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Rotational Cohomology and Total Pattern Equivariant Cohomology of Tiling Spaces Acted on by Infinite Groups

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Rotational Cohomology and Total Pattern Equivariant Cohomology of Tiling Spaces Acted on by Infinite Groups

by

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Dedicated to my wife Stephanie, and my baby daughter Amelia.

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Rotational Cohomology and Total Pattern Equivariant Cohomology of Tiling Spaces Acted on by Infinite Groups

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In 2003, Johannes Kellendonk and Ian Putnam introduced pattern equivariant cohomology for tilings. In 2006, Betseygail Rand defined a type of pattern equivariant cohomology that incorporates rotational symmetry, using representation of the rotation group. In this doctoral thesis we study the relationship between these two types of pattern equivariant cohomology, showing exactly how to calculate one from the other in the case in which the rotation group is a finitely generated abelian group of free rank 1. We apply our result by calculating the cohomology of the pinwheel tiling.

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Chapter 1

Introduction

In 2003, Johannes Kellendonk [10] invented pattern equivariant cohomology for tilings. He and Ian Putnam [11] proved that the pattern equivariant cohomology of a tiling T is isomorphic to the Čech cohomology of the associated tiling space. In her thesis [16], Betseygail Rand expanded on Kellendonk's idea. Many interesting tilings of $\mathbb{R}^N$ exhibit rotational symmetry, but Kellendonk's pattern equivariant cohomology ignores this. Rand incorporated the action of the rotation group into pattern equivariant cohomology. For each representation ρ of the rotation group G of a tiling T, she defined a cohomology $H^*_\rho(T)$ associated to the tiling. We repeatedly refer to this as "Rand cohomology", though the reader should note that this term is not in widespread use.

For a given tiling, what can Rand cohomology tell us about Kellendonk's non-rotational pattern equivariant cohomology? For the case in which G is a finite abelian group, her thesis completely resolves this issue. A wide swath of interesting tilings falls into this case, but some of the favorite tilings of tiling space theorists do not. Most notably, the famous pinwheel tiling, first discovered by John Conway and Charles Radin [15], has an infinite rotation group. The fundamental question that I address is: How does Rand cohomology relate to the non-rotational, or "total", pattern equivariant cohomology, for tilings with an infinite rotation group like that of the pinwheel? In this thesis, I specify an appropriate noncompact tiling space Ω, associated to the pinwheel tiling, that allows me to define a type of cohomology which should be properly thought of as the total pattern equivariant cohomology for pinwheel-like tilings. Then, using some basic homological algebra, representation theory, module theory and algebraic topology, I show exactly how to determine Rand cohomology from total pattern equivariant cohomology, and vice-versa.

Chapters 2, 3, and 4 provide the necessary background by defining tilings, tiling spaces, pattern equivariant cohomology, Rand cohomology, and all of the necessary nuts and bolts. Chapters 5 and 6 are dedicated to proving theorems which allow us to determine pattern equivariant cohomology from Rand cohomology, and vice-versa, for tilings like the pinwheel. Those theorems are the central results of the thesis and are presented at the end of chapter 6. In chapter 7, we put the theorems to good use by using the cutting edge Barge-Diamond technique [2],[3] to calculate the Rand cohomology of the non-compact pinwheel tiling space (for certain well chosen representations), and then deducing the total pattern equivariant cohomology of the pinwheel tiling.

Chapter 2

Background part I: Tilings

2.1 Basic Definitions

Informally, a *tiling* of $\mathbb{R}^N$ is a particular way of covering $\mathbb{R}^N$ with a collection of shapes, in a non-overlapping fashion. Take a single square of unit length; that's a tile. Now take infinitely many copies of that square and line them up edge to edge in the obvious way, so that you cover all of $\mathbb{R}^2$. That's a tiling of $\mathbb{R}^2$.

Let us be more precise.

Definition 2.1 (tile). An N-dimensional *tile* $t = (D_t, L_t)$ is a region $D_t \subset \mathbb{R}^N$ which is homeomorphic to a closed N-dimensional disk, along with an associated label or marking $L_t \subset D_t$. We may refer to D_t as the *body* of the tile.

The label is really just a book-keeping mechanism for distinguishing between tiles that are the same shape. For that reason, when we mean to refer to the body of the tile, we will usually just say "tile". For example, if we were to say that the tiles t_1 and t_2 are adjacent, we really mean that the bodies of the two tiles D_{t_1} and D_{t_2} intersect on their boundaries.

Let $E(N)$ be the Euclidean group, the group of isometries of $\mathbb{R}^N$. If $g \in E(N)$, and $t = (D_t, L_t)$ is an N-dimensional tile, then $g \cdot t = (g \cdot D_t, g \cdot L_t)$ is another N-dimensional tile, and $g \cdot t$ looks like t, but it is positioned differently and perhaps rotated by some angle. Two tiles t_1 are t_2 said to be of the same *tile type* if there exists $g \in E(N)$ such that $g \cdot t_1 = t_2$.

Definition 2.2 (tiling). A *tiling* of $\mathbb{R}^N$ is a collection of N-dimensional tiles $\{t = (D_t, L_t)\}$ such that

1. The number of tile types represented by $\{t = (D_t, L_t)\}$ is finite.

2. $\cup D_t = \mathbb{R}^N$.

3. If $t_1 \neq t_2$ then D_{t_1} and D_{t_2} intersect only on their boundaries (if at all).

The set of tilings that tiling space theorists deem worthy of study is a proper subset of all tilings, and the set of tilings that I study is a proper subset of that, so we will define some conditions on tilings that limit our scope.

First, we need to develop some notation and terminology for various isometry groups. Note that the group $\mathbb{R}^N$ acts on the space of all N-dimensional tilings by translation. Given an N-dimensional tiling tiling T, and $\mathbf{v} \in \mathbb{R}^N$ the tiling $T + \mathbf{v}$ is just that tiling in which all of the constituent tiles of T have been displaced by the vector $\mathbf{v}$. When we mean to refer to $\mathbb{R}^N$ acting in this fashion, we will call it the *translation group.* If $G \subset SO(N)$, then we say G is a *rotation* group. If G is a group of euclidean motions that contains the translation group, then we say that G is a *partial euclidean group* (this last term is non-standard).

In this line of work, the tiling of $\mathbb{R}^2$ by squares that we just mentioned is considered uninteresting, because it is periodic. By periodic, we mean that shifting the entire tiling by one unit gives us back the exact same tiling. We are interested in *non-periodic* tilings.

Definition 2.3 (non-periodic). An N-dimensional tiling T is *non-periodic* if for all $\mathbf{v} \in \mathbb{R}^N$, $T + \mathbf{v} \neq T$.

In tiling space theory, we repeatedly need to refer to a section of a tiling:

Definition 2.4 (patch). A *patch* is any finite collection of tiles that interesect at most on their boundaries. A patch of a tiling T is any finite subcollection of the tiles of T.

Given a tiling T and a region $S \subset \mathbb{R}^N$, we write $[S]_T$ (or just $[S]$, when there can be no confusion) to refer to the collection of tiles that intersect S. Let $B_r(x) \subset \mathbb{R}^N$ be the closed ball of radius r centered at the point $x \in \mathbb{R}^N$.

$[B_r(x)]_T$ is exactly the patch of tiles that contain points whose distance from from x is less than or equal to r. Note that the Euclidean group acts on the set of patches: given $g \in E(N)$, and a patch P, we can define $g \cdot P$ to be the patch in which each of the tiles of P have been moved by the Euclidean motion g.

We are principally interested in tilings that have some semblance of rotational symmetry. The following definition helps us make sense of this.

Definition 2.5 (**rotation group of a tiling**)**.** Let T be a tiling, and let $g \in SO(N)$. If for all patches P of T, we find that a translated copy of $g \cdot P$ is found somewhere in T, then we say that g is a rotational symmetry of T. The subgroup of $SO(N)$ generated by all rotational symmetries of T is called the *rotation group* of T, and denoted by G^T_{rot}.

We define $G^T_{euc} \equiv \mathbb{R}^N \rtimes G^T_{rot}$, the partial euclidean group generated by the rotations in G^T_{rot} and all translations.

Definition 2.6 (**repetitivity**)**.** Let G be a partial euclidean group. A tiling T is *repetitive* w.r.t. G, if for every patch P, there exists a radius R such that for all $x \in \mathbb{R}^N$, $[B_R(x)]$ contains a patch $g \cdot P$, for some $g \in G$. If $G = \mathbb{R}^N$, we say T is *translationally repetitive.* If $G = G^T_{euc}$ (or equivalently $E(N)$), we say T is *rotationally repetitive.*

Repetitivity ensures some level of order. For any given patch P in a repetitive tiling, the distance between nearest copies of the patch P is bounded.

Definition 2.7 (**finite local complexity**)**.** A tiling T is said to have the property of finite local complexity (FLC) with respect to the group G, if for every r there exist finitely many patches $P_1, \ldots, P_n$ such that for every $x \in \mathbb{R}^N$ we have $[B_r(x)] = g \cdot P_j$ for some $g \in G$, $1 \leq j \leq n$. When G is just the translation group, we may say that T is *translationally finite* or *translationally FLC*. When $G = G^T_{euc}$, we may say that T is *rotationally finite* or *rotationally FLC.*

Finite local complexity ensures that, for any give radius r, we have only finitely many patterns of size r to account for.

2.2 Substitution Tilings

Substitution tilings constitute a large portion of the world of tilings. The substitution process is perhaps the easiest way to generate tilings that are non-periodic, repetitive, and FLC (although certainly not all substituion tilings have these properties). In this section we will explain what we mean by substitution, and use it to build two tilings: the equithirds tiling, and the pinwheel tiling. The equithirds tiling is just an example to cut our teeth on; we could replace it with any one of a variety of elementary but interesting tilings, and the same goal would be achieved. On the other hand, for reasons which will gradually become more clear, the pinwheel tiling is of crucial importance to this thesis.

A substitution is a recursive way of describing arbitrarily large patches of a tiling.

Definition 2.8 (**substitution**)**.** Let $\mathcal{A}$ be a finite collection of N-dimensional tiles, and let $\mu > 1$. For all $t \in \mathcal{A}$, let μt denote the shape obtained by taking the tile t and stretching it by a factor of μ (so the volume of μt is the volume of t multiplied by μ^N). A *substitution*, is a prescribed way of covering μt with tiles in $\mathcal{A}$, for all $t \in \mathcal{A}$. This way of covering μt is called a first order supertile of type t.

A substitution σ then defines a higher order substitution σ^k. For example, take a tile t and stretch it by a factor of μ^2, and call the new inflated shape $\mu^2 t$. Now cover $\mu^2 t$ with first order supertiles in the exact same way that we covered μt with tiles. Since each first order supertile is made up of tiles, we have described a way of covering $\mu^2 t$ with tiles. This way of covering $\mu^2 t$ is called a second order supertile of type t, and similarly, we can define supertiles of all orders. We denote a k^{th} order supertile of type t by $\sigma^k(t)$.

Definition 2.9 (**substitution tiling**)**.** Let σ be a substitution, using tiles $t_1, \ldots, t_m$. A tiling T using the tiles $t_1, \ldots, t_m$ is a *substitution tiling* with substitution σ if every patch of T is a subpatch of a translate of $\sigma^k(t_j)$ for some $k > 0$, $1 \leq j \leq m$.

This definition raises some existential questions: Given a substitution σ, under what conditions can we be assured that there exists a tiling with

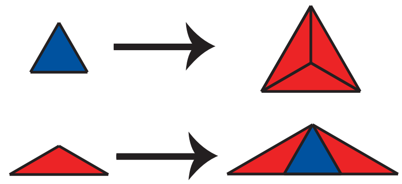

Figure 2.1: The Equithirds Substitution

substitution σ? If we can be assured of such, is there a general method for producing a tiling from the substitution? For full answers to these questions, we refer the reader to Lorenzo Sadun's book [18]. We will not directly address these particular issues, but we will show how to produce a tiling from a substitution in our two examples.

2.3 Example: The Equithirds tiling

The equithirds tiling is a two dimensional tiling. It is new to the literature, although a nearly identical substitution was discovered by Ludwig Danzer about ten years ago. The tiling consists of equilateral triangles, which we'll call *Eq* tiles, and 30°-30°-120° triangles, which we will call *Is* tiles. The equithirds substitution σ is shown in figure 2.1. As shown, σ substitutes an *Eq* tile with three *Is* tiles, and substitutes an *Is* tile with one *Eq* tile and two *Is* tiles. Figure 2.2 shows the second, third, and fourth order supertiles.

We are going to use σ to construct a tiling. To be precise, the tiling we are about to construct is not a substitution tiling with substitution σ; it is a substitution tiling with substitution σ^2. The strategy is to find a sequence of patches $\{P_n\}_{n=0}^{\infty}$ each centered at the origin such that each $P_n = \sigma^2(P_{n-1})$, and P_{n-1} is a sub-patch of P_n. Let *Is.Is* denote the patch consisting of two *Is* tiles sharing a long edge which runs vertically and is centered at the origin. Now repeatedly apply σ^2, and let T be the resulting tiling. Figure 2.3 shows *Is.Is*, σ^2(*Is.Is*), and σ^4(*Is.Is*).

By, construction $\sigma^2(T) = T$, i.e., applying σ^2 to the entire tiling T just gives us back T. Tilings possessing this property are called *self-similar* tilings. How do we know that T is non-periodic? T can also be viewed as a

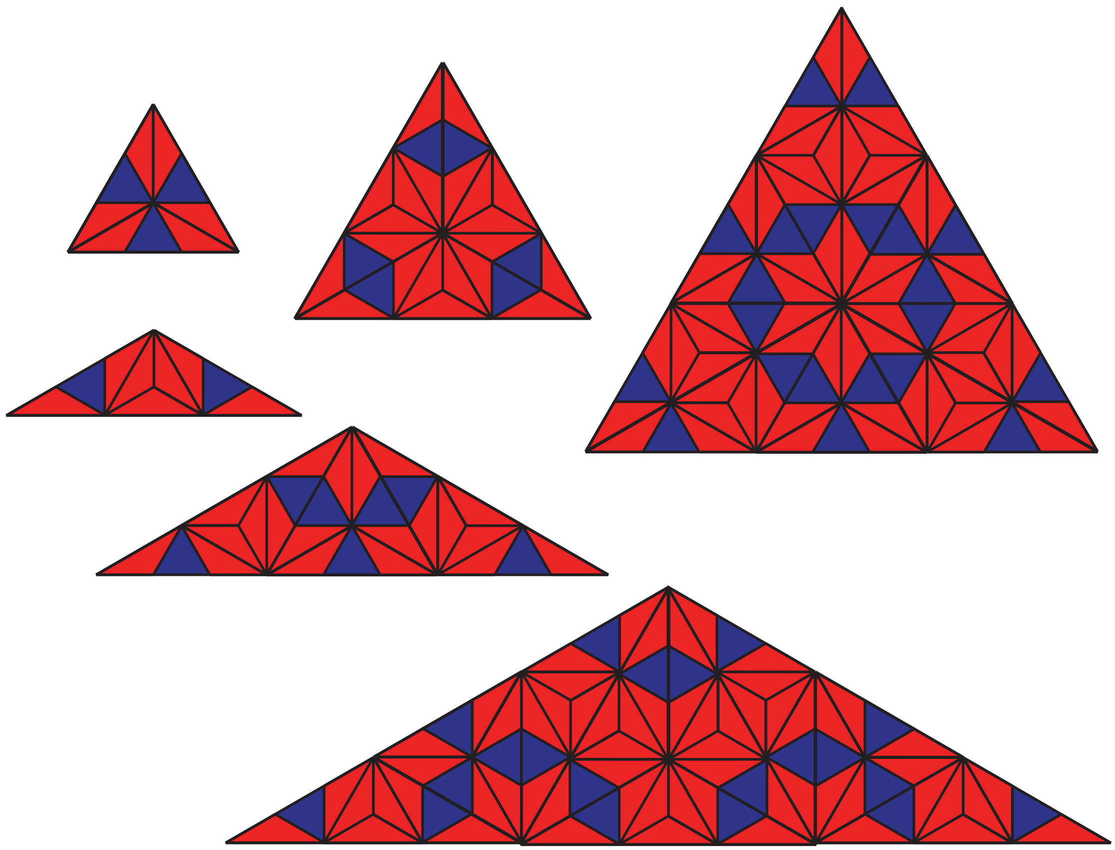

Figure 2.2: Second, third, and fourth order *Eq* and *Is* supertiles

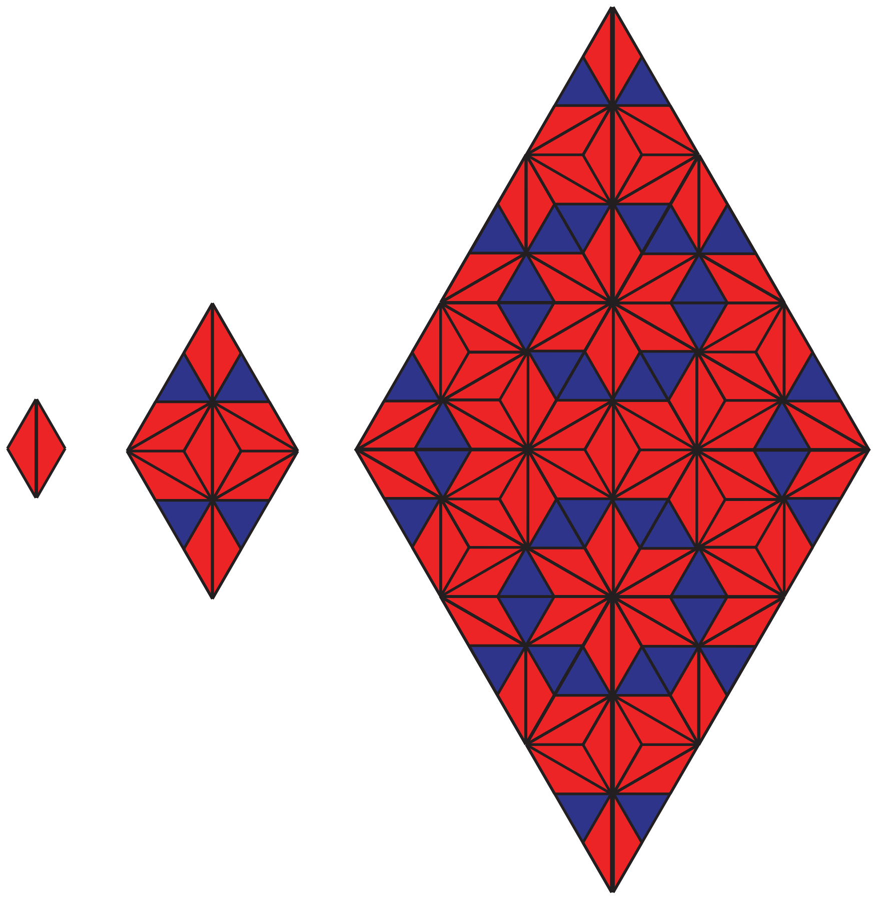

Figure 2.3: Patches $Is.Is$, $\sigma^2(Is.Is)$, and $\sigma^4(Is.Is)$. The sequence of patches $\{\sigma^{2n}(Is.Is)\}_{n=0}^{\infty}$ defines a tiling of the whole plane.

tiling of first order supertiles, and in fact there is a *unique* way to view T as a tiling of first order supertiles. This is true, because the only way that any *Eq* tile can appear in T is by having two of its sides adjacent to two *Is* tiles, forming an *Is* supertile. Thus, the *Eq* tiles mark the location of the first order *Is* supertiles. *Is* tiles occur in both types of supertiles, but observe that we can find a first order *Eq* supertile where and only where we see three *Is* tiles whose 120° angles meet at the same point. So the 120°-120°-120° junctions mark the locations of the first order *Eq* supertiles.

