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RESISTANCE OF BALANCED INCOMPLETE BLOCK DESIGNS

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Hedayat and John (1974) studied the case of resistance of a balanced incomplete block design (BIBD) to the removal of one treatment. They showed that a necessary and sufficient condition for a BIBD to be resistant (that is, retain its variance balance) upon removal of a single treatment is that the two proper subdesigns created by this removal be BIBDs. Herein their results on the structure of resistant BIBDs are generalized and the relationship of such designs to t-designs (which are of interest in combinatorial theory) is shown. The main result is a structure theorem which states that a BIBD in v treatments and k plots per block is resistant to the removal of any subset of a specific set of n treatments, $n < k < v$, if and only if all of the 2^n proper subdesigns formed when all n treatments are removed are BIBDs in all $v - n$ remaining treatments. This leads to a lower bound (of $2^n(v - n)$) on the number of blocks required for fully locally resistant BIBDs which are not essentially trivial. Examples of resistant designs are given.

1. Preliminaries. In dealing with block designs we assume the homoscedastic linear additive fixed-effects model and the basic definitions, notation, and introductory lemmas of Sections 1 and 2 of Hedayat and John (1974). All designs considered here are connected so that we may use the coefficient matrix, $\mathbf{C}$, to characterize variance balance in a block design, D (see Rao (1958), Atiqullah (1961)).

LEMMA 1.1. *If D is connected then it is variance balanced if and only if $\mathbf{C}$ is of the form $\mathbf{C} = c_1\mathbf{I} + c_2\mathbf{J}$, c_i is a scalar, $\mathbf{I}$ the identity matrix and $\mathbf{J}$ a matrix of ones.*

A binary design (the only kind we consider), D, in v treatments and b blocks, may be looked upon as the union of proper subdesigns $D_1, \cdots, D_q$, $q \leqq b$, and the elements of $\mathbf{C}$ may be written

$$(1.1) \qquad c_{ii} = r_i - \textstyle\sum_{j=1}^{q} \frac{r_i(j)}{k(j)}\,; \qquad c_{hi} = -\textstyle\sum_{j=1}^{q} \frac{\lambda_{hi}(j)}{k(j)}\,, \qquad h, i = 1, 2, \cdots, v,\ h \neq i$$

where q is the number of proper subdesigns considered, and for subdesign D_j, we let $r_i(j)$, $k(j)$, and $\lambda_{hi}(j)$ (the *pairing parameter* for treatments h and i) denote, respectively, the number of replications of treatment i, the number of plots per block, and the number of blocks in which treatments h and i both appear.

2. Definition of and notation for resistance of BIBDs. Let D be a BIBD (v, b, r, k, λ) on a set, Ω, of v treatments.

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Let S be a subset of Ω. We make the following definitions (where "$=$" is read "is defined to mean"):

D is LR (S) $=$ The block design remaining after all experimental units corresponding to the treatments in S are removed is variance balanced; in this case we say D is *locally resistant* to S.

D is LR (s) $=$ D is locally resistant of degree s.
$=$ There exists at least one subset of Ω of cardinality s to which D is locally resistant. We may specify such a subset explicitly in which case we would write D is LR $(x_1, x_2, \cdots, x_s)$ where $\{x_1, x_2, \cdots, x_s\}$ is a subset of Ω.

D is FLR $(x_1, x_2, \cdots, x_s)$ $=$ D is *fully locally resistant* to $(x_1, \cdots, x_s)$.
$=$ D is LR (all subsets of $\{x_1, \cdots, x_s\}$).

D is FLR (n) $=$ D is fully locally : sistant of degree n.
$=$ D is FLR (some treatment set of size n).

D is FLR $(s: x_1, \cdots, x_n)$ $=$ D is fully locally resistant to any subset of $\{x_1, \cdots, x_n\}$ of size less than or equal to s, $s \leqq n$.

D is FGR (m) $=$ D is *fully globally resistant* of degree m.
$=$ D is FLR (all subsets of Ω of cardinality less than or equal to m).

3. Notation regarding subdesigns. Let D be a BIBD with parameters v, b, r, k, λ whose treatments are denoted by $t_1, t_2, t_3, \cdots, t_i, \cdots, t_v$. When a subset consisting of $s(s \leqq v)$ treatments is removed, this subset will be labelled $x_1, x_2, \cdots, x_s$. For the moment, we imagine the treatments being removed successively starting with x_1. Collecting those blocks of D which contain x_1 and those which do not, we have two subdesigns; if x_1 is removed from each block in which it appears, we are left with two proper subdesigns of D. Removal of x_2 from each of these subdesigns results in formation of four subdesigns. Continuing in this manner we see that when s treatments are removed there are 2^s proper subdesigns "formed" (with block sizes ranging from max $(k - s, 0)$ to k plots per block).

We denote by $\mathscr{D}_s$ the collection of subdesigns "created" or formed by the removal of treatments $x_1, \cdots, x_s$. The notation $D(s: \hat{x}_1, \hat{x}_2, \cdots, \hat{x}_s)$ refers to a proper subdesign of the design formed when s treatments, namely $x_1, x_2, \cdots, x_s$, are removed from D; further we uniquely identify one of these subdesigns by specifying those blocks of D which, when truncated (that is, when the s treatments are removed), comprise the blocks of the desired subdesign. The subdesigns of interest may be defined in terms of which of the s treatments had to be deleted from the blocks of D in order to form the blocks of this subdesign.

We set $\hat{x}_j = \bar{x}_j$ when x_j is present, and $\hat{x}_j = x_j$ when x_j is not present in the original blocks of D prior to their removal when forming the subdesigns.

