The Dissertation Committee for Alice Harway Mark
certifies that this is the approved version of the following dissertation:

The Classification of Rank 3 Reflective Hyperbolic Lattices Over $\mathbb{Z}(\sqrt{2})$

Committee:

Daniel Allcock, Supervisor

Alan Reid

Lewis Bowen

Cameron Gordon

Ian Agol

The Classification of Rank 3 Reflective Hyperbolic Lattices Over $\mathbb{Z}(\sqrt{2})$

by

Alice Harway Mark, B.A.

DISSERTATION

Presented to the Faculty of the Graduate School of

The University of Texas at Austin

in Partial Fulfillment

of the Requirements

for the Degree of

DOCTOR OF PHILOSOPHY

THE UNIVERSITY OF TEXAS AT AUSTIN

May 2015

Dedicated to my sister Hannah.

Acknowledgments

First of all I would like to thank my advisor, Daniel Allcock, for suggesting this project, and for all of his patience with me and my questions as I worked through it. Thanks also to the other members of my committee, Alan Reid, Amir Mohammadi, Lewis Bowen, Cameron Gordon, and Ian Agol. I am grateful to the staff of the UT Math Department, especially Elisa Bass and Sandra Catlett, for all that they do to make everything run smoothly. Several long computations would not have been possible without the Math Department computing resources maintained by Maorong Zhu and Patrick Goetz. I want to thank Dakota, Richard, and Tom because life as a graduate student really is more fun when you live on a commune. And of course, to my fantastic family, thank you for all of your support and encouragement.

The Classification of Rank 3 Reflective Hyperbolic Lattices Over $\mathbb{Z}(\sqrt{2})$

Alice Harway Mark, Ph.D.
The University of Texas at Austin, 2015

Supervisor: Daniel Allcock

There are 432 strongly squarefree symmetric bilinear forms of signature $(2, 1)$ defined over $\mathbb{Z}[\sqrt{2}]$ whose integral isometry groups are generated up to finite index by finitely many reflections. We adapted Allcock's method (based on Nikulin's) of analysis for the 2-dimensional Weyl chamber to the real quadratic setting, and used it to produce a finite list of quadratic forms which contains all of the ones of interest to us as a sub-list. The standard method for determining whether a hyperbolic reflection group is generated up to finite index by reflections is an algorithm of Vinberg. However, for a large number of our quadratic forms the computation time required by Vinberg's algorithm was too long. We invented some alternatives, which we present here.

Table of Contents

List of Tables

List of Figures

Chapter 1

Introduction, statement of the main theorem

Hyperbolic reflection groups arise as discrete subgroups of automorphism groups of quadratic forms of signature $(n, 1)$. The integral automorphism group of a quadratic form has a subgroup generated by reflections. If the quadratic form has signature $(n, 1)$, those reflections act on hyperbolic space by hyperbolic reflections. The fundamental domain for this action is a hyperbolic Coxeter polyhedron. Arithmetic subgroups of algebraic groups are a particularly well studied class of discrete subgroups of algebraic groups that have finite co-area. The quadratic form Q defined over the totally real field F is arithmetic if for all nontrivial embeddings σ of F into $\mathbb{R}$, the quadratic form σQ is definite.

Let Q be an arithmetic quadratic form of signature $(n, 1)$ defined over the ring of integers in a totally real finite extension of $\mathbb{Q}$. The integral automorphism group Q is the group of isometries of Q that preserve an integral lattice L in $\mathbb{R}^{n,1}$. The automorphism group of L is a discrete subgroup of $O(Q) \cong O(n, 1)$. It acts on the model of hyperbolic space obtained as the projectivization of the future cone in $\mathbb{R}^{n,1}$. The subgroup that acts by reflections is a hyperbolic reflection group. We say that Q (or L) is reflective if its integral automorphism group is generated up to finite index by finitely many reflections.

This paper is devoted to proving the following:

Theorem 1. *There are 432 rank 3 strongly squarefree reflective arithmetic hyperbolic lattices defined over* $\mathbb{Z}[\sqrt{2}]$.

The structure of our proof is based on Allcock's in [3]. We modify several of his lemmas so that we can apply them to lattices with ground fields that are real quadratic extensions of $\mathbb{Q}$. Our modifications are inspired by the modifications Bugaenko made to Vinberg's algorithm in [5] [6] and [7], where he applied the algorithm to lattices defined over totally real finite extensions of $\mathbb{Q}$. What is new in this paper is that we do not use Vinberg's algorithm to determine reflectivity of our lattices. Instead we "walk around" the edges of the fundamental chamber in order. We will be more precise about what this means later on.

There are finitely many discrete maximal hyperbolic reflection groups with finite covolume that are arithmetic in $O(n,1)$. In 1985, Vinberg proved that there are no reflective arithmetic quadratic forms in dimension ≥ 30 [18]. In the early 1980s, Nikulin showed in a series of papers that there are only finitely many in any dimension greater than 9 [12] [13]. In their 2006 paper about genus 0 fuchsian groups, Long, Maclachlan and Reid used covolume bounds to prove finiteness in dimension 2 [10]. Using similar methods Agol proved finiteness in dimension 3 [1]. In two papers published at around the same time using different methods, Nikulin [11], and Agol, Belolipetsky, Storm, and Whyte [2] both finished the proof of finiteness by proving it for dimensions 4 through 9.

Allcock's classification of the rank 3 hyperbolic reflective lattices over $\mathbb{Z}$ is most similar to our own. He restricted his classification to the reflective ones because of their role the study of both Kac-Moody algebras and K3 surfaces. Also similar is Nikulin's classification of hyperbolic root systems of rank 3. In a paper in 3 parts, he classifies all the hyperbolic quadratic forms from a wider class of lattices which are "almost reflective." His part I contains the strongly squarefree reflective lattices, which are all of the "essential versions" of the lattices on Allcock's list. Allcock's list only contains reflective lattices, but it contains the ones that are not strongly-squarefree as well.

In Chapter 2 we give the necessary background about lattices and quadratic forms over number fields. In particular we highlight the things that are different from [3] due to the fact that we are working over a quadratic extension of $\mathbb{Q}$. Some of these differences are quite trivial while others are fairly substantial. In his very thorough book [15] on quadratic forms, O'Meara fully develops the theory of quadratic forms over Dedekind domains of arithmetic type, and we refer the reader there for any further details.

In Chapter 3 we prove versions of the lemmas from [3] that have been modified so that they now apply to lattices defined over real quadratic extensions of $\mathbb{Q}$. The fundamental chamber of the reflection part of the automorphism group of a reflective lattices is a hyperbolic polygon with finite volume and finitely many sides. All such polygons have "thin parts," which Allcock made precise by introducing three types of configurations called "short edge," "short pair," and "close pair." One of these must occur in any finite sided polygon with finite volume. We used these lemmas to generate a finite list of lattices defined over $\mathbb{Q}[\sqrt{2}]$, which we will pare down into our classification in Chapter 4.

As a consequence of our modifications of the lemmas, we get an upper bound on the discriminant of a real quadratic field over which a reflective arithmetic lattice can be defined. There are no reflective arithmetic lattices with real quadratic ground fields of discriminant larger than 27935. We prove this, and in fact we get even better bounds for the short pair and close pair cases, in Theorem 2.

In Chapter 4 we explain our method for determining which of the lattices from Chapter 3 are reflective. We introduce a method for finding an element of the automorphism group of a lattice that takes the fundamental chamber to its nearest translate along a line in hyperbolic space containing an edge of the chamber. We use this in a fast algorithm for finding all the edges of a boundary component of that fundamental domain. This algorithm is what we call "walking," and it is used to determine whether a lattice is reflective. We also describe how we resolved the few cases for which walking by itself was not fast enough.

The classification itself is in the appendix. We have organized the lattices into tables by the number of sides of the fundamental polygon. The smallest are triangles, of which there are 3. We recognize them from Takeuchi's list of arithmetic triangle fuchsian groups [16]. There are 8 triangles on Takeuchi's list with ground field $\mathbb{Q}(\sqrt{2})$. Among these eight, 3 are maximal, and these are the 3 that appear on our list. The largest polygons on our list are 24-gons with order 2 rotation. The entries in our table sorted by the norm in $\mathbb{Q}(\sqrt{2})/\mathbb{Q}$ of the determinant of the quadratic form. For each entry, we give the determinant, the shape of the fundamental domain for the action on hyperbolic space, the quadratic form, and a list of simple roots.

All of the computations were done using the PARI/GP library [17] in C/C++.

Chapter 2

Background

Throughout this paper F will be a totally real quadratic extension of $\mathbb{Q}$ whose ring of integers $\mathfrak{o}$ (or $\mathfrak{o}_F$ if it's ambiguous) is a PID. We fix an embedding of F into $\mathbb{R}$.

2.1 Lattices, quadratic forms, Minkowski space

A lattice L is a projective $\mathfrak{o}$-module[1] with an F valued symmetric bilinear form. The rank of L is the rank of its associated vector space $V = L \otimes F$. We denote the norm of a vector v with respect to the bilinear form on V by v^2. We say that L is integral if the bilinear form is $\mathfrak{o}$-valued. The scale of L is the ideal generated by all of its inner products. We say that L is unscaled if its scale is $\mathfrak{o}$. All of our lattices will be unscaled unless otherwise specified. If the bilinear form on L has signature $(n, 1)$, then $V \otimes \mathbb{R}$ is Minkowski space, which means it is a real $n+1$ dimensional vector space with a bilinear that is equivalent over $\mathbb{R}$ to the standard quadratic form of signature $(n, 1)$

$$f_0 = -x_0^2 + x_1^2 + \ldots + x_n^2 \tag{2.1}$$

[1]Since $\mathfrak{o}$ is a PID, being projective is the same as being free.

The vectors with norm 0 form a cone in Minkowski space and are called light-like. The vectors of positive norm lie outside the cone and are called space-like. The vectors of negative norm lie inside the light cone and are called time-like. The set of negative norm vectors

$$\mathfrak{C} = \{v \in V : v^2 < 0\}$$

has two components. We fix a vector p of negative norm, and declare the future cone $\mathfrak{C}^+$ to be those vectors in $\mathfrak{C}$ whose inner product with p is negative, that is

$$\mathfrak{C}^+ = \{v \in \mathfrak{C} : v \cdot p < 0\}$$

From $\mathfrak{C}^+$ we obtain a model of n-dimensional hyperbolic space Λ^n by taking the quotient of $\mathfrak{C}^+$ by the equivalence relation $v \sim \lambda v$ for all real $\lambda > 0$.

$$\Lambda^n = \mathfrak{C}^+/\sim$$

If $v \in V$ is a space-like vector, then the restriction of the bilinear form to the hyperplane $v^\perp$ has signature $(n-1, 1)$. It cuts through $\mathfrak{C}^+$, and its image in Λ^n is a geodesic hyperplane.

2.2 Units in number fields

Denote the group of units in $\mathfrak{o}$ by U (or by $U(F)$ if the field is ambiguous). Let α_0 be the fundamental unit in U. By convention we take $\alpha_0 > 0$. We have $U = \langle \alpha_0 \rangle \times \{\pm 1\}$. We define two subroups U^+ and U_1^+ to be the group

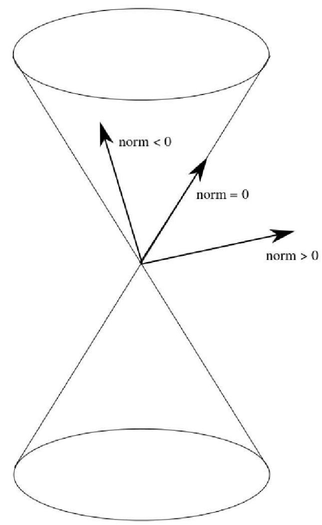


Figure 2.1: Light cone in Minkowski space

of positive units, and the group of positive units with norm 1 respectively.

$$U^+ = \{\alpha \in U : \alpha > 0\} = \langle \alpha_0 \rangle$$

$$U_1^+ = \{\alpha \in U^+ : N_{K/F}(\alpha) = 1\}$$

In a quadratic field F with norm $N_{F/\mathbb{Q}} : F \to \mathbb{Q}$, one of the following is true.

(U1) $N_{F/\mathbb{Q}}(\alpha_0) = 1$, and U_1^+ is the union of two square classes in U^+,

$$\{\alpha_0^{2k} : n \in \mathbb{Z}\} \text{ and } \{\alpha_0^{2k+1} : k \in \mathbb{Z}\}$$

(U2) $N_{F/\mathbb{Q}}(\alpha_0) = -1$, and U_1^+ is a single square-class in U^+

$$\{\alpha_0^{2k} : k \in \mathbb{Z}\}$$

We wish to draw attention to the fact that units may be positive or negative, and independently have positive or negative norm. We point this out because of what scaling by units does to vectors in lattices and to their norms. Let $v \in L$ be a lattice vector, and $\alpha \in U$ a unit. Then v and αv generate the same sublattice of L, $\langle \alpha v \rangle = \langle v \rangle$. This may mean that we want to consider αv equivalent to v. However, if direction matters then we only want to consider them equivalent if $\alpha > 0$ since if $\alpha < 0$ then αv points the opposite direction to v. For us, v and v' will be equivalent if $v' = \alpha v$ for some $\alpha > 0$ that is a unit.

The norm of αv is $\alpha^2 v^2$. We would like equivalent vectors to have equivalent norms. Since αv and v are equivalent vectors, we say that two norms are equivalent if they differ by a factor of a unit squared. The arithmeticity condition, which we will discuss later, will imply that $N_{F/\mathbb{Q}}(v^2)$ must always have the same sign as v^2, so αv^2 could only be a norm if $\alpha \in U_1^+$. Let $n \in \mathfrak{o}_F$ be a number that is a norm of a vector in L. There are two possibilities, corresponding to (U1) and (U2) above.

(N1) There are two equivalence classes of numbers in $\mathfrak{o}$ associated to the norm n, namely

$$\{\alpha n : \alpha \in U_1^+ \text{ is not a square}\} \text{ and } \{\alpha n : \alpha \in U_1^+ \text{ is a square}\}$$

(N2) There is one equivalence class of numbers in $\mathfrak{o}$ associated to the norm the norm n, namely

$$\{\alpha n : \alpha \in U_1^+\}$$

We note that when $F = \mathbb{Q}(\sqrt{2})$, (U2) and (N2) hold.

2.3 Roots

A vector $v \in L$ is called primitive if whenever we have

$$v = \alpha v'$$

for some $v' \in L$ and $\alpha \in \mathfrak{o}$, then α is a unit. A *root* is a primitive space-like vector $r \in L$ such that the reflection negating r and fixing $r^\perp$ is an automorphism of L. Reflection with respect to r is denoted R_r and is given by the formula

$$R_r(v) = v - \frac{2r \cdot v}{r^2} r \tag{2.2}$$

thus the condition for r to be a root can be stated as

$$r \cdot v \in \frac{r^2}{2}\mathfrak{o} \text{ for all } v \in L \tag{2.3}$$

2.4 Arithmeticity

Let Q be a quadratic form defined over $\mathfrak{o}_F$. The full real isometry group $O(Q)(\mathbb{R})$ is an algebraic group defined over $\mathfrak{o}$. The integral isometry group Γ is a discrete subgroup preserving a lattice. Let $\sigma \in \mathrm{Gal}(F/\mathbb{Q})$ be a nontrivial element of the Galois group. Since Q, $O(Q)(\mathbb{R})$ are defined over F, we may apply σ to Q and to any element of $O(Q)(\mathbb{R})$. If Q has signature $(2,1)$, $O(Q)(\mathbb{R}) \cong O(2,1)$. We say that Γ is arithmetic if for all non-identity elements $\sigma \in \mathrm{Gal}(F/\mathbb{Q})$, the isometry group $O(\sigma Q)(\mathbb{R})$ is isomorphic to $O(3)(\mathbb{R})$. We say that Q is arithmetic (or that the matrix for Q is an arithmetic matrix) if Q has signature $(n,1)$, and σQ is definite for all nontrivial $\sigma \in \mathrm{Gal}(F/\mathbb{Q})$.

In particular, if F is a real quadratic field, then there is only one nontrivial element of the Galois group of $F/\mathbb{Q}$. Since we can scale Q by a positive element of $\mathfrak{o}_F$ with negative galois conjugate, we may assume that σQ is positive definite. In particular, if $U(F)$ contains units with norm -1, we may assume that Q is unscaled and σQ is positive definite.

Lemma 1. *Let L be a lattice of signature $(n,1)$ defined over F whose reflection group is arithmetic.*

(i) L contains no nontrivial vectors of norm 0.

(ii) The fundamental domain P for the action of L on Λ^n has no ideal vertices.

(iii) If r is a root, then $N_{F/\mathbb{Q}}(r^2) > 0$. Equivalently, its norm r^2 has positive Galois conjugate $\overline{r}^2$.

(iv) If r and s are any vectors of L then there is a bound on the Galois conjugate of $r \cdot s$,

$$0 \leq |\overline{r \cdot s}| < \sqrt{\overline{r^2 s^2}}$$

The notation $\overline{v}^2$ and $\overline{r \cdot s}$ may seem sloppy. However, since Galois conjugation is a homomorphism, it commutes with taking inner products. So the notation is fine as long as the inner products of conjugate vectors are taken with respect to the conjugate quadratic form.

Proof. Let $v \in L$.

(i) If $v^2 = 0$, $\overline{v}^2 = 0$. Since the conjugate of the quadratic form is positive definite, this is only possible if v is the zero vector.

(ii) At any vertex of P there is a lattice vector $p \in \langle r_1, \ldots r_n \rangle^\perp$ where the r_i are roots for the stabilizer of p. If P were an ideal vertex, then p would live in $\partial\mathfrak{C}$. But by (i), L has no nontrivial vectors of norm 0.

(iii) Since σQ is positive definite, $\overline{r}^2 > 0$. Since r is a root, $r^2 > 0$. Thus

$$N_{F/\mathbb{Q}}(r^2) = r^2\overline{r}^2 > 0$$

(iv) This is a property of positive definite quadratic forms.

□

2.5 Sublattices and glue

Recall that $\mathfrak{o}$ is a PID. If the bilinear form on L is non-degenerate, then L has a generating set of size $\operatorname{rank}(L)$ which we call a basis for L.

A sublattice $M \subset L$ means an $\mathfrak{o}$-submodule of L. We say that M is saturated if we have

$$M = (M \otimes F) \cap L.$$

If $\operatorname{rank}(M) = \operatorname{rank}(L) = n$, then M has finite index in L, and there is a basis $\{v_1, \ldots, v_n\}$ for L and $a_1, \ldots a_n \in \mathfrak{o}$ such that $\{a_1 v_1, \ldots, a_n v_n\}$ is a basis for M and $(a_n) \supset \ldots \supset (a_1)$ (Theorem 81:11 in [15]). The a_i are the invariant factors of a matrix taking a basis for L to a basis for M.

Let L be a lattice with saturated sublattices M_1 and M_2, each the orthogonal complement of the other. Let π_i denote projection onto the vector space spanned by M_i. We call $v \in L$ a glue vector for M_1 and M_2 if it is not contained in the lattice $M_1 \oplus M_2$. If L is generated by M_1, M_2 and a glue vector g, we write

$$L = M_1 \oplus_g M_2$$

2.6 Duals

Every non-degenerate lattice L has a dual lattice

$$L^* = \{v \in L \otimes F : v \cdot w \in \mathfrak{o} \text{ for all } w \in L\}$$

Given a basis $v_0, \ldots v_n$ for L, the corresponding dual basis for L^* is $\hat{v}_0, \ldots, \hat{v}_n$, defined by

$$\hat{v}_i \cdot v_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \tag{2.4}$$

If L is an integral lattice, then $L \subset L^*$. In this case, we define the discriminant $\mathfrak{o}$-module to be

$$\Delta(L) := L^*/L$$

If L is non-degenerate and finite dimensional, then $\Delta(L)$ is a torsion $\mathfrak{o}$-module, and so it has an in invariant factor decomposition

$$L^*/L \cong \bigoplus_{i=1}^{n} \mathfrak{o}/a_i\mathfrak{o}$$

where the $a_i \in \mathfrak{o}$ satisfy $(a_n) \supset (a_{n-1}) \supset \ldots \supset (a_1)$. The inner product matrix Q of the basis v_i for L is the matrix that writes the vectors v_i in terms of the dual basis vectors $\hat{v}_i$. Thus all of the information about the structure of $\Delta(L)$ is encoded in Q. In particular, the ideal generated by $\det(Q)$ is the same as the ideal generated by the product of the a_i's. This ideal is independent of our choice of basis, and so we refer to it as the determinant ideal of L, denoted $\det(L)$.

Lemma 2. *Let $M \subset L$ be a sublattice of finite index such that*

$$L/M = \bigoplus_{i=1}^{n} \mathfrak{o}/b_i\mathfrak{o}$$

with $(b_n) \supset \ldots \supset (b_1)$. Let

$$b = \prod_{i=1}^{n} b_i$$

Then $\det(M) = b^2 \det(L)$.

Proof. There is a basis $\{v_1, \ldots, v_n\}$ for L such that $\{b_1 v_1, \ldots b_n v_n\}$ is a basis for M. We have

$$M \subset L \subset L^* \subset M^*$$

The corresponding dual bases for L^* and M^* are

$$\{\hat{v}_1, \ldots, \hat{v}_n\} \text{ and } \{b_1^{-1}\hat{v}_1, \ldots, b_n^{-1}\hat{v}_n\}$$

respectively. Thus M^*/L^* has the same elementary divisor decomposition as L/M, so the product of the elementary divisors of M^*/L^* is b. Thus we have $\det(M) = b^2 \det(L)$. □

2.7 Strongly squarefree lattices

A lattice L is called strongly squarefree (SSF) if the smallest generating set for $\Delta(L)$ as an $\mathfrak{o}$-module has at most $\frac{1}{2}\text{rank}(L)$ elements, and every invariant factor of $\Delta(L)$ is squarefree. The automorphism group of any non-SSF lattice is contained in the automorphism group of an SSF one. For this reason, we restrict our classification to SSF lattices, as Nikulin did in [14]. Allcock applies Vinberg's algorithm to get only the SSF lattices, and obtains the non-SSF ones from the SSF ones in other ways [3]. For rank 3, being SSF is equivalent to $\det(L)$ being squarefree.

Though we do not use the $\mathfrak{p}$-filling or $\mathfrak{p}$-duality operations in the rest of this paper, we describe them briefly here, since they are the operations by which we would turn a non-SSF lattice into an SSF one whose automorphism group contains $\text{Aut}(L)$. They serve the same purpose as the p-duality and

p-filling operations used by both Nikulin and Allcock in their respective classifications. The only difference is we replace prime numbers p with prime ideals $\mathfrak{p}$, and p-elementary abelian groups with $\mathfrak{p}$-elementary $\mathfrak{o}$-modules.

For a prime ideal $\mathfrak{p}$ of $\mathfrak{o}$, the $\mathfrak{p}$-power part of $\Delta(L)$ is the submodule annihilated by some power of the ideal $\mathfrak{p}$. The $\mathfrak{p}$-dual of a lattice is the sublattice of L^* corresponding to the $\mathfrak{p}$-power part of $\Delta(L)$, which is a direct sum $\oplus_{i=0}^{n}\mathfrak{o}/\mathfrak{p}^{a_i}$. Since L is unscaled, at least one of the summands is trivial. To get the rescaled $\mathfrak{p}$-dual of L we scale $\mathfrak{p}$-dual(L) by $\mathfrak{p}^a$ where a is the largest of the a_i. The rescaled $\mathfrak{p}$-dual is an unscaled integral lattice. Like other kinds of duals, taking the rescaled $\mathfrak{p}$-dual twice gets back the original lattice, so the lattice and its rescaled $\mathfrak{p}$-dual have the same automorphism group. All the unscaled lattices related by rescaled $\mathfrak{p}$-dualities for various primes $\mathfrak{p}$ form a duality class, all of whose members have the same automorphism groups.

If $\Delta(L)$ has any elements annihilated by $\mathfrak{p}^2 \setminus \mathfrak{p}$, then we can do an operation known as $\mathfrak{p}$-filling. Let A be the $\mathfrak{p}$-power part of $\Delta(L)$, and suppose $\mathfrak{p}^a$ is its annihilator. The $\mathfrak{p}$-filling of L is if the sublattice of L^* whose image in $\Delta(L)$ is $\mathfrak{p}^{a-1}A$. We have

$$L \subsetneq \mathfrak{p}\text{-fill}(L) \subseteq \mathfrak{p}\text{-fill}(L) \subsetneq L^*$$

so $\mathfrak{p}$-fill(L) is integral and its discriminant $\mathfrak{o}$-module is strictly smaller than that of L and $\mathrm{Aut}(L) \subseteq \mathrm{Aut}(\mathfrak{p}\text{-fill}(L))$. A finite number of $\mathfrak{p}$-filling operations followed by a finite number of $\mathfrak{p}$-duality operations turns L into an SSF lattice.

2.8 The reflection part of $\mathbf{Aut}(L)$, simple roots, and chains of roots

In [21], Vinberg gives a descrption of the action of the reflection part of $\mathrm{Aut}(L)$ on n-dimensional hyperbolic space. We repeat it here because it is important for what comes later.

The reflecting plane $V_r = r^\perp$ of the root $r \in L$ divides $V = L \otimes F$ into two halfspaces,

$$V_r^+ = \{v \in V : v \cdot r > 0\} \text{ and } V_r^- = \{v \in V : v \cdot r < 0\}$$

with $r \in V_r^+$. Let $\{r_i\}_i$ be a collection of roots of L. If we have an indexed set of roots like this we abbreviate

$$V_{r_i} = V_i$$

$$V_{r_i}^\pm = V_i^\pm$$

Let

$$P = \bigcap_i V_i^- \subset V$$

If every compact subset of $\mathfrak{C}^+$ intersects only finitely many V_i's, then we will call P a polygonal cone. Any polygonal cone can be written as the intersection of halfspaces in such a way that no V_i^- contains the intersection of all the others. When we write it this way, we say the roots whose halfspaces we intersect to get P are the roots defining P. If $P \cap \mathfrak{C}^+$ is nonempty, then the image in Λ^n of $P \cap \mathfrak{C}^+$ is nonempty and is called a polygonal cell.

The full automorphism group of L contains transformations that swap $\mathfrak{C}^+$ and $\mathfrak{C}^-$. We will not be concerned with these automorphisms, so **for us Aut(L) will the group of isometries of L that preserve $\mathfrak{C}^+$**. Let $\Gamma \subseteq \mathrm{Aut}(L)$ be the subgroup of $\mathrm{Aut}(L)$ generated by the reflections in all of the roots of L. If $r \in L$ is a root, then the image in Λ^n of $V_r \cap \mathfrak{C}^+$ is a geodesic hyperplane. The hyperplanes in Λ^n corresponding to the roots of L carve up Λ^n into polygonal cells that tile Λ^n. Each of these cells is a copy of the fundamental domain for the action of Γ on Λ^n, called the Weyl chamber of Γ. Fix a copy C of the chamber. If we write the polygonal cone whose image is C as an intersection

$$\bigcap_i V_i^- \subset V$$

such that no V_i^- contains the intersection of all the others, then the set of roots $\{r_i\}$ are a set of simple roots for L. Their reflections generate Γ, and $\mathrm{Aut}(L)$ can be written as a semidirect product

$$\mathrm{Aut}(L) = \Gamma \rtimes H$$

where H is the group of symmetries of C.

Because Γ is discrete, the angles between the $n-1$ dimensional faces of C are all of the form $\frac{\pi}{m}$ for some $m \in \mathbb{N}$. We say that L is reflective if Γ is finitely generated, or equivalently, if C has finitely many faces.

When $n = 2$, C is 2-dimensional, and its boundary is 1-dimensional. We can think of each boundary component of C as a chain of consecutive edges. More generally, we will define a chain of edges in such a way that it is not necessarily part of the boundary component of a single copy of the chamber. We start by defining a *chain of roots.*

A chain of roots is a set of roots of L that can be indexed by a set I of consecutive integers such that the following are satisfied

1. For $i, j \in I$ with $i < j$, we have $r_i \cdot r_j \leq 0$, and

$$\frac{r_i \cdot r_j}{\sqrt{r_i^2 r_j^2}} \begin{cases} > -1 & \text{if } j = i+1 \\ < -1 & \text{otherwise} \end{cases}$$

2. The intersection of the negative halfspaces

$$\bigcap_{i \in I} V_i^-$$

   is nonempty.

3. The chain is "locally simple," that is, for each consecutive pair $i, i+1 \in I$, the roots r_i, r_{i+1} are part of a system of simple roots. This means there is a copy C of the chamber where V_i^- and V_{i+1}^- are two of the halfspaces in the intersection defining C.

A chain of roots defines a polygonal cell in Λ^2, which may have infinitely many sides but is locally finite. The reflection group generated by reflections in the roots in the chain is a subgroup Γ' of Γ. Each edge of the polygonal cell is contained in a hyperplane orthogonal to a root in the chain of roots. We call the sequence of edges of the polygonal cell for a chain of roots a *chain of edges.* Each pair of consecutive roots forms a corner of a copy of the chamber for Γ that is contained in the chamber for Γ'.

A closed chain of roots is the same as a chain of roots, except it is cyclically indexed. For example, if L is a reflective lattice, and C is a copy of the chamber for Γ, a system of simple roots for L are a closed chain of roots.

Let $\Phi = \{r_i\}_{i \in I}$ be a non-closed chain of roots. If I is bounded above with $k = \max I$, we call r_k the highest root in the chain. If there exists a root r_{k+1} such that $\Phi \cup \{r_{k+1}\}$ is a chain of roots, then we say that Φ can be extended. In particular, if Φ can be extended then there is a root r_{k+1} such that $r_{k-1}, r_k,$ and r_{k+1} all bound a single copy of the chamber. We call this root the *next root* of Φ. If I is bounded below, then Φ may have a *previous root*, which we define similarly. In all of our applications, every bounded chain of roots has a next root and a previous root.

2.9 Reflective hulls and enlargements

Let L be a lattice generated by roots. The reflective hull, denoted by L^{rh}, of L is

$$L^{rh} = \left\{ v \in L \otimes F : r \cdot v \in \frac{r^2}{2}\mathfrak{o} \text{ for all roots } r \in L \right\}$$

An $\mathfrak{o}$-lattice L sits inside its reflective hull $L \subset L^{rh}$. A lattice M with

$$L \subset M \subset L^{rh}$$

is called a reflection-stable enlargement of L. Reflection-stable enlargements of L correspond to submodules of L^{rh}/L. If we take the invariant factor decomposition

$$L^{rh}/L = \bigoplus_{i=1}^{n} \mathfrak{o}/a_i\mathfrak{o}$$

with $(a_n) \supset \ldots \supset (a_1)$, that means there is a basis $x_1, \ldots x_n$ for L^{rh} such that $L = \langle a_1x_1, \ldots, a_nx_n\rangle$. We may find all reflection-stable enlargements of L by iterating over all matrices of the form

$$\begin{pmatrix} d_1 & & & \\ b_{1,2} & d_2 & & \\ \vdots & & \ddots & \\ b_{1,n} & b_{2,n} & \ldots & d_n \end{pmatrix} \tag{2.5}$$

where $(d_n) \supset (a_n)$, and $b_{i,n}$ iterates over the set of all distinct coset representatives of $\mathfrak{o}/d_i\mathfrak{o}$. For each matrix (2.5), the vectors

$$d_ix_i + \sum_{j=1}^{i-1} b_{i,j}x_j$$

$i = 1, \ldots, n$ generate a reflection-stable enlargement of L.

2.10 Root norms

Suppose we wish to list all the norms that roots of a SSF lattice L may have up to equivalence in the sense of (N1) or (N2). Let r be a root of L, $M = \langle r\rangle \oplus r^{\perp}$, and π_r, $\pi_{r^\perp}$ projection onto the F-spans of r and $r^{\perp}$ respectively. Suppose that g is a glue vector between $\langle r\rangle$ and $r^{\perp}$. Since r is a root and L is integral, we have

$$g \cdot r = \pi_r(g) \cdot r \in \frac{r^2}{2}\mathfrak{o} \cap \mathfrak{o}$$

which implies that

$$\pi_r(g) \in \langle r \rangle^{rh} \cap \langle r \rangle^*$$

Likewise, the fact that L is integral means that for all $v \in r^\perp$,

$$g \cdot v = \pi_{r^\perp}(g) \cdot v \in \mathfrak{o}$$

and so we have

$$\pi_{r^\perp}(g) \in (r^\perp)^*$$

The reflective hull of $\langle r \rangle$ is generated by $\frac{r}{2}$. Thus

$$\langle r \rangle^{rh} / \langle r \rangle \cong \mathfrak{o}/2\mathfrak{o}$$

By primitivity of r and saturatedness of $r^\perp$, the projections $\pi_r, \pi_{r^\perp}$ descend to injective $\mathfrak{o}$-module homomorphisms

$$\overline{\pi}_r : L/M \to (\langle r \rangle^{rh} \cap \langle r \rangle^*)/ \langle r \rangle$$

and

$$\overline{\pi}_{r^\perp} : L/M \to (r^\perp)^*/r^\perp = \Delta(r^\perp)$$

Thus we have established that L/M is isomorphic to an $\mathfrak{o}$-submodule of $\mathfrak{o}/2\mathfrak{o}$, and also to a submodule of $\Delta(r^\perp)$. Since $\mathfrak{o}/2\mathfrak{o}$ has 3 submodules, there are 3 cases.

1. If L/M is trivial, then

$$\det L = \det M = r^2 \det(r^\perp)$$

   In this case r^2 divides $\det L$, and so it certainly also divides $2 \det L$.

2. If $L/M \cong \mathfrak{o}/\sqrt{2}\mathfrak{o}$, then

$$2\det L = \det M = r^2 \det(r^\perp)$$

In this case r^2 divides $2\det L$.

3. If $L/M \cong \mathfrak{o}/2\mathfrak{o}$, then

$$4\det L = \det M = r^2 \det(r^\perp)$$

Since L/M injects into $\Delta(r^\perp)$, 2 divides $\det M$. Thus r^2 divides $2\det(L)$.

We see that in each case, r^2 divides $2\det L$. To list all the possible norms of roots of L, we therefore list all of the positive divisors of $2\det L$ up to whichever equivalence (N1) or (N2) applies in F. By arithmeticity, if n is the norm of a root then $N_{F/\mathbb{Q}}(n)$ must be positive. Thus when (N1) holds we throw out any norms that do not have $N_{F/\mathbb{Q}}(n) > 0$, and when (N2) holds we replace all n with $N_{F/\mathbb{Q}}(n) < 0$ by $\alpha_0 n$.

2.11 Vinberg's algorithm, Bugaenko's modification

Vinberg's algorithm, first introduced in [21], is a way of listing all simple roots of the fundamental domain of a hyperbolic reflection group starting from a known corner. Fix a corner of the chamber in Λ^n, and let $p \in \mathfrak{C}^+$ be a primitive lattice vector pointing along the corner. Let $r_1, \ldots r_n$ be simple roots for the stabilizer of p. All the faces of the chamber not passing through p correspond to roots having non-positive inner product with $p, r_1, \ldots r_n$.

New edges of the chamber can only be at certain distances from the corner. The list of these distances is discrete. The algorithm iterates over this list in increasing order, and at each possible distance checks whether there is a root. If the chamber has finitely many sides, then eventually the algorithm will find a system of simple roots.

If the chamber does not have finitely many sides, then there is always a finite stage of the algorithm at which it is possible to prove that it has infinitely many sides. Vinberg has several methods for doing this, including Proposition 1 in [21] and Proposition 4.1 in [18]. In [19], Vinberg speeds up the process of finding roots by using symmetries of the polygon. We use that idea here as Allcock did in [3]. If the chamber has infinitely many sides, then it will have a symmetry of infinite order which can be found at some finite stage of the algorithm.

Vinberg's original proof of the algorithm uses the fact that $\mathbb{Z}$ is discrete in $\mathbb{R}$ to show that the list of distances from p is discrete. In a field whose ring of integers is not discrete in the order topology on $\mathbb{R}$, there is some additional work required to get a discrete list of distances. Bugaenko's innovation, which he uses in [5], [6], and [7], was to notice that arithmeticity gives a way of restricting the inner products that roots can have with p to a discrete ordered set. The Galois conjugate of the inner product of two vectors is bounded. If we think of the elements of F as living in the plane, with $a + b\sqrt{d}$ corresponding to the point $(a, b) \in \mathbb{Q}^2$, this bound describes an upward sloping strip in the plane. The elements $a+b\sqrt{d}$ of $\mathfrak{o}$ such that the point (a, b) is inside this region are a discrete subset of $\mathbb{R}$ under the identity embedding $\mathfrak{o} \to \mathbb{R}$, and they inherit an ordering from $\mathbb{R}$.

2.12 Chamber angles over $\mathbb{Q}(\sqrt{2})$

A version of the following lemma is true for any number field. For example, the angles allowed for lattices defined over $\mathbb{Q}$ are $\frac{\pi}{n}$ with $n = 2, 3, 4,$ or 6. This is the $\mathbb{Q}(\sqrt{2})$ version.

Lemma 3. *Suppose r and s are consecutive simple roots with R and S the corresponding sides of the fundamental polygon. The angle between R and S is $\frac{\pi}{m}$ for $m = 2, 3, 4, 6,$ or 8 and up to choosing a scale for L, and replacing r or s with equivalent roots in the sense of (N2), we have that*

1. *if $m = 3$, $r^2 = s^2$.*
2. *if $m = 4$, either $r^2 = s^2$, $r^2 = \frac{s^2}{2}$, or $r^2 = 2s^2$.*
3. *if $m = 6$, either $r^2 = \frac{s^2}{3}$, or $r^2 = 3s^2$.*
4. *if $m = 8$, either $r^2 = (2 + \sqrt{2})s^2$, or $r^2 = \frac{s^2}{2+\sqrt{2}}$*

Proof. We have

$$\cos\left(\frac{\pi}{m}\right) = -\frac{r \cdot s}{\sqrt{r^2 s^2}}$$

The square of the righthand side clearly lives in $\mathbb{Q}(\sqrt{2})$. The only $m \in \mathbb{N}$ for which $\cos\left(\frac{\pi}{m}\right)$ lives in a degree 2 extension of $\mathbb{Q}(\sqrt{2})$ are $m = 2, 3, 4, 5, 6, 8$, so the angle between R and S is $\frac{\pi}{m}$ for one of these m. The $m = 5$ case does not occur, because $\cos^2\left(\frac{\pi}{5}\right)$ is not an element of $\mathbb{Q}(\sqrt{2})$

Since r and s are roots, we can write $r \cdot s$ in two ways

$$r \cdot s = -\frac{r^2\alpha}{2} \quad \text{and} \quad r \cdot s = -\frac{s^2\beta}{2}$$

where $\alpha, \beta \in \mathbb{Z}[\sqrt{2}]$ and $\alpha, \beta > 0$. We also have that

$$\cos\left(\frac{\pi}{m}\right) = -\frac{r \cdot s}{\sqrt{r^2 s^2}}$$

so

$$\cos\left(\frac{\pi}{m}\right) = \frac{\alpha}{2}\sqrt{\frac{r^2}{s^2}} = \frac{\beta}{2}\sqrt{\frac{s^2}{r^2}}$$

Now, $\cos\left(\frac{\pi}{m}\right) = \frac{\sqrt{\gamma(m)}}{2}$, where $\gamma(3) = 1$, $\gamma(4) = 2$, $\gamma(6) = 3$, $\gamma(8) = 2 + \sqrt{2}$ so

$$\gamma(m) = 4\left(\frac{\sqrt{\gamma(m)}}{2}\right)^2 = 4\left(\frac{\alpha}{2}\sqrt{\frac{r^2}{s^2}}\right)\left(\frac{\beta}{2}\sqrt{\frac{s^2}{r^2}}\right) = \alpha\beta$$

The cases listed in the statement of the lemma come from the factorizations of the various values of $\gamma(m)$ as products of positive elements of $\mathfrak{o}$ with positive field norm $N_{F/\mathbb{Q}}$.

□

2.13 A_2 corners

Let L be a lattice of signature $(2, 1)$, $\Gamma \leq \mathrm{Aut}(L)$ the reflection subgroup of the automorphism group of L, C a fixed Weyl chamber for the action of Γ

on Λ^2, and c a corner of C. Fix an orientation on $V = L \otimes F$. There are roots r, s pointing outward from walls of C that meet at c, and a primitive lattice vector $p \in V_r \cap V_s \cap \mathfrak{C}^+$ such that the basis $\{r, s, p\}$ is in the orientation on V. These conditions uniquely determine r, s, and p up to multiplication by positive units. We call this the corner basis at c, and we say that p lies along the corner c.

The roots r and s at the corner are simple roots for a rank 2 positive definite sublattice of L. The type of c is the type of that positive definite sublattce. By Lemma 3, $\langle r, s\rangle$ has type A_1^2, A_2, B_2, G_2, or $I_2(8)$.

The next several lemmas are about the structure of L at an A_2 corner. By Lemma 3, we know that $r^2 = s^2u^2$ where $u > 0$ is a unit. Since (U2) and (N2) hold when $F = \mathbb{Q}(\sqrt{2})$, we may replace s by su^{-1}, and assume that $r^2 = s^2$.

Lemma 4. *Let c be an A_2 corner of the chamber C with $\{r, s, p\}$ a corner basis at c with $r^2 = s^2$. Then we have*

$$L / \langle r, s, p\rangle \cong \mathfrak{o}/3\mathfrak{o}$$

Let bar denote passage to the quotient. If $\overline{g}$ is a generator for $L/\langle r, s, p\rangle$ then

$$L = \langle r, s\rangle \oplus_g \langle p\rangle$$

If L is strongly squarefree, then up to multiplication by a square unit, $r^2 = s^2 = 2$, and $p^2 = 3\det(L)$.

Recall that both p^2 and $\det(L)$ are both defined up to multiplication by square units. We will see that fixing a choice of square unit multiple for one of them also fixes a choice for the other.

Proof. Let $M = \langle r, s\rangle$. Then M^{rh}, the largest superlattice of M in which r and s are roots, has type G_2 and is generated over $\mathfrak{o}$ by M and $a = \frac{2r+s}{3}$. As an $\mathfrak{o}$-module, the quotient

$$M^{rh}/M \cong \mathfrak{o}/3\mathfrak{o} \cong \mathbb{F}_9$$

is isomorphic to a finite field with 9 elements, $\mathbb{F}_9$.

Let π be orthogonal projection onto $M \otimes F$. Since r and s are roots of L, we have $M \subseteq \pi(L) \subseteq M^{rh}$. If there were no glue between M and $\langle p\rangle$, the reflection that negates a and preserves $a^{\perp}$ would preserve L, and there would be a root of L in the F-span of a. But r and s are simple roots of L, so this is impossible. Therefore there must be a nontrivial glue vector g gluing M to $\langle p\rangle$. Since any nontrivial element of $\mathbb{F}_9$ generates it as an $\mathfrak{o}$-module, and $\pi(L) = M^{rh}$, we may assume that $\pi(g) = a$.

We have $g \in L$ and $3\pi(g) \in L$, thus $3(g - \pi(g)) \in L \cap (\langle p\rangle \otimes F) = \langle p\rangle$. Since $\pi(g) \notin L$, we have $g - \pi(g) \notin \langle p\rangle$. Thus

$$g - \pi(g) = \frac{\alpha p}{3}$$

for some $\alpha \in \mathfrak{o}$ representing a nontrivial coset of $\mathfrak{o}/3\mathfrak{o}$. By subtracting $\mathfrak{o}$-multiples of p, we may assume that $0 < \alpha < 3$.

We claim that $L = \langle r, s, g \rangle$. To see that this is true, let $h \in L$. If $\pi(h) \in M$ then $h - \pi(h) \in L \cap \langle p \rangle$, so $h \in \langle r, s, p \rangle \subset \langle r, s, g \rangle$. Suppose $\pi(h) \in M^{rh} \setminus M$. Then $\pi(g)$ generates all of M^{rh}/M, as an $\mathfrak{o}$-module, so there is some $z \in \mathfrak{o}$ such that $\pi(zg) = z\pi(g) = \pi(h)$. Thus $zg - h \in \langle p \rangle$, so $h \in \langle r, s, g \rangle$. Thus $L = \langle r, s, g \rangle$ and

$$L/\langle r, s, p \rangle \cong \mathfrak{o}/3\mathfrak{o}$$

is a finite $\mathfrak{o}$-module generated by $\overline{g}$.

We compute the norm of g:

$$\begin{aligned} g^2 &= \left(\frac{2r+s}{3} + \frac{\alpha p}{3}\right)^2 \\ &= \frac{4r^2 + 4r \cdot s + s^2}{9} + \frac{\alpha^2 p^2}{9} \\ &= \frac{r^2}{3} + \frac{\alpha^2 p^2}{9} \end{aligned}$$

Recall from the proof of Lemma 3 that

$$-r \cdot s = -\frac{r^2 \beta}{2} = -\frac{s^2 \delta}{2}$$

where

$$\beta\delta = \gamma(3) = 1$$

Thus r^2 is divsible by 2.

Since $g, p \in L$, we have $g \cdot p \in \mathfrak{o}$.

$$g \cdot p = \frac{\alpha p^2}{3}$$

Since α represents a nontrivial coset of $\mathfrak{o}/3\mathfrak{o}$ and 3 is prime in $\mathfrak{o}$, this implies that p^2 is divisible by 3.

Since $L = \langle r, s, g \rangle$, their inner product matrix has determinant $\det(L)$.

$$\begin{aligned}\det(L) &= \begin{vmatrix} r^2 & -\frac{r^2}{2} & \frac{r^2}{2} \\ -\frac{r^2}{2} & r^2 & 0 \\ \frac{r^2}{2} & 0 & g^2 \end{vmatrix} \\ &= \left(\frac{r^2}{2}\right)^2 \left(3g^2 - r^2\right) \\ &= \left(\frac{r^2}{2}\right)^2 \left(3\left(\frac{r^2}{3} + \frac{\alpha^2 p^2}{9}\right) - r^2\right) \\ &= \left(\frac{r^2}{2}\right)^2 \left(\frac{\alpha^2 p^2}{3}\right)\end{aligned}$$

Since 2 divides r^2 and 3 divides p^2, both $\frac{r^2}{2}$ and $\frac{\alpha p^2}{3}$ are elements of $\mathfrak{o}$. Because L is SSF, $\det(L)$ is squarefree. Thus the squared factor

$$\left(\frac{\alpha r^2}{2}\right)^2$$

must be a unit. This means we must have

$$\alpha r^2 = 2u$$

where $u \in U(F)$ is a unit. Because we know that 2 divides r^2 and $r^2, \overline{r}^2 > 0$, we can conclude that up to scaling r by a positive unit, $\alpha \in U(F)$ and $r^2 = 2$. Here we are using the fact that (U2) holds in $\mathbb{Q}(\sqrt{2})$ to say that $r^2 = 2$ and not 2 times a non-square unit.

Finally, by Lemma 2, we have

$$9 \det L = \det(M \oplus \langle p \rangle) = \det(M) \det \langle p \rangle = 3p^2$$

so

$$3\det(L) = p^2$$

□

For the rest of this section we will be working with an A_2 corner in an SSF lattice with corner basis $\{r, s, p\}$ where $r^2 = s^2 = 2$.

Lemma 5. *Let c be an A_2 corner of C, and let*

$$g_\pm = \frac{r+s}{2} \pm \frac{r-s}{6} + \frac{p}{3} \tag{2.6}$$

Exactly one of g_+ or g_- is an element of L, and

$$L = \langle r, s\rangle \oplus_{g_\pm} \langle p\rangle$$

Proof. We have that g_+ and g_- are not both elements of L, since

$$g_+ + g_- = \frac{r+s}{2} \notin L$$

On the other hand, one of $g_\pm$ must be in L, by the following argument. As we saw in the proof of Lemma 4, there is a glue vector g such that

$$L = \langle r, s\rangle \oplus_g \langle p\rangle$$

with $\pi(g) = \frac{2r+s}{3} =: a$. Also recall from the proof of Lemma 4 that

$$g = \pi(g) + \frac{\alpha p}{3} = a + \frac{\alpha p}{3}$$

where $0 < \alpha < 3$ represents some coset of $\mathfrak{o}/3\mathfrak{o}$. Since $L = \langle r, s, g \rangle$, the inner product matrix of the generators has determinant $\det(L)$.

$$\begin{aligned}
\det(L) &= \begin{vmatrix} 2 & -1 & 1 \\ -1 & 2 & 0 \\ 1 & 0 & g^2 \end{vmatrix} \\
&= 3g^2 - 2 \\
&= 3\left(a^2 + \alpha^2 \frac{p^2}{9}\right) - 2 \\
&= 3\left(\frac{2}{3} + \alpha^2 \frac{p^2}{9}\right) - 2 \\
&= -\alpha^2 \frac{p^2}{3}
\end{aligned}$$

Thus $\alpha = \pm 1$. If $\alpha = 1$, then $g = g_+$ and

$$L = \langle r, s \rangle \oplus_{g_+} \langle p \rangle \tag{2.7}$$

If $\alpha = -1$, then

$$g_- = 2g + p - r \in L$$

Since g_- projects to an element of $\langle r, s \rangle^{rh} \setminus \langle r, s \rangle$, it is a glue vector between $\langle r, s \rangle$ and $\langle p \rangle$, and

$$L = \langle r, s \rangle \oplus_{g_-} \langle p \rangle \tag{2.8}$$

□

If L has the form (2.7) (resp. (2.8)) at the A_2 corner c of the chamber C, then we say that C has positive (resp. negative) glue at c, or that the pair (c, C) has positive (resp. negative) glue. Note that this depends on our fixed choice of $\mathfrak{C}^+$ and orientation on $V = L \otimes F$.

Let $\phi \in \text{Aut}(L)$ be an automorphism of L that preserves the future cone $\mathfrak{C}^+$. Let C be a copy of the chamber, c a corner of C. Then ϕ takes C to another copy of the chamber C' and c to a corner c' of C'. Let $\{r, s, p\}$ and $\{r's', p'\}$ be corner bases at the corners c and c' respectively. Then $\phi(p) = p'$ and $\phi(\{r, s\}) = \{r', s'\}$. As we will see in this next lemma, ϕ may or may not preserve the orientation on V.

Recall that $\text{Aut}(L)$ is the group of automorphisms of L that preserve the future cone $\mathfrak{C}^+$.

Lemma 6. *Suppose C and C' are copies of the chamber for L, and c and c' are A_2 corners of C and C' respectively. Then there is an automorphism $\phi \in Aut(L)$ that takes the pair (c, C) to (c', C'). If (c, C) and (c', C') both have positive glue or both have negative glue, then ϕ preserves the orientation on $V = L \otimes F$. If they have opposite glue, then ϕ reverses the orientation on V.*

Proof. First suppose that c and c' both have positive glue. Then the linear transformation defined by

$$\phi : (r, s, p) \mapsto (r', s', p')$$

preserves the gluing, meaning $\phi(g_+) = g'_+$ where g'_+ is defined as g_+ but with all non-primed things in (2.6) primed. Since $L = \langle r, s, g_+ \rangle = \langle r', s', g' \rangle$, ϕ is an automorphism of L. The argument is the same if c and c' both have negative glue.

Now suppose that c has positive glue and c' has negative glue. Let ψ be the linear transformation defined by

$$\psi : (r, s, p) \mapsto (s', r', p')$$

Then $\psi(g_+) = g'_-$, so $\psi \in \mathrm{Aut}(L)$. The ordered basis $\{s', r', p'\}$ has the opposite orientation from V, so ψ is orientation reversing. □

Lemma 7. *Let C be a copy of the chamber for L. If C has two A_2 corners c and c' in the same boundary component of C, then either they both have positive glue, or both have negative glue.*

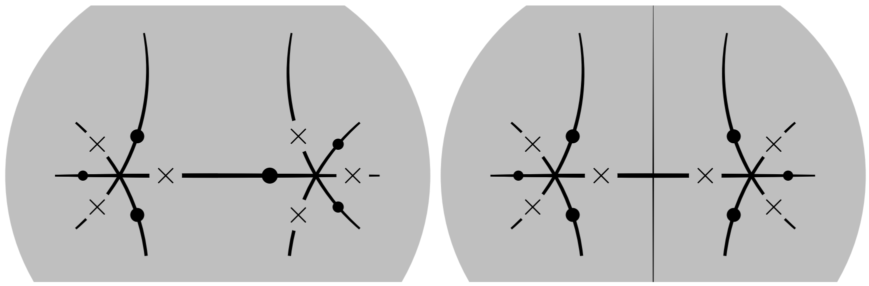
Proof. By Lemma 6, there exists $\phi \in \mathrm{Aut}(L)$ that preserves the chamber C and takes c to c'. Because c and c' are corners of the same boundary component of the chamber C, there is a chain of roots $\{r_1, \ldots, r_k\}$ with $r = r_1, s = r_2, r' = r_{k-1}$, and $s' = r_k$ such that for each i with $1 \leq i \leq k-1$, r_{i+1} is the next root after r_i.

Suppose that c and c' have opposite glue. Then ϕ is orientation reversing, and

$$\phi(r_1) = r_k, \phi(r_2) = r_{k-1}$$

Because each root r_{i+1} is the next root after r_i, the automorphism ϕ takes the chain to itself, with $\phi(r_i) = r_{k+1-i}$. Thus it is a reflection and not a glide reflection. But then there must be a root t such that $R_t = \phi$, and the reflecting plane V_t cuts through the interior of C. This is impossible since C is a chamber. Therefore c and c' cannot have opposite glue.

□



On the left is a chamber with two A_2 corners with the same glue. On the right, we see that if a chamber had 2 A_2 corners with opposite glue, a reflecting plane would cut through the chamber, making it not a chamber and giving a contradiction. The •'s show the glue vector and its orbit under order 3 rotation, projected into the plane spanned by the roots. The ×'s show where the projection of the opposite glue vector and its orbit would be if it were present.

Figure 2.2: If a chamber has 2 A_2 corners, they must have the same glue.

Corollary 1. *If a chamber C for Γ has consecutive A_2 corners, then all of the corners in the boundary component containing those two are A_2 corners.*

Proof. If C has two consecutive A_2 corners c_1 and c_2, then by Lemma 6 there is an automorphism ϕ that takes a corner basis at c_1 to a corner basis at c_2. By Lemma 7, ϕ is orientation preserving. Since ϕ preserves C, and c_1 and c_2 are in the same boundary component of C, ϕ also preserves the chain of roots for the boundary component of C containing both c_1 and c_2. Therefore ϕ preserves adjacency of roots in that chain. By applying all integer powers of ϕ to the chain, we see that all corners in that boundary component are A_2 corners. □

Chapter 3

Getting a finite list

The first step in the proof of Theorem 1 is to generate a finite list of matrices, each the inner product matrix for roots in a chain of length 3, 4, or 5. The construction of this list ensures that any reflective lattice contains some such chain. It will also contain a lot of non-reflective lattices, and some of the lattices listed will be redundant. Those will all be sorted away in the second step.

3.1 Hyperbolic polygons are thin

We take the following definitions from Allcock, whose short edges in [3] were a version of Nikulin's thin parts of polygons in [14]. Our discussion will now involve hyperbolic polygons, since the chamber of a reflective lattice has finitely many sides.

Let P be a hyperbolic polygon. At any corner of P, the angle bisector means the ray originating at the vertex that passes through the interior of P and bisects the angle at the corner[1]. Similarly the perpendicular bisector of an

[1]Allcock gives a definition of angle bisector at an ideal vertex so that this definition extends to hyperbolic polygons with ideal vertices. We do not need that here, since none of our polygons have any ideal vertices.

edge of P means a ray that passes through P that originates at the midpoint of the edge and is orthogonal to it.

An edge of P is called a *short edge* if the angle bisectors at its endpoints intersect. Theorem 1 in [3] says that any finite sided polygon has a short edge in one of three configurations. We restate the three possibilities, as we will refer to them often.

1. P has a short edge orthogonal to at most one of its neighbors such that those neighbors are not orthogonal to each other.

2. P has at least 5 edges, and *short pair* (S, T). Here S is a short edge orthogonal to both its neighbors, T is one of these neighbors, and the perpendicular bisector of S intersects the angle bisector emanating from the opposite end of T.

3. P has at least 6 edges, and a *close pair* of short edges (S, S'). Here S and S' are both short edges with a common neighbor that is not a short edge, both S and S' are orthogonal to their neighbors, and their perpendicular bisectors intersect.

We can be even more precise about how thin these thin parts are. There are bounds on the distances between the edges adjacent to a short edge, short pair, or close pair. These bounds are given in Lemmas 3-5 in [3]. Together with the following adaptation of Lemma 6 in [3], these bounds are what goes into generating our list.

Lemma 8 (Pair of Roots)**.** *Suppose $a, b \in L$ are roots whose associated unit vectors satisfy $-\hat{a} \cdot \hat{b} = k > 0$. Then the inner product matrix of a and b is an F multiple of*

$$\begin{pmatrix} 2u & -uv \\ -uv & 2v \end{pmatrix} \tag{3.1}$$

for some positive $u, v \in \mathfrak{o}$ with $uv = 4k^2$. Furthermore, if the quadratic form q on L is arithmetic, then for any non-identity element $\sigma \in Gal(F/\mathbb{Q})$ we have $\sigma(u), \sigma(v) > 0$ and $\sigma(uv) < 4$.

Proof. Because a and b are roots with $a \cdot b < 0$, we can write

$$a \cdot b = -\frac{va^2}{2} \quad \text{and} \quad a \cdot b = -\frac{ub^2}{2} \tag{3.2}$$

for some positive $u, v \in \mathfrak{o}$. Thus we have

$$-a \cdot b = \sqrt{(-a \cdot b)^2} = \sqrt{\left(\frac{va^2}{2}\right)\left(\frac{ub^2}{2}\right)} = \frac{\sqrt{a^2}\sqrt{b^2}}{2}\sqrt{uv}$$

Using the fact that

$$-\hat{a} \cdot \hat{b} = k$$

we get that

$$-a \cdot b = \sqrt{a^2}\sqrt{b^2}k$$

Thus

$$uv = 4k^2$$

We then scale the inner product so that $a \cdot b = -uv$, and then by (3.2) we have $a^2 = 2u$ and $b^2 = 2v$.

The matrix (3.1) is the matrix for q restricted to the F-span of a and b. If q is arithmetic then for any non-identity element $\sigma \in \mathrm{Gal}\,(F/\mathbb{Q})$, the quadratic form σq is positive-definite on σFL. Then for any such σ, we have $\sigma(a^2), \sigma(b^2) > 0$, which gives us

$$\sigma(u) > 0 \quad \text{and} \quad \sigma(v) > 0 \tag{3.3}$$

We also have that

$$-1 \leq -\sigma q\left(\widehat{\sigma(a)}, \widehat{\sigma(b)}\right) \leq 1 \tag{3.4}$$

The unit vectors $\hat{a}$ and $\hat{b}$ do not live in L, but they do live in the lattice EL where E is the splitting field over F of $(x^2 - a^2)(x^2 - b^2) \in F[x]$. Thus σ extends to an automorphism of E. If we fix a lift of σ to $\mathrm{Gal}\,(E/\mathbb{Q})$, the following makes sense:

$$\sigma\left(-\hat{a}\cdot\hat{b}\right) = \sigma\left(\frac{-a\cdot b}{\sqrt{a^2}\sqrt{b^2}}\right) = \frac{\sigma(-a\cdot b)}{\sqrt{\sigma(a^2)}\sqrt{\sigma(b^2)}} = \frac{\sigma(uv)}{\sqrt{4\sigma(uv)}} = \frac{\sqrt{\sigma(uv)}}{2}$$

combining this with (3.3) and (3.4) we obtain

$$0 < \sigma(uv) < 4$$

□

3.2 Good factorizations of elements of $\mathfrak{o}$

We use Lemma 8 repeatedly in the proofs of Lemmas 9-11, which play the same role in our classification that Allcock's lemmas 7-9 do in his. There are two ways in which our lemmas are different from Allcock's. First, our version of Lemma 8 involves a bound on the conjugate of the inner product of any pair of roots, and so our versions of Lemmas 9-11 do as well. Second, if there are to be finitely many factorizations of an integer $z \in \mathfrak{o}$ as a product of two integers, then factorizations need to be defined up to some sort of equivalence in the sense of (N1) or (N2). We make that precise now before we state the lemmas. The following discussion applies to any $F = \mathbb{Q}(\sqrt{d})$ with $\mathfrak{o}_F$ a PID, not $\mathbb{Q}(\sqrt{2})$ exclusively.

Given a positive number $z \in \mathfrak{o}_F$, we wish to recover all inner product matrices for pairs of roots a, b with $-a \cdot b = z$. We will call a factorization $z = uv$ a good factorization if $u, v, \sigma(u), \sigma(v) > 0$ and $\sigma(uv) < 4$ for all non-identity $\sigma \in \mathrm{Gal}(F/\mathbb{Q})$. If $z = uv$ is a good factorization and $\alpha \in U(F)$ is a unit, then $z = (u\alpha)(\alpha^{-1}v)$ is also a good factorization of z provided $\alpha > 0$ and $\sigma(\alpha) > 0$ for all $\sigma \in \mathrm{Gal}(F/\mathbb{Q})$. The group U_1^+ of positive units with norm 1 acts on the set of good factorizations of z, and thus on the set of configurations of pairs of roots a, b with $-a \cdot b = z$ by

$$\alpha . \begin{pmatrix} 2u & -uv \\ -uv & 2v \end{pmatrix} = \begin{pmatrix} 2u\alpha & -uv \\ -uv & 2v\alpha^{-1} \end{pmatrix} \tag{3.5}$$

We say that two good factorizations (u, v) and $(u\alpha, v\alpha^{-1})$ are equivalent if $\alpha \in U_1^+$ is a square. In other words, the equivalence classes are the orbits of the subgroup of U_1^+ consisting of square units. If (U1) holds in F, then there are two equivalence classes of good factorizations for z, and if (U2) holds there is one equivalence class.

3.3 The root configuration lemmas

We follow the notation used in [3]. P is the fundamental domain in Λ^2 for an arithmetic reflection group acting discretely by isometries on $\mathbb{R}^{2,1}$. We let $r, (s), t, (s'), r'$ be roots in a lattice L corresponding to consecutive faces of P, called $R, (S), T, (S'), R'$ respectively. When we have no S and S' the edge T is short. When we have S but no S', (S, T) is a short pair. When we have both S and S' then (S, S') is a close pair. We define $\mu := -\hat{r} \cdot \hat{t}$, $\mu' := -\hat{r}' \cdot \hat{t}$, $\lambda = -\hat{r} \cdot \hat{r}'$, and $K = 1 + \mu + \mu' + 2\sqrt{1+\mu}\sqrt{1+\mu'}$.

The numerical inequalities given by Lemmas 3-5 in [3] all hold. In fact, we can do slightly better for a trivial reason. The arithmeticity condition means that lattices defined over nontrivial extensions of $\mathbb{Q}$ have no isotropic vectors, so the bounds on the inner products between unit normal vectors of adjacent sides are all strict. The bounds are as follows.

Short edge: $0 < \mu < 1$, $0 \leq \mu' < 1$, and $\lambda < K < 7$

Short pair: $1 < \mu < 3$, $0 \leq \mu' < 1$, and $\lambda < K < 5 + 4\sqrt{2}$

Close pair: $1 < \mu, \mu' < 3$, and $\lambda < K < 15$

We are working in a real quadratic extension $F = \mathbb{Q}(\sqrt{d})$ of $\mathbb{Q}$. Bar denotes Galois conjugation $\bar{\cdot} : \sqrt{d} \mapsto -\sqrt{d}$.

Lemma 9 (short edge)**.** *Suppose P has consecutive edges R, T, R'. Suppose also that T is a short edge and $R \not\perp T, R'$. Then, up to scale, there are finitely many possibilities for the inner product matrix of r, t, r'.*

To list those possibilities, we let (A, B) vary over a set of representatives for the equivalence classes of all good factorizations of elements z of $\mathfrak{o}$ with $0 < z, \overline{z} < 4$. Let (A', B') vary over all those same pairs and also $(0, 0)$.

