_p$ and $g_2 g_p^l$, respectively. However, these two-cornered faces are of the same form, making $k = l = 1$, and so we would have a two-cornered face bigon, contradicting an earlier statement. Suppose instead that both sets of ghosts occur at vertices of Scharlemann cycles. Then the non-ghost 4- and 5-edges of these vertices are nice boundary $(5,4)$-edges. Hence the non-ghost 3- and 6-edges of these vertices are both $(3,6)$-edges, giving a similar contradiction. Thus we may suppose without loss of generality that the 3- and 4-ghosts occur at a vertex of a Scharlemann cycle and the 5- and 6-ghosts do not. Hence $\partial\Lambda$ appears exactly as in Figure 7.6, where the vertices v_a and v_b may be equal to v_c and v_d if there is exactly one Scharlemann cycle incident to $\partial\Lambda$. We will only give the proof assuming they are distinct, as the argument is the same (with some labels made equivalent) if there is only one

Scharlemann cycle.

To see that Γ cannot be constructed so that the labels are consistent, begin with the subgraph Γ' as in Figure 7.7, consisting of the edges of the first and last Scharlemann cycles, the (s_1, d)-edge between u_1 and u_6, the (c, c')- and (a, t)-edges and between u_6 and u_3, and (b, b')-, (d, d')-, and (t, t')-edges between u_3 and u_2. Note that there is ambiguity between whether the (t, t')-edge is above, below, or between the other two edges, but it will not be an issue for us. The only other ambiguity about the labeling of edges is the ordering of the four Scharlemann edges, but the fact that the two-cornered faces adjacent to the first and last Scharlemann cycles are parallel on Σ shows that the labeling as in Figure 7.7 may be taken, without loss of generality. This collection of edges cuts the Scharlemann subregion into two important disks, D_1 and D_2 as shown.

We consider the effect of two paths in Λ on the labeling of edges in Γ. The two paths are

$$\gamma_1 : v_d \to v_{s_1} \to v_{s_2} \to \cdots \to v_s$$

via $(6, 1)$-edges, and

$$\gamma_2 : v_t \to v_{t_1} \to v_{t_2} \to \cdots \to v_s$$

via $(6, 3)$-edges. Assuming the labeling of the vertex u_6 is consistent, we see that the edges of γ_1 must lie to the D_1-side of u_6, giving that the 6-ghost (occurring at the vertex v_s) lies to the D_1-side of u_6. However, we also see that the edges of γ_2 must lie to the D_2-side of u_6, giving that the 6-ghost must