We may simplify the above notation by arranging the individual subdesigns

TABLE 3.1

Standard order name	*Subdesign full designation*	*Contains all blocks originally having the treatments indicated**		
$D(3, 1)$	$D(3: \bar{1}\,\bar{2}\,\bar{3})$	x_3	x_2	x_1
$D(3, 2)$	$D(3: \bar{1}\,\bar{2}\,3)$		x_2	x_1
$D(3, 3)$	$D(3: \bar{1}\,2\,\bar{3})$	x_3		x_1
$D(3, 4)$	$D(3: \bar{1}\,2\,3)$			x_1
$D(3, 5)$	$D(3: 1\,\bar{2}\,\bar{3})$	x_3	x_2	
$D(3, 6)$	$D(3: 1\,\bar{2}\,3)$		x_2	
$D(3, 7)$	$D(3: 1\,2\,\bar{3})$	x_3		
$D(3, 8)$	$D(3: 1\,2\,3)$			

* Note that these treatments are then removed in forming the subdesigns.

comprising $\mathscr{D}_s$ in a standard order and denoting them by $D(s, 1), D(s, 2), \cdots, D(s, 2^s)$. The standard ordering is exemplified in Table 3.1 for the case when $s = 3$ and the special subcase when the treatments removed are x_1, x_2, and x_3. For example, $D(3, 1)$ consists of just those blocks of D which contained treatments x_1, x_2 and x_3 but with these three treatments removed.

Corresponding notation may be used to specify the parameters of a specific subdesign of D. Thus, using the standard order notation, the number of blocks in $D(s, j)$, $1 \leqq j \leqq 2^s$, will be denoted $b(s, j)$; the number of plots per block in $D(s, j)$ will be denoted $k(s, j)$; the number of replications of treatment i in $D(s, j)$ will be denoted $r_i(s, j)$; the pairing parameter of treatments h and i in $D(s, j)$ will be denoted $\lambda_{hi}(s, j)$. (All of the subdesigns of D to be considered will be proper and binary.)

4. The structure theorem.

THEOREM 4.1 (structure theorem). *A* BIBD, *D*, *with k plots per block, is* FLR $(x_1, \cdots, x_n)$, $n < k$, *if and only if all subdesigns formed by removal of each subset of* $x_1, \cdots, x_n$ *are* BIBDs *in all of the remaining treatments* (*but generally with differing block sizes*).

Theorem 4.1 focuses on the structure of those BIBDs whose resistance properties would seem to be of greatest significance in actual experimentation where any element of a certain set of treatments is potentially lethal, while the subset of these treatments which will actually be lethal is not known a priori; in general, loss of experimental units upsets the balance of the experiment in addition to complicating the analysis.

The case when $n \geqq k$ requires more groundwork and is covered in Theorem 6.6.

5. Proof of structure theorem and corollaries. For $n = 1$, the theorem has been proved by Hedayat and John (1974). We proceed by complete induction on the size of the subset removed using Lemma 1.1 to test for balance.

Assume the theorem to be true when the size of the subset removed is $s - 1$

or less, $s \leqq n$. Thus our induction hypothesis, H, is: D is FLR $(s-1: x_1, \cdots, x_n)$ if and only if *all* subdesigns formed by the removal of any subset of $x_1, \cdots, x_n$ of size $\leqq s-1$ are BIBDs. We must show that this implies: D is FLR $(s: x_1, \cdots, x_n)$ if and only if *all* designs formed by the removal of any subset of $\{x_1, \cdots, x_n\}$ of size $\leqq s$ are BIBDs. We are considering the removal of subsets of $\{x_1, \cdots, x_n\}$ of size s which may be thought of as subsets of size $s-1$ with one other treatment adjoined. By relabelling and rearrangement of blocks we may, without loss of generality, focus on the subset $\{x_1, \cdots, x_s\}$; that is, the proof will be the same for any subset of size s. In this light and because of our induction hypothesis, H, we may rephrase our goal as follows: Assuming D is FLR $(s-1: x_1, x_2, \cdots, x_s)$ if and only if *all* subdesigns formed by the removal of any subset of $x_1, \cdots, x_s$ of size $\leqq s-1$ are BIBDs, we wish to show D is LR $(x_1, \cdots, x_s)$ if and only if *all* designs formed by the removal of $x_1, \cdots, x_s$ are BIBDs. By the way in which we have defined the subdesigns of interest in terms of the original BIBD D, each of these subdesigns is binary and proper. Thus, for any two treatments h and i, our interest will center on the parameters $\lambda_{hi}(s, j)$ and $r_i(s, j)$. If, for design $D(s, j)$, the parameters $\lambda_{hi}(s, j)$ and $r_i(s, j)$ can be shown to be independent of h, i and i respectively, then we may conclude that $D(s, j)$ is a BIBD.

We shall focus on the parameters $\lambda_{hi}(s, j)$ where $j = 1, 2, \cdots, 2^s$. (A parallel argument holds when the parameters $r_i(s, j)$ are used in place of $\lambda_{hi}(s, j)$; in fact, except for a minor change in (5.5), we may identify $r_i(s, j)$ with "$\lambda_{ii}(s, j)$"). We may relate these parameters to those of the subdesigns formed by removal of subsets of size $s-1$ or less. For example, when $s = 4$, we may consider the subdesign $D(2: \bar{2}, \bar{3})$ formed when only treatments 2 and 3 are removed as the union of four subdesigns. Using the notation introduced earlier, we have

$$\begin{aligned} D(2: \bar{2}\,\bar{3}) = D(4: \bar{1}\,\bar{2}\,\bar{3}\,\bar{4}; +1, +4) &+ D(4: \bar{1}\,\bar{2}\,\bar{3}\,4; +1) \\ &+ D(4: 1\,\bar{2}\,\bar{3}\,\bar{4}; +4) + D(4: 1\,\bar{2}\,\bar{3}\,4) \end{aligned}$$

where the notation "$+1, +4$" indicates that treatments 1 and 4 have been appended to each block of the design; the notations "$+1$" and "$+4$" are defined similarly. Then, restricting h and i to be distinct treatments which are not among those whose removal is considered we have

$$\lambda_{hi}(2: \bar{2}\,\bar{3}) = \lambda_{hi}(4, 1) + \lambda_{hi}(4, 2) + \lambda_{hi}(4, 9) + \lambda_{hi}(4, 10)\,.$$