Given such an A, B, A', B', let (C, C') vary over a set of representatives for the equivalence classes of all good factorizations of all $z \in \mathfrak{o}$ with $0 < z < 4K^2$, $0 < \overline{z} < 4$. If $A', B' > 0$, we only consider pairs (C, C') satisfying

$$\frac{AB'C'}{A'BC} = u^2 \tag{3.6}$$

where $u \in U_1^+$ is a square unit. Let $\beta = A'BCu$. For some such A, B, A', B', C, C', the the inner product matrix of r, t, r' is a F multiple of

$$\begin{pmatrix} 2AB' & -ABB' & -\beta \\ -ABB' & 2BB' & -A'B'B \\ -\beta & -A'B'B & 2A'B \end{pmatrix} \quad \text{if } A', B' > 0 \tag{3.7}$$

$$\begin{pmatrix} 2AC & -ABC & -ACC' \\ -ABC & 2BC & 0 \\ -ACC' & 0 & 2AC' \end{pmatrix} \quad \text{if } A', B' = 0 \tag{3.8}$$

which we keep only if the matrix has signature $(2, 1)$ and its Galois conjugate is positive definite.

Proof. We know that $0 < \mu < 1$ since R and T intersect inside of Λ^2. By Lemma 8, we know that there exist $A, B \in \mathfrak{o}$ satisfying $A, B, \overline{A}, \overline{B} > 0$, $AB < 4$, and $\overline{AB} < 4$ such that the inner product matrix of r and t is an F multiple of

$$\begin{pmatrix} 2A & -AB \\ -AB & 2B \end{pmatrix} \tag{3.9}$$

If $T \not\angle R'$ then we apply Lemma 8 again to get that there exist $A', B' \in \mathfrak{o}$ satisfying the same conditions as A, B such that the inner product matrix of r' and t has the same form as (3.9) with A', B' in place of A, B.

We apply Lemma 8 once again to r and r', using the fact that $\lambda < K^2$ to get that there exist $C, C' \in \mathfrak{o}$ satisfying $C, C', \overline{C}, \overline{C'} > 0$, $CC' < 4K^2$, and $\overline{CC'} < 4$ such that the inner product matrix of r and r' is and F multiple of

$$\begin{pmatrix} 2C & -CC' \\ -CC' & 2C' \end{pmatrix} \tag{3.10}$$

We have the following:

$$\frac{2C'}{2C} = \frac{r'^2}{r^2} = \frac{r'^2 t^2}{r^2 t^2} = \frac{(2A')(2B)}{(2A)(2B')}$$

so in order to put these matrices together in a sensible way, we need

$$AB'C' = A'BC \tag{3.11}$$

Our choice of C may C' fail to satisfy (3.11) in two possible ways. The ratio

$$\frac{AB'C'}{A'BC} \tag{3.12}$$

either is or is not a square unit. If (3.12) is a square unit, then all hope is not lost. As discussed in 3.2, we get equivalent configurations of the roots r and r' if instead of CC' we choose the factorization $(Cu)(u^{-1}C')$ where u is a square unit.

Making this substitution into (3.11), we get (3.6). So if a square unit can be found that makes (3.6) hold, we can combine the 3 2×2 matrices into a 3×3 matrix of the form (3.7) by letting $\beta = A'BCu$ and choosing the scale at which $t^2 = 2BB'$.

In the case where $T \perp R'$, we take $A' = B' = 0$. As before r, t have inner product matrix an F multiple of (3.9) and r, r' have inner product matrix an F multiple of (3.10). The condition (3.6) is trivially true since both sides of the equation are 0. We put together the two matrices by choosing the scale at which $r^2 = 2AC$. The resulting inner product matrix is (3.8).

□

Lemma 10 (short pair)**.** *Suppose that P has at least 5 edges, and R, S, T, R' are consecutive edges with (S, T) a short pair. Then the inner product matrix of r, s, t, r' is one of finitely many possibilities up to scale.*

To list those possibilities, let (A, B) vary over a set of representatives for the equivalence classes of all good factorizations of all $z \in \mathfrak{o}$ with $4 < z < 36$ and $\overline{z} < 4$. Let (A', B') vary over a set of representatives for the equivalence classes of all good factorizations of all $z \in \mathfrak{o}$ with $z, \overline{z} < 4$, and also $(0, 0)$.

For fixed A, B, A', B', let (C, C') vary over a set of representatives for the equivalence classes of all good factorizations of all $z \in \mathfrak{o}$ such that $4 < z < 4K^2$, $0 < \overline{z} < 4$, there exists a square unit u such that (3.6) holds if $A', B' > 0$, and

$$N := 4 + 4\frac{CC' + \beta + A'B'}{AB - 4} \tag{3.13}$$

is an element of $\mathfrak{o}$. As in Lemma 9, $\beta = AB'C'u^{-1}$.

The inner product matrix for r, t, r' has the form (3.7) *or* (3.8). *Since we only want arithmetic quadratic forms of signature* $(2,1)$, *we keep it on our list only if it has signature* $(2,1)$ *and its galois conjugate is positive definite.*

Fixing A, B, A', B', C, C', let k vary over all positive elements of $\mathfrak{o}$ dividing N (also up to equivalence in the sense of (U1) or (U2)). For some such A, B, A', B', C, C', k, the inner product matrix of r, s, t, r' is an F multiple of

$$\begin{pmatrix} 2AB' & 0 & -ABB' & -\beta \\ 0 & 2A'B\frac{N}{k^2} & 0 & -A'B\frac{N}{k} \\ -ABB' & 0 & 2BB' & -A'B'B \\ -\beta & -A'B\frac{N}{k} & -A'B'B & 2A'B \end{pmatrix} \quad \text{if } A', B' > 0 \tag{3.14}$$

$$\begin{pmatrix} 2AC & 0 & -ABC & -ACC' \\ 0 & 2AC'\frac{N}{k} & 0 & -AC'\frac{N}{k} \\ -ABC & 0 & 2BC & 0 \\ -ACC' & -AC'\frac{N}{k} & 0 & 2AC' \end{pmatrix} \quad \text{if } A', B' = 0 \tag{3.15}$$

Proof. We apply the same argument as in the proof of Lemma 9 with different bounds on μ, and λ to get that the inner product matrix of r, t, r' is one of (3.7) or (3.8) up to scale. What changes here is that $1 < \mu < 3$, so $4 < AB < 36$, and $\lambda > 1$ so $4 < CC' < 4k^2$. Otherwise everything is identical.

Given a matrix of the form (3.7) or (3.8) we wish to build the possible 4×4 inner product matrix for r, s, t, r' of which it is a submatrix. Consider the root s. Let s_0 be the projection of $\hat{r}'$ onto the F-span of s, so that the projection of r' is $\sqrt{r'^2}s_0$. Since s is a root, this lies in $\frac{s}{2}\mathfrak{o}$. Therefore there exists some $k \in \mathfrak{o}$ with $k > 0$ such that $\sqrt{r'^2}s_0 = -\frac{ks}{2}$. This can be rearranged to get that

$$s = -2\frac{\sqrt{r'^2}s_0}{k} \tag{3.16}$$

We also have that r' is a root, so $s \cdot r' \in \frac{r'^2}{2}\mathfrak{o}$, so there exists $M \in \mathfrak{o}$ with $M < 0$ such that $s \cdot r' = \frac{Mr'^2}{2}$. We have

$$\begin{aligned}
M &= 2\frac{s \cdot r'}{r'^2} \\
&= 2\frac{\sqrt{r'^2}s \cdot s_0}{r'^2} \quad \text{using the projection} \\
&= 4\frac{r'^2}{kr'^2}s_0^2 \quad \text{using (3.16)} \\
&= -\frac{4}{k}\left(1 + \frac{\lambda^2 + 2\lambda\mu\mu' + \mu'^2}{\mu^2 - 1}\right) \quad \text{using Lemma 4 from [3]}
\end{aligned}$$

Writing λ, μ, μ' in terms of A, B, A', B', C, C' shows that $M = -\frac{N}{k}$ where N is as in (3.13). Since $M, k \in \mathfrak{o}$, we have $N \in \mathfrak{o}$ with k dividing N. Now we can fill in the rest of the matrix. Since (S, T) is a short pair, we have $s \cdot r = s \cdot t = 0$. We have

$$s \cdot r' = \frac{Mr'^2}{2} = -\frac{Nr'^2}{2k}$$

Finally, we can compute s^2:

$$s^2 = \left(\frac{2}{k}\sqrt{r'^2}s_0\right)^2 = \frac{4r'^2}{k^2}s_0^2 = \frac{4r'^2}{k^2}\left(-\frac{Mk}{4}\right) = -\frac{r'^2}{k}\left(-\frac{N}{k}\right) - \frac{Nr'^2}{k^2}$$

□

Lemma 11 (close pair). *Suppose that P has at least 6 edges, and R, S, T, S', R' are consecutive edges with (S, S') a close pair. Then the inner product matrix of r, s, t, s', r' is one of finitely many possibilities up to scale.*

To list those possibilities, let (A, B) vary over a set of representatives for the equivalence classes of all good factorizations of all $z \in \mathfrak{o}$ with $4 < z < 36$ and $0 < \overline{z} < 4$. Let (A', B') vary over the same set of pairs.

For fixed $A, B, A'B'$, let (C, C') vary over a set of representatives for the equivalence classes of all good factorizations of all $z \in \mathfrak{o}$ such that $4 < z < 4K^2$, $\overline{z} < 4$, there exists a square unit u such that (3.6) *holds, and $N, N' \in \mathfrak{o}$. N is defined as in* (3.13) *and N' is similarly defined but with primed and non-primed letters swapped. As in Lemma 9, $\beta = AB'C'u^{-1}$.*

The inner product matrix for r, t, r' has the form (3.7) *or* (3.8)*. Since we only want arithmetic quadratic forms of signature* $(2, 1)$*, we keep only those matrices with* $(2, 1)$ *where the galois conjugate is positive definite.*

Fixing A, B, A', B', C, C', let k and k' vary over all positive elements (up to equivalence in the sense of (U1) or (U2)) of $\mathfrak{o}$ dividing N and N' respectively such that

$$\frac{\gamma k^2}{A'BN} \quad \text{and} \quad \frac{\gamma k'^2}{AB'N'} \tag{3.17}$$

are both elements of $\mathfrak{o}$ where

$$\gamma = \frac{2\beta}{kk'}\left(2 + \frac{\beta}{CC'} - \frac{(2CC' + \beta)^3}{(AB-4)(A'B'-4)(CC')^2}\right) \tag{3.18}$$

For some such $A, B, A', B', C, C', k, k'$, the inner product matrix of r, s, t, s', r' is an F multiple of

$$\begin{pmatrix} 2AB' & 0 & -ABB' & -AB'\frac{N'}{k'} & -\beta \\ 0 & 2A'B\frac{N}{k^2} & 0 & \gamma & -A'B\frac{N}{k} \\ -ABB' & 0 & 2BB' & 0 & -A'B'B \\ -AB'\frac{N'}{k'} & \gamma & 0 & 2AB'\frac{N'}{k'^2} & 0 \\ -\beta & -A'B\frac{N}{k} & -A'B'B & 0 & 2A'B \end{pmatrix} \tag{3.19}$$

Proof. We repeat the argument from Lemma 10 using the bounds $1 < \mu, \mu' < 3$ and $1 < \lambda < K$ to get A, B, A', B', C, C', u satisfying (3.6) so that the inner product matrix of r, t, r' has the form (3.7) up to scale. The same argument in Lemma 10 that gets us N and k satisfying (3.13) gets us N, k, N', k' here, by replacing primed letters by unprimed ones. Thus the only entries remaining to be filled in to get the matrix (3.19) are the $s \cdot s'$ ones. As in Lemma 10 we use the fact that s and s' are both roots, so we may write them as (3.16) for s, and similarly for s' with primed and unprimed letters switched. We compute:

$$\begin{aligned} s \cdot s_0' &= \frac{4}{kk'}\sqrt{r^2 r'^2} s_0 \cdot s_0' \\ &= \frac{4}{kk'}\sqrt{r^2 r'^2}\left(\lambda + 4 - \frac{(\lambda+4)^3}{(\mu^2-2)(\mu'^2-1)}\right) \quad \text{using Lemma 5 from [3]} \end{aligned}$$

Writing λ, μ, μ' in terms of A, B, A', B', C, C' shows that $s \cdot s' = \gamma$ as defined by (3.18). We get the condition (3.17) from the fact that s and s' are both roots, and so $\gamma = s \cdot s'$ lies in both $\frac{s^2}{2}\mathfrak{o}$ and $\frac{s'^2}{2}\mathfrak{o}$. □

3.4 The box picture

The bounds on μ, μ', and λ can be used to show that for d large enough, there are no reflective arithmetic lattices of signature $(2,1)$ defined over $\mathbb{Q}(\sqrt{d})$. The argument in Theorem 2 could be adapted to get similar bounds on the discriminant for a field $F/\mathbb{Q}$ of any fixed degree.

Theorem 2 (Discriminant bounds). *Suppose $d > 0$ is a square-free integer, and $F = \mathbb{Q}(\sqrt{d})$ with $\mathfrak{o}_F$ a PID.*

1. *If $d > 27935$, or if $d \equiv 2$ or $3 \mod 4$ and $d > 6983$ then there are no no polygons that are fundamental chambers for reflective lattices with ground field F, so there can be no such lattices.*

2. If $d > 1296$, or if $d \equiv 2$ or $3 \mod 4$ and $d > 324$ then there are no polygons with a short pair or close pair that are fundamental chambers for reflective hyperbolic lattices with ground field F, so there can be no such lattices.

Proof. The ring of integers $\mathfrak{o}$ in a quadratic extension of $\mathbb{Q}$ can be thought of as a lattice in $\mathbb{R}^2$. Each element $z \in \mathfrak{o}$ can be written as

$$z = a + b\sqrt{d}$$

If $d \equiv 2$ or $3 \mod 4$ then $a, b \in \mathbb{Z}$. If $d \equiv 1 \mod 4$ then either $a, b \in \mathbb{Z}$ or $a, b \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$. The point in $\mathbb{R}^2$ corresponding to $a + b\sqrt{d}$ will be $(a, b\sqrt{d})$.

Let D_i, $i = 0, 1, 2, 3$ be the coset of $i \mod 4$. Suppose $d > d'$, and $\mathfrak{o}, \mathfrak{o}'$ are the rings of integers in $\mathbb{Q}(\sqrt{d}), \mathbb{Q}(\sqrt{d'})$ respectively. Notice that if $d, d' \in D_1$, then the vertical spacing between the lattice points of $\mathfrak{o}$ is greater than for $\mathfrak{o}'$. This is also true for $d, d' \in D_2 \cup D_3$.

The inequalities used to find A, B, A', B', C, C' in Lemmas 9, 10, and 11 describe lines in the plane that specify regions where the lattice points corresponding to $AB, A'B'$, and CC' must lie. For example, the condition $0 < z < 4$ says that the lattice point corresponding to $z = a + b\sqrt{d}$ lies between two downward sloping lines given by the equations $a + b = 0$ and $a + b = 4$. The condition $0 < \overline{z} < 4$ says that the lattice point corresponding to z lies between two upward sloping lines given by $a - b = 0$ and $a - b = 4$. In this way, upper and lower bounds on both z and its conjugate $\overline{z}$ define a box in which the corresponding lattice point in $\mathbb{R}^2$ lives. The bounds do not vary with d, since they come from the geometry of a hyperbolic polygon. For large values of d, the boxes have very few or no points in them. In the example in Figure 3.1, the only points the only points left in the box for $d > 5$ will be $1, 2,$ and 3.

The box in which the point corresponding to AB lives for both the short pair and close pair cases is shown in Figure 3.2, and is empty for the values of d stated in (2).

The box in which the point corresponding to CC' lives for the short edge case is never empty because it always contains the points corresponding to $1, 2,$ and 3. However, for the values of d specified in (1), these are the only 3 points in that box. Having $CC' = 1, 2,$ or 3 means that R and R' intersect, so P is a triangle. All arithmetic triangles were described by Takeuchi in [16]. For all the ones whose ground field is a quadratic extension of $\mathbb{Q}$, that ground field is one of $\mathbb{Q}(\sqrt{2})$, $\mathbb{Q}(\sqrt{3})$, $\mathbb{Q}(\sqrt{5})$, $\mathbb{Q}(\sqrt{6})$ and $2, 3, 5,$ and 6 are all smaller than 27935.

□

3.5 Results of this chapter

We wrote a C program using the PARI library that iterates over all the points in the boxes in order to build all matrices of the form (3.7), (3.8), (3.14), (3.15), and (3.19) with entries in $\mathbb{Q}(\sqrt{2})$. We removed all matrices that were non-arithmetic or did not have rank 3. We then computed all reflection stable enlargements of each root configuration, and separated out the squarefree ones from the non-squarefree ones. We also removed any where the root sequence was not a chain of roots due to not being locally simple. The number of things on each list is summarized in Table 3.1. We did not take care at this point to prevent redundancies in our enumeration, and so the numbers in the table are large overestimates. These computations took varying amounts of time to run on a personal laptop. The close pair cases took about 10 days, the short pair non-orthogonal cases took about 2, and everything else took a few hours or even less.

Table 3.1: Summary of root configurations on our list

matrix type	root configs	enlargements	squarefree enlargements
(3.7)	282	1489	130
(3.8)	1736	7181	341
(3.14)	5777	3653	233
(3.15)	88836	38871	1096
(3.19)	97526	13688	581

In Table 3.1, the numbers in "root configs" column are how many root configurations there are of each type. The types come from Lemmas 9, 10, and 11. The number in the "enlargements" column is how many reflection stable enlargements therre are for each type of root configuration became. In the short pair and close pair cases, it is smaller than the total number of root configurations. This is because reflection-stable enlargments of a subset of the roots were discarded when the remaining roots were not primitive. The number in the “squarefree enlargements” column is the number of enlargements that are strongly squarefree.

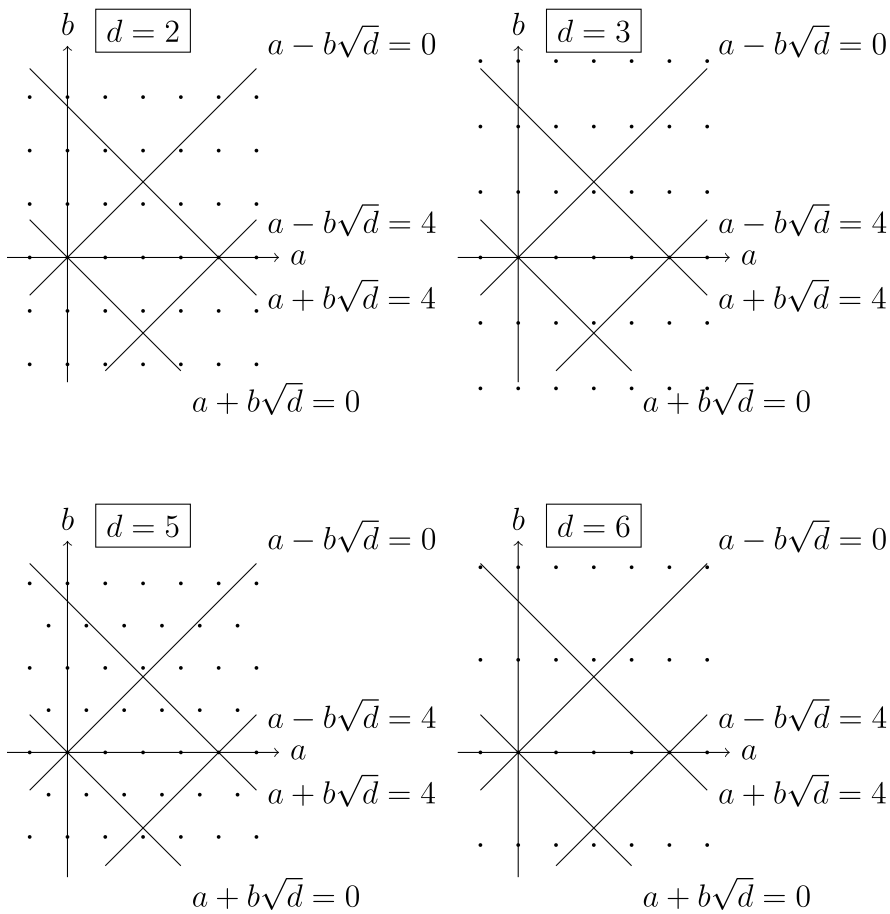


Figure 3.1: The box picture for bounds $0 < AB, \overline{AB} < 4$ with $d = 2$, $d = 3$, $d = 5$, $d = 6$.

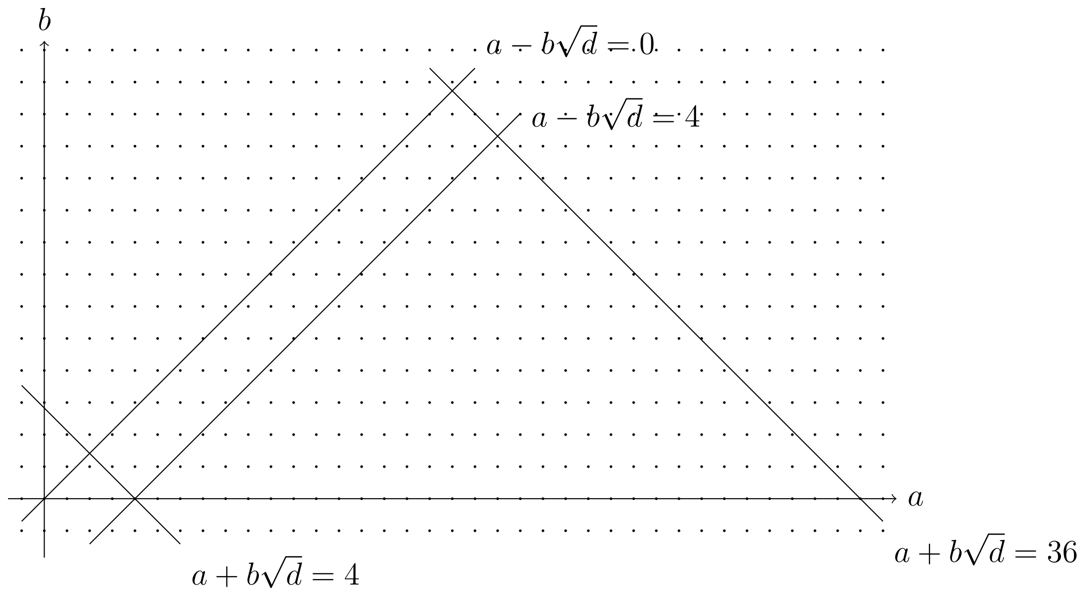


Figure 3.2: The box picture for the bounds $4 < AB < 36$, $0 < \overline{AB} < 4$ from Lemma 10 and 11 shown with $d = 2$

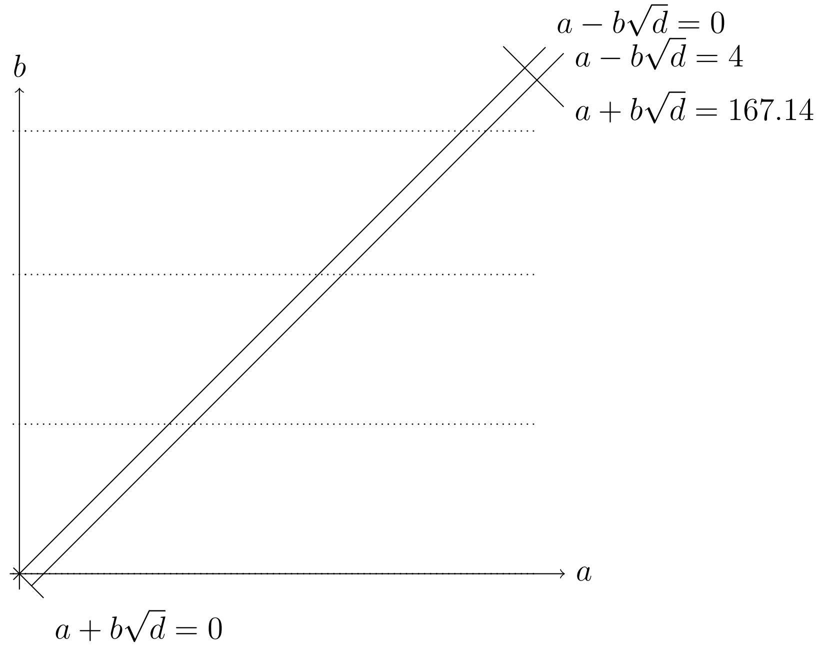


Figure 3.3: The box picture for the bounds $0 < CC' < 167.14$, $0 < \overline{CC'} < 4$ from Lemma 9 shown with the large value of d, $d = 611$

Chapter 4

Determining Reflectivity

The next step in the proof of Theorem 1 is to determine which of the quadratic forms on the finite list from the previous section (summarized in Table 3.1) are reflective and which are not. The tool that is usually used for this sort of computation is Vinberg's algorithm. We wrote and implemented a version of his algorithm for our problem, but found that it would be impossible to use it to do the entire calculation, since that would have taken an unreasonable amount of time. We therefore came up with a new way to structure the search for new roots, which we think of as "walking around the chamber." The key difference between Vinberg's algorithm and walking is the search space. Whereas Vinberg's algorithm searches for new roots inside an n-dimensional polygonal cone, walking has a much more restricted search space. The roots we seek satisfy additional inner product conditions that make the search space 1-dimensional. The idea is to build a chain of roots for a boundary component of the chamber starting from a single corner. The new technique that makes walking fast enough to finish the computation was a method for finding the nearest translate of a corner of the chamber along an edge. This technique also gave us a way of estimating computation times for searches done by walking.

With walking and finding the nearest translate, we were able to determine reflectivity for all but 48 of the strongly squarefree lattices on our list, and we were able to give time estimates for how long the 48 remaining cases would take. These time estimates were all unreasonably long (as high as 10^{44} days). The methods we used to resolve these cases were more technical, and only worked because a portion of the chamber had already been found by walking. As one might reasonably suspect, none of the remaining 48 lattices were reflective. A description of walking and the method for finding the nearest translate make up the bulk of this section. The resolution of the final 48 cases comes at the end.

Because we now have a finite list of lattices, we know the following.

Corollary 2 (Corollary of Lemma 1)**.** *All of the chambers for the lattices listed at the end of Section 3 enumerated in Table 3.1 have at least one boundary component with no consecutive A_2 corners.*

Proof. The data for each of the lattices on our list is a quadratic form and a chain of $3, 4,$ or 5 (not necessarily simple) roots. At least one corner in each chain is not an A_2 corner. By Lemma 1, consecutive A_2 corners will never occur in the boundary component of that chamber containing that corner. □

For the remainder of this section, L will be a SSF lattice from the list tabulated in Table 3.1. We fix a choice of future cone $\mathfrak{C}^+$ and an orientation on $V = L \otimes F$. For each lattice, we have a chain of $3, 4,$ or 5 roots. This chain of roots usually does not bound a single chamber. Since we wish to find a chain of roots that does bound a single copy of the chamber, we begin our searches with a consecutive pair of roots from our chain, which we call r_1 and r_2. Since they are consecutive we know that they are part of a system of simple roots for L.

4.1 Simple and non-simple roots at a corner

Recall that the roots at a corner of the chamber are simple roots for a positive definite lattice of type $A_1^2, A_2, B_2, G_2,$ or $I_2(8)$. A lattice of type B_2 also contains a lattice of type A_1^2. The simple roots of the A_1^2 sublattice are roots of the B_2 lattice, though they are not simple roots of it. Similarly, a lattices of type G_2 contain sublattices of type A_1^2 and A_2. Lattices of type $I_2(8)$ contain sublattices of type A_1^2 and B_2.

Sometimes we will need to work with roots at a corner that are not simple roots of L, but are simple roots for a sublattice of L. In particular, at $B_2, G_2,$ and $I_2(8)$ corners we would sometimes like to work with the simple roots of the A_1^2 sublattice.

Conversely, if we have a pair of roots that span a sublattice of type A_1^2, we would like to have a way of determining whether they are contained in a sublattice of L of type $B_2, G_2,$ or $I_2(8)$, or none of these. That is, we would like to be able to say whether or not they are simple roots of L.

The following lemma gives us a way to tell whether roots whose mirrors intersect are simple roots of L, and a way to go back and forth between simple and non-simple roots at a corner.

Lemma 12. *Let r and s be roots of L such that $V_r \cap V_s$ is negative definite.*

1. *Suppose r and s are simple roots such that the angle between the associated lines R and S in Λ^2 is $\frac{\pi}{m}$ with $m = 4, 6,$ or 8. Then there is a root r' such that $V_{r'} \cap V_s = V_r \cap V_s$ and the line associated to r' in Λ^2 is orthogonal to S.*

2. *If r and s are roots that are orthogonal but possibly not simple, then there is a way to find the root r' such that $V_{r'} \cap V_s = V_r \cap V_s$ and r' and s are simple roots.*

Proof. These constructions are, in some sense, opposites of each other.

1. For each m, there are a few reflections we can apply to r and/or s to obtain r':

   If $m = 4$, then $r' = R_r(s)$. Note that in this case $r'^2 = s^2$

   If $m = 6$, let $s' = R_r(s)$. Then $r' = R_{s'}(r)$. Note that in this case $r'^2 = r^2$, and by Lemma 3, r^2 is equal to either $3s^2$ or $\frac{s^2}{3}$.

   If $m = 8$, let $r'' = R_r(s)$. Then $r' = R_{r''}(s)$ Note that in this case $r'^2 = s^2$

2. By part (1), if r and s are not simple, then up to scaling r or s by a unit one of the following is true:

   (a) $r^2 = s^2$ and the angle between the lines R' and S is one of $\frac{\pi}{4}$ or $\frac{\pi}{8}$

(b) r^2 is equal to either $3s^2$ or $\frac{s^2}{3}$ and the angle between the lines R' and S is $\frac{\pi}{6}$

We do not know whether r and s are simple roots or not. If their norms do not satisfy (a) or (b), they must be simple and there is nothing further to check.

If $r^2 = s^2$, we let t be a primitive lattice vector in the span of $r - s$. We check whether t is a root by checking whether the reflection R_t preserves L. If t is not a root, then r and s were simple. If t is a root, we must check whether or not s and t are simple. If $t^2 \neq s^2$, then there is no mirror bisecting the angle between t's mirror and s's mirror, so t is the desired root r'. If $t^2 = s^2$, we let t' be a primitive lattice vector in the span of $t - s$, and we check whether t' is a root by checking whether the reflection $R_{t'}$ preserves L. If it is a root, then $r' = t'$, and if not then $r' = t$.

If $r^2 = 3s^2$, then we let t be a primitive lattice vector in the span of $\frac{r-3s}{2}$. If t is a root, then t is the desired simple root. Otherwise r and s are simple roots. If $r^2 = \frac{s^2}{3}$ then we switch the labels on r and s and run the same argument.

□

Lemma 13. *Let r and s be simple roots at a corner of the chamber C. Let q be the projection of r onto V_s*

$$q = r - \frac{r \cdot q}{q^2} q$$

Then the image of r under reflection with respect to q is a root.

Proof. If the sublattice spanned by r and s has any type except A_2, then by Lemma 12 there is a root t in the span of q. The reflection R_t is an automorphism of L, and so $R_q(r) = R_t(r)$ is a root.

If the sublattice spanned by r and s has type A_2, then there is no root in the span of q since if there were the corner would have type G_2 and r and s would not be simple roots. Recall that we may assume $r^2 = s^2$. The composition of reflections

$$R_r \circ R_s$$

is an order 3 rotation preserving the vector p that lies along the corner, and

$$R_q(r) = R_r \circ R_s(r)$$

Thus $R_q(r)$ is a root, even though R_q is not an element of $\mathrm{Aut}(L)$. $\square$

4.2 Finding the shortest translation along a line

Recall that the hyperbolic plane Λ^2 is tiled by chambers for the reflection subgroup Γ of $\mathrm{Aut}(L)$. If r is a root of L, then the image of $V_r \cap \mathfrak{C}^+$ in Λ^2 is a line R that contains an edge of a copy of the chamber. Let $H_r \leq \mathrm{Aut}(L)$ be the subgroup consisting of all translations along R. Fix a chamber C with an edge contained in R. We call the image of C under an element of H_r a translate of C along R. Since $\mathrm{Aut}(L)$ is discrete, if H_r is nontrivial then C has a nearest translate in either direction along R. When H_r is nontrivial, it is isomorphic to $\mathbb{Z}$. We will give a method for computing a generator of H_r in this case.

Let $\{r_1, r_2, p_1\}$ be a corner basis at a corner c_1 of C, suppose that H_{r_2} is nontrivial, and let ϕ be a generator of H_{r_2}. Then ϕ and ϕ^{-1} translate in opposite directions along R_2. We will establish a convention by which one direction is positive and the other negative. Intuitively, the positive direction is to the side of R_1 where the next edge of the chamber C would go. To make this precise, consider $\phi(r_1)$. Let s be the projection of $\phi(r_1)$ onto $r_2^{\perp}$. By Lemma 13, we know that

$$r_1' = R_s(\phi(r_1))$$

is a root of L. We say that ϕ is a translation in the positive direction if $\{r_1, r_2, r_3 := r_1'\}$ is a chain of roots.

We will now describe the process by which we can write down a matrix for ϕ. We make use of the classical correspondence, due to Gauss and Dedekind, between quadratic forms defined over a field and ideals in quadratic extensions of that field. For an exposition, see [9] Chapter 9. Let

$$q = 2\left(r_1 - \frac{r_1 \cdot r_2}{r_2^2}\right)$$

be twice the projection of r_1 onto $r_2^{\perp}$. The lattice spanned by q and p is a rank 2 integral sublattice of L. If $D = -q^2p^2$ is squarefree in F, then $\langle q, p\rangle$ is isometric to the ring of integers $\mathfrak{o}_K$, which is a lattice in the quartic field

$$K = F(\sqrt{D}) = \mathbb{Q}(\sqrt{2}, \sqrt{D})$$

with quadratic form given by the norm $N_{K/F}$. Define $\varphi : \langle p, q \rangle_F \to K$ by

$$\varphi : \begin{array}{rl} q & \mapsto 1 \\ q^2 p & \mapsto \sqrt{-p^2 q^2} \end{array} \tag{4.1}$$

The inner product matrix for $q, q^2 p \in L$ is q^2 times the one for $1, -p^2 q^2 \in K$.

Lemma 14. *Let*

$$G = \{u = a + b\sqrt{D} \in U(K) : a, b \in F, a > 0, N_{K/F}(u) = 1\} \tag{4.2}$$

(Note that a and b might not be elements of $\mathfrak{o}_F$, since 1 and $\sqrt{D}$ may not be an integral basis for K/F.) Then G is a rank 1 free subgroup of $U(K)$, and H_{r_2} is isomorphic to a (finite index) subgroup $H \leq G$ such that under the correspondence given by (4.1), the translation taking the corner c_1 to its nearest translate along R_2 corresponds to a generator of H.

Proof. Let φ be the map defined by (4.1). Suppose m is an isometry of K. Then $M = \varphi^{-1} \circ m \circ \varphi$ is an isometry of the subspace $r_2^{\perp}$ of $V = L \otimes F$. M can be extended to all of V by declaring that it fixes the subspace spanned by r_2. We are interested in the isometries of K that induce isometries of V preserving L, $\mathfrak{C}^+$, and the orientation on V. These isometries descend to hyperbolic translations that preserve the line in Λ^2 containing R_2.

Let m be an isometry of K defined on the basis $1, \sqrt{D}$ for K by

$$\left\{ \begin{array}{ccc} 1 & \mapsto & a + b\sqrt{D} \\ \sqrt{D} & \mapsto & c + d\sqrt{D} \end{array} \right\}$$

Since m is an isometry we have

$$1 = N_{K/F}(m(1)) = a^2 - b^2 D \tag{4.3}$$

and

$$-D = N_{K/F}(m(\sqrt{D})) = c^2 - d^2\sqrt{D} \tag{4.4}$$

Alternative forms of (4.3) and (4.4) that will be useful for our calculations are

$$b^2D = a^2 - 1 \text{ and } c^2 = D(d^2 - 1) \tag{4.5}$$

The fact that m is an isometry also means that $m(1)$ and $m(\sqrt{D})$ must have inner product 0. We compute their inner product.

$$\begin{aligned}
m(1) \cdot m(\sqrt{D}) &= \frac{N_{K/F}(m(1) + m(\sqrt{D})) - N_{K/F}(m(1)) - N_{K/F}(m(\sqrt{D}))}{2} \\
&= \frac{N_{K/F}(a + c + (b + d)\sqrt{D}) - 1 + D}{2} \\
&= \frac{(a + c)^2 - (b + d)^2D - 1 + D}{2} \\
&= \frac{a^2 + 2ac + c^2 - b^2D - 2bdD - d^2D - 1 + D}{2} \\
&= \frac{a^2 + 2ac + D(d^2 - 1) - (a^2 - 1) - 2bdD - d^2D - 1 + D}{2} \quad \text{using (4.5)} \\
&= ac - bdD
\end{aligned}$$

Thus we need

$$ac = bdD \tag{4.6}$$

If we square both sides of (4.6) and make substitutions from (4.5), we get that we need

$$\begin{aligned}
a^2D(d^2 - 1) &= (a^2 - 1)Dd^2 \\
a^2d^2 - a^2 &= a^2d^2 - d^2 \\
a^2 &= d^2 \\
a &= \pm d
\end{aligned}$$

By (4.6), if $d = a$ then $c = bD$, and if $d = -a$ then $c = -bD$. Thus if m is an isometry of K, then m acts on K either by scaling by u, or by scaling 1 by u and scaling $\sqrt{D}$ by $-u$ where $u = a + b\sqrt{d} \in U(K)$ has norm 1. We will show that if the map induced by m on V preserves the positive cone $\mathfrak{C}^+$ and the orientation on V then only the first of these options is possible, and also we must have $a > 0$.

If the map m induces on V is to preserve the future cone $\mathfrak{C}^+$, then $m(\sqrt{D})$ needs to have negative inner product with $\sqrt{D}$. We compute their inner product.

$$\begin{aligned}
\sqrt{D} \cdot m_u(\sqrt{D}) &= \frac{N_{K/F}(\sqrt{D} + m_u(\sqrt{D})) - N_{K/F}(\sqrt{D}) - N_{K/F}(m_u(\sqrt{D}))}{2} \\
&= \frac{N_{K/F}(\sqrt{D} + c + d\sqrt{D}) + 2D}{2} \\
&= \frac{c^2 - (d+1)^2 D + 2D}{2} \\
&= \frac{c^2 - d^2 D - 2dD - D + 2D}{2} \\
&= \frac{N_{K/F}(c + d\sqrt{D}) - 2dD + D}{2} \\
&= \frac{-D - 2dD + D}{2} \\
&= -dD
\end{aligned}$$

Since D is positive, d must be positive in order for this to be negative. If the map m induces on V also preserves the orientation on V, then we need $m(1)$ to have positive inner product with 1. A similar calculation to the one above shows that we must have $a > 0$. Thus $a = d > 0$, and m is multiplication by $u = a + b\sqrt{D}$. Together, $N_{F/K}(u) = 1$ and $a > 0$ imply that $u, u^{-1} > 0$. The maps induced by inverses are translations in opposite directions by the same amount.

The element u preserves a finitely generated $\mathfrak{o}_F$-submodule of K that contains the $\mathfrak{o}_F$-module generated by 1 and $\sqrt{D}$. In particular, u is contained in a finitely generated $\mathfrak{o}_F$ module, so $\mathfrak{o}_F[u]$ is finitely generated as an $\mathfrak{o}_F$-module. Therefore $u \in \mathfrak{o}_K$, so u is an element of the unit group $U(K)$.

We now show that

$$G = \{u = a + b\sqrt{D} : N_{K/F}(u) = 1 \text{ and } a > 0\}$$

is a rank 1 subgroup of $U(K)$. First we show that G is a subgroup. Let $v = a + b\sqrt{D}, w = a' + b'\sqrt{D} \in G$. Since elements of G have norm 1, the inverse of v is its conjugate, $a - b\sqrt{D}$, which is also an element of G. We have

$$vw = (a + b\sqrt{D})(a' + b'\sqrt{D}) = aa' + bb'D + (ab' + ba')\sqrt{D}$$

Since v and w both have norm 1, their product also has norm 1. We want to show

$$aa' + bb'D > 0 \tag{4.7}$$

If b and b' have the same sign, then (4.7) is true since the lefthand side is a sum of positive numbers. If b and b' have opposite signs, then we use the fact that

$$a^2 - b^2 D = 1 > 0$$

from which it follows that

$$a > |b|\sqrt{D}$$

and likewise

$$a' > |b'|\sqrt{D}$$

Thus,

$$aa' > |bb'|D$$

so we have

$$aa' - |bb'|D > 0$$

Since $bb' < 0$, $|bb'| = -bb'$, so (4.7) holds.

We now show that G has rank 1. Because the elements of G have norm 1, they live in the kernel of the restriction of $N_{K/F}$ to $U(K)$. Dirichlet's Unit Theorem says that the rank of the unit group of K is $s_1 + s_2 - 1$, where s_1 is the number of real embeddings of K into R, and s_2 is the number of conjugate pairs of complex embeddings of K into $\mathbb{C}$. By arithmeticity,

$$-p^2q^2 > 0 \text{ and } -\overline{p^2q^2} < 0,$$

so K has exactly 2 real embeddings and one pair of complex embeddings. Thus $U(K)$ has rank 2. If $u \in U(F) \setminus \{\pm 1\}$, then $N_{K/F}(u) = u^2$. Thus if we restrict $N_{K/F}$ to $U(K)$, its image in $U(F)$ is nontrivial. Therefore the image has rank 1, and so the kernel also has rank 1. The kernel consists of all units of norm 1, and it isomorphic to $\mathbb{Z} \times \mathbb{Z}/2$. The subgroup G consisting of only those units $a + b\sqrt{D}$ with $a > 0$ is isomorphic to $\mathbb{Z}$.

Let H be the subgroup of G whose induced maps on V preserve L. If H is nontrivial, then it has finite index in G, and is generated by some power of u.

□

4.3 Walking

We begin with a lemma that will give us the termination condition for the walking algorithm.

Lemma 15. *Let $\Phi = \{r_i\}_{i \in I}$ be a chain of roots all of whose edges bound a single copy C of the chamber for L. Let c_i be the corner of the chamber formed by the roots $r_i, r_{i+1} \in \Phi$. If Φ is large enough, then there is a lattice automorphism ψ taking a corner basis at c_i to a corner basis at c_j for some $i \neq j$. By applying all powers of ψ to the roots in Φ we obtain all of the roots whose mirrors make up a single boundary component of C. In particular, if C has a boundary component with at least 2 edges, then it has infinitely many edges if and only if it has an automorphism of infinite order.*

Proof. By Lemma 3 there are finitely many m for which $\frac{\pi}{m}$ could be an angle of the chamber C. Each corner c_i has corner basis $\{r_i, r_{i+1}, p_i\}$ with

$$L/\langle r_i, r_{i+1}, p_i \rangle$$

a finite $\mathfrak{o}$-module. As we showed for the A_2 case in Lemma 5, there are finitely many non-isomorphic ways to glue $\langle r_i, r_{i+1} \rangle$ to $\langle p_i \rangle$. The same is true for the other possible corner angles. If c_i and c_j are two corners of the same type with the same gluing, then the linear transformation defined by $(r_i, r_{i+1}, p_i) \mapsto (r_j, r_{j+1}, p_j)$ is an automorphism of L that preserves $\mathfrak{C}^+$ and the orientation on $L \otimes F$.

If I is large enough then by the pigeonhole principle there are two corners in the boundary component of C that have the same type and the same glue. Let ψ be the automorphism taking one to the other. Since the mirrors of the roots in Φ all bound a single copy of the chamber, ψ preserves adjacency of roots. If ψ has finite order k, then if we apply $\psi, \psi^2, \ldots, \psi^{k-1}$ to the roots of Φ and adjoin them to Φ we get a closed chain of roots Φ that bound a chamber with finitely many sides.

If ψ has infinite order, then

$$\bigcup_{m \in \mathbb{Z}} \psi^m(\Phi)$$

is an infinite chain of roots whose mirrors make up an entire boundary component of C. □

We are now ready to discuss walking. Walking is a method for extending a bounded chain of roots by finding the next root if it exists. There is no built in way of detecting whether or not there is always a next root. However, in all of our lattices, we were able to prove that there is always a next root before even if it is hard to find it.

Like Vinberg's algorithm, the walking algorithm does a search for new roots in discrete batches ordered by a quantity called *height*. The height of a potential new root is directly related to the hyperbolic distance from the root's mirror to a known corner of the chamber.

The inputs to Vinberg's algorithm are a quadratic form Q for a lattice L, and a pair of simple roots r_1 and r_2 at one corner of the chamber. Recall that a choice of future cone $\mathfrak{C}^+$ and orientation on V determines a corner basis $\{r_1, r_2, p_1\}$ that is in the orientation on V. As described in section 2.9, we can use Q to get a list of possible norms for roots. The height of a vector v with respect to p_1 is

$$\mathrm{ht}(v) = \frac{(v \cdot p_1)^2}{v^2}$$

The algorithm searches for roots in batches ordered by increasing height with respect to p_1. We call this kind of search a *batch search.* If we require the roots to satisfy certain additional inner product conditions, we call it a restricted batch search. Walking involves two kinds of restricted batch searches. Suppose we have a chain of roots $\{r_i\}_{i \in I}$ with $\max(I) = k + 1$.

1. If we are looking for an A_2 corner, so that $r_k^2 = 2$, we seek roots of norm 2 whose inner product with r_{k+1} is equal to -1. We call this a batch search of type I.

2. If we are looking for a non-A_2 corner, we look for roots whose inner product with r_{k+1} is 0. (In this case we will be able to apply Lemma 12 to get the next simple root.) We call this a batch search of type II.

Walking lets us take advantage of the things we know about a partial chamber to deduce information about the next root, which we need in order to justify restricting a batch search. The next lemma tells us precisely how we extend a bounded chain of roots when the next root exists.

Lemma 16. *Let $\Phi = \{r_1, \ldots, r_{k+1}\}$ be a bounded chain of roots of L all of whose mirrors bound a single copy of the chamber, and assume the next root* *of Φ exists. For any pair of consecutive roots r_i and r_{i+1}, let $\frac{\pi}{m_i}$ be the angle between the corresponding edges R_i and R_{i+1}. Then one of the following is true:*

1. *m_k is even and $r_{k+1}^2 \neq 2u^2$ for any unit $u \in U(F)$.*

2. *m_k is even and $r_{k+1}^2 = 2u^2$ for some unit $u \in U(F)$*

3. *$m_k = 3$*

In each case we are able to find the next root r_{k+2} by a combination of translations and restricted batch searches.

Proof. In all 3 cases, we may apply Lemma 14 to find the nearest translate of r_k along R_{k+1}. Let ϕ be a generator for $H_{r_{k+1}}$, chosen so that ϕ is a translation the positive direction as defined 4.2 (here $i = 1$).

Suppose either (1) or (2) holds. Let R'_k be the line in Λ^2 associated to the root $r'_k = \phi(r_k)$. Since ϕ is an isometry, the angle between R_{k+1} and R'_k is $\frac{\pi}{m_k}$. Since (1) or (2) is true, by Lemma 12, there is a root s such that $s^\perp \cap r_{k+1}^\perp = {r'_k}^\perp \cap r_{k+1}^\perp$, and s is orthogonal to r_{k+1}. Let $r''_k = R_s(r'_k)$. The composition $R_s \circ \phi$ is a lattice isometry that preseves $\mathfrak{C}^+$ and reverses the orientation on V. We have that

$$R_s \circ \phi(r''_k - r_k) = R_s \circ \phi(r''_k) - R_s(r'_k) = R_s \circ \phi(r''_k) - r''_k$$

But $R_s \circ \phi(r''_k) = r_k$, so $R_s \circ \phi$ negates the subspace spanned by $r''_k - r_k$. Thus it must be a a reflection and not a glide reflection. Let t be a primitive lattice vector in the span of $r''_k - r_k$. Then t is a root orthogonal to r_{k+1}. Let c be the corner formed by t and r_{k+1}.

The root t has the property that if there is a root whose reflecting plane intersects R_{k+1} between c_k and c, it makes an A_2 corner with r_{k+1}. If this were not the case, then ϕ would not take C to its nearest translate along R_{k+1}. In other words, t is the nearest root to c_k whose mirror makes an angle of $\frac{\pi}{2}$ with R_k. Using Lemma 12 we can find the simple root t' satisfying $t' \cap r_k = t \cap r_k$

If (1) holds, then t' is the next root.

If (2) holds, then we do a restricted batch search of type I to see if there is a root r_{k+2} of height less that $\mathrm{ht}(t')$ that makes an A_2 corner with r_{k+1}. If our restricted batch search finds a root before passing the height of t', then that is the next root. Otherwise t' is the next root.

If (3) holds, then we know that m_{k+1} will be even by Lemma 1. Let r_{k+2} be the next root. By Lemma 12, there is a root r'_{k+1} such that $r'^{\perp}_{k+1} \cap r^{\perp}_k = r^{\perp}_{k+1} \cap r^{\perp}_k$, and r'_{k+1} is orthogonal to r_k. A restricted batch search of type II will always find r'_{k+1}. Then we can find r_{k+1} by part (2) of Lemma 12.

□

Now we have all that we need to state the walking theorem. We assume from here on out that all of our bounded chains of roots have a next root and a previous root.

Theorem 3 (Walking). *Let $\Phi = \{r_1, \ldots, r_{k+1}\}$ be a bounded chain of roots all of whose mirrors bound a single copy of the chamber C, and let c_i be the corner* *formed by r_i and r_{i+1}. Φ can be extended by repeatedly adjoining the next root. After adjoining a finite number of roots, either Φ will become a closed chain of roots, or else there will be an orientation preserving automorphism ψ that takes a basis at some corner c_i to a basis at the last corner c_k. After applying all powers of ψ to the roots in Φ and adjoining all these roots to Φ, Φ completely describes a boundary component of C.*

Proof. From a corner c_i, the only inputs needed to either find the nearest translate or do a restricted batch search are the roots r_i and r_{i+1}. Thus if we know at least one corner, we can always find the next root by Lemma 16. By Lemma 15, if Φ never becomes a closed chain then after adjoining some finite number of roots to Φ there will be an automorphism ψ taking a basis at some lower corner c_i to a basis at the highest corner c_k. In fact, we will always have $i = 1$, because if $i > 1$, then since ψ is an orientation preserving automorphism that preserves adjacency of roots in the chain, it takes c_1 to a corner lower than c_k, so we would have already found ψ before getting to c_k.

If ψ has finite order, then applying all powers of ψ to the roots in Φ and adjoining their images to Φ makes Φ a closed chain. Otherwise applying all powers of ψ to the roots of Φ and adjoining their images to Φ makes Φ into a chain of roots whose mirrors are all part of a single boundary component of C with infinitely many sides.

□

4.4 An example

This example demonstrates how walking works in practice. The norm-angle sequence for the polygon that we build in this example is

$$(2+\sqrt{2})_8(6+4\sqrt{2})_3(6+4\sqrt{2})_2(50+35\sqrt{2})_2$$

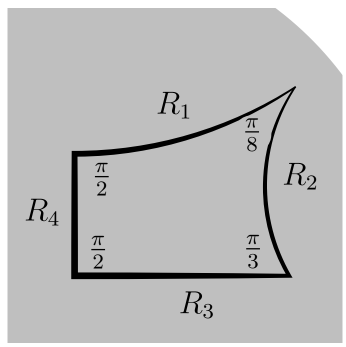


Figure 4.1: An 8,3,2,2 polygon

Our code took less than a second to do these computations. All the numbers involved in the computation are in Table 4.1. In the table, elements $a+b\sqrt{2} \in \mathfrak{o}_F$ are represented as pairs $(a\ b)$. Vectors are represented as triples of pairs $(a_1\ b_1|a_2\ b_2|a_3\ b_3)$. Some vectors appear with their norms. Elements $(a_1+b_1\sqrt{2})+(a_2+b_2\sqrt{2})\sqrt{D} \in \mathfrak{o}_K$ are represented as pairs of pairs $(a_1\ b_1|a_2\ b_2)$.

To start off, we know the quadratic form Q and the roots r_1 and r_2 at the corner c_1 of the chamber C. Their mirrors make an angle of $\frac{\pi}{8}$. The vector $p_1 \in \mathfrak{C}^+$ is a primitive lattice vector in $\langle r_1, r_2\rangle^\perp$ for which we compute coordinates. Let $u_0 = 1+\sqrt{2}$ be a fundamental unit in $U(F)$. Since $r_2^2 = 2u_0^2$,

Table 4.1: Numbers involved in the computation for the example.

matrices

$$Q = \left(\begin{array}{cc|cc|cc} 60 & -5 & 20 & -20 & 10 & -10 \\ 20 & -20 & 18 & -12 & 9 & -6 \\ 10 & -10 & 9 & -6 & 6 & -4 \end{array}\right)$$

$$M_u = \left(\begin{array}{cc|cc|cc} -701 & -490 & 156 & 114 & 78 & 57 \\ -8150 & -5770 & 1859 & 1310 & 929 & 655 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right)$$

	vectors						norms	
r_1	1	0	0	1	3	2	2	1
r_2	0	0	3	2	-6	-4	6	4
r_3	-22	-16	-100	-70	3	2	6	4
r_4	-73	-52	-340	-240	35	25	50	35
p_1	-11	-8	-50	-35	0	0	-120	-85

	vectors						formulae
q	2	0	3	4	0	0	$2\pi_{r_2^\perp}(r_1)$
q'	-22	-14	-243	-174	0	0	$M_u(q)$
r_1'	11	7	120	86	3	2	$M_u(r_1)$
t	-7	-5	-85	-60	0	0	$\frac{1}{\sqrt{2}}R_q(r_1 - r_1')$

numbers

D	240	170		
u_0	1	1		
u_1	1	1	0	0
u_2	-19	-13	1	$\frac{1}{2}$
u	579	410	-26	-19

we are in situation (2) of the Lemma 16, so it is possible that the next corner will be an A_2.

As outlined in the proof of Lemma 16 we will get a height bound for a type I restricted batch search. To find the root t whose mirror makes the next non-A_2 corner along R_2, we first find a generator for H_{r_2}. Twice the projection of r_1 onto $r_2^{\perp}$ is given by

$$q = 2\left(r_1 - \frac{r_1 \cdot r_2}{r_2^2} r_2\right)$$

By Lemma 4.1

$$\langle p, q\rangle \cong K = \mathbb{Q}(\sqrt{2}, \sqrt{D})$$

Fundamental units in K are u_1 and u_2. The group G defined by (4.2) is generated by $u = u_1^{-1}u_2^2$. We let M_u be the matrix for the translation corresponding to u. It turns out that M_u preserves L, so H_{r_2} is nontrivial, and is in fact all of G.

Since $G = \langle u\rangle$, we know that M_u takes the chamber to its nearest translate in some direction. The vector $M_u(q)$ is the projection of $M_u(r)$ onto $r_2^{\perp}$. Let

$$r_1' = R_{M_u(q)}(M_u(r_1))$$

The root t is a primitive lattice vector in the span of $r_1' - r_1$ whose inner product with p is positive. We compute t, and then we check that $\{r_1, r_2, r_3 = t\}$ is a chain of roots. It is not, since t has positive inner product with r_1. Thus we replace t with $R_q(t)$, which is what we would have gotten by translating in the opposite direction. (This just means we chose the wrong generator for G.)

Since t and r_2 are simple roots at their corner, we know that t is the nearest root whose mirror makes a non-A_2 corner with R_2, and so if it is not the next root of Φ, the next root has height less than $\mathrm{ht}(t)$.

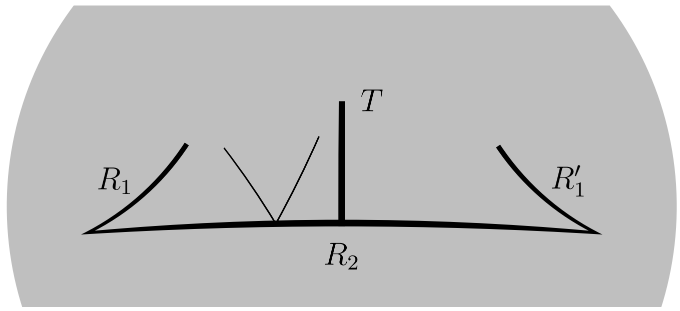


Figure 4.2: R_1, R_2, R_1', R_1'', T, and the still hypothetical $\frac{\pi}{3}$ corner.

The height of the next root is less than or equal to the height of t. The height of t is

$$\mathrm{ht}(t) = \frac{1}{t^2}\left(\frac{t \cdot p_1}{p_1^2}\right)^2 \approx 2.41$$

We do a restricted batch search looking only for vectors of norm 2 whose inner product with r_2 is $-\frac{1}{2}$. We find one such vector in the batch of height ≈ 0.086 which extends Φ. We call it r_3, and adjoin it to Φ.

By Lemma 1, we know that the next root r_4 makes a non-A_2 corner r_3. We find it by doing a batch search for roots of any possible norm that satisfy

$$r_3 \cdot r_4 = 0$$

We know that the height will be less than the height of a $\frac{\pi}{3}$ corner further along R_3, which we find by taking the nearest translate. We also know this bound is not very good (Figure 4.3), since there are several corners between an A_2 corner and its nearest translate. We use this bound for time estimates only. As a measure of just how bad this bound is, it tells us that we are looking

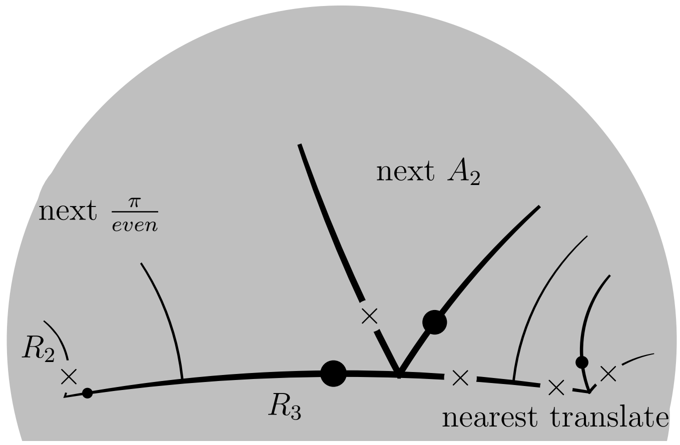


Figure 4.3: The bound coming from taking the nearest translate of an A_2 corner is not very good.

for a root of height less than 12578.6. The actual next root that we find via batch search has height ≈ 0.059.

At this point we check whether R_1 and R_4 intersect inside hyperbolic space by checking that $V_1 \cap V_2$ is negative definite. It is, and the angle between them is $\frac{\pi}{2}$. Thus $\{r_1, r_2, r_3, r_4\}$ is a closed chain of roots for the fundamental chamber of a reflective lattice.

4.5 Algorithmic descriptions

Let $\Phi = \{r_i\}_{1 \leq i \leq k+1}$ be a chain of roots all of whose mirrors bound a single chamber C. The line containing the edge of C corresponding to r_i is R_i. The corner of C formed by r_i and r_{i+1} is c_i. The primitive lattice vector lying along the corner c_i is p_i. The point in Λ^2 corresponding to p_i is P_i. The angle at the corner c_i is $\frac{\pi}{m_i}$. (Figure 4.4).

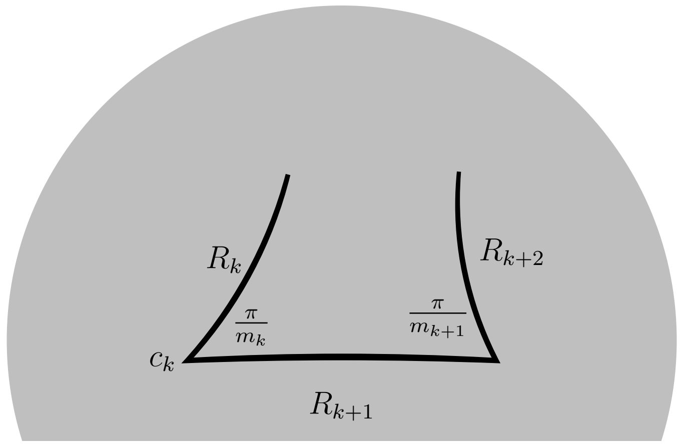


Figure 4.4: The next edge of Φ is R_{k+2}.

Figure 4.5 shows 3 examples of the possibilities outlined in Lemma 16. On the left, m_k is even and $r_{k+1}^2 \neq 2u^2$, so m_{k+1} must be even. In the middle, m_k is even and $r_{k+1}^2 = 2u^2$, so m_{k+1} could be 3, and there is a $\frac{\pi}{even}$ further along R_{k+1}. On the right $m_k = 3$ so m_{k+1} must be even.

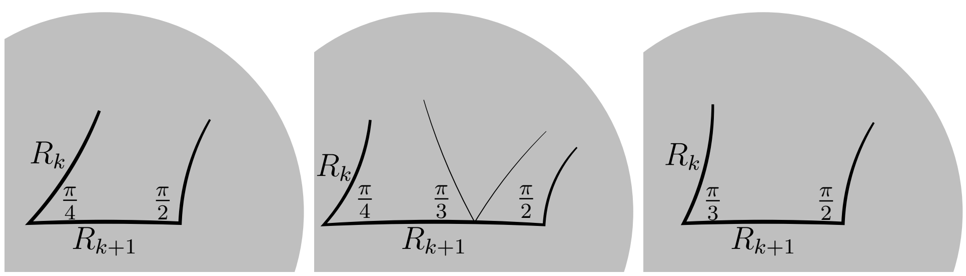


Figure 4.5: Three cases

Algorithm 1. *This algorithm finds the positive generator for $H_{r_{k+1}}$. Let*

$$q = 2\left(r_k - \frac{r_k \cdot r_{k+1}}{r_{k+1}^2} r_{k+1}\right)$$

be twice the projection of r_k *onto* V_{k+1}*. Let* $D = -q^2p^2$*,* $K = F(\sqrt{D})$*. Let* φ *be the isometry from Lemma 4.1.*

1. *Recall from the proof of Lemma 14 that* $U(K)$ *has two generators. Call them* u_1 *and* u_2*.*

2. *The map* $N_{K/F} : U(K) \to U(K)$ *is given by a matrix which we can write in terms of the basis* u_1*,* u_2*, for the rank* 2 *free subgroup of* $U(K)$*. Compute the kernel of that matrix to get a generator* u *for the free subroup of* $U(K)$ *consisting of units of norm* 1*. Let* $u = a + b\sqrt{d}$*. If* $a < 0$*, replace* u *by* $-u$*.*

3. *Let* $p' = \frac{1}{q^2}\varphi^{-1}(u\sqrt{D})$*,* $q' = \varphi^{-1}(u)$*. Let* M_u *be the matrix defined by*

$$(p, q, r_{k+1}) \mapsto (p', q', r_{k+1})$$

4. *Check whether* M_u *preserves* L *by writing it in terms of a basis for* L *and checking whether all its entries are in* $\mathfrak{o}$*. Since* M_u *has determinant* 1*, we know that if it preserves* L *its inverse will also preserve* L *and so it is an automorphism. Let* j *be the smallest natural number such that* M_u^j *preserves* L*. Since we assumed* Φ *has a next root,* $j > 0$ *exists. Let* $\phi = M_u^j$*.*

5. *Using Lemma 14, we know that* ϕ *is a generator for* $H_{r_{k+1}}$*. The final step is to check whether it is the positive or negative generator with repect to the chain of roots* Φ*. Let*

$$r'_k = R_{\phi(q)}(\phi(r_k))$$

if the intersection

$$V_k^- \cap V_{k'} \cap V_{k+1}$$

is nonempty then ϕ is a positive generator. Otherwise ϕ was a negative generator, so ϕ^{-1} is a positive generator.

For the next algorithm, we suppose m_k is even.

Algorithm 2. *This algorithm tells us how to find the next root t whose mirror T intersects R_{k+1} at a non-A_2 corner. The root t may or may not be the next root of Φ.*

1. *Follow Algorithm 14 to find r'_k as defined in step (5).*

2. *Let $t' = \gamma(r'_k - r_k)$ where γ is a scalar that makes t' a primitive lattice vector.*

3. *Let t be a simple root at the corner formed by t' and s (Lemma 12 already gives an algorithmic description of how to find this).*

The next algorithm is a full description of walking.