Iterating the same reasoning shows that T can be viewed as a tiling of second order supertiles, in a unique way. How does this show that T is non-periodic? Let $\mathbf{v} \in \mathbb{R}^2$, assume that $\mathbf{v}$ is shorter than the lengths of all of the second order supertiles, and let $p \in \mathbb{R}^2$ be the location of the vertex of a second order supertile. Then, the tiling $T - p$ cannot be the same as $T - p - \mathbf{v}$, because if it were it would be possible to view T as a tiling of second order supertiles which has a vertex at $p + \mathbf{v}$. This shows that if $T = T + \mathbf{v}$, then $\mathbf{v}$ must be at least as long as an edge of a second order supertile. Now, we can reason that T can be viewed as a tiling of fourth order supertiles, in a unique way, and using the same argument, we know that $\mathbf{v}$ must be at least as long as a fourth order supertile, and so forth. We are left only to conclude that $T \neq T + \mathbf{v}$ for any $\mathbf{v} \in \mathbb{R}^2$, i.e., T is non-periodic. This argument that the ability to uniquely recognize a tiling as a tiling of higher order supertiles implies that the tiling is non-periodic is due to Mossè [12] and also to Solomyak [19].

Repetitivity is built right in. Inspection shows that the patch *Is.Is* is contained in all second order supertiles. Given a patch P in T, by construction P must be contained in $\sigma^{2n}(Is.Is)$ for some n. Thus P is contained all $2n+2$ order supertiles. As long as r is big enough that any ball of radius r is guaranteed to fully contain a $2n+2$ order supertile, then for all x, $[B_r(x)]_T$ must contain a copy of P.

We can see T is translationally FLC. There are only two types of tiles in T, and they only occur in finitely many orientations, and they always meet full edge to full edge. With these restrictions, there are only finitely many ways that one can piece together a patch of radius r. Perhaps we should probe a bit more deeply. Given a radius r, we can choose n to be large enough that a ball of radius r never touches more than 12 n^{th} order supertiles. There are only finitely many ways in which 12 or fewer n^{th} order supertiles can sit adjacent to each other, and every $[B_r(x)]$ is a subpatch of one of these configurations. Thus, there are only finitely many patches of the form $[B_r(x)]$. If we wanted to list all the possible patches (up to translation) of radius r, we could take a ball of radius r and place it at each point in all n^{th} order supertiles and record the patch. If the point is within r of a super-edge, we can place all the possible adjacent supertiles along that edge, and then record the patch, and similarly if the point is within r of a super-vertex. This process records all of the possible patches of radius r.

We haven't mentioned rotational symmetries yet, because T is already repetitive and FLC at the level of translations. We can easily see that any patch in T rotated by $60°$ degrees is another patch in T, and conversely only rotations by $n \cdot 60°$ send patches to other patches. Thus, the rotation group of the equithirds tiling is $\mathbb{Z}/6\mathbb{Z}$.

2.4 The Pinwheel Tiling

It is difficult to overstate the importance of the pinwheel tiling to this thesis. First discovered by John Conway and Charles Radin [15], it is the first and still the the simplest known example of a tiling which is rotationally FLC but not translationally FLC. It has infinite rotation group, and tilings with infinite rotation group are of central concern to us. We will revisit the pinwheel and perform a detailed calculation of the cohomology of the pinwheel tiling space in chapter 7. The pinwheel tiling consists of two tiles L and R, shown in figure 2.4, both of which are 1-2-$\sqrt{5}$ triangles, but they are mirror images of each other. We call L the *left-handed* triangle and R the *right-handed* triangle.

The substitution σ is shown in figure 2.5.

Again, we will construct a self-similar substitution tiling with substitution σ^2 (not σ). Start with eight tiles forming a union jack centered at the origin. Call this patch *UJ*. Now, apply σ^2 as illustrated in figure 2.6. Let T be the resulting tiling.

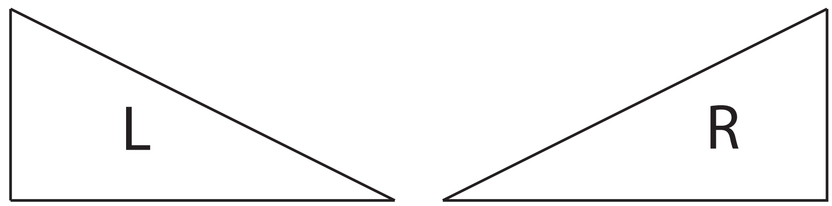

Figure 2.4: The L-tile (left) and R-tile (right) of the pinwheel tiling.

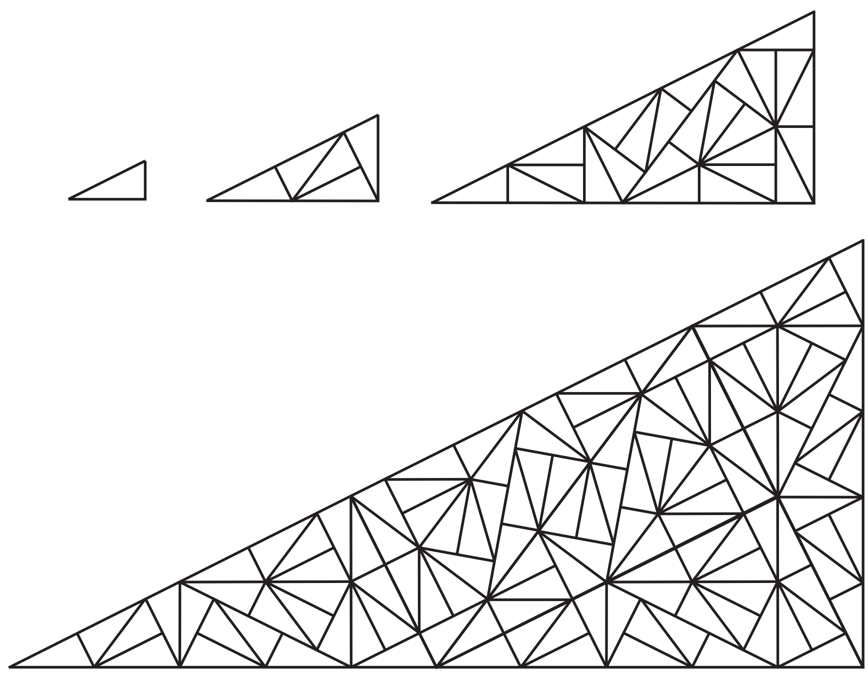

Figure 2.5: The pinwheel substition: here we see an R-tile, and first, second, and third order R-supertiles. The L tiles and supertiles are mirror images of these.

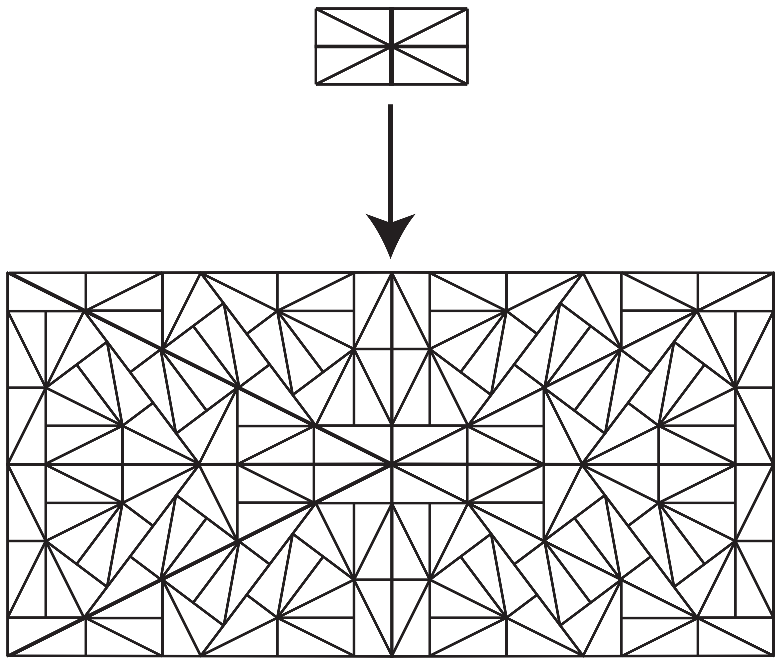

Figure 2.6: The union jack patch, UJ, and the twice substituted union jack $\sigma^2(UJ)$. Notice that UJ sits inside $\sigma^2(UJ)$, and thus $\sigma^{2n}(UJ)$ sits inside $\sigma^{2n+2}(UJ)$, so the sequence of patches $UJ, \sigma^2(UJ), \sigma^4(UJ), \ldots$ defines a tiling of the entire plane.

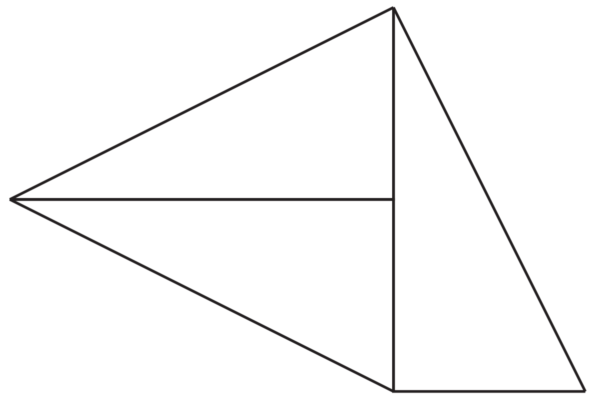

Figure 2.7: Patches of the form shown (and its mirror image) mark the first order supertiles in the pinwheel tiling.

The previous example shows that if there exists a unique way of viewing T as a tiling of first order supertiles, then we can be assured that T is non-periodic. To see this, we observe that the three tile patch shown in figure 2.7 marks the first order supertiles. In other words, any time that we see that patch, we can be assured that the three tiles of that configuration all must be in the same first order supertile. So, there can be only one way of organizing the tiles of T into first order supertiles. Following our earlier reasoning, T must be non-periodic.

What is the rotation group? Consider the R and L tiles shown in figure 2.4 to be in standard orientation. Observe that a second order R supertile in standard orientation contains R and L tiles in standard orientation and R and L tiles that have been rotated by angle $\beta \equiv 2\tan^{-1}(1/2)$. Similarly, a second order L supertile in standard orientation contains R and L tiles in standard orientation and R and L tiles that have been rotated by $-\beta$. This implies that a fourth order R supertile in standard orientation contains standard R and L tiles, and R and L tiles rotated by $-\beta, \beta$, and 2β. Similarly, a fourth order L supertile in standard orientation contains standard R and L tiles, and R and L tiles rotated by $\beta, -\beta$, and -2β. In general, a $(2n)^{th}$ order R supertile in standard orientation contains R and L tiles rotated by $k\beta$, if and only if $(1-n) \leq k \leq n$. β is not a rational multiple of 2π, so in the pinwheel tiling, the tiles occur in infinitely many orientations. It is also true that fourth order supertiles contain R and L rotated by $90°$, $180°$, and $270°$. Thus, we see that in the pinwheel tiling, we can find the R and L tiles rotated by angle θ if and only if $\theta = m\beta + n(\pi/2)$, where $m, n \in \mathbb{Z}$. Thus, the rotation group is isomorphic to $\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$. We call this rotation group *the pinwheel group*, and denote it by $\mathcal{P}$.

Since the pinwheel group is infinite, the pinwheel tiling is not translationally repetitive. To see this, let r be given, and let n be large enough that any $(2n)^{th}$ order supertile is big enough to hold a ball of radius r. Now, find an R supertile of order $2n$, which has been rotated an angle $2n\beta$ from the standard orientation. This supertile contains R tiles that have been rotated by angles $\beta, 2\beta, \ldots, 2n\beta$, but it does not contain any R tiles in standard orientation. So, no matter how large a radius r we choose, we can always find balls of radius r that do not contain standard R tiles.

On the other hand, the pinwheel tiling is rotationally repetitive. Inspection shows that a translate of the UJ patch can be found in both the R and L sixth order supertiles. By construction, any patch P in T can be found in $\sigma^{2n}(UJ)$ for some n. So, if r is large enough that all balls of radius r are guaranteed to contain a supertile of order $2n+6$, then all balls of radius r will contain a rotated copy of $\sigma^{2n}(UJ)$, thus all balls of radius r are gauaranteed to contain a copy of the patch P.

The infinite rotation group also breaks translational finite local complexity. For any given radius r, there exists n so that $[B_r(0)]$ is contained in $\sigma^{2n}(UJ)$. Since T contains a translate of $(m\beta)\sigma^{2n}(UJ)$ for all m, it also contains a translate of $(m\beta)[B_r(0)]$, for all m. So we have infinitely many patches for any given radius. However, the pinwheel is rotationally finite. Given r, we can choose n large enough that all balls of radius r touch at most 8 supertiles of order $2n$. Up to rotation, there are only finitely many possible configurations of 8 supertiles and only finitely many subpatches of these configurations. So, up to rotation, there are only finitely many patches of radius r or less.

Chapter 3

Background part II: Tiling Spaces

A tiling space is a topological space whose points are tilings. We can use a single tiling to generate a tiling space, and we can also use a substitution to generate a tiling space. When rotations are in play, there are a few different tiling spaces that we can generate from a given tiling or substitution. We will develop a framework which encompasses all of them.

3.1 Definitions

We begin by defining a metric on tilings. In this section, G will always be a partial Euclidean group.

Definition 3.1 (tiling metric). The *G-metric on tilings* is given by

$$d_G(T_1, T_2) = \min(1, \bar{d}_G(T_1, T_2)) \tag{3.1}$$

where T_1 and T_2 are tilings of $\mathbb{R}^N$, and

$$\bar{d}_G(T_1, T_2) = \min\{|g| \text{ such that } [B_{1/|g|}(0)]_{gT_1} = [B_{1/|g|}(0)]_{T_2} \text{ or } [B_{1/|g|}(0)]_{T_1} = [B_{1/|g|}(0)]_{gT_2}\}. \tag{3.2}$$

(Here, $|g|$ means the norm of g as an element of the euclidean group.) The big idea is that two tilings T_1 and T_2 should be considered close to each other if a small euclidean motion $g \in G$ acting on T_1 produces a tiling which agrees with T_2 out to some large radius. This is the concept that defines tiling space topology. Notice that if G is a dense subgroup of $\bar{G}$, then the G-metric produces the same open sets as the $\bar{G}$-metric. For this reason, we henceforth assume that G is a closed subset of $E(N)$.

This idea of putting a topology on a family of tilings is originally due to Radin and Berend [4]. In principle, any random collection of tilings is a tiling space, using the topology induced by this metric. But that is silly. Our tiling spaces always consist of tilings that all look similar to some specific tiling T. The following two definitions make this notion rigorous.

Definition 3.2 (**orbit**)**.** Given a tiling T, the *G-orbit of T*, denoted by O_G^T is the set of tilings T' such that $T' = gT$ for some $g \in G$.

Definition 3.3 (**orbit closure**)**.** Given a tiling T, the tiling space X_G^T is the metric closure of O_G^T, under the d_G metric. It is called the *G-orbit closure* of T.