Since similar equations may be written for any other pair of removed treatments or, in fact, for any subset of size less than or equal to $4 - 1 = 3$, the "trick" lies in selecting a correct (linearly independent) subset of these relations for consideration. In general, just such a linearly independent subset is given by the matrix equation (5.1):

$$\mathbf{M}_s \boldsymbol{\lambda}_{hi}(s) = \boldsymbol{\lambda}_{hi}(\leqq s-1)\,. \tag{5.1}$$

where

$$\boldsymbol{\lambda}_{hi}(s) = (\lambda_{hi}(s, 1), \lambda_{hi}(s, 2), \cdots, \lambda_{hi}(s, 2^s))'$$

$$\lambda_{hi}(\leqq s-1) = [\lambda, \lambda_{hi}(1:\bar{s}), \lambda_{hi}(1:\overline{s-1}), \lambda_{hi}(2:\bar{s}, \overline{s-1}), \cdots, \\ \lambda_{hi}(s-1:\overline{s-1}, \overline{s-2}, \cdots, \bar{1})]'$$

The matrix $\mathbf{M}_s$ is defined by four submatrices as in equation 5.2:

$$\mathbf{M}_s = \begin{bmatrix} \mathbf{M}_{s1} & \mathbf{M}_{s2} \\ \mathbf{M}_{s3} & \mathbf{M}_{s4} \end{bmatrix}. \tag{5.2}$$

These submatrices are defined recursively as follows, for $j = 1, \cdots, s-1$,

$$\begin{aligned} &\mathbf{M}_1 = [\mathbf{M}_{11} \quad \mathbf{M}_{12}] = [1 \quad 1] \\ &\mathbf{M}_{j+1,1} = \mathbf{M}_j \quad \text{with a row added consisting of a } 1 \text{ followed by } 2^j - 1 \text{ zeros} \\ &\mathbf{M}_{j+1,2} = \mathbf{M}_{j+1,1} \\ &\mathbf{M}_{j+1,3} = \mathbf{M}_j \\ &\mathbf{M}_{j+1,4} = \mathbf{0}_{((2^j-1)\times 2^j)}, \quad \text{that is, a } (2^j - 1) \times 2^j \text{ matrix of zeroes.} \end{aligned} \tag{5.3}$$

For demonstration purposes, when $s = 3$, Equation (5.1) becomes

$$\mathbf{M}_3 \boldsymbol{\lambda}_{hi}(3) = \ ' \quad \leqq 2) \tag{5.4}$$

where

$$\mathbf{M}_3 = \left[\begin{array}{cccc:cccc} 1 & 1 & 1 & 1 & 1 & 1 & & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ \hdashline 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] = \left[\begin{array}{c:c} \mathbf{M}_{31} & \mathbf{M}_{32} \\ \hdashline \mathbf{M}_{33} & \mathbf{M}_{34} \end{array}\right].$$

Equation (5.1) provides a set of $2^s - 1$ linearly independent equations in the 2^s parameters $\lambda_{hi}(s, 1), \lambda_{hi}(s, 2), \cdots, \lambda_{hi}(s, 2^s)$. Recalling equation (1.1) to describe the (h, i) entry of the coefficient matrix corresponding to the design remaining after removal of the treatments $\{x_1, x_2, \cdots, x_s\}$, we have relation (5.5):

$$\textstyle\sum_{j=1}^{2^s} \frac{\lambda_{hi}(s, j)}{k(s, j)} = -c_{hi}. \tag{5.5}$$

Appending this equation to equation (5.1), we obtain:

$$\hat{\mathbf{M}}_s \boldsymbol{\lambda}_{hi}(s) = \hat{\boldsymbol{\lambda}}_{hi}(\leqq s-1) \equiv \begin{bmatrix} \boldsymbol{\lambda}_{hi}(\leqq s-1) \\ -c_{hi} \end{bmatrix}$$

where

$$\hat{\mathbf{M}}_s = \left[\begin{array}{c} \mathbf{M}_s \\ \hdashline \frac{1}{k(s, 1)} \quad \frac{1}{k(s, 2)} \cdots \frac{1}{k(s, 2^s)} \end{array}\right].$$

For $s < n$ we have:

(i) From equation (5.6), $\boldsymbol{\lambda}_{hi}(s)$ is constant if and only if $\hat{\boldsymbol{\lambda}}_{hi}(\leqq s-1)$ is constant since rank $(\hat{\mathbf{M}}_s) = 2^s$ (see Lemma 5.7).

(ii) Saying that $\boldsymbol{\lambda}_{hi}(s)$ (and, using a parallel argument, $\mathbf{r}_i(s)$) is a constant independent of h and i is equivalent to saying that each subdesign in $\mathscr{D}_s$ is a BIBD.

(iii) By the definition of $\hat{\boldsymbol{\lambda}}_{hi}(\leqq s-1)$, $\hat{\boldsymbol{\lambda}}_{hi}(\leqq s-1)$ is constant if and only if $\boldsymbol{\lambda}_{hi}(\leqq s-1)$ is constant and c_{hi} is constant.