Algorithm 3. *This algorithm tells us how to extend Φ to a chain of roots whose mirrors make up an entire boundary component of C. The 3 cases we refer to are the ones from Lemma 16.*

1. *In cases (1) and (2), use find the root t defined in Algorithm 2.*

2. *In case (1), $r_{k+2} = t$, proceed to step (5).*

3. *In case (2), compute* h*, the height of* t*. Do a restricted batch search of type I for roots of norm 2 whose mirrors intersect* R_{k+1} *in an angle of* $\frac{\pi}{3}$*. If a root is found this way, it is the next root* r_{k+2}*. If the height* h *is passed and no root found, then* $r_{k+1} = t$*. Proceed to step (5).*

4. *In case (3), do a restricted batch search search for roots whose mirrors intersect* R_{k+1} *in an angle of* $\frac{\pi}{2}$*. When this search finds a root* t*, use Lemma 12 to find the simple root at that corner. This is the next root* r_{k+2}*.*

5. *Let* $\Phi' = \Phi \cup \{r_{k+2}\}$*. Check whether* Φ' *is a closed chain by checking whether* $V_1 \cap V_{r_{k+2}}$ *is negative definite. If* Φ' *is a closed chain then we are finished.*

6. *If* Φ' *is not a closed chain, we check to see whether the linear transformation defined by*

$$\psi : (r_1, r_2, p_1) \mapsto (r_{i+1}, r_{i+2}, p_{i+1})$$

is a lattice automorphism. If it is, then we are done. If it is not, then repeat from step (1) with $\Phi = \Phi'$*.*

If we exit Algorithm 3 from step (5), then L is a reflective lattice. If we exit the algorithm from step (6), we have lattice automorphism ψ which may have finite or infinite order. If ψ has finite order, then

$$\bigcup_{j \in Z} \psi^j(\Phi) \tag{4.8}$$

is a finite closed chain of roots, so L is reflective. If ψ has infinite order, then (4.8) is an infinite chain of roots whose mirrors are an entire boundary component of C, and we therefore know L is not reflective.

In all but 48 of the 2381 squarefree lattices on our list, these algorithms were enough to determine reflectivity in under two days (in fact, most of them took under a minute, and all but two of them took under a day). In the 48 remaining cases, the algorithm had reached a corner from which, according to Algorithm 3, the next corner would have to be found using a restricted batch search (either case (2) or (3) of Lemma 16). The height bound given by translation along the side was so big that the estimated time it would take to do a batch search up to that height was absurdly long (in the worst case it was 10^{44} days, in the best it was about 100 days).

4.6 The last 48 cases

The remaining 48 cases fall into two categories, which we will call type I and type II. The chains of roots of type I are those with a known A_2 corner. The chains of roots of type II are those with no known A_2 corner.

4.6.1 Partial simple systems of type I

We consider a chain of roots $\Phi = \{r_1, \dots r_k\}$ whose edges bound a chamber with one A_2 corner and several non-A_2 corners. An example of such a chamber appears in Figure 4.6. We may assume that the A_2 corner is the first one, formed by r_1 and r_2. We say that Φ has type I if $k \geq 3$, and $r_k^2 = 2u^2$

for some unit $u \in U(F)$.

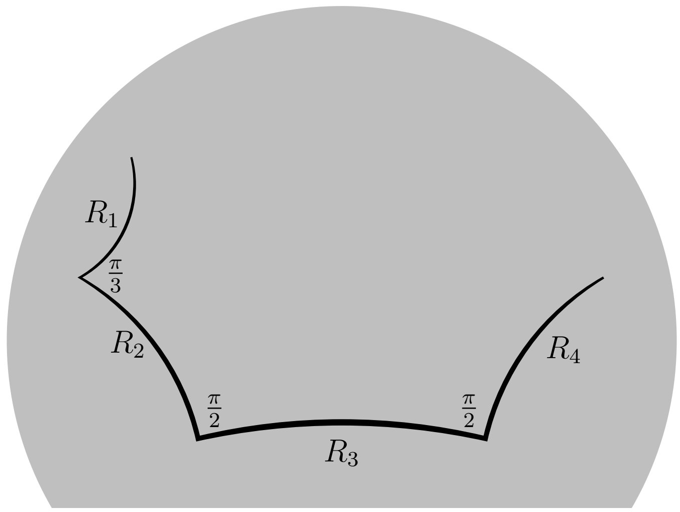


Figure 4.6: An A_2 corner followed by a bunch of non A_2's

If we were walking around the polygon, we would first use Algorithm 2 to find the nearest root in the positive direction along R_k whose mirror makes a non-A_2 corner with R_k. We would then do a restricted batch search of type I to determine if t is the next root, or if the next root is between r_{k-1} and t and makes an A_2 corner with r_k.

Suppose that if the batch search were allowed to run for as much time as needed, it eventually would find that the next root r_{k+1} makes an A_2 corner with r_k. Then by Lemma 7 the linear transformation ψ defined by

$$\psi : (r_1, r_2, p_1) \mapsto (r_k, r_{k+1}, p_k)$$

is an automorphism of L. The strategy here is to produce ψ without first knowing r_{k+1} and p_k. Let p be the primitive lattice vector that lies along the corner formed by the roots t and r_k.

Lemma 17. *The automorphism ψ of L factors as the product of an order 3 rotation fixing p_1, and an automorphism of L that takes the corner basis $\{r_2, r_3, p_2\}$ to the corner basis $\{r_k, t, p\}$.*

Proof. First we will write down the order 3 rotation fixing p_1 as a product of 2 reflections. Let $r_2' = R_{r_1}(r_2)$. The composition

$$\rho = R_{r_2} \circ R_{r_2'}$$

is a rotation by $\frac{2\pi}{3}$ fixing the vector p_1.

Let $\tau = \psi \circ \rho^{-1}$. Then τ is a lattice automorphism preserving a chamber for a sublattice of L that takes the basis (r_2, r_3, p_2) to (r_k, t, p). The desired factorization of ψ is

$$\psi = \tau \circ \rho$$

□

In our application of Lemma 17, we will not know ψ. But if it exists, we will be able to find both τ and ρ. Indeed, the next corner of Φ is an A_2 if and only if τ from this factorization exists, for if it does not then there can be no ψ. The following algorithm describes what we do with chains of roots of type I.

Algorithm 4. *Let $\Phi = \{r_1, \ldots, r_k\}$ be a chain of roots of type I. Let ρ be the order 3 rotation defined in Lemma 17.*

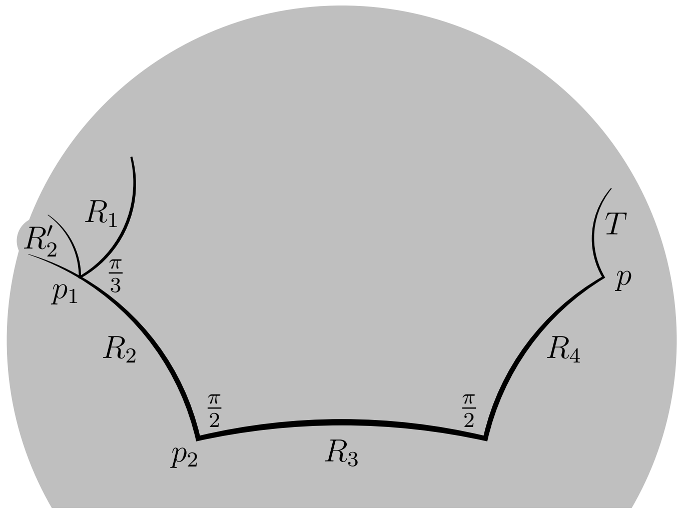


Figure 4.7: The polygon from Figure 4.6, now showing R_2' and T as well.

1. *Follow Algorithm 2 to find the next root t that makes a non-A_2 corner with R_k. Note that this is the same way we would begin if we were going to attempt to find r_{k+1} by a restricted batch search, and t would be the root giving the height bound. Let p be a primitive lattice vector such that $\{r_k, t, p\}$ is a corner basis.*

2. *Check for a lattice automorphism τ taking (r_2, r_3, p_2) to (r_k, t, p). If there is no automorphism, then there can be no A_2 corner involving r_k. Then the next root r_{k+1} is t. Let $\Phi' = \Phi \cup \{r_{k+1}\}$. If Φ' is a closed chain, then we are done. If Φ' is not a closed chain, we continue with walking (Algorithm 3).*

3. *If in step (2), we do find an automorphism τ, then let $\psi = \tau \circ \rho$. Then $r_{k+1} = \psi(r_2)$ is the next root, and we are done.*

If we exit the algorithm at step (3), we are finished, because (4.8) describes an entire boundary component of the chamber. In all of our type I lattices, the algorithm exited at step (3) with an infinite order automorphism ψ, so they were all non-reflective.

4.6.2 Partial simple systems of type II

Now consider a chain of roots $\Phi = \{r_1, \dots r_k\}$ that bound a chamber with no A_2 corners such that $k > 3$ and $r_1^2 = 2u_1^2$, $r_k^2 = 2u_k^2$ for some units $u_1, u_k \in U(F)$. If Φ satisfies all of these properties we will call Φ a chain of roots of type II. An example of such a chamber with $k = 4$ is shown in Figure 4.8.

With the type I chains, we had a known A_2 corner, and were therefore able to write down an order 3 rotation about that corner. Here we don't have that, but we will still be able to use Lemma 17. We will show that whether or not there is an A_2 corner, the chamber has a symmetry of infinite order.

There is an isomorphism between $\mathrm{Isom}(\Lambda^2)$ and $SO(2,1)^+$. In one direction, it is given by restricting the action of $SO(2,1)$ on $\mathbb{R}^{2,1}$ to the positive cone, and looking at the action on the quotient Λ^2. In the other direction, it is given by the adjoint representation of $PSL_2\mathbb{R} \cong \mathrm{Isom}(\Lambda^2)$ acting on the lie algebra $\mathfrak{sl}_2\mathbb{R}$ with the killing form. The Lie algebra $\mathfrak{sl}_2\mathbb{R}$ has dimension 3 and the killing form has signature $(2,1)$. If $\theta \in SL_2\mathbb{R}$ has trace $\pm a$, then the corresponding element of $\mathrm{ad}(PSL_2\mathbb{R})$ has trace $|a| + 1$.

We may think of elements of $\mathrm{Aut}(L)$ as isometries of Λ^2. The isometry group $\mathrm{Isom}(\Lambda^2) \cong PSL_2\mathbb{R}$ contains three types of orientation preserving isometries, called elliptic, parabolic, and hyperbolic. These 3 types are characterized by the absolute values of their traces as elements of $SL_2\mathbb{R}$. Let $\theta \in SL_2\mathbb{R}$, and let

$$\pi : SL_2\mathbb{R} \to PSL_2\mathbb{R}$$

be the standard projection.

1. If $|\mathrm{tr}(\theta)| < 2$, then $\pi(\theta)$ is elliptic and θ has a fixed point in the interior of Λ^2.

2. If $|\mathrm{tr}(\theta)| = 2$, then $\pi(\theta)$ is parabolic and θ has no fixed points in the interior of Λ^2 but a single fixed point on $\partial\Lambda^2$.

3. If $|\mathrm{tr}(\theta)| > 2$, then $\pi(\theta)$ is hyperbolic, and θ stabilizes a line in Λ^2 that joins two points of $\partial\Lambda^2$ that are fixed by θ.

Since $\mathrm{Aut}(L)$ is discrete, any elliptic element of $\mathrm{Aut}(L)$ must have finite order. This means that if $\psi \in \mathrm{Aut}(L)$ is an orientation preserving isometry of infinite order, it must be a parabolic or hyperbolic element of $\mathrm{Isom}(\Lambda^2)$. Parabolic and hyperbolic elements of $PSL_2\mathbb{R}$ come from elements of $SL_2\mathbb{R}$ whose trace in absolute value is greater then or equal to 2, so as matrices acting on $R^{n,1}$ they have trace greater than or equal to 3.

Let t be the next non-A_2 root along R_k, and let q be the primitive lattice vector such that $\{r_k, t, q\}$ is the corner basis at the corner formed by t and r_k. If both the previous corner and the next corner of Φ are A_2's, then

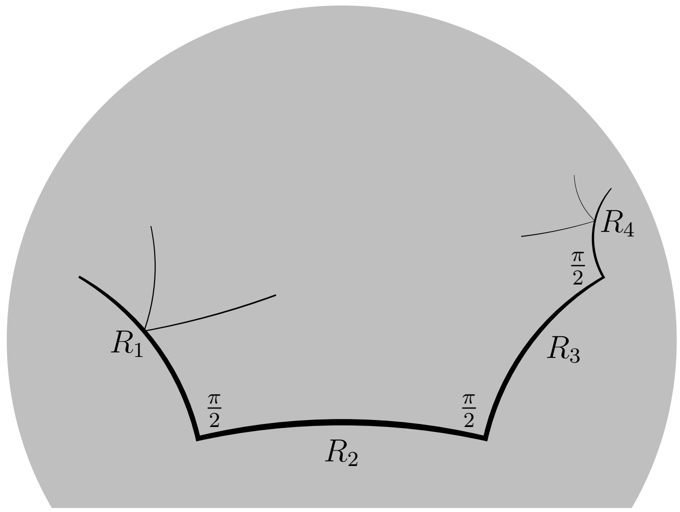


Figure 4.8: A sequence of non-A_2 corners, on either end the potential for an A_2 corner

there is a lattice automorphism ψ that takes the corner along one to the other. In this scenario let r_{k+1} be the next root, r_0 the previous root, and ψ be given by

$$\psi : (r_k, r_{k+1}, p_k) \mapsto (r_0, r_1, p_0)$$

By Lemma 17, ψ factors as the product of an order 3 rotation ρ fixing p_k and the lattice automorphism τ defined by

$$\tau : (r_1, r_2, p_1) \mapsto (r_k, t, q)$$

Introduce a parameter $x \in [0, 1]$, and let $v(x) = (1-x)p_{k-1} + xq$. Then we have

$$v(0) = p_{k-1}, v(1) = q$$

and the image of $v(x)$ in Λ^2 is a point on the segment of R_k joining the two corners. Let $w(x)$ be the projection of $p_{k-1} - q$ onto $v^\perp$. If $w(x) \cdot r_k > 0$, we replace $w(x)$ by $-w(x)$, so that our directional conventions work out. When coding this algorithm, we want to be precise and avoid rounding. Thus instead of computing the square root, we introduce a function

$$z(x) = \sqrt{\frac{6}{w(x)^2}}$$

Then $zw(x)$ is a vector of norm 6 orthogonal to both $v(x)$ and s.

If there is an A_2 corner somewhere along R_k, then for some $x \in (0,1)$, $v(x)$ points along that corner, and order 3 rotation fixing $v(x)$ is a lattice automorphism. Recall that u_k is a unit such that $(u_k^{-1} r_k)^2 = 2$. Order 3 rotation fixing $v(x)$ is a function of x and is given by

$$\rho(x) : \left(u_k^{-1} r_k, zw(x), v\right) \mapsto \left(\frac{zw(x) - u_k^{-1} r_k}{2}, \frac{-zw(x) - 3u_k{-}1 r_k}{2}, v(x)\right)$$

The chamber symmetry ψ taking the A_2 corner along R_k to the one along R_1 would then be the composition

$$\psi = \tau^{-1} \circ \rho(x)$$

The trace of ψ is then a function of x, and ψ is parabolic or hyperbolic if and only if $\operatorname{tr}(\psi)(x) \geq 3$. There are two possibilities. Either the chamber has an A_2 corner, and $\psi(x)$ is a lattice automorphism for some $x \in (0,1)$, or else there are no A_2 corners and τ is a lattice automorphism. If we establish that τ has infinite order and $\psi(x)$ has infinite order for all $x \in (0,1)$, then the chamber has an infinite order automorphism whether or not it has an A_2 corner. Thus, to show that the chamber has infinitely many sides, it suffices to show that τ has infinite order, and $\text{tr}(\psi(x)) > 3$ for all $x \in [0, 1]$.

For each of the type II chains of roots, we computed $\text{tr}(\psi(x)) - 3$ and $z(x)^2$ and found that they always had the same form.

$$\begin{aligned} \text{tr}(\psi(x)) - 3 &= \tfrac{f_1(x)z(x)^2 + f_2(x)z(x) + f_3(x)}{f_4(x)z(x)} \\ z(x)^2 &= f_5(x) \end{aligned} \tag{4.9}$$

where $f_1, f_2, f_3, f_4,$ and f_5 are polynomials with coefficients in $\mathbb{Q}(\sqrt{2})$, f_1 has degree 1, f_2, f_4, f_5 have degree 2, and f_3 has degree 3.

Sturm's theorem on polynomials in one variable gives an algorithm for determining how many real roots a polynomial has on a given closed interval. A good explanation of Sturm's theorem can be found in Bartlett's notes [4]. We apply it here to show that certain polynomials have no roots. Once we know that a polynomial has no roots in $[0, 1]$, we can say whether it is strictly positive or strictly negative by evaluating it at any point in the interval.

Our goal is to show that $\text{tr}(\psi(z, x)) > 3$ for all $x, \in (0, 1)$. We check that the (4.9) is defined for all $x \in (0, 1)$ by checking that both $f_5(x)$ and $f_4(x)$ have no roots in $[0, 1]$. Then we check and find that $f_5(x) > 0$ for all $x \in (0, 1)$, so that $z(x) \in \mathbb{R}$. If these conditions are met, then $\text{tr}(\psi(x)) - 3$ has no solution in $(0, 1)$ if and only if the numerator

$$f_1(x)z(x)^2 + f_2(x)z(x) + f_3(x)$$

has no zeros in the interval $(0,1)$. We have:

$$f_1(x)z^2 + f_2(x)z + f_3(x) = 0$$

$$\Leftrightarrow$$

$$(f_1(x)f_5(x) + f_3(x)) + f_2(x)z = 0$$

Let $h(x) = f_1(x)f_5(x)+f_3(x)$. If $h(x)$ and $f_2(x)$ are both either strictly positive or strictly negative on the interval $(0,1)$, then since $z(x) > 0$ we conclude that

$$h(x) + f_2(x)z(x) = 0$$

has no solution with $x \in (0,1)$. Therefore $\mathrm{tr}(\psi(x)) > 3$ for all $x \in (0,1)$

Here is an algorithmic description of what we do with type II chains of roots.

Algorithm 5. *Let $\Phi = \{r_1, \ldots, r_k\}$ be a chain of roots of type II.*

1. *Initialize an list $I = \{1\}$. We will keep track in I of the indices of all the edges that might span multiple chambers.*

2. *Find the the root t that makes the next non-A_2 corner along R_t. Let q be a primitive vector pointing along the corner formed by r_k and t, so that $\{r_k, t, q\}$ is a corner basis. For each $i \in I$, check whether the linear transformation τ_{ik} defined by*

$$\tau_{ik} : (r_i, r_{i+1}, p_i) \mapsto (r_k, t, q)$$

is a lattice automorphism.

3. *If τ_{ik} is not an automorphism of L, then we store k in the list I, and let $\Phi' = \{r_1, \ldots, r_k, r_{k+1} = t\}$. Note that at this point, the roots in Φ' may not all be part of a system of simple roots. We continue to extend Φ' by walking as in Algorithm 3 until it once again true that the highest root r_k satisfies $r_k^2 = 2u_k^2$ for some unit u_k, and then we return to step (1).*

4. *If τ_{ik} is an automorphism of L, we introduce the parameters x and z, and compute the polynomials f_1, f_2, f_3, f_4, f_5 as defined above. If all of the following hold for $x \in (0, 1)$, then C has an infinite order symmetry.*

   (a) τ_{ik} *has infinite order*

   (b) $f_5(x) > 0$

   (c) $|f_4(x)| > 0$

   (d) $f_2(x)$ *and* $(f_1 f_5 + f_3)(x)$ *both have no roots, and have the same sign.*

For all of our lattices of type II, conditions (a)-(d) in step (4) all hold, so none of these lattices are reflective. This finishes the proof of Theorem 1.

Appendix

Appendix A

The table

A.1 Description of tables

The entries in the table each describe an $\mathfrak{o}$-lattice L. A generator δ for the determinant ideal is given as an element of $\mathfrak{o}$, with $w = \sqrt{2}$. We also give the norm $N_{F/\mathbb{Q}}(\delta)$ because those are easier to compare.

Below that is a sequence of numbers, either undecorated or inside of a $(\cdot)^2$. This is the sequence of angles between the closed chain of edges for the polygon. For example, 842 describes a triangle with angles $\frac{\pi}{8}, \frac{\pi}{4}, \frac{\pi}{2}$, and $(842)^2$ describes a hexagon with an order 2 rotations whose angles are $\frac{\pi}{8}, \frac{\pi}{4}, \frac{\pi}{2}, \frac{\pi}{8}, \frac{\pi}{4}, \frac{\pi}{2}$.

Below the angle sequence are the quadratic form for L and a list of roots along with their norms. The quadratic form and the roots are both written with respect to the same basis for L. The entries in the matrices and vectors are pairs $(a\ b)$, standing for elements $a + b\sqrt{2} \in \mathfrak{o}$.

A.2 Tables

Table A.1: Triangles

det $= 1 - w$	det norm $= -1$
824	

quadratic form

$$\left(\begin{array}{cc|cc|cc} 1 & -1 & -2 & 2 & -1 & 0 \\ -2 & 2 & 6 & -4 & 1 & -1 \\ -1 & 0 & 1 & -1 & 1 & 0 \end{array}\right)$$

root list

	roots						norms	
r_1	0	0	−1	−1	−2	0	2	0
r_2	−1	1	0	1	1	0	2	−1
r_3	0	0	0	0	−1	1	3	−2

det $= -24 - 17w$	det norm $= -2$
382	

quadratic form

$$\left(\begin{array}{cc|cc|cc} 180 & 127 & 388 & 274 & 8 & 5 \\ 388 & 274 & 838 & 592 & 17 & 11 \\ 8 & 5 & 17 & 11 & 2 & -1 \end{array}\right)$$

root list

	roots						norms	
r_1	0	0	−1	0	20	14	6	4
r_2	−3	−1	2	0	1	1	6	4
r_3	1	1	−1	0	−2	−2	2	1

det $= -3w$	det norm $= -18$
826	

quadratic form

$$\begin{pmatrix} 600 & 423 & 36 & 27 & -24 & -18 \\ 36 & 27 & 6 & -1 & -3 & 0 \\ -24 & -18 & -3 & 0 & 2 & 0 \end{pmatrix}$$

root list

roots							norms	
r_1	1	0	−9	−6	−2	0	2	0
r_2	0	0	1	0	1	0	2	−1
r_3	−1	0	9	6	3	0	6	0

Table A.2: Quadrilaterals

det $= -24 - 17w$	det norm $= -2$
2224	

quadratic form

$$\left(\begin{array}{cc|cc|cc} 2 & -1 & -6 & 4 & -2 & 1 \\ -6 & 4 & 16 & -17 & -1 & -9 \\ -2 & 1 & -1 & -9 & -5 & -6 \end{array}\right)$$

root list

	roots						norms	
r_1	4	2	0	0	1	0	3	2
r_2	9	5	2	1	0	0	10	7
r_3	0	0	1	1	0	−1	10	7
r_4	−7	−6	−1	−1	0	0	34	24

det $= -5 - 4w$	det norm $= -7$
8222	

quadratic form

$$\left(\begin{array}{cc|cc|cc} -31 & -22 & 0 & 0 & 0 & 0 \\ 0 & 0 & 34 & -24 & 3 & -2 \\ 0 & 0 & 3 & -2 & 2 & 1 \end{array}\right)$$

root list

	roots						norms	
r_1	2	0	−34	−24	1	0	2	1
r_2	0	0	3	2	−2	0	6	4
r_3	−23	−16	741	524	0	0	3	2
r_4	−53	−37	1704	1205	3	1	13	9

det $= -2209 - 1562w$	det norm $= -7$
2228	

quadratic form

$$\begin{pmatrix} 3401 & 2392 & -5439 & -3843 & 270 & 169 \\ -5439 & -3843 & 8721 & 6166 & -413 & -287 \\ 270 & 169 & -413 & -287 & 46 & -5 \end{pmatrix}$$

root list

	roots						norms	
r_1	11	7	13	2	29	20	10	7
r_2	21	13	21	7	65	46	92	65
r_3	−21	−15	−18	−10	−60	−42	3	2
r_4	−25	−16	−21	−11	−70	−50	6	4

det = $-31 - 22w$ det norm = -7

2224

quadratic form

$$\begin{pmatrix} 17 & 12 & 0 & 0 & 0 & 0 \\ 0 & 0 & 14 & -10 & 1 & -1 \\ 0 & 0 & 1 & -1 & 6 & 4 \end{pmatrix}$$

root list

	roots						norms	
r_1	0	2	21	15	2	−1	2	0
r_2	−1	2	14	10	0	0	1	0
r_3	−10	−8	−160	−113	−1	−2	26	18
r_4	−5	−5	−92	−65	−1	0	1	0

det = $-65 - 46w$ det norm = -7

4242

quadratic form

$$\begin{pmatrix} 85 & 58 & -22 & -16 & 10 & 6 \\ -22 & -16 & 6 & 4 & -3 & -2 \\ 10 & 6 & -3 & -2 & 2 & 0 \end{pmatrix}$$

root list

	roots						norms	
r_1	2	0	1	4	−2	0	6	4
r_2	−1	0	0	−2	1	0	3	2
r_3	0	0	0	0	1	1	6	4
r_4	3	2	9	7	0	0	3	2

det $= -1055 - 746w$ det norm $= -7$

2224

quadratic form

$$\left(\begin{array}{cc|cc|cc} -103 & -84 & -62 & -58 & -209 & -149 \\ -62 & -58 & -34 & -42 & -135 & -97 \\ -209 & -149 & -135 & -97 & -359 & -254 \end{array}\right)$$

root list

	roots						norms	
r_1	−29	−19	44	30	1	0	34	24
r_2	−61	−44	98	70	0	0	17	12
r_3	0	0	7	5	−2	−2	874	618
r_4	15	9	−23	−15	0	0	17	12

det $= -31 - 22w$ det norm $= -7$

2224

quadratic form

$$\left(\begin{array}{cc|cc|cc} 809 & 572 & 12 & 10 & 28 & 20 \\ 12 & 10 & 34 & -24 & 5 & -3 \\ 28 & 20 & 5 & -3 & 2 & 0 \end{array}\right)$$

Table A.2, cont.

root list

roots							norms	
r_1	0	0	7	5	−5	−4	6	4
r_2	3	−3	20	14	0	0	3	2
r_3	0	2	−85	−60	30	21	150	106
r_4	3	2	−99	−70	4	3	3	2

det $= -379 - 268w$ det norm $= -7$

2228

quadratic form

$$\left(\begin{array}{cc|cc|cc} 1 & 0 & 1 & 0 & -3 & 1 \\ 1 & 0 & -61 & -44 & 100 & 74 \\ -3 & 1 & 100 & 74 & -157 & -125 \end{array}\right)$$

root list

roots							norms	
r_1	7	4	−1	0	0	0	6	4
r_2	11	3	2	0	2	0	3	2
r_3	−59	−41	0	0	−3	−2	92	65
r_4	−65	−43	−6	−3	−7	−4	10	7

det $= -5 - 4w$ det norm $= -7$

4222

quadratic form

$$\left(\begin{array}{cc|cc|cc} 17 & 8 & 1 & 5 & -6 & -2 \\ 1 & 5 & 5 & -2 & 1 & -2 \\ -6 & -2 & 1 & -2 & 2 & 0 \end{array}\right)$$

root list

	roots						norms	
r_1	1	0	−3	−1	2	−2	2	0
r_2	0	0	1	0	−1	1	1	0
r_3	−1	−1	5	4	3	1	26	18
r_4	1	0	−2	−2	0	0	1	0

det $= -31 - 22w$ det norm $= -7$

4222

quadratic form

$$\left(\begin{array}{cc|cc|cc} 9 & 4 & -14 & -8 & 2 & -4 \\ -14 & -8 & 22 & 14 & 1 & 5 \\ 2 & -4 & 1 & 5 & 10 & -6 \end{array}\right)$$

root list

	roots						norms	
r_1	2	0	1	0	−2	−1	6	4
r_2	−1	0	0	0	−1	−1	3	2
r_3	−10	−8	−10	−8	55	39	150	106
r_4	−3	−1	−4	−2	20	14	3	2

det $= -31 - 22w$ det norm $= -7$

2224

quadratic form

$$\left(\begin{array}{cc|cc|cc} 361 & 96 & 336 & 22 & 12 & -10 \\ 336 & 22 & 342 & -50 & 17 & -13 \\ 12 & -10 & 17 & -13 & 6 & 2 \end{array}\right)$$

Table A.2, cont.

root list

	roots						norms	
r_1	48	44	−69	−56	2	−1	2	0
r_2	31	32	−46	−40	0	0	1	0
r_3	−418	−298	563	400	−1	−2	26	18
r_4	−241	−177	327	236	−1	0	1	0

det = $-21 - 15w$ det norm = -9

4228

quadratic form

$$\left(\begin{array}{cc|cc|cc} 33 & 21 & 0 & 9 & 21 & 15 \\ 0 & 9 & 28 & -18 & 5 & 3 \\ 21 & 15 & 5 & 3 & 17 & 12 \end{array}\right)$$

root list

	roots						norms	
r_1	0	1	−3	−2	−6	4	2	0
r_2	0	0	0	0	3	−2	1	0
r_3	−1	0	0	0	−3	3	6	3
r_4	−1	1	−2	−1	3	−2	2	−1

det = $-3 - 3w$ det norm = -9

8224

quadratic form

$$\left(\begin{array}{cc|cc|cc} -3 & -3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 6 & 4 & -1 & -3 \\ 0 & 0 & -1 & -3 & 18 & -11 \end{array}\right)$$

root list

	roots						norms	
r_1	4	2	−11	−10	−30	−21	2	0
r_2	0	0	−5	4	1	0	2	−1
r_3	−2	−2	9	6	21	15	6	3
r_4	1	1	−7	−2	−11	−8	1	0

det $= -21 - 15w$ det norm $= -9$

6262

quadratic form

$$\left(\begin{array}{cc|cc|cc} -7 & -5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 2 & 0 \end{array}\right)$$

root list

	roots						norms	
r_1	2	0	−3	−2	1	0	2	0
r_2	0	0	1	1	−2	−2	18	12
r_3	−10	−6	25	17	0	0	2	0
r_4	−144	−102	377	266	11	8	18	12

det $= -123 - 87w$ det norm $= -9$

6262

quadratic form

$$\left(\begin{array}{cc|cc|cc} -1 & -1 & 4 & 4 & -2 & -2 \\ 4 & 4 & 2 & -4 & -1 & 2 \\ -2 & -2 & -1 & 2 & 2 & 0 \end{array}\right)$$

root list

	roots						norms	
r_1	0	0	−1	0	−2	0	6	4
r_2	2	2	2	2	3	3	102	72
r_3	0	0	0	0	1	1	6	4
r_4	−2	−2	−11	−8	0	0	102	72

det = $-1294 - 915w$ det norm = -14

2232

quadratic form

$$\begin{pmatrix} -122 & -95 & 20 & 32 & -136 & -90 \\ 20 & 32 & 18 & -24 & 39 & 15 \\ -136 & -90 & 39 & 15 & -90 & -68 \end{pmatrix}$$

root list

	roots						norms	
r_1	7	0	21	14	5	−1	92	65
r_2	1	1	4	3	0	0	10	7
r_3	2	−3	−9	−7	2	−3	34	24
r_4	−2	0	−3	−2	−1	1	34	24

det = $-122362 - 86523w$ det norm = -14

2223

quadratic form

$$\begin{pmatrix} 9570 & 6767 & -372 & -264 & -594 & -420 \\ -372 & -264 & 70 & -28 & 23 & 18 \\ -594 & -420 & 23 & 18 & 34 & 24 \end{pmatrix}$$

root list

	roots						norms	
r_1	0	0	−17	−12	7	6	198	140
r_2	−1	1	0	0	2	2	58	41
r_3	4	−3	48	34	−23	−17	874	618
r_4	−2	0	34	24	−27	−20	198	140

det $= -106 - 75w$ det norm $= -14$

2223

quadratic form

$$\begin{pmatrix} 1730 & 1223 & -72 & -52 & -36 & -26 \\ -72 & -52 & 6 & 0 & 3 & 0 \\ -36 & -26 & 3 & 0 & 2 & 0 \end{pmatrix}$$

root list

	roots						norms	
r_1	2	0	19	14	3	2	6	4
r_2	3	−1	16	12	0	0	2	1
r_3	−2	0	−18	−13	−5	−4	26	18
r_4	−2	1	−6	−5	1	1	6	4

det $= 6 - 5w$ det norm $= -14$

2223

quadratic form

$$\begin{pmatrix} 298 & 209 & 8 & 12 & 4 & 6 \\ 8 & 12 & 18 & -12 & 9 & -6 \\ 4 & 6 & 9 & -6 & 6 & -4 \end{pmatrix}$$

root list

	roots						norms	
r_1	0	1	−22	−15	5	3	2	0
r_2	−1	1	−6	−4	0	0	2	−1
r_3	0	−3	65	45	−13	−9	6	2
r_4	−2	−2	71	50	−9	−6	2	0

det $= -106 - 75w$ det norm $= -14$

2223

quadratic form

$$\begin{pmatrix} 2 & 1 & -4 & 2 & -2 & -1 \\ -4 & 2 & 38 & -30 & -3 & -7 \\ -2 & -1 & -3 & -7 & -12 & -9 \end{pmatrix}$$

root list

	roots						norms	
r_1	3	1	0	0	1	0	6	4
r_2	−1	6	2	2	0	0	2	1
r_3	0	0	3	2	−2	0	26	18
r_4	0	−3	−1	−1	0	0	6	4

det $= -6 - 5w$ det norm $= -14$

8222

quadratic form

$$\begin{pmatrix} -6 & -5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 6 & 4 & -1 & -3 \\ 0 & 0 & -1 & -3 & 18 & -11 \end{pmatrix}$$

root list

	roots						norms	
r_1	4	2	−15	−13	−40	−28	2	0
r_2	0	0	−5	4	1	0	2	−1
r_3	−47	−32	218	154	526	372	2	−1
r_4	−154	−109	723	515	1754	1240	10	6

det $= -256206 - 181165w$ det norm $= -14$

2223

quadratic form

$$\left(\begin{array}{cc|cc|cc} 16214 & 11465 & -7584 & -5363 & 274 & 194 \\ -7584 & -5363 & 3564 & 2517 & -131 & -90 \\ 274 & 194 & -131 & -90 & 6 & 2 \end{array}\right)$$

root list

	roots						norms	
r_1	1	5	11	7	60	42	198	140
r_2	−3	3	2	1	14	10	58	41
r_3	−15	−6	−30	−22	−179	−127	536	379
r_4	−8	−5	−20	−14	−129	−91	198	140

det $= -18 - 13w$ det norm $= -14$

2223

quadratic form

$$\left(\begin{array}{cc|cc|cc} 10 & 7 & -4 & -2 & 0 & 0 \\ -4 & -2 & 22 & -14 & 5 & -3 \\ 0 & 0 & 5 & -3 & 6 & 2 \end{array}\right)$$

root list

	roots						norms	
r_1	−8	2	−11	−7	2	−1	2	0
r_2	−3	0	−6	−4	0	0	2	−1
r_3	2	3	13	9	−1	0	6	2
r_4	6	−4	1	0	−1	1	2	0

det = $-18 - 13w$ det norm = -14

2232

quadratic form

$$\left(\begin{array}{cc|cc|cc} 3226 & -2007 & -522 & 519 & -54 & 38 \\ -522 & 519 & 188 & -51 & 11 & -8 \\ -54 & 38 & 11 & -8 & 2 & 0 \end{array}\right)$$

root list

	roots						norms	
r_1	−13	−10	28	16	2	0	2	1
r_2	−37	−30	87	43	0	0	26	18
r_3	95	68	−181	−124	−6	−4	6	4
r_4	29	22	−61	−36	−1	0	6	4

det = $-18 - 13w$ det norm = -14

8222

quadratic form

$$\left(\begin{array}{cc|cc|cc} -106 & -75 & 0 & 0 & 0 & 0 \\ 0 & 0 & 34 & -24 & 3 & -2 \\ 0 & 0 & 3 & -2 & 2 & 1 \end{array}\right)$$

root list

	roots						norms	
r_1	0	1	−44	−31	1	0	2	1
r_2	0	0	3	2	−2	0	6	4
r_3	−71	−51	4295	3037	0	0	44	31
r_4	−29	−20	1714	1212	2	1	10	7

det $= 6 - 5w$ det norm $= -14$

2223

quadratic form

$$\left(\begin{array}{cc|cc|cc} 2 & 1 & 2 & -2 & 0 & 0 \\ 2 & -2 & 22 & -14 & -1 & 2 \\ 0 & 0 & -1 & 2 & 10 & -6 \end{array}\right)$$

root list

	roots						norms	
r_1	0	0	−3	−2	3	2	2	0
r_2	5	−5	−10	−6	8	6	2	−1
r_3	−4	1	−5	−4	4	3	6	2
r_4	−2	2	5	3	−4	−3	2	0

det $= -6 - 5w$ det norm $= -14$

8222

quadratic form

$$\left(\begin{array}{cc|cc|cc} 6 & 3 & -8 & 6 & -4 & -3 \\ -8 & 6 & 82 & -58 & -1 & 1 \\ -4 & -3 & -1 & 1 & 10 & 7 \end{array}\right)$$

Table A.2, cont.

root list

roots							norms	
r_1	0	0	0	0	1	−1	2	1
r_2	0	1	−3	−2	0	1	6	4
r_3	−32	−22	147	104	−14	−11	54	38
r_4	−9	−7	44	31	−4	−4	2	1

det $= -6 - 5w$ det norm $= -14$

3222

quadratic form

$$\begin{pmatrix} 94 & 41 & 8 & 16 & 4 & 8 \\ 8 & 16 & 6 & 0 & 3 & 0 \\ 4 & 8 & 3 & 0 & 2 & 0 \end{pmatrix}$$

root list

roots							norms	
r_1	2	0	−4	−6	1	0	2	0
r_2	0	0	1	0	−2	0	2	0
r_3	−3	−4	29	16	0	0	4	1
r_4	−1	−1	8	4	0	1	2	−1

det $= -32 - 23w$ det norm $= -34$

8282

quadratic form

$$\begin{pmatrix} -188 & -133 & 0 & 0 & 0 & 0 \\ 0 & 0 & 34 & -24 & 3 & -2 \\ 0 & 0 & 3 & -2 & 2 & 1 \end{pmatrix}$$

Table A.2, cont.

root list

	roots						norms	
r_1	0	1	−58	−41	1	0	2	1
r_2	0	0	3	2	−2	0	6	4
r_3	−17	−11	1301	920	0	0	2	1
r_4	−37	−27	2998	2120	3	1	6	4

det = $-120 - 85w$ det norm = -50

2283

quadratic form

$$\begin{pmatrix} 760 & 505 & -40 & -40 & -20 & -20 \\ -40 & -40 & 6 & 0 & 3 & 0 \\ -20 & -20 & 3 & 0 & 2 & 0 \end{pmatrix}$$

root list

	roots						norms	
r_1	2	0	7	9	6	4	6	4
r_2	1	1	15	10	0	0	50	35
r_3	−1	−1	−13	−8	−6	−5	2	1
r_4	−4	−2	−35	−27	−13	−9	6	4

det = $-700 - 495w$ det norm = -50

8223

quadratic form

$$\begin{pmatrix} 556 & 393 & -180 & -128 & -70 & -50 \\ -180 & -128 & 62 & 40 & 25 & 15 \\ -70 & -50 & 25 & 15 & 10 & 5 \end{pmatrix}$$

root list

	roots						norms	
r_1	0	0	−1	−1	2	2	6	4
r_2	2	−1	0	1	1	0	2	1
r_3	−40	−30	0	0	−271	−191	50	35
r_4	−11	−10	−2	−1	−79	−56	6	4

det = $-7w$ det norm = −98

2282

quadratic form

$$\left(\begin{array}{cc|cc|cc} 92 & 65 & 0 & 0 & 0 & 0 \\ 0 & 0 & 82 & -58 & 7 & -5 \\ 0 & 0 & 7 & -5 & 2 & 0 \end{array}\right)$$

root list

	roots						norms	
r_1	−4	1	110	78	3	−1	10	−6
r_2	3	−3	54	38	0	0	8	−5
r_3	−2	2	−35	−25	−2	1	6	−4
r_4	−1	0	42	30	1	0	10	−7

det = $-7w$ det norm = −98

2228

quadratic form

$$\left(\begin{array}{cc|cc|cc} 16 & 11 & -10 & 8 & 0 & 0 \\ -10 & 8 & 514 & -362 & -1 & 2 \\ 0 & 0 & -1 & 2 & 10 & -6 \end{array}\right)$$

root list

	roots						norms	
r_1	4	−1	−14	−10	13	9	2	0
r_2	−21	25	−86	−60	76	54	4	1
r_3	16	−11	−5	−4	4	3	6	2
r_4	19	−14	5	3	−4	−3	2	−1

det $= -168 - 119w$ det norm $= -98$

2228

quadratic form

$$\begin{pmatrix} 92 & 65 & 22 & 16 & 0 & 0 \\ 22 & 16 & 22 & -8 & 5 & -3 \\ 0 & 0 & 5 & -3 & 6 & 2 \end{pmatrix}$$

root list

	roots						norms	
r_1	4	6	−23	−16	1	0	6	4
r_2	35	25	−130	−92	0	0	16	11
r_3	4	−1	−5	−4	−2	0	26	18
r_4	−1	−2	7	5	0	0	2	1

det $= -7w$ det norm $= -98$

8222

quadratic form

$$\begin{pmatrix} 560 & 371 & 14 & 21 & -28 & -14 \\ 14 & 21 & 6 & -3 & 1 & -2 \\ -28 & -14 & 1 & -2 & 2 & 0 \end{pmatrix}$$

root list

roots							norms	
r_1	1	0	−13	−9	2	−2	2	0
r_2	0	0	1	0	−1	1	2	−1
r_3	−1	0	13	9	−1	2	6	2
r_4	1	1	−32	−22	0	0	4	1

Table A.3: Pentagons

$\det = -11 - 8w$	det norm $= -7$
22222	

quadratic form

$$\begin{pmatrix} 11 & -10 & -6 & 5 & -3 & 10 \\ -6 & 5 & 4 & -2 & -1 & -7 \\ -3 & 10 & -1 & -7 & 5 & -5 \end{pmatrix}$$

root list

	roots						norms	
r_1	3	1	3	1	1	0	2	1
r_2	−1	1	−2	2	0	0	1	0
r_3	−6	−3	−5	−3	−2	0	10	6
r_4	−1	−1	−1	−1	0	0	1	0
r_5	2	0	3	−1	1	1	5	3

$\det = -379 - 268w$	det norm $= -7$
22222	

quadratic form

$$\begin{pmatrix} 433 & -296 & 66 & -78 & 6 & -2 \\ 66 & -78 & 86 & 35 & -1 & -2 \\ 6 & -2 & -1 & -2 & 10 & 7 \end{pmatrix}$$

root list

	roots						norms	
r_1	−12	−10	−9	1	2	−1	2	1
r_2	−11	−11	−15	5	0	0	5	3
r_3	17	11	1	6	5	−4	1	0
r_4	132	94	41	26	−1	−2	10	6
r_5	35	28	22	0	2	−2	1	0

$\det = -11 - 8w$	det norm $= -7$
22222	

quadratic form

$$\left(\begin{array}{cc|cc|cc} 17 & 12 & 0 & 0 & 0 & 0 \\ 0 & 0 & 14 & -10 & 1 & -1 \\ 0 & 0 & 1 & -1 & 2 & 1 \end{array}\right)$$

root list

	roots						norms	
r_1	2	0	14	10	3	2	2	1
r_2	3	−1	10	7	1	1	5	3
r_3	1	−1	−3	−2	0	−1	1	0
r_4	0	0	3	2	2	1	10	6
r_5	−1	2	14	10	4	2	1	0

$\det = -379 - 268w$ det norm $= -7$

22222

quadratic form

$$\left(\begin{array}{cc|cc|cc} 67 & 47 & 2 & 4 & -22 & -16 \\ 2 & 4 & 14 & -9 & -3 & 1 \\ -22 & -16 & -3 & 1 & 5 & 3 \end{array}\right)$$

root list

	roots						norms	
r_1	2	0	−3	−2	0	2	10	7
r_2	1	2	0	0	4	2	27	19
r_3	0	−2	3	2	−3	−1	3	2
r_4	−15	−9	16	11	−21	−18	54	38
r_5	−3	−4	3	2	−8	−5	3	2

$\det = -65 - 46w$ det norm $= -7$

22222

quadratic form

$$\left(\begin{array}{cc|cc|cc} -195 & -138 & -6 & -5 & -11 & -8 \\ -6 & -5 & 4 & -3 & 1 & -1 \\ -11 & -8 & 1 & -1 & 1 & 0 \end{array}\right)$$

root list

	roots						norms	
r_1	1	0	−13	−9	3	2	10	7
r_2	−1	1	−4	−3	0	0	3	2
r_3	−1	0	16	11	−5	−3	54	38
r_4	0	0	0	0	1	1	3	2
r_5	2	0	−27	−19	11	8	27	19

det $= -11 - 8w$ det norm $= -7$

22222

quadratic form

$$\left(\begin{array}{cc|cc|cc} 173 & 122 & -9 & -4 & 16 & 11 \\ -9 & -4 & 19 & -13 & -3 & 1 \\ 16 & 11 & -3 & 1 & 2 & 1 \end{array}\right)$$

root list

	roots						norms	
r_1	1	0	11	8	−2	1	2	1
r_2	2	0	27	19	0	0	5	3
r_3	0	0	1	1	−1	1	1	0
r_4	−1	0	−11	−8	3	−1	10	6
r_5	−1	1	4	3	0	0	1	0

det $= -379 - 268w$ det norm $= -7$

22222

quadratic form

$$\begin{pmatrix} 97 & 68 & 34 & 26 & 0 & 0 \\ 34 & 26 & 22 & 9 & -3 & -2 \\ 0 & 0 & -3 & -2 & 2 & 1 \end{pmatrix}$$

root list

	roots						norms	
r_1	−2	−4	11	7	12	8	10	7
r_2	−7	1	6	6	8	6	3	2
r_3	0	2	−5	−3	−5	−3	54	38
r_4	1	0	−1	−1	−1	−1	3	2
r_5	−1	−6	15	9	16	11	27	19

det $= -75041 - 53062w$ det norm $= -7$

22222

quadratic form

$$\begin{pmatrix} 1485 & 1050 & 90 & 61 & 249 & 176 \\ 90 & 61 & 98 & -61 & 19 & 10 \\ 249 & 176 & 19 & 10 & 27 & 19 \end{pmatrix}$$

root list

	roots						norms	
r_1	0	0	−7	−5	0	2	58	41
r_2	1	−1	0	0	0	1	17	12
r_3	5	−4	11	8	1	−2	314	222
r_4	6	−4	−3	−2	−1	1	17	12
r_5	3	−1	−27	−19	4	2	157	111

det $= -11 - 8w$ det norm $= -7$

22222

quadratic form

$$\begin{pmatrix} -43 & -32 & -37 & 4 & 21 & 14 \\ -37 & 4 & 399 & -290 & -4 & 13 \\ 21 & 14 & -4 & 13 & 5 & 3 \end{pmatrix}$$

root list

	roots						norms	
r_1	−14	−14	47	33	1	0	2	1
r_2	−3	−3	10	7	0	0	1	0
r_3	11	22	−59	−41	0	−1	10	6
r_4	−8	0	11	8	2	−1	1	0
r_5	−55	−33	141	100	3	1	5	3

det $= -11 - 8w$ det norm $= -7$

22222

quadratic form

$$\begin{pmatrix} 607 & 423 & -332 & -230 & -16 & -11 \\ -332 & -230 & 182 & 125 & 9 & 6 \\ -16 & -11 & 9 & 6 & 1 & 0 \end{pmatrix}$$

root list

	roots						norms	
r_1	−10	−6	−19	−12	18	12	2	1
r_2	−27	−17	−52	−34	54	38	5	3
r_3	14	11	28	21	−29	−21	1	0
r_4	163	115	319	225	−335	−236	10	6
r_5	55	37	107	73	−112	−79	1	0

det $= -106 - 75w$ det norm $= -14$

22224

quadratic form

$$\begin{pmatrix} -106 & -75 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 \end{pmatrix}$$

root list

	roots						norms	
r_1	0	1	−10	−7	−7	−5	3	2
r_2	1	1	−18	−13	−18	−13	44	31
r_3	−1	0	7	5	4	3	10	7
r_4	1	1	−13	−9	0	0	44	31
r_5	2	0	−13	−9	−6	−4	6	4

det $= -38 - 27w$ det norm $= -14$

22242

quadratic form

$$\begin{pmatrix} 10 & 7 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & -2 & -3 & 1 \\ 0 & 0 & -3 & 1 & 3 & -1 \end{pmatrix}$$

root list

	roots						norms	
r_1	−2	−4	−11	−8	−8	−6	10	6
r_2	−1	0	0	−2	0	−2	2	−1
r_3	2	4	11	8	9	6	3	−1
r_4	5	0	6	6	5	5	3	−2
r_5	2	1	7	2	6	2	6	−4

det $= -106 - 75w$ det norm $= -14$

42222

quadratic form

$$\left(\begin{array}{cc|cc|cc} 44 & 31 & -62 & -44 & 3 & 1 \\ -62 & -44 & 6 & 4 & 1 & -1 \\ 3 & 1 & 1 & -1 & 17 & -12 \end{array}\right)$$

root list

	roots						norms	
r_1	0	0	0	0	−7	−5	3	2
r_2	0	0	1	0	14	10	6	4
r_3	1	0	0	0	0	0	44	31
r_4	−1	1	0	−1	−24	−17	10	7
r_5	1	0	−3	−1	−106	−75	44	31

det $= -222 - 157w$ det norm $= -14$

22224

quadratic form

$$\left(\begin{array}{cc|cc|cc} 874 & 618 & 130 & 92 & -27 & -19 \\ 130 & 92 & 22 & 13 & -2 & -4 \\ -27 & -19 & -2 & -4 & 3 & -1 \end{array}\right)$$

root list

	roots						norms	
r_1	1	0	−2	−1	6	4	6	4
r_2	3	−1	0	0	20	14	54	38
r_3	0	0	1	0	2	2	2	1
r_4	0	0	0	0	1	1	5	3
r_5	3	−2	−1	0	1	0	1	0

det $= -6 - 5w$ det norm $= -14$

22224

quadratic form

$$\left(\begin{array}{cc|cc|cc} 27 & 19 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right)$$

root list

	roots						norms	
r_1	1	0	11	8	−4	−3	1	0
r_2	2	0	27	19	−11	−8	5	3
r_3	0	0	3	2	−2	−1	2	1
r_4	−1	−1	−27	−19	11	8	54	38
r_5	1	0	7	5	−1	−1	6	4

det $= -7 - 6w$ det norm $= -23$

34222

quadratic form

$$\left(\begin{array}{cc|cc|cc} 75 & -54 & 6 & -8 & 3 & -4 \\ 6 & -8 & 2 & 0 & 1 & 0 \\ 3 & -4 & 1 & 0 & 1 & 0 \end{array}\right)$$

root list

	roots						norms	
r_1	1	0	−5	3	1	0	2	0
r_2	0	0	1	0	−2	0	2	0
r_3	0	0	0	0	1	0	1	0
r_4	5	3	−10	−2	19	13	38	26
r_5	1	1	0	−2	4	2	1	0

det $= -1533 - 1084w$ det norm $= -23$

22243

quadratic form

$$\begin{pmatrix} 17 & 12 & 14 & 10 & -7 & -5 \\ 14 & 10 & -2 & -4 & 1 & 2 \\ -7 & -5 & 1 & 2 & 1 & 0 \end{pmatrix}$$

root list

	roots						norms	
r_1	0	0	−1	0	−2	0	6	4
r_2	−1	1	0	0	0	0	3	2
r_3	−1	−1	5	4	7	6	218	154
r_4	−2	0	1	1	1	1	3	2
r_5	−1	−3	2	1	1	0	6	4

det $= -263 - 186w$ det norm $= -23$

22243

quadratic form

$$\begin{pmatrix} 17 & 12 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & -2 & -1 & -1 \\ 0 & 0 & -1 & -1 & 2 & 0 \end{pmatrix}$$

root list

	roots						norms	
r_1	2	0	2	2	1	0	2	0
r_2	−1	2	2	2	0	0	1	0
r_3	0	0	1	1	−2	−2	38	26
r_4	1	−1	−1	0	0	0	1	0
r_5	2	0	3	1	1	1	2	0

det $= 7 - 7w$ det norm $= -49$

84222

quadratic form

$$\left(\begin{array}{cc|cc|cc} 49 & -35 & 14 & -14 & 7 & -7 \\ 14 & -14 & 6 & -4 & 3 & -2 \\ 7 & -7 & 3 & -2 & 3 & -2 \end{array}\right)$$

root list

	roots						norms	
r_1	1	0	−3	0	3	2	2	1
r_2	0	0	3	2	−6	−4	6	4
r_3	0	0	0	0	3	2	3	2
r_4	1	1	−10	−8	31	22	44	31
r_5	3	2	−16	−11	38	27	27	19

det $= -1673 - 1183w$ det norm $= -49$

82224

quadratic form

$$\left(\begin{array}{cc|cc|cc} 4733 & 3337 & -1427 & -1003 & 230 & 163 \\ -1427 & -1003 & 431 & 301 & -69 & -49 \\ 230 & 163 & -69 & -49 & 10 & 7 \end{array}\right)$$

root list

	roots						norms	
r_1	−3	−3	−11	−9	2	0	6	4
r_2	0	0	0	0	−1	1	2	1
r_3	131	95	432	309	−52	−35	5	3
r_4	187	131	611	430	−73	−50	8	5
r_5	56	36	179	121	−19	−16	1	0

det $= 10 - 9w$ det norm $= -62$

22322

quadratic form

$$\begin{pmatrix} 342 & 213 & -32 & 4 & -16 & 2 \\ -32 & 4 & 18 & -12 & 9 & -6 \\ -16 & 2 & 9 & -6 & 6 & -4 \end{pmatrix}$$

root list

	roots						norms	
r_1	1	0	8	5	6	4	2	1
r_2	2	1	33	23	0	0	66	46
r_3	0	−2	−21	−17	−14	−10	2	0
r_4	−2	−1	−29	−20	−13	−9	2	0
r_5	−5	−5	−105	−76	−33	−23	20	13

det $= -1574 - 1113w$ det norm $= -62$

22232

quadratic form

$$\begin{pmatrix} 110 & 77 & -4 & 4 & -6 & -4 \\ -4 & 4 & 42 & -30 & 3 & 0 \\ -6 & -4 & 3 & 0 & 6 & 4 \end{pmatrix}$$

root list

	roots						norms	
r_1	0	0	−7	−5	−2	0	66	46
r_2	−3	2	−2	−1	0	0	2	1
r_3	9	−5	57	39	3	2	112	79
r_4	−2	2	15	11	1	0	6	4
r_5	8	−5	13	8	0	0	6	4

det $= -9174 - 6487w$ det norm $= -62$

22223

quadratic form

$$\begin{pmatrix} 30 & 13 & -300 & -204 & -12 & -14 \\ -300 & -204 & 3694 & 2604 & 203 & 149 \\ -12 & -14 & 203 & 149 & 14 & 6 \end{pmatrix}$$

root list

	roots						norms	
r_1	4	2	1	0	−6	−4	34	24
r_2	−1	4	−1	4	−48	−34	652	461
r_3	−1	−1	0	0	−2	−1	10	7
r_4	0	0	0	0	3	2	382	270
r_5	8	−3	6	−4	−1	0	6	4

det $= -270 - 191w$ det norm $= -62$

32222

quadratic form

$$\begin{pmatrix} 358 & 253 & -220 & -156 & 26 & 20 \\ -220 & -156 & 138 & 96 & -21 & -9 \\ 26 & 20 & -21 & -9 & 18 & -10 \end{pmatrix}$$

root list

	roots						norms	
r_1	0	0	−1	0	−6	−4	6	4
r_2	0	1	2	0	−1	−1	6	4
r_3	0	0	0	0	7	5	382	270
r_4	−1	−1	−2	−2	−4	−3	10	7
r_5	−13	−10	−33	−23	−116	−82	652	461

det $= -53470 - 37809w$ det norm $= -62$

22232

quadratic form

$$\begin{pmatrix} 338 & 239 & 2 & 2 & 362 & 256 \\ 2 & 2 & 198 & -140 & 3 & -9 \\ 362 & 256 & 3 & -9 & -920 & -651 \end{pmatrix}$$

root list

	roots						norms	
r_1	0	0	−75	−53	1	0	652	461
r_2	−3	2	0	0	0	0	10	7
r_3	28	−19	151	107	4	−5	382	270
r_4	−8	6	27	19	−2	1	6	4
r_5	37	−26	1	1	7	−5	6	4

det $= -53470 - 37809w$ det norm $= -62$

22232

quadratic form

$$\begin{pmatrix} 18026 & 12745 & -23212 & -16412 & 276 & 198 \\ -23212 & -16412 & 29890 & 21134 & -355 & -254 \\ 276 & 198 & -355 & -254 & 8 & -1 \end{pmatrix}$$

root list

	roots						norms	
r_1	−68	−38	−55	−28	−2	−2	382	270
r_2	−1	−7	0	−6	0	0	10	7
r_3	0	0	0	0	7	5	652	461
r_4	11	0	10	−1	1	1	34	24
r_5	−16	−7	−13	−5	0	0	34	24

det $= -6 - 7w$ det norm $= -62$

22223

quadratic form

$$\begin{pmatrix} 4406 & 3115 & -158 & -112 & -66 & -46 \\ -158 & -112 & 34 & -16 & -1 & 4 \\ -66 & -46 & -1 & 4 & 2 & 0 \end{pmatrix}$$

root list

	roots						norms	
r_1	4	−1	33	23	8	3	2	0
r_2	7	6	191	135	46	33	20	13
r_3	3	−2	2	1	4	−1	2	−1
r_4	−4	1	−33	−23	−7	−3	14	6
r_5	6	−4	5	3	5	−3	6	−4

det $= -53470 - 37809w$ det norm $= -62$

22223

quadratic form

$$\begin{pmatrix} 90 & 59 & -1012 & -711 & -12 & -14 \\ -1012 & -711 & 12600 & 8905 & 203 & 149 \\ -12 & -14 & 203 & 149 & 8 & -1 \end{pmatrix}$$

root list

	roots						norms	
r_1	3	1	1	0	−26	−18	34	24
r_2	−1	4	−1	4	−164	−116	2226	1574
r_3	−1	−1	0	0	−6	−4	58	41
r_4	0	0	0	0	17	12	3800	2687
r_5	5	1	2	−1	−3	−2	198	140

det $= -168 - 119w$ det norm $= -98$

32222

Table A.3, cont.