Any tiling space we would ever consider is the orbit closure of some particular tiling. Whenever the group G is understood, we will just write O^T and X^T. Let's try to make some sense of X_G^T. X_G^T is the set of tilings T' such that, for all $r > 0$, there exists $g_r \in G$ such that T' and $g_r T$ agree out to radius r. In other words, a tiling T' is in X^T if and only if every patch of T' is also a patch of T, up to a euclidean motion in G. In the case where T is repetitive, something stronger is true:

Lemma 3.1. *If T is repetitive and $T' \in X_G^T$, then every patch of T is a patch in T', up to a euclidean motion in G.*

Proof. Let P be a patch in T. By repetitivity, there exists r such that each $[B_r(x)]_T$ contains a copy of P, which also means that for all $g \in G$, the patch $[B_r(0)]_{gT}$ contains a copy of P. Since $T' \in X_G^T$, there exists a sequence $g_0T, g_1T, g_2T, \ldots$ which converges to the tiling T'. This implies that there exists n_r such that for all $n \geq n_r$, g_nT agrees with T' out to a radius r. Since T is repetitive, $[B_r(0)]_{T'} = [B_r(0)]_{g_nT}$ contains a copy of P. $\square$

As an immediate consequence, we have the following theorem.

Theorem 3.2. *If T is repetitive w.r.t. G, then $T \in X^{T'}$ if and only if $T' \in X^T$.*

In other words, if T is repetitive, then X^T is the orbit closure of any of its tilings. We can also generate a tiling space from a substitution σ. The resulting tiling spaces are exactly the same as the ones that you get by taking a σ substitution tiling T and generating the space X^T, but in most cases it is better to think in terms of the substitution.

Definition 3.4 (substitution tiling space). Let σ be a tiling substitution. Let X_G^σ be the space of tilings such that every patch of the tiling is contained is some $g \cdot \sigma^k(t)$, where $g \in G$, and $\sigma^k(t)$ is a k^{th} order supertile. Endow X_G^σ with the topology obtained via the metric d_G. We call X_G^σ a *substitution tiling space.*

Again, when G is understood, we omit it and just write X^σ.

If we choose any substitution tiling T that contains all patches of all supertiles of the substitution σ, in all orientations admitted by G, then $X_G^T = X_G^\sigma$. In particular, this is true of the tilings in the previous section.

Examine the case where G_{rot}^T is finite. Let us consider the differences between, $X_{\mathbb{R}^N}^T$, $X_{G_{euc}^T}^T$, and $X_{E(N)}^T$. Assume T is rotationally repetitive. Since the rotation group is finite, T must also be translationally repetitive. In this case, $X_{\mathbb{R}^N}^T$ and $X_{G_{euc}^T}^T$ are the same. If we let T be our equithirds tiling, we can see that rotating T by $60°$ gives us another tiling with all the same patches as T, up to translation. So, as sets, $X_{\mathbb{R}^N}^T = X_{G_{euc}^T}^T$, and they are the same as topological spaces as well, because they have the same small open sets. Rotations in G_{rot}^T cannot be of arbitrarily small size, so two tilings differ by a small euclidean motion if and only if they differ by a small translation.

$X_{E(N)}^T$ is a different beast. It consists of all the tilings in $X_{\mathbb{R}^N}^T$, rotated to arbitrary angles.

When I study the tiling space generated by T, where T has a finite rotation group, I am usually thinking of $X^T = X_{\mathbb{R}^N}^T$, but I am constantly keeping in mind the fact that X^T comes equipped with a group action by the group G_{rot}^T. If we take any tiling $T' \in X^T$, and $g \in G$, then gT' is again in X^T.

Now, let us consider the situation where T is a tiling of the plane such that G_{rot}^T is a finitely generated and dense subgroup of $SO(2)$, e.g. the pinwheel tiling. In this case, G_{euc}^T is dense in $E(N)$, so $X_{G_{euc}^T}^T$ the same as $X_{E(N)}^T$. There is some argument that $X_{E(N)}^T$ is the "right" tiling space for

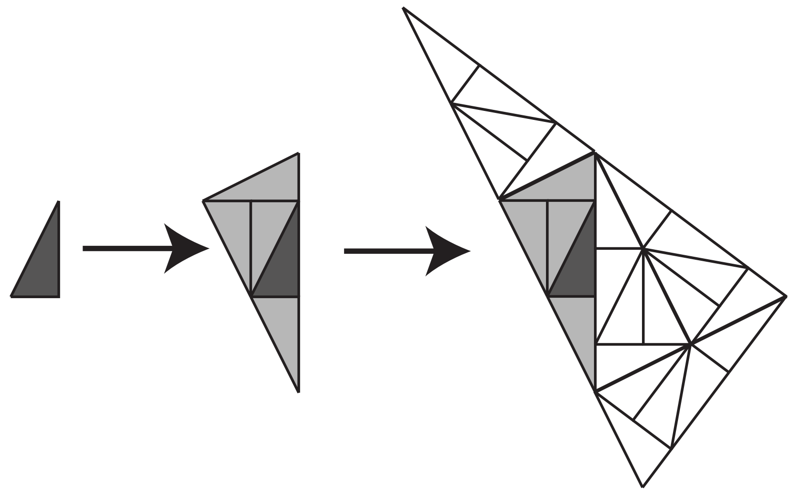

Figure 3.1: Continuing this sequence of patches defines a peculiar pinwheel tiling.

studying the pinwheel. If two tilings are off by only a tiny rotation, shouldn't we regard them as being close together? In $X^T_{\mathbb{R}^N}$, we certainly have tilings that differ by only a tiny angle, but we do not view them as being close together.

When I study the pinwheel space, I am studying $X^T = X^T_{\mathbb{R}^N}$. There is an interesting ambiguity here, coming from the fact that pinwheel tilings are not translationally repetitive. If we start with the pinwheel tiling T from the previous section, then T contains the R and L tiles and supertiles of all orders in all possible orientations allowed by the pinwheel group $\mathcal{P}$. Also, if σ is the substitution as shown in figure 2.5, then every R and L supertile of every order and in every possible orientation sits in side $\sigma^k(R)$ and $\sigma^k(L)$, for some k. So, $X^T = X^\sigma =$ the space of tilings whose patches lie in some supertile in some orientation allowed by $\mathcal{P}$.

However, consider the sequence of patches shown in figure 3.1, which defines a tiling $T' \in X^\sigma$. T' only contains tiles and supertiles that have been rotated by $m\beta + n(\pi/2)$ where m is a non-negative integer. Thus, $X^{T'}_{\mathbb{R}^N}$ is a proper subset of $X^T_{\mathbb{R}^N}$. In particular, $T \notin X^{T'}_{\mathbb{R}^N}$ because, for example, T contains the patch consisting of a single R tile rotated by $-\beta$, but that patch is nowhere to be found in T'. $X^{T'}_{\mathbb{R}^N}$ and other similar subspaces of $X^T_{\mathbb{R}^N}$ are potentially interesting in their own right, but we will not grant them any further consideration in this thesis. We want our tiling space to include all orientations of R and L that are allowed by the action $\mathcal{P}$, so that we have a nice group action of $\mathcal{P}$ on the tiling space.

Definition 3.5 (the pinwheel tiling space). In the rest of this paper, Ω will refer to the pinwheel tiling space, which is defined as

$$\Omega = X^{\sigma^2}_{\mathbb{R}^N} \tag{3.3}$$

where σ is the substitution shown in figure 2.5. Ω admits an action by the pinwheel group $\mathcal{P} \cong \mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$.

3.2 The Local Structure of a Tiling Space

Let $\bar{T}$ be a non-periodic tiling in which tiles meet full edge to full edge, and assume that for every patch P occuring in $\bar{T}$, there are multiple translated copies of P in $\bar{T}$. Let $X = X^{\bar{T}}_{\mathbb{R}^N}$. What is the local structure of X? Let T be a tiling in X, let $\epsilon > 0$ be small, and let U^T_ϵ be the small open set in X consisting of all of those tilings T' such that $d_{\mathbb{R}^N}(T, T') < \epsilon$. U^T_ϵ contains a set of nearby translates of T, provided that they are translated by a vector smaller than ϵ. Topologically, this set of translates of T is just an open ball of radius ϵ in $\mathbb{R}^N$. If T' is one of these translates of T, then U^T_ϵ also contains all tilings that agree with T' out to a radius $1/\epsilon$. Let $V^{T'}_\epsilon$ be this set of tilings. What is the topology of $V^{T'}_\epsilon$? Observe the following:

1. $V^{T'}_\epsilon$ is totally disconnected. Essentially, this is because the set of patches containing any given patch is a discrete set. In other words, if we start with the patch $P = [B_r(0)]_{T'}$ (let $r \gg 1/\epsilon$), then for any $\delta > 0$ there are only finitely many ways that we can add more tiles out to radius $r + \delta$, since there are only finitely many tile types and tiles must meet full edge to full edge. This means that there must exist r_1, r_2 with $r < r_1 < r_2 < r + \delta$, such that all tilings in $V^{T'}_\epsilon$ either agree with T' out to a radius strictly greater than r_2, or fail to agree with T' out to a radius r_1. This is exactly a decomposition of $V^{T'}_\epsilon$ into two open sets, and in the exact same fashion, we can decompose *any* open subset of $V^{T'}_\epsilon$ into two open sets.

2. $V_\epsilon^{T'}$ is a perfect space. In other words, any tiling $\tilde{T} \in V_\epsilon^{T'}$ is a limit of a sequence of tilings in $V_\epsilon^{T'}$, where each tiling in the sequence is distinct from $\tilde{T}$. Since $\tilde{T} \in X^T$, it is a limit of translates of $\bar{T}$ that agree with $\tilde{T}$ out to large radii. So, if $\tilde{T}$ is not itself a translate of $\bar{T}$ it must be limit of a sequence of tilings in $V_\epsilon^{T'}$, that are not the same as $\tilde{T}$. If $\tilde{T} = \bar{T} - x$, then since patches occur multiple times, we can find a sequence $\{x_n\}$ with $x_n \neq x$, such that $[B_n(0)]_{\tilde{T}} = [B_n(x_n)]_{\bar{T}}$, for all natural numbers n. Thus, the sequence $\{\bar{T} - x_n\}$ converges to $\tilde{T}$ and since $\bar{T}$ is non-periodic, we also know that $\bar{T} - x_n \neq \tilde{T}$.

3. $V_\epsilon^{T'}$ is compact. This is a consequence of the fact that the set of patches of a bounded radius and containing a given patch is finite. Let $\{T_n\}$ be an arbitrary sequence of tilings in $V_\epsilon^{T'}$. Let $r_j = j + (1/\epsilon)$. By definition, all tilings in $V_\epsilon^{T'}$ agree with T' out to a radius r_0. There are only finitely many patches of radius r_1 containing patch P as a subpatch. Thus, there must exist a patch P_1 and an infinite subsequence $\{T_n^1\}$ of $\{T_n\}$ such that $[B_{r_1}(0)]_{T_n^1} = P_1$ for all n. Similarly, there must exist a patch P_2 and an infinite subsequence $\{T_n^2\}$ of $\{T_n^1\}$ such that $[B_{r_1}(0)]_{T_n^2} = P_2$ for all n, and so forth. We can then take the sequence $\{T_n^n\} = T_1^1, T_2^2, T_3^3, \ldots$. This is a Cauchy sequence, because for all $m, n > M$, T_m^m and T_n^n agree out to radius at least $M + (1/\epsilon)$, thus $d(T_n^n, T_n^m) < 1/M$, and since $X^{\bar{T}}$ is complete, $\{T_n^n\}$ is a convergent subsequence of $\{T_n\}$.

So, $V_\epsilon^{T'}$ is a totally disconnected, perfect, compact metric space. Thus, it is homeomorphic to a Cantor set. So, to summarize, the local neighborhood U_ϵ^T around the point T is homeomorpic to $\mathbb{R}^N \times \mathcal{C}$, where $\mathcal{C}$ is a Cantor set, i.e., "a tiling space is locally $\mathbb{R}^N\times$ a Cantor set". We often envision U_ϵ^T as a bunch of small euclidean balls of radius $(1/\epsilon)$ stacked on top of each other. Each of these balls is called a *leaf* or *translational leaf*. The spaces $V_\epsilon^{T'}$ are then called the *vertical spaces*. So, two tilings in the same leaf are translates of each other, and two tilings in the same vertical space have the same local pattern.

3.3 Tiling Spaces are Inverse Limit Spaces

A *directed set* I is a set with a partial order relation $<$, with the added property that for all $\alpha, \beta \in I$, there exists γ such that $\alpha < \gamma$ and $\beta < \gamma$. Let I be a directed set which we use to index a family of topological spaces $\{\Gamma_\alpha\}_{\alpha \in I}$. For each $\alpha < \beta \in I$, we have a continuous map $\sigma_{\beta\alpha} : \Gamma_\beta \to \Gamma_\alpha$, such that if $\alpha < \beta < \gamma$, then $\sigma_{\gamma\alpha} = \sigma_{\gamma\beta} \circ \sigma_{\beta\alpha}$.

Let $\prod \Gamma$ be the product of all the Γ, endowed with the product topology. Define the following subset of $\prod \Gamma$

$$\varprojlim \Gamma \equiv \{\{a\}_{\alpha \in I} | \sigma_{\beta\alpha}(a_\beta) = a_\alpha\}. \tag{3.4}$$

This is called *inverse limit* of the Γ, and we call the Γ_α the *approximants* to the inverse limit. Let $\Theta = \varprojlim \Gamma$. For our purposes, the big idea is that a point in the approximant Γ_α represents a collection of closely related points in Θ. If $\alpha < \beta$, then a point in Γ_β represents a smaller, more closely related collection of points in Θ.

The truth is that tiling spaces are a type of space that most topologists find nasty: They are locally totally disconnected! But as we will see, especially in the next chapter, the fact that tiling spaces are inverse limit spaces makes them amenable to many of the usual tools of algebraic topology.

As an example, we will use the Anderson-Putnam approximant scheme [1] to show how we can view the equithirds tiling space as an inverse limit of CW complexes. The Anderson-Putnam inverse limit structure can be applied to any substitution tiling space, and it has the nice feature that all the approximants are isomorphic, so we just call the approximant the Anderson Putnam complex. Furthermore, the σ maps are basically just the substitution maps.

Let Θ be the equithirds tiling space. We start by describing the approximants and σ maps in a heuristic fashion, then we will come back and fill in the important topological details. Let the points in Γ_0 be the possible ways of placing a tile so that some point of the tile lies at the origin. Let the points in Γ_k be the possible ways of placing a $(2k)^{th}$ order supertile at the origin. There is an obvious map $\sigma_{kk'} : \Gamma_k \to \Gamma_{k'}$ when $k > k'$. Each $(2k)^{th}$ order supertile is made up of supertiles of order $(2k')$ in some unique fashion, so placing a $(2k)^{th}$ order supertile so that some point of it lies at the origin also specifies a way of placing a supertile of order $(2k')$ so that some point of it lies at the origin.

A given tile can have several different possible configurations of neighboring tiles. An *Is* tile must have another *Is* along its long edge, but we cannot say for certain which types of tiles sit along the other edges. An *Eq* tile must have another *Eq* tile sitting along exactly one of of its three sides, but we cannot say for certain which one. However, in the equithirds substitution, every second order supertile has a *unique* configuration of neighboring (zeroeth order) tiles. Figure 3.2 illustrates this. Thus, every $(2k+2)$-order supertile has a unique configuration of $(2k)^{th}$ order supertiles. (Substitutions that have the property that neighboring tiles of an n^{th} order supertile is completely determined by the substitution are said to "force the border". This terminology is originally due to Kellendonk [9].) As a result, specifying a way of placing a $(2k+2)$-order supertile so that some point of the supertile lies at the origin describes a way of tiling the plane out to a radius of 3^k, even if we choose to put the origin near the edge of a supertile. Let $\{a_k\}_{k=0}^{\infty}$ be a point in the inverse limit. Then a_k describes a way of tiling the plane out to a radius 3^{k-2}, which agrees with $a_{k'}$ for $k' < k$. Hence, $\{a_k\}_{k=0}^{\infty}$ defines a way of tiling the entire plane. Clearly, all points in Θ define a unique way of placing $(2k)^{th}$ order supertiles at the origin, for each k. Thus, we identify the inverse limit of the Anderson Putnam approximants with Θ.

What is the topology of these approximants? Every point in the interior of each tile type (in every possible orientation) gives us a point in Γ_0, so we can think of each tile as giving us two-cell that sits inside Γ_0. So, the equithirds substitution gives Γ_0 eight two-cells, corresponding to the six orientations of the *Is* tile and the two orientations of the *Eq* tile. If two tiles t_1 and t_2 can meet along an edge, then we glue those two tiles along that edge. This makes sense because putting the origin at point near an edge should be considered "close" to putting the origin at a point in a different tile right across the edge. Another way to look at it is that this gluing is required if the σ maps are to be continuous. Consider the situation where we place a second order supertile so that the origin sits in the interior of the supertile but on or near the edge of a (zeroeth order) tile. Examine figure 3.3. Let $\bar{p}, \bar{q} \in \Gamma_2$ mean placing the tile so that the origin is at p and q, respectively. As far as Γ_2 is concerned, $\bar{p}$

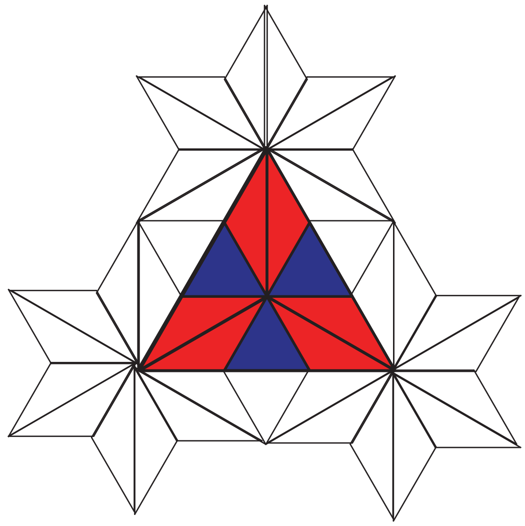

Figure 3.2: Here we see a second order Eq supertile, and all of its neighboring tiles. In the equithirds substitution, this is the *only* way to configure tiles around the border of an Eq supertile. As a result, we say that the Equithird substitution forces the border.

and $\bar{q}$ are close to each other, thus their images under σ_{10} must also be close. This requires us to glue along the nearby edge in Γ_0.