(iv) $\boldsymbol{\lambda}_{hi}(\leqq s-1)$ is constant if and only if D is FLR $(s-1: x_1, x_2, \cdots, x_s)$ by the induction assumption.

(v) c_{hi} is constant if and only if D is LR $(x_1, x_2, \cdots, x_s)$ by Lemma 1.1.

Combining these, we have:

$\boldsymbol{\lambda}_{hi}(s)$ is constant if and only if D is FLR $(s: x_1, x_2, \cdots, x_s)$
which is the desired result.

LEMMA 5.7. *For $s < k$, the rank of $\mathbf{M}_s$ is 2^s.*

PROOF. Denote by E_s the determinant of $\hat{\mathbf{M}}_s$. We wish to show $E_s \neq 0$; we will, in fact, show $E_s > 0$. Because of the pattern of elements in $\mathbf{M}_s$, $\hat{\mathbf{M}}_s$ may by elementary column operations be reduced (see Most (1973)) to an upper triangular matrix with all diagonal elements being one except for the entry in the last row. This entry, and hence E_s is given by (5.8).

$$E_s = \textstyle\sum_{j=0}^{s} (-1)^j \binom{s}{j} \dfrac{1}{k-(s-j)} . \tag{5.8}$$

To show $E_s > 0$, consider advancing (mth order) finite differences of a positive real valued function $f(x)$ with unit, "1," incrementation of x, $\Delta_1{}^m f(x)$, $m = 0, 1, 2, \cdots$, defined recursively as follows:

$$\begin{aligned}
\Delta_1{}^0 f(x) &= \Delta_1{}^0 f(x) = f(x)\\
\Delta_1{}^1 f(x) &= f(x+1) - f(x)\\
\Delta_1{}^2 f(x) &= \Delta_1{}^1 f(x+1) - \Delta_1{}^1 f(x) = f(x+2) - 2f(x+1) + f(x)\\
&\ \ \vdots \qquad\qquad\qquad\qquad\quad \vdots\\
\Delta_1{}^m f(x) &= \Delta_1{}^{m-1} f(x+1) - \Delta_1{}^{m-1} f(x) = \textstyle\sum_{j=0}^{m} (-1)^j \binom{m}{j} f(x+m-j) .
\end{aligned}$$

When $f(x) = (a+x)^{-1}$, $x = 0$, $m = s$, $a = k - s$ with k an integer greater than s, we have, after some reparametrization (namely, in E_s, note that $\binom{s}{j} = \binom{s}{s-j}$; let $u = s - j$; then let $u = j$ and compare term by term with $\Delta_1{}^s f(0)$),

$$E_s = (-1)^s \Delta_1{}^s f(0) . \tag{5.9}$$

But, by repeated use of the mean value theorem of calculus, we may write

$$\Delta^s f(x) = f^{(s)}(x + \xi_s) \qquad \text{for some} \quad \xi_s \quad \text{in} \quad (0, s) \tag{5.10}$$

(valid for all $x > -1$ since $f(x)$ is infinitely differentiable for $x > -1$) where the superscript (s) represents the sth order derivative. Using $x = 0$ in (5.10) and substituting in (5.9) we have

$$(5.11) \qquad E_s = (-1)^s \Delta_1^s f(0) = s!\,(k - s + \xi_s)^{-(s+1)} \qquad \text{for some } \xi_s \text{ in } (0, s)\,.$$

Recalling $k > s > 0$, we conclude $E_s > 0$, which completes the proof of Lemma 5.7. □

COROLLARY 5.12 (to structure theorem). BIBD *D in k plots per block is* FLR $(x_1, x_2, \cdots, x_n)$, $n < k$, *if and only if for each* $s \leqq n$, *every* $(s + 2)$*-tuple containing any subset of* $x_1, \cdots, x_n$ *of size s appears in the same number of blocks of D.*

PROOF. The condition on the $(s + 2)$-tuples is clearly equivalent to requiring that all of the subdesigns formed when any subset of $x_1, \cdots, x_n$ is removed have the same pairing parameter, λ_{hi}, for any two remaining treatments h and i and that these subdesigns are equireplicate; these subdesigns are thus BIBDs. Hence the result follows immediately from the Structure Theorem (4.1). □

6. Essentially trivial designs and degeneracy. An *unreduced* (or *trivial*) block design for v treatments in blocks of size k $(k < v)$ is obtained by taking as blocks all possible combinations of k out of v treatments; we will consider multiples of unreduced designs to be unreduced designs. The resultant BIBD has parameters

$$(v, c\tbinom{v}{k}, c\tbinom{v-1}{k-1}, k, c\tbinom{v-2}{k-2})\,, \qquad c \quad \text{a positive integer.}$$

We shall say that a BIBD is *essentially trivial* if its parameters are identical to those of an unreduced design. (We note that while the combinatorial properties of trivial (unreduced) designs and essentially trivial designs may differ considerably, the two are identical insofar as analysis of experimental data using our assumed model is concerned.)

To illustrate, consider the following BIBD: *ABE*, *CDE*, *ACF*, *BDF*, *ADG*, *BCG*, *EFG* with parameters $(v, b, r, k, \lambda) = (7, 7, 3, 3, 1)$. This design is not trivial. Yet, if we form a new design, P, by uniting five copies of this one, the resultant design has parameters (7, 35, 15, 3, 5) exactly those of a trivial design for seven treatments in blocks of size three; here, P is essentially trivial but not trivial. Obviously, from the definition, a trivial design is also essentially trivial.