quadratic form

$$\left(\begin{array}{cc|cc|cc} 164 & 113 & -92 & -68 & -14 & -14 \\ -92 & -68 & 58 & 38 & 13 & 5 \\ -14 & -14 & 13 & 5 & 6 & -2 \end{array}\right)$$

root list

	roots						norms	
r_1	0	0	−1	0	2	1	6	4
r_2	0	1	2	0	1	1	6	4
r_3	0	0	0	0	3	2	54	38
r_4	−5	−3	−10	−8	18	13	44	31
r_5	−1	−1	−3	−2	4	3	10	7

det $= -33292 - 23541w$ det norm $= -98$

22223

quadratic form

$$\left(\begin{array}{cc|cc|cc} 6776 & 4739 & -10332 & -7294 & 106 & 79 \\ -10332 & -7294 & 15826 & 11188 & -167 & -119 \\ 106 & 79 & -167 & -119 & 2 & 1 \end{array}\right)$$

root list

	roots						norms	
r_1	4	2	5	0	20	14	34	24
r_2	3	−2	13	−9	6	4	10	7
r_3	−1	−1	0	−1	8	5	44	31
r_4	−3	2	−13	9	−5	−3	54	38
r_5	1	2	−3	4	7	4	6	4

det $= -980 - 693w$ det norm $= -98$

22232

quadratic form

$$\begin{pmatrix} 8 & 5 & -6 & -2 & -22 & -12 \\ -6 & -2 & 6 & -4 & 7 & -8 \\ -22 & -12 & 7 & -8 & 2 & -19 \end{pmatrix}$$

root list

	roots						norms	
r_1	0	0	−3	−1	1	0	10	7
r_2	1	1	0	0	0	0	44	31
r_3	0	1	7	3	−2	0	54	38
r_4	0	0	3	2	0	0	6	4
r_5	1	−1	−1	0	1	0	6	4

det $= -1854 - 1311w$ det norm $= -126$

22222

quadratic form

$$\begin{pmatrix} 3822 & 2667 & 6708 & 4728 & -156 & -108 \\ 6708 & 4728 & 11818 & 8350 & -273 & -192 \\ -156 & -108 & -273 & -192 & 6 & 4 \end{pmatrix}$$

root list

	roots						norms	
r_1	0	2	3	−3	8	3	6	4
r_2	−17	−11	12	6	36	24	30	21
r_3	−6	−6	1	5	−3	−1	26	18
r_4	1	1	0	−1	0	−1	2	1
r_5	29	21	−15	−12	24	15	132	93

det $= -1854 - 1311w$ det norm $= -126$

22222

quadratic form

$$\begin{pmatrix} 23134 & 16357 & 12734 & 9007 & -330 & -229 \\ 12734 & 9007 & 7016 & 4955 & -173 & -132 \\ -330 & -229 & -173 & -132 & 16 & -5 \end{pmatrix}$$

root list

	roots						norms	
r_1	15	5	−19	−15	−2	−2	26	18
r_2	−3	5	−6	−1	0	0	2	1
r_3	−73	−57	141	98	9	6	132	93
r_4	−22	−11	33	25	1	1	6	4
r_5	−89	−68	170	118	0	0	30	21

det $= 18 - 15w$ det norm $= -126$

22222

quadratic form

$$\begin{pmatrix} 222 & 123 & -102 & 99 & 18 & 6 \\ -102 & 99 & 616 & -433 & -23 & 18 \\ 18 & 6 & -23 & 18 & 2 & 0 \end{pmatrix}$$

root list

	roots						norms	
r_1	−1	−1	13	9	2	−2	2	0
r_2	−10	−6	93	66	0	0	24	15
r_3	1	2	−20	−14	2	−1	2	−1
r_4	16	13	−176	−124	−1	2	6	2
r_5	49	34	−492	−348	0	0	6	3

det $= -367086 - 259569w$ det norm $= -126$

22226

quadratic form

$$\left(\begin{array}{cc|cc|cc} 13814 & 9755 & -20380 & -14398 & -462 & -336 \\ -20380 & -14398 & 30070 & 21250 & 687 & 495 \\ -462 & -336 & 687 & 495 & 18 & 6 \end{array}\right)$$

root list

	roots						norms	
r_1	30	18	21	12	−22	−15	594	420
r_2	5	−3	5	−3	−10	−7	256	181
r_3	−3	−3	−2	−2	−2	−1	58	41
r_4	0	0	0	0	17	12	15282	10806
r_5	10	11	6	8	−1	−1	198	140

det $= -1854 - 1311w$ det norm $= -126$

22222

quadratic form

$$\left(\begin{array}{cc|cc|cc} 2486 & 1757 & 824 & 578 & -102 & -69 \\ 824 & 578 & 290 & 180 & -45 & -15 \\ -102 & -69 & -45 & -15 & 12 & -3 \end{array}\right)$$

root list

	roots						norms	
r_1	0	0	−3	−2	−30	−21	34	24
r_2	−3	−3	0	0	−110	−78	174	123
r_3	1	−2	5	4	27	19	150	106
r_4	1	0	−2	−1	−2	−1	10	7
r_5	9	3	−39	−27	−188	−133	768	543

det $= -318 - 225w$ det norm $= -126$

62222

quadratic form

$$\begin{pmatrix} 174 & 123 & -60 & -42 & 8 & 5 \\ -60 & -42 & 30 & 12 & -13 & 5 \\ 8 & 5 & -13 & 5 & 16 & -11 \end{pmatrix}$$

root list

	roots						norms	
r_1	0	0	−1	0	−12	−8	6	4
r_2	1	1	2	2	−9	−6	102	72
r_3	0	0	0	0	7	5	44	31
r_4	−1	0	−2	−1	−6	−4	10	7
r_5	−8	−5	−31	−22	−246	−174	2622	1854

det $= -1854 - 1311w$ det norm $= -126$

22222

quadratic form

$$\begin{pmatrix} 57774 & 38571 & 6060 & 4554 & -24 & 6 \\ 6060 & 4554 & 718 & 476 & 1 & -2 \\ -24 & 6 & 1 & -2 & 6 & 4 \end{pmatrix}$$

root list

	roots						norms	
r_1	−40	−30	381	255	2	0	6	4
r_2	−99	−71	903	630	0	0	132	93
r_3	9	5	−64	−57	−2	1	2	1
r_4	2	2	−25	−13	−1	2	26	18
r_5	−281	−199	2532	1788	12	6	30	21

det $= -1854 - 1311w$ det norm $= -126$

22262

quadratic form

$$\begin{pmatrix} 58 & 41 & 2 & 2 & 62 & 44 \\ 2 & 2 & 34 & -24 & 7 & -7 \\ 62 & 44 & 7 & -7 & -48 & -37 \end{pmatrix}$$

root list

	roots						norms	
r_1	0	0	−13	−9	1	0	44	31
r_2	1	−1	0	0	0	0	10	7
r_3	2	3	181	128	−10	−7	2622	1854
r_4	0	2	27	19	−2	−1	34	24
r_5	5	5	17	12	−3	−2	594	420

det $= -318 - 225w$ det norm $= -126$

22622

quadratic form

$$\begin{pmatrix} -106 & -75 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 2 & 0 \end{pmatrix}$$

root list

	roots						norms	
r_1	1	0	−6	−4	2	1	2	1
r_2	1	0	−5	−4	0	0	8	5
r_3	−6	−6	83	59	−22	−16	18	12
r_4	0	−3	24	17	−5	−4	2	0
r_5	−6	−6	80	57	−13	−9	78	54

det $= -318 - 225w$ det norm $= -126$

22222

quadratic form

$$\left(\begin{array}{cc|cc|cc} 10 & 7 & -8 & -6 & 0 & 0 \\ -8 & -6 & 14 & 2 & 9 & 3 \\ 0 & 0 & 9 & 3 & 24 & 15 \end{array}\right)$$

root list

	roots						norms	
r_1	−2	−3	−5	−3	2	0	2	0
r_2	−9	−6	−12	−9	2	2	6	3
r_3	7	0	5	4	−1	−1	6	2
r_4	3	−3	−2	0	2	−1	2	−1
r_5	−21	−21	−39	−27	10	6	24	15

det = $-318 - 225w$ det norm = -126

22222

quadratic form

$$\left(\begin{array}{cc|cc|cc} 174 & 123 & 0 & 0 & 0 & 0 \\ 0 & 0 & 14 & -10 & 5 & -3 \\ 0 & 0 & 5 & -3 & 6 & 2 \end{array}\right)$$

root list

	roots						norms	
r_1	2	0	41	29	−4	−3	6	4
r_2	3	1	93	66	−12	−9	132	93
r_3	1	−1	−8	−6	0	1	2	1
r_4	0	0	0	0	1	1	26	18
r_5	7	5	288	204	−24	−18	30	21

det = $-318 - 225w$ det norm = -126

22222

quadratic form

$$\begin{pmatrix} -282 & -213 & \vline & 18 & 6 & \vline & 108 & 78 \\ 18 & 6 & \vline & 2 & -2 & \vline & -3 & -1 \\ 108 & 78 & \vline & -3 & -1 & \vline & 2 & 0 \end{pmatrix}$$

root list

	roots						norms	
r_1	0	0	0	0	1	1	6	4
r_2	1	1	60	42	0	0	30	21
r_3	2	−1	11	13	−3	−1	26	18
r_4	1	−1	−12	−5	0	1	2	1
r_5	−1	−3	−126	−84	15	12	132	93

det $= -10806 - 7641w$ det norm $= -126$

62222

quadratic form

$$\begin{pmatrix} 310 & 219 & \vline & -618 & -438 & \vline & 30 & 24 \\ -618 & -438 & \vline & 1246 & 876 & \vline & -75 & -39 \\ 30 & 24 & \vline & -75 & -39 & \vline & 30 & -18 \end{pmatrix}$$

root list

	roots						norms	
r_1	0	−3	−3	0	−22	−15	102	72
r_2	6	−3	−2	2	−1	0	6	4
r_3	0	0	0	0	7	5	450	318
r_4	−3	1	2	−2	0	−1	2	1
r_5	−9	4	1	−2	−4	−3	8	5

det $= -54 - 39w$ det norm $= -126$

22222

quadratic form

$$\left(\begin{array}{cc|cc|cc} 5710 & 4037 & 144 & 249 & 162 & 117 \\ 144 & 249 & 26432 & -18675 & 447 & -306 \\ 162 & 117 & 447 & -306 & 12 & -3 \end{array}\right)$$

root list

	roots						norms	
r_1	36	−9	−277	−196	13	9	6	4
r_2	75	−36	−288	−204	14	10	30	21
r_3	−3	−22	406	287	−19	−13	26	18
r_4	−17	25	−218	−154	10	7	2	1
r_5	249	69	−4119	−2913	189	134	132	93

det = $-10806 - 7641w$ det norm = -126

22222

quadratic form

$$\left(\begin{array}{cc|cc|cc} 26 & -5 & -344 & -220 & 6 & -12 \\ -344 & -220 & 13298 & 9380 & 219 & 171 \\ 6 & -12 & 219 & 171 & 12 & -3 \end{array}\right)$$

root list

	roots						norms	
r_1	4	2	1	0	−30	−21	34	24
r_2	−3	6	−3	6	−188	−133	768	543
r_3	−1	−1	0	0	−2	−1	10	7
r_4	−5	3	−5	3	27	19	150	106
r_5	21	18	0	3	−110	−78	174	123

det = $-1854 - 1311w$ det norm = -126

22222

quadratic form

$$\begin{pmatrix} 1878 & 1293 & -864 & -600 & -90 & -72 \\ -864 & -600 & 398 & 278 & 43 & 33 \\ -90 & -72 & 43 & 33 & 6 & 2 \end{pmatrix}$$

root list

	roots						norms	
r_1	4	2	9	4	−2	−1	34	24
r_2	3	1	9	3	−30	−21	768	543
r_3	−1	−1	−2	−2	−2	−1	10	7
r_4	0	0	0	0	3	2	150	106
r_5	25	17	54	36	0	0	174	123

det $= -54 - 39w$ det norm $= -126$

22222

quadratic form

$$\begin{pmatrix} 30 & 21 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & -3 & -1 \\ 0 & 0 & -3 & -1 & 6 & 2 \end{pmatrix}$$

root list

	roots						norms	
r_1	2	0	22	15	9	6	2	0
r_2	3	1	54	39	21	15	24	15
r_3	1	−1	−4	−3	−2	−1	2	−1
r_4	0	0	−3	−1	−1	0	6	2
r_5	7	5	144	102	60	42	6	3

det $= -54 - 39w$ det norm $= -126$

22226

Table A.3, cont.

quadratic form

$$\begin{pmatrix} 2030 & 1435 & 144 & 90 & 78 & 54 \\ 144 & 90 & 242 & -158 & 27 & -12 \\ 78 & 54 & 27 & -12 & 6 & 0 \end{pmatrix}$$

root list

	roots						norms	
r_1	−18	6	39	27	53	36	6	0
r_2	27	−22	15	12	23	17	4	−1
r_3	−9	6	2	1	4	2	2	−1
r_4	18	−6	−39	−27	−52	−36	78	54
r_5	−2	1	3	2	2	2	2	0

Table A.4: Hexagons

det $= -2-3w$	det norm $= -14$
222222	

quadratic form

$$\left(\begin{array}{cc|cc|cc} 13 & 9 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right)$$

root list

	roots						norms	
r_1	1	0	7	5	−2	−2	1	0
r_2	0	1	13	9	−5	−4	3	1
r_3	0	−2	−21	−15	8	5	2	−1
r_4	−3	−3	−57	−40	21	15	2	0
r_5	−15	−9	−225	−159	85	61	6	2
r_6	−4	−1	−45	−32	18	12	2	−1

det $= -122362-86523w$	det norm $= -14$
222222	

quadratic form

$$\left(\begin{array}{cc|cc|cc} -816 & -577 & 0 & 0 & -31 & -22 \\ 0 & 0 & 34 & 24 & 3 & 1 \\ -31 & -22 & 3 & 1 & 27 & -19 \end{array}\right)$$

root list

	roots						norms	
r_1	0	0	0	0	−17	−12	75	53
r_2	17	−12	5	−3	−6	−4	10	7
r_3	0	0	1	0	0	0	34	24
r_4	−16	11	−5	3	6	4	150	106
r_5	−3	2	−1	0	0	0	10	7
r_6	−17	12	−6	4	−1	−1	3	2

det $= -222 - 157w$ det norm $= -14$

222222

quadratic form

$$\begin{pmatrix} 92 & 65 & 0 & 0 & 27 & 19 \\ 0 & 0 & 2 & -2 & -1 & -2 \\ 27 & 19 & -1 & -2 & 1 & 0 \end{pmatrix}$$

root list

	roots						norms	
r_1	0	0	0	0	1	1	3	2
r_2	1	0	7	5	0	0	10	7
r_3	0	1	11	8	−2	−2	34	24
r_4	−1	0	0	0	0	0	92	65
r_5	−1	0	−6	−4	2	1	10	7
r_6	−3	−1	−27	−19	16	11	92	65

det $= -38 - 27w$ det norm $= -14$

222222

quadratic form

$$\begin{pmatrix} 92 & 65 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & -3 & 1 & -1 \\ 0 & 0 & 1 & -1 & 1 & 0 \end{pmatrix}$$

root list

	roots						norms	
r_1	0	1	17	12	2	1	2	1
r_2	2	0	27	19	0	0	16	11
r_3	−3	−2	−71	−50	−6	−5	6	4
r_4	−7	−3	−140	−99	−10	−7	2	1
r_5	−3	−2	−73	−52	−5	−3	1	0
r_6	−15	−12	−406	−287	−22	−16	16	11

Table A.4, cont.

det $= -122362 - 86523w$	det norm $= -14$
222222	

quadratic form

$$\begin{pmatrix} -1055 & -746 & 0 & 0 & 0 & 0 \\ 0 & 0 & 17 & 12 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 1 \end{pmatrix}$$

root list

	roots						norms	
r_1	0	1	−2	−1	−17	−12	58	41
r_2	1	1	0	0	−31	−22	437	309
r_3	−4	−2	7	5	82	58	99	70
r_4	−28	−20	48	34	683	483	338	239
r_5	−87	−61	137	97	2110	1492	5094	3602
r_6	−7	−5	7	5	174	123	1154	816

det $= -38 - 27w$	det norm $= -14$
222222	

quadratic form

$$\begin{pmatrix} 20 & -9 & 8 & -12 & 0 & 0 \\ 8 & -12 & 12 & -5 & -7 & -5 \\ 0 & 0 & -7 & -5 & -15 & -11 \end{pmatrix}$$

root list

	roots						norms	
r_1	4	4	5	2	−3	−1	1	0
r_2	53	34	36	30	−22	−16	16	11
r_3	−28	−19	−21	−16	12	9	2	1
r_4	−702	−499	−555	−389	314	222	16	11
r_5	−295	−210	−233	−163	132	93	6	4
r_6	−107	−72	−78	−60	46	33	2	1

Table A.4, cont.

det $= -1294 - 915w$ det norm $= -14$

222222

quadratic form

$$\left(\begin{array}{cc|cc|cc} 58 & 39 & 14 & 14 & 0 & 0 \\ 14 & 14 & 26 & 10 & -7 & -5 \\ 0 & 0 & -7 & -5 & 3 & 2 \end{array}\right)$$

root list

	roots						norms	
r_1	−4	1	3	4	8	4	6	4
r_2	5	−5	6	0	4	4	2	1
r_3	−2	1	0	1	1	1	1	0
r_4	−7	4	−2	3	4	1	16	11
r_5	−1	1	−1	0	0	−1	2	1
r_6	−7	3	3	6	10	6	16	11

det $= -7542 - 5333w$ det norm $= -14$

222222

quadratic form

$$\left(\begin{array}{cc|cc|cc} -379 & -268 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 1 \end{array}\right)$$

root list

	roots						norms	
r_1	0	1	−3	−2	−10	−7	17	12
r_2	2	1	0	0	−27	−19	536	379
r_3	−1	0	3	2	7	5	58	41
r_4	3	2	−27	−19	−27	−19	536	379
r_5	3	2	−21	−15	−34	−24	198	140
r_6	2	2	−14	−10	−31	−22	58	41

Table A.4, cont.

det = $-7542 - 5333w$ det norm = -14

222222

quadratic form

$$\left(\begin{array}{cc|cc|cc} -140 & -99 & 0 & 0 & 14 & 10 \\ 0 & 0 & 10 & 7 & 1 & 1 \\ 14 & 10 & 1 & 1 & 3 & -2 \end{array}\right)$$

root list

	roots						norms	
r_1	0	0	0	0	−7	−5	17	12
r_2	1	−1	6	4	−48	−34	536	379
r_3	0	0	1	1	0	0	58	41
r_4	2	0	−1	−1	10	7	536	379
r_5	1	0	−3	−2	0	0	198	140
r_6	−1	1	−2	−1	−10	−7	58	41

det = $-6 - 5w$ det norm = -14

222222

quadratic form

$$\left(\begin{array}{cc|cc|cc} 162 & 114 & -12 & -10 & -11 & -8 \\ -12 & -10 & 12 & -7 & 3 & -1 \\ -11 & -8 & 3 & -1 & 1 & 0 \end{array}\right)$$

root list

	roots						norms	
r_1	1	0	6	5	2	−2	2	0
r_2	−1	1	3	2	0	0	2	−1
r_3	0	0	0	0	−1	1	3	−2
r_4	1	−1	−5	−3	4	1	4	1
r_5	0	0	−1	0	2	−1	2	−1
r_6	2	0	11	8	0	0	4	1

det $= -72 - 51w$ det norm $= -18$

222222

quadratic form

$$\left(\begin{array}{cc|cc|cc} 174 & 123 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & -3 & -1 & 1 \\ 0 & 0 & -1 & 1 & 1 & 0 \end{array}\right)$$

root list

	roots						norms	
r_1	0	1	21	15	−6	−3	6	3
r_2	3	−2	2	2	−2	0	2	−1
r_3	2	−2	−12	−9	5	1	3	−2
r_4	−5	−5	−180	−126	48	36	6	−3
r_5	4	−4	−25	−17	4	7	10	−7
r_6	1	−1	−5	−5	4	0	6	−4

det $= -2448 - 1731w$ det norm $= -18$

222224

quadratic form

$$\left(\begin{array}{cc|cc|cc} -420 & -297 & 0 & 0 & 21 & 15 \\ 0 & 0 & 2 & 1 & 0 & 1 \\ 21 & 15 & 0 & 1 & 3 & -2 \end{array}\right)$$

root list

	roots						norms	
r_1	0	0	0	0	−7	−5	17	12
r_2	1	−1	9	6	−21	−15	297	210
r_3	0	0	3	2	0	0	58	41
r_4	−1	1	−2	−1	4	3	58	41
r_5	1	1	−21	−15	0	0	594	420
r_6	−1	1	−5	−3	−6	−4	34	24

det $= -12 - 9w$	det norm $= -18$
222242	

quadratic form

$$\begin{pmatrix} 150 & 106 & -6 & -4 & 7 & 5 \\ -6 & -4 & 16 & -11 & 2 & -2 \\ 7 & 5 & 2 & -2 & 1 & 0 \end{pmatrix}$$

root list

	roots						norms	
r_1	1	0	12	9	2	−2	6	0
r_2	3	−2	3	2	0	0	2	−1
r_3	0	0	1	1	2	−1	2	−1
r_4	−3	2	0	0	1	1	3	0
r_5	0	0	0	0	−1	1	3	−2
r_6	3	−2	2	1	0	0	6	−4

det $= -72 - 51w$	det norm $= -18$
224222	

quadratic form

$$\begin{pmatrix} -72 & -51 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 \end{pmatrix}$$

root list

	roots						norms	
r_1	1	0	−6	−4	−4	−3	2	1
r_2	0	1	−9	−6	−9	−6	9	6
r_3	0	0	0	0	−1	0	1	0
r_4	0	0	1	0	2	0	2	0
r_5	2	0	−9	−6	0	0	18	12
r_6	1	0	−5	−4	−2	−2	2	1

det $= -1195 - 845w$ det norm $= -25$

422822

quadratic form

$$\begin{pmatrix} -41 & -29 & 0 & 0 & 7 & 5 \\ 0 & 0 & 6 & 4 & 1 & 0 \\ 7 & 5 & 1 & 0 & 3 & -2 \end{pmatrix}$$

root list

	roots						norms	
r_1	0	0	0	0	−3	−2	3	2
r_2	0	0	1	0	0	0	6	4
r_3	2	0	−1	0	6	4	170	120
r_4	−2	2	−1	−1	0	0	6	4
r_5	3	−2	−1	0	−1	−1	2	1
r_6	2	−1	−1	−2	−13	−9	15	10

det $= -205 - 145w$ det norm $= -25$

228224

quadratic form

$$\begin{pmatrix} -205 & -145 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 \end{pmatrix}$$

root list

	roots						norms	
r_1	0	1	−14	−10	−11	−8	3	2
r_2	2	1	−35	−25	−35	−25	85	60
r_3	−1	0	10	7	7	5	10	7
r_4	4	3	−75	−53	−38	−27	34	24
r_5	26	19	−495	−350	−290	−205	990	700
r_6	4	3	−79	−56	−52	−37	34	24

det $= -35 - 25w$ det norm $= -25$

228224

quadratic form

$$\begin{pmatrix} -205 & -145 & -15 & -10 & -205 & -145 \\ -15 & -10 & 159 & -113 & -12 & 4 \\ -205 & -145 & -12 & 4 & 1 & 0 \end{pmatrix}$$

root list

	roots						norms	
r_1	0	0	0	0	1	0	1	0
r_2	5	1	−205	−145	15	10	15	10
r_3	0	1	−45	−32	5	0	2	1
r_4	−1	−1	79	56	−6	−2	6	4
r_5	−13	−8	785	555	−40	−30	170	120
r_6	−1	−2	123	87	−6	−4	6	4

det $= 5 - 5w$ det norm $= -25$

224228

quadratic form

$$\begin{pmatrix} 115 & 75 & 5 & 0 & -10 & -10 \\ 5 & 0 & 7 & -5 & 1 & -2 \\ -10 & -10 & 1 & -2 & -2 & -3 \end{pmatrix}$$

root list

	roots						norms	
r_1	1	1	−31	−22	5	3	2	1
r_2	2	1	−35	−25	5	5	15	10
r_3	−1	0	14	10	−1	−2	1	0
r_4	−1	−1	31	22	−4	−3	2	0
r_5	−3	−3	85	60	−10	−5	30	20
r_6	1	0	−15	−11	2	3	2	0

det $= -205 - 145w$ det norm $= -25$

822422

quadratic form

$$\left(\begin{array}{cc|cc|cc} 949 & 671 & -10 & -10 & 674 & 476 \\ -10 & -10 & 170 & -120 & 25 & -30 \\ 674 & 476 & 25 & -30 & 490 & 337 \end{array}\right)$$

root list

	roots						norms	
r_1	0	0	21	15	1	0	10	7
r_2	2	0	23	16	−2	0	34	24
r_3	−140	−100	−9191	−6499	0	0	990	700
r_4	−60	−42	−3797	−2685	2	2	34	24
r_5	−73	−51	−4544	−3213	5	3	17	12
r_6	−465	−330	−28939	−20463	35	25	495	350

det $= -7 - 7w$ det norm $= -49$

222222

quadratic form

$$\left(\begin{array}{cc|cc|cc} 525 & 371 & -35 & -21 & 22 & 16 \\ -35 & -21 & 79 & -53 & 7 & -7 \\ 22 & 16 & 7 & -7 & 2 & 0 \end{array}\right)$$

root list

	roots						norms	
r_1	1	0	11	8	5	1	2	0
r_2	−2	2	11	8	5	3	3	−1
r_3	0	0	1	1	4	−1	5	−3
r_4	1	−1	−5	−3	2	−3	6	−2
r_5	5	−3	7	5	4	−1	10	−6
r_6	5	−2	22	16	8	2	13	−9

det $= -287 - 203w$	det norm $= -49$
222222	

quadratic form

$$\begin{pmatrix} 45 & 27 & 2 & 4 & 7 & 7 \\ 2 & 4 & 2 & 0 & 3 & 1 \\ 7 & 7 & 3 & 1 & 3 & 1 \end{pmatrix}$$

root list

	roots						norms	
r_1	1	−2	12	4	−7	−6	6	2
r_2	−1	−1	12	10	−12	−10	3	−1
r_3	−1	2	−13	−4	9	6	2	0
r_4	38	26	−401	−284	399	280	3	−1
r_5	24	22	−304	−204	295	207	5	−3
r_6	15	13	−182	−126	179	127	6	−2

det $= -287 - 203w$	det norm $= -49$
222222	

quadratic form

$$\begin{pmatrix} 157 & 111 & 0 & 0 & 0 & 0 \\ 0 & 0 & 14 & -10 & 5 & -3 \\ 0 & 0 & 5 & -3 & 6 & 2 \end{pmatrix}$$

root list

	roots						norms	
r_1	1	−2	36	26	−3	−4	3	1
r_2	−1	−1	49	35	−6	−5	5	3
r_3	6	2	−177	−125	22	15	6	4
r_4	73	54	−3010	−2128	378	266	5	3
r_5	46	34	−1898	−1342	239	169	26	18
r_6	14	12	−628	−444	81	57	54	38

det = $-56833 - 40187w$ det norm = -49

222222

quadratic form

$$\left(\begin{array}{cc|cc|cc} 389 & 275 & -200 & -140 & -76 & -54 \\ -200 & -140 & 130 & 60 & 37 & 32 \\ -76 & -54 & 37 & 32 & 10 & 6 \end{array}\right)$$

root list

	roots						norms	
r_1	0	0	-2	-1	1	3	34	24
r_2	3	-1	0	0	2	2	27	19
r_3	-2	2	3	2	-2	-3	150	106
r_4	-2	-4	-11	-8	8	5	314	222
r_5	-7	-4	-25	-18	24	15	75	53
r_6	-7	-7	-38	-27	37	26	157	111

det = $-9751 - 6895w$ det norm = -49

222222

quadratic form

$$\left(\begin{array}{cc|cc|cc} 45 & 15 & -332 & -218 & 14 & 14 \\ -332 & -218 & 3166 & 2222 & -171 & -125 \\ 14 & 14 & -171 & -125 & 10 & 6 \end{array}\right)$$

root list

	roots						norms	
r_1	4	2	1	0	2	1	34	24
r_2	-5	4	-5	4	4	3	27	19
r_3	-1	-1	0	0	3	1	13	9
r_4	0	0	0	0	1	1	54	38
r_5	16	-2	10	-6	3	1	26	18
r_6	17	6	6	-2	4	4	5	3

det $= -331247 - 234227w$	det norm $= -49$
222222	

quadratic form

$$\begin{pmatrix} 19939 & 14099 & 574 & 406 & -314 & -222 \\ 574 & 406 & 34 & 16 & -7 & -7 \\ -314 & -222 & -7 & -7 & 5 & 3 \end{pmatrix}$$

root list

	roots						norms	
r_1	2	0	-3	-2	46	32	198	140
r_2	3	-1	0	0	40	28	157	111
r_3	2	-4	4	3	-85	-61	75	53
r_4	-9	-4	11	8	-351	-249	314	222
r_5	-11	-6	11	8	-473	-335	150	106
r_6	-19	-3	11	8	-568	-402	27	19

det $= -56833 - 40187w$	det norm $= -49$
222222	

quadratic form

$$\begin{pmatrix} 3421 & 2419 & 1296 & 916 & 184 & 130 \\ 1296 & 916 & 506 & 344 & 73 & 48 \\ 184 & 130 & 73 & 48 & 10 & 6 \end{pmatrix}$$

root list

	roots						norms	
r_1	0	0	-3	-2	16	11	150	106
r_2	-3	1	0	0	12	8	27	19
r_3	-8	6	2	1	-13	-9	34	24
r_4	11	22	16	11	-397	-281	27	19
r_5	-1	23	11	7	-290	-205	13	9
r_6	0	14	5	3	-172	-121	54	38

det $= -9751 - 6895w$ det norm $= -49$

222222

quadratic form

$$\left(\begin{array}{cc|cc|cc} 28973 & 20487 & 6190 & 4377 & -157 & -111 \\ 6190 & 4377 & 1324 & 936 & -33 & -24 \\ -157 & -111 & -33 & -24 & 3 & -1 \end{array}\right)$$

root list

	roots						norms	
r_1	0	1	−3	−1	34	24	26	18
r_2	−5	4	0	0	48	34	5	3
r_3	−1	0	1	1	−35	−25	6	4
r_4	−14	−7	7	7	−1483	−1049	5	3
r_5	−13	−3	5	4	−1094	−774	3	1
r_6	0	−7	1	2	−664	−470	10	6

det $= 7 - 7w$ det norm $= -49$

222222

quadratic form

$$\left(\begin{array}{cc|cc|cc} 27 & 19 & 0 & 0 & 0 & 0 \\ 0 & 0 & 82 & -58 & 7 & -5 \\ 0 & 0 & 7 & -5 & 2 & 0 \end{array}\right)$$

root list

	roots						norms	
r_1	0	2	65	46	1	0	2	0
r_2	5	2	184	130	0	0	3	−1
r_3	0	0	3	2	−2	0	6	2
r_4	0	−2	−65	−46	0	0	10	6
r_5	3	1	99	70	2	2	3	1
r_6	7	4	287	203	5	3	5	3

det = $-195 - 138w$ det norm = -63

222622

quadratic form

$$\left(\begin{array}{cc|cc|cc} 101 & 68 & -142 & -97 & 16 & 10 \\ -142 & -97 & 200 & 138 & -23 & -15 \\ 16 & 10 & -23 & -15 & 2 & 0 \end{array}\right)$$

root list

	roots						norms	
r_1	18	9	13	6	−2	0	30	18
r_2	−7	5	−7	5	0	0	3	−2
r_3	−3	0	−2	0	1	0	3	0
r_4	0	0	0	0	1	0	2	0
r_5	24	15	17	10	2	1	18	12
r_6	5	2	4	1	0	0	1	0

det = $-1137 - 804w$ det norm = -63

222226

quadratic form

$$\left(\begin{array}{cc|cc|cc} 351 & 240 & -45 & -27 & -48 & -33 \\ -45 & -27 & 7 & 2 & 5 & 3 \\ -48 & -33 & 5 & 3 & 3 & 2 \end{array}\right)$$

root list

	roots						norms	
r_1	1	0	5	3	2	−2	6	4
r_2	1	0	6	3	0	0	9	6
r_3	0	0	0	0	−1	1	1	0
r_4	−2	0	−15	−9	3	6	162	114
r_5	0	0	−1	0	2	−1	1	0
r_6	0	1	3	3	−3	3	18	12

det = $-543 - 384w$ det norm = -63

622222

quadratic form

$$\left(\begin{array}{cc|cc|cc} -5 & -4 & 10 & 8 & -5 & -4 \\ 10 & 8 & -2 & -4 & 1 & 2 \\ -5 & -4 & 1 & 2 & 1 & 0 \end{array}\right)$$

root list

	roots						norms	
r_1	0	0	−1	0	−2	0	6	4
r_2	1	1	2	2	3	3	102	72
r_3	0	0	0	0	1	1	3	2
r_4	−2	−1	−7	−5	9	6	51	36
r_5	−3	−2	−13	−9	5	4	150	106
r_6	−5	−4	−28	−20	0	0	51	36

det = $-38625 - 27312w$ det norm = -63

222262

quadratic form

$$\left(\begin{array}{cc|cc|cc} -379 & -268 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 9 & 6 \end{array}\right)$$

root list

	roots						norms	
r_1	1	0	−3	−2	−3	−2	17	12
r_2	3	3	0	0	−27	−19	5490	3882
r_3	0	0	1	1	0	0	17	12
r_4	0	0	0	0	3	2	297	210
r_5	1	0	−7	−5	0	0	198	140
r_6	9	6	−72	−51	−41	−29	3462	2448

det $= -18447 - 13044w$ det norm $= -63$

222622

quadratic form

$$\begin{pmatrix} 9 & 2 & -132 & -89 & 6 & 0 \\ -132 & -89 & 4772 & 3370 & -129 & -87 \\ 6 & 0 & -129 & -87 & 6 & 0 \end{pmatrix}$$

root list

	roots						norms	
r_1	4	1	1	0	11	9	26	18
r_2	9	−6	9	−6	7	5	9	6
r_3	1	−1	0	0	−1	1	1	0
r_4	0	0	0	0	1	1	18	12
r_5	−6	5	−7	5	0	1	2	0
r_6	21	−9	18	−12	12	8	3	0

det $= -38625 - 27312w$ det norm $= -63$

222226

quadratic form

$$\begin{pmatrix} 4198871 & 2969050 & 485644 & 343402 & -10818 & -7650 \\ 485644 & 343402 & 56170 & 39718 & -1251 & -885 \\ -10818 & -7650 & -1251 & -885 & 30 & 18 \end{pmatrix}$$

root list

	roots						norms	
r_1	−8	−2	49	32	13	9	34	24
r_2	3	−3	3	6	15	11	51	36
r_3	3	−1	−8	−4	1	1	3	2
r_4	0	0	0	0	3	2	942	666
r_5	−7	2	22	10	2	1	3	2
r_6	−18	−18	186	135	25	18	102	72

det = −93 − 66w   det norm = −63

622222

quadratic form

$$\begin{pmatrix} -31 & -22 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 2 & 0 \end{pmatrix}$$

root list

	roots						norms	
r_1	2	0	−6	−4	1	0	2	0
r_2	0	0	1	1	−2	−2	18	12
r_3	−9	−7	53	37	0	0	1	0
r_4	−159	−111	877	620	7	5	9	6
r_5	−206	−146	1143	808	13	9	26	18
r_6	−363	−258	2014	1424	28	20	9	6

det = 3 − 6w   det norm = −63

222226

quadratic form

$$\begin{pmatrix} -117 & -84 & -6 & 0 & -9 & -6 \\ -6 & 0 & 14 & -10 & 1 & -1 \\ -9 & -6 & 1 & -1 & 1 & 0 \end{pmatrix}$$

root list

	roots						norms	
r_1	1	0	−17	−12	1	0	2	0
r_2	1	2	−60	−42	0	0	3	0
r_3	0	0	3	2	−2	0	6	2
r_4	1	−1	9	6	0	0	3	0
r_5	0	0	0	0	−1	1	3	−2
r_6	1	1	−42	−30	3	3	6	0

det $= -1137 - 804w$	det norm $= -63$
222262	

quadratic form

$$\begin{pmatrix} 348 & 108 & 96 & 87 & 9 & -3 \\ 96 & 87 & 49 & 32 & 1 & 2 \\ 9 & -3 & 1 & 2 & 1 & 0 \end{pmatrix}$$

root list

	roots						norms	
r_1	0	−1	7	−2	−2	−1	3	2
r_2	−4	−3	15	9	−33	−24	942	666
r_3	4	2	−6	−10	9	7	3	2
r_4	29	21	−90	−60	93	66	51	36
r_5	17	12	−51	−36	59	42	34	24
r_6	28	20	−87	−60	123	87	594	420

det $= -543 - 384w$	det norm $= -63$
226222	

quadratic form

$$\begin{pmatrix} -687 & -486 & 18 & 12 & 51 & 36 \\ 18 & 12 & 2 & -2 & -1 & -1 \\ 51 & 36 & -1 & -1 & 3 & 2 \end{pmatrix}$$

root list

	roots						norms	
r_1	1	−2	−30	−21	1	0	6	2
r_2	−3	1	−24	−18	0	0	3	0
r_3	0	1	23	16	−2	1	2	0
r_4	107	77	3399	2403	−15	−9	18	12
r_5	10	8	335	237	−1	−1	1	0
r_6	38	28	1218	861	−3	−3	9	6

det $= -195 - 138w$ det norm $= -63$

622222

quadratic form

$$\begin{pmatrix} 4063 & 2834 & 1547 & 1078 & 98 & 91 \\ 1547 & 1078 & 589 & 410 & 37 & 35 \\ 98 & 91 & 37 & 35 & 12 & -4 \end{pmatrix}$$

root list

	roots						norms	
r_1	3	1	−7	−3	−2	−2	6	4
r_2	0	−1	0	2	9	6	102	72
r_3	−1	0	2	0	4	3	3	2
r_4	8	6	−28	−21	81	57	942	666
r_5	3	1	−8	−3	3	2	3	2
r_6	14	9	−37	−24	0	0	51	36

det $= -543 - 384w$ det norm $= -63$

222262

quadratic form

$$\begin{pmatrix} 297 & 210 & -9 & -6 & 0 & 0 \\ -9 & -6 & 23 & -16 & 5 & -3 \\ 0 & 0 & 5 & -3 & 6 & 2 \end{pmatrix}$$

root list

	roots						norms	
r_1	−5	3	−30	−21	1	0	3	2
r_2	−2	−1	−123	−87	0	0	51	36
r_3	−5	3	−13	−9	−2	−2	150	106
r_4	1	−1	0	0	0	0	51	36
r_5	3	−2	3	2	1	1	34	24
r_6	−6	−7	−666	−471	21	15	594	420

Table A.4, cont.

det $= 3 - 6w$ det norm $= -63$

222262

quadratic form

$$\left(\begin{array}{cc|cc|cc} 51 & 36 & 0 & 0 & 0 & 0 \\ 0 & 0 & 82 & -58 & 7 & -5 \\ 0 & 0 & 7 & -5 & 2 & 0 \end{array}\right)$$

root list

	roots						norms	
r_1	1	0	31	22	1	0	1	0
r_2	1	2	123	87	0	0	9	6
r_3	0	0	7	5	−2	−2	26	18
r_4	1	−1	0	0	0	0	9	6
r_5	0	0	0	0	1	1	6	4
r_6	12	8	717	507	21	15	102	72

det $= -1137 - 804w$ det norm $= -63$

226222

quadratic form

$$\left(\begin{array}{cc|cc|cc} -379 & -268 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 3 & 0 \\ 0 & 0 & 3 & 0 & 6 & 0 \end{array}\right)$$

root list

	roots						norms	
r_1	1	0	−18	−13	7	5	3	2
r_2	3	0	−51	−36	17	12	51	36
r_3	−6	−4	209	148	−79	−56	34	24
r_4	−102	−72	3606	2550	−1325	−937	594	420
r_5	−13	−10	478	338	−174	−123	17	12
r_6	−84	−60	2940	2079	−1045	−739	5490	3882

det $= -18447 - 13044w$	det norm $= -63$
222226	

quadratic form

$$\left(\begin{array}{cc|cc|cc} 3875 & 2740 & 454 & 320 & 195 & 138 \\ 454 & 320 & 74 & 24 & 21 & 18 \\ 195 & 138 & 21 & 18 & 9 & 6 \end{array}\right)$$

root list

	roots						norms	
r_1	0	0	−3	−2	4	4	34	24
r_2	−3	0	0	0	22	16	51	36
r_3	3	−1	7	5	−23	−17	150	106
r_4	30	18	21	15	−449	−317	51	36
r_5	10	3	4	3	−113	−80	3	2
r_6	69	48	30	21	−1069	−758	102	72

det $= -93 - 66w$	det norm $= -63$
226222	

quadratic form

$$\left(\begin{array}{cc|cc|cc} -31 & -22 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 3 & 0 \\ 0 & 0 & 3 & 0 & 6 & 0 \end{array}\right)$$

root list

	roots						norms	
r_1	4	0	−21	−14	8	5	6	2
r_2	3	3	−36	−24	12	8	3	0
r_3	0	0	1	0	−1	0	2	0
r_4	0	0	0	0	1	1	18	12
r_5	−1	1	−2	−2	1	1	1	0
r_6	3	3	−39	−27	16	11	9	6

Table A.4, cont.

det $= -82 - 59w$ det norm $= -238$

222222

quadratic form

$$\begin{pmatrix} 36946 & 8617 & -466 & 1679 & 258 & 26 \\ -466 & 1679 & 200 & -89 & -7 & 13 \\ 258 & 26 & -7 & 13 & 2 & 0 \end{pmatrix}$$

root list

	roots						norms	
r_1	3	1	−49	−45	−2	−9	2	0
r_2	21	13	−489	−362	−82	−59	118	82
r_3	−5	−3	114	86	10	16	2	−1
r_4	−47	−31	1143	828	165	121	14	8
r_5	−30	−21	759	538	119	77	4	1
r_6	−51	−35	1272	909	194	140	6	1

det $= -164350 - 116213w$ det norm $= -238$

222222

quadratic form

$$\begin{pmatrix} 51838 & 36655 & 9628 & 6808 & 642 & 454 \\ 9628 & 6808 & 1794 & 1268 & 119 & 85 \\ 642 & 454 & 119 & 85 & 10 & 4 \end{pmatrix}$$

root list

	roots						norms	
r_1	0	0	−3	−2	38	27	150	106
r_2	−7	4	0	0	48	34	78	55
r_3	10	−7	0	1	−13	−9	6	4
r_4	57	33	78	55	−4739	−3351	344	243
r_5	−1	5	4	3	−272	−192	2	1
r_6	−10	13	5	2	−352	−249	46	32

det $= -830 - 587w$ det norm $= -238$

222222

quadratic form

$$\left(\begin{array}{cc|cc|cc} 454 & 321 & 0 & 0 & 0 & 0 \\ 0 & 0 & 14 & -10 & 5 & -3 \\ 0 & 0 & 5 & -3 & 6 & 2 \end{array}\right)$$

root list

	roots						norms	
r_1	1	0	34	24	−4	−3	2	1
r_2	7	7	587	415	−78	−55	344	243
r_3	0	0	1	1	0	−1	6	4
r_4	1	−1	0	0	0	0	78	55
r_5	0	0	0	0	3	2	150	106
r_6	2	1	110	78	−9	−7	266	188

det $= 10 - 13w$ det norm $= -238$

222222

quadratic form

$$\left(\begin{array}{cc|cc|cc} 78 & 55 & 0 & 0 & 0 & 0 \\ 0 & 0 & 82 & -58 & 7 & -5 \\ 0 & 0 & 7 & -5 & 2 & 0 \end{array}\right)$$

root list

	roots						norms	
r_1	0	1	55	39	1	0	2	0
r_2	9	5	642	454	0	0	14	9
r_3	0	0	7	5	−2	−2	26	18
r_4	−2	−1	−133	−94	0	0	46	32
r_5	1	1	92	65	2	2	2	1
r_6	39	27	2967	2098	55	39	344	243

det $= -830 - 587w$ det norm $= -238$

222222

quadratic form

$$\left(\begin{array}{cc|cc|cc} 454 & 321 & 28 & 18 & 0 & 0 \\ 28 & 18 & 102 & -70 & 5 & -3 \\ 0 & 0 & 5 & -3 & 6 & 2 \end{array}\right)$$

root list

	roots						norms	
r_1	12	−3	−54	−38	1	0	6	4
r_2	45	33	−642	−454	0	0	78	55
r_3	10	−6	−13	−9	−2	−2	150	106
r_4	−12	−5	133	94	0	0	266	188
r_5	−3	11	−86	−61	2	2	10	7
r_6	203	147	−2834	−2004	55	39	2004	1417

det $= -16378 - 11581w$ det norm $= -238$

222222

quadratic form

$$\left(\begin{array}{cc|cc|cc} 606242 & 428455 & -226452 & -160053 & -1380 & -1080 \\ -226452 & -160053 & 84588 & 59789 & 519 & 401 \\ -1380 & -1080 & 519 & 401 & 38 & -22 \end{array}\right)$$

root list

	roots						norms	
r_1	9	3	25	8	−77	−54	34	24
r_2	13	5	39	15	−573	−405	3974	2810
r_3	−3	0	−8	0	−2	−2	10	7
r_4	−5	2	−15	6	64	45	430	304
r_5	16	11	43	30	−84	−59	92	65
r_6	53	40	144	109	−428	−303	126	89

det $= -482 - 341w$	det norm $= -238$
222222	

quadratic form

$$\begin{pmatrix} 74 & 15 & 36 & -40 & 6 & -16 \\ 36 & -40 & 86 & -54 & 29 & -15 \\ 6 & -16 & 29 & -15 & 10 & -4 \end{pmatrix}$$

root list

	roots						norms	
r_1	0	0	−1	−1	0	1	6	4
r_2	5	−2	−6	−1	6	4	22	15
r_3	−1	0	5	3	−5	−3	16	11
r_4	−14	−8	59	41	−79	−57	74	52
r_5	−3	−5	20	16	−30	−22	2	1
r_6	−66	−45	259	182	−384	−271	682	482

det $= -82 - 59w$	det norm $= -238$
222222	

quadratic form

$$\begin{pmatrix} 11010 & 7785 & -362 & -253 & 104 & 74 \\ -362 & -253 & 56 & -23 & 3 & -7 \\ 104 & 74 & 3 & -7 & 2 & 0 \end{pmatrix}$$

root list

	roots						norms	
r_1	1	0	15	11	0	−3	2	0
r_2	−1	3	52	37	0	0	6	1
r_3	0	0	1	1	5	1	4	1
r_4	−1	0	−15	−11	1	3	14	8
r_5	−1	1	6	4	−4	0	2	−1
r_6	9	4	215	152	−52	−37	118	82

det $= -95458 - 67499w$ det norm $= -238$

222222

quadratic form

$$\begin{pmatrix} 2590 & 1809 & 16502 & 11678 & 1980 & 1398 \\ 16502 & 11678 & 105882 & 74866 & 12697 & 8979 \\ 1980 & 1398 & 12697 & 8979 & 1532 & 1083 \end{pmatrix}$$

root list

	roots						norms	
r_1	−154	−112	32	15	−19	−14	92	65
r_2	−131	−89	16	20	−18	−13	126	89
r_3	181	127	−29	−23	23	16	34	24
r_4	15599	11031	−2673	−1888	2057	1455	3974	2810
r_5	1039	739	−186	−121	138	98	10	7
r_6	2001	1412	−338	−246	269	190	430	304

det $= -482 - 341w$ det norm $= -238$

222222

quadratic form

$$\begin{pmatrix} 3186 & 2137 & -988 & -765 & 60 & 78 \\ -988 & -765 & 372 & 225 & -43 & -10 \\ 60 & 78 & -43 & -10 & 10 & -4 \end{pmatrix}$$

root list

	roots						norms	
r_1	3	1	5	6	0	1	6	4
r_2	1	3	13	5	16	11	682	482
r_3	−5	−2	−10	−10	−2	−2	2	1
r_4	−30	−22	−96	−67	−27	−20	74	52
r_5	−15	−13	−56	−35	−21	−14	16	11
r_6	−23	−18	−80	−53	−38	−26	22	15

det $= -82 - 59w$ det norm $= -238$

222222

quadratic form

$$\left(\begin{array}{cc|cc|cc} 1222 & 861 & -36 & -6 & -60 & -44 \\ -36 & -6 & 94 & -66 & -5 & 7 \\ -60 & -44 & -5 & 7 & -20 & -15 \end{array}\right)$$

root list

	roots						norms	
r_1	2	2	116	82	9	4	4	1
r_2	9	4	356	252	18	20	6	1
r_3	0	1	35	25	2	2	2	0
r_4	−2	0	−37	−26	0	0	118	82
r_5	1	−1	−10	−7	−4	2	2	−1
r_6	1	0	22	15	5	−2	14	8

det $= 10 - 13w$ det norm $= -238$

222222

quadratic form

$$\left(\begin{array}{cc|cc|cc} 334 & 137 & 580 & -397 & 6 & 16 \\ 580 & -397 & 4652 & -3289 & -67 & 48 \\ 6 & 16 & -67 & 48 & 2 & 0 \end{array}\right)$$

root list

	roots						norms	
r_1	−3	1	23	16	−6	4	2	0
r_2	−8	−7	243	172	0	0	60	41
r_3	−3	2	2	1	−4	3	2	−1
r_4	3	−1	−23	−16	7	−4	10	4
r_5	1	−3	49	35	5	−3	6	2
r_6	−5	−7	220	156	0	0	6	−1

det $= -482 - 341w$	det norm $= -238$

222222

quadratic form

$$\left(\begin{array}{cc|cc|cc} 658 & 465 & 394 & 284 & -356 & -243 \\ 394 & 284 & 318 & 116 & -85 & -240 \\ -356 & -243 & -85 & -240 & 400 & -19 \end{array}\right)$$

root list

	roots						norms	
r_1	−16	17	−19	−10	−11	−9	16	11
r_2	13	−3	−28	−21	−26	−18	22	15
r_3	−16	10	−1	2	0	−1	6	4
r_4	2	−3	5	3	4	3	682	482
r_5	25	−16	0	−4	−2	0	2	1
r_6	9	6	−25	−18	−13	−9	74	52

det $= -82 - 59w$	det norm $= -238$

222222

quadratic form

$$\left(\begin{array}{cc|cc|cc} 126 & 89 & 0 & 0 & 0 & 0 \\ 0 & 0 & 6 & -4 & -3 & 1 \\ 0 & 0 & -3 & 1 & 4 & 1 \end{array}\right)$$

root list

	roots						norms	
r_1	1	0	24	17	6	4	2	1
r_2	6	5	341	241	74	52	682	482
r_3	0	0	3	2	0	0	6	4
r_4	1	−1	0	0	0	0	22	15
r_5	0	0	0	0	1	1	16	11
r_6	1	1	52	37	15	11	74	52

Table A.4, cont.

det $= -142 - 101w$ det norm $= -238$

222222

quadratic form

$$\left(\begin{array}{cc|cc|cc} 78 & 55 & 0 & 0 & 0 & 0 \\ 0 & 0 & 14 & -10 & 1 & -1 \\ 0 & 0 & 1 & -1 & 6 & 4 \end{array}\right)$$

root list

	roots						norms	
r_1	−2	−3	102	72	1	0	6	2
r_2	−1	−4	110	78	0	0	6	−1
r_3	2	−1	−9	−7	−3	2	6	−4
r_4	−9	−3	211	149	5	2	16	3
r_5	3	−3	20	14	2	−1	10	−7
r_6	−4	0	64	46	7	−4	14	−8

det $= -4838 - 3421w$ det norm $= -238$

222222

quadratic form

$$\left(\begin{array}{cc|cc|cc} 68590 & 48481 & 25132 & 17776 & -732 & -524 \\ 25132 & 17776 & 9214 & 6514 & -271 & -190 \\ -732 & -524 & -271 & -190 & 10 & 4 \end{array}\right)$$

root list

	roots						norms	
r_1	4	2	−7	−8	4	3	34	24
r_2	5	2	−3	−8	106	75	2004	1417
r_3	−3	−3	12	5	−10	−7	10	7
r_4	−32	−23	83	57	−185	−131	266	188
r_5	−50	−34	117	88	−393	−278	150	106
r_6	−91	−67	232	154	−860	−608	78	55

det = −164350 − 116213w det norm = −238

222222

quadratic form

$$\begin{pmatrix} 8218 & 5811 & -2164 & -1530 & -266 & -188 \\ -2164 & -1530 & 584 & 401 & 75 & 47 \\ -266 & -188 & 75 & 47 & 10 & 4 \end{pmatrix}$$

root list

	roots						norms	
r_1	0	0	−2	−1	11	7	34	24
r_2	−1	3	0	0	38	27	454	321
r_3	−1	0	11	8	−80	−57	874	618
r_4	−23	−16	55	39	−878	−621	1550	1096
r_5	−15	−12	30	21	−562	−397	58	41
r_6	−271	−190	454	321	−9177	−6489	11680	8259

det = −830 − 587w det norm = −238

222222

quadratic form

$$\begin{pmatrix} -4682 & -3311 & -46 & -32 & -1472 & -1041 \\ -46 & -32 & 2 & -2 & -19 & -13 \\ -1472 & -1041 & -19 & -13 & 160 & 113 \end{pmatrix}$$

root list

	roots						norms	
r_1	−1	−1	102	72	−1	2	6	2
r_2	3	−1	−64	−46	0	0	6	−1
r_3	2	−1	−25	−17	−6	4	6	−4
r_4	−3	2	9	7	0	0	16	3
r_5	1	−1	18	12	−4	3	10	−7
r_6	−5	2	92	64	7	−4	14	−8

Table A.4, cont.

det $= -4838 - 3421w$ det norm $= -238$

222222

quadratic form

$$\left(\begin{array}{cc|cc|cc} 8894 & 6289 & -492 & -348 & -266 & -188 \\ -492 & -348 & 30 & 18 & 13 & 12 \\ -266 & -188 & 13 & 12 & 10 & 4 \end{array}\right)$$

root list

	roots						norms	
r_1	0	0	−3	−2	4	3	34	24
r_2	5	2	0	0	116	82	454	321
r_3	−2	−1	17	12	−75	−53	874	618
r_4	−64	−46	55	39	−1997	−1412	1550	1096
r_5	−45	−32	24	17	−1376	−973	58	41
r_6	−761	−539	321	227	−23104	−16337	11680	8259

det $= -680 - 481w$ det norm $= -322$

222282

quadratic form

$$\left(\begin{array}{cc|cc|cc} 8 & 5 & 2 & 3 & -18 & -13 \\ 2 & 3 & 2 & -3 & 3 & 3 \\ -18 & -13 & 3 & 3 & 2 & 1 \end{array}\right)$$

root list

	roots						norms	
r_1	1	−1	−1	−2	2	−2	10	2
r_2	−1	1	0	0	0	0	4	−1
r_3	3	−2	2	−1	3	−2	6	−4
r_4	−4	4	2	3	3	1	18	−1
r_5	0	0	0	0	3	−2	10	−7
r_6	−3	2	−1	0	0	0	6	−4

det $= -796 - 563w$	det norm $= -322$
222228	

quadratic form

$$\left(\begin{array}{cc|cc|cc} 13052 & -5697 & -704 & 1070 & 132 & 61 \\ -704 & 1070 & 182 & -36 & 15 & 14 \\ 132 & 61 & 15 & 14 & 10 & 7 \end{array}\right)$$

root list

	roots						norms	
r_1	−4	−4	37	16	−34	24	2	0
r_2	−56	−40	349	243	0	0	194	136
r_3	−1	0	−1	5	34	−24	2	−1
r_4	1	−1	12	−7	−26	19	8	3
r_5	1	0	1	−5	−31	22	6	−2
r_6	−2	−1	7	10	−41	29	10	−7

det $= -918700 - 649619w$	det norm $= -322$
222822	

quadratic form

$$\left(\begin{array}{cc|cc|cc} 7290736 & 5155295 & 2276810 & 1609966 & -221134 & -156346 \\ 2276810 & 1609966 & 711038 & 502770 & -69047 & -48834 \\ -221134 & -156346 & -69047 & -48834 & 6718 & 4739 \end{array}\right)$$

root list

	roots						norms	
r_1	84	49	−235	−185	−29	−21	58	41
r_2	289	218	−984	−671	−104	−73	1212	857
r_3	−79	−39	196	169	27	20	314	222
r_4	36	19	−94	−78	−15	−11	10	7
r_5	91	38	−201	−190	−31	−23	34	24
r_6	1401	975	−4487	−3201	−573	−406	6562	4640

det $= -136 - 97w$	det norm $= -322$
222282	

quadratic form

$$\left(\begin{array}{cc|cc|cc} 76 & 39 & -172 & -133 & -36 & -2 \\ -172 & -133 & 506 & 349 & 41 & 47 \\ -36 & -2 & 41 & 47 & 34 & -16 \end{array}\right)$$

root list

	roots						norms	
r_1	3	−1	−3	1	9	6	10	6
r_2	−3	−4	−8	−3	38	27	36	25
r_3	−3	1	2	−2	1	1	2	1
r_4	0	0	0	0	7	5	1126	796
r_5	4	0	−2	2	3	2	6	4
r_6	2	3	1	−1	11	8	2	1

det $= -20 - 19w$	det norm $= -322$
222282	

quadratic form

$$\left(\begin{array}{cc|cc|cc} 892 & 599 & -112 & -88 & -22 & -14 \\ -112 & -88 & 18 & 10 & 3 & 2 \\ -22 & -14 & 3 & 2 & 2 & 0 \end{array}\right)$$

root list

	roots						norms	
r_1	0	−2	−15	−4	2	−2	6	−2
r_2	−9	−6	−64	−47	0	0	8	3
r_3	1	2	17	8	−1	1	2	−1
r_4	464	328	3453	2442	11	7	194	136
r_5	46	32	339	241	1	0	2	0
r_6	43	31	325	228	0	0	2	−1

Table A.4, cont.

det $= -20 - 19w$ det norm $= -322$

222228

quadratic form

$$\begin{pmatrix} 1432 & 997 & -60 & -57 & 40 & 38 \\ -60 & -57 & 14 & -5 & -9 & 3 \\ 40 & 38 & -9 & 3 & 6 & -2 \end{pmatrix}$$

root list

	roots						norms	
r_1	1	0	9	7	−1	−1	2	0
r_2	5	3	97	68	−3	−4	194	136
r_3	0	0	1	0	1	0	2	−1
r_4	1	−1	0	0	8	3	8	3
r_5	0	0	0	0	1	0	6	−2
r_6	−1	1	2	4	−5	3	10	−7

det $= -136 - 97w$ det norm $= -322$

228222

quadratic form

$$\begin{pmatrix} -796 & -563 & 0 & 0 & 0 & 0 \\ 0 & 0 & 34 & -24 & 3 & -2 \\ 0 & 0 & 3 & -2 & 2 & 1 \end{pmatrix}$$

root list

	roots						norms	
r_1	1	0	−85	−60	2	0	2	1
r_2	8	6	−1359	−961	0	0	1126	796
r_3	0	0	3	2	−2	0	6	4
r_4	0	0	0	0	1	0	2	1
r_5	1	0	−92	−65	5	3	54	38
r_6	11	8	−1922	−1359	36	25	208	147

Table A.4, cont.

det $= -116 - 83w$ det norm $= -322$

222822

quadratic form

$$\begin{pmatrix} 1802096 & 1274269 & -14232 & -9728 & -746 & -531 \\ -14232 & -9728 & 15126 & -10542 & -151 & 115 \\ -746 & -531 & -151 & 115 & 2 & -1 \end{pmatrix}$$

root list

	roots						norms	
r_1	−5	−1	−413	−292	−9	−7	6	4
r_2	1	−7	−574	−406	0	0	44	31
r_3	10	4	1007	712	38	26	218	154
r_4	10	−9	−175	−124	−8	−6	6	4
r_5	2	−6	−417	−295	−13	−11	2	1
r_6	−62	−71	−10449	−7389	−315	−224	282	199

det $= -136 - 97w$ det norm $= -322$

822222

quadratic form

$$\begin{pmatrix} 568 & 213 & 104 & 16 & 4 & 36 \\ 104 & 16 & 22 & -2 & -3 & 8 \\ 4 & 36 & -3 & 8 & 6 & -2 \end{pmatrix}$$

root list

	roots						norms	
r_1	0	0	−1	−1	2	1	6	4
r_2	−1	1	2	−4	1	1	2	1
r_3	−22	−16	272	194	−241	−170	54	38
r_4	−507	−359	6124	4332	−5350	−3783	208	147
r_5	−55	−39	663	469	−578	−409	10	7
r_6	−910	−644	10911	7717	−9488	−6709	6562	4640

Table A.4, cont.

det $= -3964 - 2803w$ det norm $= -322$

222282

quadratic form

$$\begin{pmatrix} 8 & 5 & 0 & 0 & -88 & -62 \\ 0 & 0 & 2 & -2 & 23 & 17 \\ -88 & -62 & 23 & 17 & 6 & 4 \end{pmatrix}$$

root list

	roots						norms	
r_1	4	−4	−1	−5	14	−10	10	2
r_2	−1	1	0	0	0	0	4	−1
r_3	6	−4	5	−2	17	−12	6	−4
r_4	1	1	39	27	5	−3	18	−1
r_5	7	−5	5	−1	17	−12	10	−7
r_6	−2	1	1	1	0	0	6	−4

det $= -116 - 83w$ det norm $= -322$

222228

quadratic form

$$\begin{pmatrix} 1328 & -843 & 140 & -133 & 2 & -18 \\ 140 & -133 & 30 & -9 & 5 & 1 \\ 2 & -18 & 5 & 1 & 6 & 4 \end{pmatrix}$$

root list

	roots						norms	
r_1	−2	−2	−9	−2	−2	−2	2	−1
r_2	−50	−34	−115	−92	−50	−33	50	33
r_3	−1	−1	−5	−1	2	−3	2	0
r_4	−1	0	2	−4	0	0	8	5
r_5	3	2	7	6	5	1	38	26
r_6	−3	−1	1	−8	−3	−1	2	0

Table A.4, cont.

det $= -20 - 19w$ det norm $= -322$

822222

quadratic form

$$\left(\begin{array}{cc|cc|cc} 12676 & 8951 & 250 & 157 & -116 & -78 \\ 250 & 157 & 30 & -15 & -7 & 2 \\ -116 & -78 & -7 & 2 & 2 & 0 \end{array}\right)$$

root list

	roots						norms	
r_1	1	0	−27	−19	0	−2	2	0
r_2	0	0	1	0	3	−1	2	−1
r_3	−1	0	27	19	1	2	10	6
r_4	11	8	−624	−441	−58	−39	36	25
r_5	2	1	−95	−67	−8	−6	2	1
r_6	45	32	−2485	−1757	−194	−136	1126	796

det $= -23104 - 16337w$ det norm $= -322$

222282

quadratic form

$$\left(\begin{array}{cc|cc|cc} 1164 & 823 & -1604 & -1135 & -50 & -33 \\ -1604 & -1135 & 2226 & 1565 & 47 & 60 \\ -50 & -33 & 47 & 60 & 58 & -39 \end{array}\right)$$

root list

	roots						norms	
r_1	1	−9	−5	−1	−80	−56	218	154
r_2	1	−2	0	0	−48	−34	44	31
r_3	8	−3	−1	2	33	24	6	4
r_4	101	57	33	25	2329	1647	282	199
r_5	15	−4	−1	3	129	92	2	1
r_6	9	0	1	1	150	106	6	4

Table A.4, cont.

det $= -796 - 563w$ det norm $= -322$

222822

quadratic form

$$\begin{pmatrix} -28304 & -1715811 & \vline & -180538 & 10435 & \vline & -12282 & -1863 \\ -180538 & 10435 & \vline & 2354 & -9581 & \vline & -111 & -634 \\ -12282 & -1863 & \vline & -111 & -634 & \vline & -22 & -43 \end{pmatrix}$$

root list

	roots						norms	
r_1	−19	−17	347	204	45	−17	10	7
r_2	−59	−42	876	616	36	25	208	147
r_3	29	25	−512	−310	−59	22	54	38
r_4	14	15	−302	−154	−53	25	2	1
r_5	4	−2	31	−34	47	−35	6	4
r_6	−224	−158	3301	2339	97	68	1126	796

det $= -680 - 481w$ det norm $= -322$

822222

quadratic form

$$\begin{pmatrix} 4140 & 2927 & \vline & -596 & -418 & \vline & 302 & 215 \\ -596 & -418 & \vline & 106 & 46 & \vline & -35 & -37 \\ 302 & 215 & \vline & -35 & -37 & \vline & 26 & 13 \end{pmatrix}$$

root list

	roots						norms	
r_1	0	0	4	3	9	6	10	7
r_2	0	1	7	5	4	3	34	24
r_3	−128	−90	−2765	−1955	−3910	−2765	1270	898
r_4	−223	−157	−4732	−3346	−6630	−4688	256	181
r_5	−39	−27	−812	−574	−1131	−800	34	24
r_6	−320	−225	−6630	−4688	−9177	−6489	1642	1161

det = $-796 - 563w$ det norm = -322

222282

quadratic form

$$\begin{pmatrix} 2014900 & 1422415 & -122958 & -86710 & -1288 & -966 \\ -122958 & -86710 & 7506 & 5284 & 77 & 60 \\ -1288 & -966 & 77 & 60 & 2 & 0 \end{pmatrix}$$

root list

	roots						norms	
r_1	50	33	809	548	35	22	10	6
r_2	47	31	760	515	36	25	36	25
r_3	−39	−26	−632	−431	−27	−18	2	1
r_4	−4882	−3453	−79996	−56575	−3503	−2478	1126	796
r_5	−426	−303	−6988	−4959	−309	−219	6	4
r_6	−363	−257	−5949	−4210	−265	−188	2	1

det = $-116 - 83w$ det norm = -322

222228

quadratic form

$$\begin{pmatrix} -680 & -481 & -34 & -16 & -962 & -680 \\ -34 & -16 & 794 & -562 & 15 & -20 \\ -962 & -680 & 15 & -20 & 62 & 43 \end{pmatrix}$$

root list

	roots						norms	
r_1	−2	−1	205	145	−3	−4	2	1
r_2	4	3	−481	−340	17	8	282	199
r_3	1	2	−229	−162	3	5	6	4
r_4	−23	−16	2728	1929	−62	−44	44	31
r_5	−55	−38	6504	4599	−147	−103	218	154
r_6	−13	−9	1540	1089	−33	−25	6	4

det $= -680 - 481w$ det norm $= -322$

222228

quadratic form

$$\begin{pmatrix} 44 & 31 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 5 & 1 \\ 0 & 0 & 5 & 1 & 10 & 2 \end{pmatrix}$$

root list

	roots						norms	
r_1	1	0	5	3	−3	−2	2	1
r_2	7	6	83	58	−57	−40	282	199
r_3	0	0	1	1	−1	−1	6	4
r_4	−1	0	0	0	0	0	44	31
r_5	0	0	0	0	3	2	218	154
r_6	0	1	6	4	−3	−2	6	4

det $= -134660 - 95219w$ det norm $= -322$

222822

quadratic form

$$\begin{pmatrix} 44 & 31 & 52 & 36 & -26 & -18 \\ 52 & 36 & -26 & -24 & 13 & 12 \\ -26 & -18 & 13 & 12 & 2 & 0 \end{pmatrix}$$

root list

	roots						norms	
r_1	0	0	−1	0	−2	0	34	24
r_2	1	1	0	0	0	0	256	181
r_3	−2	−1	5	4	7	6	1270	898
r_4	−4	−2	3	2	3	2	34	24
r_5	−3	−1	1	1	0	1	10	7
r_6	−35	−23	13	9	0	0	1642	1161

Table A.4, cont.