So, Γ_0 is the cell complex consisting of eight two-cells, one for each tile in each possible orientation. A pair of two-cells are glued along an edge, whenever it is possible for the corresponding tiles to meet along that edge. Clearly, all of the Γ_k are the same CW-complex, and the map $\sigma_{k,k-1}$ is just the substitution map. For example, if we let $(\theta)t$ be the two-cell corresponding to the tile t rotated by an angle θ, then $\sigma_{k,k-1}$ maps Eq three-to-one onto itself and one-to-one onto each of Is, $(\pi/3)Is$, $(2\pi/3)Is$, $(\pi)Is$, $(4\pi/3)Is$, $(5\pi/3)Is$.

In the next chapter we will define various cohomologies of tiling spaces, and we will show how the inverse limit structure allows us to calculate the cohomology of the tiling space from the cohomology of the approximants. As far as I know, it is impossible to actually compute the cohomology of tiling space without appealing to some sort of inverse limit structure. We will use the Anderson-Putnam complex to calculate the cohomology of the equithirds tiling space.

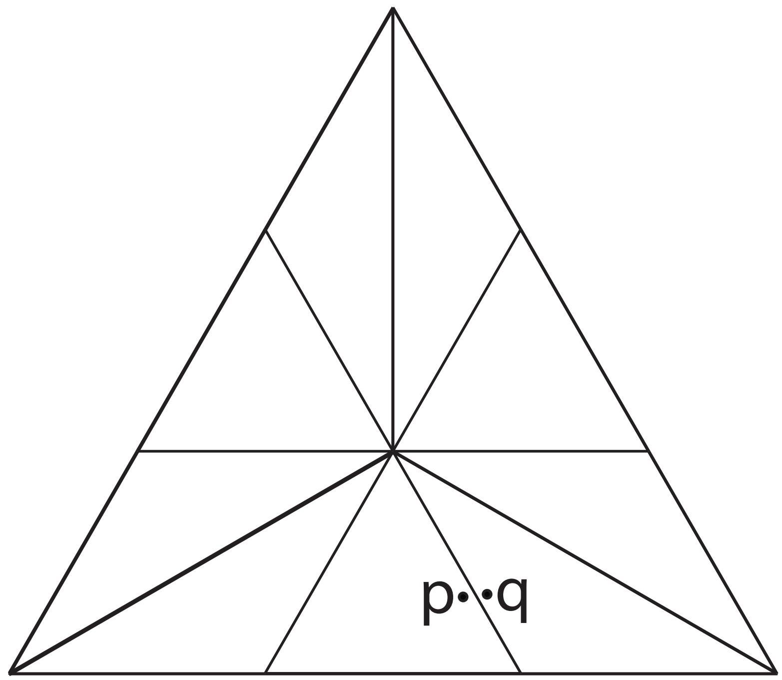

Figure 3.3: Two nearby points p and q that are in the same second order supertile, but different tiles.

The combination of the equithirds substitution and the Anderson Putnam complex was chosen for sake of expository expediency: it is easy to see the inverse limit structure on Θ, and relatively easy to perform the cohomology calculations. There are other tiling space approximant schemes out there, but they are all fundamentally quite similar, in the following sense: If h is a large number, then a single point in Γ_h refers to a whole collection of similar tilings that look the same, at least to some large radius r_h. This basic tenet holds in the case of the Anderson-Putnam approach, because as we have seen in the case of the equithirds tiling space, a point in Γ_k defines a way of tiling the plane out to a radius of at least 3^{k-2}, thus a point in Γ_k can be thought of as a collection of tilings, with those tilings all agreeing out to a radius of at least 3^{k-2}.

For most substitution tiling spaces, the Anderson-Putnam approach runs into a significant technical hurdle. When a substitution forces the border, the natural map from the tiling space Θ to the inverse limit of the Anderson-Putnam approximants bijects. Not all substitutions force the border, which is problematic. Tiling space theorists have come up with a few techniques to patch up (pun intended) the Anderson-Putnam construction. In practice, these techniques can be quite cumbersome, sometimes prohibitively so. The latest and greatest inverse limit structure is the Barge-Diamond approximant scheme [2]. It is a bit more difficult to explain, but ultimately much more powerful, and if we want to calculate the cohomology of the pinwheel tiling space, we cannot do without it. Thus, when we go to calculate the cohomology of the pinwheel tiling space we will give a detailed explanation of the Barge-Diamond technique.

Chapter 4

Background part III: Tiling Space Cohomology

In this chapter, we will define pattern equivariant cohomology, which is arguably the best way to understand cohomology on tilings and tiling spaces. We will also define Rand cohomology, which is a type of pattern equivariant cohomology that incorporates the rotation group. These two cohomologies are the major players in this thesis, and the goal of this thesis is to understand how these two relate.

4.1 Čech cohomology

In this section, we provide an overview of Čech cohomology (see Bott and Tu [5] for a comprehensive treatment). Along the way, it will be instructive to relate Čech cohomology to other familiar cohomologies. Let A be an abelian group. For any CW-complex Γ, we can define $H^*_{CW}(\Gamma : A)$, the cellular cohomology of Γ with coefficients in A. For any smooth manifold M, we can define $H^k_{dR}(M : A)$, the k^{th} deRham cohomology of M (clearly, in this context A must be a $\mathbb{R}$-vector space). Čech cohomology is in some sense more topological than deRham or cellular cohomology, because in order to define $\check{H}^k(X : A)$, the only requirement on X is that it be a topological space. Furthemore, $\check{H}^k(X : A)$ is a homeomorphism invariant, and it is usually the same as your favorite category-specific cohomology. In particular, $\check{H}^k(\Gamma : A) \cong H^*_{CW}(\Gamma : A)$, for any abelian group A, and $\check{H}^k(M : \mathbb{R}) \cong H^*_{dR}(M : \mathbb{R})$.

In the next section we will define the pattern equivariant cohomology $H^*_{PE}(T : A)$ of a tiling T and it will turn out to be isomorphic to $\check{H}^*(X^T : A)$. Relating pattern equivariant cohomology to a more general functor like Čech cohomology makes it easy if not trivial to prove properties that help convince us of the robustness of pattern equivariant cohomology. For example, if T_1 and T_2 are two tilings that generate the same tiling space X, then $H^*_{PE}(T_1 : A) \cong H^*_{PE}(T_2 : A)$.

However, for our purposes, the most important reason to relate Čech cohomology to pattern equivariant cohomology is that Čech cohomology works well with inverse limits, as we'll see. In the last chapter, we saw how tiling spaces are inverse limit spaces. Specifically, we saw how the equithirds tiling space is the inverse limit of the Anderson-Putnam complex. So, if we can relate pattern equivariant cohomology to Čech cohomology, we can use the inverse limit structure to help us calculate pattern equivariant cohomology.

4.1.1 The direct limit of groups

One key ingredient in defining Čech cohomology is the direct limit of groups. The direct limit of groups is doubly important for us, because not only is it a part of the definition of Čech cohomology, it is also what allows us to calculate the cohomology of an inverse limit space from the cohomology of the approximants.

Definition 4.1 (group direct limit). Let I be a directed set, used to index a family of groups $\{G_\alpha\}_{\alpha \in I}$. Suppose that for each $\alpha, \beta \in I$ where $\alpha < \beta$, we have a map $\zeta_{\beta\alpha} : G_\alpha \to G_\beta$, with the property that if $\alpha < \beta < \gamma$, then $\zeta_{\gamma\alpha} = \zeta_{\gamma\beta} \circ \zeta_{\beta\alpha}$. Then the *direct limit* of $\{G_\alpha\}_{\alpha \in I}$, denoted by $\varinjlim(G_\alpha, \zeta_{\beta\alpha})$ or just $\varinjlim G_\alpha$, is the disjoint union of all the G_α modulo the relation that for $x \in G_\alpha$, $\alpha < \beta$, $x \sim \zeta_{\beta\alpha}(x)$.

The fact that I is a directed set gives us a natural multiplication on $\varinjlim G_\alpha$. If $g_\alpha \in G_\alpha$, $g_\beta \in G_\beta$, then to evaluate $g_\alpha g_\beta$, find γ such that $\alpha < \gamma$, $\beta < \gamma$ and define $g_\alpha g_\beta \equiv \zeta_{\gamma\alpha}(g_\alpha)\zeta_{\gamma\beta}(g_\beta)$.

Here is a simple example of a direct limit, which we will see again in the cohomology of the equithirds tiling: Let I be the natural numbers, let $G_n = \mathbb{Z}$ for all n, and let $\zeta_{n+1,n}$ be the group homomorphism given as multiplication by 3:

$$\mathbb{Z} \xrightarrow{\times 3} \mathbb{Z} \xrightarrow{\times 3} \mathbb{Z} \xrightarrow{\times 3} \mathbb{Z} \xrightarrow{\times 3} \cdots \tag{4.1}$$

Let $H = \varinjlim(\mathbb{Z}, \times 3)$ be the corresponding direct limit. Let $n_m \in H$ refer to the integer $n \in G_m \cong \mathbb{Z}$. In H, the element $1_0 \in G_0$ is identified with $3_1 \in G_1$. Examining the element $1_1 \in G_1$, we have $1_1 + 1_1 + 1_1 = 3_1 = 1_0$. So, 1_1 is essentially one-third of 1_0. Similarly, $1_2 \in G_2$ is one-ninth of 1_0, $1_3 \in G_2$ is one-twenty-seventh of 1_0, etc. So, this direct limit is like starting with $\mathbb{Z}$, and then adjoining $1/3, 1/9, 1/27$, etc. As abelian groups under addition, $H \cong \mathbb{Z}[1/3]$, under the map that sends $n_m \in H$ to $n/3^m \in \mathbb{Z}[1/3]$.

4.1.2 Definition of Čech cohomology

Definition 4.2 (**Nerve of an open cover**)**.** An open cover $\mathcal{U}$ of a topological space X defines a simplicial complex $N(\mathcal{U})$ called the *nerve* of $\mathcal{U}$: We have a zero-simplex (vertex) for each nonempty open set of $\mathcal{U}$, a one-simplex (edge) for every non-empty intersection of two open sets of $\mathcal{U}$, and in general, an n-simplex for every nonempty intersection of $n+1$ open sets of $\mathcal{U}$.

Definition 4.3 (**Čech cohomology of an open cover**)**.** Let A be an abelian group. Let $\mathcal{U}$ be an open cover of a topological space X. The Čech cohomology $\check{H}^*(X, \mathcal{U} : A)$ of the open cover $\mathcal{U}$ with coefficients in A is the simplicial cohomology (with coefficients in A) of the nerve of $\mathcal{U}$.

Having defined the Čech cohomology of an open cover, we use the Čech cohomology $\check{H}^*(X, \mathcal{U} : A)$ of all of the open covers $\mathcal{U}$ of X to define the Čech cohomology of X. We say that an open cover $\mathcal{V}$ is a refinement of $\mathcal{U}$, if every open set in $\mathcal{V}$ is contained in some open set in $\mathcal{U}$. Notice that the set of all open covers is a directed set under refinement. We write $\mathcal{U} < \mathcal{V}$ whenever $\mathcal{V}$ is a refinement of $\mathcal{U}$. Given two open covers $\mathcal{U}$ and $\mathcal{V}$, we can build a new cover $\mathcal{W}$ by letting $\mathcal{W}$ be the collection of sets of the form $U \cap V$ where $U \in \mathcal{U}$, $V \in \mathcal{V}$. $\mathcal{W}$ is a refinement of both $\mathcal{U}$ and $\mathcal{V}$, i.e. $\mathcal{U} < \mathcal{W}$ and $\mathcal{V} < \mathcal{W}$.

Suppose $\mathcal{U}_\beta$ is a refinement of $\mathcal{U}_\alpha$. By associating to each open set $U \in \mathcal{U}_\beta$ one of the open sets $U' \in \mathcal{U}_\alpha$ which contains U, we can build a map $\zeta_{\beta\alpha} : N(\mathcal{U}_\beta) \to N(\mathcal{U}_\alpha)$. We will leave the details to Bott and Tu [5], but the important points are the following:

1. $\zeta_{\beta\alpha}$ is a simplicial map, and thus it induces a well-defined pullback on

cohomology:

$$\zeta_{\beta\alpha}^* : \check{H}^k(X, \mathfrak{U}_\alpha : A) \to \check{H}^k(X, \mathfrak{U}_\beta : A). \tag{4.2}$$

2. The pullback $\zeta_{\beta\alpha}^*$ is independent of which $U \supset U' \in \mathfrak{U}_\alpha$ we chose to associate to $U \in \mathfrak{U}_\beta$.

This allows us to define the Čech cohomology of X.

Definition 4.4 (Čech cohomology)**.** The *Čech cohomology* $\check{H}^k(X : A)$ of a topological space X is the direct limit of the Čech cohomology of the open covers:

$$\check{H}^k(X) \cong \varinjlim \check{H}^k(X, \mathfrak{U}_\alpha : A) \tag{4.3}$$

where we take the direct limit via the maps $\zeta_{\beta\alpha}^*$ in equation (4.2).

As promised, defining the Čech cohomology $\check{H}^k(X : A)$ requires no structure other than a topology on X. We will also make use of the following properties of Čech cohomology, which we state without proof.

Theorem 4.1. *Let $\{X_\alpha\}_{\alpha \in I}$ be a family of topological spaces indexed by the directed set I, and let*

$$X = \varprojlim X_\alpha, \tag{4.4}$$

where the inverse limit is taken via maps $\sigma_{\beta\alpha} : X_\beta \to X_\alpha$. Then

$$\check{H}^k(X : A) \cong \varinjlim \check{H}^k(X_\alpha : A), \tag{4.5}$$

where the direct limit is taken via the pullbacks $\sigma_{\beta\alpha}^* : \check{H}^k(X_\alpha) \to \check{H}^k(X_\beta)$.

Also, as we mentioned, Čech cohomology is the same as cellular cohomology.

Theorem 4.2. *Let Γ be a CW-complex, then $\check{H}^k(\Gamma : A) \cong H^k_{CW}(\Gamma : A)$.*

4.2 Pattern Equivariant Cohomology

The idea of pattern equivariance and pattern equivariant cohomology is originally due to Johannes Kellendonk [10]. First, some simple terminology: let T be a tiling of $\mathbb{R}^N$, and let $x, x' \in \mathbb{R}^N$ and $R > 0$. If $[B_R(x)] = [B_R(x')] + (x - x')$ then we say that the local patterns of T at x and at x' agree to radius R. To define pattern equivariant cohomology, we first need to define pattern equivariant functions.

Definition 4.5 (pattern equivariant function). Let T be a tiling of $\mathbb{R}^N$ and let $f : \mathbb{R}^N \to S$ where S is any set, and let $R > 0$. Suppose that whenever the local patterns at x and x' agree to a radius R, we have that $f(x) = f(x')$. Then we say that f is *pattern equivariant with radius R*. If f is pattern equivariant with some radius R, we may just say that f is pattern equivariant.

For any given tiling T on $\mathbb{R}^N$ we have the associated class of pattern equivariant functions f defined on $\mathbb{R}^N$. We can tweak this definition a bit, and think of a pattern equivariant function as a function defined on the tiling space $X^T_{\mathbb{R}^N}$. Given the tiling T, we naturally associate each point $x' \in \mathbb{R}^N$ with the tiling $T' = T - x'$, so that T' is that tiling whose pattern at the origin is exactly the same as the pattern of T at the point x'. If f is pattern equivariant with radius R, then whenever the local patterns at x' and x'' agree to a radius R, we have that $f(x') = f(x'')$. So, if we want to think of f as a function on tiling spaces, then we ought to have that $f(T') = f(T'')$, whenever the local pattern of T' (at the origin) agrees with the local pattern of T'' (at the origin) to radius R. This is the definition of a pattern equivariant function $f : X \to S$ defined on the tiling space X.

In the last chapter, we examined the structure of a tiling space, and we saw that (under certain conditions on the tiling T) the tiling space $X = X^T_{\mathbb{R}^N}$ is locally $\mathbb{R}^N \times$ (a Cantor set). More precisely, an ϵ neighborhood of T' is just $B_{1/\epsilon}(0) \times V^{T'}_\epsilon$, where we called $V^{T'}_\epsilon$ the vertical space and we saw that $V^{T'}_\epsilon$ is a Cantor set. Taking this viewpoint, we see that a pattern equivariant function is a function that is *locally constant on the vertical spaces.* A function $f : X \to S$ that is pattern equivariant with radius R takes the same value on T' and T'' whenever T' and T'' agree out to radius R. This is exactly the same as saying that f is constant on the vertical space $V^{T'}_{1/R}$.

We also saw how tiling spaces are inverse limit spaces, $X^T_{\mathbb{R}^N} = \varprojlim \Gamma_k$. We specifically examined the Anderson-Putnam approximant scheme, but we noted that all tiling space approximant schemes are the same in spirit: a point in the approximant Γ_k refers to a collection of tilings in X that all agree out to some large radius R_k (where R_k increases, unbounded, as k increases). Taking this viewpoint we see that a pattern equivariant function is a function that descends to a well defined function on approximants. If $f : X \to S$ is pattern equivariant with radius $R_{\tilde{k}}$, then f takes the same value on all tilings represented by the same point in $\Gamma_{\tilde{k}}$, and f descends to a well defined map $\tilde{f}$ on $\Gamma_{\tilde{k}}$.