We shall say that a BIBD is *degenerate upon removal of p treatments* if for some q, $1 \leqq q \leqq p$, removal of q treatments results in making at least one of the 2^q proper subdesigns formed a complete block design in all of the unremoved treatments. (In this context, a void design is considered to be complete.) Clearly, degeneracy upon removal of p treatments implies degeneracy upon removal of $p' > p$ treatments. Degeneracy, upon removal of n treatments, occurs if (1) less than 2^n nonvoid subdesigns are formed, and/or (2) some of the subdesigns formed are complete block designs which may be considered as a special type of BIBD but which need not satisfy Fisher's inequality. Both of these possibilities are exemplified in the following unreduced design, U, for five treatments in blocks of size three: *ABC ABD ABE ACD ACE ADE BCD BCE BDE CDE.* This design is FGR (3) and hence is certainly FLR (A, B, C). Yet, when A and B are removed we have CDE as one of the subdesigns formed (situation (2) above) and when A, B, and C are removed, $D(3, 1)$ is void (situation (1)).

The following lemmas prepare us for the proofs of Theorems 6.6 and 7.1. For Lemmas 6.1 through 6.3, let A be an LR (x) BIBD and let A_1 and A_2 be the BIBDs formed when x is removed from A (see Figure 6.1). It is readily shown (Most (1973), Appendix C) that if the parameters of any one of designs A, A_1, and A_2 are known, then those of the other two are determined; the equations for computing these parameters will be referred to below as the "LR (x) relations"; for example, we have: If A is (v, b, r, k, λ), then A_1 is $(v-1, r, \lambda, k-1, \lambda(k-2)/(v-2))$ and A_2 is $(v-1, b-r, r-\lambda, k, (r-\lambda)(k-1)/(v-2))$.

LEMMA 6.1. *If A_2 is complete in v treatments (and not void) then A_1 is trivial in v treatments, $v-1$ plots per block.*

(Note that since A_1 has one less plot per block than A_2 and both subdesigns contain the same treatments, it would be impossible for A_1 to be complete unless A_2 were void.)

PROOF. A_2 is complete implies that $v_2 = k_2$. In terms of the parameters of A_1 this may be stated $v_1 = k_1 + 1$ or $k_1 = v_1 - 1$ so that the blocks of A_1 are formed by sets of all but one of the treatments. Identifying each block by the treatment it lacks, we see that $b_1 = v_1 = \binom{v_1}{v_1-1}$ = the number of combinations of $v_1 - 1$ out of v_1 treatments, yielding a trivial design. □

In the proofs of the next two lemmas we consider only essentially trivial designs with the smallest possible number of blocks, $\binom{v}{k}$; the proofs still go through for finite unions of such designs.

LEMMA 6.2. *A_1 is essentially trivial in v treatments and $(k-1)$ plots if and only if A_2 is essentially trivial in v treatments and k plots.*

PROOF. Suppose A_2 is essentially trivial in v treatments, k plots. This implies that $b_2 = \binom{v}{k}$ and $r_2 = b_2k/v = (k/v)\binom{v}{k}$. We must show that $b_1 = \binom{v}{k-1}$ (that is, that the number of blocks in A_1 is equal to that of a trivial design). Now the LR (x) relations imply $b_1 = r_2v/(v-k+1) = (k/v)\binom{v}{k}v/(v-k+1) = \binom{v}{k}k/(v-(k-1)) = \binom{v}{k-1}$.

On the other hand, suppose A_1 is essentially trivial in v treatments, $k-1$ plots. This implies that $b_1 = \binom{v}{k-1}$ and $r_1 = (k-1)/v\binom{v}{k-1}$. We must show that $b_2 = \binom{v}{k}$. The LR (x) relations imply that

$$b_2 = \frac{v_1(b_1 - r_1)}{k_1 + 1} = \frac{v\binom{v}{k-1} - ((k-1)/v)(v/(k-1))}{k}$$
$$= \frac{v}{k} \cdot \frac{v-k+1}{v} \binom{v}{k-1} = \binom{v}{k}\,. \qquad \square$$

LEMMA 6.3. *A is essentially trivial if and only if A_1 and A_2 are essentially trivial.*

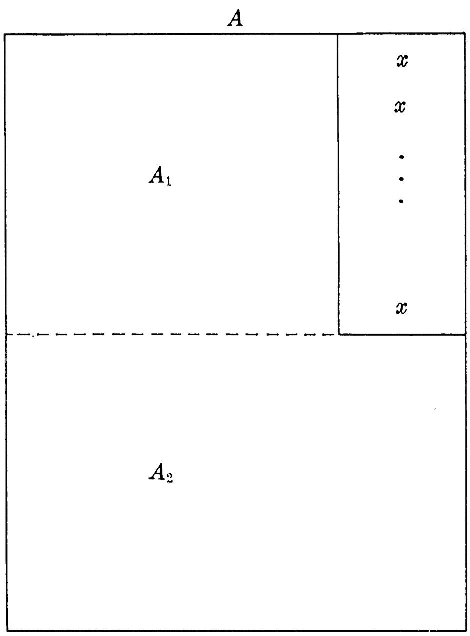

FIG. 6.1.

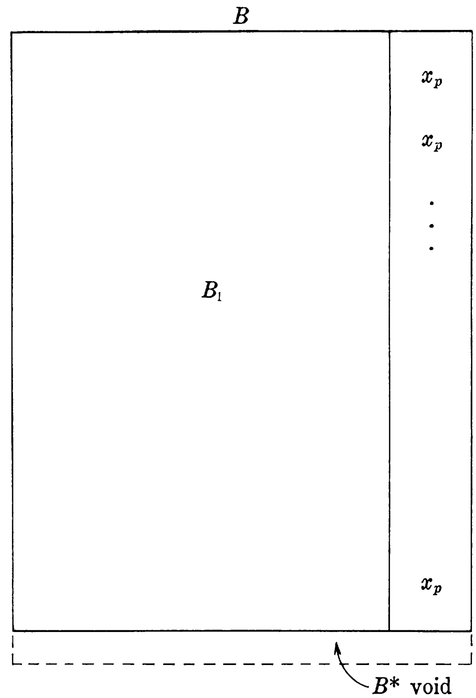

FIG. 6.2.