det $= -181 - 128w$	det norm $= -7$
$(222)^2$	

quadratic form

$$\left(\begin{array}{cc|cc|cc} 17 & 12 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & -1 \\ 0 & 0 & -1 & -1 & 2 & 1 \end{array}\right)$$

root list

	roots						norms	
r_1	2	0	4	3	1	0	2	1
r_2	1	0	2	2	0	0	1	0
r_3	0	0	1	1	0	-1	8	5

det $= -35839 - 25342w$	det norm $= -7$
$(222)^2$	

quadratic form

$$\left(\begin{array}{cc|cc|cc} 99 & 70 & 44 & 31 & 44 & 31 \\ 44 & 31 & 18 & 9 & 35 & 21 \\ 44 & 31 & 35 & 21 & 4 & -1 \end{array}\right)$$

root list

	roots						norms	
r_1	2	3	-5	-4	-2	-1	256	181
r_2	1	-1	0	0	0	0	17	12
r_3	-2	-3	6	4	1	1	58	41

det $= -5 - 4w$	det norm $= -7$
$(222)^2$	

quadratic form

$$\begin{pmatrix} 17 & 12 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & -2 & 1 & -2 \\ 0 & 0 & 1 & -2 & 4 & -1 \end{pmatrix}$$

root list

	roots						norms	
r_1	1	0	10	7	6	4	1	0
r_2	4	−1	31	22	17	12	8	5
r_3	0	0	3	2	1	1	2	1

det = $-4179 - 2955w$ det norm = -9

$(222)^2$

quadratic form

$$\begin{pmatrix} -831 & -594 & 240 & 177 & 417 & 297 \\ 240 & 177 & -66 & -55 & -121 & -88 \\ 417 & 297 & -121 & -88 & -180 & -128 \end{pmatrix}$$

root list

	roots						norms	
r_1	11	7	36	25	1	0	17	12
r_2	53	38	186	132	0	0	297	210
r_3	0	0	1	1	0	−1	58	41

det = $-1 - 3w$ det norm = -17

$(842)^2$

quadratic form

$$\begin{pmatrix} 845 & 597 & -14 & -8 & -7 & -4 \\ -14 & -8 & 6 & -4 & 3 & -2 \\ -7 & -4 & 3 & -2 & 3 & -2 \end{pmatrix}$$

root list

	roots						norms	
r_1	1	0	24	17	3	2	2	1
r_2	0	0	3	2	−6	−4	6	4
r_3	−46	−32	−2328	−1646	3	2	3	2

det $= -9 - 7w$ det norm $= -17$

$(322)^2$

quadratic form

$$\begin{pmatrix} 85 & 51 & -20 & -8 & -10 & -4 \\ -20 & -8 & 6 & 0 & 3 & 0 \\ -10 & -4 & 3 & 0 & 2 & 0 \end{pmatrix}$$

root list

	roots						norms	
r_1	2	0	5	2	1	0	2	0
r_2	0	0	1	0	−2	0	2	0
r_3	−148	−106	−623	−443	0	0	10	4

det $= -89 - 63w$ det norm $= -17$

$(223)^2$

quadratic form

$$\begin{pmatrix} 2461 & 1739 & 88 & 60 & 44 & 30 \\ 88 & 60 & 6 & 0 & 3 & 0 \\ 44 & 30 & 3 & 0 & 2 & 0 \end{pmatrix}$$

root list

	roots						norms	
r_1	2	0	−35	−24	6	4	6	4
r_2	0	2	−45	−33	0	0	74	52
r_3	0	0	1	1	−2	−2	6	4

det $= -519 - 367w$	det norm $= -17$
$(223)^2$	

quadratic form

$$\left(\begin{array}{cc|cc|cc} 14339 & 10139 & -208 & -148 & -104 & -74 \\ -208 & -148 & 6 & 0 & 3 & 0 \\ -104 & -74 & 3 & 0 & 2 & 0 \end{array}\right)$$

root list

	roots						norms	
r_1	2	0	59	42	6	4	6	4
r_2	6	−2	97	70	0	0	74	52
r_3	0	0	1	1	−2	−2	6	4

det $= -55 - 39w$	det norm $= -17$
$(322)^2$	

quadratic form

$$\left(\begin{array}{cc|cc|cc} 259 & 183 & -204 & -144 & -18 & -14 \\ -204 & -144 & 162 & 114 & 13 & 12 \\ -18 & -14 & 13 & 12 & 14 & -8 \end{array}\right)$$

root list

	roots						norms	
r_1	0	0	−1	0	4	3	6	4
r_2	2	0	2	0	3	2	6	4
r_3	−82	−60	0	0	−983	−695	46	32

det $= -89 - 63w$	det norm $= -17$
$(322)^2$	

quadratic form

$$\begin{pmatrix} 21 & 11 & -16 & -20 & -12 & -2 \\ -16 & -20 & 42 & 10 & -1 & 14 \\ -12 & -2 & -1 & 14 & 10 & -4 \end{pmatrix}$$

root list

	roots						norms	
r_1	0	0	−1	0	0	1	6	4
r_2	0	2	2	0	1	0	6	4
r_3	−30	−22	0	0	−49	−35	74	52

det $= -420 - 297w$ det norm $= -18$

$(222)^2$

quadratic form

$$\begin{pmatrix} -474 & -357 & 126 & 114 & 177 & 120 \\ 126 & 114 & -22 & -44 & -55 & -33 \\ 177 & 120 & -55 & -33 & -52 & -38 \end{pmatrix}$$

root list

	roots						norms	
r_1	11	7	36	25	0	1	10	7
r_2	53	38	186	132	0	0	174	123
r_3	0	0	1	1	−2	0	34	24

det $= -147 - 104w$ det norm $= -23$

$(324)^2$

quadratic form

$$\begin{pmatrix} -9 & -12 & 6 & 8 & -3 & -4 \\ 6 & 8 & -2 & -4 & 1 & 2 \\ -3 & -4 & 1 & 2 & 1 & 0 \end{pmatrix}$$

root list

	roots						norms	
r_1	0	0	−1	0	−2	0	6	4
r_2	1	0	2	0	1	0	6	4
r_3	0	0	0	0	1	1	3	2

det $= -5234 - 3701w$ det norm $= -46$

$(222)^2$

quadratic form

$$\left(\begin{array}{cc|cc|cc} 58 & 41 & -32 & -22 & 34 & 24 \\ -32 & -22 & 42 & -6 & -15 & -8 \\ 34 & 24 & -15 & -8 & 6 & 4 \end{array}\right)$$

root list

	roots						norms	
r_1	−10	−2	−13	−9	−2	0	218	154
r_2	1	−1	0	0	0	0	10	7
r_3	6	3	10	7	1	0	34	24

det $= -5234 - 3701w$ det norm $= -46$

$(222)^2$

quadratic form

$$\left(\begin{array}{cc|cc|cc} 2 & -1 & -4 & 2 & -2 & 1 \\ -4 & 2 & -470 & -342 & -235 & -171 \\ -2 & 1 & -235 & -171 & -108 & -79 \end{array}\right)$$

root list

	roots						norms	
r_1	11	7	0	0	1	0	6	4
r_2	19	−3	6	−4	0	0	2	1
r_3	0	0	1	0	−2	0	38	26

Table A.4, cont.

det $= -2926 - 2069w$	det norm $= -46$
$(222)^2$	

quadratic form

$$\begin{pmatrix} 4334 & 3057 & -544 & -388 & 222 & 158 \\ -544 & -388 & 70 & 48 & -29 & -20 \\ 222 & 158 & -29 & -20 & 6 & 4 \end{pmatrix}$$

root list

	roots						norms	
r_1	4	2	25	20	−2	0	34	24
r_2	1	1	11	7	0	0	208	147
r_3	−3	−3	−30	−19	0	1	10	7

det $= -17054 - 12059w$	det norm $= -46$
$(222)^2$	

quadratic form

$$\begin{pmatrix} 46942 & 33193 & 12268 & 8675 & -582 & -412 \\ 12268 & 8675 & 3212 & 2263 & -165 & -99 \\ -582 & -412 & -165 & -99 & 34 & -14 \end{pmatrix}$$

root list

	roots						norms	
r_1	−5	−1	13	8	−2	−2	34	24
r_2	−5	2	4	3	0	0	10	7
r_3	−1	2	−2	−3	11	7	208	147

det $= -26 - 19w$	det norm $= -46$
$(222)^2$	

quadratic form

$$\begin{pmatrix} 58 & 41 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & -2 & 1 & -1 \\ 0 & 0 & 1 & -1 & 6 & -4 \end{pmatrix}$$

root list

	roots						norms	
r_1	−2	−5	48	34	13	9	38	26
r_2	−1	0	6	4	0	0	2	1
r_3	0	1	−7	−5	−3	−2	6	4

det = $-86 - 61w$ det norm = -46

$(222)^2$

quadratic form

$$\begin{pmatrix} 78 & 55 & -30 & -21 & 8 & 6 \\ -30 & -21 & 12 & 7 & -3 & -3 \\ 8 & 6 & -3 & -3 & 2 & 0 \end{pmatrix}$$

root list

	roots						norms	
r_1	1	0	1	1	0	−1	2	0
r_2	−1	1	0	1	0	0	2	−1
r_3	0	0	1	1	3	1	8	3

det = $-898 - 635w$ det norm = -46

$(222)^2$

quadratic form

$$\begin{pmatrix} 10 & 7 & 12 & 8 & -6 & -4 \\ 12 & 8 & 2 & -4 & -1 & 2 \\ -6 & -4 & -1 & 2 & 2 & 0 \end{pmatrix}$$

root list

roots							norms	
r_1	0	0	−1	0	−2	0	6	4
r_2	−1	1	0	0	0	0	2	1
r_3	0	−1	3	1	5	1	38	26

det = $-26 - 19w$ det norm = -46

$(222)^2$

quadratic form

$$\begin{pmatrix} 2 & 1 & -2 & 2 & 0 & 0 \\ -2 & 2 & 26 & -18 & -5 & -1 \\ 0 & 0 & -5 & -1 & 38 & 26 \end{pmatrix}$$

root list

roots							norms	
r_1	8	−3	−4	−4	3	−3	6	−4
r_2	−11	10	−6	−2	−4	2	10	−7
r_3	−14	11	−5	−1	−8	5	22	−14

det = $4 - 7w$ det norm = -82

$(228)^2$

quadratic form

$$\begin{pmatrix} -16 & -13 & 0 & 0 & 0 & 0 \\ 0 & 0 & 34 & -24 & 3 & -2 \\ 0 & 0 & 3 & -2 & 2 & 1 \end{pmatrix}$$

root list

roots							norms	
r_1	1	0	−13	−9	2	−1	2	−1
r_2	3	2	−71	−50	0	0	10	3
r_3	0	0	1	0	−6	4	6	−4

Table A.4, cont.

det $= -1376 - 973w$	det norm $= -82$
$(822)^2$	

quadratic form

$$\left(\begin{array}{cc|cc|cc} 2256 & 1595 & -1184 & -836 & -196 & -138 \\ -1184 & -836 & 626 & 436 & 105 & 71 \\ -196 & -138 & 105 & 71 & 18 & 11 \end{array}\right)$$

root list

	roots						norms	
r_1	0	0	−1	−1	6	5	6	4
r_2	−2	2	0	1	1	0	2	1
r_3	0	0	0	0	1	1	98	69

det $= -16 - 13w$	det norm $= -82$
$(228)^2$	

quadratic form

$$\left(\begin{array}{cc|cc|cc} -100 & -71 & 0 & 0 & 0 & 0 \\ 0 & 0 & 34 & -24 & 3 & -2 \\ 0 & 0 & 3 & -2 & 2 & 1 \end{array}\right)$$

root list

	roots						norms	
r_1	2	0	−61	−43	2	0	2	0
r_2	3	2	−171	−121	0	0	42	29
r_3	−3	−2	177	125	−4	−1	2	1

det $= -115636 - 81767w$	det norm $= -82$
$(822)^2$	

quadratic form

$$\left(\begin{array}{cc|cc|cc} 8360 & 5361 & -8566 & -5554 & -21576 & -15211 \\ -8566 & -5554 & 8782 & 5750 & 22221 & 15671 \\ -21576 & -15211 & 22221 & 15671 & 58278 & 41205 \end{array}\right)$$

root list

	roots						norms	
r_1	−40	−29	−40	−29	1	0	58	41
r_2	−10	−7	−7	−5	−2	0	198	140
r_3	25797	18242	24971	17658	0	0	8218	5811

det $= -40 - 29w$ det norm $= -82$

$(228)^2$

quadratic form

$$\left(\begin{array}{cc|cc|cc} 6328 & 4473 & 84 & 65 & -80 & -58 \\ 84 & 65 & 18 & -11 & -5 & 2 \\ -80 & -58 & -5 & 2 & 2 & 0 \end{array}\right)$$

root list

	roots						norms	
r_1	1	0	−37	−26	−2	0	6	4
r_2	2	2	−167	−118	0	0	98	69
r_3	0	0	1	1	0	1	2	1

det $= -40 - 29w$ det norm $= -82$

$(228)^2$

quadratic form

$$\left(\begin{array}{cc|cc|cc} -236 & -167 & -14 & -4 & -334 & -236 \\ -14 & -4 & 998 & -706 & 23 & -22 \\ -334 & -236 & 23 & -22 & 34 & 23 \end{array}\right)$$

root list

	roots						norms	
r_1	-2	-1	123	87	-1	-2	2	1
r_2	2	2	-167	-118	7	2	98	69
r_3	7	4	-457	-323	11	2	6	4

det $= -666 - 471w$ det norm $= -126$

$(622)^2$

quadratic form

$$\begin{pmatrix} 290 & 205 & -124 & -88 & -22 & -16 \\ -124 & -88 & 58 & 38 & 13 & 5 \\ -22 & -16 & 13 & 5 & 6 & -2 \end{pmatrix}$$

root list

	roots						norms	
r_1	0	0	-1	0	2	1	6	4
r_2	2	1	2	2	9	6	102	72
r_3	-12	-10	0	0	-133	-94	54	38

det $= -768618 - 543495w$ det norm $= -126$

$(222)^2$

quadratic form

$$\begin{pmatrix} 160850 & 113653 & -56084 & -39704 & -468 & -320 \\ -56084 & -39704 & 19606 & 13838 & 157 & 117 \\ -468 & -320 & 157 & 117 & 2 & 0 \end{pmatrix}$$

root list

	roots						norms	
r_1	4	2	9	8	-104	-73	198	140
r_2	5	-3	-7	7	-92	-65	536	379
r_3	-3	-2	-8	-6	-12	-9	1014	717

det $= -114 - 81w$	det norm $= -126$
$(622)^2$	

quadratic form

$$\left(\begin{array}{cc|cc|cc} -38 & -27 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 2 & 0 \end{array}\right)$$

root list

	roots						norms	
r_1	0	1	−5	−3	1	0	2	0
r_2	0	0	1	1	−2	−2	18	12
r_3	−24	−19	157	111	0	0	10	6

det $= -18 - 15w$	det norm $= -126$
$(222)^2$	

quadratic form

$$\left(\begin{array}{cc|cc|cc} 642 & 453 & -42 & -33 & 24 & 18 \\ -42 & -33 & 16 & -7 & -5 & 1 \\ 24 & 18 & -5 & 1 & 2 & 0 \end{array}\right)$$

root list

	roots						norms	
r_1	1	0	7	5	0	−1	2	0
r_2	1	2	30	21	0	0	6	3
r_3	0	0	1	1	1	1	4	1

det $= -114 - 81w$	det norm $= -126$
$(222)^2$	

quadratic form

$$\begin{pmatrix} 114 & 3 & 36 & -24 & 6 & -6 \\ 36 & -24 & 22 & -16 & 5 & -3 \\ 6 & -6 & 5 & -3 & 2 & 0 \end{pmatrix}$$

root list

roots							norms	
r_1	0	0	−1	−1	−2	0	6	4
r_2	1	1	−12	−9	0	0	30	21
r_3	1	0	−4	−2	1	2	16	11

det $= -131874 - 93249w$ det norm $= -126$

$(222)^2$

quadratic form

$$\begin{pmatrix} -15774 & -11163 & -2304 & -1446 & 3444 & 2433 \\ -2304 & -1446 & 2314 & -2052 & 433 & 352 \\ 3444 & 2433 & 433 & 352 & -736 & -521 \end{pmatrix}$$

root list

roots							norms	
r_1	0	0	−31	−22	−15	−17	536	379
r_2	−1	−1	0	0	−6	−3	1014	717
r_3	−1	1	24	17	19	9	198	140

det $= -114 - 81w$ det norm $= -126$

$(222)^2$

quadratic form

$$\begin{pmatrix} 174 & 123 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 3 & -1 \\ 0 & 0 & 3 & -1 & 8 & -5 \end{pmatrix}$$

root list

roots							norms	
r_1	1	2	−65	−46	68	48	16	11
r_2	1	2	−72	−51	72	51	30	21
r_3	−1	0	16	11	−17	−12	6	4

Table A.5: Septagons

det $= -22387 - 15830w$	det norm $= -31$
2228222	

quadratic form

$$\left(\begin{array}{cc|cc|cc} 17037 & 12038 & 2307 & 1633 & -152 & -112 \\ 2307 & 1633 & 313 & 221 & -21 & -14 \\ -152 & -112 & -21 & -14 & 2 & -1 \end{array}\right)$$

root list

	roots						norms	
r_1	3	1	−13	−14	−2	−1	58	41
r_2	1	1	−9	−5	0	0	932	659
r_3	−1	0	2	4	2	1	17	12
r_4	0	0	0	0	7	5	58	41
r_5	5	3	−33	−26	7	5	198	140
r_6	8	6	−63	−43	3	2	99	70
r_7	56	40	−423	−297	0	0	1591	1125

det $= -113 - 80w$	det norm $= -31$
2222282	

quadratic form

$$\left(\begin{array}{cc|cc|cc} 2805 & 1982 & 40 & 36 & -75 & -52 \\ 40 & 36 & 34 & -23 & 3 & -4 \\ -75 & -52 & 3 & -4 & -13 & -10 \end{array}\right)$$

root list

	roots						norms	
r_1	2	1	−102	−72	3	0	1	0
r_2	11	7	−621	−438	16	2	9	5
r_3	−6	−1	221	156	−8	1	2	−1
r_4	−154	−111	9245	6538	−134	−102	8	1
r_5	−27	−22	1728	1221	−28	−17	3	−2
r_6	−65	−47	3907	2763	−57	−43	2	−1
r_7	−23	−15	1314	928	−27	−9	2	0

det = $-22387 - 15830w$ det norm = -31

2222282

quadratic form

$$\left(\begin{array}{cc|cc|cc} -239 & -169 & 0 & 0 & 24 & 17 \\ 0 & 0 & 10 & 7 & 1 & 1 \\ 24 & 17 & 1 & 1 & 3 & -2 \end{array}\right)$$

root list

	roots						norms	
r_1	0	0	0	0	−7	−5	17	12
r_2	−3	2	4	3	−34	−24	273	193
r_3	0	0	1	0	0	0	10	7
r_4	−2	2	−1	0	4	3	160	113
r_5	3	−2	−1	0	0	0	3	2
r_6	−1	1	−2	−1	−3	−2	10	7
r_7	3	−2	−1	−1	−7	−5	34	24

det = $-287 - 203w$ det norm = -49

2222222

quadratic form

$$\left(\begin{array}{cc|cc|cc} 437 & 309 & 0 & 0 & 0 & 0 \\ 0 & 0 & 14 & -10 & 1 & -1 \\ 0 & 0 & 1 & -1 & 2 & 1 \end{array}\right)$$

root list

	roots						norms	
r_1	1	0	34	24	5	4	3	2
r_2	2	4	256	181	31	22	406	287
r_3	0	0	0	0	−1	−1	10	7
r_4	1	−1	0	0	0	0	75	53
r_5	0	0	17	12	10	7	314	222
r_6	2	0	75	53	18	13	150	106
r_7	1	2	133	94	24	17	27	19

$\det = -7 - 7w$ $\quad$ det norm $= -49$

2222222

quadratic form

$$\left(\begin{array}{cc|cc|cc} 13 & 9 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & -2 & 1 & -1 \\ 0 & 0 & 1 & -1 & 2 & -1 \end{array}\right)$$

root list

	roots						norms	
r_1	2	0	5	4	1	0	2	−1
r_2	5	3	26	18	0	0	3	1
r_3	0	0	1	1	−2	−2	10	6
r_4	−2	0	−5	−4	0	0	6	2
r_5	1	2	9	7	4	2	3	−1
r_6	−1	2	5	3	1	1	3	−2
r_7	12	10	67	48	13	9	14	7

$\det = -1673 - 1183w$ $\quad$ det norm $= -49$

2222222

quadratic form

$$\begin{pmatrix} 15021 & 10621 & 7914 & 5597 & -182 & -126 \\ 7914 & 5597 & 4172 & 2948 & -91 & -70 \\ -182 & -126 & -91 & -70 & 14 & -7 \end{pmatrix}$$

root list

	roots						norms	
r_1	0	−7	11	6	12	8	26	18
r_2	5	−4	3	−1	6	4	5	3
r_3	5	−3	0	−1	1	0	1	0
r_4	0	0	0	0	3	2	70	49
r_5	0	−2	4	1	1	1	2	1
r_6	−25	−20	52	36	34	24	13	9
r_7	−20	−19	47	31	34	24	54	38

det $= -331247 - 234227w$ det norm $= -49$

2222222

quadratic form

$$\begin{pmatrix} -1393 & -985 & 0 & 0 & 58 & 41 \\ 0 & 0 & 26 & 18 & 3 & 1 \\ 58 & 41 & 3 & 1 & 3 & -2 \end{pmatrix}$$

root list

	roots						norms	
r_1	0	0	0	0	−17	−12	99	70
r_2	−3	2	4	3	−48	−34	915	647
r_3	0	0	7	5	0	0	5094	3602
r_4	0	2	−3	−2	34	24	10666	7542
r_5	3	1	−17	−12	0	0	2547	1801
r_6	−1	1	−3	−2	−7	−5	338	239
r_7	−1	2	−17	−12	−181	−128	13790	9751

det $= -56833 - 40187w$ det norm $= -49$

2222222

quadratic form

$$\begin{pmatrix} -239 & -169 & 0 & 0 & -65 & -46 \\ 0 & 0 & 14 & 7 & 7 & 7 \\ -65 & -46 & 7 & 7 & -5 & -6 \end{pmatrix}$$

root list

	roots						norms	
r_1	−2	3	3	2	−11	−8	157	111
r_2	9	−6	1	1	−2	−1	17	12
r_3	0	0	7	5	0	0	2366	1673
r_4	−8	5	−1	−1	2	1	58	41
r_5	−5	−4	−24	−17	0	0	437	309
r_6	−7	0	−21	−15	−11	−8	1830	1294
r_7	5	−3	−3	−2	−13	−9	874	618

det $= -49 - 35w$ det norm $= -49$

2222222

quadratic form

$$\begin{pmatrix} 143 & 101 & -132 & -94 & 10 & 5 \\ -132 & -94 & 130 & 86 & 5 & -15 \\ 10 & 5 & 5 & -15 & 42 & -29 \end{pmatrix}$$

root list

	roots						norms	
r_1	0	0	−1	1	5	3	2	1
r_2	3	1	4	3	34	24	13	9
r_3	2	2	3	3	16	11	54	38
r_4	0	0	1	0	4	3	26	18
r_5	−3	0	−1	0	16	11	5	3
r_6	−1	0	1	−1	5	4	1	0
r_7	−6	−2	−1	−1	65	46	70	49

det $= -9751 - 6895w$ det norm $= -49$

2222222

quadratic form

$$\left(\begin{array}{cc|cc|cc} 344 & 243 & 28 & 21 & -49 & -35 \\ 28 & 21 & 9 & 0 & -4 & -1 \\ -49 & -35 & -4 & -1 & 5 & 3 \end{array}\right)$$

root list

	roots						norms	
r_1	1	0	−2	−1	0	2	10	7
r_2	1	2	0	0	8	5	406	287
r_3	0	0	1	0	2	−1	3	2
r_4	0	0	0	0	1	1	27	19
r_5	1	0	−5	−4	3	1	150	106
r_6	5	3	−27	−19	13	10	314	222
r_7	7	4	−31	−22	18	13	75	53

det $= -287 - 203w$ det norm $= -49$

2222222

quadratic form

$$\left(\begin{array}{cc|cc|cc} 2225 & 1573 & -504 & -357 & -287 & -203 \\ -504 & -357 & 116 & 80 & 65 & 46 \\ -287 & -203 & 65 & 46 & 37 & 26 \end{array}\right)$$

root list

	roots						norms	
r_1	1	0	1	1	1	2	2	1
r_2	3	1	−5	−3	27	19	70	49
r_3	0	0	−1	−1	3	1	1	0
r_4	1	−1	−5	−3	6	5	5	3
r_5	0	0	−1	−1	2	1	26	18
r_6	4	2	11	8	6	5	54	38
r_7	5	4	16	11	14	10	13	9

det $= -386 - 273w$ det norm $= -62$

8222222

quadratic form

$$\left(\begin{array}{cc|cc|cc} -1726 & -1231 & 124 & 62 & 310 & 217 \\ 124 & 62 & 38 & -36 & -15 & -16 \\ 310 & 217 & -15 & -16 & -46 & -33 \end{array}\right)$$

root list

	roots						norms	
r_1	4	2	56	40	1	0	10	7
r_2	0	0	3	2	−2	0	34	24
r_3	−111	−79	−1884	−1332	0	0	10	7
r_4	−2188	−1547	−37045	−26195	10	9	546	386
r_5	−526	−372	−8911	−6301	4	3	34	24
r_6	−6697	−4735	−113479	−80242	66	47	932	659
r_7	−621	−439	−10526	−7443	8	5	58	41

det $= -1137 - 804w$ det norm $= -63$

2222282

quadratic form

$$\begin{pmatrix} 17 & 12 & -10 & -7 & 0 & 0 \\ -10 & -7 & 18 & -4 & -3 & -6 \\ 0 & 0 & -3 & -6 & 48 & 33 \end{pmatrix}$$

root list

	roots						norms	
r_1	3	3	5	4	2	−1	1	0
r_2	9	4	11	8	1	0	5	3
r_3	0	3	3	3	3	−2	6	3
r_4	−3	2	0	0	0	0	1	0
r_5	0	0	0	0	1	0	48	33
r_6	4	−1	2	1	−1	1	2	1
r_7	10	8	15	11	2	0	6	4

det $= -3165 - 2238w$ det norm $= -63$

2222222

quadratic form

$$\begin{pmatrix} 303 & 214 & 7 & -1 & -217 & -154 \\ 7 & -1 & 99 & -70 & 10 & -6 \\ -217 & -154 & 10 & -6 & -59 & -43 \end{pmatrix}$$

root list

	roots						norms	
r_1	0	0	31	22	1	0	44	31
r_2	−7	5	2	1	0	0	3	2
r_3	5	−3	−119	−84	−2	0	450	318
r_4	0	0	−17	−12	0	0	3	2
r_5	6	−5	−282	−199	3	2	225	159
r_6	−3	2	−14	−10	1	0	10	7
r_7	1	−1	14	10	2	1	51	36

Table A.5, cont.

det $= -3165 - 2238w$ det norm $= -63$

2222222

quadratic form

$$\left(\begin{array}{cc|cc|cc} 1663 & -1134 & -2848 & 282 & 210 & 220 \\ -2848 & 282 & 52914 & 35134 & -10895 & -7775 \\ 210 & 220 & -10895 & -7775 & 2338 & 1651 \end{array}\right)$$

root list

	roots						norms	
r_1	−36	−26	−5	−4	−19	−16	10	7
r_2	−263	−178	−41	−21	−137	−96	225	159
r_3	−5	−3	−1	0	2	−4	3	2
r_4	−20	−14	−3	−2	−12	−8	450	318
r_5	−3	−10	4	−5	−4	−6	3	2
r_6	−48	−31	−10	−4	−29	−24	44	31
r_7	−131	−90	−22	−13	−82	−55	51	36

det $= -15 - 12w$ det norm $= -63$

2222222

quadratic form

$$\left(\begin{array}{cc|cc|cc} 294 & 63 & -48 & 54 & 9 & -18 \\ -48 & 54 & 39 & -26 & -11 & 7 \\ 9 & -18 & -11 & 7 & 3 & -2 \end{array}\right)$$

root list

	roots						norms	
r_1	1	0	2	1	6	7	2	1
r_2	3	2	39	27	78	54	39	27
r_3	0	−1	−4	−4	−15	−9	1	0
r_4	−56	−39	−543	−384	−1263	−897	78	54
r_5	−8	−6	−85	−60	−194	−135	1	0
r_6	−17	−13	−191	−136	−425	−298	8	5
r_7	−16	−11	−183	−129	−390	−276	9	6

det $= -93 - 66w$ det norm $= -63$

2222222

quadratic form

$$\left(\begin{array}{cc|cc|cc} 9 & 6 & -6 & -3 & -9 & -6 \\ -6 & -3 & 4 & -3 & 1 & -1 \\ -9 & -6 & 1 & -1 & 1 & 0 \end{array}\right)$$

root list

	roots						norms	
r_1	1	0	−3	−2	3	1	8	5
r_2	1	0	0	0	0	0	9	6
r_3	0	0	1	1	0	−1	2	1
r_4	1	−1	9	6	0	0	39	27
r_5	0	0	0	0	1	0	1	0
r_6	1	1	−12	−9	15	12	78	54
r_7	−1	1	−2	−1	2	1	1	0

det $= -33 - 24w$ det norm $= -63$

2282222

quadratic form

$$\left(\begin{array}{cc|cc|cc} -195 & -138 & 0 & 0 & 0 & 0 \\ 0 & 0 & 34 & -24 & 3 & -2 \\ 0 & 0 & 3 & -2 & 2 & 1 \end{array}\right)$$

root list

	roots						norms	
r_1	1	0	−43	−30	3	−1	3	−1
r_2	3	−2	−7	−5	0	0	3	−2
r_3	0	0	1	0	−6	4	6	−4
r_4	0	0	0	0	3	−2	10	−7
r_5	−6	5	−48	−33	9	−3	24	−15
r_6	3	−2	−8	−5	2	−1	17	−12
r_7	−8	6	−21	−15	−3	3	30	−21

det $= -38625 - 27312w$ det norm $= -63$

2222282

quadratic form

$$\left(\begin{array}{cc|cc|cc} -211 & -163 & 214 & 162 & 376 & 269 \\ 214 & 162 & -216 & -161 & -377 & -269 \\ 376 & 269 & -377 & -269 & -559 & -396 \end{array}\right)$$

root list

	roots						norms	
r_1	18	12	18	12	1	0	17	12
r_2	27	20	28	21	0	0	157	111
r_3	0	0	1	1	0	−1	174	123
r_4	−3	−2	−3	−2	0	0	17	12
r_5	3	0	−2	−4	5	3	1608	1137
r_6	11	7	10	6	1	1	58	41
r_7	59	43	57	42	3	2	198	140

det $= -93 - 66w$ det norm $= -63$

2222222

Table A.5, cont.

quadratic form

$$\left(\begin{array}{cc|cc|cc} 9 & 6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 2 & -5 & -4 \\ 0 & 0 & -5 & -4 & 8 & 5 \end{array}\right)$$

root list

	roots						norms	
r_1	1	0	3	2	2	1	1	0
r_2	5	4	39	27	21	15	39	27
r_3	0	0	1	1	1	0	2	1
r_4	−1	0	0	0	0	0	9	6
r_5	0	0	0	0	1	0	8	5
r_6	1	0	2	2	2	1	1	0
r_7	8	5	39	27	24	18	78	54

det $= -3165 - 2238w$ det norm $= -63$

2222222

quadratic form

$$\left(\begin{array}{cc|cc|cc} -420 & -297 & 0 & 0 & 21 & 15 \\ 0 & 0 & 3 & 2 & 1 & 1 \\ 21 & 15 & 1 & 1 & 3 & -2 \end{array}\right)$$

root list

	roots						norms	
r_1	0	0	0	0	−7	−5	17	12
r_2	−2	0	21	15	−51	−36	2622	1854
r_3	0	0	1	1	0	0	17	12
r_4	1	0	−4	−3	10	7	256	181
r_5	2	1	−21	−15	0	0	297	210
r_6	1	0	−7	−5	−7	−5	58	41
r_7	3	2	−42	−30	−123	−87	1311	927

Table A.5, cont.

det $= -543 - 384w$	det norm $= -63$
2222222	

quadratic form

$$\left(\begin{array}{cc|cc|cc} 9 & 6 & 0 & 0 & -42 & -30 \\ 0 & 0 & 4 & -3 & 7 & 4 \\ -42 & -30 & 7 & 4 & 2 & 1 \end{array}\right)$$

root list

	roots						norms	
r_1	−4	2	−7	−4	4	−3	8	5
r_2	1	0	0	0	0	0	9	6
r_3	−2	2	6	5	3	−2	2	1
r_4	1	2	144	102	3	0	39	27
r_5	−3	2	9	7	3	−2	1	0
r_6	−2	−5	99	69	−3	3	78	54
r_7	−1	0	4	3	0	0	1	0

det $= -3165 - 2238w$	det norm $= -63$
2222222	

quadratic form

$$\left(\begin{array}{cc|cc|cc} -831 & -594 & 240 & 177 & 417 & 297 \\ 240 & 177 & -66 & -55 & -121 & -88 \\ 417 & 297 & -121 & -88 & -187 & -133 \end{array}\right)$$

root list

	roots						norms	
r_1	11	7	36	25	1	0	10	7
r_2	23	15	78	54	0	0	51	36
r_3	0	0	1	1	0	−1	44	31
r_4	−1	−1	−4	−3	0	0	3	2
r_5	−1	2	−3	0	3	3	450	318
r_6	2	3	9	8	1	0	3	2
r_7	98	70	330	234	9	6	225	159

det = $-543 - 384w$ det norm = −63

2222222

quadratic form

$$\begin{pmatrix} -585 & -419 & 94 & 78 & 172 & 125 \\ 94 & 78 & 0 & -25 & -25 & -25 \\ 172 & 125 & -25 & -25 & -45 & -34 \end{pmatrix}$$

root list

	roots						norms	
r_1	4	2	20	14	1	0	3	2
r_2	37	25	220	155	0	0	225	159
r_3	0	0	1	1	0	−1	10	7
r_4	−3	−2	−17	−12	0	0	51	36
r_5	1	0	0	0	3	1	44	31
r_6	3	3	19	14	2	1	3	2
r_7	53	37	295	208	15	12	450	318

det = $-5712 - 4039w$ det norm = −98

2222222

quadratic form

$$\begin{pmatrix} 2632 & 1861 & 2348 & 1658 & -578 & -400 \\ 2348 & 1658 & 2130 & 1456 & -647 & -265 \\ -578 & -400 & -647 & -265 & 652 & -285 \end{pmatrix}$$

root list

	roots						norms	
r_1	−44	24	18	8	39	27	92	65
r_2	−7	−9	22	16	48	34	44	31
r_3	72	−52	−5	5	0	1	6	4
r_4	10	−8	1	2	4	3	238	168
r_5	−39	28	3	−2	2	1	2	1
r_6	73	−50	−4	6	14	11	16	11
r_7	−10	7	2	1	7	5	8	5

det = $-168 - 119w$ det norm = -98

2222222

quadratic form

$$\left(\begin{array}{cc|cc|cc} -61504 & -52265 & 16076 & 16806 & 1842 & 880 \\ 16076 & 16806 & -3478 & -5830 & -639 & -190 \\ 1842 & 880 & -639 & -190 & -8 & -26 \end{array}\right)$$

root list

	roots						norms	
r_1	−63	−36	−201	−132	37	−21	4	1
r_2	−53	−42	−186	−139	6	2	4	−1
r_3	40	15	115	66	−58	38	6	−4
r_4	2838	2009	9661	6837	−232	−138	14	0
r_5	343	244	1171	828	−4	−34	10	−7
r_6	1265	880	4282	3012	−164	−19	8	−5
r_7	173	113	577	393	−13	−9	16	−11

det = $-168 - 119w$ det norm = -98

2222222

quadratic form

$$\begin{pmatrix} 256 & 181 & 0 & 0 & 0 & 0 \\ 0 & 0 & 6 & -4 & -3 & 1 \\ 0 & 0 & -3 & 1 & 4 & 1 \end{pmatrix}$$

root list

	roots						norms	
r_1	1	0	35	25	8	6	2	1
r_2	2	4	287	203	62	44	238	168
r_3	0	0	3	2	0	0	6	4
r_4	1	−1	0	0	0	0	44	31
r_5	0	0	0	0	3	2	92	65
r_6	0	1	44	31	13	9	44	31
r_7	5	3	314	222	78	55	92	65

det $= -168 - 119w$ det norm $= -98$

2222222

quadratic form

$$\begin{pmatrix} 1304 & 921 & 28 & 21 & 98 & 70 \\ 28 & 21 & 2 & -1 & 3 & 1 \\ 98 & 70 & 3 & 1 & 8 & 5 \end{pmatrix}$$

root list

	roots						norms	
r_1	1	0	3	3	−6	−7	2	0
r_2	3	1	21	14	−38	−27	42	28
r_3	0	0	1	0	−2	1	2	−1
r_4	1	−1	0	0	4	1	4	1
r_5	0	0	0	0	−1	1	4	−1
r_6	2	0	6	5	−13	−12	4	1
r_7	3	1	16	10	−34	−24	4	−1

Table A.5, cont.

det $= -5712 - 4039w$	det norm $= -98$
2222222	

quadratic form

$$\begin{pmatrix} 372 & 263 & -84 & -60 & 26 & 18 \\ -84 & -60 & 30 & 12 & -3 & -8 \\ 26 & 18 & -3 & -8 & 4 & -1 \end{pmatrix}$$

root list

	roots						norms	
r_1	0	0	−1	−1	−12	−8	34	24
r_2	3	1	0	0	−68	−48	256	181
r_3	−2	−1	7	5	119	84	536	379
r_4	−16	−10	13	9	581	411	256	181
r_5	−95	−67	58	41	3438	2431	536	379
r_6	−25	−17	13	9	868	614	58	41
r_7	−186	−132	75	53	6378	4510	8078	5712

det $= -28 - 21w$	det norm $= -98$
2222222	

quadratic form

$$\begin{pmatrix} 44 & 31 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 3 & -1 \\ 0 & 0 & 3 & -1 & 8 & -5 \end{pmatrix}$$

root list

	roots						norms	
r_1	−2	0	−17	−13	18	13	2	0
r_2	−5	−3	−88	−62	88	62	8	5
r_3	0	0	−5	−3	3	2	16	11
r_4	0	1	13	9	−13	−9	8	5
r_5	−5	−3	−70	−49	78	55	16	11
r_6	−3	−1	−35	−25	38	27	2	1
r_7	−32	−22	−525	−371	556	393	238	168

det $= -5712 - 4039w$ det norm $= -98$

2222222

quadratic form

$$\begin{pmatrix} 228 & 161 & -120 & -80 & -32 & -22 \\ -120 & -80 & 140 & -10 & 27 & 5 \\ -32 & -22 & 27 & 5 & 4 & 1 \end{pmatrix}$$

root list

	roots						norms	
r_1	0	0	−1	−1	1	3	10	7
r_2	1	2	0	0	8	6	92	65
r_3	−2	0	6	4	−19	−11	44	31
r_4	−20	−12	38	27	−165	−117	92	65
r_5	−37	−28	68	48	−318	−225	44	31
r_6	−1	−4	5	3	−27	−16	6	4
r_7	−15	−9	11	8	−85	−61	238	168

det $= -114 - 81w$ det norm $= -126$

2282222

quadratic form

$$\begin{pmatrix} -666 & -471 & 0 & 0 & 0 & 0 \\ 0 & 0 & 34 & -24 & 3 & -2 \\ 0 & 0 & 3 & -2 & 2 & 1 \end{pmatrix}$$

root list

	roots						norms	
r_1	1	0	−78	−55	2	0	2	1
r_2	2	3	−471	−333	0	0	162	114
r_3	0	0	3	2	−2	0	6	4
r_4	0	0	0	0	1	0	2	1
r_5	−1	1	−34	−24	2	1	2	1
r_6	1	1	−195	−138	6	5	16	11
r_7	0	2	−225	−159	6	3	18	12

det $= -666 - 471w$ det norm $= -126$

2282222

quadratic form

$$\begin{pmatrix} -666 & -471 & 0 & 0 & 0 & 0 \\ 0 & 0 & 6 & 4 & -1 & -3 \\ 0 & 0 & -1 & -3 & 18 & -11 \end{pmatrix}$$

root list

	roots						norms	
r_1	1	1	−61	−45	−146	−103	16	11
r_2	−1	1	−10	−7	−24	−17	2	1
r_3	0	0	3	−1	1	1	2	1
r_4	0	0	−1	0	0	0	6	4
r_5	2	3	−177	−123	−390	−276	162	114
r_6	1	0	−28	−19	−62	−44	2	1
r_7	0	2	−75	−54	−174	−123	18	12

Table A.6: Octagons

det = $-32-23w$	det norm = -34
22222222	

quadratic form

$$\left(\begin{array}{cc|cc|cc} 46 & 32 & 0 & 0 & -9 & -7 \\ 0 & 0 & 0 & -1 & 2 & 1 \\ -9 & -7 & 2 & 1 & -1 & -2 \end{array}\right)$$

root list

	roots						norms	
r_1	−1	0	−6	−4	−3	−3	1	0
r_2	−6	−3	−55	−39	−32	−23	14	9
r_3	3	2	33	23	20	14	2	1
r_4	155	110	1706	1206	1018	720	46	32
r_5	77	54	840	594	500	354	6	4
r_6	599	424	6553	4634	3898	2756	78	55
r_7	68	48	741	524	440	311	10	7
r_8	150	106	1628	1151	963	681	133	94

det = $4-5w$	det norm = -34
22222222	

quadratic form

$$\left(\begin{array}{cc|cc|cc} 26 & 10 & -10 & 4 & 1 & 3 \\ -10 & 4 & 16 & -11 & 2 & -2 \\ 1 & 3 & 2 & -2 & 1 & 0 \end{array}\right)$$

Table A.6, cont.

root list

	roots						norms	
r_1	1	0	4	3	2	−2	2	0
r_2	1	1	15	11	0	0	6	1
r_3	0	0	1	1	2	−1	2	−1
r_4	1	−1	0	0	5	−2	5	−2
r_5	0	0	0	0	−1	1	3	−2
r_6	6	3	37	26	6	1	6	1
r_7	1	1	9	6	−2	2	2	−1
r_8	7	5	52	37	0	0	14	8

det = $-304 - 215w$ det norm = -34

22222222

quadratic form

$$\left(\begin{array}{cc|cc|cc} -173 & -145 & -42 & -41 & 0 & 0 \\ -42 & -41 & -9 & -12 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 1 \end{array}\right)$$

root list

	roots						norms	
r_1	−4	−1	15	7	−4	−3	3	2
r_2	−13	−10	60	44	−15	−11	37	26
r_3	24	15	−106	−71	31	22	10	7
r_4	331	234	−1505	−1064	430	304	126	89
r_5	95	66	−429	−301	119	84	34	24
r_6	1121	793	−5079	−3592	1379	975	430	304
r_7	181	130	−823	−586	222	157	10	7
r_8	744	526	−3366	−2380	897	634	126	89

det = $-37232 - 26327w$ det norm = -34

22222222

quadratic form

$$\begin{pmatrix} 14862 & 10509 & -830 & -587 & 243 & 172 \\ -830 & -587 & 56 & 39 & -14 & -9 \\ 243 & 172 & -14 & -9 & 5 & 2 \end{pmatrix}$$

root list

	roots						norms	
r_1	−3	−1	−3	−2	110	78	198	140
r_2	−9	−7	0	0	512	362	2646	1871
r_3	8	5	7	5	−386	−273	338	239
r_4	243	172	133	94	−12759	−9022	4517	3194
r_5	118	83	58	41	−6197	−4382	577	408
r_6	1284	908	587	415	−67762	−47915	15422	10905
r_7	143	101	58	41	−7566	−5350	1970	1393
r_8	431	305	133	94	−22964	−16238	52654	37232

$\det = -4 - 5w$ $\quad$ det norm $= -34$

22222222

quadratic form

$$\begin{pmatrix} 42 & 10 & -6 & 1 & 7 & -4 \\ -6 & 1 & 2 & -1 & -2 & 1 \\ 7 & -4 & -2 & 1 & 3 & -2 \end{pmatrix}$$

root list

	roots						norms	
r_1	1	0	11	8	2	3	2	1
r_2	2	1	55	39	18	14	14	9
r_3	0	0	2	2	1	1	1	0
r_4	1	−1	0	0	−1	3	5	2
r_5	0	0	−1	0	0	0	2	−1
r_6	−1	2	9	7	6	−1	6	−1
r_7	−1	1	4	1	4	−2	6	−4
r_8	3	−1	14	9	−2	6	14	−8

$\det = -8 - 7w$ $\qquad$ det norm $= -34$

22222222

quadratic form

$$\left(\begin{array}{cc|cc|cc} 184 & 127 & -20 & -9 & 8 & 7 \\ -20 & -9 & 10 & -5 & 1 & -2 \\ 8 & 7 & 1 & -2 & 1 & 0 \end{array}\right)$$

root list

	roots						norms	
r_1	1	0	5	4	2	−2	2	0
r_2	1	1	15	11	0	0	14	8
r_3	0	0	1	0	−2	2	2	−1
r_4	1	−1	0	0	6	1	6	1
r_5	0	0	0	0	−1	1	3	−2
r_6	3	0	15	11	5	−2	5	−2
r_7	1	1	12	9	2	−1	2	−1
r_8	6	5	67	48	0	0	6	1

$\det = -8 - 7w$ $\qquad$ det norm $= -34$

22222222

quadratic form

$$\left(\begin{array}{cc|cc|cc} 14 & 8 & 0 & 0 & -1 & -3 \\ 0 & 0 & 0 & -1 & 0 & 1 \\ -1 & -3 & 0 & 1 & 3 & -2 \end{array}\right)$$

root list

	roots						norms	
r_1	0	0	0	0	−1	−1	1	0
r_2	−2	−2	−7	−4	−22	−15	22	15
r_3	1	0	1	1	8	6	2	1
r_4	29	21	74	52	326	230	74	52
r_5	11	8	28	20	116	82	6	4
r_6	143	102	363	256	1438	1016	22	15
r_7	26	19	67	47	262	185	2	1
r_8	33	23	82	59	319	226	7	4

$\det = -304 - 215w$ $\quad$ det norm $= -34$

22222222

quadratic form

$$\left(\begin{array}{cc|cc|cc} -82 & -58 & 0 & 0 & 7 & 5 \\ 0 & 0 & 2 & 1 & 1 & 0 \\ 7 & 5 & 1 & 0 & 3 & -2 \end{array}\right)$$

root list

	roots						norms	
r_1	0	0	0	0	−3	−2	3	2
r_2	1	−1	7	5	−24	−17	126	89
r_3	0	0	1	1	0	0	10	7
r_4	1	1	−4	−3	14	10	430	304
r_5	1	0	−4	−3	0	0	34	24
r_6	6	5	−59	−42	−102	−72	126	89
r_7	1	1	−11	−8	−24	−17	10	7
r_8	3	0	−14	−10	−41	−29	37	26

det = $-1096 - 775w$ det norm = -34

22222222

quadratic form

$$\begin{pmatrix} -82 & -58 & 0 & 0 & -5 & -4 \\ 0 & 0 & 10 & 7 & 1 & 0 \\ -5 & -4 & 1 & 0 & 17 & -12 \end{pmatrix}$$

root list

	roots						norms	
r_1	0	0	0	0	−17	−12	17	12
r_2	−1	1	5	4	−106	−75	454	321
r_3	0	0	1	1	0	0	58	41
r_4	−3	−1	−2	−1	34	24	1550	1096
r_5	−1	−1	−4	−3	0	0	198	140
r_6	−10	−7	−45	−32	−198	−140	2646	1871
r_7	−1	−1	−7	−5	−58	−41	338	239
r_8	−3	−2	−20	−14	−379	−268	4517	3194

det = $-32 - 23w$ det norm = -34

22222222

quadratic form

$$\left(\begin{array}{cc|cc|cc} 2160 & 1527 & -96 & -69 & 32 & 23 \\ -96 & -69 & 10 & -1 & -3 & 0 \\ 32 & 23 & -3 & 0 & 1 & 0 \end{array}\right)$$

root list

	roots						norms	
r_1	1	0	11	8	−2	−2	2	0
r_2	−1	2	23	16	0	0	10	4
r_3	0	0	1	0	2	0	2	−1
r_4	−3	2	0	0	4	5	6	−1
r_5	0	0	0	0	−1	1	3	−2
r_6	−1	1	5	2	1	−3	7	−4
r_7	3	−2	2	1	0	−1	10	−7
r_8	0	1	13	12	−12	2	22	−15

det $= -304 - 215w$ det norm $= -34$

22222222

quadratic form

$$\left(\begin{array}{cc|cc|cc} 37 & 26 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 2 \end{array}\right)$$

root list

	roots						norms	
r_1	0	1	7	5	−2	−2	2	1
r_2	1	1	15	11	−7	−4	22	15
r_3	−3	−2	−30	−21	11	7	6	4
r_4	−171	−121	−1846	−1305	675	478	74	52
r_5	−37	−27	−407	−288	149	106	2	1
r_6	−194	−138	−2113	−1494	778	549	22	15
r_7	−3	−3	−40	−28	14	11	1	0
r_8	−6	−5	−74	−52	29	19	7	4

det $= -350848 - 248087w$ det norm $= -34$

22222222

quadratic form

$$\left(\begin{array}{cc|cc|cc} -1393 & -985 & 0 & 0 & 0 & 0 \\ 0 & 0 & 10 & 7 & 0 & 0 \\ 0 & 0 & 0 & 0 & 7 & 4 \end{array}\right)$$

root list

	roots						norms	
r_1	0	1	−3	−2	−10	−7	58	41
r_2	5	−2	0	0	−17	−12	734	519
r_3	−1	0	3	2	7	5	198	140
r_4	13	5	−89	−63	−99	−70	2506	1772
r_5	3	2	−24	−17	−31	−22	58	41
r_6	16	14	−141	−100	−198	−140	734	519
r_7	4	−2	−4	−3	−7	−5	17	12
r_8	1	3	−15	−11	−34	−24	215	152

det $= -1096 - 775w$ det norm $= -34$

22222222

quadratic form

$$\begin{pmatrix} 1410 & 997 & 32 & 23 & -78 & -55 \\ 32 & 23 & 8 & -4 & 1 & -3 \\ -78 & -55 & 1 & -3 & 5 & 2 \end{pmatrix}$$

root list

	roots						norms	
r_1	1	0	−3	−2	6	4	10	7
r_2	−1	3	0	0	22	16	78	55
r_3	0	0	1	1	1	0	3	2
r_4	0	0	0	0	1	1	23	16
r_5	3	−2	−2	−1	2	0	2	1
r_6	−1	3	−23	−16	18	12	14	9
r_7	2	−1	−3	−3	2	3	2	0
r_8	−8	7	−9	−7	8	9	10	4

det $= -4 - 5w$ det norm $= -34$

22222222

quadratic form

$$\begin{pmatrix} 14 & 9 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & -3 & -1 & 1 \\ 0 & 0 & -1 & 1 & 1 & 0 \end{pmatrix}$$

root list

	roots						norms	
r_1	1	0	4	3	−2	0	2	−1
r_2	1	1	9	7	−5	−2	5	2
r_3	0	0	0	0	−1	0	1	0
r_4	−1	0	0	0	0	0	14	9
r_5	0	0	1	1	0	1	2	1
r_6	5	4	55	39	0	0	46	32
r_7	3	2	27	19	−2	−2	6	4
r_8	24	17	211	149	−32	−23	78	55

det $= -1096 - 775w$ det norm $= -34$

22222222

quadratic form

$$\left(\begin{array}{cc|cc|cc} -353 & -250 & 124 & 87 & 0 & 0 \\ 124 & 87 & -42 & -31 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 2 \end{array}\right)$$

root list

	roots						norms	
r_1	−2	1	−1	−2	−4	−3	2	1
r_2	1	1	0	0	−14	−9	23	16
r_3	0	1	8	5	21	15	3	2
r_4	4	2	165	117	596	422	78	55
r_5	−1	0	33	23	131	93	10	7
r_6	−13	−10	220	156	995	704	266	188
r_7	−5	−4	42	30	221	156	34	24
r_8	−31	−22	133	94	853	603	454	321

det $= -4 - 5w$ det norm $= -34$

22222222

quadratic form

$$\begin{pmatrix} 78 & 55 & 0 & 0 & -23 & -16 \\ 0 & 0 & 14 & -10 & -3 & 3 \\ -23 & -16 & -3 & 3 & 3 & 0 \end{pmatrix}$$

root list

	roots						norms	
r_1	0	−1	−18	−13	−3	−3	1	0
r_2	0	−3	−55	−39	−9	−7	5	2
r_3	3	1	57	40	12	7	2	−1
r_4	43	34	1174	830	220	156	6	−1
r_5	4	9	215	153	38	30	6	−4
r_6	30	17	697	493	128	92	14	−8
r_7	9	−3	62	43	14	6	10	−7
r_8	11	2	179	126	32	23	22	−15

det $= -10328 - 7303w$ det norm $= -34$

22222222

quadratic form

$$\begin{pmatrix} 1014 & 717 & 52 & 37 & -89 & -63 \\ 52 & 37 & 12 & -3 & -6 & -1 \\ -89 & -63 & -6 & -1 & 7 & 4 \end{pmatrix}$$

root list

	roots						norms	
r_1	1	0	−3	−2	2	2	34	24
r_2	5	−2	0	0	8	6	126	89
r_3	0	0	1	1	0	1	10	7
r_4	0	0	0	0	1	1	37	26
r_5	3	−2	−1	−1	1	0	3	2
r_6	8	7	−89	−63	44	31	126	89
r_7	1	2	−18	−13	10	6	10	7
r_8	13	5	−89	−63	48	34	430	304

det $= -10328 - 7303w$ det norm $= -34$

22222222

quadratic form

$$\left(\begin{array}{cc|cc|cc} -816 & -577 & 0 & 0 & 41 & 29 \\ 0 & 0 & 2 & 1 & 0 & 1 \\ 41 & 29 & 0 & 1 & 3 & -2 \end{array}\right)$$

root list

	roots						norms	
r_1	0	0	0	0	−7	−5	17	12
r_2	−3	2	7	5	−17	−12	215	152
r_3	0	0	3	2	0	0	58	41
r_4	1	0	−4	−3	10	7	734	519
r_5	−1	1	−7	−5	0	0	198	140
r_6	3	2	−113	−80	−246	−174	2506	1772
r_7	1	0	−20	−14	−58	−41	58	41
r_8	0	3	−85	−60	−314	−222	734	519

det $= -6627 - 4686w$ det norm $= -63$

22222222

quadratic form

$$\begin{pmatrix} -23061 & -16308 & -840 & -603 & -8421 & -5955 \\ -840 & -603 & 10 & -51 & -307 & -220 \\ -8421 & -5955 & -307 & -220 & -2807 & -1985 \end{pmatrix}$$

root list

	roots						norms	
r_1	−61	−45	1698	1201	1	0	92	65
r_2	−35	−24	942	666	0	0	51	36
r_3	17	11	−443	−313	1	−1	34	24
r_4	−119	−85	3237	2289	3	3	594	420
r_5	−164	−115	4431	3133	3	3	17	12
r_6	−1419	−1004	38523	27240	30	21	471	333
r_7	−38	−29	1073	759	0	1	10	7
r_8	−21	−15	574	406	2	−1	3	2

det $= -3 - 6w$ det norm $= -63$

22222222

quadratic form

$$\begin{pmatrix} 9 & 6 & -6 & -3 & 12 & 9 \\ -6 & -3 & 8 & -4 & 1 & -3 \\ 12 & 9 & 1 & -3 & 2 & -1 \end{pmatrix}$$

root list

	roots						norms	
r_1	0	0	0	0	1	0	2	−1
r_2	−1	1	2	1	0	1	3	−2
r_3	−2	0	−9	−6	−9	−7	4	1
r_4	−7	−5	−60	−42	−42	−30	3	0
r_5	−4	−1	−23	−16	−14	−10	2	0
r_6	−18	−13	−153	−108	−84	−60	18	12
r_7	−7	−6	−65	−46	−34	−24	1	0
r_8	−49	−34	−408	−288	−207	−147	15	9

det $= -33 - 24w$ det norm $= -63$

22222222

quadratic form

$$\left(\begin{array}{cc|cc|cc} 801 & 156 & -444 & -399 & 18 & -12 \\ -444 & -399 & 510 & 343 & 3 & -3 \\ 18 & -12 & 3 & -3 & 2 & -1 \end{array}\right)$$

root list

	roots						norms	
r_1	72	53	82	48	3	2	16	11
r_2	51	38	60	33	0	0	9	6
r_3	−16	−10	−11	−14	−1	−1	6	4
r_4	86	62	93	60	21	15	102	72
r_5	139	98	141	101	21	15	3	2
r_6	1231	873	1269	885	174	123	81	57
r_7	36	26	39	25	4	3	2	1
r_8	23	14	14	21	2	1	1	0

det $= -33 - 24w$ det norm $= -63$

22222222

quadratic form

$$\left(\begin{array}{cc|cc|cc} 9 & 6 & -6 & 3 & 0 & 0 \\ -6 & 3 & 126 & -89 & 3 & -1 \\ 0 & 0 & 3 & -1 & 16 & 11 \end{array}\right)$$

root list

	roots						norms	
r_1	−8	5	−12	−8	3	−2	2	−1
r_2	1	−1	−4	−3	0	0	3	−2
r_3	−8	5	−5	−3	4	−3	4	1
r_4	1	−1	0	0	0	0	3	0
r_5	−4	3	1	1	3	−2	2	0
r_6	−6	1	−81	−57	3	0	18	12
r_7	1	−4	−69	−49	−1	2	1	0
r_8	−23	−11	−552	−390	9	3	15	9

det $= -195 - 138w$ det norm $= -63$

22222222

quadratic form

$$\left(\begin{array}{cc|cc|cc} 2259 & -284 & -1676 & -776 & 108 & 195 \\ -1676 & -776 & 2134 & 1420 & -269 & -216 \\ 108 & 195 & -269 & -216 & 46 & 25 \end{array}\right)$$

root list

	roots						norms	
r_1	−4	−4	0	−5	−1	−2	10	7
r_2	−181	−127	−132	−89	−59	−41	471	333
r_3	−23	−16	−17	−11	−8	−5	17	12
r_4	−48	−34	−35	−25	−24	−17	594	420
r_5	−26	−18	−21	−13	−18	−13	34	24
r_6	−217	−152	−170	−114	−152	−106	51	36
r_7	−126	−89	−97	−68	−87	−62	92	65
r_8	−13	−10	−8	−9	−8	−7	3	2

det = $-33 - 24w$ det norm = -63

22222222

quadratic form

$$\left(\begin{array}{cc|cc|cc} 315 & 210 & -27 & 9 & 18 & 15 \\ -27 & 9 & 47 & -33 & 3 & -3 \\ 18 & 15 & 3 & -3 & 2 & 1 \end{array}\right)$$

root list

	roots						norms	
r_1	1	0	21	15	4	−3	2	1
r_2	4	3	195	138	0	0	81	57
r_3	0	0	3	2	−1	1	3	2
r_4	−1	−1	−51	−36	3	0	102	72
r_5	1	1	45	32	1	0	6	4
r_6	15	11	594	420	0	3	9	6
r_7	10	7	390	276	3	−1	16	11
r_8	1	1	48	34	0	0	1	0

det = $-6627 - 4686w$ det norm = -63

22222222

quadratic form

$$\begin{pmatrix} 4557 & 3222 & -1635 & -1155 & -117 & -84 \\ -1635 & -1155 & 589 & 412 & 38 & 32 \\ -117 & -84 & 38 & 32 & 7 & -1 \end{pmatrix}$$

root list

	roots						norms	
r_1	3	1	7	4	−2	−2	34	24
r_2	3	4	12	9	0	0	51	36
r_3	1	−1	0	−1	5	3	92	65
r_4	1	−1	0	−1	2	1	3	2
r_5	2	−1	1	0	3	2	10	7
r_6	59	42	153	108	81	57	471	333
r_7	8	7	23	17	8	6	17	12
r_8	19	13	51	36	0	0	594	420

det $= -38625 - 27312w$ det norm $= -63$

22222222

quadratic form

$$\begin{pmatrix} 17969 & 12706 & -4474 & -3164 & 16586 & 11728 \\ -4474 & -3164 & 1290 & 664 & -4091 & -2938 \\ 16586 & 11728 & -4091 & -2938 & 15428 & 10901 \end{pmatrix}$$

Table A.6, cont.

root list

	roots						norms	
r_1	72	48	228	162	−11	−12	92	65
r_2	93	71	314	222	−20	−14	51	36
r_3	−102	−74	−337	−238	22	14	34	24
r_4	−7192	−5086	−23389	−16538	1472	1038	594	420
r_5	−5405	−3819	−17567	−12422	1104	782	17	12
r_6	−41729	−29505	−135669	−95932	8537	6034	471	333
r_7	−766	−547	−2503	−1769	161	109	10	7
r_8	−201	−147	−664	−470	40	31	3	2

det $= -3 - 6w$ det norm $= -63$

22222222

quadratic form

$$\begin{pmatrix} 1095 & -756 & 366 & -270 & 42 & -30 \\ 366 & -270 & 134 & -88 & 15 & -10 \\ 42 & -30 & 15 & -10 & 2 & -1 \end{pmatrix}$$

root list

	roots						norms	
r_1	−2	0	1	−3	2	−2	2	0
r_2	−3	−4	−12	−3	0	0	3	0
r_3	−2	−1	−3	−3	3	−1	4	1
r_4	−3	2	8	−6	2	−1	3	−2
r_5	0	0	0	0	1	0	2	−1
r_6	−11	−9	−18	−9	15	9	15	9
r_7	−1	−2	−5	0	0	2	1	0
r_8	−4	−4	−9	−3	0	0	18	12

det $= -33292 - 23541w$ det norm $= -98$

22222422

quadratic form

$$\begin{pmatrix} 35122 & 24835 & \vline & -406 & -287 & \vline & 287 & 203 \\ -406 & -287 & \vline & 8 & 3 & \vline & -2 & -3 \\ 287 & 203 & \vline & -2 & -3 & \vline & 3 & 1 \end{pmatrix}$$

root list

	roots						norms	
r_1	1	−2	−13	−9	74	52	150	106
r_2	−3	2	0	0	8	6	10	7
r_3	0	0	1	1	2	1	10	7
r_4	0	0	0	0	3	2	75	53
r_5	−2	1	−11	−8	22	16	92	65
r_6	1	−1	−7	−5	14	10	34	24
r_7	−2	1	−7	−5	21	15	17	12
r_8	−4	−2	−65	−46	260	184	536	379

$\det = -980 - 693w$ det norm $= -98$

22224222

quadratic form

$$\begin{pmatrix} 92 & 65 & \vline & 0 & 0 & \vline & 0 & 0 \\ 0 & 0 & \vline & 2 & -4 & \vline & -3 & -1 \\ 0 & 0 & \vline & -3 & -1 & \vline & -1 & -1 \end{pmatrix}$$

root list

	roots						norms	
r_1	1	0	7	5	−2	−2	2	1
r_2	−1	1	4	3	−2	−1	2	1
r_3	0	0	4	3	−3	−1	13	9
r_4	1	−1	0	0	0	0	16	11
r_5	0	0	−1	−1	2	0	6	4
r_6	1	0	4	3	−1	0	3	2
r_7	9	6	92	65	−22	−16	92	65
r_8	8	5	89	63	−26	−18	150	106

det $= -67 - 48w$ det norm $= -119$

42222222

quadratic form

$$\begin{pmatrix} 495 & 312 & 5 & 32 & 18 & 20 \\ 5 & 32 & 17 & -10 & 5 & -2 \\ 18 & 20 & 5 & -2 & 2 & 0 \end{pmatrix}$$

root list

	roots						norms	
r_1	1	0	−7	−4	−2	−9	2	0
r_2	0	0	1	0	−3	1	1	0
r_3	−1	−1	15	11	21	12	74	52
r_4	1	1	−19	−13	−11	−8	6	4
r_5	47	33	−682	−482	−520	−370	37	26
r_6	10	7	−142	−101	−117	−81	26	18
r_7	27	19	−378	−267	−327	−233	37	26
r_8	4	3	−57	−40	−52	−37	3	2

det $= -393 - 278w$ det norm $= -119$

42222222

quadratic form

$$\left(\begin{array}{cc|cc|cc} 573 & 310 & 178 & 148 & -14 & -42 \\ 178 & 148 & 78 & 50 & -19 & -6 \\ -14 & -42 & -19 & -6 & 10 & -4 \end{array}\right)$$

root list

	roots						norms	
r_1	2	0	1	−4	0	1	6	4
r_2	−1	0	0	2	1	0	3	2
r_3	−30	−22	58	38	−81	−57	430	304
r_4	−22	−16	36	24	−81	−57	34	24
r_5	−445	−315	652	460	−1788	−1264	215	152
r_6	−72	−50	97	72	−305	−216	150	106
r_7	−141	−100	185	130	−651	−460	215	152
r_8	−15	−11	19	12	−78	−55	17	12