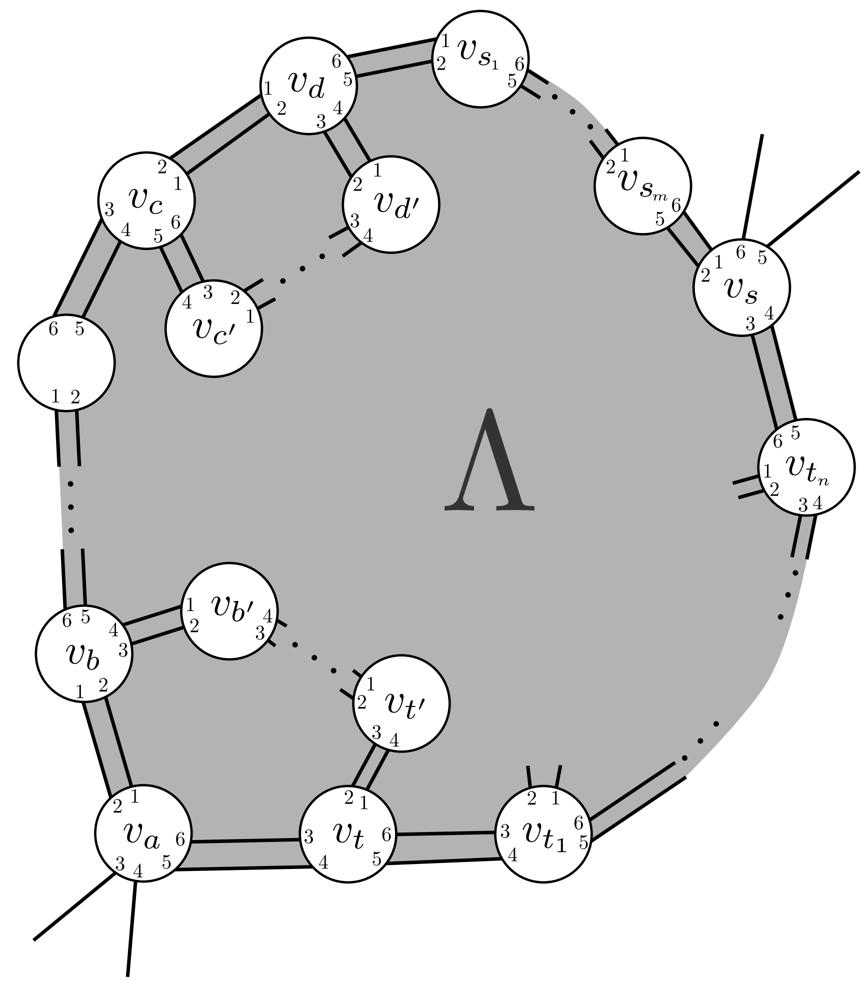


Figure 7.6: The boundary of Λ in the $p = 6$ case

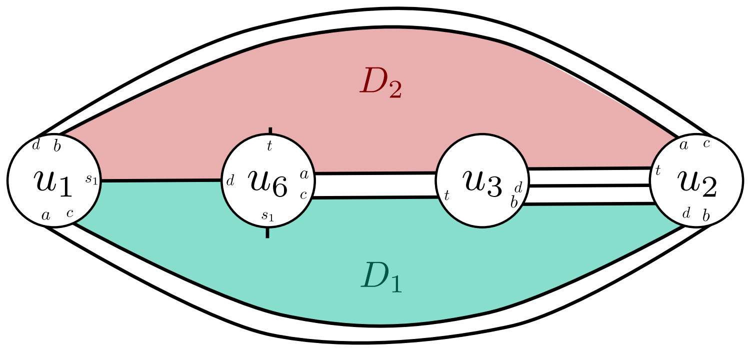


Figure 7.7: Constructing the graph Γ in the $p = 6$ case

occur on the D_2-side of u_6. This is a contradiction to the labeling at u_6 being consistent, as we have found two distinct places where the label s must be placed. This completes the proof in the case that $p = 6$, and hence gives the desired result. $\square$

Appendix

Appendix 1

Existence of great webs

In this Appendix, we discuss the proof of the following Theorem and how it appears in the literature.

Theorem 1.0.1. *Let X be a compact, orientable, irreducible manifold with ∂X a torus. Suppose Q is a properly embedded, connected planar surface, and P is a properly embedded, orientable surface without closed components chosen so that Q and P intersect essentially. Let $p = |\partial P|$, and assume that $\Delta > 1 - \frac{\chi(\widehat{P})}{p}$, where Δ is the geometric intersection number between the slopes given by ∂Q and ∂P. Then either G_P represents all types or G_Q contains a great $(p - \chi(\widehat{P}))$-web.*

Note that this is a generalization of Theorem 4.0.1. We are especially interested in the situation where P is a disconnected planar surface (note that we have $Q \cap P_i \neq \emptyset$ for each connected component so long as P has no closed components). We stress that the proof does occur in the literature, but not in any specific place. Thus the purpose of this Appendix is not to prove Theorem 1.0.1 from first principles, but rather to discuss how to complete the proof from what is found in the literature.

Theorem 1.0.1 is equivalent to Theorem 5.6 of the article [6]. There, Theorem 5.6 is not proven in complete detail, rather the following statement (their Theorem 5.1) is proven, which is the first step in proving Theorem 1.0.1. We assume the reader is familiar with the language and techniques of that article, as we will continue the arguments of the paper.

Theorem 1.0.2. *4.0.1 Let $\Delta > 1 - \chi(\widehat{P})/p$. If $\widehat{Q}$ is a sphere, then G_P represents all types or G_Q contains a $(p - \chi(\widehat{P}))$-web.*

The difference between the two statements is that the web Λ is not assumed to be great, *i.e.* it satisfies condition (2) of Definition 4.0.2, but has a weakened version of condition (1), which is simply that all vertices of Λ are of the same sign (with no conditions on the underlying surface). It is clear, then, that the way to complete the proof of Theorem 1.0.1 is to see how to deal with the existence of vertices not in Λ in more than one complementary region of Λ. We suppress the integer $p - \chi(\widehat{P})$ from the notation, all webs (great or otherwise) will be assumed to be $(p - \chi(\widehat{P}))$-webs.

To begin, let U and be a disk in $\widehat{Q}$ and Λ be a graph in G_Q which satisfy one of the following:

1. U is disjoint from G_Q (*i.e.* U is a small disk in some face of G_Q) and $\Lambda = \emptyset$, or
2. Λ is a web and U is a component of $\widehat{Q} \setminus N(\Lambda)$.

Let D be the disk $\widehat{Q} \setminus U$, and define L to be the set of vertices of $D \setminus \Lambda$. Then D contains Λ, and if L is empty then we are in case (2) and Λ (is a nonempty web, and) is great. This is the base case of an induction on $|L|$: one proceeds to prove by induction on $|L|$ that either G_Q contains a great web or $G(L)$ represents all L-types (recall $G(L)$ is the subgraph of G_P consisting of all edges with at least one label in L). The final case of this induction is then the statement that G_Q contains a great web or G_P represents all types, because then $L = \mathbf{q} = V(G_Q)$.

The reader is now directed to the proof of Theorem 2.5 of [10], which describes the induction step in the case that $\widehat{P}$ is a torus. It turns out that nothing needs to be changed in the argument to give the case of general $\widehat{P}$. However, the language of this proof is that of the proof of the knot complement problem [9], instead of that of the article [6]. We now describe most of the work in proving the induction step by updating the language of the proof of Theorem 2.5 of [10].