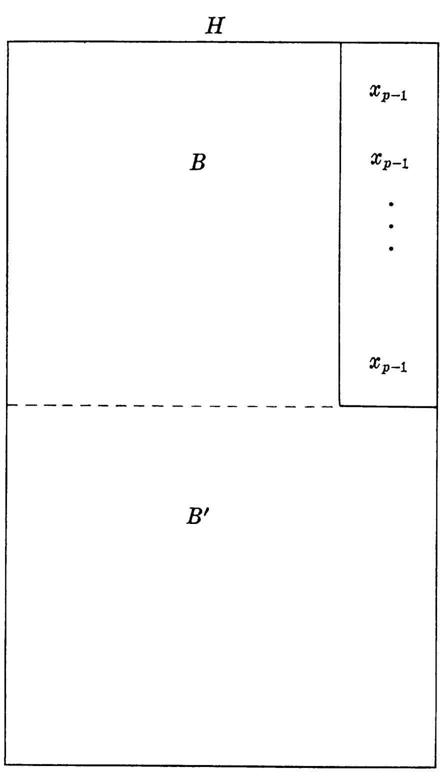

FIG. 6.3a.

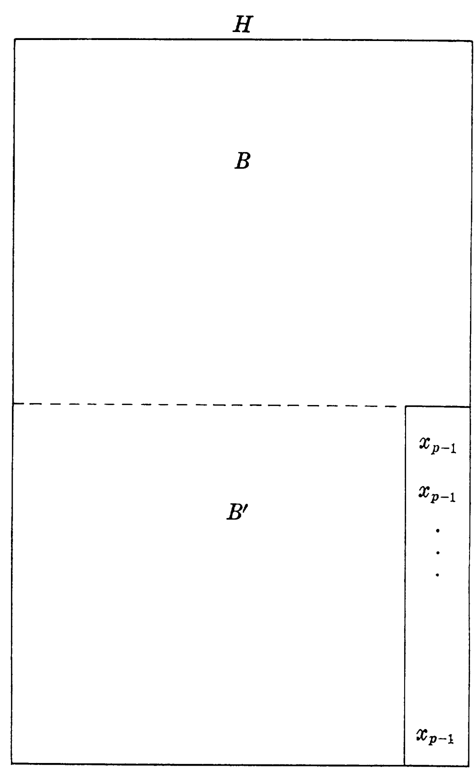

FIG. 6.3b.

PROOF. Let the parameters of A be (v, b, r, k, λ). Suppose A is essentially trivial. Then $b = \binom{v}{k}$ and $r = \binom{v-1}{k-1}$. By the LR(x) relations we have $b_1 = r = \binom{v-1}{k-1}$ so that A_1 is essentially trivial, and $b_2 = b - r = \binom{v}{k} - \binom{v-1}{k-1} = \binom{v-1}{k}$ so that A_2 is essentially trivial. On the other hand, suppose A_1 and A_2 are essentially trivial. Then $b_1 = \binom{v-1}{k-1}$ and $b_2 = \binom{v-1}{k}$. Hence $b = b_1 + b_2 = \binom{v}{k}$ so that A is essentially trivial. □

LEMMA 6.4. *If* BIBD *D is* FLR(n) *with k plots per block and is degenerate upon removal of n treatments, $n < k$, then D is essentially trivial.*

PROOF. Suppose the smallest integer for which D is degenerate is p. This means that when any $p - 1$ treatments of the n to which D is FLR are removed there are 2^{p-1} nonvoid, noncomplete BIBDs formed; also, when a pth treatment is removed, one of the 2^p subdesigns formed is degenerate (that is, either void or complete in $v - p$ treatments). Denote this degenerate design by B^*. B^* must be a subdesign of one design, say B, of the 2^{p-1} nonvoid, noncomplete designs.

We observe that B^* cannot be void, else we would have the situation depicted in Figure 6.2. That is, B^* void would imply that BIBD B contained variety x_p in each block which would imply that B were complete, a possibility ruled out by our definition of p. Thus, B^* is complete and not void.

By Lemma 6.1, B is trivial and, a fortiori, essentially trivial. Now, either $B = D$, in which case we are done, or B is embedded in H, a subdesign of D, in the manner depicted in Figure 6.3a (case a) or Figure 6.3b (case b). (This embedding arises as a consequence of the formation of B when the $(p - 1)$th treatment is removed.) In either case, by Lemmas 6.2 and 6.3, H is essentially trivial. If b_h, b are the number of blocks in H and B respectively, we observe that b_h is strictly greater than b; this is so because B' cannot be void, again by our definition of p.