$\det = -9 - 10w$ det norm $= -119$

22222422

quadratic form

$$\left(\begin{array}{cc|cc|cc} -2217 & -1568 & -14 & -8 & -37 & -26 \\ -14 & -8 & 14 & -10 & 1 & -1 \\ -37 & -26 & 1 & -1 & 1 & 0 \end{array}\right)$$

root list

	roots						norms	
r_1	1	0	−79	−56	1	0	2	0
r_2	5	2	−608	−430	0	0	7	4
r_3	0	0	3	2	−2	0	6	2
r_4	1	−1	37	26	0	0	7	4
r_5	0	0	0	0	1	0	1	0
r_6	−1	1	−34	−24	1	1	2	0
r_7	2	0	−161	−114	3	2	1	0
r_8	7	5	−1127	−797	15	11	74	52

det $= -3882 - 2745w$ det norm $= -126$

22222222

quadratic form

$$\left(\begin{array}{cc|cc|cc} -1110 & -785 & 272 & 194 & 186 & 131 \\ 272 & 194 & -42 & -56 & -47 & -25 \\ 186 & 131 & -47 & -25 & -26 & -21 \end{array}\right)$$

root list

	roots						norms	
r_1	0	0	−5	−4	9	5	30	21
r_2	3	−2	0	0	0	1	2	1
r_3	3	−3	17	12	−32	−22	48	33
r_4	−6	4	1	2	−6	−2	2	0
r_5	−7	4	4	2	−8	−10	2	−1
r_6	−8	−3	25	18	−92	−65	10	6
r_7	−3	−12	36	25	−142	−100	6	3
r_8	−2	1	1	0	−1	−4	2	−1

det $= -666 - 471w$ det norm $= -126$

22222222

quadratic form

$$\begin{pmatrix} 1806 & 1277 & -718 & -508 & -42 & -30 \\ -718 & -508 & 290 & 200 & 21 & 9 \\ -42 & -30 & 21 & 9 & 6 & -3 \end{pmatrix}$$

root list

	roots						norms	
r_1	−2	1	−1	−2	14	11	10	6
r_2	−3	3	0	0	20	14	6	3
r_3	0	0	1	0	−7	−4	2	−1
r_4	−12	−3	15	9	−437	−308	6	3
r_5	−7	−8	14	9	−464	−327	2	−1
r_6	−63	−57	99	72	−3576	−2530	12	3
r_7	0	−2	−1	3	−64	−48	6	−4
r_8	−15	10	2	−1	−18	−12	10	−7

det $= -114 - 81w$ det norm $= -126$

22222222

quadratic form

$$\begin{pmatrix} 298 & -55 & -204 & -86 & 36 & 9 \\ -204 & -86 & 250 & 164 & -39 & -24 \\ 36 & 9 & -39 & -24 & 6 & 3 \end{pmatrix}$$

root list

	roots						norms	
r_1	4	2	5	0	0	1	6	4
r_2	3	3	3	6	16	11	276	195
r_3	−3	−2	−2	−1	2	1	10	7
r_4	0	0	0	0	3	2	174	123
r_5	16	11	12	7	3	2	10	7
r_6	531	375	372	261	90	64	174	123
r_7	324	229	227	160	54	38	314	222
r_8	35	25	24	18	6	4	10	7

det $= -114 - 81w$ det norm $= -126$

22222222

quadratic form

$$\begin{pmatrix} -150 & -129 & -72 & -66 & -36 & -33 \\ -72 & -66 & -34 & -34 & -17 & -17 \\ -36 & -33 & -17 & -17 & -6 & -7 \end{pmatrix}$$

root list

	roots						norms	
r_1	4	2	−8	−4	1	0	2	1
r_2	25	17	−48	−33	0	0	30	21
r_3	0	0	1	1	−2	−2	54	38
r_4	−1	−1	2	2	0	0	2	1
r_5	0	1	−1	−2	2	1	6	4
r_6	265	187	−561	−396	132	93	276	195
r_7	41	29	−86	−61	18	13	10	7
r_8	64	45	−132	−93	21	15	174	123

det $= -18 - 15w$ det norm $= -126$

22222222

quadratic form

$$\begin{pmatrix} 30 & 21 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & -2 & 1 & -1 \\ 0 & 0 & 1 & -1 & 2 & -1 \end{pmatrix}$$

root list

	roots						norms	
r_1	0	1	5	4	2	1	2	0
r_2	−1	1	2	1	0	0	2	−1
r_3	−10	−6	−71	−50	−20	−14	10	6
r_4	−31	−23	−246	−174	−60	−42	6	3
r_5	−4	−1	−21	−15	−5	−3	2	−1
r_6	−14	−10	−111	−78	−21	−15	6	3
r_7	−9	−4	−58	−41	−10	−7	2	−1
r_8	−43	−30	−339	−240	−54	−39	12	3

det $= -99876 - 70623w$ det norm $= -882$

22222222

quadratic form

$$\begin{pmatrix} 26696 & 18057 & -5084 & -3958 & 616 & 406 \\ -5084 & -3958 & 1206 & 692 & -113 & -93 \\ 616 & 406 & -113 & -93 & 10 & 6 \end{pmatrix}$$

root list

	roots						norms	
r_1	4	2	16	13	1	3	34	24
r_2	3	−1	0	7	5	4	406	287
r_3	−3	−2	−14	−10	3	0	174	123
r_4	0	0	0	0	3	2	314	222
r_5	31	25	168	112	24	15	768	543
r_6	3	1	9	9	2	0	10	7
r_7	10	7	49	35	6	2	150	106
r_8	175	126	884	620	102	72	1608	1137

det $= -2940 - 2079w$ det norm $= -882$

22222222

quadratic form

$$\left(\begin{array}{cc|cc|cc} 88 & 57 & 86 & 67 & 18 & 15 \\ 86 & 67 & 94 & 59 & 21 & 12 \\ 18 & 15 & 21 & 12 & 6 & 3 \end{array}\right)$$

root list

	roots						norms	
r_1	−5	−1	3	5	−6	−7	6	2
r_2	−45	−30	66	48	−108	−76	48	33
r_3	6	1	−3	−6	7	8	2	0
r_4	417	292	−635	−451	1023	723	14	7
r_5	531	378	−819	−576	1315	927	6	3
r_6	1052	744	−1613	−1140	2593	1833	10	6
r_7	4215	2985	−6468	−4569	10396	7349	24	15
r_8	88	63	−137	−95	220	153	2	−1

det $= -17136 - 12117w$ det norm $= -882$

22222222

quadratic form

$$\begin{pmatrix} 10108 & 6363 & 6734 & 5103 & 252 & 147 \\ 6734 & 5103 & 5198 & 3527 & 159 & 126 \\ 252 & 147 & 159 & 126 & 6 & 3 \end{pmatrix}$$

root list

	roots						norms	
r_1	1	−1	3	−1	−28	−11	6	4
r_2	−15	−12	30	18	−260	−184	276	195
r_3	1	2	−6	−2	83	51	26	18
r_4	30	24	−62	−38	627	437	2	1
r_5	1485	1050	−2676	−1893	28436	20100	132	93
r_6	365	259	−659	−464	6940	4906	54	38
r_7	180	129	−327	−228	3394	2392	30	21
r_8	134	97	−245	−168	2464	1743	70	49

det = $-582120 - 411621w$ det norm = -882

22222222

quadratic form

$$\begin{pmatrix} 31332 & 22155 & -2436 & -1722 & 276 & 195 \\ -2436 & -1722 & 198 & 136 & -23 & -15 \\ 276 & 195 & -23 & -15 & 2 & 1 \end{pmatrix}$$

Table A.6, cont.

root list

	roots						norms	
r_1	0	0	−3	−2	−60	−42	198	140
r_2	5	4	0	0	−1332	−942	9372	6627
r_3	−1	0	7	5	269	190	874	618
r_4	−10	−8	13	9	2925	2068	58	41
r_5	−497	−351	522	369	134730	95268	4476	3165
r_6	−122	−87	119	84	33032	23357	1830	1294
r_7	−61	−43	51	36	16254	11493	1014	717
r_8	−45	−33	27	19	11998	8484	2366	1673

det $= -10905 - 7711w$ det norm $= -17$

$(2222)^2$

quadratic form

$$\left(\begin{array}{cc|cc|cc} 58115 & 41049 & -28300 & -20022 & 928 & 650 \\ -28300 & -20022 & 13798 & 9754 & -449 & -319 \\ 928 & 650 & -449 & -319 & 14 & 9 \end{array}\right)$$

root list

	roots						norms	
r_1	−16	−10	−27	−24	11	7	58	41
r_2	−9	−7	−19	−11	49	35	775	548
r_3	39	27	76	56	−37	−26	99	70
r_4	1174	830	2334	1651	−1731	−1224	2646	1871

det $= -519 - 367w$ det norm $= -17$

$(2222)^2$

quadratic form

$$\left(\begin{array}{cc|cc|cc} -156 & -111 & -15 & -11 & 0 & 0 \\ -15 & -11 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 2 \end{array}\right)$$

Table A.6, cont.

root list

	roots						norms	
r_1	1	0	−2	−1	−2	−2	2	1
r_2	1	1	0	0	−8	−7	22	15
r_3	0	0	0	1	−1	0	1	0
r_4	1	−1	1	3	0	0	7	4

det $= -9 - 7w$ det norm $= -17$

$(2222)^2$

quadratic form

$$\left(\begin{array}{cc|cc|cc} 1367 & 863 & -591 & -437 & -26 & -7 \\ -591 & -437 & 285 & 198 & 7 & 7 \\ -26 & -7 & 7 & 7 & 2 & -1 \end{array}\right)$$

root list

	roots						norms	
r_1	−9	−7	−23	−13	12	8	2	1
r_2	−52	−38	−121	−79	78	55	23	16
r_3	−3	−2	−6	−5	7	5	3	2
r_4	0	−1	−6	1	23	16	78	55

det $= -55 - 39w$ det norm $= -17$

$(2222)^2$

quadratic form

$$\left(\begin{array}{cc|cc|cc} 133 & 94 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & -3 & 1 & -1 \\ 0 & 0 & 1 & -1 & 1 & 0 \end{array}\right)$$

root list

	roots						norms	
r_1	1	0	14	10	2	1	3	2
r_2	2	1	55	39	0	0	78	55
r_3	−4	−2	−99	−70	−10	−7	10	7
r_4	−65	−46	−1926	−1362	−156	−110	133	94

det $= -3025 - 2139w$ det norm $= -17$

$(2222)^2$

quadratic form

$$\begin{pmatrix} 7 & 1 & -96 & -64 & -8 & -7 \\ -96 & -64 & 2254 & 1590 & 223 & 159 \\ -8 & -7 & 223 & 159 & 22 & 15 \end{pmatrix}$$

root list

	roots						norms	
r_1	4	2	1	0	−5	−3	10	7
r_2	5	−2	5	−2	−13	−9	37	26
r_3	−1	−1	0	0	−1	0	3	2
r_4	0	0	0	0	1	1	126	89

det $= -55 - 39w$ det norm $= -17$

$(2222)^2$

quadratic form

$$\begin{pmatrix} 79 & 35 & 26 & -10 & -22 & -19 \\ 26 & -10 & 28 & -19 & 1 & -4 \\ -22 & -19 & 1 & -4 & -2 & -2 \end{pmatrix}$$

root list

	roots						norms	
r_1	1	1	−11	−8	1	0	1	0
r_2	7	3	−51	−34	6	−1	5	2
r_3	2	−1	−3	−1	−4	3	2	−1
r_4	−2	1	5	2	0	0	6	−1

det $= -9 - 7w$ det norm $= -17$

$(2222)^2$

quadratic form

$$\begin{pmatrix} 633 & 447 & -37 & -25 & 32 & 23 \\ -37 & -25 & 7 & -2 & -1 & -2 \\ 32 & 23 & -1 & -2 & 2 & 1 \end{pmatrix}$$

root list

	roots						norms	
r_1	1	0	11	8	4	0	2	−1
r_2	−1	2	23	16	4	5	6	−1
r_3	0	0	1	0	−2	2	3	−2
r_4	−3	2	0	0	−1	3	7	−4

det $= -1 - 3w$ det norm $= -17$

$(2222)^2$

quadratic form

$$\begin{pmatrix} 57 & 35 & -11 & 1 & 8 & 7 \\ -11 & 1 & 15 & -10 & 1 & -2 \\ 8 & 7 & 1 & -2 & 2 & 1 \end{pmatrix}$$

root list

	roots						norms	
r_1	1	0	5	4	4	−3	2	−1
r_2	1	1	15	11	0	0	6	1
r_3	0	0	1	0	−4	3	3	−2
r_4	1	−1	0	0	5	−2	5	−2

det $= -10905 - 7711w$ det norm $= -17$

$(2222)^2$

quadratic form

$$\begin{pmatrix} -239 & -169 & 0 & 0 & 17 & 12 \\ 0 & 0 & 10 & 7 & 1 & 1 \\ 17 & 12 & 1 & 1 & 3 & -2 \end{pmatrix}$$

root list

	roots						norms	
r_1	0	0	0	0	−7	−5	17	12
r_2	−3	2	3	2	−24	−17	133	94
r_3	0	0	1	0	0	0	10	7
r_4	2	−1	−1	0	4	3	78	55

det $= -17631 - 12467w$ det norm $= -17$

$(2222)^2$

quadratic form

$$\begin{pmatrix} 7064 & 4995 & 304 & 215 & 215 & 152 \\ 304 & 215 & 17 & 10 & 8 & 7 \\ 215 & 152 & 8 & 7 & 7 & 4 \end{pmatrix}$$

root list

	roots						norms	
r_1	1	0	−2	−1	−18	−12	10	7
r_2	5	−2	0	0	−44	−31	126	89
r_3	0	0	1	0	0	−1	3	2
r_4	0	0	0	0	1	1	37	26

det $= -1533 - 1084w$ det norm $= -23$
$(2228)^2$

quadratic form

$$\begin{pmatrix} 13665 & 9518 & -2523 & -1742 & -1094 & -780 \\ -2523 & -1742 & 467 & 318 & 201 & 144 \\ -1094 & -780 & 201 & 144 & 98 & 69 \end{pmatrix}$$

root list

	roots						norms	
r_1	−6	−5	−34	−27	3	0	10	7
r_2	−49	−35	−270	−192	7	6	109	77
r_3	−2	−3	−13	−15	2	−1	3	2
r_4	−5	−3	−27	−17	0	1	6	4

det $= -857 - 606w$ det norm $= -23$
$(2228)^2$

quadratic form

$$\begin{pmatrix} 9381 & 6620 & -3902 & -2757 & 122 & 86 \\ -3902 & -2757 & 1624 & 1148 & -51 & -36 \\ 122 & 86 & -51 & -36 & 2 & 1 \end{pmatrix}$$

Table A.6, cont.

root list

	roots						norms	
r_1	−6	−5	−13	−14	−27	−20	2	1
r_2	−22	−18	−49	−50	−111	−79	36	25
r_3	13	10	30	27	58	42	1	0
r_4	32	27	69	76	154	110	2	0

det $= -7 - 6w$ det norm $= -23$

$(2282)^2$

quadratic form

$$\begin{pmatrix} -7 & -6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 6 & 4 & -1 & -3 \\ 0 & 0 & -1 & -3 & 18 & -11 \end{pmatrix}$$

root list

	roots						norms	
r_1	1	1	−5	−6	−16	−11	1	0
r_2	9	6	−45	−32	−109	−77	19	13
r_3	0	0	3	−1	1	1	2	1
r_4	0	0	−1	0	0	0	6	4

det $= -45 - 32w$ det norm $= -23$

$(2228)^2$

quadratic form

$$\begin{pmatrix} -263 & -186 & -14 & -12 & -526 & -372 \\ -14 & -12 & 1142 & -808 & 25 & -26 \\ -526 & -372 & 25 & -26 & 54 & 37 \end{pmatrix}$$

root list

	roots						norms	
r_1	−2	−2	181	128	−3	−3	2	1
r_2	3	0	−109	−77	5	1	19	13
r_3	3	−1	−59	−42	−6	6	1	0
r_4	−4	−2	255	180	−14	2	2	0

det = $-29113 - 20586w$ det norm = -23

$(8222)^2$

quadratic form

$$\begin{pmatrix} 17525 & 12392 & 10948 & 7742 & 330 & 233 \\ 10948 & 7742 & 6850 & 4836 & 203 & 148 \\ 330 & 233 & 203 & 148 & 8 & 3 \end{pmatrix}$$

root list

	roots						norms	
r_1	0	0	−1	−1	36	25	34	24
r_2	−8	5	0	1	1	1	10	7
r_3	116	78	0	0	−5389	−3810	208	147
r_4	27	11	0	−1	−992	−702	3	2

det = $-147 - 104w$ det norm = -23

$(2282)^2$

quadratic form

$$\begin{pmatrix} -147 & -104 & 0 & 0 & 0 & 0 \\ 0 & 0 & 6 & 4 & -1 & -3 \\ 0 & 0 & -1 & -3 & 18 & -11 \end{pmatrix}$$

root list

	roots						norms	
r_1	1	1	−28	−21	−68	−48	3	2
r_2	6	5	−147	−104	−355	−251	208	147
r_3	0	0	1	2	3	2	10	7
r_4	0	0	−1	−1	0	0	34	24

det $= -113 - 80w$ det norm $= -31$

$(2242)^2$

quadratic form

$$\left(\begin{array}{cc|cc|cc} -113 & -80 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 \end{array}\right)$$

root list

	roots						norms	
r_1	1	0	−7	−5	−4	−3	3	2
r_2	3	2	−47	−33	−47	−33	546	386
r_3	0	0	0	0	−1	−1	3	2
r_4	0	0	1	1	2	2	6	4

det $= -659 - 466w$ det norm $= -31$

$(4222)^2$

quadratic form

$$\left(\begin{array}{cc|cc|cc} 21 & 12 & -134 & -92 & 2 & 8 \\ -134 & -92 & 1022 & 720 & -49 & -41 \\ 2 & 8 & -49 & -41 & 14 & -6 \end{array}\right)$$

Table A.6, cont.

root list

	roots						norms	
r_1	2	0	1	0	6	4	6	4
r_2	−1	0	0	0	1	1	3	2
r_3	0	0	0	0	7	5	546	386
r_4	−1	3	−2	2	4	3	3	2

det $= -4 - 5w$ det norm $= -34$

$(2222)^2$

quadratic form

$$\begin{pmatrix} 88 & -9 & 72 & -46 & -16 & -3 \\ 72 & -46 & 94 & -66 & -9 & 5 \\ -16 & -3 & -9 & 5 & 0 & -1 \end{pmatrix}$$

root list

	roots						norms	
r_1	1	1	−11	−8	0	1	2	−1
r_2	7	3	−51	−34	−2	6	6	−1
r_3	2	−1	−3	−1	6	−4	6	−4
r_4	−2	1	5	2	0	0	14	−8

det $= -1096 - 775w$ det norm $= -34$

$(2222)^2$

quadratic form

$$\begin{pmatrix} 34132 & 23983 & -11744 & -8278 & 556 & 372 \\ -11744 & -8278 & 4044 & 2855 & -189 & -130 \\ 556 & 372 & -189 & -130 & 10 & 4 \end{pmatrix}$$

Table A.6, cont.

root list

	roots						norms	
r_1	−16	−10	−48	−27	11	7	34	24
r_2	−9	−7	−22	−19	49	35	454	321
r_3	39	27	112	76	−37	−26	58	41
r_4	1174	830	3302	2334	−1731	−1224	1550	1096

det $= -304 - 215w$ det norm $= -34$

$(2222)^2$

quadratic form

$$\begin{pmatrix} 12 & -5 & -32 & -32 & -2 & -6 \\ -32 & -32 & 664 & 463 & 95 & 64 \\ -2 & -6 & 95 & 64 & 14 & 8 \end{pmatrix}$$

root list

	roots						norms	
r_1	4	2	0	1	−5	−3	6	4
r_2	5	−2	−4	5	−13	−9	22	15
r_3	−3	−3	−2	0	7	5	2	1
r_4	−74	−52	−30	−22	231	164	74	52

det $= -86 - 61w$ det norm $= -46$

$(2228)^2$

quadratic form

$$\begin{pmatrix} 5522 & 3859 & -1612 & -1145 & 72 & 50 \\ -1612 & -1145 & 476 & 336 & -21 & -15 \\ 72 & 50 & -21 & -15 & 2 & 0 \end{pmatrix}$$

root list

	roots						norms	
r_1	−6	−5	−28	−13	−27	−20	2	0
r_2	−22	−18	−100	−49	−111	−79	22	14
r_3	13	10	54	30	58	42	2	−1
r_4	27	16	69	76	110	77	2	−1

det $= -86 - 61w$ det norm $= -46$

$(2282)^2$

quadratic form

$$\begin{pmatrix} -502 & -355 & 0 & 0 & 0 & 0 \\ 0 & 0 & 34 & -24 & 3 & -2 \\ 0 & 0 & 3 & -2 & 2 & 1 \end{pmatrix}$$

root list

	roots						norms	
r_1	1	0	−68	−48	2	0	2	1
r_2	4	1	−355	−251	0	0	122	86
r_3	0	0	3	2	−2	0	6	4
r_4	0	0	0	0	1	0	2	1

det $= -2 - 5w$ det norm $= -46$

$(2282)^2$

quadratic form

$$\begin{pmatrix} -26 & -19 & 0 & 0 & 0 & 0 \\ 0 & 0 & 34 & -24 & 3 & -2 \\ 0 & 0 & 3 & -2 & 2 & 1 \end{pmatrix}$$

root list

	roots						norms	
r_1	1	0	−16	−11	2	−1	2	−1
r_2	3	3	−109	−77	0	0	12	7
r_3	0	0	1	1	2	−2	2	0
r_4	0	0	0	0	−1	1	2	−1

det $= -17054 - 12059w$ det norm $= -46$

$(2228)^2$

quadratic form

$$\begin{pmatrix} 338 & 239 & -6 & -4 & -150 & -106 \\ -6 & -4 & 34 & -24 & 7 & -8 \\ -150 & -106 & 7 & -8 & -70 & -53 \end{pmatrix}$$

root list

	roots						norms	
r_1	0	0	−13	−9	1	0	58	41
r_2	−1	1	0	0	0	0	58	41
r_3	−2	−1	219	155	−12	−9	4138	2926
r_4	0	−2	65	46	−4	−3	198	140

det $= -5234 - 3701w$ det norm $= -46$

$(2228)^2$

quadratic form

$$\begin{pmatrix} 201578 & 142523 & -37072 & -26210 & 604 & 429 \\ -37072 & -26210 & 6818 & 4820 & -111 & -79 \\ 604 & 429 & -111 & -79 & 2 & 1 \end{pmatrix}$$

Table A.6, cont.

root list

	roots						norms	
r_1	4	2	21	12	20	14	34	24
r_2	1	2	9	11	90	64	372	263
r_3	−3	−3	−18	−16	−30	−21	10	7
r_4	−16	−11	−91	−63	−205	−145	10	7

det = $-26-19w$ det norm = -46

$(2228)^2$

quadratic form

$$\left(\begin{array}{cc|cc|cc} -154 & -109 & 0 & 0 & 0 & 0 \\ 0 & 0 & 34 & -24 & 3 & -2 \\ 0 & 0 & 3 & -2 & 2 & 1 \end{array}\right)$$

root list

	roots						norms	
r_1	2	0	−75	−53	2	0	2	0
r_2	3	0	−109	−77	0	0	12	7
r_3	1	−1	16	11	−2	1	2	−1
r_4	2	−1	−23	−16	3	−1	2	−1

det = $-195-138w$ det norm = -63

$(4222)^2$

quadratic form

$$\left(\begin{array}{cc|cc|cc} 957 & 672 & -426 & -300 & 36 & 30 \\ -426 & -300 & 190 & 134 & -17 & -13 \\ 36 & 30 & -17 & -13 & 6 & -2 \end{array}\right)$$

root list

	roots						norms	
r_1	2	0	5	0	2	1	6	4
r_2	−1	0	−2	0	1	1	3	2
r_3	0	0	0	0	3	2	54	38
r_4	7	5	16	12	10	7	3	2

det $= -33 - 24w$ det norm $= -63$

$(4222)^2$

quadratic form

$$\begin{pmatrix} 741 & 516 & 51 & 39 & -30 & -18 \\ 51 & 39 & 5 & 2 & -1 & -2 \\ -30 & -18 & -1 & -2 & 2 & 0 \end{pmatrix}$$

root list

	roots						norms	
r_1	1	0	−5	−3	5	1	2	0
r_2	0	0	1	0	0	1	1	0
r_3	−1	0	5	3	−4	−1	10	6
r_4	1	1	−14	−9	6	4	1	0

det $= -26566 - 18785w$ det norm $= -94$

$(2222)^2$

quadratic form

$$\begin{pmatrix} 4701758 & 3324643 & 544668 & 385136 & -12400 & -8770 \\ 544668 & 385136 & 63098 & 44614 & -1435 & -1017 \\ -12400 & -8770 & -1435 & -1017 & 34 & 22 \end{pmatrix}$$

root list

	roots						norms	
r_1	−8	−2	49	32	16	11	34	24
r_2	−5	−6	61	45	168	119	1888	1335
r_3	5	3	−40	−28	4	3	58	41
r_4	0	0	0	0	7	5	6446	4558

det $= -134 - 95w$ det norm $= -94$

$(2222)^2$

quadratic form

$$\begin{pmatrix} 4782 & 3285 & 120 & -44 & 92 & 54 \\ 120 & -44 & 122 & -86 & 11 & -7 \\ 92 & 54 & 11 & -7 & 2 & 0 \end{pmatrix}$$

root list

	roots						norms	
r_1	0	0	7	5	−6	−5	6	4
r_2	3	0	−73	−50	−134	−95	324	229
r_3	−1	1	−18	−14	−10	−7	10	7
r_4	2	1	−190	−134	−39	−28	1106	782

det $= -134 - 95w$ det norm $= -94$

$(2222)^2$

quadratic form

$$\begin{pmatrix} 1118 & 787 & -74 & -70 & 78 & 56 \\ -74 & -70 & 70 & -40 & -9 & -2 \\ 78 & 56 & -9 & -2 & 6 & 4 \end{pmatrix}$$

root list

	roots						norms	
r_1	0	1	9	6	2	−3	2	0
r_2	4	2	39	28	−12	−5	34	22
r_3	3	−2	0	1	−4	2	2	−1
r_4	−1	1	0	0	−7	1	12	5

det = −154838 − 109487w det norm = −94

$(2222)^2$

quadratic form

$$\begin{pmatrix} 2231086 & 1577611 & 693276 & 490223 & -4626 & -3267 \\ 693276 & 490223 & 215428 & 152329 & -1435 & -1017 \\ -4626 & -3267 & -1435 & -1017 & 12 & 5 \end{pmatrix}$$

root list

	roots						norms	
r_1	−7	0	5	13	60	42	34	24
r_2	−7	1	3	13	238	168	1106	782
r_3	3	−1	2	−5	2	2	10	7
r_4	0	0	0	0	3	2	324	229

det = 4 − 9w det norm = −146

$(2238)^2$

quadratic form

$$\begin{pmatrix} 3412 & 2397 & 56 & 20 & 28 & 10 \\ 56 & 20 & 18 & -12 & 9 & -6 \\ 28 & 10 & 9 & -6 & 6 & -4 \end{pmatrix}$$

root list

	roots						norms	
r_1	1	0	−49	−34	6	4	2	1
r_2	4	2	−315	−222	0	0	210	148
r_3	−4	−2	329	232	−34	−24	6	4
r_4	−10	−7	949	671	−79	−56	6	4

det $= -864 - 611w$ det norm $= -146$

$(3228)^2$

quadratic form

$$\begin{pmatrix} 816 & 575 & -1112 & -792 & -76 & -48 \\ -1112 & -792 & 1542 & 1074 & 87 & 78 \\ -76 & -48 & 87 & 78 & 18 & -4 \end{pmatrix}$$

root list

	roots						norms	
r_1	0	0	−1	0	6	5	6	4
r_2	0	2	2	0	1	0	6	4
r_3	0	0	0	0	3	2	210	148
r_4	−1	−1	−2	0	1	2	2	1

det $= 24 - 19w$ det norm $= -146$

$(3822)^2$

quadratic form

$$\begin{pmatrix} 2592 & -1833 & 296 & -210 & 24 & -19 \\ 296 & -210 & 34 & -24 & 3 & -2 \\ 24 & -19 & 3 & -2 & 2 & 1 \end{pmatrix}$$

Table A.6, cont.

root list

	roots						norms	
r_1	1	0	−5	3	3	−2	6	−4
r_2	0	0	1	0	−6	4	6	−4
r_3	0	0	0	0	3	−2	10	−7
r_4	37	27	−52	−44	35	24	18	4

det $= -24 - 19w$ det norm $= -146$

$(2283)^2$

quadratic form

$$\begin{pmatrix} 2448 & 1719 & 8 & 55 & -48 & -38 \\ 8 & 55 & 150 & -105 & -13 & 8 \\ -48 & -38 & -13 & 8 & 2 & 0 \end{pmatrix}$$

root list

	roots						norms	
r_1	1	0	−29	−20	−6	4	2	0
r_2	1	2	−105	−74	0	0	38	24
r_3	0	0	1	0	6	−4	2	−1
r_4	0	0	0	0	1	0	2	0

det $= 24 - 19w$ det norm $= -146$

$(8322)^2$

quadratic form

$$\begin{pmatrix} 2608 & -1505 & 192 & -152 & 96 & -76 \\ 192 & -152 & 18 & -12 & 9 & -6 \\ 96 & -76 & 9 & -6 & 6 & -4 \end{pmatrix}$$

root list

	roots						norms	
r_1	1	0	1	9	3	2	2	1
r_2	0	0	3	2	−6	−4	6	4
r_3	0	0	0	0	3	2	6	4
r_4	12	8	122	92	201	142	402	284

det = $-45590 - 32237w$ det norm = -238

$(2222)^2$

quadratic form

$$\begin{pmatrix} -1322 & -935 & -312 & -210 & -308 & -217 \\ -312 & -210 & 342 & -312 & -21 & -56 \\ -308 & -217 & -21 & -56 & -48 & -37 \end{pmatrix}$$

root list

	roots						norms	
r_1	0	0	−5	−4	1	7	44	31
r_2	−3	2	0	0	0	1	10	7
r_3	−1	2	57	40	−72	−46	556	393
r_4	6	−4	5	3	−10	−2	6	4

det = $230 - 163w$ det norm = -238

$(2222)^2$

quadratic form

$$\begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 82 & -58 & 13 & -9 \\ 0 & 0 & 13 & -9 & 6 & -2 \end{pmatrix}$$

root list

roots							norms	
r_1	−4	−1	−27	−19	1	0	6	−2
r_2	−1	0	−6	−4	0	0	10	−7
r_3	0	0	−11	−8	−2	0	22	2
r_4	2	−1	3	2	0	0	6	−4

det $= -2005690 - 1418237w$ det norm $= -238$

$(2222)^2$

quadratic form

$$\begin{pmatrix} 338 & 239 & 12 & 8 & -198 & -140 \\ 12 & 8 & 78 & -58 & 33 & 25 \\ -198 & -140 & 33 & 25 & 34 & 24 \end{pmatrix}$$

root list

roots							norms	
r_1	0	0	−17	−12	−2	0	2458	1738
r_2	3	−2	0	0	0	0	10	7
r_3	18	−13	14	10	1	0	314	222
r_4	−22	14	20	14	1	0	34	24

det $= -50 - 37w$ det norm $= -238$

$(2222)^2$

quadratic form

$$\begin{pmatrix} 3042 & 1023 & 360 & -281 & 8 & 44 \\ 360 & -281 & 180 & -127 & -13 & 9 \\ 8 & 44 & -13 & 9 & 2 & 0 \end{pmatrix}$$

root list

	roots						norms	
r_1	1	−1	−19	−10	5	1	2	0
r_2	1	−4	−209	−138	37	25	74	50
r_3	−3	2	−2	−10	0	2	2	−1
r_4	−1	1	19	10	−4	−1	10	6

det $= -1342 - 949w$ det norm $= -238$

$(2222)^2$

quadratic form

$$\begin{pmatrix} 20342 & 14327 & 3284 & 2338 & 194 & 123 \\ 3284 & 2338 & 538 & 376 & 27 & 23 \\ 194 & 123 & 27 & 23 & 4 & -1 \end{pmatrix}$$

root list

	roots						norms	
r_1	4	2	−19	−16	−12	−8	34	24
r_2	5	4	−29	−19	−178	−126	3240	2291
r_3	−9	−6	52	38	44	31	58	41
r_4	−44	−31	261	185	369	261	256	181

det $= -298 - 211w$ det norm $= -238$

$(2222)^2$

quadratic form

$$\begin{pmatrix} 58258 & 41177 & -8316 & -5873 & 300 & 222 \\ -8316 & -5873 & 1188 & 837 & -41 & -33 \\ 300 & 222 & -41 & -33 & 6 & -2 \end{pmatrix}$$

Table A.6, cont.

root list

	roots						norms	
r_1	3	1	17	10	2	1	6	4
r_2	1	2	13	12	32	23	422	298
r_3	−5	−2	−30	−18	−6	−4	2	1
r_4	−22	−16	−158	−113	−51	−36	54	38

det = $-1738 - 1229w$ det norm = -238

$(2222)^2$

quadratic form

$$\begin{pmatrix} 10 & 7 & -4 & -4 & 14 & 10 \\ -4 & -4 & 18 & -14 & 1 & 3 \\ 14 & 10 & 1 & 3 & 6 & 4 \end{pmatrix}$$

root list

	roots						norms	
r_1	0	0	−3	−2	−2	0	54	38
r_2	1	−1	0	0	0	0	2	1
r_3	4	1	17	12	5	2	422	298
r_4	0	2	3	2	1	0	6	4

det = $-564 - 399w$ det norm = -306

$(6222)^2$

quadratic form

$$\begin{pmatrix} -4 & -5 & 8 & 10 & -4 & -5 \\ 8 & 10 & 2 & -8 & -1 & 4 \\ -4 & -5 & -1 & 4 & 2 & -1 \end{pmatrix}$$

Table A.6, cont.

root list

roots							norms	
r_1	0	0	-1	0	-2	0	6	4
r_2	1	1	2	2	3	3	102	72
r_3	0	0	0	0	3	2	10	7
r_4	-10	-7	-32	-23	69	48	1362	963

det $= -912 - 645w$ det norm $= -306$

$(2226)^2$

quadratic form

$$\left(\begin{array}{cc|cc|cc} -304 & -215 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 3 & 0 \\ 0 & 0 & 3 & 0 & 6 & 0 \end{array}\right)$$

root list

roots							norms	
r_1	2	0	-33	-23	13	9	6	4
r_2	3	3	-111	-78	37	26	378	267
r_3	1	-1	7	5	-3	-2	2	1
r_4	6	3	-177	-126	83	59	18	12

det $= -912 - 645w$ det norm $= -306$

$(6222)^2$

quadratic form

$$\left(\begin{array}{cc|cc|cc} 296 & 209 & -432 & -303 & -24 & -21 \\ -432 & -303 & 642 & 433 & 15 & 45 \\ -24 & -21 & 15 & 45 & 42 & -27 \end{array}\right)$$

root list

	roots						norms	
r_1	−3	0	−3	0	4	4	18	12
r_2	−3	3	−2	2	−1	1	2	0
r_3	−42	−24	0	0	−285	−202	66	45
r_4	−8	2	−2	1	−17	−11	2	−1

det $= -1424 - 1007w$ det norm $= -322$

$(8222)^2$

quadratic form

$$\begin{pmatrix} 236 & 153 & -100 & 26 & 82 & 67 \\ -100 & 26 & 482 & -334 & 31 & -41 \\ 82 & 67 & 31 & -41 & 38 & 21 \end{pmatrix}$$

root list

	roots						norms	
r_1	0	0	1	1	3	−1	10	7
r_2	0	1	7	5	0	1	34	24
r_3	0	0	17	12	4	3	314	222
r_4	−17	−13	64	45	52	38	372	263

det $= -380 - 269w$ det norm $= -322$

$(8222)^2$

quadratic form

$$\begin{pmatrix} 1712 & 1171 & -676 & -472 & 20 & 19 \\ -676 & -472 & 270 & 190 & -9 & -7 \\ 20 & 19 & -9 & -7 & 2 & -1 \end{pmatrix}$$

root list

	roots						norms	
r_1	0	−1	3	−4	12	8	6	4
r_2	0	0	0	0	1	1	2	1
r_3	31	20	64	33	−586	−415	8	5
r_4	71	50	133	93	−1409	−997	22	14

det $= -40 - 31w$ det norm $= -322$

$(2282)^2$

quadratic form

$$\begin{pmatrix} 2552 & 1803 & 30 & 29 & -52 & -38 \\ 30 & 29 & 42 & -29 & -7 & 4 \\ -52 & -38 & -7 & 4 & 2 & 0 \end{pmatrix}$$

root list

	roots						norms	
r_1	1	0	−37	−25	−6	4	6	−2
r_2	−1	2	−64	−45	0	0	8	−3
r_3	0	0	−1	1	−7	5	10	−7
r_4	0	0	0	0	−1	1	6	−4

det $= -2557304 - 1808287w$ det norm $= -322$

$(2228)^2$

quadratic form

$$\begin{pmatrix} 1492 & 1055 & -26 & -18 & -662 & -468 \\ -26 & -18 & 34 & -24 & -7 & -18 \\ -662 & -468 & -7 & -18 & -514 & -367 \end{pmatrix}$$

root list

roots							norms	
r_1	0	0	−13	−9	1	0	338	239
r_2	1	0	0	0	0	0	1492	1055
r_3	0	−1	91	64	−6	−3	4138	2926
r_4	0	−2	65	46	−4	−3	198	140

Table A.7: Decagons

det = −194040 − 137207w det norm = −98

2222222222

quadratic form

$$\left(\begin{array}{cc|cc|cc} -1055 & -746 & 0 & 0 & 0 & 0 \\ 0 & 0 & 10 & 7 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 & 3 \end{array}\right)$$

root list

	roots						norms	
r_1	0	1	−3	−2	−10	−7	58	41
r_2	1	2	0	0	−31	−22	2366	1673
r_3	−1	0	3	2	7	5	198	140
r_4	3	3	−31	−22	−31	−22	874	618
r_5	21	14	−157	−111	−205	−145	1830	1294
r_6	11	7	−78	−55	−109	−77	58	41
r_7	118	82	−861	−609	−1236	−874	2366	1673
r_8	3	1	−16	−11	−24	−17	17	12
r_9	4	1	−18	−13	−31	−22	75	53
r_{10}	1	2	−11	−8	−24	−17	157	111

det = −33292 − 23541w det norm = −98

2222222222

quadratic form

$$\left(\begin{array}{cc|cc|cc} -138 & -100 & -106 & -55 & -915 & -646 \\ -106 & -55 & 56 & -125 & -595 & -429 \\ -915 & -646 & -595 & -429 & -4795 & -3391 \end{array}\right)$$

root list

	roots						norms	
r_1	−104	−75	160	113	1	0	157	111
r_2	−25	−21	43	30	0	0	58	41
r_3	0	0	7	5	0	−1	2366	1673
r_4	29	18	−42	−30	0	0	198	140
r_5	−295	−208	434	307	4	2	874	618
r_6	−1487	−1049	2210	1563	14	10	1830	1294
r_7	−736	−521	1099	777	6	5	58	41
r_8	−8141	−5754	12159	8598	70	50	2366	1673
r_9	−151	−104	223	158	1	1	17	12
r_{10}	−175	−119	258	183	1	1	75	53

det $= -5712 - 4039w$ det norm $= -98$

2222222222

quadratic form

$$\left(\begin{array}{cc|cc|cc} 2878 & 2035 & 168 & 119 & -119 & -84 \\ 168 & 119 & 12 & 7 & -6 & -5 \\ -119 & -84 & -6 & -5 & 5 & 3 \end{array}\right)$$

root list

	roots						norms	
r_1	1	0	−3	−2	6	4	34	24
r_2	1	2	0	0	36	26	406	287
r_3	0	0	1	1	2	1	10	7
r_4	0	0	0	0	1	1	27	19
r_5	1	0	−5	−4	5	4	13	9
r_6	0	1	−7	−5	7	5	3	2
r_7	48	33	−455	−322	468	331	406	287
r_8	5	3	−44	−31	46	32	10	7
r_9	11	8	−103	−73	110	78	314	222
r_{10}	3	3	−31	−22	36	26	150	106

det $= -367086 - 259569w$	det norm $= -126$
2222222222	

quadratic form

$$\left(\begin{array}{cc|cc|cc} 18208 & 12875 & 816 & 577 & 239 & 169 \\ 816 & 577 & 58 & 39 & 10 & 7 \\ 239 & 169 & 10 & 7 & 3 & 2 \end{array}\right)$$

root list

	roots						norms	
r_1	1	−2	7	5	46	32	150	106
r_2	−6	3	7	5	48	34	174	123
r_3	0	0	0	0	1	1	17	12
r_4	0	0	−7	−5	24	17	4476	3165
r_5	−3	2	0	0	4	3	58	41
r_6	−39	−27	232	164	1830	1294	4476	3165
r_7	−9	−3	41	29	314	222	198	140
r_8	−24	−18	157	111	1174	830	1014	717
r_9	−4	−6	41	29	297	210	437	309
r_{10}	−2	0	7	5	48	34	58	41

det $= -10806 - 7641w$	det norm $= -126$
2222222222	

quadratic form

$$\left(\begin{array}{cc|cc|cc} 39936 & 28239 & 954 & 675 & 399 & 282 \\ 954 & 675 & 34 & 9 & 6 & 9 \\ 399 & 282 & 6 & 9 & 5 & 2 \end{array}\right)$$

root list

	roots						norms	
r_1	1	0	7	5	−74	−52	34	24
r_2	2	0	21	15	−162	−114	174	123
r_3	−3	−1	−34	−24	331	234	75	53
r_4	−6	−2	−75	−53	676	478	10	7
r_5	−57	−40	−981	−694	8730	6172	150	106
r_6	−182	−129	−3165	−2238	28044	19830	174	123
r_7	−37	−27	−655	−463	5789	4094	17	12
r_8	−820	−579	−14319	−10125	126282	89295	4476	3165
r_9	−7	−5	−126	−89	1090	771	58	41
r_{10}	−65	−45	−1188	−840	10038	7098	4476	3165

det $= -10806 - 7641w$ det norm $= -126$

2222222222

quadratic form

$$\begin{pmatrix} 4954 & 3503 & 642 & 454 & -26 & -18 \\ 642 & 454 & 6 & 4 & 1 & -2 \\ -26 & -18 & 1 & -2 & 27 & -19 \end{pmatrix}$$

root list

	roots						norms	
r_1	0	0	0	0	−7	−5	13	9
r_2	−7	5	0	2	34	24	30	21
r_3	0	0	1	0	14	10	6	4
r_4	−5	3	−1	2	−14	−10	132	93
r_5	7	−5	1	−1	−10	−7	2	1
r_6	−7	0	−32	−20	−1390	−983	132	93
r_7	11	−8	0	−2	−65	−46	1	0
r_8	7	−6	−4	−7	−320	−226	6	3
r_9	−16	11	−3	−1	−102	−72	6	2
r_{10}	−17	12	−6	4	−8	−6	2	−1

Table A.7, cont.

det $= -6 - 9w$ det norm $= -126$

2222222222

quadratic form

$$\left(\begin{array}{cc|cc|cc} 114 & 78 & -6 & -6 & -21 & -15 \\ -6 & -6 & 8 & -5 & 1 & 1 \\ -21 & -15 & 1 & 1 & 3 & 2 \end{array}\right)$$

root list

	roots						norms	
r_1	1	0	8	6	−6	4	10	−6
r_2	1	0	9	6	0	0	6	−3
r_3	0	0	0	0	3	−2	3	−2
r_4	−1	0	−21	−15	6	3	24	15
r_5	0	0	−1	−1	2	−1	2	−1
r_6	16	12	153	108	18	12	24	15
r_7	3	2	30	21	2	2	2	0
r_8	11	8	123	87	6	6	6	3
r_9	3	2	35	25	1	1	3	1
r_{10}	1	0	7	5	0	0	2	−1

det $= -1854 - 1311w$ det norm $= -126$

2222222222

quadratic form

$$\left(\begin{array}{cc|cc|cc} 21506 & 15207 & 318 & 225 & -349 & -247 \\ 318 & 225 & 6 & 3 & -6 & -3 \\ -349 & -247 & -6 & -3 & 7 & 3 \end{array}\right)$$

root list

	roots						norms	
r_1	1	0	0	0	28	20	10	7
r_2	6	3	31	22	318	225	768	543
r_3	0	0	1	1	1	1	3	2
r_4	0	0	−1	−1	0	0	30	21
r_5	−1	1	−5	−4	8	5	26	18
r_6	2	−1	−3	−2	14	10	2	1
r_7	4	1	−18	−13	137	97	13	9
r_8	12	9	−71	−50	636	450	30	21
r_9	3	3	−19	−13	188	133	6	4
r_{10}	21	18	−106	−75	1218	861	132	93

det $= -1854 - 1311w$ det norm $= -126$

2222222222

quadratic form

$$\left(\begin{array}{cc|cc|cc} 174 & 123 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 3 & 1 \\ 0 & 0 & 3 & 1 & 3 & 1 \end{array}\right)$$

root list

	roots						norms	
r_1	2	0	15	11	−22	−16	6	4
r_2	3	1	39	27	−60	−42	132	93
r_3	1	−1	−3	−2	4	3	2	1
r_4	21	14	264	186	−336	−237	132	93
r_5	1	1	16	11	−21	−14	1	0
r_6	9	4	96	69	−126	−90	6	3
r_7	2	3	41	30	−54	−40	6	2
r_8	1	0	8	4	−10	−6	2	−1
r_9	4	−1	18	13	−25	−18	3	1
r_{10}	3	2	42	30	−60	−42	6	3

det = $-1854 - 1311w$	det norm = -126

2222222222

quadratic form

$$\left(\begin{array}{cc|cc|cc} 6 & 3 & 6 & 6 & -42 & -30 \\ 6 & 6 & 14 & 4 & -45 & -31 \\ -42 & -30 & -45 & -31 & 3 & 2 \end{array}\right)$$

root list

	roots						norms	
r_1	2	3	-1	-3	14	-10	6	2
r_2	1	0	0	0	0	0	6	3
r_3	-2	-2	1	2	-7	5	1	0
r_4	-199	-141	144	102	0	3	132	93
r_5	-13	-10	9	7	-4	3	2	1
r_6	-1073	-759	717	507	6	6	132	93
r_7	-158	-112	105	74	2	0	6	4
r_8	-511	-361	336	237	6	0	30	21
r_9	-101	-71	65	46	1	0	13	9
r_{10}	-7	-5	4	3	0	0	2	1

det = $-4995 - 3532w$	det norm = -23

$(22222)^2$

quadratic form

$$\left(\begin{array}{cc|cc|cc} -502 & -355 & 0 & 0 & 25 & 18 \\ 0 & 0 & 3 & 2 & 1 & 1 \\ 25 & 18 & 1 & 1 & 3 & -2 \end{array}\right)$$

Table A.7, cont.

root list

	roots						norms	
r_1	0	0	0	0	−7	−5	17	12
r_2	2	−2	11	7	−25	−18	710	502
r_3	0	0	1	1	0	0	17	12
r_4	−1	1	−2	−1	4	3	58	41
r_5	5	3	−61	−43	0	0	355	251

det = $-4995 - 3532w$ det norm = -23

$(22222)^2$

quadratic form

$$\begin{pmatrix} 194485 & 137360 & -72819 & -51435 & 4032 & 2613 \\ -72819 & -51435 & 27265 & 19260 & -1505 & -982 \\ 4032 & 2613 & -1505 & -982 & 326 & -121 \end{pmatrix}$$

root list

	roots						norms	
r_1	57	30	157	79	29	20	58	41
r_2	130	96	351	261	93	66	355	251
r_3	−66	−51	−177	−139	−42	−30	17	12
r_4	−3405	−2412	−9215	−6529	−2206	−1560	710	502
r_5	−1009	−703	−2734	−1901	−654	−462	17	12

det = $-25 - 18w$ det norm = -23

$(22222)^2$

quadratic form

$$\begin{pmatrix} -192 & -141 & -18 & -24 & -5 & 1 \\ -18 & -24 & 15 & -14 & -7 & 5 \\ -5 & 1 & -7 & 5 & 3 & -2 \end{pmatrix}$$

root list

	roots						norms	
r_1	1	0	8	6	−64	−45	2	1
r_2	3	0	39	29	−280	−197	11	7
r_3	0	−1	−14	−10	105	73	1	0
r_4	−34	−27	−829	−584	5941	4203	22	14
r_5	−10	−8	−249	−176	1782	1259	1	0

det = $-29113 - 20586w$ det norm = -23

$(22222)^2$

quadratic form

$$\begin{pmatrix} 81 & -40 & -82 & 31 & 27 & 32 \\ -82 & 31 & 84 & -22 & -13 & -21 \\ 27 & 32 & -13 & -21 & -81 & -59 \end{pmatrix}$$

root list

	roots						norms	
r_1	3	1	3	1	1	0	58	41
r_2	5	−4	6	−4	0	0	17	12
r_3	−14	−9	−11	−7	−2	0	710	502
r_4	−3	−2	−3	−2	0	0	17	12
r_5	−26	−22	−31	−26	3	3	355	251

det = $-345 - 244w$ det norm = -47

$(22432)^2$

quadratic form

$$\begin{pmatrix} -345 & -244 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 \end{pmatrix}$$

Table A.7, cont.

root list

	roots						norms	
r_1	1	0	−13	−9	−10	−7	3	2
r_2	3	1	−59	−42	−59	−42	286	202
r_3	0	0	0	0	−1	−1	3	2
r_4	0	0	1	1	2	2	6	4
r_5	1	0	−10	−7	−1	−1	6	4

det $= -345 - 244w$ det norm $= -47$

$(42223)^2$

quadratic form

$$\begin{pmatrix} 983 & -362 & 510 & 254 & -18 & 78 \\ 510 & 254 & 810 & 562 & 79 & 65 \\ -18 & 78 & 79 & 65 & 14 & 2 \end{pmatrix}$$

root list

	roots						norms	
r_1	−6	−2	17	−8	−8	−3	6	4
r_2	3	1	−8	4	1	−1	3	2
r_3	218	152	−178	−106	619	436	286	202
r_4	43	30	−36	−22	138	97	3	2
r_5	176	124	−144	−97	599	424	6	4

det $= -59 - 42w$ det norm $= -47$

$(32224)^2$

quadratic form

$$\begin{pmatrix} 48999 & 34562 & 29829 & 21141 & 222 & 166 \\ 29829 & 21141 & 18243 & 12872 & 143 & 96 \\ 222 & 166 & 143 & 96 & 2 & 0 \end{pmatrix}$$

root list

	roots						norms	
r_1	25	16	−37	−29	−2	−2	6	4
r_2	0	0	0	0	1	1	6	4
r_3	−7	−6	14	8	4	3	3	2
r_4	−38	−29	66	43	143	101	286	202
r_5	11	6	−14	−13	23	16	3	2

det = $-59 - 42w$ det norm = -47

$(32224)^2$

quadratic form

$$\left(\begin{array}{cc|cc|cc} 359 & 246 & 18 & 16 & 9 & 8 \\ 18 & 16 & 2 & 0 & 1 & 0 \\ 9 & 8 & 1 & 0 & 1 & 0 \end{array}\right)$$

root list

	roots						norms	
r_1	1	0	−13	−11	1	0	2	0
r_2	0	0	1	0	−2	0	2	0
r_3	−3	−1	60	45	0	0	1	0
r_4	−106	−75	2953	2087	34	25	50	34
r_5	−19	−13	519	368	8	5	1	0

det = $-9751 - 6895w$ det norm = -49

$(22222)^2$

quadratic form

$$\left(\begin{array}{cc|cc|cc} 87 & 51 & 404 & 290 & -12 & -10 \\ 404 & 290 & 2080 & 1469 & -73 & -51 \\ -12 & -10 & -73 & -51 & 2 & 1 \end{array}\right)$$

root list

	roots						norms	
r_1	−12	−10	5	0	−2	−1	92	65
r_2	−3	2	−7	5	0	0	3	2
r_3	79	53	−10	−14	5	4	119	84
r_4	18	14	−6	−1	1	1	10	7
r_5	114	77	−16	−20	5	4	44	31

det $= -1673 - 1183w$ det norm $= -49$

$(22222)^2$

quadratic form

$$\begin{pmatrix} 1465 & 1011 & -424 & -286 & 182 & 126 \\ -424 & -286 & 124 & 80 & -53 & -36 \\ 182 & 126 & -53 & -36 & 16 & 11 \end{pmatrix}$$

root list

	roots						norms	
r_1	4	2	13	8	−1	1	10	7
r_2	1	2	7	7	3	1	119	84
r_3	−1	−1	−4	−3	−1	1	3	2
r_4	0	0	0	0	1	1	92	65
r_5	20	14	70	49	3	1	44	31

det $= -15 - 12w$ det norm $= -63$

$(82222)^2$

quadratic form

$$\begin{pmatrix} -15 & -12 & 0 & 0 & 0 & 0 \\ 0 & 0 & 6 & 4 & -1 & -3 \\ 0 & 0 & -1 & -3 & 18 & -11 \end{pmatrix}$$

root list

	roots						norms	
r_1	2	2	−17	−14	−44	−31	2	0
r_2	0	0	−5	4	1	0	2	−1
r_3	−72	−51	533	376	1285	909	4	−1
r_4	−242	−170	1773	1263	4299	3039	6	−3
r_5	−219	−155	1614	1144	3904	2760	3	−2

det = $-1137 - 804w$ det norm = -63

$(22222)^2$

quadratic form

$$\left(\begin{array}{cc|cc|cc} 2079 & 1464 & -648 & -456 & 114 & 81 \\ -648 & -456 & 202 & 142 & -35 & -25 \\ 114 & 81 & -35 & -25 & 2 & 1 \end{array}\right)$$

root list

	roots						norms	
r_1	−12	−10	−39	−32	−2	0	54	38
r_2	−1	0	−3	0	0	0	9	6
r_3	11	8	35	26	1	0	5	3
r_4	14	10	45	32	1	0	2	1
r_5	253	177	810	567	6	6	9	6

det = $-33 - 24w$ det norm = -63

$(22222)^2$

quadratic form

$$\left(\begin{array}{cc|cc|cc} -9 & -12 & -48 & 42 & -15 & -9 \\ -48 & 42 & 722 & -514 & 21 & -7 \\ -15 & -9 & 21 & -7 & 5 & 3 \end{array}\right)$$

root list

	roots						norms	
r_1	1	2	4	3	3	−2	2	−1
r_2	1	8	12	9	0	0	3	0
r_3	−6	2	−5	−3	2	−2	10	6
r_4	7	−7	−3	−3	0	0	3	0
r_5	0	0	0	0	−1	1	3	−1

det $= -3165 - 2238w$ det norm $= -63$

$(82222)^2$

quadratic form

$$\begin{pmatrix} 273 & 192 & -72 & -54 & 24 & 15 \\ -72 & -54 & 30 & 12 & -3 & -8 \\ 24 & 15 & -3 & -8 & 4 & -1 \end{pmatrix}$$

root list

	roots						norms	
r_1	0	0	−1	−1	−12	−8	34	24
r_2	2	−1	0	1	−1	−1	10	7
r_3	0	0	0	0	3	2	44	31
r_4	2	−3	−6	−3	−9	−6	30	21
r_5	−1	−1	−6	−4	−14	−10	3	2

det $= -6627 - 4686w$ det norm $= -63$

$(22222)^2$

quadratic form

$$\begin{pmatrix} 237 & -162 & -285 & 129 & 24 & -27 \\ -285 & 129 & 1009 & 382 & 113 & 124 \\ 24 & -27 & 113 & 124 & 38 & 21 \end{pmatrix}$$

root list

	roots						norms	
r_1	−1	−2	1	−1	3	−1	10	7
r_2	−4	−8	3	−3	1	1	27	19
r_3	36	26	3	3	−12	−6	51	36
r_4	175	126	15	13	−44	−30	314	222
r_5	399	284	36	27	−90	−66	51	36

det $= -5712 - 4039w$ det norm $= -98$

$(22222)^2$

quadratic form

$$\begin{pmatrix} 15620 & 11045 & -1642 & -1161 & 328 & 232 \\ -1642 & -1161 & 174 & 121 & -33 & -25 \\ 328 & 232 & -33 & -25 & 8 & 4 \end{pmatrix}$$

root list

	roots						norms	
r_1	−1	−2	13	9	150	106	54	38
r_2	4	−3	1	1	12	8	2	1
r_3	14	7	−81	−58	−944	−667	70	49
r_4	5	1	−23	−16	−258	−183	6	4
r_5	21	14	−149	−106	−1668	−1179	26	18

det $= -168 - 119w$ det norm $= -98$

$(22222)^2$

quadratic form

$$\begin{pmatrix} 908 & 557 & -148 & -138 & 112 & 70 \\ -148 & -138 & 44 & 18 & -19 & -17 \\ 112 & 70 & -19 & -17 & 10 & 6 \end{pmatrix}$$

root list

	roots						norms	
r_1	4	2	16	13	−1	1	6	4
r_2	1	2	14	7	3	1	70	49
r_3	−1	−1	−6	−4	−1	1	2	1
r_4	0	0	0	0	1	1	54	38
r_5	20	14	98	70	3	1	26	18

det $= -318 - 225w$ det norm $= -126$

$(82222)^2$

quadratic form

$$\begin{pmatrix} 1278 & 903 & -168 & -114 & 60 & 45 \\ -168 & -114 & 46 & -2 & 5 & -15 \\ 60 & 45 & 5 & -15 & 10 & -3 \end{pmatrix}$$

root list

	roots						norms	
r_1	0	0	1	1	5	3	10	7
r_2	0	1	7	5	4	3	34	24
r_3	−36	−26	−489	−346	−692	−489	150	106
r_4	−128	−89	−1689	−1194	−2346	−1659	102	72
r_5	−117	−82	−1546	−1093	−2138	−1512	10	7

det $= -54 - 39w$ det norm $= -126$

$(82222)^2$

quadratic form

$$\begin{pmatrix} -318 & -225 & 0 & 0 & 0 & 0 \\ 0 & 0 & 34 & -24 & 3 & -2 \\ 0 & 0 & 3 & -2 & 2 & 1 \end{pmatrix}$$

Table A.7, cont.

root list

	roots						norms	
r_1	0	1	−75	−53	1	0	2	1
r_2	0	0	3	2	−2	0	6	4
r_3	−72	−51	7491	5297	0	0	26	18
r_4	−242	−170	25053	17715	6	6	18	12
r_5	−219	−155	22752	16088	8	6	2	1

det = $-10806 - 7641w$ det norm = -126

$(22228)^2$

quadratic form

$$\begin{pmatrix} 2782 & 1967 & 1868 & 1318 & -470 & -327 \\ 1868 & 1318 & 1294 & 866 & -379 & -177 \\ -470 & -327 & -379 & -177 & 198 & -29 \end{pmatrix}$$

root list

	roots						norms	
r_1	0	0	10	6	39	27	58	41
r_2	−1	−1	10	7	34	24	58	41
r_3	−2	−3	7	6	12	9	594	420
r_4	0	0	3	2	10	7	874	618
r_5	6	3	7	5	58	41	198	140

det = $-182 - 129w$ det norm = -158

$(22322)^2$

quadratic form

$$\begin{pmatrix} 2550 & 1797 & -92 & -60 & -46 & -30 \\ -92 & -60 & 6 & 0 & 3 & 0 \\ -46 & -30 & 3 & 0 & 2 & 0 \end{pmatrix}$$

root list

	roots						norms	
r_1	1	0	12	8	2	1	2	1
r_2	5	2	99	68	0	0	76	53
r_3	0	0	1	0	−2	0	2	0
r_4	0	0	0	0	1	0	2	0
r_5	12	5	198	136	129	91	46	30

det $= -182 - 129w$ det norm $= -158$

$(22223)^2$

quadratic form

$$\left(\begin{array}{cc|cc|cc} 6878 & 2387 & -642 & 598 & 72 & -8 \\ -642 & 598 & 318 & -222 & -17 & 13 \\ 72 & -8 & -17 & 13 & 2 & 0 \end{array}\right)$$

root list

	roots						norms	
r_1	0	1	−37	−24	2	−2	6	4
r_2	11	9	−615	−432	0	0	440	311
r_3	−3	−2	148	105	0	1	10	7
r_4	−256	−181	13131	9285	23	15	1502	1062
r_5	−50	−36	2593	1832	3	2	34	24

det $= -6190 - 4377w$ det norm $= -158$

$(22223)^2$

quadratic form

$$\left(\begin{array}{cc|cc|cc} 43214 & 30203 & 80132 & 56624 & -1360 & -990 \\ 80132 & 56624 & 149358 & 105608 & -2571 & -1821 \\ -1360 & -990 & -2571 & -1821 & 46 & 30 \end{array}\right)$$

root list

	roots						norms	
r_1	−16	−10	17	0	23	16	34	24
r_2	−23	−15	25	6	269	190	2564	1813
r_3	39	27	−26	−13	−82	−58	58	41
r_4	1942	1373	−1161	−819	−6494	−4592	8754	6190
r_5	334	236	−203	−142	−1256	−888	198	140

det = $-36078 - 25511w$ det norm = -158

$(22223)^2$

quadratic form

$$\begin{pmatrix} 2141862 & 1514491 & -1045076 & -738989 & 5282 & 3723 \\ -1045076 & -738989 & 509936 & 360577 & -2571 & -1821 \\ 5282 & 3723 & -2571 & -1821 & 16 & 7 \end{pmatrix}$$

root list

	roots						norms	
r_1	−11	−6	−15	−17	84	59	34	24
r_2	−7	−8	−25	−6	380	269	1502	1062
r_3	19	11	28	29	−170	−120	10	7
r_4	470	334	942	659	−5383	−3806	440	311
r_5	31	20	54	45	−363	−256	6	4

det = $-182 - 129w$ det norm = -158

$(22223)^2$

quadratic form

$$\begin{pmatrix} 430 & 277 & 76 & -105 & -46 & -30 \\ 76 & -105 & 664 & -465 & -7 & 10 \\ -46 & -30 & -7 & 10 & 6 & 4 \end{pmatrix}$$

Table A.7, cont.

root list

	roots						norms	
r_1	-1	2	11	7	14	-10	2	0
r_2	7	2	53	38	0	0	46	30
r_3	3	-2	0	1	-14	10	2	-1
r_4	-1	1	0	0	-11	10	16	7
r_5	0	0	0	0	3	-2	6	-4

det $= 2 - 11w$ det norm $= -238$

$(22222)^2$

quadratic form

$$\left(\begin{array}{cc|cc|cc} 126 & 89 & 0 & 0 & 0 & 0 \\ 0 & 0 & 82 & -58 & 27 & -19 \\ 0 & 0 & 27 & -19 & 10 & -6 \end{array}\right)$$

root list

	roots						norms	
r_1	-2	-3	267	189	-30	-22	14	8
r_2	1	-1	18	13	-2	-2	2	-1
r_3	0	0	1	1	0	-1	2	0
r_4	3	-2	0	0	0	0	6	1
r_5	0	0	0	0	1	1	6	2

det $= -1342 - 949w$ det norm $= -238$

$(22222)^2$

quadratic form

$$\left(\begin{array}{cc|cc|cc} 79438 & 56159 & 65040 & 45996 & -310 & -216 \\ 65040 & 45996 & 53266 & 37662 & -251 & -179 \\ -310 & -216 & -251 & -179 & 2 & 0 \end{array}\right)$$

Table A.7, cont.