In other words, if Θ is a tiling space with $\Theta = \varprojlim \Gamma_k$, and if $\tilde{\pi} : \varprojlim \Gamma_k \to \Gamma_{\tilde{k}}$ is the natural projection, then $f : \Theta \to S$ is pattern equivariant with radius $R_{\tilde{k}}$ if and only if there exists a map $\tilde{f} : \Gamma_{\tilde{k}} \to S$ such that the following diagram commutes:

$$\begin{tikzcd} \Theta \arrow[r, "\tilde{\pi}"] \arrow[dr, "f"'] & \Gamma_{\tilde{k}} \arrow[d, "\tilde{f}"] \\ & S \end{tikzcd} \tag{4.6}$$

Pattern equivariance is really the key idea. To go from pattern equivariant functions to pattern equivariant cohomology, we just extend this idea of pattern equivariance to differential forms (or cochains), restrict our attention to pattern equivariant objects, and then do cohomology in the usual way.

Let T be a tiling of $\mathbb{R}^2$. Let ω be differential form on $\mathbb{R}^2$, and let $R > 0$. Suppose that whenever there exist $x, x' \in \mathbb{R}$ and a translation g such that

$$[B_R(x')] = g[B_R(x)], \tag{4.7}$$

then we also have that

$$\omega|_{x'} = (g^{-1})^* \omega|_x. \tag{4.8}$$

In these circumstances, we say that ω is a pattern equivariant differential form with radius r. More concretely, a pattern equivariant 0-form is a smooth pattern equivariant function $f : \mathbb{R}^2 \to \mathbb{R}$ (or $\mathbb{C}$). A pattern equivariant 1-form is a differential form $f_1 dx + f_2 dy$, where f_1 and f_2 are pattern equivariant functions, and a pattern equivariant 2-form is a differential form $f dxdy$ where f

is pattern equivariant. Let d^k_{PE} be the usual deRham differential, but restricted to pattern equivariant k-forms.

Definition 4.6 (pattern equivariant cohomology with differential forms). The *pattern equivariant cohomology* $H^k_{PE}(T)$ is defined as

$$H^k_{PE}(T) = \frac{\ker d^k_{PE}}{\operatorname{Im} d^{k-1}_{PE}}. \tag{4.9}$$

Johannes Kellendonk [10] presented the concept of pattern equivariance in 2003, along with a theorem, proven by himself and Ian Putnam [11], relating pattern equivariant cohomology and Čech cohomology[1].

Theorem 4.3. *Let T be a tiling of $\mathbb{R}^N$. Let $X = X^T_{\mathbb{R}^N}$, then $H^k_{PE}(T) \cong \check{H}(X : \mathbb{R})$.*

The only conspicuous shortcoming of Kellendonk and Putnam's work is that we necessarily must work over $\mathbb{R}$ (or $\mathbb{C}$, or some real vector space). Sadun [17] presents an approach which allows us to work over $\mathbb{Z}$ (or any abelian group). Given a tiling T of $\mathbb{R}^2$, interpret $\mathbb{R}^2$ as a CW-complex in which the tiles are the 2-cells, the edges of the tiles are the 1-cells, and vertices of the tiles are the 0-cells. Let σ be a $\mathbb{Z}$-valued 2-cochain defined on the CW-complex in the usual way. For each tile t, let p_t be the point at the tile's center of mass. We say that σ is a *pattern equivariant* 2*-cochain* with radius r, if $\sigma(t') = \sigma(t)$ whenever $[B_r(p_{t'})] = [B_r(p_t)] + (p_{t'} - p_t)$. We can similarly define the pattern equivariant 0-cochains, and 1-cochains. Let δ^n_{PE} be the usual coboundary map from k-cochains to $k+1$-cochains, but restricted to pattern equivariant cochains.

[1]Kellendonk and Putnam and Sadun assume that T has finite local complexity, but in fact that assumption is more than is necessary. A close inspection of Sadun's proof reveals that it suffices to assume that there are finitely many tiles, up to rotation, and finitely many ways that tiles can meet. That allows us to show that the approximants are CW-complexes (or branched manifolds). Under these slightly relaxed conditions, the approximants might be non-compact, but we need not require compactness.

Definition 4.7 (**pattern equivariant cohomology (integer version)**). The *integer-valued pattern equivariant cohomology* $H^k_{PE}(T : \mathbb{Z})$ is

$$H^k_{PE}(T : \mathbb{Z}) = \frac{\ker \delta^k_{PE}}{\operatorname{Im} \delta^{k-1}_{PE}}. \tag{4.10}$$

Moreover, Sadun shows

Theorem 4.4. *Let T be a tiling of $\mathbb{R}^N$. Let $X = X^T_{\mathbb{R}^N}$, then $H^k_{PE}(T : \mathbb{Z}) \cong \check{H}(X : \mathbb{Z})$.*

Tiling space theorists had been studying the Čech cohomology of tiling spaces before pattern equivariant cohomology was invented, but thinking in terms of pattern equivariant cohomology yields insight into what our results actually mean, when we crank through a cohomology calculation.

4.3 Rand cohomology

In her thesis [16], Betseygail Rand's basic idea was to do pattern equivariant cohomology while bringing rotations into the fold. Let T be a tiling of $\mathbb{R}^2$, let G^T_{rot} be its associated rotation group, and assume that if P is any patch that can be found in T, then for all $g \in G^T_{rot}$, a translation of the patch gP can be found somewhere in T. Now, let A be an abelian group, and let $\rho : G^T_{rot} \to Aut(A)$ be a group action, so that group elements act on A as automorphisms. In practice, A is usually $\mathbb{Z}^n, \mathbb{R}^n$, or $\mathbb{C}^n$. In the case where A is a vector space (i.e. $\mathbb{R}^n$ or $\mathbb{C}^n$), ρ is just a representation of the group G^T_{rot}. Even when A is not a vector space, we will abuse terminology and just call ρ a representation of G^T_{rot}.

Let $f : \mathbb{R}^2 \to A$, $R > 0$. Suppose that whenever $x, x' \in \mathbb{R}^2$ are such that

$$[B_R(x')]_T - x' = g \cdot ([B_R(x)]_T - x) \tag{4.11}$$

for some $g \in G^T_{rot}$, then we also have that

$$f(x') = \rho(g) f(x). \tag{4.12}$$

Then, we say that σ is *pattern equivariant to radius R with representation ρ* or just *ρ-invariant* for short. More informally, a function $f : \mathbb{R}^2 \to A$ is ρ-invariant if, whenever the local patterns at x and x' differ only by a rotation $g \in G^T_{rot}$, the values of $f(x)$ and $f(x')$ differ by $\rho(g)$.

With this new idea of pattern equivariance that takes into account the rotations, we proceed in analogy to the previous section. As before, we can alter this definition a bit, and think of a ρ-invariant function as a function defined on a tiling space X. Namely, a function $f : X \to A$ is ρ-invariant, if $f(T') = \rho(g)f(T)$ whenever the local pattern (at the origin) of the rotated tiling gT agrees with the local pattern of T'.

We can define the concept of a ρ-invariant differential form. Let ω be a differential form on $\mathbb{R}^2$, taking values in a representation ρ. Suppose that whenever there exist $x, x' \in \mathbb{R}^2$, and $g = g_r g_t$ with $g_r \in G^T_{rot}$ and g_t a translation such that

$$[B_R(x')]_T = g \cdot [B_R(x)]_T \tag{4.13}$$

then we also have

$$\omega|_{x'} = \rho(g_r)(g^{-1})^* \omega|_x. \tag{4.14}$$

Under these circumstances, we say that ω is ρ-invariant. We use this to define Rand cohomology.

Definition 4.8 (**Rand cohomology with differential forms**)**.** Let d_ρ be the usual De Rham differential, but restricted to ρ-invariant forms of the tiling T. The *Rand cohomology of the tiling T, with representation ρ* is

$$H^k_\rho(T) = \frac{\ker d^k_\rho}{\operatorname{Im} d^{k-1}_\rho}. \tag{4.15}$$

We can also define Rand cohomology using cochains. Given a tiling T of $\mathbb{R}^2$, again we can interpret $\mathbb{R}^2$ as a CW-complex in which the tiles are the 2-cells, the edges of the tiles are the 1-cells, and vertices of the tiles are the 0-cells. Let σ be a 2-cochain defined on the CW-Complex, but taking values in A. For each tile t, let p_t be the point at the tile's center of mass. We say that σ is a *ρ-invariant* 2-*cochain* with radius R, if $\sigma(t') = \rho(g)\sigma(t)$ whenever $[B_R(p_{t'})] - p_{t'} = g \cdot ([B_R(p_t)] - p_t)$ for some $g \in G^T_{rot}$. We can similarly define the ρ-invariant 1-cochains and 0-cochains. Let δ_ρ^k be the usual coboundary map from k-cochains to $(k+1)$-cochains, but restricted to pattern equivariant cochains.

Definition 4.9 (Rand cohomology with cochains). The *Rand cohomology of the tiling T, with representation ρ* is

$$H_\rho^k(T) = \frac{\ker \delta_\rho^k}{\operatorname{Im} \delta_\rho^{k-1}}. \tag{4.16}$$

Rand introduced her cohomology as an object specifically tailored to the world of tilings and tiling spaces. In fact, the fundamental idea of weaving a group action into the cohomology via a representation is broadly applicable. For example, let Γ be a CW-complex. Let $\mathcal{K}^n$ be the space of n-cells, and let

$$\cdots \longleftarrow C_{n-1} \xleftarrow{\partial_n} C_n \xleftarrow{\partial_{n+1}} C_{n+1} \longleftarrow \cdots \tag{4.17}$$

be the corresponding chain complex. Now suppose that a group G acts on Γ, in a way that respects the CW-complex structure. This means that G_Γ permutes the elements of $\mathcal{K}_n$ for each n. This induces an action of G on C_n, and, if G truly respects CW-complex structure, it must do so in such a way that the boundary map commutes with the action of G_Γ. In other words, for all $g \in G$, and for all $\alpha \in C_n$, if

$$\partial_n \alpha = c^1 \beta_1 + \cdots + c^k \beta_k \tag{4.18}$$

Then

$$\partial_n (g\alpha) = c^1 (g\beta_1) + \cdots + c^k (g\beta_k) \tag{4.19}$$

Another way of saying this is that each element $g \in G$ is a cellular map on Γ. Now, let $\rho : G \times A \to A$ be a representation, as above. We say that a map $\tau : C_n \to A$ is a *ρ-invariant n-cochain* if it commutes with the action of G_Γ, i.e. $\tau(gv) = \rho(g)\tau(v)$, and we denote the space of all such maps as C_ρ^n. Now, we can define the ρ-invariant cochain complex

$$\cdots \longrightarrow C_\rho^{n-1} \xrightarrow{\delta_\rho^{n-1}} C_\rho^n \xrightarrow{\delta_\rho^n} C_\rho^{n+1} \longrightarrow \cdots \tag{4.20}$$

where $d_\rho^n \tau = \tau \circ \partial_{n+1}$. We define

$$H_\rho^n(\Gamma) = \frac{\ker \delta_\rho^n}{\operatorname{Im} \delta_\rho^{n-1}}. \tag{4.21}$$

We will refer to this as the n^{th} Rand cohomology of Γ with representation ρ. More generally, if we have any type of structure on a space X that allows us to define some particular type of cohomology, then if we have a G action on X that respects the structure, we can define a ρ-invariant version of that cohomology, for each representation ρ of G. We defined the pattern equivariant cohomology $H_{PE}^k(T)$ associated to a tiling T, and if T has an associated rotation group G_{rot}^T, then for any representation ρ of G_{rot} we can define the Rand cohomology $H_\rho^k(T)$. On a CW-complex Γ we have cellular cohomology, and if a group G acts on Γ via cellular maps, we can define $H_\rho^n(\Gamma)$. Similarly, any smooth manifold X has de Rham cohomology. If a group G acts on X via diffeomorphisms, then we can define ρ-invariant de Rham cohomology on X.

What if X is just a topological space with a G-action? Well, as long as G acts via homeomorphisms, we can define a ρ-invariant version of Čech cohomology as well. In order to do this, we need to introduce the idea of a G-respectful open cover $\mathcal{U}_G$, which is just an open cover of X with the property that if U is an open set in the cover, then gU is also an open set in the cover. The nerve $N(\mathcal{U}_G)$ of $\mathcal{U}_G$ then naturally admits a G action, so for any given representation we can define the ρ-invariant Čech cohomology $\check{H}_\rho^*(X, \mathcal{U}_G)$ of the cover $\mathcal{U}_G$. The collection of all G-respectful open covers is a directed set, and if $\mathcal{V}_G$ is a refinement of $\mathcal{U}_G$, we have a natural map $\sigma : H_\rho^*(X, \mathcal{U}_G) \to H_\rho^*(X, \mathcal{V}_G)$. So we can define the Rand or ρ-invariant Čech cohomology $\check{H}_\rho^*(X)$ to be the direct limit of $H_\rho^*(X, \mathcal{U}_G)$ over all G-respectful open covers.

N.B.:We are being more than a bit cavalier in our use of the term "Rand cohomology". Although the term is not in widespread use, we can justify the use of the term when working with pattern equivariant cohomology of tilings and tiling spaces, since she did introduce the idea in her thesis. However, when we use the term in a broader context, we only mean to use the term as a shorthand. The fundamental idea is not new to algebraic topology. See, for example, the appendix on homology and cohomology with *local coefficients* in Hatcher's book [8].

All of the statements of the previous two sections that relate one type of cohomology to another have direct analogs here. Suppose that I is a directed set and that $\{X_\alpha\}_{\alpha \in I}$ is a collection of topological spaces, each of which admits a G-action. Suppose that for each $\alpha < \beta \in I$ we have a map $\sigma_{\beta\alpha} : X_\beta \to X_\alpha$, so that $\sigma_{\gamma\alpha} = \sigma_{\gamma\beta} \circ \sigma_{\beta\alpha}$ whenever $\alpha < \beta < \gamma$. Furthermore assume that $\sigma_{\beta\alpha}$ always commutes with the action of G. If we let $X = \varprojlim X_\alpha$, then X admits a G action as well, and we have

$$\check{H}^*_\rho(X) \cong \varinjlim \check{H}^*_\rho(X_\alpha) \tag{4.22}$$

for any representation ρ of G. It is understood that the direct limit in (4.22) is obtained via the pullbacks $\sigma^*_{\beta\alpha}$. Let Γ be a CW-complex with a G action, then

$$H^n_\rho(\Gamma) \cong \check{H}^n_\rho(\Gamma), \tag{4.23}$$

where $H^n_\rho(\Gamma)$ is the cellular ρ-invariant cohomology, as in equation (4.21). Finally, if T is a tiling with associated rotation group G^T_{rot}, then $X = X^T_{\mathbb{R}^N}$ admits a G^T_{rot} action, and we have

$$H^k_\rho(T) \cong \check{H}^k_\rho(X). \tag{4.24}$$

With this machinery in place, we are now ready to actually calculate some tiling space cohomology. Before we proceed, we want to introduce/clarify some terminology that we will use throughout the rest of this thesis. Whether we are talking about pattern equivariant cohomology, cellular cohomology, or Čech cohomology, whenever we mean to refer to the ρ-invariant version of the cohomology, we will call it the Rand cohomology, but when we mean to refer to the standard, non-Rand version of the cohomology, we will call it the **Total** cohomology.

4.4 The Cohomology of the Equithirds Tiling Space

Let T be the equithirds tiling, and let $G = \mathbb{Z}/6\mathbb{Z}$ be its rotation group, generated by r which is a rotation by $60°$. In this section, we will calculate

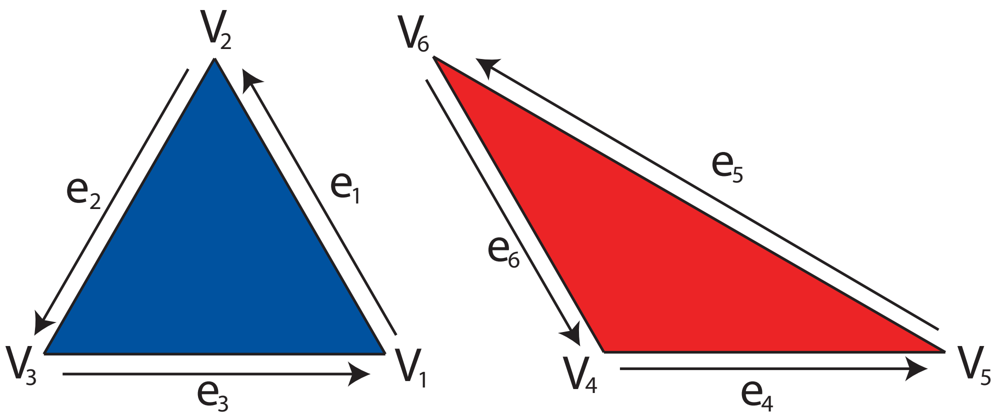

Figure 4.1: Labeling of vertices and edges in the Anderson Putnam complex of the equithirds tiling space. The face corresponding to the Eq tile in the orientation shown is called f_1, and the face corresponding to the Is tile in the orientation shown is called f_2.

the total cohomology $H^*(T)$ and the Rand cohomology $H^*_\rho(T)$ for irreducible representations ρ. In both instances, we achieve this by calculating the cellular cohomology $H^*(\Gamma)$ or $H^*_\rho(\Gamma)$ of the Anderson-Putnam complex, and then taking a direct limit via the pullback of the (squared) substition map.