Now we repeat our reasoning. Either $H = D$ or H is embedded in D in one of the two possible ways illustrated in Figures 6.3a and 6.3b for the case of B's embedding in H which is contained in D. Thus, H is contained in an essentially trivial design, say J, having a number of blocks strictly greater than that of H. We continue in this way to consider larger and larger (in number of blocks) subdesigns of D. Since D is of finite block size, the process terminates at which point the essentially trivial design, analogous to H or J, must be D itself. □

LEMMA 6.5. *Let D be a* BIBD *with k plots per block which is* FLR(n), $n < k$. *If any one of the subdesigns formed when $q \leqq n$ treatments are removed is essentially trivial, then D is essentially trivial.*

PROOF. Same as for Lemma 6.4 after the point at which we concluded that subdesign B is essentially trivial. □

We may now complete the structural description of FLR(n) BIBDs begun in the Structure Theorem (4.1) by considering the case $n \geqq k$:

THEOREM 6.6. *If* BIBD D *with* k *plots per block is* FLR (n), $n \geqq k$, *then* D *is essentially trivial.*

PROOF. Observe that if D is degenerate upon removal of $k-1$ treatments (and since D is FLR (k) implies D is FLR $(k-1)$) then (by Lemma 6.4) D is essentially trivial and we are done. Hence, assume that D is not degenerate upon removal of $x_1, \cdots, x_{k-1}$ and, therefore, that this removal results in 2^{k-1} nonvoid, noncomplete BIBD subdesigns being formed which contain all of the remaining treatments. Specifically, $D(k-1, 2)$ is a BIBD with two plots per block which is readily shown to be essentially trivial. Hence, by Lemma 6.5, D is essentially trivial. □

7. Further structure results.

THEOREM 7.1. *Let* D *be a* FLR (n) BIBD *with parameters* (v, b, r, k, λ). *Then either*

(i) $b \geqq 2^n(v-n)$ *or*
(ii) D *is an essentially trivial design.*

PROOF. If $n \geqq k$, D is essentially trivial by Theorem 6.6; hence, consider below only the case $n < k$. If D is not degenerate upon removal of n treatments, then Fisher's inequality (that is, the number of blocks is at least as great as the number of treatments) and Theorem 4.1 imply (i) of Theorem 7.1. To complete the proof, it suffices to prove that degeneracy occurs only in the case of an essentially trivial design; but this is exactly the statement of Lemma 6.4. □

For the essentially trivial FLR (2) design, U, given near the beginning of Section 6, $b = 10$ while $2^n(v-n)$ is 12 for $n = 2$.

The following theorem describes the structure of an FLR (n) design which is not essentially trivial and in which the lower bound on the number of blocks of Theorem 4.2 is actually attained.

THEOREM 7.2. *If* D *is an* FLR (n) BIBD (v, b, r, k, λ), $n \leqq k$, *with* $b = 2^n(v-n)$ *and* D *is not essentially trivial, then each of the* 2^n *proper subdesigns formed when the* n *treatments to which* D *is resistant are removed is a symmetric* BIBD *in* $v-n$ *treatments.*

PROOF. By hypothesis, $b = \sum_{i=1}^{2^n} b(n, i) = 2^n(v-n)$. Thus if $b(n, j) > v-n$ for some j then $b(n, k) < v-n$ for some k, $k \neq j$, $k, j = 1, 2, \cdots, 2^n$. However, since by Lemma 6.4 each subdesign is a nonvoid BIBD in $v-n$ treatments, $b(n, k) < v-n$ violates Fisher's inequality. Therefore $b(n, j) = v-n$ for each $j, j = 1, \cdots, 2^n$. □

FGR (n) designs are intimately related to t-designs (or tactical configurations, Definition 4.1 in Hedayat and John (1974)) as follows:

THEOREM 7.3. *A* BIBD, D, *with* k *plots per block is* FGR (n), $n < k$, *if and only if* D *is an* $(n+2)$*-design.*

PROOF. Follows directly from Corollary 5.12 and the definition of a t-design. □

Some of the consequences of Theorem 7.3 are discussed in Most (1973).

8. Existence and construction of BIBDs resistant to more than one treatment. Hedayat and John (1974) present a significant number of resistant designs of degree one. Aside from a few examples of strictly LR (1) designs, the examples they give are FGR (1) designs which, by Theorem 7.3, are t-designs with $t \geqq 3$. When we consider resistance of degree two or more, we leave behind the relatively simple world in which the only types of resistance to be considered are LR (1) and GR (1). If we consider the potential removal of several, say n, treatments from a BIBD, the types of resistance to be considered increase rapidly. We may consider resistance to any subset of s treatments, $s \leqq n$; again, local as well as global resistance are topics of interest. In addition, it is possible to consider sequential resistance.

By Theorem 7.3, the existence of an $(n + 2)$-design implies the existence of an FGR (n) design which is, a fortiori, an FLR (n) design. Since infinite classes of nontrivial t-designs are known to exist for $t = 3, 4$, and 5 (see Alltop 1969, 1971, 1972), we may conclude that infinite classes of fully globally resistant BIBDs of degrees 1, 2, and 3 exist. Unfortunately, to our knowledge, no nontrivial t-designs for $t \geqq 6$ have yet been found. Existence of a nontrivial t-design for "high" values of t necessitates finding (perhaps not explicitly) many nontrivial BIBDs which "fit" together in just the right way—a very difficult task in general. Such considerations lead us to anticipate that the nonexistence of certain BIBDs implies the nonexistence of certain t-designs—a topic which we shall not pursue further at this time.

We now state two existence theorems for fully locally resistant BIBDs of degree 2; proofs may be found in Most (1973).

THEOREM 8.1. *Let A be a* BIBD *which is resistant to the removal of a single treatment, y. Further assume that the parameters of A may be written* $(2k + 1, m(2k + 1), mk, k, m(k - 1)/2)$, *for positive integers m, k. Then, there exists an* FLR (2) BIBD *with parameters* $(2(k + 1), 2m(2k + 1), m(2k + 1), k + 1, mk)$.