Let τ be an L-type. We will show that there is a face of $G(L)$ representing τ. There are two cases, given by whether or not τ is the trivial type ($++\cdots+$ or $--\cdots-$).

Suppose first that τ is trivial. The proof of this case in [10] carries over nearly verbatim to the general case, and the language is exactly the same as [6]. One defines a graph $\widehat{\Lambda}$ which is a component of the graph on the vertices in L which are of opposite sign to Λ. If $\widehat{\Lambda}$ is a web, then the induction hypothesis applies to find a face representing τ or a great web. Otherwise, $\widehat{\Lambda}$ has too many edges leaving it, that is to say more than $p - \chi(P)$ edges connect $\widehat{\Lambda}$ to vertices of opposite sign to those in $\widehat{\Lambda}$. Let Σ be the subgraph of G_P consisting of all vertices of G_P and only these edges. Then using Σ to count $\chi(\widehat{P})$, one sees that the sum of the Euler characteristics of the faces of Σ is positive, and hence as P has no closed components there are no spheres regions in this decomposition of $\widehat{P}$ by Σ. This Σ contains a disk face, and such a disk face represents τ.

Hence we may suppose that τ is nontrivial. The proof of this case in [10] uses quite different language from [6], so we will be more careful here. First, we will require the definition of a relative derivative of a type. We define this here for the reader's convenience. Let L_0 be a subset of the labels $\mathbf{q}$. The *derivative of τ relative to L_0* is defined to be the $(C(\tau) \cup L_0)$-type $d_0\tau$ given by the following: if I is a $(C(\tau) \cup L_0)$-interval, then

1. if there is an element a of $A(\tau)$ in I, then $d_0\tau|I = d\tau|I = \text{sign } a$, as in the definition of a usual (non-relative) derivative d of a type, and

2. if there is no such element, then all the edges of $\Gamma(\tau)$ incoming to positive vertices in G_P within the interval I have the same orientation; define $d_0\tau|I$ to be this sign ($+$ for inwards, $-$ for outwards at a positive vertex, opposite this for a negative vertex).

Thus $d = d_0$ for $L_0 = \emptyset$. We now proceed as on page 283 of [6] to take derivatives of the type τ, constructing a sequence of types $\tau_0, \tau_1, \tau_2, \ldots, \tau_n$, satisfying

1. $\tau = \pm\tau_0$,

2. $\tau_i = \pm d\tau_{i-1}$ is a nontrivial $C(\tau_{i-1})$-type, $1 \leq i \leq n$, and

3. τ_n has properties defined (*a) and (*c) as stated in [6], namely that

   - all elements of $C(\tau_n)$ have the same sign, and

   - all elements of $A(\tau_n)$ have the same sign.

By Proposition 5.5 of [6], we have that (we are done, or) there are no faces of $G(L_n)$ representing τ_n. Hence Lemma 5.4 of [6] gives that there is a web Ω whose vertices are contained in $C(\tau_n)$. We know that Ω is in D and is disjoint from Λ. Let D_Ω be the subdisk of D bounded by Λ, and let L_0 be the set of vertices of $G_Q \setminus \Omega$. We may assume $L_0 \neq \emptyset$, otherwise Ω is great and we are done.

Let $\tau_0', \tau_1', \ldots, \tau_n'$ be a new sequence of types corresponding to $\tau_0, \tau_1, \ldots, \tau_n$ except this time obtained by taking derivatives relative to L_0. That is, $\tau_0' = \tau_0 = \pm\tau$, and $\tau_i' = \pm d_0\tau_{i-1}'$, and τ_n' has the properties stated in (3) above. We next need to discuss the construction of a type τ_0 which has the property that any disk face of $G(L_0)$ representing τ_0 contains a face of $G(C(\tau_{n-1}) \cup L_0)$ representing τ_n'. If such a type τ_0 exists, the induction hypothesis applies to conclude that either G_Q contains a great web or ($G(L_0)$ represents τ_0, and hence) there is a face of $G(C(\tau_{n-1}) \cup L_0)$ representing τ_n'. An argument analogous to the proof of Proposition 5.5 of [6] then shows the following.

Proposition 1.0.3. *Let τ be a nontrivial L-type, and $L_0 \subset \mathbf{q}$. Then any face of $G(C(\tau) \cup L_0)$ representing $d_0\tau$ contains a face of $G(L \cup L_0)$ representing τ.*

Repeated applications of this Proposition will complete the proof of the induction step, and hence of Theorem 1.0.1. Hence it suffices to construct the type τ_0 with the desired property; we briefly describe this. Note first that $G(L_0)$ is a subgraph of $G(C(\tau'_{n-1}) \cup L_0)$, so within any L_0 interval I there may be some switches of τ'_n. One defines I to be good or bad based on these switches, and how the corners in I lie locally near $\widehat{Q}$ (See Section 2.6 of [9]). Then $\tau_0|I$ is defined to be $+$ if I is good, and $-$ is I is bad. This allows for an index count in any disk face F of $G(L_0)$ representing τ_0 (Lemma 2.7.1 and 2.3.3 of [10]), in order to find a face of $G(C(\tau'_{n-1}) \cup L_0)$ representing τ'_n.

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