root list

	roots						norms	
r_1	−36	−26	45	31	−2	−2	26	18
r_2	−107	−72	122	94	0	0	22	15
r_3	36	26	−45	−31	1	1	6	4
r_4	551	391	−676	−475	2	1	2	1
r_5	8054	5697	−9837	−6952	15	11	74	52

det $= -11690018 - 8266091w$ det norm $= -238$

$(22222)^2$

quadratic form

$$\begin{pmatrix} 250634 & 177225 & -16492 & -11662 & -2052 & -1451 \\ -16492 & -11662 & 1206 & 696 & 141 & 93 \\ -2052 & -1451 & 141 & 93 & 16 & 11 \end{pmatrix}$$

root list

	roots						norms	
r_1	0	0	−17	−12	108	76	1550	1096
r_2	3	−2	0	0	8	6	58	41
r_3	−31	22	3	2	−13	−9	198	140
r_4	−17	0	−222	−157	624	441	2646	1871
r_5	−9	3	−65	−46	192	136	536	379

det $= -10130 - 7163w$ det norm $= -238$

$(22222)^2$

quadratic form

$$\begin{pmatrix} 7378 & 5217 & 768 & 542 & 146 & 103 \\ 768 & 542 & 106 & 40 & 21 & 7 \\ 146 & 103 & 21 & 7 & 4 & 1 \end{pmatrix}$$

root list

	roots						norms	
r_1	0	0	−3	−2	12	8	34	24
r_2	1	−1	0	0	8	6	10	7
r_3	−3	1	6	4	9	7	266	188
r_4	0	0	0	0	3	2	92	65
r_5	5	1	−44	−31	78	55	454	321

det $= 2 - 11w$ det norm $= -238$

$(22222)^2$

quadratic form

$$\begin{pmatrix} 2254 & 1543 & 8662 & 6084 & -174 & -114 \\ 8662 & 6084 & 33626 & 23742 & -657 & -458 \\ -174 & -114 & -657 & -458 & 14 & 8 \end{pmatrix}$$

root list

	roots						norms	
r_1	6	2	−3	0	−16	−16	10	−6
r_2	−7	−4	0	0	−90	−58	14	−9
r_3	−4	1	3	−1	19	21	6	−4
r_4	25	19	2	1	440	315	10	−7
r_5	402	288	21	12	6497	4603	10	−4

det $= -96 - 69w$ det norm $= -306$

$(82222)^2$

quadratic form

$$\begin{pmatrix} 420 & 291 & 108 & 72 & 30 & 12 \\ 108 & 72 & 30 & 18 & 11 & 1 \\ 30 & 12 & 11 & 1 & 14 & -8 \end{pmatrix}$$

root list

	roots						norms	
r_1	0	0	−1	−1	4	3	6	4
r_2	−1	1	0	−2	3	2	2	1
r_3	0	0	0	0	3	2	46	32
r_4	−1	0	0	0	9	6	6	3
r_5	−5	−2	−4	−5	85	60	14	9

det = $-24 - 21w$ det norm = -306

$(22822)^2$

quadratic form

$$\left(\begin{array}{cc|cc|cc} -156 & -111 & 0 & 0 & 0 & 0 \\ 0 & 0 & 34 & -24 & 3 & -2 \\ 0 & 0 & 3 & -2 & 2 & 1 \end{array}\right)$$

root list

	roots						norms	
r_1	1	0	−39	−27	6	−3	6	−3
r_2	1	0	−37	−26	0	0	6	1
r_3	0	0	1	1	2	−2	2	0
r_4	0	0	0	0	−1	1	2	−1
r_5	3	2	−230	−163	9	10	14	8

det = $-12 - 15w$ det norm = -306

$(82222)^2$

quadratic form

$$\left(\begin{array}{cc|cc|cc} 1092 & 651 & -36 & 57 & -24 & -30 \\ -36 & 57 & 42 & -29 & -7 & 4 \\ -24 & -30 & -7 & 4 & 2 & 0 \end{array}\right)$$

root list

	roots						norms	
r_1	1	0	−23	−16	−6	4	2	0
r_2	0	0	1	0	3	−2	2	−1
r_3	−1	0	23	16	7	−4	10	4
r_4	0	1	−33	−24	9	−6	6	−3
r_5	8	6	−385	−273	7	−4	6	−1

det = $-180588 - 127695w$ det norm = -306

$(22228)^2$

quadratic form

$$\begin{pmatrix} 96 & -33 & -612 & -504 & -6 & -18 \\ -612 & -504 & 21546 & 15185 & 537 & 370 \\ -6 & -18 & 537 & 370 & 14 & 8 \end{pmatrix}$$

root list

	roots						norms	
r_1	4	2	0	1	−31	−21	34	24
r_2	−9	7	14	−9	−31	−22	126	89
r_3	−11	−11	−6	0	129	93	30	21
r_4	−70	−40	−6	−18	691	490	74	52
r_5	−43	−9	14	−20	317	222	2	1

det = $-5316 - 3759w$ det norm = -306

$(82222)^2$

quadratic form

$$\begin{pmatrix} 20688 & 13857 & -11322 & -7773 & 5502 & 3921 \\ -11322 & -7773 & 6230 & 4335 & -3053 & -2168 \\ 5502 & 3921 & -3053 & -2168 & 1522 & 1075 \end{pmatrix}$$

Table A.7, cont.

root list

	roots						norms	
r_1	−24	−18	−42	−33	1	0	10	7
r_2	−9	−6	−17	−11	−2	0	34	24
r_3	3552	2510	6407	4525	0	0	126	89
r_4	3528	2495	6363	4500	0	3	30	21
r_5	9905	7005	17865	12636	8	7	74	52

det = $-30 - 25w$ det norm = -350

$(22222)^2$

quadratic form

$$\begin{pmatrix} 3150 & 2227 & -1340 & -966 & -160 & -110 \\ -1340 & -966 & 1278 & -80 & -55 & 135 \\ -160 & -110 & -55 & 135 & 30 & -10 \end{pmatrix}$$

root list

	roots						norms	
r_1	0	0	−7	−5	60	42	2	0
r_2	5	−5	−50	−35	400	284	10	5
r_3	5	−4	−5	−3	31	24	4	1
r_4	−3	2	0	0	−2	−1	2	−1
r_5	0	0	0	0	1	1	50	30

det = $-190 - 135w$ det norm = -350

$(22222)^2$

quadratic form

$$\begin{pmatrix} 58 & 41 & 0 & 0 & 0 & 0 \\ 0 & 0 & 382 & -270 & -179 & 125 \\ 0 & 0 & -179 & 125 & 88 & -55 \end{pmatrix}$$

Table A.7, cont.

root list

	roots						norms	
r_1	-1	-2	43	30	14	9	16	11
r_2	-5	0	64	45	22	15	50	35
r_3	1	0	-10	-7	-3	-2	6	4
r_4	-215	-150	4088	2891	979	693	1570	1110
r_5	-27	-20	530	375	128	91	10	7

det $= -37710 - 26665w$ det norm $= -350$

$(22222)^2$

quadratic form

$$\begin{pmatrix} 129450 & 91515 & -31620 & -22355 & 600 & 430 \\ -31620 & -22355 & 7724 & 5461 & -147 & -105 \\ 600 & 430 & -147 & -105 & 4 & 1 \end{pmatrix}$$

root list

	roots						norms	
r_1	3	1	13	4	12	8	34	24
r_2	5	3	25	15	170	120	9150	6470
r_3	-1	-1	-4	-4	6	4	58	41
r_4	0	0	0	0	7	5	536	379
r_5	41	29	170	120	120	85	1690	1195

det $= -6470 - 4575w$ det norm $= -350$

$(22222)^2$

quadratic form

$$\begin{pmatrix} 58 & 41 & 2 & 2 & 62 & 44 \\ 2 & 2 & 34 & -24 & 5 & -9 \\ 62 & 44 & 5 & -9 & -124 & -91 \end{pmatrix}$$

root list

	roots						norms	
r_1	0	0	−13	−9	1	0	92	65
r_2	1	−1	0	0	0	0	10	7
r_3	4	1	129	91	−8	−5	1570	1110
r_4	4	−2	11	8	0	−1	6	4
r_5	135	93	1680	1188	−118	−84	50	35

det = $-6470 - 4575w$ det norm = -350

$(22222)^2$

quadratic form

$$\left(\begin{array}{cc|cc|cc} 12950 & 9155 & 1080 & 775 & -190 & -140 \\ 1080 & 775 & 136 & 33 & -39 & 4 \\ -190 & -140 & -39 & 4 & 14 & -6 \end{array}\right)$$

root list

	roots						norms	
r_1	1	−2	11	7	−2	−2	34	24
r_2	−3	−4	50	35	0	0	290	205
r_3	3	−2	0	−1	5	3	92	65
r_4	−1	1	−2	−1	4	3	10	7
r_5	−3	3	5	5	55	40	1570	1110

det = $-78 - 57w$ det norm = -414

$(22262)^2$

quadratic form

$$\left(\begin{array}{cc|cc|cc} 10 & 7 & 0 & 0 & 0 & 0 \\ 0 & 0 & 24 & -17 & -3 & 4 \\ 0 & 0 & -3 & 4 & 12 & 7 \end{array}\right)$$

Table A.7, cont.

root list

	roots						norms	
r_1	−5	−1	−45	−32	1	0	12	7
r_2	1	−1	−4	−3	0	0	2	−1
r_3	0	0	−19	−13	−2	1	36	21
r_4	3	−2	1	1	0	0	6	−4
r_5	−12	−9	−119	−84	5	3	6	0

det = $-91518 - 64713w$ det norm = -414

$(22226)^2$

quadratic form

$$\left(\begin{array}{cc|cc|cc} 11906 & 8123 & -17672 & -12202 & -636 & -486 \\ -17672 & -12202 & 26282 & 18292 & 963 & 717 \\ -636 & -486 & 963 & 717 & 36 & 21 \end{array}\right)$$

root list

	roots						norms	
r_1	30	18	21	12	−20	−11	102	72
r_2	11	−7	11	−7	−8	−5	64	45
r_3	−21	−6	−16	−3	14	5	2	1
r_4	−450	−297	−315	−201	349	250	192	135
r_5	−21	−4	−17	−1	11	9	2	0

det = $-6 - 15w$ det norm = -414

$(22226)^2$

quadratic form

$$\left(\begin{array}{cc|cc|cc} 1590 & 1123 & 72 & 42 & 38 & 26 \\ 72 & 42 & 62 & -40 & 3 & 0 \\ 38 & 26 & 3 & 0 & 2 & 0 \end{array}\right)$$

root list

	roots						norms	
r_1	−6	2	39	27	2	−2	6	0
r_2	−3	1	19	13	0	0	8	−3
r_3	7	−5	0	1	−4	3	10	−7
r_4	−3	2	0	0	5	1	24	−9
r_5	0	0	0	0	3	−2	34	−24

det = $-3108918 - 2198337w$ det norm = -414

$(22262)^2$

quadratic form

$$\begin{pmatrix} -17434 & -12331 & -15438 & -11316 & -13338 & -9429 \\ -15438 & -11316 & 20078 & -34206 & -12153 & -8304 \\ -13338 & -9429 & -12153 & -8304 & -9972 & -7053 \end{pmatrix}$$

root list

	roots						norms	
r_1	48	51	−93	−66	43	14	6504	4599
r_2	1	−1	0	0	2	−1	58	41
r_3	−129	−81	191	135	−70	−45	2168	1533
r_4	−300	−207	471	333	−170	−121	3462	2448
r_5	−183	−120	283	200	−101	−77	198	140

det = $-462 - 327w$ det norm = -414

$(22226)^2$

quadratic form

$$\begin{pmatrix} 1746 & 429 & -756 & -276 & -96 & -102 \\ -756 & -276 & 338 & 156 & 51 & 47 \\ -96 & -102 & 51 & 47 & 12 & 7 \end{pmatrix}$$

Table A.7, cont.

root list

	roots						norms	
r_1	4	2	9	4	−4	−1	6	4
r_2	7	6	21	18	−54	−39	1116	789
r_3	−3	−2	−6	−4	0	−1	10	7
r_4	0	0	0	0	3	2	372	263
r_5	55	39	114	81	−9	−6	594	420

det = $-236028 - 166897w$ det norm = -434

$(22228)^2$

quadratic form

$$\begin{pmatrix} 16992 & 12015 & -5424 & -3832 & 448 & 316 \\ -5424 & -3832 & 1786 & 1192 & -155 & -93 \\ 448 & 316 & -155 & -93 & 14 & 6 \end{pmatrix}$$

root list

	roots						norms	
r_1	0	0	−3	−2	−46	−32	198	140
r_2	13	10	0	0	−656	−464	3800	2687
r_3	−4	0	7	5	207	146	874	618
r_4	−264	−184	79	56	13903	9831	2226	1574
r_5	−155	−112	41	29	8213	5808	58	41

det = $-2496 - 1765w$ det norm = -434

$(82222)^2$

quadratic form

$$\begin{pmatrix} 760 & 393 & -1220 & -830 & 36 & 23 \\ -1220 & -830 & 2194 & 1544 & -63 & -44 \\ 36 & 23 & -63 & -44 & 2 & 1 \end{pmatrix}$$

root list

	roots						norms	
r_1	−2	−2	1	−2	14	10	6	4
r_2	0	0	0	0	1	0	2	1
r_3	3	2	2	1	4	3	112	79
r_4	−1	1	−5	4	5	3	54	38
r_5	−55	−38	−21	−11	365	258	382	270

det = $-204 - 145w$ det norm = -434

$(82222)^2$

quadratic form

$$\begin{pmatrix} 2388 & 1685 & 240 & 164 & -64 & -54 \\ 240 & 164 & 30 & 12 & 3 & -12 \\ -64 & -54 & 3 & -12 & 18 & -10 \end{pmatrix}$$

root list

	roots						norms	
r_1	0	0	−1	−1	−6	−4	6	4
r_2	−1	1	−2	−2	−1	−1	2	1
r_3	0	0	0	0	3	2	66	46
r_4	0	−2	10	8	−13	−9	26	18
r_5	−33	−23	224	158	−430	−304	112	79

det = $-428 - 303w$ det norm = -434

$(22228)^2$

quadratic form

$$\begin{pmatrix} -428 & -303 & 0 & 0 & 0 & 0 \\ 0 & 0 & 6 & 4 & -1 & -3 \\ 0 & 0 & -1 & -3 & 18 & -11 \end{pmatrix}$$

root list

	roots						norms	
r_1	−1	−1	−49	−35	−116	−82	10	7
r_2	−7	−5	−270	−191	−652	−461	652	461
r_3	13	9	520	368	1229	869	314	222
r_4	625	442	24802	17538	59035	41744	2226	1574
r_5	209	148	8290	5862	19741	13959	34	24

det $= -32 - 27w$ det norm $= -434$

$(82222)^2$

quadratic form

$$\left(\begin{array}{cc|cc|cc} 4760 & 3311 & -40 & 129 & -64 & -54 \\ -40 & 129 & 326 & -229 & -19 & 12 \\ -64 & -54 & -19 & 12 & 2 & 0 \end{array}\right)$$

root list

	roots						norms	
r_1	1	0	−33	−23	−8	5	2	0
r_2	0	0	1	0	9	−6	2	−1
r_3	−1	0	33	23	9	−5	14	6
r_4	1	1	−83	−58	−9	4	6	2
r_5	41	29	−2766	−1956	−52	−40	20	13

det $= -428 - 303w$ det norm $= -434$

$(22822)^2$

quadratic form

$$\left(\begin{array}{cc|cc|cc} 26220 & 18531 & 130 & 69 & -172 & -118 \\ 130 & 69 & 42 & -29 & -7 & 4 \\ -172 & -118 & -7 & 4 & 2 & 0 \end{array}\right)$$

root list

	roots						norms	
r_1	−1	2	−211	−149	−6	4	10	6
r_2	1	0	−112	−79	0	0	20	13
r_3	−3	2	20	14	−7	5	2	−1
r_4	1	0	−117	−83	3	−2	2	0
r_5	447	314	−103871	−73448	21	9	66	46

det $= -504 - 357w$ det norm $= -882$

$(82222)^2$

quadratic form

$$\begin{pmatrix} 8428 & 4963 & 588 & 140 & 210 & 84 \\ 588 & 140 & 74 & -24 & 21 & -3 \\ 210 & 84 & 21 & -3 & 6 & 0 \end{pmatrix}$$

root list

	roots						norms	
r_1	0	0	−1	−1	1	2	6	4
r_2	−1	1	0	−6	1	0	2	1
r_3	−18	−12	420	294	−139	−97	18	12
r_4	−102	−75	2466	1754	−801	−566	10	6
r_5	−1320	−933	31356	22170	−10127	−7158	78	54

det $= -2940 - 2079w$ det norm $= -882$

$(82222)^2$

quadratic form

$$\begin{pmatrix} 6372 & 4505 & -42 & -42 & 3854 & 2724 \\ -42 & -42 & 162 & -114 & -15 & -33 \\ 3854 & 2724 & -15 & -33 & 2338 & 1651 \end{pmatrix}$$

root list

	roots						norms	
r_1	0	0	31	22	1	0	10	7
r_2	0	1	13	9	−2	0	34	24
r_3	−264	−186	−27983	−19787	0	0	2622	1854
r_4	−268	−190	−27921	−19743	10	6	314	222
r_5	−1302	−921	−134571	−95156	60	42	594	420

Table A.8: Undecagons

det $= -13114 - 9273w$	det norm $= -62$
22242222222	

quadratic form

$$\left(\begin{array}{cc|cc|cc} 94 & 66 & 38 & 28 & -19 & -14 \\ 38 & 28 & 14 & 7 & -2 & 0 \\ -19 & -14 & -2 & 0 & 1 & 0 \end{array}\right)$$

root list

	roots						norms	
r_1	1	0	−2	−1	−2	0	34	24
r_2	1	1	0	0	0	0	546	386
r_3	0	0	1	0	2	0	10	7
r_4	0	0	0	0	1	1	3	2
r_5	−1	1	−2	−1	4	2	6	4
r_6	2	−1	−3	−1	0	4	2	1
r_7	4	1	−19	−14	19	14	47	33
r_8	1	0	−4	−2	1	3	3	2
r_9	6	6	−47	−33	28	19	160	113
r_{10}	1	0	−3	−2	0	1	10	7
r_{11}	5	2	−19	−14	0	0	160	113

det $= -2250 - 1591w$	det norm $= -62$
22222422222	

quadratic form

$$\left(\begin{array}{cc|cc|cc} 6224 & 4401 & 94 & 66 & 226 & 160 \\ 94 & 66 & 8 & -3 & 1 & 4 \\ 226 & 160 & 1 & 4 & 9 & 5 \end{array}\right)$$

root list

	roots						norms	
r_1	0	1	−3	−2	−22	−15	10	7
r_2	1	4	0	0	−108	−76	160	113
r_3	−2	0	3	2	31	22	3	2
r_4	−38	−28	66	47	1225	867	47	33
r_5	−9	−8	16	11	322	227	2	1
r_6	−23	−16	33	23	724	511	6	4
r_7	−23	−14	28	20	679	481	3	2
r_8	−49	−36	62	44	1588	1123	10	7
r_9	−423	−297	499	353	13410	9483	546	386
r_{10}	−47	−33	51	36	1492	1055	34	24
r_{11}	−282	−198	273	193	8966	6340	932	659

det $= -10 - 9w$ det norm $= -62$

22222422222

quadratic form

$$\begin{pmatrix} 80 & 41 & -28 & 12 & 7 & -3 \\ -28 & 12 & 48 & -33 & -12 & 8 \\ 7 & -3 & -12 & 8 & 3 & -2 \end{pmatrix}$$

root list

	roots						norms	
r_1	1	0	18	13	12	11	2	1
r_2	2	2	113	80	94	66	28	19
r_3	0	−1	−28	−20	−23	−15	1	0
r_4	−27	−20	−1186	−838	−959	−674	9	5
r_5	−6	−6	−313	−222	−254	−179	2	−1
r_6	−17	−11	−709	−502	−572	−409	2	0
r_7	−15	−11	−671	−474	−547	−384	1	0
r_8	−36	−25	−1573	−1112	−1280	−905	2	1
r_9	−301	−213	−13321	−9419	−10854	−7673	94	66
r_{10}	−33	−24	−1489	−1053	−1216	−859	6	4
r_{11}	−201	−142	−9000	−6364	−7362	−5206	160	113

det $= -22626 - 15999w$ det norm $= -126$

22242222222

quadratic form

$$\begin{pmatrix} 58 & 41 & 38 & 27 & -24 & -17 \\ 38 & 27 & 24 & 13 & -5 & -3 \\ -24 & -17 & -5 & -3 & 3 & 2 \end{pmatrix}$$

root list

	roots						norms	
r_1	3	0	−3	−2	−2	0	102	72
r_2	−1	1	0	0	0	0	10	7
r_3	−15	−9	24	17	5	3	471	333
r_4	−5	−3	7	5	1	1	17	12
r_5	−29	−21	41	29	6	4	34	24
r_6	−651	−462	895	633	120	86	942	666
r_7	−79	−54	106	75	14	10	10	7
r_8	−117	−81	157	111	21	14	51	36
r_9	−93	−67	126	89	16	11	92	65
r_{10}	−4	−5	7	5	0	1	10	7
r_{11}	−6	−5	7	5	0	0	92	65

det $= -22626 - 15999w$ det norm $= -126$

22242222222

quadratic form

$$\left(\begin{array}{cc|cc|cc} -140 & -99 & 0 & 0 & 51 & 36 \\ 0 & 0 & 30 & 18 & 15 & 9 \\ 51 & 36 & 15 & 9 & -9 & -8 \end{array}\right)$$

Table A.8, cont.

root list

	roots						norms	
r_1	0	-3	3	2	-9	-6	51	36
r_2	-7	4	1	1	-2	-2	10	7
r_3	0	0	3	2	0	0	942	666
r_4	6	-3	-1	-1	2	2	34	24
r_5	-2	3	-3	-2	-1	-1	17	12
r_6	15	9	-55	-39	-81	-57	471	333
r_7	-5	6	-9	-6	-16	-12	10	7
r_8	0	3	-17	-12	-42	-30	102	72
r_9	5	-4	-8	-6	-32	-22	92	65
r_{10}	-1	0	0	0	-4	-3	10	7
r_{11}	1	-5	3	2	-16	-11	92	65

$\det = -131874 - 93249w$ det norm $= -126$

22222422222

quadratic form

$$\begin{pmatrix} 174538 & 123417 & 2940 & 2079 & -1359 & -961 \\ 2940 & 2079 & 54 & 33 & -24 & -15 \\ -1359 & -961 & -24 & -15 & 11 & 7 \end{pmatrix}$$

root list

	roots						norms	
r_1	1	0	14	10	90	64	58	41
r_2	0	1	27	19	146	103	536	379
r_3	0	0	3	2	9	6	297	210
r_4	0	0	−3	−2	−6	−4	58	41
r_5	3	0	−27	−19	114	81	5490	3882
r_6	1	0	3	2	64	45	198	140
r_7	2	1	31	22	267	189	99	70
r_8	39	27	785	555	6237	4410	2745	1941
r_9	6	5	137	97	1066	754	58	41
r_{10}	15	9	301	213	2286	1617	594	420
r_{11}	9	5	184	130	1348	953	536	379

det $= -131874 - 93249w$ det norm $= -126$

22222222242

quadratic form

$$\left(\begin{array}{cc|cc|cc} 1492 & 1055 & 420 & 297 & 123 & 87 \\ 420 & 297 & 118 & 83 & 30 & 21 \\ 123 & 87 & 30 & 21 & 9 & 6 \end{array}\right)$$

root list

roots							norms	
r_1	−3	−6	21	15	−2	−2	942	666
r_2	−2	1	1	1	0	0	10	7
r_3	0	0	0	0	1	1	51	36
r_4	0	0	−1	−1	4	3	92	65
r_5	1	−1	0	0	2	1	10	7
r_6	−3	−6	10	7	20	14	92	65
r_7	−9	−9	21	15	32	22	102	72
r_8	−8	−2	11	8	14	10	10	7
r_9	−33	−24	72	51	79	56	471	333
r_{10}	−2	−1	4	3	3	2	17	12
r_{11}	1	−2	3	2	0	0	34	24

det $= -114 - 81w$ det norm $= -126$

22222222242

quadratic form

$$\begin{pmatrix} 26 & 18 & -6 & -4 & -21 & -15 \\ -6 & -4 & 4 & -1 & 3 & 3 \\ -21 & -15 & 3 & 3 & 9 & 6 \end{pmatrix}$$

Table A.8, cont.

root list

	roots						norms	
r_1	3	0	6	6	−6	4	18	−6
r_2	3	−2	1	0	0	0	10	−7
r_3	0	0	0	0	3	−2	9	−6
r_4	−3	2	−1	−1	2	−1	8	−5
r_5	0	0	−1	0	−4	3	10	−7
r_6	0	1	−5	−2	0	2	8	−5
r_7	3	0	−6	−3	−4	6	18	−12
r_8	3	−1	−1	−2	2	0	10	−7
r_9	6	3	−9	−6	7	3	9	−3
r_{10}	2	−1	0	0	−1	1	3	−2
r_{11}	−1	1	0	1	0	0	6	−4

Table A.9: Dodecagons

det $= -461 - 326w$	det norm $= -31$
$(222222)^2$	

quadratic form

$$\left(\begin{array}{cc|cc|cc} 8 & -1 & 8 & -1 & -13 & -10 \\ 8 & -1 & 7 & -2 & -3 & -3 \\ -13 & -10 & -3 & -3 & 3 & 2 \end{array}\right)$$

root list

	roots						norms	
r_1	1	0	−2	0	4	−3	2	1
r_2	3	2	0	0	0	0	112	79
r_3	0	0	1	1	2	−1	3	2
r_4	−2	−2	13	10	7	3	382	270
r_5	0	0	0	0	1	0	3	2
r_6	15	11	−46	−33	13	10	191	135

det $= -91279 - 64544w$	det norm $= -31$
$(222222)^2$	

quadratic form

$$\left(\begin{array}{cc|cc|cc} 20737 & 13792 & -26349 & -17772 & -3511 & -2471 \\ -26349 & -17772 & 33531 & 22862 & 4488 & 3162 \\ -3511 & -2471 & 4488 & 3162 & -977 & -691 \end{array}\right)$$

root list

	roots						norms	
r_1	−634	−440	−497	−343	1	0	652	461
r_2	−7	−14	−4	−12	0	0	17	12
r_3	1969	1393	1539	1089	2	−2	2226	1574
r_4	342	244	267	191	0	0	17	12
r_5	14272	10089	11158	7887	1	3	1113	787
r_6	1961	1381	1534	1079	1	0	58	41

det $= -461 - 326w$ det norm $= -31$

$(222222)^2$

quadratic form

$$\left(\begin{array}{cc|cc|cc} 112 & 79 & 0 & 0 & 13 & 10 \\ 0 & 0 & -1 & -1 & 2 & 1 \\ 13 & 10 & 2 & 1 & 3 & -2 \end{array}\right)$$

root list

	roots						norms	
r_1	0	0	0	0	−3	−2	3	2
r_2	1	2	−13	−10	−33	−23	382	270
r_3	−1	1	−1	−1	−2	−1	3	2
r_4	1	0	0	0	0	0	112	79
r_5	−3	2	2	0	0	1	2	1
r_6	−4	−2	33	23	0	0	33	23

det $= -15661 - 11074w$ det norm $= -31$

$(222222)^2$

quadratic form

$$\left(\begin{array}{cc|cc|cc} -816 & -577 & 0 & 0 & 41 & 29 \\ 0 & 0 & 3 & 2 & 1 & 1 \\ 41 & 29 & 1 & 1 & 3 & -2 \end{array}\right)$$

root list

	roots						norms	
r_1	0	0	0	0	−7	−5	17	12
r_2	2	−2	17	12	−41	−29	2226	1574
r_3	0	0	1	1	0	0	17	12
r_4	1	0	−4	−3	10	7	652	461
r_5	3	−2	−2	−1	0	0	10	7
r_6	3	1	−51	−36	−68	−48	191	135

Table A.9, cont.

det $= -15661 - 11074w$ det norm $= -31$

$(222222)^2$

quadratic form

$$\left(\begin{array}{cc|cc|cc} 1439 & 1014 & -18657 & -13189 & 1372 & 970 \\ -18657 & -13189 & 242551 & 171506 & -17849 & -12621 \\ 1372 & 970 & -17849 & -12621 & 1314 & 929 \end{array}\right)$$

root list

	roots						norms	
r_1	−18	−13	−4	−3	−37	−26	10	7
r_2	−138	−93	−32	−18	−249	−176	652	461
r_3	−9	−6	−2	−1	−14	−10	17	12
r_4	−31	−22	−7	−5	−64	−45	2226	1574
r_5	−12	−5	−5	0	−24	−17	17	12
r_6	−844	−597	−209	−148	−1957	−1384	1113	787

det $= -11 - 9w$ det norm $= -41$

$(228224)^2$

quadratic form

$$\left(\begin{array}{cc|cc|cc} -69 & -49 & 0 & 0 & 0 & 0 \\ 0 & 0 & 34 & -24 & 3 & -2 \\ 0 & 0 & 3 & -2 & 2 & 1 \end{array}\right)$$

root list

	roots						norms	
r_1	2	0	−51	−36	2	0	2	0
r_2	4	2	−167	−118	0	0	58	40
r_3	0	0	1	1	2	−2	2	0
r_4	0	0	0	0	−1	1	2	−1
r_5	9	9	−570	−403	29	20	7	2
r_6	1	1	−63	−45	−1	5	3	−2

det $= -29 - 21w$	det norm $= -41$
$(224228)^2$	

quadratic form

$$\left(\begin{array}{cc|cc|cc} -171 & -121 & -6 & -10 & -342 & -242 \\ -6 & -10 & 650 & -460 & 13 & -16 \\ -342 & -242 & 13 & -16 & 42 & 29 \end{array}\right)$$

root list

	roots						norms	
r_1	−2	−2	147	104	−1	−3	2	1
r_2	3	2	−171	−121	3	5	71	50
r_3	7	4	−386	−273	5	6	3	2
r_4	150	108	−9283	−6564	150	103	6	4
r_5	8364	5914	−513091	−362810	8128	5743	142	100
r_6	3376	2388	−207147	−146475	3280	2317	6	4

det $= 3 - 5w$	det norm $= -41$
$(224228)^2$	

quadratic form

$$\left(\begin{array}{cc|cc|cc} 497 & 347 & 18 & 11 & 7 & 2 \\ 18 & 11 & 4 & -2 & -1 & 1 \\ 7 & 2 & -1 & 1 & 3 & -2 \end{array}\right)$$

root list

	roots						norms	
r_1	1	0	−15	−10	3	3	2	1
r_2	2	2	−69	−49	29	20	29	20
r_3	0	0	0	0	1	1	1	0
r_4	0	0	−1	−1	2	1	2	0
r_5	4	2	−127	−89	40	29	58	40
r_6	2	0	−35	−24	8	7	2	0

det $= -3 - 5w$ det norm $= -41$

$(228224)^2$

quadratic form

$$\left(\begin{array}{cc|cc|cc} -29 & -21 & -3 & -5 & -29 & -21 \\ -3 & -5 & 29 & -21 & 2 & -3 \\ -29 & -21 & 2 & -3 & 1 & 0 \end{array}\right)$$

root list

	roots						norms	
r_1	0	0	0	0	1	0	1	0
r_2	3	2	−71	−50	13	8	13	8
r_3	−2	1	7	5	−1	−2	2	−1
r_4	−5	−4	129	91	−26	−17	2	0
r_5	−133	−95	3233	2286	−610	−429	26	16
r_6	−49	−33	1157	818	−216	−153	2	0

det $= -9751 - 6895w$ det norm $= -49$

$(222222)^2$

quadratic form

$$\left(\begin{array}{cc|cc|cc} 13 & 9 & 16 & 10 & -18 & -13 \\ 16 & 10 & 2 & -8 & 7 & 3 \\ -18 & -13 & 7 & 3 & 2 & 1 \end{array}\right)$$

root list

	roots						norms	
r_1	0	0	−1	−1	−2	0	34	24
r_2	1	1	0	0	0	0	75	53
r_3	0	−1	3	2	1	2	92	65
r_4	−4	−2	3	2	1	1	10	7
r_5	−13	−10	9	6	1	3	3	2
r_6	−65	−43	39	27	10	8	13	9

Table A.9, cont.

det $= -9751 - 6895w$ det norm $= -49$

$(222222)^2$

quadratic form

$$\begin{pmatrix} 305221 & 215817 & -50568 & -35756 & 742 & 525 \\ -50568 & -35756 & 8378 & 5924 & -123 & -87 \\ 742 & 525 & -123 & -87 & 2 & 1 \end{pmatrix}$$

root list

	roots						norms	
r_1	4	2	25	12	20	14	34	24
r_2	1	0	7	0	26	18	75	53
r_3	−1	−1	−6	−6	3	2	17	12
r_4	0	0	0	0	3	2	58	41
r_5	58	41	352	249	157	111	536	379
r_6	195	138	1184	838	512	362	437	309

det $= -93 - 66w$ det norm $= -63$

$(222222)^2$

quadratic form

$$\begin{pmatrix} 17 & -20 & -17 & 20 & 7 & -7 \\ -17 & 20 & 23 & -16 & -10 & 5 \\ 7 & -7 & -10 & 5 & 5 & -1 \end{pmatrix}$$

root list

	roots						norms	
r_1	3	0	3	−1	6	1	6	0
r_2	−1	1	0	0	4	−2	3	−2
r_3	−3	0	−4	2	−9	1	2	−1
r_4	−26	−19	−8	−5	−71	−47	3	1
r_5	−36	−27	−7	−9	−92	−72	3	0
r_6	−13	−8	−3	−2	−30	−24	2	0

Table A.9, cont.

det = $-93 - 66w$ det norm = -63

$(222222)^2$

quadratic form

$$\left(\begin{array}{cc|cc|cc} 271 & -178 & -115 & 27 & 82 & -59 \\ -115 & 27 & 175 & 92 & -23 & 18 \\ 82 & -59 & -23 & 18 & 26 & -19 \end{array}\right)$$

root list

	roots						norms	
r_1	-6	-12	15	-14	26	20	3	0
r_2	-13	-4	-14	7	21	14	3	1
r_3	18	21	-14	17	-55	-40	2	-1
r_4	79	49	30	1	-174	-120	3	-2
r_5	435	306	95	63	-1004	-709	6	0
r_6	117	92	3	35	-286	-201	6	-4

det = $-2967 - 2098w$ det norm = -119

$(222242)^2$

quadratic form

$$\left(\begin{array}{cc|cc|cc} 3 & 2 & 10 & 6 & 27 & 19 \\ 10 & 6 & 10 & -8 & -1 & -2 \\ 27 & 19 & -1 & -2 & 3 & 2 \end{array}\right)$$

root list

	roots						norms	
r_1	-3	1	-2	-1	7	-4	10	4
r_2	-3	2	0	0	0	0	3	-2
r_3	-5	4	1	0	1	-1	6	-4
r_4	4	6	5	5	9	-10	7	-4
r_5	4	1	2	2	5	-5	3	-2
r_6	0	8	5	3	-4	0	6	-4

det $= -509 - 360w$ det norm $= -119$

$(224222)^2$

quadratic form

$$\begin{pmatrix} 4729 & 3340 & 89 & 73 & 74 & 50 \\ 89 & 73 & 21 & -12 & -3 & 4 \\ 74 & 50 & -3 & 4 & 2 & 0 \end{pmatrix}$$

root list

	roots						norms	
r_1	1	0	−19	−13	−20	−11	6	4
r_2	0	1	−23	−16	−32	−23	23	16
r_3	−1	−1	44	31	45	32	3	2
r_4	−9	−6	309	219	343	244	6	4
r_5	−53	−39	1903	1346	2145	1521	46	32
r_6	−17	−12	596	422	676	480	1	0

det $= -2967 - 2098w$ det norm $= -119$

$(222224)^2$

quadratic form

$$\begin{pmatrix} 99 & 70 & 0 & 0 & 0 & 0 \\ 0 & 0 & 30 & 18 & -13 & -12 \\ 0 & 0 & -13 & -12 & 10 & 4 \end{pmatrix}$$

root list

	roots						norms	
r_1	−3	1	10	7	17	12	3	2
r_2	1	−3	23	16	38	27	23	16
r_3	2	4	−49	−35	−84	−59	6	4
r_4	25	14	−294	−208	−498	−352	3	2
r_5	174	124	−2302	−1628	−3895	−2754	266	188
r_6	98	68	−1284	−908	−2171	−1535	34	24

Table A.9, cont.

det $= -10806 - 7641w$ det norm $= -126$

$(222222)^2$

quadratic form

$$\begin{pmatrix} 4910 & 3361 & -1396 & -918 & 124 & 81 \\ -1396 & -918 & 406 & 244 & -37 & -22 \\ 124 & 81 & -37 & -22 & 2 & 1 \end{pmatrix}$$

root list

	roots						norms	
r_1	4	2	13	8	−2	0	34	24
r_2	1	0	3	1	0	0	256	181
r_3	−33	−22	−116	−80	6	3	174	123
r_4	−10	−5	−33	−20	1	1	10	7
r_5	−522	−368	−1879	−1327	57	39	174	123
r_6	−377	−269	−1362	−967	40	28	10	7

det $= -54 - 39w$ det norm $= -126$

$(222222)^2$

quadratic form

$$\begin{pmatrix} 486 & -335 & -164 & 83 & -18 & 15 \\ -164 & 83 & 132 & 35 & 3 & -6 \\ -18 & 15 & 3 & -6 & 6 & 3 \end{pmatrix}$$

root list

	roots						norms	
r_1	−33	−27	0	−12	8	1	6	−3
r_2	−28	−16	−15	2	2	3	4	−1
r_3	69	54	5	21	−12	−5	6	−4
r_4	223	157	54	37	−24	−24	10	−7
r_5	918	648	222	153	−121	−83	6	−3
r_6	266	180	82	30	−35	−23	10	−7

det = $-11284 - 7979w$ det norm = -226

$(822222)^2$

quadratic form

$$\begin{pmatrix} -9200 & -7107 & 124 & 228 & -290 & -176 \\ 124 & 228 & 18 & -20 & 11 & 1 \\ -290 & -176 & 11 & 1 & 2 & 0 \end{pmatrix}$$

root list

	roots						norms	
r_1	0	0	−1	−1	−2	0	34	24
r_2	1	−1	−4	−10	1	0	10	7
r_3	14	10	664	470	3	2	34	24
r_4	2148	1520	101254	71602	291	207	2738	1936
r_5	187	133	8834	6250	25	17	10	7
r_6	6591	4662	310562	219607	843	595	802	567

det = $-4 - 11w$ det norm = -226

$(228222)^2$

quadratic form

$$\begin{pmatrix} -56 & -41 & 0 & 0 & 0 & 0 \\ 0 & 0 & 34 & -24 & 3 & -2 \\ 0 & 0 & 3 & -2 & 2 & 1 \end{pmatrix}$$

root list

	roots						norms	
r_1	1	0	−23	−16	2	−1	2	−1
r_2	5	4	−235	−166	0	0	26	15
r_3	0	0	1	1	2	−2	2	0
r_4	0	0	0	0	−1	1	2	−1
r_5	1	0	−24	−17	1	1	2	0
r_6	75	53	−3540	−2503	123	84	82	56

Table A.9, cont.

det $= -1320232 - 933545w$ det norm $= -226$

$(822222)^2$

quadratic form

$$\begin{pmatrix} -643228 & -454963 & 90290 & 63865 & -26162 & -18511 \\ 90290 & 63865 & -12674 & -8965 & 3673 & 2599 \\ -26162 & -18511 & 3673 & 2599 & -502 & -356 \end{pmatrix}$$

root list

	roots						norms	
r_1	−44	−31	−313	−221	1	0	338	239
r_2	55	39	391	278	−2	0	1154	816
r_3	782	553	5573	3941	0	0	338	239
r_4	46472	32861	331201	234198	32	25	16098	11383
r_5	3581	2532	25521	18046	2	3	198	140
r_6	81361	57531	579863	410025	82	57	9430	6668

det $= -32 - 25w$ det norm $= -226$

$(228222)^2$

quadratic form

$$\begin{pmatrix} 26248 & 18557 & 410 & 285 & -164 & -114 \\ 410 & 285 & 14 & -1 & -5 & 0 \\ -164 & -114 & -5 & 0 & 2 & 0 \end{pmatrix}$$

root list

	roots						norms	
r_1	1	0	−33	−23	−2	−2	2	0
r_2	3	1	−139	−98	0	0	50	32
r_3	−2	0	65	46	2	4	2	−1
r_4	−6	−4	376	266	17	12	2	0
r_5	−5	−3	298	210	13	8	2	−1
r_6	−151	−105	9626	6807	367	262	18	7

Table A.9, cont.

det $= -23104 - 16337w$	det norm $= -322$
$(222222)^2$	

quadratic form

$$\begin{pmatrix} 7248 & 5125 & -796 & -564 & -218 & -154 \\ -796 & -564 & 98 & 56 & 23 & 18 \\ -218 & -154 & 23 & 18 & 6 & 4 \end{pmatrix}$$

root list

	roots						norms	
r_1	0	0	−3	−2	7	6	34	24
r_2	3	3	0	0	90	64	372	263
r_3	0	−1	7	5	−36	−26	150	106
r_4	−24	−18	21	15	−674	−476	34	24
r_5	−1890	−1336	1379	975	−50852	−35958	5606	3964
r_6	−101	−71	71	50	−2702	−1912	10	7

det $= -680 - 481w$	det norm $= -322$
$(222222)^2$	

quadratic form

$$\begin{pmatrix} 372 & 263 & 0 & 0 & 0 & 0 \\ 0 & 0 & 14 & -10 & 5 & -3 \\ 0 & 0 & 5 & -3 & 6 & 2 \end{pmatrix}$$

root list

	roots						norms	
r_1	−4	1	80	57	−9	−8	6	2
r_2	−7	−4	398	282	−52	−38	12	7
r_3	4	−1	−81	−57	11	7	2	0
r_4	15	9	−871	−617	113	83	2	−1
r_5	4004	2835	−252021	−178206	33335	23572	166	116
r_6	438	311	−27609	−19522	3653	2582	2	0

det $= -157624 - 111457w$ det norm $= -322$

$(222222)^2$

quadratic form

$$\left(\begin{array}{cc|cc|cc} -64 & -51 & -18 & -15 & 52 & 34 \\ -18 & -15 & 14 & 9 & 27 & 18 \\ 52 & 34 & 27 & 18 & 2 & 0 \end{array}\right)$$

root list

	roots						norms	
r_1	1	−2	−3	3	−2	0	122	86
r_2	3	−2	−8	5	0	0	16	11
r_3	2	−1	−3	2	−1	1	2	1
r_4	5	−1	−7	6	1	1	6	4
r_5	136	99	93	60	91	63	330	233
r_6	10	3	−3	9	5	3	2	1

det $= -784856 - 554977w$ det norm $= -322$

$(222222)^2$

quadratic form

$$\left(\begin{array}{cc|cc|cc} 32956 & 23303 & -2488 & -1758 & 282 & 199 \\ -2488 & -1758 & 198 & 136 & -23 & -15 \\ 282 & 199 & -23 & -15 & 2 & 1 \end{array}\right)$$

root list

	roots						norms	
r_1	0	0	−3	−2	−60	−42	198	140
r_2	1	1	0	0	−308	−218	2168	1533
r_3	1	−1	8	5	207	146	874	618
r_4	−7	−6	30	21	2579	1824	198	140
r_5	−593	−418	2040	1443	192279	135962	32674	23104
r_6	−32	−22	106	75	10193	7207	58	41

Table A.9, cont.

det $= -4640 - 3281w$ det norm $= -322$

$(222222)^2$

quadratic form

$$\left(\begin{array}{cc|cc|cc} 2036 & 1279 & -2196 & -1578 & 112 & 42 \\ -2196 & -1578 & 2554 & 1802 & -95 & -73 \\ 112 & 42 & -95 & -73 & 10 & -2 \end{array}\right)$$

root list

	roots						norms	
r_1	4	2	0	5	11	8	34	24
r_2	3	4	18	1	119	84	1922	1359
r_3	−3	−3	−8	0	−17	−12	10	7
r_4	−122	−86	−150	−108	−911	−644	710	502
r_5	−97	−67	−116	−92	−796	−563	92	65
r_6	−25	−17	−29	−25	−218	−154	10	7

det $= -5354576 - 3786257w$ det norm $= -322$

$(222222)^2$

quadratic form

$$\left(\begin{array}{cc|cc|cc} 144 & 85 & -32 & -29 & 614 & 430 \\ -32 & -29 & 2 & -1 & 39 & 26 \\ 614 & 430 & 39 & 26 & -502 & -356 \end{array}\right)$$

root list

	roots						norms	
r_1	0	0	5	4	1	0	338	239
r_2	−1	−1	6	5	0	0	3124	2209
r_3	−3	−2	−51	−36	−2	−2	24118	17054
r_4	0	0	−17	−12	0	0	338	239
r_5	12	8	−589	−415	16	11	65290	46167
r_6	1	1	−29	−21	2	1	1154	816

det $= -3964 - 2803w$ det norm $= -322$

$(222222)^2$

quadratic form

$$\begin{pmatrix} 12 & 7 & 14 & 12 & -26 & -19 \\ 14 & 12 & 10 & 4 & -7 & -4 \\ -26 & -19 & -7 & -4 & 2 & 1 \end{pmatrix}$$

root list

	roots						norms	
r_1	0	1	−3	−1	−2	0	26	18
r_2	1	1	0	0	0	0	64	45
r_3	−1	0	1	1	1	0	6	4
r_4	−2	−3	3	1	1	0	2	1
r_5	−869	−615	481	340	71	51	962	680
r_6	−95	−67	51	36	7	5	6	4

det $= -190 - 135w$ det norm $= -350$

$(222322)^2$

quadratic form

$$\begin{pmatrix} 54 & -13 & 24 & -20 & 8 & -2 \\ 24 & -20 & 18 & -14 & 5 & -2 \\ 8 & -2 & 5 & -2 & 2 & 0 \end{pmatrix}$$

root list

	roots						norms	
r_1	0	0	−1	−1	−2	0	10	6
r_2	−3	2	4	−3	0	0	2	−1
r_3	1	2	17	10	5	5	80	55
r_4	−2	2	5	−1	1	0	2	0
r_5	4	2	10	8	1	1	2	0
r_6	26	18	76	54	5	5	30	20

det $= -1110 - 785w$ det norm $= -350$

$(222322)^2$

quadratic form

$$\begin{pmatrix} 118 & 83 & -48 & -33 & -32 & -22 \\ -48 & -33 & 20 & 11 & 15 & 9 \\ -32 & -22 & 15 & 9 & 10 & 6 \end{pmatrix}$$

root list

	roots						norms	
r_1	−1	−2	−5	−3	0	−2	10	6
r_2	1	−1	0	−1	0	0	2	−1
r_3	25	15	60	43	15	8	80	55
r_4	4	1	8	5	1	1	2	0
r_5	15	12	47	33	5	4	2	0
r_6	115	80	337	239	37	24	30	20

det $= 10 - 15w$ det norm $= -350$

$(222223)^2$

quadratic form

$$\begin{pmatrix} 2230 & 1555 & -370 & 275 & 50 & 30 \\ -370 & 275 & 9324 & -6593 & -93 & 66 \\ 50 & 30 & -93 & 66 & 2 & 0 \end{pmatrix}$$

root list

	roots						norms	
r_1	1	−1	27	19	5	1	2	0
r_2	−6	0	325	230	55	40	80	55
r_3	7	−5	2	1	2	0	2	−1
r_4	−1	1	−27	−19	−4	−1	10	6
r_5	−7	−3	905	640	80	55	30	20
r_6	−5	−1	509	360	46	31	2	0

det $= -84 - 63w$ det norm $= -882$

$(222222)^2$

quadratic form

$$\begin{pmatrix} 616 & 49 & -182 & 21 & 0 & 21 \\ -182 & 21 & 58 & -17 & 3 & -6 \\ 0 & 21 & 3 & -6 & 6 & 3 \end{pmatrix}$$

root list

	roots						norms	
r_1	−4	−4	−19	−16	4	0	2	−1
r_2	−180	−126	−777	−546	70	49	210	147
r_3	25	17	107	74	−10	−7	2	0
r_4	575	405	2482	1751	−228	−162	4	1
r_5	900	636	3888	2748	−359	−252	6	3
r_6	2521	1782	10892	7700	−1001	−707	42	28

det $= -504 - 357w$ det norm $= -882$

$(222222)^2$

quadratic form

$$\begin{pmatrix} 21140 & 14443 & 2704 & 2152 & 336 & 308 \\ 2704 & 2152 & 478 & 224 & 77 & 21 \\ 336 & 308 & 77 & 21 & 14 & 0 \end{pmatrix}$$

root list

	roots						norms	
r_1	0	0	−1	−1	7	6	6	4
r_2	−1	−2	14	7	12	9	16	11
r_3	−3	−1	14	12	3	3	30	21
r_4	0	0	0	0	3	2	238	168
r_5	1	1	−10	−6	5	2	2	1
r_6	161	112	−1232	−875	573	405	1218	861

det = $-17136 - 12117w$ det norm = -882

$(222222)^2$

quadratic form

$$\begin{pmatrix} 21072 & 14865 & 7662 & 5406 & -546 & -384 \\ 7662 & 5406 & 2786 & 1966 & -199 & -140 \\ -546 & -384 & -199 & -140 & 6 & 4 \end{pmatrix}$$

root list

	roots						norms	
r_1	−50	−37	137	102	−2	0	238	168
r_2	1	−1	−3	3	0	0	30	21
r_3	69	47	−190	−129	0	1	16	11
r_4	52	36	−143	−99	1	0	6	4
r_5	3077	2178	−8463	−5991	27	18	1218	861
r_6	63	45	−173	−124	1	0	2	1

det = $-84 - 63w$ det norm = -882

$(222222)^2$

quadratic form

$$\begin{pmatrix} -22204 & -15701 & -42 & -28 & -1120 & -791 \\ -42 & -28 & 14 & -10 & 9 & -10 \\ -1120 & -791 & 9 & -10 & 22 & 1 \end{pmatrix}$$

root list

	roots						norms	
r_1	0	−1	322	227	3	3	6	3
r_2	−1	1	−92	−65	0	0	4	1
r_3	0	1	−323	−228	−4	−3	2	0
r_4	67	47	−30709	−21714	−504	−357	210	147
r_5	−1	3	−749	−528	−12	−10	2	−1
r_6	28	19	−12691	−8974	−238	−168	42	28

Table A.10: Tetradecagons

det = $-2011 - 1422w$	det norm = -47
$(8222222)^2$	

quadratic form

$$\left(\begin{array}{cc|cc|cc} 447 & 316 & -160 & -114 & 84 & 59 \\ -160 & -114 & 66 & 36 & -27 & -24 \\ 84 & 59 & -27 & -24 & 16 & 9 \end{array}\right)$$

root list

	roots						norms	
r_1	0	0	−1	−1	−4	−2	6	4
r_2	2	−1	0	1	1	−1	2	1
r_3	−68	−50	0	0	521	369	84	59
r_4	−13	−3	0	−1	62	45	1	0
r_5	−40	−31	−5	−4	299	213	2	1
r_6	−751	−537	−109	−76	5373	3798	25	17
r_7	−61	−46	−9	−7	446	317	1	0

det = $-68315 - 48306w$	det norm = -47
$(2228222)^2$	

quadratic form

$$\left(\begin{array}{cc|cc|cc} 2196567 & 1553192 & -295767 & -209136 & 15748 & 11126 \\ -295767 & -209136 & 39825 & 28160 & -2121 & -1498 \\ 15748 & 11126 & -2121 & -1498 & 118 & 77 \end{array}\right)$$

Table A.10, cont.

root list

	roots						norms	
r_1	19	10	139	78	29	20	58	41
r_2	68	48	513	362	143	101	833	589
r_3	−22	−17	−167	−127	−42	−30	17	12
r_4	−237	−172	−1791	−1292	−466	−330	34	24
r_5	−1972	−1387	−14850	−10457	−3851	−2722	10	7
r_6	−15250	−10787	−114912	−81276	−29879	−21128	488	345
r_7	−497	−357	−3752	−2685	−982	−695	3	2

det $= -543 - 384w$ det norm $= -63$

$(2222222)^2$

quadratic form

$$\begin{pmatrix} 17 & 12 & 0 & 0 & 0 & 0 \\ 0 & 0 & -18 & -14 & -15 & -5 \\ 0 & 0 & -15 & -5 & 12 & -17 \end{pmatrix}$$

root list

	roots						norms	
r_1	2	−4	−9	−3	16	12	6	2
r_2	1	−1	−2	0	2	2	3	−2
r_3	0	0	−7	0	7	6	24	15
r_4	−4	3	1	0	−1	−1	2	−1
r_5	0	−3	−5	−2	11	8	6	3
r_6	−9	−3	−15	−12	43	30	3	1
r_7	−3	−5	−13	−9	34	24	1	0

det $= -233242 - 164927w$ det norm $= -94$

$(2228222)^2$

quadratic form

$$\begin{pmatrix} 495186 & 350129 & -66572 & -47070 & -1078 & -765 \\ -66572 & -47070 & 8950 & 6328 & 145 & 103 \\ -1078 & -765 & 145 & 103 & 2 & 1 \end{pmatrix}$$

root list

	roots						norms	
r_1	4	2	29	16	−60	−42	198	140
r_2	3	1	23	10	−286	−202	2844	2011
r_3	−1	−1	−8	−7	−2	−1	58	41
r_4	0	0	0	0	3	2	58	41
r_5	44	30	327	225	−132	−93	198	140
r_6	364	257	2717	1919	−1178	−833	9710	6866
r_7	13	10	98	74	−48	−34	58	41

det $= -823 - 582w$ det norm $= -119$

$(3222222)^2$

quadratic form

$$\left(\begin{array}{cc|cc|cc} 411 & 290 & -194 & -136 & -23 & -18 \\ -194 & -136 & 94 & 64 & 9 & 10 \\ -23 & -18 & 9 & 10 & 5 & -2 \end{array}\right)$$

root list

	roots						norms	
r_1	0	0	−1	0	2	2	6	4
r_2	1	0	2	0	1	0	6	4
r_3	0	0	0	0	3	2	37	26
r_4	−1	0	−1	−1	2	1	3	2
r_5	−52	−37	−141	−100	222	157	682	482
r_6	−7	−4	−18	−12	28	20	3	2
r_7	−37	−26	−104	−74	173	122	430	304

det $= -141 - 100w$ det norm $= -119$

$(2232222)^2$

quadratic form

$$\begin{pmatrix} 1121 & 660 & 40 & 52 & 20 & 26 \\ 40 & 52 & 6 & 0 & 3 & 0 \\ 20 & 26 & 3 & 0 & 2 & 0 \end{pmatrix}$$

root list

	roots						norms	
r_1	1	0	−10	−11	3	2	3	2
r_2	1	1	−29	−19	0	0	37	26
r_3	0	0	1	1	−2	−2	6	4
r_4	0	0	0	0	1	1	6	4
r_5	20	14	−534	−378	237	167	74	52
r_6	7	6	−212	−144	90	64	1	0
r_7	102	70	−2689	−1913	1164	823	118	82

det $= -243 - 172w$ det norm $= -119$

$(2222223)^2$

quadratic form

$$\begin{pmatrix} 3135 & 2172 & 88 & 76 & 44 & 38 \\ 88 & 76 & 6 & 0 & 3 & 0 \\ 44 & 38 & 3 & 0 & 2 & 0 \end{pmatrix}$$

root list

	roots						norms	
r_1	2	0	−36	−30	3	2	6	4
r_2	3	2	−110	−78	0	0	23	16
r_3	−4	−2	131	95	−8	−5	26	18
r_4	−475	−337	18147	12828	−344	−243	23	16
r_5	−185	−131	7059	4991	−131	−93	3	2
r_6	−4892	−3460	186523	131889	−3421	−2419	266	188
r_7	−3912	−2766	149125	105448	−2721	−1924	34	24

Table A.10, cont.

det = $-8259 - 5840w$	det norm = -119

$$(3222222)^2$$

quadratic form

$$\left(\begin{array}{cc|cc|cc} 10183 & 7200 & -14056 & -9940 & -120 & -82 \\ -14056 & -9940 & 19414 & 13726 & 159 & 118 \\ -120 & -82 & 159 & 118 & 14 & -8 \end{array}\right)$$

root list

	roots						norms	
r_1	0	0	−1	0	52	37	34	24
r_2	0	2	2	0	3	2	34	24
r_3	0	0	0	0	7	5	266	188
r_4	−1	−1	−2	0	15	11	3	2
r_5	−17	−17	−19	−11	280	198	23	16
r_6	−18	−14	−15	−12	264	186	26	18
r_7	−437	−304	−366	−262	6186	4374	23	16

det = $-612 - 433w$	det norm = -434

$$(2222322)^2$$

quadratic form

$$\left(\begin{array}{cc|cc|cc} 1068 & 755 & 56 & 38 & 1020 & 718 \\ 56 & 38 & 14 & -6 & 75 & 21 \\ 1020 & 718 & 75 & 21 & 1018 & 655 \end{array}\right)$$

Table A.10, cont.

root list

	roots						norms	
r_1	0	0	−12	−7	1	0	2	1
r_2	1	4	−76	−54	0	0	28	19
r_3	2	0	1	−2	−2	0	10	6
r_4	0	0	3	2	0	0	94	66
r_5	−2	−1	−11	−9	2	2	6	4
r_6	−1	−2	−19	−13	3	2	6	4
r_7	−9	−6	−129	−92	15	11	54	38

det $= -9932 - 7023w$ det norm $= -434$

$(3222222)^2$

quadratic form

$$\left(\begin{array}{cc|cc|cc} 760 & 529 & 492 & 356 & -22 & -26 \\ 492 & 356 & 342 & 234 & -27 & -9 \\ -22 & -26 & -27 & -9 & 10 & -6 \end{array}\right)$$

root list

	roots						norms	
r_1	0	0	−1	0	−22	−15	34	24
r_2	0	−1	2	0	−1	−1	34	24
r_3	0	0	0	0	7	5	150	106
r_4	1	2	−4	−2	−15	−11	10	7
r_5	282	200	−508	−358	−2617	−1850	546	386
r_6	528	372	−946	−670	−4919	−3478	150	106
r_7	2557	1808	−4594	−3248	−23912	−16908	160	113

det $= -9932 - 7023w$ det norm $= -434$

$(2222223)^2$

quadratic form

$$\left(\begin{array}{cc|cc|cc} 6224 & 4401 & -776 & -548 & -320 & -226 \\ -776 & -548 & 110 & 60 & 45 & 25 \\ -320 & -226 & 45 & 25 & 18 & 10 \end{array}\right)$$

root list

	roots						norms	
r_1	0	0	−3	−2	6	4	34	24
r_2	9	5	0	0	136	96	932	659
r_3	−2	−2	17	12	−75	−53	874	618
r_4	−188	−132	113	80	−3401	−2405	3182	2250
r_5	−93	−65	41	29	−1649	−1166	58	41
r_6	−5562	−3932	2233	1579	−98715	−69802	874	618
r_7	−15444	−10921	6159	4355	−274055	−193786	198	140

det = $-612 - 433w$ det norm = -434

$(3222222)^2$

quadratic form

$$\begin{pmatrix} 1336 & 943 & -868 & -616 & -58 & -46 \\ -868 & -616 & 570 & 400 & 45 & 25 \\ -58 & -46 & 45 & 25 & 14 & -6 \end{pmatrix}$$

root list

	roots						norms	
r_1	0	0	−1	0	6	4	6	4
r_2	0	1	2	0	1	1	6	4
r_3	0	0	0	0	3	2	94	66
r_4	−4	0	−2	−4	11	8	10	6
r_5	−19	−14	−38	−28	122	87	28	19
r_6	−1	−2	−5	−2	13	9	2	1
r_7	−30	−22	−65	−46	233	165	54	38

det = $-337396 - 238575w$ det norm = -434

$(2232222)^2$

quadratic form

$$\left(\begin{array}{cc|cc|cc} 36276 & 25651 & 10448 & 7388 & 772 & 546 \\ 10448 & 7388 & 3018 & 2130 & 225 & 156 \\ 772 & 546 & 225 & 156 & 18 & 10 \end{array}\right)$$

root list

	roots						norms	
r_1	0	0	−3	−2	34	24	150	106
r_2	−7	3	0	0	56	40	160	113
r_3	−14	10	0	1	−11	−8	6	4
r_4	8	12	13	9	−661	−467	6	4
r_5	28	35	37	27	−2023	−1431	26	18
r_6	55	7	31	21	−1681	−1189	2	1
r_7	3592	2557	3333	2358	−186299	−131734	94	66

det $= 4 - 15w$ det norm $= -434$

$(2222223)^2$

quadratic form

$$\left(\begin{array}{cc|cc|cc} 160 & 113 & 0 & 0 & 0 & 0 \\ 0 & 0 & 82 & -58 & 7 & -5 \\ 0 & 0 & 7 & -5 & 2 & 0 \end{array}\right)$$

root list

	roots						norms	
r_1	2	0	113	80	1	0	2	0
r_2	17	11	1864	1318	0	0	28	19
r_3	0	0	7	5	−2	−2	26	18
r_4	−2	−2	−273	−193	0	0	94	66
r_5	1	1	133	94	2	2	2	1
r_6	124	88	13797	9756	184	130	26	18
r_7	354	251	39383	27848	518	366	6	4