Theorem 8.1 may be considered as a means of extending a design in v treatments which is resistant to one treatment to a design in $v + 1$ treatments which is resistant to two treatments. Alltop (1972, page 393) proves an Extension Theorem for t-designs which is similar in form to Theorem 8.1. In our notation, Alltop's Extension Theorem states that given a $t - (2k + 1, k, \mu)$ design, with t even, there exists a $(t + 1) - (2k + 2, k + 1, \mu)$ design. Thus, Alltop, starting with a t-design of special form, extends it to a $(t + 1)$-design. We, starting with an FLR (1) design of special form extend it to an FLR (2) design. For certain parameter values, Theorem 8.1 expresses a special case of Alltop's theorem. However, Theorem 8.1 provides an extension of FLR (1) designs which are not also FGR (1) designs (and hence not covered in Alltop's theorem) to FLR (2)

Overall BIBD: (12, 66, 33, 6, 15): FLR (x_1, x_2)

We describe the design in terms of the subdesigns formed when x_1 and x_2 are removed.

$D(2, 1)$ is the following BIBD (10, 15, 6, 4, 2):

1	2	3	4	1	6	8	10	3	5	9	10
1	2	5	6	2	3	6	9	3	6	7	10
1	3	7	8	2	4	7	10	3	4	5	8
1	4	9	10	2	5	8	10	4	5	6	7
1	5	7	9	2	7	8	9	4	6	8	9

$D(2, 2)$ and $D(2, 3)$ are each the following BIBD (10, 18, 9, 5, 4):

1	2	3	4	5	1	4	5	6	10	2	5	6	8	10
1	2	3	6	7	1	4	8	9	10	2	6	7	9	10
1	2	4	6	9	1	5	7	9	10	3	4	6	7	9
1	2	5	7	8	2	3	4	8	10	3	4	5	7	9
1	3	6	8	9	2	3	5	9	10	3	5	6	8	9
1	3	7	8	10	2	4	7	8	9	4	5	6	7	8

$D(2, 4)$ is the following BIBD (10, 15, 9, 6, 5):

1	2	4	5	8	9	2	3	4	6	8	10	1	4	5	7	8	10
5	6	7	8	9	10	1	2	6	7	9	10	1	2	3	5	7	10
2	4	5	6	9	10	1	3	5	6	8	9	2	3	5	6	7	8
1	2	4	6	7	8	1	2	3	8	9	10	1	3	4	5	6	10
3	4	7	8	9	10	2	3	4	5	7	9	1	3	4	6	7	9

FIG. 8.1. (Source: Hedayat and John (1974)).

designs. An example of such an FLR (1) design corresponding to A of Theorem 8.1 is known to exist for $k = 5$, $m = 3$. The corresponding FLR (2) design is given in Figure 8.1. Design A of Theorem 8.1 is a BIBD (11, 33, 15, 5, 6) formed by uniting $D(2, 1)$ and $D(2, 2)$ and adjoining an additional treatment to each block of $D(2, 1)$.

Another extension theorem of the same genre as the two discussed above is given by Hedayat and John (1974) (their Theorem 5.1). Their theorem states that starting with a BIBD (v, b, r, k, λ) such that $b + 2\lambda = 3r$ we may construct a GR (1) design with parameters $(v + 1, 2b, b, (v + 1)/2, r)$.

To our knowledge, the three extension theorems mentioned above were independently motivated. Yet, the basic rationale in all cases was the same: Unite a design with its complement and add a treatment to each block of the original design; then, see what conditions are needed on the original design parameters to make the resultant design an "extension" of the original design.

THEOREM 8.2. *If there exists a* BIBD, $D(2, 1)$, *with parameters* (v, b, r, k, λ) *such that* $b + 2\lambda = 3r$ *and a second* BIBD, $D(2, 4)$, *with parameters* $(v, v(b - r)(v - k - 1)/(k + 2)(k + 1), (b - r)(v - k - 1)/(k + 1), k + 2, b - 2r + \lambda)$

then there exists an FLR (2) *design for* $v + 2$ *varieties in blocks of size* $k + 2$ *with* (BIBD) *pairing parameter* b.

At least ten FLR (2) designs have been found corresponding to Theorem 8.2. These are described precisely in Most (1973) which also contains a variety of examples of other FLR (2) designs. To illustrate, we describe an FLR (x_1, x_2) BIBD (11, 66, 36, 6, 18) in terms of the subdesigns formed when treatments x_1 and x_2 are removed:

$D(2, 1)$ is the following BIBD (9, 18, 8, 4, 3):

1 2 3 4	1 4 8 9	2 5 6 8
1 2 5 6	1 5 7 9	3 5 8 9
1 2 7 8	2 3 8 9	4 6 7 9
1 3 5 7	2 4 5 9	3 4 5 6
1 4 6 8	2 6 7 9	3 6 7 8
1 3 6 9	2 3 4 7	4 5 7 8

$D(2, 2)$ and $D(2, 3)$ are each the following BIBD (9, 18, 10, 5, 5):

1 3 6 7 8	1 3 4 7 9	1 3 4 5 8
2 3 4 6 8	1 2 3 6 9	1 2 4 6 7
2 4 5 7 8	1 4 5 6 7	1 2 3 5 8
5 6 7 8 9	1 2 4 5 9	2 3 5 7 9
3 4 5 6 9	3 4 7 8 9	1 2 7 8 9
2 4 6 8 9	2 3 5 6 7	1 5 6 8 9

$D(2, 4)$ is the following BIBD (9, 12, 8, 6, 5):

1 2 4 5 7 8	1 2 5 6 7 9	1 3 5 6 7 8	4 5 6 7 8 9
2 3 5 6 8 9	1 3 4 5 8 9	1 2 4 6 8 9	1 2 3 4 5 6
1 3 4 6 7 9	2 3 4 6 7 8	2 3 4 5 7 9	1 2 3 7 8 9

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