Let's start with the total cohomology. The Anderson-Putnam complex Γ has eight 2-cells, namely the Eq tile (equilateral) in two different orientations, and the Is tile (30°-30°-120°) in six different orientations. Figure 4.1 shows the Eq and Is tiles, with associated edges and vertices labeled. Let f_1 refer to the Eq tile in the orientation shown in figure 4.1, and let f_2 refer to the Is tile in the orientation shown. The eight 2-cells are $f_1, rf_1, f_2, rf_2, r^2f_2, r^3f_2, r^4f_2, r^5f_2$. A priori, it seems that Γ can have as many as 24 1-cells, on account of the fact that each of the eight 2-cells has three edges, but recall that the Anderson Putnam construction requires that we glue along an edge, every time two tiles can meet at that edge. So, for example it is possible that Eq and Is are adjacent to each other so that e_6 lines up (oppositely oriented) with e_1, thus $e_6 = -e_1$. Similarly, we see that it is possible to line up an Eq tile with a sixty degree rotation of Eq , and in that configuration, we see that e_1 lines up with

$-re_2$. When we examine all possibilities, we get

$$e_2 = r^2 e_1, \quad e_3 = -re_1, \quad e_4 = -re_1, \quad e_6 = -e_1, \quad e_j = -r^3 e_j. \tag{4.25}$$

So after all these identifications, we have only six 1-cells, which we can choose to be $e_1, re_1, r^2e_1, e_5, re_5, r^2e_5$. Now we do the same thing with the vertices. Again, a priori it seems we may have as many as 24 0-cells, until we identify the vertices where tiles can meet, but there is only one 0-cell after all the identifications. All zero cells are identified with v_1.

To summarize, if we let C_n be the space of n-chains, we have

$$C_0 = \operatorname{span}\{v_1\} \cong \mathbb{Z}, \tag{4.26a}$$
$$C_1 = \operatorname{span}\{e_1, re_1, r^2e_1, e_5, re_5, r^2e_5\} \cong \mathbb{Z}^6, \tag{4.26b}$$
$$C_2 = \operatorname{span}\{f_1, rf_1, f_2, rf_2, r^2f_2, r^3f_2, r^4f_2, r^5f_2\} \cong \mathbb{Z}^8. \tag{4.26c}$$

Inspection of figure 4.1 allows us to write down the boundary maps:

$$\partial e_1 = v_2 - v_1 = 0, \tag{4.27a}$$
$$\partial e_5 = v_6 - v_5 = 0, \tag{4.27b}$$
$$\partial f_1 = e_1 + e_2 + e_3 = (1 + r^2 - r)e_1, \tag{4.27c}$$
$$\partial f_2 = e_4 + e_5 + e_6 = (-1 - r)e_1 + e_5. \tag{4.27d}$$

Since ∂ commutes with the action of G, equations (4.27) completely define the boundary map ∂. This gives us that the coboundary map δ^0 is identically zero, which immediately implies that $H^0(\Gamma) \cong \mathbb{Z}$ and that δ^1 is given by:

$$\delta^1(e_1^*) = f_1^* - (rf_1)^* - f_2^* + (r^2f_2)^* + (r^3f_2)^* - (r^5f_2)^*, \tag{4.28a}$$
$$\delta^1((re_1)^*) = -f_1^* + (rf_1)^* - f_2^* - (r^2f_2)^* + (r^3f_2)^* + (r^4f_2)^*, \tag{4.28b}$$
$$\delta^1((r^2e_1)^*) = f_1^* - (rf_1)^* - (rf_2)^* - (r^2f_2)^* + (r^4f_2)^* + (r^5f_2)^*, \tag{4.28c}$$
$$\delta^1(e_5^*) = f_2^* - (r^3f_2)^*, \tag{4.28d}$$
$$\delta^1((re_5)^*) = (rf_2)^* - (r^4f_2)^*, \tag{4.28e}$$
$$\delta^1((r^2e_5)^*) = (r^2f_2)^* - (r^5f_2)^*. \tag{4.28f}$$

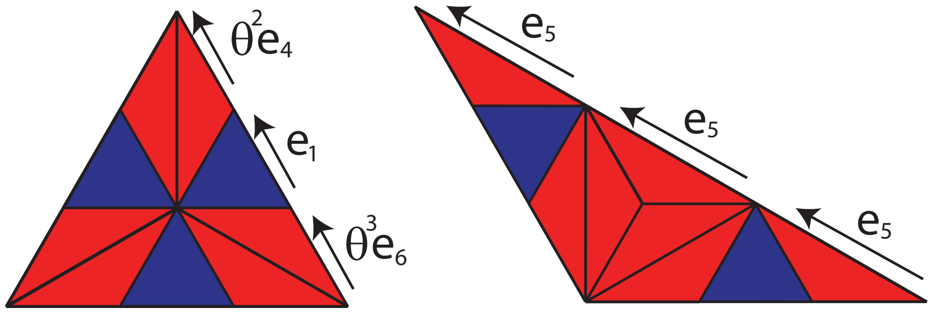

Figure 4.2: An illustration of how σ^2 acts on the Anderson Putnam complex.

The kernel of δ^1 is spanned by

$$\tau_1 = e_1^* - (r^2 e_1)^* + e_5^* - 2(re_5)^* - (r^2 e_5)^*, \tag{4.29a}$$
$$\tau_2 = (re_1)^* + (re_2)^* + e_5^* + (re_5)^* + 2(r^2 e_5)^*. \tag{4.29b}$$

Since $\operatorname{Im}\delta^0 = 0$, this shows that $H^1(\Gamma) \cong \mathbb{Z}^2$.

The image of δ^1 is spanned by $f_1^* - (rf_1)^*$, $f_2^* - (r^3 f_2)^*$, $(rf_2)^* - (r^4 f_2)^*$, and $(r^2 f_2)^* - (r^5 f_2)^*$. This gives us a convenient basis for $H^2(\Gamma)$, namely $\{f_1^*, f_2^*, (rf_2)^*, (r^2 f_2)^*\}$, and where $f_1^* = (rf_1)^*$, $f_2^* = (r^3 f_2)^*$, $(rf_2)^* = (r^4 f_2)^*$, and $(r^2 f_2)^* = (r^5 f_2)^*$, in cohomology. So, $H^2(\Gamma) \cong \mathbb{Z}^4$. The total cohomology of the Anderson Putnam complex is

$$H^0(\Gamma) \cong 0, \qquad H^1(\Gamma) \cong \mathbb{Z}^2, \qquad H^2(\Gamma) \cong \mathbb{Z}^4. \tag{4.30}$$

In order to calculate the pattern equivariant cohomology $H^*(T)$, we need to examine the squared substitution map σ^2. Examine figure 4.2. Substitution just sends the vertex v_0 to itself, thus $(\sigma^2)^* : H^0(\Gamma) \to H^0(\Gamma)$ is the identity, so $H^0(T) = \varinjlim H^0(\Gamma) \cong \mathbb{Z}$. Now examine the action of σ^2 on the edges. Clearly, $\sigma^2(e_5) = 3e_5$, but also $\sigma^2(e_1) = r^2 e_4 + e_1 + r^3 e_6 = 3e_1$. Thus, $(\sigma^2)^* : H^1(\Gamma) \to H^1(\Gamma)$ is multiplication by 3, so $H^1(T) = \mathbb{Z}[1/3]^2$. Finally we examine the action of σ^2 on f_1 and f_2. We have

$$\sigma^2(f_1) = 3f_1 + f_2 + rf_2 + r^2 f_2 + r^3 f_2 + r^4 f_2 + r^5 f_2, \tag{4.31a}$$
$$\sigma^2(f_2) = f_1 + rf_1 + 3f_2 + rf_2 + r^2 f_2 + r^4 f_2 + r^5 f_2. \tag{4.31b}$$

Writing $(\sigma^2)^* : H^2(\Gamma) \to H^2(\Gamma)$ in terms of our basis, we have

$$(\sigma^2)^*(f_1^*) = 3f_1^* + 2f_2^* + 2(rf_2)^* + 2(r^2f_2)^*, \tag{4.32a}$$
$$(\sigma^2)^*(f_2^*) = 2f_1^* + 3f_2^* + 2(rf_2)^* + 2(r^2f_2)^*, \tag{4.32b}$$
$$(\sigma^2)^*((rf_2)^*) = 2f_1^* + 2f_2^* + 3(rf_2)^* + 2(r^2f_2)^*, \tag{4.32c}$$
$$(\sigma^2)^*((r^2f_2)^*) = 2f_1^* + 2f_2^* + 2(rf_2)^* + 3(r^2f_2)^*. \tag{4.32d}$$

$(\sigma^2)^*$ has a one-dimensional eigenspace V_9 with eigenvalue $\lambda = 9$ where

$$V_9 = \mathrm{span}\{f_1^* + f_2^* + (rf_2)^* + (r^2f_2)^*\}. \tag{4.33}$$

So, $\varinjlim V_9 \cong \mathbb{Z}[1/9]$. $(\sigma^*)^2$ also has a three-dimensional eigenspace V_1 with eigenvalue $\lambda = 1$ where

$$V_1 = \{c_1f_1^* + c_2f_2^* + c_3(rf_2)^* + c_4(r^2f_2)^* | c_1 + c_2 + c_3 + c_4 \neq 0\}. \tag{4.34}$$

Claim. $\varinjlim(H^2(\Gamma)) \cong \mathbb{Z}[1/3] \oplus \mathbb{Z}^3$.

Proof. For $\tau \in H^2(\Gamma)$, let $(\tau)^k \in \varinjlim H^2(\Gamma)$, refer to the element $\tau \in \Gamma_k \cong \Gamma$. So, (4.32) shows that in $\varinjlim H^2(\Gamma)$ we have

$$(f_1^*)^0 = (3f_1^* + 2f_2^* + 2(rf_2)^* + 2(r^2f_2)^*)^1, \tag{4.35a}$$
$$(f_2^*)^0 = (2f_1^* + 3f_2^* + 2(rf_2)^* + 2(r^2f_2)^*)^1, \tag{4.35b}$$
$$((rf_2)^*)^0 = (2f_1^* + 2f_2^* + 3(rf_2)^* + 2(r^2f_2)^*)^1. \tag{4.35c}$$

We prove the claim by showing that

$$\varinjlim H^2(\Gamma) = \mathrm{span}\{(f_1^*)^0, (f_2^*)^0, ((rf_2)^*)^0\} \oplus \varinjlim V_9. \tag{4.36}$$

(4.35) shows that $(f_1^*)^0 = (f_1^*)^1 + 2(\omega)^1$, where $\omega = f_1^* + f_2^* + (rf_2)^* + (r^2f_2)^*$, and similarly for $(f_2^*)^0$ and $((rf_2)^*)^0$. But $\omega \in V_9$, thus $(\omega)^k = 9(\omega)^{k+1}$, so inductively we can see that

$$(f_1^*)^0 = (f_1^*)^k + s_k(\omega)^k, \tag{4.37a}$$
$$(f_2^*)^0 = (f_2^*)^k + s_k(\omega)^k, \tag{4.37b}$$
$$((rf_2)^*)^0 = ((rf_2)^*)^k + s_k(\omega)^k. \tag{4.37c}$$

for some integer s_k. So, we have

$$
\begin{aligned}
(f_1^*)^k &= (f_1^*)^0 - s_k(\omega)^k, && \text{(4.38a)}\\
(f_2^*)^k &= (f_2^*)^0 - s_k(\omega)^k, && \text{(4.38b)}\\
((rf_2)^*)^k &= ((rf_2)^*)^0 - s_k(\omega)^k, && \text{(4.38c)}\\
((r^2 f_2)^*)^k &= (3s_k+1)(\omega)^k - (f_1^*)^0 - (f_2^*)^0 - ((rf_2)^*)^0. && \text{(4.38d)}
\end{aligned}
$$

since $(\omega)^k \in \varinjlim V_9$, equations 4.38 show that any element in the direct limit $\varinjlim H^2(\Gamma)$ is in $(\mathrm{span}\{(f_1^*)^0, (f_2^*)^0, ((rf_2)^*)^0\}) \oplus \varinjlim V_9$. □

So,

$$H^0(T) \cong \mathbb{Z}, \qquad H^1(T) \cong \mathbb{Z}[1/3]^2, \qquad H^2(T) \cong \mathbb{Z}[1/3] \oplus \mathbb{Z}^3. \tag{4.39}$$

Now, let's calculate the Rand cohomology $H^*_\rho(T)$ for all irreducible representations ρ of G. In any representation of G, we must have $r^6 - 1 = 0$. This equation factors over $\mathbb{Z}$ or $\mathbb{R}$ into $(r^6-1) = (r-1)(r+1)(r^2+r+1)(r^2-r+1) = 0$. This gives us four irreducible real representations:

$$\frac{\mathbb{R}[r]}{(r-1)}, \quad \frac{\mathbb{R}[r]}{(r+1)}, \quad \frac{\mathbb{R}[r]}{(r^2-r+1)}, \quad \frac{\mathbb{R}[r]}{(r^2+r+1)}. \tag{4.40}$$

So, we have two one-dimensional real representations $(r = 1)$ and $(r = -1)$ and two two-dimensional real representations $(r^2+r+1 = 0)$ and $(r^2-r+1 = 0)$. We will calculate the integer Rand cohomology $H^*_\rho(T)$ where ρ is each of the four analogous integer representations:

$$\frac{\mathbb{Z}[r]}{(r-1)}, \quad \frac{\mathbb{Z}[r]}{(r+1)}, \quad \frac{\mathbb{Z}[r]}{(r^2-r+1)}, \quad \frac{\mathbb{Z}[r]}{(r^2+r+1)}. \tag{4.41}$$

Let us clarify how the ring $\mathbb{Z}[r]/(F(r))$, where F is a monic polynomial in $\mathbb{Z}[r]$, should be thought of as a representation. For example, examine $R = \mathbb{Z}[r]/(r^2-r+1)$. As an abelian group, $R \cong \mathbb{Z}^2$, but the group action is also apparent: the element $r \in G$ sends $f \in \mathbb{Z}[r]/(r^2-r+1)$ to rf. This also makes it easy to refer to cochains. When we write e_1^*, we mean the ρ-invariant cochain which evaluates to $1 \in \mathbb{Z}[r]/(r^2-r+1)$ on e_1, and evaluates to zero on e_5, and similarly for e_5^*, f_1^*, f_2^*.

For any given representation ρ, let $C^n_\rho(\Gamma)$ be the space of ρ-invariant n-cochains on Γ.

For the $r = 1$ representation, we have $C^0_\rho(\Gamma) = \text{span}\{v_1^*\} \cong \mathbb{Z}$. However, any ρ-invariant 1-cochain must vanish every everywhere. Since $e_j = -r^3 e_j$, if τ is a ρ-invariant 1-cochain then $\tau(e_j) = \tau(-r^3 e_j) = -r^3\tau(e_j) = -\tau(e_j)$, thus $\tau \equiv 0$. So, $C^1_\rho(\Gamma) \cong 0$. Finally, $C^2_\rho = \text{span}\{f_1^*, f_2^*\} \cong \mathbb{Z}^2$. So, trivially, we have $H^0_\rho(\Gamma) \cong \mathbb{Z}$, $H^1_\rho(\Gamma) \cong 0$, $H^2_\rho(\Gamma) \cong \mathbb{Z}^2$.

To calculate $H^0_\rho(T)$, we need to examine σ^2. On vertices, we have $[(\sigma^2)^*(v_1^*)](v_1) = v_1^*(\sigma^2(v_1)) = v_1^*(v_1) = 1$. Thus $(\sigma^2)^*(v_1^*) = v_1^*$, so $H^0_\rho(\Gamma) = \mathbb{Z}$. We also have

$$\begin{aligned}[(\sigma^2)^*(f_1^*)](f_1) &= f_1^*(\sigma^2(f_1)) \\ &= f_1^*(3f_1 + f_2 + rf_2 + r^2 f_2 + r^3 f_2 + r^4 f_2 + r^5 f_2) = 3,\end{aligned} \tag{4.42a}$$

$$\begin{aligned}[(\sigma^2)^*(f_1^*)](f_2) &= f_1^*(\sigma^2(f_2)) \\ &= f_1^*(f_1 + rf_1 + 3f_2 + rf_2 + r^2 f_2 + r^4 f_2 + r^5 f_2) = 2,\end{aligned} \tag{4.42b}$$

$$\begin{aligned}[(\sigma^2)^*(f_2^*)](f_1) &= f_2^*(\sigma^2(f_1)) \\ &= f_2^*(3f_1 + f_2 + rf_2 + r^2 f_2 + r^3 f_2 + r^4 f_2 + r^5 f_2) = 6,\end{aligned} \tag{4.42c}$$

$$\begin{aligned}[(\sigma^2)^*(f_2^*)](f_2) &= f_2^*(\sigma^2(f_2)) \\ &= f_2^*(f_1 + rf_1 + 3f_2 + rf_2 + r^2 f_2 + r^4 f_2 + r^5 f_2) = 7.\end{aligned} \tag{4.42d}$$

So,

$$(\sigma^2)^*(f_1^*) = 3f_1^* + 2f_2^*, \tag{4.43a}$$

$$(\sigma^2)^*(f_2^*) = 6f_1^* + 7f_2^*. \tag{4.43b}$$

We see that $(\sigma^2)^*$ has eigenvalues 9 and 1, with one-dimensional eigenspaces V_9 and V_1, respectively. Similarly, to the situation in total cohomology, we have $H^2_\rho(\Gamma) \cong \mathbb{Z} \oplus \mathbb{Z}[1/9]$.