Table A.11: Hexadecagons

det $= -5712 - 4039w$	det norm $= -98$
$(22222222)^2$	

quadratic form

$$\left(\begin{array}{cc|cc|cc} 58 & 41 & -44 & -31 & 10 & 7 \\ -44 & -31 & 32 & 20 & -3 & -1 \\ 10 & 7 & -3 & -1 & 1 & 0 \end{array}\right)$$

root list

	roots						norms	
r_1	−1	−2	−3	−2	−2	−1	92	65
r_2	1	−1	0	0	0	0	10	7
r_3	14	7	17	12	5	4	119	84
r_4	2	3	4	3	1	1	3	2
r_5	55	37	68	48	18	13	44	31
r_6	22	13	25	18	6	5	6	4
r_7	420	301	523	370	132	93	238	168
r_8	35	29	47	33	12	8	2	1

det $= -5712 - 4039w$	det norm $= -98$
$(22222222)^2$	

quadratic form

$$\left(\begin{array}{cc|cc|cc} -618 & -437 & 0 & 0 & 31 & 22 \\ 0 & 0 & 2 & 1 & 0 & 1 \\ 31 & 22 & 0 & 1 & 3 & -2 \end{array}\right)$$

root list

	roots						norms	
r_1	0	0	0	0	−7	−5	17	12
r_2	1	−1	13	9	−31	−22	693	490
r_3	0	0	3	2	0	0	58	41
r_4	1	0	−4	−3	10	7	536	379
r_5	1	0	−10	−7	0	0	58	41
r_6	11	7	−243	−172	−106	−75	8078	5712
r_7	1	1	−31	−22	−24	−17	198	140
r_8	8	6	−225	−159	−256	−181	1492	1055

det $= -509 - 360w$ det norm $= -119$

$(82222222)^2$

quadratic form

$$\begin{pmatrix} 605 & 426 & -112 & -98 & 160 & 117 \\ -112 & -98 & 166 & -80 & -63 & -4 \\ 160 & 117 & -63 & -4 & 50 & 27 \end{pmatrix}$$

root list

	roots						norms	
r_1	0	0	4	3	3	3	10	7
r_2	2	0	7	5	2	1	34	24
r_3	−92	−64	−1401	−991	−820	−581	266	188
r_4	−45	−34	−703	−497	−408	−289	3	2
r_5	−356	−254	−5383	−3806	−3122	−2207	6	4
r_6	−3925	−2777	−59060	−41762	−34238	−24212	23	16
r_7	−1288	−911	−19372	−13698	−11229	−7940	16	11
r_8	−1540	−1092	−23178	−16389	−13431	−9497	78	55

det $= -4797 - 3392w$ det norm $= -119$

$(22222222)^2$

quadratic form

$$\begin{pmatrix} 135 & 76 & 744 & 534 & -18 & -20 \\ 744 & 534 & 4646 & 3282 & -147 & -101 \\ -18 & -20 & -147 & -101 & 6 & 1 \end{pmatrix}$$

root list

	roots						norms	
r_1	−12	−10	5	0	8	6	54	38
r_2	−5	2	−9	7	6	4	37	26
r_3	5	3	0	−1	1	1	27	19
r_4	0	0	0	0	3	2	126	89
r_5	−7	−2	−4	4	2	1	3	2
r_6	−882	−623	149	107	283	200	1164	823
r_7	−108	−80	25	9	35	25	10	7
r_8	−579	−411	102	68	184	130	37	26

det $= -141 - 100w$ det norm $= -119$

$(22222222)^2$

quadratic form

$$\begin{pmatrix} 215 & 152 & 0 & 0 & 0 & 0 \\ 0 & 0 & 14 & -10 & 1 & -1 \\ 0 & 0 & 1 & -1 & 2 & 1 \end{pmatrix}$$

root list

	roots						norms	
r_1	1	0	24	17	4	3	3	2
r_2	6	5	304	215	37	26	1164	823
r_3	0	0	0	0	−1	−1	10	7
r_4	1	−1	0	0	0	0	37	26
r_5	0	0	7	5	4	3	54	38
r_6	5	2	215	152	52	37	37	26
r_7	5	3	246	174	57	40	27	19
r_8	14	10	734	519	163	115	126	89

det $= -393 - 278w$ det norm $= -119$

$(22822222)^2$

quadratic form

$$\begin{pmatrix} -2291 & -1620 & 0 & 0 & 0 & 0 \\ 0 & 0 & 34 & -24 & 3 & -2 \\ 0 & 0 & 3 & -2 & 2 & 1 \end{pmatrix}$$

root list

	roots						norms	
r_1	2	0	−289	−204	6	1	14	8
r_2	−7	5	−10	−7	0	0	3	−2
r_3	−4	2	169	119	−4	0	6	−4
r_4	6	−6	357	252	−9	1	10	−7
r_5	−16	4	1483	1049	−15	−11	14	−9
r_6	−11	4	765	542	3	−13	27	−19
r_7	−17	2	2031	1435	−26	−10	23	−16
r_8	−2	0	287	202	−14	6	34	−24

det $= -41 - 30w$ det norm $= -119$

$(22222222)^2$

quadratic form

$$\begin{pmatrix} 101 & 71 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & -3 & -1 & 1 \\ 0 & 0 & -1 & 1 & 1 & 0 \end{pmatrix}$$

root list

	roots						norms	
r_1	1	0	11	8	−4	−2	1	0
r_2	5	3	101	71	−41	−30	202	142
r_3	−2	−2	−54	−38	17	12	1	0
r_4	−244	−172	−5425	−3836	1761	1245	101	71
r_5	−21	−15	−470	−332	153	109	2	1
r_6	−449	−318	−9997	−7069	3288	2325	23	16
r_7	−536	−379	−11923	−8431	3928	2777	44	31
r_8	−1056	−747	−23493	−16612	7750	5480	78	55

det $= -23 - 18w$ det norm $= -119$

$(22222222)^2$

quadratic form

$$\begin{pmatrix} 9325 & 282 & -265 & 786 & -38 & 22 \\ -265 & 786 & 143 & -49 & 5 & -4 \\ -38 & 22 & 5 & -4 & 2 & 1 \end{pmatrix}$$

root list

	roots						norms	
r_1	−13	−10	160	108	2	−1	1	0
r_2	−209	−147	2397	1700	0	0	200	141
r_3	95	68	−1105	−776	−2	−1	2	1
r_4	1247	881	−14356	−10156	−16	−14	7	4
r_5	647	457	−7447	−5269	−7	−7	10	6
r_6	2044	1444	−23529	−16646	−23	−18	7	4
r_7	1716	1214	−19775	−13979	−18	−15	5	3
r_8	3929	2779	−45269	−32005	−44	−30	22	15

det $= -17293 - 12228w$ det norm $= -119$

$(22228222)^2$

quadratic form

$$\begin{pmatrix} 15183 & 10736 & -5060 & -3578 & -188 & -133 \\ -5060 & -3578 & 1694 & 1196 & 65 & 43 \\ -188 & -133 & 65 & 43 & 6 & -1 \end{pmatrix}$$

root list

	roots						norms	
r_1	0	0	−1	−1	28	20	34	24
r_2	−1	3	0	0	116	82	133	94
r_3	0	−1	3	2	−119	−84	92	65
r_4	−28	−18	23	16	−2451	−1733	454	321
r_5	−26	−18	17	12	−2243	−1586	58	41
r_6	−332	−236	195	138	−28444	−20113	198	140
r_7	−890	−628	509	360	−75690	−53521	1550	1096
r_8	−165	−118	93	66	−14084	−9959	17	12

det $= -8259 - 5840w$ det norm $= -119$

$(22222222)^2$

quadratic form

$$\left(\begin{array}{cc|cc|cc} -618 & -437 & 0 & 0 & 31 & 22 \\ 0 & 0 & 3 & 2 & 1 & 1 \\ 31 & 22 & 1 & 1 & 3 & -2 \end{array}\right)$$

root list

	roots						norms	
r_1	0	0	0	0	−7	−5	17	12
r_2	−2	0	31	22	−75	−53	6842	4838
r_3	0	0	1	1	0	0	17	12
r_4	1	0	−4	−3	10	7	454	321
r_5	1	1	−18	−13	0	0	256	181
r_6	4	1	−45	−32	−24	−17	133	94
r_7	1	0	−9	−6	−7	−5	10	7
r_8	21	17	−406	−287	−437	−309	587	415

$\det = -393 - 278w$ det norm $= -119$

$(22222222)^2$

quadratic form

$$\left(\begin{array}{cc|cc|cc} 43 & 30 & 12 & 11 & 21 & 0 \\ 12 & 11 & 4 & -14 & -73 & 48 \\ 21 & 0 & -73 & 48 & 417 & -295 \end{array}\right)$$

root list

	roots						norms	
r_1	−4	1	−17	6	10	10	6	2
r_2	−3	2	5	−4	2	0	3	−2
r_3	−2	11	−3	34	−69	−43	11	−1
r_4	7	−4	−11	11	−9	−3	10	−7
r_5	4	6	9	24	−65	−43	5	−3
r_6	13	0	35	7	−64	−48	3	−2
r_7	230	163	785	572	−2346	−1656	58	38
r_8	9	6	38	16	−88	−64	3	−2

det $= -8259 - 5840w$ det norm $= -119$

$(22222222)^2$

quadratic form

$$\left(\begin{array}{cc|cc|cc} 2407 & 1702 & -896 & -634 & -78 & -55 \\ -896 & -634 & 366 & 218 & 19 & 28 \\ -78 & -55 & 19 & 28 & 6 & -1 \end{array}\right)$$

root list

	roots						norms	
r_1	0	0	−1	−1	11	7	10	7
r_2	−1	3	0	0	38	27	133	94
r_3	−2	−1	14	10	−163	−115	256	181
r_4	−28	−18	78	55	−1309	−926	454	321
r_5	−17	−13	44	31	−800	−566	17	12
r_6	−412	−294	963	681	−18160	−12841	6842	4838
r_7	−17	−12	37	26	−722	−511	17	12
r_8	−537	−378	1096	775	−22199	−15697	3421	2419

det $= -4797 - 3392w$ det norm $= -119$

$(22222222)^2$

quadratic form

$$\begin{pmatrix} 159 & 112 & 230 & 159 & 36 & 23 \\ 230 & 159 & 352 & 218 & 65 & 25 \\ 36 & 23 & 65 & 25 & 16 & -3 \end{pmatrix}$$

root list

	roots						norms	
r_1	−2	3	−1	−2	4	4	54	38
r_2	−1	−3	0	0	8	6	37	26
r_3	0	0	1	1	−5	−4	10	7
r_4	128	95	59	41	−675	−477	1164	823
r_5	9	8	4	2	−48	−33	3	2
r_6	336	243	96	67	−1501	−1061	126	89
r_7	367	258	98	70	−1597	−1130	27	19
r_8	815	575	215	152	−3536	−2500	37	26

det $= -41 - 30w$ det norm $= -119$

$(22222222)^2$

quadratic form

$$\begin{pmatrix} 537 & 361 & 51 & 17 & -16 & -3 \\ 51 & 17 & 17 & -8 & -7 & 4 \\ -16 & -3 & -7 & 4 & 3 & -2 \end{pmatrix}$$

root list

	roots						norms	
r_1	1	0	9	8	38	29	2	1
r_2	5	3	142	101	486	344	101	71
r_3	0	−1	−18	−11	−63	−43	1	0
r_4	−144	−101	−3847	−2722	−13665	−9665	202	142
r_5	−14	−10	−383	−270	−1354	−956	1	0
r_6	−63	−46	−1757	−1240	−6196	−4379	14	9
r_7	−61	−43	−1683	−1191	−5926	−4192	8	5
r_8	−88	−63	−2463	−1740	−8654	−6117	5	2

det = $-67 - 48w$ det norm = -119

$(22222222)^2$

quadratic form

$$\left(\begin{array}{cc|cc|cc} 1085 & 756 & -31 & -8 & 38 & 29 \\ -31 & -8 & 15 & -10 & 1 & -2 \\ 38 & 29 & 1 & -2 & 2 & 1 \end{array}\right)$$

root list

	roots						norms	
r_1	1	0	31	22	5	−1	2	1
r_2	6	4	393	278	29	19	163	115
r_3	0	0	3	2	0	1	3	2
r_4	−1	−1	−75	−53	−2	−3	150	106
r_5	3	2	160	113	8	6	3	2
r_6	183	129	10113	7151	518	364	1898	1342
r_7	10	7	553	391	28	20	3	2
r_8	22	15	1205	852	63	42	13	9

det = $-13353 - 9442w$ det norm = -119

$(82222222)^2$

quadratic form

$$\begin{pmatrix} 663 & 466 & -356 & -240 & -58 & -38 \\ -356 & -240 & 222 & 108 & 39 & 15 \\ -58 & -38 & 39 & 15 & 6 & 1 \end{pmatrix}$$

root list

	roots						norms	
r_1	0	0	−1	−1	4	4	34	24
r_2	−2	2	0	1	1	0	10	7
r_3	0	0	0	0	3	2	126	89
r_4	3	−4	−2	−3	5	4	13	9
r_5	−7	−4	−15	−11	22	18	37	26
r_6	−6	2	−5	−2	6	4	6	4
r_7	−16	−14	−45	−33	68	50	74	52
r_8	3	−7	−8	−7	16	8	1	0

$\det = -453609 - 320750w$ det norm $= -119$

$(22222222)^2$

quadratic form

$$\begin{pmatrix} -3025 & -2139 & 0 & 0 & 0 & 0 \\ 0 & 0 & 17 & 12 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 1 \end{pmatrix}$$

root list

	roots						norms	
r_1	1	0	−3	−2	−17	−12	99	70
r_2	5	4	0	0	−215	−152	64474	45590
r_3	0	0	1	1	0	0	99	70
r_4	0	0	0	0	7	5	437	309
r_5	−1	1	−3	−2	0	0	58	41
r_6	18	13	−230	−163	−215	−152	5531	3911
r_7	2	1	−21	−15	−24	−17	99	70
r_8	5	4	−62	−44	−99	−70	5094	3602

det $= -77827 - 55032w$ det norm $= -119$

$(82222222)^2$

quadratic form

$$\begin{pmatrix} -560049 & -396018 & 8686 & 6134 & 25297 & 17887 \\ 8686 & 6134 & -122 & -104 & -391 & -278 \\ 25297 & 17887 & -391 & -278 & -1093 & -773 \end{pmatrix}$$

root list

	roots						norms	
r_1	11	7	674	477	1	0	58	41
r_2	0	0	3	2	−2	0	198	140
r_3	−829	−586	−53600	−37901	0	0	99	70
r_4	−23240	−16433	−1502913	−1062720	22	15	14606	10328
r_5	−13578	−9601	−878095	−620907	18	13	1154	816
r_6	−248661	−175830	−16081229	−11371146	378	267	7303	5164
r_7	−132888	−93966	−8594050	−6076911	207	146	2547	1801
r_8	−404495	−286021	−26159288	−18497410	645	456	24934	17631

det $= -393 - 278w$ det norm $= -119$

$(22222222)^2$

quadratic form

$$\left(\begin{array}{cc|cc|cc} 83249 & 58820 & 13419 & 9504 & 510 & 340 \\ 13419 & 9504 & 2171 & 1530 & 75 & 60 \\ 510 & 340 & 75 & 60 & 10 & -3 \end{array}\right)$$

root list

	roots						norms	
r_1	−1	−2	15	6	5	3	10	7
r_2	−96	−68	593	419	89	63	949	671
r_3	−8	−5	45	34	4	3	17	12
r_4	−11	−8	69	48	10	7	874	618
r_5	5	4	−36	−24	48	34	17	12
r_6	391	276	−2529	−1790	2758	1950	11062	7822
r_7	22	16	−145	−101	144	102	17	12
r_8	50	35	−319	−227	303	214	75	53

$\det = -23 - 18w$ det norm $= -119$

$(22222222)^2$

quadratic form

$$\left(\begin{array}{cc|cc|cc} -11841 & -8408 & -10 & 106 & -1515 & -1077 \\ -10 & 106 & 258 & -183 & -5 & 15 \\ -1515 & -1077 & -5 & 15 & -133 & -95 \end{array}\right)$$

root list

	roots						norms	
r_1	9	6	2130	1506	31	21	2	1
r_2	15	12	3900	2758	58	38	7	4
r_3	−41	−28	−9821	−6944	−143	−97	10	6
r_4	−1000	−706	−243631	−172273	−3479	−2464	7	4
r_5	−1134	−801	−276352	−195410	−3951	−2792	5	3
r_6	−3355	−2373	−818166	−578531	−11693	−8270	22	15
r_7	−681	−482	−166130	−117472	−2372	−1681	1	0
r_8	−63748	−45077	−15544099	−10991338	−222190	−157114	200	141

det $= -141 - 100w$ det norm $= -119$

$(22222222)^2$

quadratic form

$$\left(\begin{array}{cc|cc|cc} 93 & 58 & 9 & 10 & -74 & -52 \\ 9 & 10 & 3 & 0 & -11 & -8 \\ -74 & -52 & -11 & -8 & 10 & 7 \end{array}\right)$$

root list

	roots						norms	
r_1	−1	−1	9	5	−6	4	6	−2
r_2	−1	−2	14	8	0	0	5	−2
r_3	1	1	−9	−5	3	−2	2	−1
r_4	149	106	−993	−701	1	3	36	23
r_5	11	7	−68	−50	−4	3	3	−2
r_6	330	234	−2194	−1550	1	3	6	1
r_7	351	249	−2334	−1649	5	0	3	−1
r_8	778	550	−5161	−3651	2	6	5	−2

det $= -17293 - 12228w$ det norm $= -119$

$(22222822)^2$

quadratic form

$$\begin{pmatrix} -10301 & -7296 & -1068 & -672 & 1409 & 992 \\ -1068 & -672 & 306 & -356 & 115 & 111 \\ 1409 & 992 & 115 & 111 & -135 & -97 \end{pmatrix}$$

root list

	roots						norms	
r_1	−5	−4	57	40	5	−1	92	65
r_2	−5	−1	32	23	0	0	133	94
r_3	4	1	−29	−21	2	−3	34	24
r_4	7	5	−75	−53	−2	−1	17	12
r_5	318	227	−3389	−2396	−60	−41	1550	1096
r_6	434	307	−4599	−3252	−74	−53	198	140
r_7	245	172	−2585	−1828	−41	−28	58	41
r_8	1987	1405	−21035	−14874	−321	−227	454	321

det $= -6767 - 4785w$ det norm $= -161$

$(22222222)^2$

quadratic form

$$\begin{pmatrix} 153517 & 106953 & 41744 & 29702 & -436 & -262 \\ 41744 & 29702 & 11522 & 8126 & -107 & -81 \\ -436 & -262 & -107 & -81 & 2 & 0 \end{pmatrix}$$

root list

	roots						norms	
r_1	4	2	−7	−12	36	25	34	24
r_2	3	−1	17	−15	64	45	109	77
r_3	−3	−2	10	8	5	4	75	53
r_4	0	0	0	0	3	2	34	24
r_5	47	34	−176	−118	180	128	109	77
r_6	16	14	−81	−34	62	44	150	106
r_7	118	80	−395	−309	436	308	218	154
r_8	77	52	−256	−202	290	204	13	9

det $= -229879 - 162549w$ det norm $= -161$

$(22222222)^2$

quadratic form

$$\begin{pmatrix} 3701 & 2617 & -1824 & -1290 & -150 & -106 \\ -1824 & -1290 & 926 & 628 & 67 & 55 \\ -150 & -106 & 67 & 55 & 6 & 2 \end{pmatrix}$$

root list

	roots						norms	
r_1	−4	−10	−13	−9	−152	−107	1270	898
r_2	−5	3	0	0	−20	−14	75	53
r_3	6	0	4	3	53	38	34	24
r_4	65	59	92	65	1571	1111	109	77
r_5	63	42	75	53	1317	931	75	53
r_6	54	56	81	57	1453	1027	34	24
r_7	851	598	1024	724	18640	13180	109	77
r_8	186	118	212	150	3899	2757	150	106

det $= -649 - 459w$ det norm $= -161$

$(22222222)^2$

quadratic form

$$\begin{pmatrix} 1315 & 887 & 202 & 116 & 44 & 28 \\ 202 & 116 & 38 & 10 & 7 & 3 \\ 44 & 28 & 7 & 3 & 6 & 4 \end{pmatrix}$$

root list

	roots						norms	
r_1	−2	−2	17	12	−6	4	6	2
r_2	−11	−5	66	42	0	0	5	−1
r_3	0	2	−7	−9	−7	5	6	−4
r_4	406	288	−2821	−1998	−7	6	26	4
r_5	86	66	−617	−444	3	−2	6	−4
r_6	1295	916	−8989	−6358	−7	6	5	−1
r_7	475	332	−3282	−2315	−1	1	5	−3
r_8	2362	1672	−16400	−11601	−7	6	10	−2

det = $-39441 - 27889w$ det norm = -161

$(22222222)^2$

quadratic form

$$\begin{pmatrix} 635 & 449 & -436 & -308 & 26 & 18 \\ -436 & -308 & 342 & 234 & -27 & -9 \\ 26 & 18 & -27 & -9 & 10 & -6 \end{pmatrix}$$

root list

	roots						norms	
r_1	0	0	−1	0	−22	−15	34	24
r_2	−1	2	0	0	−48	−34	75	53
r_3	−2	−2	5	4	353	250	1270	898
r_4	−20	−16	13	9	1659	1173	874	618
r_5	−599	−424	280	198	43142	30506	635	449
r_6	−98	−68	44	31	6927	4898	198	140
r_7	−243	−172	106	75	17177	12146	437	309
r_8	−683	−481	291	206	47895	33867	635	449

det = $-749017 - 529635w$ det norm = -161

$(22222222)^2$

quadratic form

$$\begin{pmatrix} 3522763 & 2490775 & -1319052 & -932646 & 8622 & 6114 \\ -1319052 & -932646 & 493902 & 349220 & -3229 & -2289 \\ 8622 & 6114 & -3229 & -2289 & 22 & 14 \end{pmatrix}$$

root list

	roots						norms	
r_1	12	6	33	16	63	44	198	140
r_2	3	4	9	12	195	138	2069	1463
r_3	−39	−24	−107	−65	−282	−199	437	309
r_4	−582	−408	−1599	−1120	−5992	−4237	4138	2926
r_5	−710	−500	−1953	−1375	−7724	−5461	874	618
r_6	−16083	−11370	−44258	−31288	−178542	−126248	355	251
r_7	−1450	−1032	−3989	−2841	−16181	−11441	34	24
r_8	−10876	−7678	−29935	−21130	−121217	−85712	3134	2216

det = $-3783 - 2675w$ det norm = -161

$(22222222)^2$

quadratic form

$$\begin{pmatrix} 268797 & 189783 & -2100 & -868 & -1358 & -952 \\ -2100 & -868 & 954 & -660 & 21 & -3 \\ -1358 & -952 & 21 & -3 & 6 & 4 \end{pmatrix}$$

root list

	roots						norms	
r_1	0	0	7	5	−11	−12	6	4
r_2	1	−2	−100	−72	−94	−64	61	43
r_3	−2	2	2	3	127	89	150	106
r_4	310	222	31282	22121	37733	26682	710	502
r_5	329	234	33178	23461	39725	28091	75	53
r_6	2641	1870	265961	188064	317733	224671	355	251
r_7	602	424	60487	42770	72189	51045	34	24
r_8	12890	9118	1298359	918080	1548195	1094741	3134	2216

det $= -649 - 459w$ det norm $= -161$

$(22222222)^2$

quadratic form

$$\begin{pmatrix} 355 & 251 & 0 & 0 & 0 & 0 \\ 0 & 0 & 14 & -10 & 5 & -3 \\ 0 & 0 & 5 & -3 & 6 & 2 \end{pmatrix}$$

root list

	roots						norms	
r_1	2	0	59	42	−6	−5	6	4
r_2	6	2	269	190	−36	−25	538	380
r_3	0	0	1	1	0	−1	6	4
r_4	1	−1	0	0	0	0	61	43
r_5	0	0	0	0	3	2	150	106
r_6	36	26	2130	1506	−183	−129	710	502
r_7	39	27	2260	1598	−201	−142	75	53
r_8	309	219	18119	12812	−1628	−1151	355	251

det = $-3783 - 2675w$ det norm = -161

$(22222222)^2$

quadratic form

$$\begin{pmatrix} 451123 & 318099 & 25170 & 17966 & -26 & -4 \\ 25170 & 17966 & 1442 & 988 & 1 & -2 \\ -26 & -4 & 1 & -2 & 6 & 4 \end{pmatrix}$$

root list

	roots						norms	
r_1	−60	−44	1095	766	2	0	6	4
r_2	−288	−206	5171	3644	0	0	538	380
r_3	298	210	−5299	−3751	−6	−2	6	4
r_4	7567	5349	−134830	−95348	−100	−72	61	43
r_5	2266	1602	−40379	−28554	−29	−19	150	106
r_6	23738	16786	−423063	−299147	−269	−190	710	502
r_7	20251	14319	−360897	−255196	−225	−159	75	53
r_8	150869	106681	−2688749	−1901230	−1664	−1176	355	251

det = $-3783 - 2675w$ det norm = -161

$(22222222)^2$

quadratic form

$$\begin{pmatrix} 3093 & 2187 & -2612 & -1846 & 222 & 158 \\ -2612 & -1846 & 2214 & 1554 & -179 & -139 \\ 222 & 158 & -179 & -139 & 26 & 4 \end{pmatrix}$$

root list

	roots						norms	
r_1	−8	2	−3	−1	9	6	26	18
r_2	−5	1	0	0	22	16	11	7
r_3	10	−6	1	0	−4	−3	2	0
r_4	76	86	17	15	−1028	−727	94	64
r_5	36	4	5	1	−226	−159	2	0
r_6	287	214	39	29	−3248	−2297	11	7
r_7	89	88	12	11	−1183	−837	3	1
r_8	518	384	64	47	−5917	−4184	22	14

det $= -22049 - 15591w$ det norm $= -161$

$(22222222)^2$

quadratic form

$$\begin{pmatrix} 10437 & 7379 & 7420 & 5236 & 982 & 696 \\ 7420 & 5236 & 5342 & 3670 & 687 & 502 \\ 982 & 696 & 687 & 502 & 94 & 64 \end{pmatrix}$$

root list

	roots						norms	
r_1	0	0	−3	−2	20	15	34	24
r_2	−11	−7	0	0	104	74	355	251
r_3	−6	−2	13	9	−47	−33	874	618
r_4	222	158	−441	−312	917	648	4138	2926
r_5	245	175	−499	−353	1093	773	437	309
r_6	1997	1413	−4077	−2883	9064	6409	2069	1463
r_7	458	322	−935	−661	2092	1479	198	140
r_8	9846	6964	−20213	−14293	45486	32163	18266	12916

det $= -1342 - 949w$ det norm $= -238$

$(82222222)^2$

quadratic form

$$\begin{pmatrix} 2770 & 1957 & -604 & -444 & 578 & 398 \\ -604 & -444 & 258 & 12 & -51 & -144 \\ 578 & 398 & -51 & -144 & 166 & 49 \end{pmatrix}$$

root list

	roots						norms	
r_1	0	0	−11	−7	−11	−9	10	7
r_2	2	0	3	2	−2	−1	34	24
r_3	0	0	3	2	4	3	430	304
r_4	−5	−1	−31	−22	−18	−13	44	31
r_5	−15	−11	−164	−115	−104	−75	126	89
r_6	−4	−1	−31	−22	−21	−15	10	7
r_7	−30	−22	−387	−274	−281	−198	126	89
r_8	−19	−15	−264	−186	−194	−138	10	7

det $= -298 - 211w$ det norm $= -238$

$(22228222)^2$

quadratic form

$$\left(\begin{array}{cc|cc|cc} 242 & 171 & -192 & -134 & 44 & 29 \\ -192 & -134 & 178 & 92 & -63 & -4 \\ 44 & 29 & -63 & -4 & 42 & -19 \end{array}\right)$$

root list

	roots						norms	
r_1	0	0	0	1	3	3	2	1
r_2	−1	3	6	5	22	16	14	9
r_3	0	−1	−5	−4	−22	−16	10	6
r_4	−28	−18	−127	−91	−468	−332	46	32
r_5	−26	−18	−119	−84	−430	−304	6	4
r_6	−96	−70	−445	−314	−1599	−1130	2	1
r_7	−262	−183	−1185	−837	−4255	−3008	14	9
r_8	−165	−118	−754	−532	−2704	−1911	2	1

$\det = -45590 - 32237w$ $\qquad$ det norm $= -238$

$(82222222)^2$

quadratic form

$$\left(\begin{array}{cc|cc|cc} -200461174 & -141751343 & 5304908 & 3748997 & -575982 & -407097 \\ 5304908 & 3748997 & -139472 & -99799 & 15163 & 10823 \\ -575982 & -407097 & 15163 & 10823 & -1648 & -1174 \end{array}\right)$$

root list

	roots						norms	
r_1	11	7	393	280	−10	1	34	24
r_2	0	0	3	−1	33	−13	10	7
r_3	−1	−1	−46	−32	0	0	10	7
r_4	1	0	14	8	−31	−59	126	89
r_5	5	3	168	121	−57	−26	10	7
r_6	179	126	6600	4670	−1376	−949	126	89
r_7	102	72	3769	2668	−764	−509	44	31
r_8	331	235	12285	8686	−2316	−1627	430	304

det $= -10130 - 7163w$ det norm $= -238$

$(82222222)^2$

quadratic form

$$\begin{pmatrix} 13086 & 9253 & -4700 & -3324 & -174 & -124 \\ -4700 & -3324 & 1694 & 1196 & 65 & 43 \\ -174 & -124 & 65 & 43 & 6 & -1 \end{pmatrix}$$

root list

	roots						norms	
r_1	0	0	−1	−1	28	20	34	24
r_2	2	−1	0	1	3	2	10	7
r_3	0	0	0	0	3	2	78	55
r_4	1	−2	−4	−2	24	17	10	7
r_5	−6	−4	−22	−16	161	114	10	7
r_6	−91	−67	−358	−252	2594	1834	78	55
r_7	−32	−21	−119	−84	866	612	54	38
r_8	−36	−28	−147	−103	1072	758	266	188

det $= -10130 - 7163w$ det norm $= -238$

$(22222822)^2$

quadratic form

$$\begin{pmatrix} 41354 & 29227 & -18856 & -13336 & 280 & 197 \\ -18856 & -13336 & 8602 & 6082 & -127 & -90 \\ 280 & 197 & -127 & -90 & 2 & 1 \end{pmatrix}$$

root list

	roots						norms	
r_1	−10	−8	−27	−14	−2	−2	54	38
r_2	−21	−16	−52	−31	0	0	78	55
r_3	10	8	27	14	1	1	10	7
r_4	141	99	306	220	4	3	10	7
r_5	2432	1720	5343	3776	55	39	78	55
r_6	3482	2461	7641	5409	75	53	10	7
r_7	6854	4847	15055	10643	144	102	34	24
r_8	56546	39985	124189	87810	1174	830	266	188

det $= -993741 - 702681w$ det norm $= -441$

$(22222222)^2$

quadratic form

$$\begin{pmatrix} 3701 & 2617 & 406 & 287 & 106 & 75 \\ 406 & 287 & 116 & 78 & 7 & 7 \\ 106 & 75 & 7 & 7 & 3 & 1 \end{pmatrix}$$

root list

	roots						norms	
r_1	0	3	−3	−2	−142	−100	594	420
r_2	5	−3	0	0	−28	−20	75	53
r_3	−12	−3	10	7	547	387	471	333
r_4	−7	1	3	2	191	135	34	24
r_5	−303	−207	277	196	20542	14526	471	333
r_6	−45	−36	44	31	3311	2341	150	106
r_7	−105	−84	100	71	7737	5471	942	666
r_8	−11	−13	13	9	1018	719	13	9

det = $-21 - 21w$ det norm = -441

$(22222222)^2$

quadratic form

$$\begin{pmatrix} 855 & 597 & -63 & 0 & -30 & -18 \\ -63 & 0 & 199 & -139 & 15 & -9 \\ -30 & -18 & 15 & -9 & 2 & 0 \end{pmatrix}$$

root list

	roots						norms	
r_1	1	0	21	15	−11	−12	2	0
r_2	2	1	81	57	−63	−42	15	9
r_3	0	0	3	2	−6	−2	3	1
r_4	−1	0	−21	−15	12	12	18	12
r_5	3	1	78	55	−36	−26	3	1
r_6	39	28	1413	999	−696	−489	162	114
r_7	31	22	1127	797	−566	−401	26	18
r_8	275	195	10038	7098	−5100	−3606	81	57

det = $-29253 - 20685w$ det norm = -441

$(22222222)^2$

quadratic form

$$\begin{pmatrix} 31759 & 22457 & -4588 & -3244 & -942 & -666 \\ -4588 & -3244 & 674 & 462 & 141 & 93 \\ -942 & -666 & 141 & 93 & 30 & 18 \end{pmatrix}$$

root list

	roots						norms	
r_1	0	0	−3	−2	13	9	34	24
r_2	3	6	0	0	178	126	471	333
r_3	−2	0	7	5	−62	−44	150	106
r_4	−132	−96	81	57	−4512	−3190	942	666
r_5	−45	−39	24	17	−1660	−1174	13	9
r_6	−186	−132	81	57	−6140	−4341	102	72
r_7	−457	−328	187	132	−15114	−10687	13	9
r_8	−1857	−1320	747	528	−61079	−43189	81	57

det = $-29253 - 20685w$ det norm = −441

$(22222222)^2$

quadratic form

$$\begin{pmatrix} 19417 & 13529 & -6572 & -4758 & 594 & 432 \\ -6572 & -4758 & 2338 & 1592 & -213 & -144 \\ 594 & 432 & -213 & -144 & 18 & 12 \end{pmatrix}$$

root list

	roots						norms	
r_1	4	2	9	8	1	2	34	24
r_2	3	0	3	6	11	8	471	333
r_3	−3	−2	−8	−6	3	1	75	53
r_4	0	0	0	0	3	2	594	420
r_5	47	31	130	96	18	13	75	53
r_6	738	522	2160	1527	281	198	5490	3882
r_7	556	394	1630	1151	207	146	874	618
r_8	4827	3414	14130	9990	1776	1256	2745	1941

det $= -28 - 35w$ det norm $= -1666$

$(22822222)^2$

quadratic form

$$\begin{pmatrix} 364 & 231 & -28 & 14 & -14 & -7 \\ -28 & 14 & 34 & -24 & 3 & -2 \\ -14 & -7 & 3 & -2 & 2 & 1 \end{pmatrix}$$

root list

	roots						norms	
r_1	1	0	24	17	7	−4	14	−8
r_2	−1	1	10	8	0	0	16	−11
r_3	0	0	−1	1	14	−10	34	−24
r_4	0	0	0	0	−7	5	58	−41
r_5	−3	3	24	22	13	−5	118	−82
r_6	−1	1	10	6	−28	21	256	−181
r_7	1	0	25	14	−16	14	126	−89
r_8	−4	4	35	28	−14	14	238	−168

det $= -12404 - 8771w$ det norm $= -1666$

$(22222228)^2$

quadratic form

$$\begin{pmatrix} 860 & 607 & -52 & -48 & -118 & -82 \\ -52 & -48 & 86 & -54 & -1 & 14 \\ -118 & -82 & -1 & 14 & 14 & 8 \end{pmatrix}$$

root list

	roots						norms	
r_1	0	0	−3	−2	0	1	34	24
r_2	7	4	0	0	32	22	1164	823
r_3	0	−1	7	5	−5	−4	150	106
r_4	−44	−30	37	26	−227	−161	430	304
r_5	−321	−228	195	138	−1673	−1183	406	287
r_6	−467	−330	267	189	−2425	−1713	126	89
r_7	−514	−364	287	203	−2669	−1887	54	38
r_8	−439	−314	243	172	−2289	−1620	2	1

det $= 28 - 35w$ det norm $= -1666$

$(22222228)^2$

quadratic form

$$\begin{pmatrix} 200 & 141 & 0 & 0 & 0 & 0 \\ 0 & 0 & 82 & -58 & 7 & -5 \\ 0 & 0 & 7 & -5 & 2 & 0 \end{pmatrix}$$

root list

	roots						norms	
r_1	0	1	89	63	1	0	2	0
r_2	13	9	1646	1164	0	0	36	23
r_3	0	0	3	2	−2	0	6	2
r_4	0	−1	−89	−63	0	0	14	8
r_5	5	4	655	463	12	10	14	7
r_6	9	8	1253	886	22	15	6	1
r_7	12	9	1527	1080	26	17	6	−2
r_8	11	8	1379	975	20	17	10	−7

det $= -7672 - 5425w$ det norm $= -1666$

$(22222282)^2$

quadratic form

$$\left(\begin{array}{cc|cc|cc} 140780 & 99543 & -38432 & -27169 & -1022 & -728 \\ -38432 & -27169 & 10498 & 7411 & 273 & 203 \\ -1022 & -728 & 273 & 203 & 14 & 0 \end{array}\right)$$

root list

	roots						norms	
r_1	−19	−11	−65	−45	30	21	202	142
r_2	−5	4	−2	3	2	2	4	1
r_3	6	16	49	41	−31	−22	14	9
r_4	49	49	217	161	−167	−118	42	28
r_5	81	46	279	192	−229	−161	10	4
r_6	145	130	610	446	−532	−376	4	−1
r_7	45	31	167	119	−146	−102	6	−4
r_8	18	12	67	46	−57	−42	10	−7

det $= -421372 - 297955w$ det norm $= -1666$

$(22822222)^2$

quadratic form

$$\begin{pmatrix} 5008 & 3541 & 972 & 684 & 282 & 200 \\ 972 & 684 & 234 & 108 & 49 & 45 \\ 282 & 200 & 49 & 45 & 14 & 8 \end{pmatrix}$$

root list

	roots						norms	
r_1	0	0	−3	−2	6	5	150	106
r_2	−1	−3	0	0	32	22	1164	823
r_3	−4	3	2	1	−5	−4	34	24
r_4	11	9	17	12	−183	−129	10	7
r_5	190	141	265	187	−2967	−2098	314	222
r_6	1809	1279	2417	1709	−27469	−19424	734	519
r_7	6949	4911	9253	6543	−105431	−74551	2366	1673
r_8	13070	9239	17379	12289	−198263	−140193	2506	1772

det $= -1519028 - 1074115w$ det norm $= -1666$

$(22282222)^2$

quadratic form

$$\begin{pmatrix} 26932 & 19039 & 11984 & 8386 & -3954 & -2798 \\ 11984 & 8386 & 6418 & 2926 & -1719 & -1254 \\ -3954 & -2798 & -1719 & -1254 & -686 & -486 \end{pmatrix}$$

root list

	roots						norms	
r_1	−31	−43	103	74	1	0	454	321
r_2	7	−9	6	5	0	0	92	65
r_3	266	188	−601	−425	−2	0	6842	4838
r_4	59	44	−137	−97	0	0	58	41
r_5	398	292	−917	−649	1	1	198	140
r_6	2149	1514	−4854	−3432	8	6	1492	1055
r_7	1980	1404	−4487	−3173	10	7	9034	6388
r_8	5658	3996	−12799	−9050	38	27	47082	33292

Table A.12: Octadecagons

det $= -417 - 295w$ det norm $= -161$

$(223222222)^2$

quadratic form

$$\begin{pmatrix} 9253 & 6515 & 160 & 124 & 80 & 62 \\ 160 & 124 & 6 & 0 & 3 & 0 \\ 80 & 62 & 3 & 0 & 2 & 0 \end{pmatrix}$$

root list

	roots						norms	
r_1	2	0	−63	−48	6	4	6	4
r_2	8	6	−519	−366	0	0	2014	1424
r_3	0	0	1	1	−2	−2	6	4
r_4	0	0	0	0	1	1	6	4
r_5	−1	1	−18	−8	5	3	5	3
r_6	1	2	−133	−91	26	19	19	13
r_7	4	2	−229	−164	38	27	54	38
r_8	108	76	−7191	−5086	988	699	218	154
r_9	167	119	−11184	−7906	1516	1072	27	19

det $= -46167 - 32645w$ det norm $= -161$

$(222284222)^2$

quadratic form

$$\begin{pmatrix} 366263 & 258975 & 114148 & 80722 & -1904 & -1358 \\ 114148 & 80722 & 35582 & 25156 & -601 & -418 \\ -1904 & -1358 & -601 & -418 & 18 & 1 \end{pmatrix}$$

root list

	roots						norms	
r_1	−8	−2	13	16	22	16	34	24
r_2	−5	1	1	9	56	40	355	251
r_3	11	8	−37	−26	−55	−39	17	12
r_4	1126	796	−3747	−2650	−6839	−4836	11202	7921
r_5	90	66	−309	−214	−593	−419	58	41
r_6	246	174	−823	−582	−1704	−1205	198	140
r_7	309	217	−1030	−731	−2223	−1572	99	70
r_8	1326	937	−4446	−3145	−9823	−6946	3124	2209
r_9	9549	6752	−32064	−22673	−72320	−51138	12059	8527

det $= -6767 - 4785w$ det norm $= -161$

$(222228224)^2$

quadratic form

$$\begin{pmatrix} -13087 & -9338 & 1366 & 1046 & 2412 & 1729 \\ 1366 & 1046 & -100 & -147 & -247 & -197 \\ 2412 & 1729 & -247 & -197 & -415 & -300 \end{pmatrix}$$

root list

	roots						norms	
r_1	18	12	162	114	1	0	17	12
r_2	271	191	2514	1777	0	0	2803	1982
r_3	0	0	1	1	0	−1	58	41
r_4	−17	−12	−157	−111	0	0	437	309
r_5	5	3	19	13	19	13	2168	1533
r_6	27	19	235	166	11	8	58	41
r_7	63	44	558	394	17	12	198	140
r_8	955	675	8582	6068	199	141	32674	23104
r_9	61	43	552	390	9	7	198	140

det $= -233 - 165w$ det norm $= -161$

$(422222228)^2$

quadratic form

$$\begin{pmatrix} 4491 & 3159 & -1686 & -1190 & 40 & 38 \\ -1686 & -1190 & 634 & 448 & -17 & -13 \\ 40 & 38 & -17 & -13 & 8 & -5 \end{pmatrix}$$

root list

	roots						norms	
r_1	2	0	9	−2	12	8	6	4
r_2	−1	0	−4	1	1	1	3	2
r_3	−22	−16	−78	−58	−581	−411	92	65
r_4	−421	−298	−1570	−1112	−12708	−8986	355	251
r_5	−62	−44	−233	−166	−1936	−1369	34	24
r_6	−1379	−975	−5239	−3704	−44306	−31329	355	251
r_7	−347	−245	−1321	−932	−11205	−7923	17	12
r_8	−14602	−10325	−55641	−39343	−473937	−335124	11202	7921
r_9	−728	−515	−2777	−1965	−23729	−16779	58	41

det $= -2431 - 1719w$ det norm $= -161$

$(322222222)^2$

quadratic form

$$\begin{pmatrix} 3987 & 2819 & -5536 & -3916 & -204 & -142 \\ -5536 & -3916 & 7698 & 5434 & 271 & 206 \\ -204 & -142 & 271 & 206 & 26 & -4 \end{pmatrix}$$

root list

	roots						norms	
r_1	0	0	−1	0	12	9	6	4
r_2	0	2	2	0	1	0	6	4
r_3	0	0	0	0	7	5	2014	1424
r_4	−6	0	−2	−2	7	5	6	4
r_5	−49	−35	−42	−28	140	100	5	3
r_6	−94	−74	−83	−58	289	204	38	26
r_7	−66	−48	−55	−40	199	140	10	6
r_8	−717	−516	−607	−425	2171	1537	5	1
r_9	−473	−323	−389	−274	1396	989	3	−1

det $= -1359 - 961w$ det norm $= -161$

$(222284222)^2$

quadratic form

$$\begin{pmatrix} 63085 & 44515 & 4253 & 3018 & 214 & 155 \\ 4253 & 3018 & 289 & 203 & 15 & 10 \\ 214 & 155 & 15 & 10 & 2 & 1 \end{pmatrix}$$

root list

	roots						norms	
r_1	−7	−6	123	75	−6	4	6	4
r_2	−24	−17	355	251	0	0	61	43
r_3	20	14	−293	−209	−1	1	3	2
r_4	2569	1817	−37993	−26860	19	10	1922	1359
r_5	225	159	−3325	−2352	1	1	10	7
r_6	653	462	−9659	−6827	4	1	34	24
r_7	857	606	−12670	−8959	3	2	17	12
r_8	3800	2687	−56178	−39724	11	8	536	379
r_9	28055	19838	−414754	−293274	72	50	2069	1463

det $= -1161 - 821w$ det norm $= -161$

$(822222422)^2$

quadratic form

$$\begin{pmatrix} 185 & 127 & -52 & -50 & 50 & 33 \\ -52 & -50 & 54 & -8 & -11 & -16 \\ 50 & 33 & -11 & -16 & 12 & 7 \end{pmatrix}$$

root list

	roots						norms	
r_1	0	0	−1	−1	−4	−1	6	4
r_2	2	−1	0	1	1	−1	2	1
r_3	0	0	0	0	1	1	64	45
r_4	−1	−2	−8	−5	−2	−3	13	9
r_5	−4	−1	−11	−8	−5	−5	2	1
r_6	−85	−63	−365	−257	−209	−149	83	58
r_7	−5	−2	−16	−12	−11	−6	1	0
r_8	−10	−6	−39	−28	−24	−17	2	0
r_9	−374	−268	−1609	−1136	−994	−707	166	116

det $= -233 - 165w$ det norm $= -161$

$(228422222)^2$

quadratic form

$$\begin{pmatrix} -1359 & -961 & -39 & -29 & -1359 & -961 \\ -39 & -29 & 555 & -393 & 8 & -15 \\ -1359 & -961 & 8 & -15 & 63 & 44 \end{pmatrix}$$

root list

	roots						norms	
r_1	−1	−1	205	145	−3	−4	3	2
r_2	8	6	−1359	−961	39	29	1922	1359
r_3	2	1	−287	−203	5	6	10	7
r_4	−3	−2	495	350	−12	−8	34	24
r_5	−16	−11	2670	1888	−61	−44	17	12
r_6	−109	−77	18430	13032	−423	−300	536	379
r_7	−1099	−777	185880	131437	−4260	−3012	2069	1463
r_8	−129	−91	21793	15410	−498	−353	198	140
r_9	−2304	−1629	389671	275539	−8900	−6293	2069	1463

det $= 1 - 9w$ det norm $= -161$

$(224228222)^2$

quadratic form

$$\begin{pmatrix} 4479 & 3159 & 119 & 56 & 1526 & 1078 \\ 119 & 56 & 283 & -197 & 43 & 18 \\ 1526 & 1078 & 43 & 18 & 562 & 397 \end{pmatrix}$$

root list

	roots						norms	
r_1	−20	−14	1072	758	−11	−8	2	1
r_2	−195	−138	10532	7447	−117	−74	83	58
r_3	30	21	−1609	−1138	11	16	1	0
r_4	595	421	−32109	−22704	348	230	2	0
r_5	37523	26533	−2024345	−1431428	21230	15010	166	116
r_6	5113	3616	−275867	−195067	2904	2038	2	0
r_7	5292	3742	−285504	−201882	2989	2121	2	−1
r_8	15304	10822	−825676	−583841	8663	6121	12	7
r_9	3885	2747	−209596	−148207	2192	1559	3	1

det $= -417 - 295w$ det norm $= -161$

$(322222222)^2$

quadratic form

$$\begin{pmatrix} 1019 & 719 & -474 & -334 & -31 & -20 \\ -474 & -334 & 222 & 156 & 15 & 9 \\ -31 & -20 & 15 & 9 & 3 & -1 \end{pmatrix}$$

root list

	roots						norms	
r_1	0	0	−1	0	6	4	6	4
r_2	1	0	2	0	1	1	6	4
r_3	0	0	0	0	1	1	5	3
r_4	−3	−2	−7	−6	26	19	19	13
r_5	−6	−5	−17	−13	58	41	54	38
r_6	−228	−161	−635	−449	2168	1533	218	154
r_7	−357	−252	−996	−704	3408	2410	27	19
r_8	−205	−145	−573	−405	1963	1388	34	24
r_9	−18977	−13419	−53065	−37523	182029	128714	11738	8300

det = $-20598 - 14565w$ det norm = -846

$(222222622)^2$

quadratic form

$$\begin{pmatrix} 36734 & 25169 & 8132 & 5808 & -816 & -600 \\ 8132 & 5808 & 1850 & 1304 & -189 & -132 \\ -816 & -600 & -189 & -132 & 18 & 12 \end{pmatrix}$$

root list

	roots						norms	
r_1	4	2	−7	−16	1	2	34	24
r_2	9	6	−27	−24	59	42	8532	6033
r_3	−1	−1	8	2	0	1	10	7
r_4	0	0	0	0	1	1	102	72
r_5	65	46	−286	−202	25	17	488	345
r_6	19	13	−78	−61	6	5	10	7
r_7	50	35	−215	−157	17	12	34	24
r_8	408	288	−1785	−1269	138	98	594	420
r_9	698	493	−3057	−2169	236	168	1666	1178

det $= -120054 - 84891w$ det norm $= -846$

$(222222226)^2$

quadratic form

$$\left(\begin{array}{cc|cc|cc} 84190 & 59249 & 9684 & 6684 & 558 & 360 \\ 9684 & 6684 & 1158 & 724 & 75 & 33 \\ 558 & 360 & 75 & 33 & 6 & 0 \end{array}\right)$$

root list

	roots						norms	
r_1	0	0	−1	−1	15	12	34	24
r_2	−1	1	0	−3	4	3	10	7
r_3	3	0	−16	−8	9	8	488	345
r_4	0	0	0	0	3	2	102	72
r_5	−3	1	10	2	12	7	10	7
r_6	−57	−39	453	318	749	530	8532	6033
r_7	−4	1	15	4	17	12	34	24
r_8	−16	−10	119	82	227	160	1666	1178
r_9	−18	−15	147	108	329	232	594	420

det $= -20598 - 14565w$ det norm $= -846$

$(622222222)^2$

quadratic form

$$\begin{pmatrix} 1526 & 1079 & -1004 & -712 & 118 & 84 \\ -1004 & -712 & 742 & 418 & -99 & -41 \\ 118 & 84 & -99 & -41 & 14 & 2 \end{pmatrix}$$

root list

	roots						norms	
r_1	0	0	−1	0	−8	−3	6	4
r_2	2	1	2	2	−3	0	102	72
r_3	−68	−50	0	0	1259	890	286	202
r_4	−40	−31	−2	−2	733	516	6	4
r_5	−5713	−4039	−404	−286	98745	69822	1464	1035
r_6	−131	−99	−10	−7	2338	1653	2	1
r_7	−216	−154	−17	−12	3729	2637	18	12
r_8	−713	−498	−59	−42	12139	8583	84	59
r_9	−93	−67	−8	−6	1604	1133	2	1

Table A.13: Icosagons

det = $-666-471w$	det norm = -126
$(2222222222)^2$	

quadratic form

$$\left(\begin{array}{cc|cc|cc} 174 & 123 & 12 & 9 & 0 & 0 \\ 12 & 9 & 20 & -13 & 3 & -1 \\ 0 & 0 & 3 & -1 & 5 & 3 \end{array}\right)$$

root list

	roots						norms	
r_1	1	2	−20	−14	1	0	3	2
r_2	3	1	−24	−17	0	0	10	7
r_3	1	2	−27	−19	−2	−2	314	222
r_4	−1	0	0	0	0	0	174	123
r_5	0	0	0	0	3	2	157	111
r_6	3	3	−34	−24	6	4	58	41
r_7	13	9	−123	−87	14	10	198	140
r_8	682	482	−6627	−4686	594	420	9372	6627
r_9	88	62	−857	−606	72	51	338	239
r_{10}	1121	793	−10980	−7764	870	615	9372	6627

det = $-114-81w$	det norm = -126
$(2222222222)^2$	

quadratic form

$$\left(\begin{array}{cc|cc|cc} 48 & 33 & 0 & 0 & -15 & -9 \\ 0 & 0 & 2 & -2 & 3 & 0 \\ -15 & -9 & 3 & 0 & 3 & -2 \end{array}\right)$$

root list

	roots						norms	
r_1	0	0	0	0	−1	−1	1	0
r_2	1	−3	−15	−9	−18	−15	48	33
r_3	−1	1	2	1	6	5	2	1
r_4	29	18	228	162	486	342	48	33
r_5	2	3	27	18	52	38	2	0
r_6	9	4	61	44	120	84	2	−1
r_7	21	10	147	105	285	199	3	−1
r_8	35	29	321	225	606	432	6	−3
r_9	44	20	301	217	580	404	6	−2
r_{10}	13	1	59	44	118	78	10	−7

det $= -3882 - 2745w$ det norm $= -126$

$(2222222222)^2$

quadratic form

$$\begin{pmatrix} 1812 & 1281 & -30 & -21 & 111 & 78 \\ -30 & -21 & 2 & 1 & -2 & -1 \\ 111 & 78 & -2 & -1 & 7 & 4 \end{pmatrix}$$

root list

	roots						norms	
r_1	−1	−2	13	9	30	22	54	38
r_2	−2	0	9	6	18	12	30	21
r_3	5	3	−32	−23	−75	−53	27	19
r_4	14	10	−103	−73	−232	−164	10	7
r_5	39	28	−291	−206	−650	−460	34	24
r_6	1835	1297	−13674	−9669	−30420	−21510	1608	1137
r_7	229	162	−1710	−1209	−3800	−2687	58	41
r_8	2842	2009	−21243	−15021	−47166	−33351	1608	1137
r_9	105	74	−785	−555	−1741	−1231	17	12
r_{10}	406	287	−3045	−2153	−6746	−4770	58	41

Table A.13, cont.

det $= -131874 - 93249w$	det norm $= -126$
$(2222222222)^2$	

quadratic form

$$\left(\begin{array}{cc|cc|cc} -39396 & -27861 & -4926 & -3498 & -14385 & -10173 \\ -4926 & -3498 & -578 & -466 & -1799 & -1277 \\ -14385 & -10173 & -1799 & -1277 & -4795 & -3391 \end{array}\right)$$

root list

	roots						norms	
r_1	−104	−75	838	593	1	0	157	111
r_2	−81	−59	657	465	0	0	174	123
r_3	0	0	1	1	2	−2	314	222
r_4	5	3	−37	−26	0	0	10	7
r_5	−3	2	1	0	3	−2	3	2
r_6	−323	−226	2547	1800	6	9	276	195
r_7	−51	−36	404	286	4	−1	10	7
r_8	−775	−542	6120	4326	18	12	276	195
r_9	−48	−35	387	274	2	0	6	4
r_{10}	−63	−50	531	376	−6	6	2	1

det $= -666 - 471w$	det norm $= -126$
$(2222222222)^2$	

quadratic form

$$\left(\begin{array}{cc|cc|cc} 102 & -51 & 72 & 15 & 0 & -3 \\ 72 & 15 & 152 & 97 & -3 & -3 \\ 0 & -3 & -3 & -3 & 3 & 2 \end{array}\right)$$

root list

	roots						norms	
r_1	15	9	−8	0	1	0	5	3
r_2	13	11	0	−6	0	0	6	3
r_3	−71	−51	19	16	−4	0	10	6
r_4	−65	−50	10	21	0	−2	2	−1
r_5	−56	−40	16	12	−5	2	3	−2
r_6	−3743	−2649	1092	780	−66	−51	12	3
r_7	−466	−333	129	103	−10	−5	2	−1
r_8	−5824	−4116	1713	1203	−108	−72	12	3
r_9	−309	−212	105	52	−8	−2	6	−4
r_{10}	−347	−244	104	70	2	−10	10	−7

$\det = -114 - 81w$ $\quad$ det norm $= -126$

$(2222222222)^2$

quadratic form

$$\left(\begin{array}{cc|cc|cc} 1050 & 735 & -36 & -9 & 18 & 15 \\ -36 & -9 & 28 & -19 & 3 & -3 \\ 18 & 15 & 3 & -3 & 1 & 0 \end{array}\right)$$

root list

	roots						norms	
r_1	1	0	21	15	2	−2	6	4
r_2	4	3	195	138	0	0	276	195
r_3	0	0	3	2	0	1	10	7
r_4	−1	0	0	0	18	15	276	195
r_5	0	0	0	0	1	1	3	2
r_6	1	1	44	31	6	6	10	7
r_7	13	9	471	333	54	38	314	222
r_8	23	16	840	594	78	54	174	123
r_9	17	12	628	444	49	35	157	111
r_{10}	13	9	478	338	32	22	58	41

Table A.13, cont.

---

det $= -666 - 471w$ det norm $= -126$

$(2222222222)^2$

---

quadratic form

$$\left(\begin{array}{cc|cc|cc} 48 & -15 & 48 & 54 & 3 & 6 \\ 48 & 54 & 286 & 194 & 15 & 9 \\ 3 & 6 & 15 & 9 & -15 & -11 \end{array}\right)$$

root list

	roots						norms	
r_1	−42	−28	7	9	−10	−6	10	6
r_2	−35	−22	3	9	−6	−6	6	3
r_3	−10	−6	0	3	−1	−2	5	3
r_4	−1	1	−3	2	0	0	2	1
r_5	6	5	−3	0	2	0	6	4
r_6	135	95	−24	−18	0	0	276	195
r_7	11	7	0	−2	−2	−1	10	7
r_8	27	19	9	6	−48	−33	276	195
r_9	−3	−3	3	0	−3	−2	3	2
r_{10}	−39	−28	13	8	−18	−12	10	7

---

det $= -666 - 471w$ det norm $= -126$

$(2222222222)^2$

---

quadratic form

$$\left(\begin{array}{cc|cc|cc} 174 & 123 & 18 & 12 & 0 & 0 \\ 18 & 12 & 14 & -8 & -3 & 1 \\ 0 & 0 & -3 & 1 & 3 & -1 \end{array}\right)$$

root list

	roots						norms	
r_1	6	−2	−9	−6	−6	−4	6	4
r_2	−9	7	−2	−2	−2	−2	2	1
r_3	0	0	0	0	−1	−1	5	3
r_4	−3	2	0	0	0	0	6	3
r_5	−18	13	−1	−2	2	0	10	6
r_6	9	−6	−2	−1	0	0	2	−1
r_7	−25	18	0	−2	1	−1	3	−2
r_8	47	−11	−96	−66	−24	−15	12	3
r_9	−13	12	−11	−9	−2	−3	2	−1
r_{10}	13	26	−147	−105	−42	−30	12	3

Table A.14: 24-gons

det = −10369 − 7332w det norm = −287

$(422222222222)^2$

quadratic form

$$\begin{pmatrix} 43879 & 31022 & -4894 & -3460 & -204 & -152 \\ -4894 & -3460 & 546 & 386 & 23 & 17 \\ -204 & -152 & 23 & 17 & 10 & -6 \end{pmatrix}$$

root list

	roots						norms	
r_1	2	0	13	4	−22	−15	34	24
r_2	−1	0	−6	−2	−1	−1	17	12
r_3	−26	−18	−254	−178	1157	818	874	618
r_4	−175	−124	−1750	−1238	9229	6526	973	688
r_5	−21	−14	−207	−143	1112	786	17	12
r_6	−522	−370	−5261	−3723	29466	20836	8590	6074
r_7	−19	−13	−190	−133	1092	772	17	12
r_8	−208	−146	−2109	−1487	12580	8895	1946	1376
r_9	−34	−24	−347	−245	2134	1509	34	24
r_{10}	−889	−628	−9098	−6430	57044	40336	167	118
r_{11}	−196	−140	−2015	−1430	12747	9014	150	106
r_{12}	−15	−12	−160	−119	1050	743	3	2

det = −60435 − 42734w det norm = −287

$(222222222242)^2$

quadratic form

$$\begin{pmatrix} 103397 & 73104 & -67596 & -47789 & 2134 & 1500 \\ -67596 & -47789 & 44192 & 31240 & -1397 & -979 \\ 2134 & 1500 & -1397 & -979 & 50 & 26 \end{pmatrix}$$

Table A.14, cont.

root list

	roots						norms	
r_1	0	−7	−3	−8	11	9	26	18
r_2	−13	8	−13	8	13	9	29	20
r_3	5	−3	4	−2	−1	1	1	0
r_4	0	0	0	0	1	1	254	178
r_5	15	−12	14	−12	0	1	1	0
r_6	−36	−22	−54	−33	29	22	58	40
r_7	2	−16	−4	−19	9	8	2	0
r_8	−393	−247	−578	−376	380	270	7	2
r_9	−112	−49	−155	−82	96	65	6	2
r_{10}	15	−23	12	−27	10	6	3	−2
r_{11}	−40	13	−45	9	12	8	6	−4
r_{12}	−31	−41	−55	−55	47	36	3	−2

det $= -1779 - 1258w$ det norm $= -287$

$(222222222242)^2$

quadratic form

$$\begin{pmatrix} 7 & 2 & 7 & 2 & 0 & 0 \\ 7 & 2 & 5 & 0 & 7 & 5 \\ 0 & 0 & 7 & 5 & 17 & 12 \end{pmatrix}$$

root list

roots							norms	
r_1	1	0	−1	0	−6	4	6	2
r_2	1	0	0	0	0	0	7	2
r_3	0	0	0	0	−7	5	3	−2
r_4	2	−2	−7	−2	−3	5	50	26
r_5	0	0	−1	0	−4	3	3	−2
r_6	0	1	−11	−9	−3	5	14	4
r_7	2	−1	−5	−1	−9	7	6	−4
r_8	12	8	−109	−78	8	14	13	−8
r_9	3	2	−26	−19	15	−6	10	−6
r_{10}	2	−1	−5	0	2	−1	17	−12
r_{11}	5	−3	−6	0	−5	4	34	−24
r_{12}	−1	3	−10	−10	−6	6	17	−12

$\det = -1779 - 1258w$ $\qquad$ $\det \text{norm} = -287$

$(222242222222)^2$

quadratic form

$$\left(\begin{array}{cc|cc|cc} 93183 & 65890 & 2814 & 1993 & -1641 & -1160 \\ 2814 & 1993 & 108 & 44 & -47 & -37 \\ -1641 & -1160 & -47 & -37 & 29 & 20 \end{array}\right)$$

Table A.14, cont.

root list

	roots						norms	
r_1	−2	−1	55	39	−2	−2	34	24
r_2	−9	−8	334	236	0	0	167	118
r_3	1	0	−14	−10	5	4	150	106
r_4	−1	1	−6	−4	2	1	3	2
r_5	1	−1	10	7	5	3	6	4
r_6	−6	−4	215	152	31	22	3	2
r_7	−53	−36	1899	1343	251	177	150	106
r_8	−194	−137	7047	4983	875	619	167	118
r_9	−20	−12	670	474	81	57	3	2
r_{10}	−381	−269	13760	9730	1601	1131	1474	1042
r_{11}	−11	−7	376	266	42	29	3	2
r_{12}	−85	−58	2988	2113	305	216	334	236

det $= -1779 - 1258w$ det norm $= -287$

$(222224222222)^2$

quadratic form

$$\begin{pmatrix} 167 & 118 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & -2 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{pmatrix}$$

root list

roots							norms	
r_1	1	0	3	2	-4	-3	3	2
r_2	5	4	29	20	-69	-49	1474	1042
r_3	0	0	0	0	-1	-1	3	2
r_4	-1	0	0	0	0	0	167	118
r_5	0	0	3	2	6	4	150	106
r_6	1	0	4	3	0	0	3	2
r_7	1	1	9	6	-3	-2	6	4
r_8	5	4	37	26	-18	-13	3	2
r_9	111	79	765	541	-399	-282	150	106
r_{10}	627	443	4295	3037	-2280	-1612	167	118
r_{11}	37	26	252	178	-137	-97	34	24
r_{12}	405	286	2752	1946	-1543	-1091	1946	1376

Bibliography

[1] Ian Agol. Finiteness of arithmetic Kleinian reflection groups. In *Proceedings of the International Congress of Mathematicians: Madrid, August 22-30, 2006: invited lectures*, pages 951–960, 2006.

[2] Ian Agol, Mikhail Belolipetsky, Peter Storm, and Kevin Whyte. Finiteness of arithmetic hyperbolic reflection groups. *arXiv preprint math.GT/0612132*, 2006.

[3] D. Allcock. *The Reflective Lorentzian Lattices of Rank 3*. American Mathematical Soc., 2012.

[4] Padraic Bartlett. Finding *All* the roots: Sturm's theorem. *Mathcamp*, 2013.

[5] V.O. Bugaenko. Groups of automorphisms of unimodular hyperbolic quadratic forms over the ring $\mathbb{Z}\left[\frac{\sqrt{5}+1}{2}\right]$. *Moscow Univ. Math. Bull*, 39:6–14, 1984.

[6] V.O. Bugaenko. On reflective unimodular hyperbolic quadratic forms. *Selecta Math. Soviet*, 9(3):263–271, 1990.

[7] V.O. Bugaenko. Arithmetic crystallographic groups generated by reflections, and reflective hyperbolic lattices. *Lie groups, their discrete subgroups, and invariant theory*, 8:33–55, 1992.

[8] Harold S.M. Coxeter. Discrete groups generated by reflections. *Annals of Mathematics*, pages 588–621, 1934.

[9] H. Koch. *Number Theory: Algebraic Numbers and Functions*. Graduate studies in mathematics. American Mathematical Society, 2000.

[10] D.D. Long, C. Maclachlan, and A.W. Reid. Arithmetic fuchsian groups of genus zero. *Pure and Applied Mathematics Quarterly*, 2(2):569–99, 2006.

[11] V. V. Nikulin. Finiteness of the number of arithmetic groups generated by reflections in Lobachevsky spaces. *Izvestiya: Mathematics*, 71(1):53, 2007.

[12] V.V. Nikulin. On arithmetic groups generated by reflections in Lobachevsky spaces. *Mathematics of the USSR-Izvestiya*, 16(3):573, 1981.

[13] V.V. Nikulin. On the classification of arithmetic groups generated by reflections in lobachevsky spaces. *Mathematics of the USSR-Izvestiya*, 18(1):99, 1982.

[14] V.V. Nikulin. *On the classification of hyperbolic root systems of rank three*. Proceedings of the Steklov Institute of Mathematics. Maik Nauka/Interperiodica Pub., 2000.

[15] O.T. O'Meara. *Introduction to Quadratic Forms*. Classics in Mathematics. Springer Berlin Heidelberg, 1999.

[16] Kisao Takeuchi. Arithmetic triangle groups. *Journal of the Mathematical Society of Japan*, 29(1):91–106, 1977.

[17] The PARI Group, Bordeaux. *PARI/GP version* `2.5.4`, 2014. available from `http://pari.math.u-bordeaux.fr/`.

[18] È. B. Vinberg. Hyperbolic reflection groups. *Russian Mathematical Surveys*, 40(1):31, 1985.

[19] É. B. Vinberg and I.M. Kaplanskaja. On The Groups $O_{18,1}(\mathbb{Z})$ and $O_{19,1}(\mathbb{Z})$. *(Russian) Dokl. Akad.*, 238:1273–1275, 1978.

[20] È.B. Vinberg. Discrete groups generated by reflections in Lobachevskii spaces. *Matematicheskii Sbornik*, 114(3):471–488, 1967.

[21] È.B. Vinberg. On groups of unit elements of certain quadratic forms. *Sbornik: Mathematics*, 16(1):17–35, 1972.

Vita

Alice Harway Mark grew up in New York City. She recieved a Bachelor of Arts in Mathematics from the University of Chicago in June of 2009. In August of 2009 she moved to Austin to begin graduate school.

Permanent address: amark@math.utexas.edu

This dissertation was typeset with LaTeX[†] by the author.

[†]LaTeX is a document preparation system developed by Leslie Lamport as a special version of Donald Knuth's TeX Program.