For the $r = -1$ representation, all ρ-invariant 0-cochains vanish identically, because if τ is a ρ-invariant 1-cochain then $\tau(v_1) = \tau(rv_1) = -\tau(v_1)$, thus $C^0_\rho(\Gamma) \cong 0$. $C^1_\rho = \text{span}\{e_1^*, e_5^*\} \cong \mathbb{Z}^2$ and $C^2_\rho = \text{span}\{f_1^*, f_2^*\} \cong \mathbb{Z}^2$. Because $r = -1$, 4.27 gives us

$$\delta^1_\rho(e_1^*) = 3f_1^* \qquad \text{and} \qquad \delta^1_\rho(e_5^*) = f_2^*. \tag{4.44}$$

So, we have $H^0_\rho(\Gamma) \cong 0$, $H^1_\rho(\Gamma) \cong 0$, $H^2_\rho(\Gamma) \cong \mathbb{Z}/3\mathbb{Z}$. But, $(\sigma^2)^*(f_1^*) = 3f_1^* = 0$, so $H^2_\rho(T) = 0$.

For the $r^2 - r + 1 = 0$ representation, notice that $r^2 - r + 1 = 0$ implies $r^3 = -1$. Since $r^3 v_1 = v_1$, again we have that $C^0_\rho(\Gamma) \cong 0$. For all edges e, we have $r^3 e = -e$, so we do have nontrivial 1-cochains $e_1^*, r(e_1^*), r^2(e_1^*), e_5^*, r(e_5^*), r^2(e_5^*)$, but $r^2(e_j^*) = r(e_j^*) - e_j^*$ for $j = 1, 5$, thus $C^1_\rho(\Gamma) \cong \mathbb{Z}^4$. It is easy to check that any ρ-invariant 2-cochain must vanish on f_1, but we do have nontrivial cochains $r^k f_2^*$. Since $r^2 - r + 1 = 0$, any of these can be expressed as a linear combination of f_2^* and rf_2^*, so $C^1_\rho(\Gamma) \cong \mathbb{Z}^4$.

Notice that $\delta^1_\rho(e_5^*) = f_2^*$ and $\delta^1_\rho(r(e_5^*)) = rf_2^*$, so δ^1_ρ surjects. Thus $H^0_\rho(\Gamma) = 0$, $H^1_\rho(\Gamma) = \mathbb{Z}^2$, $H^2_\rho(\Gamma) = 0$. We saw that $\sigma^2(e_j) = 3e_j$, for $j = 1, 5$. So, $(\sigma^2)^* : H^1_\rho(\Gamma) \to H^1_\rho(\Gamma)$ is multiplication by 3, thus $H^1_\rho(T) \cong \mathbb{Z}[1/3]^2$.

Finally, for the $r^2 - r + 1 = 0$ representation, we again have $C^0_\rho(\Gamma) \cong 0$. All 1-cochains must vanish identically, so $C^1_\rho(\Gamma) \cong 0$, but we do have two linear independent 2-cochains f_2^* and rf_2^*, so $C^2_\rho(\Gamma) \cong \mathbb{Z}^2$. Trivially, $H^0_\rho(\Gamma) = 0$, $H^1_\rho(\Gamma) = \mathbb{Z}^2$, $H^2_\rho(\Gamma) = \mathbb{Z}^2$, and it is not difficult to check that $(\sigma^2)^*$ is the identity on $H^2_\rho(\Gamma) = \mathbb{Z}^2$.

Table (4.45) summarizes the Rand cohomology $H^*_\rho(\Gamma)$.

ρ	$H^0_\rho(\Gamma)$	$H^1_\rho(\Gamma)$	$H^2_\rho(\Gamma)$
$r = 1$	$\mathbb{Z}$	0	$\mathbb{Z}[1/3] \oplus \mathbb{Z}$
$r = -1$	0	0	0
$r^2 - r + 1 = 0$	0	$\mathbb{Z}[1/3]^2$	0
$r^2 + r + 1 = 0$	0	0	$\mathbb{Z}^2$

(4.45)

4.5 Rand cohomology versus total cohomology

Comparing 4.45 to 4.39, we are led to conjecture that the total cohomology $H^*(T)$ is just direct sum of the Rand cohomologies $H^*_\rho(T)$ over all irreducible representations ρ of G^T_{rot}. If we allow ourselves to work over the integers, this conjecture is false. There are examples in which the sum of the Rand cohomologies for the irreducible factors (e.g. $r - 1$, $r + 1$, $r^2 - r + 1$, $r^2 + r + 1$ in the equithirds example) is a proper, finite index subgroup of the total pattern equivariant cohomology. (N.B. : this discrepancy is of some interest, but for the purposes of this thesis, it is an issue we would rather avoid tripping over. In the following chapters, we will work exclusively over the complex numbers.)

More precisely, it is always true is that if ρ, ρ_1, ρ_2 are representations of G and $\rho = \rho_1 \oplus \rho_2$, then

$$H^*_\rho = H^*_{\rho_1} \oplus H^*_{\rho_2}. \tag{4.46}$$

The total cohomology $H^*(\Gamma)$ is the same as the Rand cohomology $H^*_{\rho_{reg}}(\Gamma)$, where ρ_{reg} is the regular representation of G. Let R be a ring, and G a finite group. The group ring $R[G]$ is the ring of formal expressions of the form

$$\sum_{g \in G} r_g g \tag{4.47}$$

where $r_g \in R$. The ring structure is defined in the natural way, with $rg + sg = (r+s)g$, and $(rg_1)(sg_2) = (rs)(g_1 g_2)$. $R[G]$ is clearly a module over R, and it is naturally a representation of G, with elements of G acting on elements of $R[G]$ by left multiplication. This representation is called the *left regular representation of G over R.*

Let τ be an n-cochain. We can associate to τ a ρ_{reg}-invariant cochain $\bar{\tau}$ defined by

$$\bar{\tau}(\alpha) = \sum_{g \in G} \tau(g\alpha) g \tag{4.48}$$

where α is any simple n-chain. It is not difficult to check that (4.48) defines an isomorphism between $C^*(\Gamma)$ and $C^*_\rho(\Gamma)$, and since the boundary map ∂ commutes with the G action, we have

$$H^*(\Gamma) \cong H^*_{\rho_{reg}}(\Gamma). \tag{4.49}$$

A basic result from representation theory is that for any finite abelian group G, the regular representation of G over $\mathbb{C}$ is the direct sum of its complex irreducible representations (the analogous statement for integer representations is not true.) So, if T is a tiling, G^T_{rot} is a finite group, and $\rho_1, \ldots, \rho_k$ are its complex irreducible representations, we have

$$H^*(T) = H^*_{\rho_1}(T) \oplus \cdots \oplus H^*_{\rho_k}(T). \tag{4.50}$$

The central question of this thesis is the following: What does Rand cohomology tell us about the total cohomology when we are working with a tiling like the pinwheel tiling, which has an infinite rotation group?

4.5.1 Example: the real line with a $\mathbb{Z}$-action.

The example that we present in this subsection is the simplest possible example of an infinite discrete group acting on a non-compact space, but for our purposes it is quite instructive, and it clearly shows that we cannot simply declare the total cohomology to be the direct sum of the irreducible Rand cohomology.

Let $\Gamma = \mathbb{R}$. We consider $\mathbb{R}$ as an infinite CW-complex with a $\mathbb{Z}$-action where the 1-cells are the unit intervals $[n, n+1]$ where $n \in \mathbb{Z}$, and the 0-cells are the endpoints of those intervals. The element $m \in \mathbb{Z}$ acts on $\mathbb{R}$ by just shifting by m units, i.e. m sends the interval $[n, n+1]$ to $[m+n, m+n+1]$. We have

$$\partial[n, n+1] = [n+1] - [n], \tag{4.51}$$

so,

$$\delta[n]^* = [n-1, n]^* - [n+1, n]^*. \tag{4.52}$$

If $\delta(\sum_{n\in\mathbb{Z}} a_n[n]^*) = 0$, then clearly $a_n = a_{n+1}$, for all n. So, any element in the kernel of δ is of the form

$$c \sum_{n\in\mathbb{Z}} [n]^*. \tag{4.53}$$

thus $H^0(\mathbb{R}) = \mathbb{C}$. On the other hand, let $\beta = \sum_{n\in\mathbb{Z}} b_n[n, n+1]^*$ be an arbitrary 1-cochain. Then

$$\beta = \delta \sum_{n\in\mathbb{Z}} \left(\sum_{k=0}^{n} -b_k[k]^* \right). \tag{4.54}$$

So, δ surjects, thus $H^2(\mathbb{R}) = 0$. This is no surprise: the zeroeth order cohomology of $\mathbb{R}$ is one-dimensional, and the first order cohomology is zero. What about the Rand cohomology? An irreducible complex representation of $\mathbb{Z}$ is a one-dimensional representation, characterized by a nonzero complex number

λ so that $m \in \mathbb{Z}$ acts as multiplication by λ^m. Refer to this representation as ρ_λ. A ρ_λ-invariant 1-cochain is determined entirely by the value it takes on $[0,1]$. Any ρ_λ-invariant 1-cochain is invariant is of the form

$$c \sum_{n \in \mathbb{Z}} \lambda^n [n+1, n]^*, \tag{4.55}$$

and a ρ_λ-invariant 1-cochain is invariant is of the form

$$c \sum_{n \in \mathbb{Z}} \lambda^n [n]^*. \tag{4.56}$$

So, $C^1_{\rho_\lambda} \cong C^0_{\rho_\lambda} \cong \mathbb{C}$. Immediately this shows that for all λ we either have $H^1_{\rho_\lambda} \cong H^0_{\rho_\lambda} \cong 0$, or $H^1_{\rho_\lambda} \cong H^0_{\rho_\lambda} \cong \mathbb{C}$. In other words, δ_{ρ_λ} either has a zero-dimensional kernel or a one-dimensional kernel. If δ_{ρ_λ} has a zero-dimensional kernel, then it also must have a zero-dimensional cokernel, so $H^1_{\rho_\lambda} \cong H^0_{\rho_\lambda} \cong 0$. If δ has a one-dimensional kernel, then it also must have a one-dimensional cokernel, so $H^1_{\rho_\lambda} \cong H^0_{\rho_\lambda} \cong \mathbb{C}$.

More precisely, we have

$$\delta_{\rho_1} \sum_{n \in \mathbb{Z}} [n]^* = 0. \tag{4.57}$$

So, δ_{ρ_1} kills ρ_1-invariant 0-cochains, thus $H^0_{\rho_1} \cong H^1_{\rho_1} \cong \mathbb{C}$. However, for $\lambda \neq 1$, nonzero ρ_λ-invariant 0-cochains are not in the kernel, so δ_{ρ_λ} is a map of full rank. So, for $\lambda \neq 1$, $H^0_{\rho_\lambda} \cong H^1_{\rho_\lambda} \cong 0$.

Notice that $H^0(\mathbb{R})$ is the direct sum of $H^0_{\rho_\lambda}(\mathbb{R})$ over all λ. In fact if we compare 4.57 to 4.53, we see that the kernel of δ_{ρ_1} is exactly the kernel of δ. However $H^1(\mathbb{R})$ is not the direct sum of $H^1_{\rho_\lambda}(\mathbb{R})$. The one-dimensional cokernel that manifests itself in $H^1_{\rho_\lambda}(\mathbb{R})$ disappears in $H^1(\mathbb{R})$. This is because the element $\sum_{n \in \mathbb{Z}} [n+1, n]^*$ (which should be viewed as an element in both $C^1_{\rho_1}(\mathbb{R})$ and $C^1(\mathbb{R})$) is not the image of any element in $C^0_{\rho_1}(\mathbb{R})$, but it is the image of some element in $C^0(\mathbb{R})$.

So we can say for certain that the total cohomology is not always just the direct sum of the Rand cohomology over all irreducible representations. However, in this example we can see that the total cohomology is the direct sum of the Rand cohomology that comes from the kernels of δ_ρ, while the Rand cohomology that comes from the cokernels of δ_ρ is not reflected in the total cohomology. This will be a key idea as we move forward.

Chapter 5

Toward an understanding of what $\check{H}^*(\Omega)$ should look like: The Cohomology of Infinite CW-Complexes with Group Actions

Previously, we discussed how a tiling space Θ can be thought of as an inverse limit of a CW-complex, and we showed exactly how this works for the equithirds tiling space. In chapter seven we will show that the same is true for the pinwheel tilings space Ω, but the approximants are infinite CW-complexes (meaning that they have infinitely many cells) equipped with an action by the infinite discrete group $\mathcal{P}$. This is object is unusual enough that we need to dedicate some effort just to determining what its cohomology should look like.

5.1 Introduction

Let Γ be a countable CW complex, so that if $\mathcal{K}_n$ denotes the set of n-cells, then $\mathcal{K}_n$ is countable, for all n. In the usual manner, we let C_n be the complex vector space of chains:

$$C_n = \bigoplus_{\alpha \in \mathcal{K}_n} \mathbb{C} \tag{5.1}$$

giving us the chain complex:

$$\cdots \longleftarrow C_{n-1} \xleftarrow{\partial_n} C_n \xleftarrow{\partial_{n+1}} C_{n+1} \longleftarrow \cdots \tag{5.2}$$

where $\partial_n \circ \partial_{n+1} = 0$. Let C^n be the complex vector space of cochains, i.e., the dual of C_n. We have the corresponding cochain complex

$$\cdots \longrightarrow C^{n-1} \xrightarrow{\delta^{n-1}} C^n \xrightarrow{\delta^n} C^{n+1} \longrightarrow \cdots \tag{5.3}$$

where δ_n is the pullback of ∂_{n+1}, i.e, $\delta^n \tau = \tau \circ \partial_{n+1}$, and consequently $\delta^n \circ \delta^{n-1} = 0$.

Let $G_\Gamma = G_\mathbb{Z} \times G_f$ be an abelian group, where $G_\mathbb{Z} \cong \mathbb{Z}$ and G_f is finite. We impose the following conditions.

1. G_Γ acts on Γ. To clarify, this implies that the action of G_Γ not only respects the topological structure of Γ, it also respects the CW-complex structure, as discussed in chapter 4.

2. The action of G_Γ on $\mathcal{K}_n$ has only finitely many orbits. In other words, there is a finite set $\mathcal{A} \subset \mathcal{K}_n$ such that every cell $\Delta \in \mathcal{K}_n$ is equal to $g\Delta_\mathcal{A}$ for some $g \in G_\Gamma$ and some $\Delta_\mathcal{A} \in \mathcal{A}$. This also implies that the group action is cocompact, i.e., the space Γ / G_Γ is compact.

3. The action of $G_\mathbb{Z}$ is free, which means that if $g\Delta = \Delta$ and $g \in G_\mathbb{Z}$ then g is the identity element: no cells are fixed by non-identity elements of $G_\mathbb{Z}$. It is important to note that we do not require that the action of G_f to be free, and in fact the action of G_f is not free in the case where Γ is the Barge-Diamond approximant of the pinwheel space. However, this condition does imply that if $g_\mathbb{Z} g_f \Delta = \Delta$, where $g_\mathbb{Z} \in G_\mathbb{Z}$, and $g_f \in G_f$ then $g_\mathbb{Z}$ is the identity.

We can see how the example of $\mathbb{R}$ with a $\mathbb{Z}$ action that we introduced at the end of last chapter fits this setup. $\mathcal{K}_1$ is the collection of unit intervals $[n, n+1]$ where $n \in \mathbb{Z}$, and $\mathcal{K}_0$ is the set of endpoints to these intervals, in other words, $\mathbb{Z} \subset \mathbb{R}$. $\mathbb{Z}$ acts on $\mathbb{R}$ by addition, in other words, if $m \in \mathbb{Z}$ then $m(r) = m + r$. $\mathbb{Z}$ permutes $\mathcal{K}_1$ and $\mathcal{K}_0$, with the element $m \in \mathbb{Z}$ sending the one-cell $[k, k+1]$ to the one-cell $[m+k, m+k+1]$, and sending the zero-cell $[k]$ to the zero-cell $[m+k]$. Clearly, both $\mathcal{K}_1$ and $\mathcal{K}_0$ have only one orbit under this action (condition 2 above), and the action is free (condition 3) since only zero sends a one-cell or a zero-cell to itself. Also, for all $m \in \mathbb{Z}$ we have

$$\begin{aligned} \partial m([k, k+1]) &= \partial([m+k, m+k+1]) = [m+k+1] - [m+k] \\ &= m([k+1] - [k]) = m(\partial[k, k+1]). \end{aligned} \tag{5.4}$$

So, the group action commutes with the boundary map ∂, satisfying condition 1 above.

Returning to the general case, note that C^n inherits a group action from C_n via

$$g\tau = \tau \circ g^{-1}. \tag{5.5}$$

We want to understand

$$H^n(\Gamma) = \frac{\ker \delta^n}{\operatorname{Im} \delta^{n-1}}. \tag{5.6}$$

For the rest of this chapter, we will usually supress the (Γ) in $H^n(\Gamma)$, but in later chapters we will often need to specify the underlying space.

N.B.: In this section, as in much of the rest of this paper, we are working in the category of vector spaces equipped with a group action, i.e., representations. If V_1 and V_2 are vector spaces equipped with G_Γ actions, then when write $V_1 \cong V_2$ we mean that there exists a vector space isomorphism between V_1 and V_2 that commutes with the G_Γ action, in other words, an isomorphism of G_Γ representations.

We will often need to introduce specific representations of G_f, $G_\mathbb{Z}$, or G_Γ. We write a representation of a group G as (V, ρ) where V is the underlying vector space, and ρ is the homomorphism of the group G into the automorphism group of V, i.e., $\rho : G \times V \to V$ is the group action. For convenience, we may just write ρ to refer to the representation.

5.2 Separating the $\mathbb{Z}$ action from the G_f action

In practice, when we try to calculate H^n, we isolate a single irreducible representation of G_f, and then we focus our attention on calculating the cohomology as a $\mathbb{Z}$ representation. This section explains what that means, and why it is justified. The next section focuses on the $\mathbb{Z}$ action.

Let $|G_f| = N$. Let (V^{ρ_f}, ρ_f) be an irreducible representation of G_f. Since G_f is abelian, V^{ρ_f} must be one dimensional, so for each $g \in G_f$, let $\chi(g)$ be the complex number satisfying

$$\rho_f(g)(v) = \chi(g)(v) \tag{5.7}$$

for $v \in V^{\rho_f}$. Note that under this circumstance $\chi : G_f \to \mathbb{C}$ is exactly the *character* of the representation ρ_f. Define $C^n[\rho_f, \cdot, \cdot] \subset C^n$ to be the subspace of cochains τ satisying

$$\tau(g\alpha) = \chi(g)\tau(\alpha) \tag{5.8}$$

for all $\alpha \in C_n$. We say that such a cochain *transforms ρ_f-invariantly.* (The notation $C^n[\rho_f, \cdot, \cdot]$ surely seems odd at this point. I beg the reader's patience; this notation will make much more sense in the next chapter). In what follows, $\rho_f^1, \ldots, \rho_f^N$ are the irreducible representations of G_f. Let χ_j be the character of the representation ρ_f^j.

Theorem 5.1.

$$C^n = C^n[\rho_f^1, \cdot, \cdot] \oplus \cdots \oplus C^n[\rho_f^N, \cdot, \cdot]. \tag{5.9}$$

Proof. This proof is a standard application of rudimentary representation theory. Examine the map $P_j : C^n \to C^n$ defined by

$$P_j\tau(\alpha) = \frac{1}{N} \sum_{g \in G_f} \chi_j(g^{-1})\tau(g\alpha). \tag{5.10}$$

Ultimately, we can show that P_j is the projection to $C^n[\rho_f^1, \cdot, \cdot]$, under the decomposition (5.9), but first we claim that the image of P_j is exactly $C^n[\rho_f^j, \cdot, \cdot]$. Let $\tau \in C^n$, $\alpha \in C_n$, and $h \in G_f$, then

$$P_j\tau(h\alpha) = \frac{1}{N} \sum_{g \in G_f} \chi_j(g^{-1})\tau(gh\alpha). \tag{5.11}$$

Reindex the summation by letting $\hat{g} = gh$, and we get

$$\begin{aligned} P_j\tau(h\alpha) &= \frac{1}{N} \sum_{\hat{g} \in G_f} \chi_j((\hat{g}h^{-1})^{-1})\tau(\hat{g}\alpha) \\ &= \chi_j(h)\frac{1}{N} \sum_{\hat{g} \in G_f} \chi_j(\hat{g}^{-1})\tau(\hat{g}\alpha) = \chi_j(h)P_j\tau(\alpha). \end{aligned} \tag{5.12}$$

Equation (5.12) shows that $P_j\tau$ transforms ρ_f^j-invariantly, thus the image of P_j is contained in $C^n[\rho_f^j, \cdot, \cdot]$. Conversely, let $\tau \in C^n[\rho_f^j, \cdot, \cdot]$, then

$$P_j\tau = \frac{1}{N} \sum_{g \in G_f} \chi_j(g^{-1})\tau \circ g = \frac{1}{N} \sum_{g \in G_f} \chi_j(g^{-1})\chi_j(g)\tau = \frac{1}{N} \sum_{g \in G_f} \tau = \tau. \tag{5.13}$$

Now, suppose that $\tau \in C^n[\rho_f^i, \cdot, \cdot]$ and $i \neq j$, then

$$P_j \tau = \frac{1}{N} \sum_{g \in G_f} \chi_j(g^{-1}) \tau \circ g = \frac{1}{N} \sum_{g \in G_f} \chi_j(g^{-1}) \chi_i(g) \tau$$

$$= \left(\frac{1}{N} \sum_{g \in G_f} \chi_j \bar{(g)} \chi_i(g) \right) \tau = 0. \quad (5.14)$$

Here, we have called upon the fact that the characters of irreducible representations are orthogonal under the inner product $\langle \gamma, \delta \rangle = \sum_{g \in G_f} \gamma \bar{(g)} \omega(g)$, where ω and γ are maps from the conjugacy classes of a group into the complex numbers.

So, τ is in the image of P_j, hence the image of P_j is $C^n[\rho_f^j, \cdot, \cdot]$. Now, we claim that for all $\tau \in C^n$, we have that

$$\tau = P_1 \tau + \cdots + P_n \tau. \quad (5.15)$$

For a finite abelian group, the sum of the characters of the irreducible representation applied to any given group element is zero, unless that group element is the identity, in which case the sum is equal to the order of the group. So,

$$\begin{aligned} P_1 \tau + \cdots + P_n \tau &= \sum_{j=1}^{N} \frac{1}{N} \sum_{g \in G_f} \chi_j(g^{-1}) \tau \circ g \\ &= \frac{1}{N} \sum_{g \in G_f} \left(\sum_{j=1}^{N} \chi_j(g^{-1}) \right) \tau \circ g \qquad (5.16) \\ &= \frac{1}{N} \sum_{g \in G_f} N \delta_e(g) \tau \circ g = \tau \circ e = \tau. \end{aligned}$$

where e is the identity element and $\delta_e(g) = 1$ if $g = e$ and $\delta_e(g) = 0$ otherwise. So, we have shown that C^n is the sum of the $C^n[\rho_f^j, \cdot, \cdot]$. It remains to show that the sum is a direct sum. Equations (5.12) and (5.14) show that P_j is the identity on $C^n[\rho_f^j, \cdot, \cdot]$ and vanishes on $C^n[\rho_f^i, \cdot, \cdot]$ for $j \neq i$. Hence, if τ is in $C^n[\rho_f^j, \cdot, \cdot]$, and is also in the span of the other $C^n[\rho_f^i, \cdot, \cdot]$, it must be zero. So, the sum is direct. □

Now, we note the following:

Lemma 5.2. *The coboundary map* δ^n *sends* $C^n[\rho_f^j, \cdot, \cdot]$ *to* $C^{n+1}[\rho_f^j, \cdot, \cdot]$.

Proof. This is a simple consequence of the fact that δ^n commutes with the group action. If $\tau \in C^n[\rho_f, \cdot, \cdot]$ then

$$\delta^n \tau(g\alpha) = \tau(\partial g\alpha) = \tau(g\partial\alpha) = \chi(g)\tau(\partial\alpha) = \chi(g)(\delta^n\tau)\alpha. \tag{5.17}$$

□

Let δ_j^n be δ^n restricted to $C^n[\rho_f^j, \cdot, \cdot]$, and let

$$K^n[\rho_f^j, \cdot, \cdot] = \ker \delta_j^n \tag{5.18}$$

and

$$H^n[\rho_f^j, \cdot, \cdot] = \frac{\ker \delta_j^n}{\operatorname{Im} \delta_j^{n-1}}. \tag{5.19}$$

As a trivial consequence of lemma 5.2 and theorem 5.1, we have that

Theorem 5.3.

$$H^n = H^n[\rho_f^1, \cdot, \cdot] \oplus \cdots \oplus H^n[\rho_f^N, \cdot, \cdot]. \tag{5.20}$$

Theorems 5.1 and 5.3 are what allow us to analyze H^n *one-G_f-representation-at-a-time*. We know how G_f acts on $C^n[\rho_f^j, \cdot, \cdot]$: If $\tau \in C^n[\rho_f^j, \cdot, \cdot]$, and $g \in G_f$ then

$$g\tau = \tau \circ g^{-1} = \chi_j(g^{-1})\tau \tag{5.21}$$

So G_f acts on each $C^n[\rho_f^j, \cdot, \cdot]$ via scalar multiplication of characters, and so it also acts on each $H^n[\rho_f^j, \cdot, \cdot]$ via scalar multiplication of characters. Notice that since the action of G_f commutes with the action of $G_{\mathbb{Z}}$, it is also true that $G_{\mathbb{Z}}$ maps each $H^n[\rho_f^j, \cdot, \cdot]$ to itself. Thus, it remains only to analyze how each $G_{\mathbb{Z}}$ acts on each of the $H^n[\rho_f^j, \cdot, \cdot]$.

5.3 Understanding the cohomology of $\mathbb{Z}$ representations

In this section, we have only $\mathbb{Z}$ actions to concern ourselves with. We modify our notation to deal with this simplified situation. Let

$$C_n = \bigoplus_{\mathbb{Z}} \mathbb{C}^{l_n}. \tag{5.22}$$

We can think of C_n as being the space of finite sequences of vectors in $\mathbb{C}^{l_n}$, where the element $m \in \mathbb{Z}$ acts by shifting a sequence m spots to the right. Another useful characterization is to think of C_n as

$$C_n = \mathbb{C}[z, z^{-1}]^{l_n} \tag{5.23}$$

where the $\mathbb{C}[z, z^{-1}]$ is the space of complex valued polynomials, (including terms with positive and negative exponents, but only finitely many) and the element $m \in \mathbb{Z}$ acts by multiplying a polynomial by z^m. Importantly, this allows us to view C_n as a module over $\mathbb{C}[z, z^{-1}]$, which is a principal ideal domain. In fact, any complex vector space with a $\mathbb{Z}$ action is naturally viewed as a module over $\mathbb{C}[z, z^{-1}]$ where multiplication by the ring element z^m acts the same as the group element $m \in \mathbb{Z}$. In this context, maps that commute with $\mathbb{Z}$ are just $\mathbb{C}[z, z^{-1}]$ module homomorphisms. So, the chain complex

$$\cdots \longleftarrow C_{n-1} \xleftarrow{\partial_n} C_n \xleftarrow{\partial_{n+1}} C_{n+1} \longleftarrow \cdots \tag{5.24}$$

is properly thought of as a complex of $\mathbb{C}[z, z^{-1}]$ modules, with the ∂_j being $\mathbb{C}[z, z^{-1}]$ module homomorphisms. The cochain side of things looks like this:

$$C^n = \prod_{\mathbb{Z}} \mathbb{C}^{l_n} = \mathbb{C}[[z, z^{-1}]]^{l_n} \tag{5.25}$$

where $\mathbb{C}[[z, z^{-1}]]$ is the space of formal bi-infinite Taylor-Laurent series (the notation is not standard), and again the element $m \in \mathbb{Z}$ acts as by multiplication by z^m, so C^n is also a $\mathbb{C}[z, z^{-1}]$ module. The maps δ^j in the cochain complex

$$\cdots \longrightarrow C^{n-1} \xrightarrow{\delta^{n-1}} C^n \xrightarrow{\delta^n} C^{n+1} \longrightarrow \cdots \tag{5.26}$$

can again be viewed as $\mathbb{C}[z, z^{-1}]$ module homomorphisms. It is our goal to understand

$$H^n = \frac{\ker \delta^n}{\operatorname{Im} \delta^{n-1}}. \tag{5.27}$$

It is important to note that C_n is a finitely generated module over the principal ideal domain $\mathbb{C}[z, z^{-1}]$, while C^n is not finitely generated. The theory of finitely generated modules over principal ideal domains is well understood (see any elementary algebra text, e.g. [7]), so it is much easier to understand

$$H_n = \frac{\ker \partial_n}{\operatorname{Im} \partial_{n+1}}. \tag{5.28}$$

Any submodule of a free module of finite rank is also a free module of finite rank. So $\ker \partial_n$ is also finitely generated. If we quotient by a submodule, clearly the resulting quotient is also finitely generated. So, H_n is also a finitely generated module over $\mathbb{C}[z, z^{-1}]$. By the fundamental theorem of finitely generated modules over principal ideal domains, we have that

$$H_n \cong \mathbb{C}[z, z^{-1}]^b \oplus \frac{\mathbb{C}[z]}{(z - \lambda_1)^{p_1}} \oplus \cdots \oplus \frac{\mathbb{C}[z]}{(z - \lambda_m)^{p_m}} \tag{5.29}$$

for some positive integers $b, p_1, p_2, \ldots, p_m$ and some $\lambda_1, \ldots, \lambda_m \in \mathbb{C} - \{0\}$. So, the situation in homology is easy, or at least the work has already been done for us. If we had a nice way of relating H^n to H_n, we would be finished. In that regard, we have the following theorem.

Theorem 5.4. *Let H^n and H_n be defined as above, with the chain complex consisting of direct sums, while the cochain complex consists of direct products. Then*

$$H^n \cong (H_n)^*. \tag{5.30}$$

In order to prove this, we will need the following technical lemma.

Lemma 5.5. *Let B be a free, finitely generated $\mathbb{C}[z, z^{-1}]$ module, and let $A \subseteq B$ be a submodule. There exists a complex vector space $W \subset B$ which is a vector space complement to A in B, i.e.*

$$B = A \oplus W. \tag{5.31}$$

Proof. We know that B/A is a finitely generated module over the PID $\mathbb{C}[z, z^{-1}]$. Let

$$\varphi : B/A \longrightarrow Y_1 \oplus \cdots \oplus Y_q \oplus Z_1 \oplus \cdots \oplus Z_s \tag{5.32}$$

be an isomorphism, where $Y_j \cong \mathbb{C}[z, z^{-1}]$, and $Z_j \cong \mathbb{C}[z]/(z-\alpha_j)^{p_j}$ (for some $\alpha_j \in \mathbb{C}-\{0\}$ and $p_j \geq 1$) and thus Z_j is finite dimensional as a complex vector space. Then, as vector spaces, we can certainly pick a basis for each of the Y_j and each of the Z_j, so in fact we have a basis $\mathcal{B}$ of B/A. For each element $\beta \in \mathcal{B}$, β is a coset in B/A, so pick an element $\beta' \in B$ representing that coset. Then let W be the span of all the β'. We know that $B = A + W$, because if we take any element α, and look at its image $\bar{\alpha}$ in the quotient B/A then

$$\bar{\alpha} = \beta_1 + \cdots + \beta_r. \tag{5.33}$$

This means that

$$\alpha = \beta_1' + \cdots + \beta_r' + \zeta \tag{5.34}$$

where β_j' is the representative of the coset $\beta_j \in B/A$ and $\zeta \in A$. On the other hand, if $A \cap W$ were non-trivial, then some linear combination of the β' would land in A, but that would mean that some combination of the β would equal zero in B/A, which contradicts the fact that we chose $\mathcal{B}$ to be a basis. $\square$

Proof of Theorem 5.4. Define

$$\begin{aligned} \Phi : \quad \frac{\ker \delta^n}{\operatorname{Im} \delta^{n-1}} &\to \left(\frac{\ker \partial_n}{\operatorname{Im} \partial_{n+1}}\right)^* \\ \tau + \operatorname{Im} \delta^{n-1} &\mapsto \quad \tau|_{\ker \partial_n}. \end{aligned} \tag{5.35}$$

We want to show that Φ is an isomorphism. We first need to check that Φ is well-defined. Suppose $\tau_1 \sim \tau_2 \bmod \operatorname{Im} \delta^{n-1}$. Then $\tau_1 - \tau_2 = \delta^{n-1}\bar{\tau}$. Then, for all $z \in \ker \partial_n$, $(\tau_1 - \tau_2)(z) = \bar{\tau}(\partial_n z) = 0$. Thus τ_1 and τ_2 agree on $\ker \partial_n$. So, Φ is well defined. We also need to check that $\Phi(\tau + \operatorname{Im} \delta^{n-1})$ hits the proposed codomain, which requires us to show that τ vanishes on $\operatorname{Im} \partial_{n+1}$. This is true because $\delta^n \tau = 0$, so $\tau(\partial_{n+1} v) = 0$, for all v.

By Lemma 5.5, we can choose B' such that $B' \oplus \operatorname{Im} \partial_n = C_{n-1}$, and choose B'' such that $B'' \oplus \ker \partial_n = C_n$.

We need to show Φ injects. If $\Phi(\tau) = 0$, then $\tau|_{\ker \partial_n} = 0$. Define $\beta \in C^{n-1}$, by letting $\beta(B') = 0$, and for $x \in \operatorname{Im} \partial_n$, let $\beta(x) = \tau(a)$, where $\partial_n(a) = x$, $a \in B''$. Let $y \in C_n$, $y = a + z$, where $a \in B''$, $z \in \ker \partial_n$. Then,

$$(\delta^{n-1}\beta)(y) = \beta(\partial_n y) = \beta(\partial_n a) = \tau(a) = \tau(a+z) = \tau(y). \tag{5.36}$$

So, $\tau = \delta^{n-1}\beta$, thus $\tau \sim 0$ modulo $\operatorname{Im}\delta^{n-1}$. Hence Φ injects. Now, we need to show that Φ surjects. Let $\tau \in (\ker\partial_n/\operatorname{Im}\partial_{n+1})^*$, so τ vanishes on $\operatorname{Im}\partial_{n+1}$. Define $\bar{\tau} \in A_1^*$ by $\tau(B'') = 0$, $\bar{\tau}(z) = \tau(z)$ for $z \in \ker\partial_n$. Since $\bar{\tau}$ vanishes on $\operatorname{Im}\partial_{n+1}$, $\bar{\tau} \in \ker\delta^n$, and $\Phi(\bar{\tau} + \operatorname{Im}\delta^{n-1}) = \tau$. So, Φ surjects.

□

We are now in a position to prove the following theorem, which summarizes the results of this chapter, and lays the ground work for comparing the total cohomology to Rand cohomology, which is the topic of the next chapter.

Theorem 5.6. *Let Γ be a CW complex with an action by the group $G_\Gamma = G_{\mathbb{Z}} \times G_f$, with the conditions outlined in section 5.1. Then*

$$H^n \cong H^n[\rho_f^1, \cdot, \cdot] \oplus \cdots \oplus H^n[\rho_f^N, \cdot, \cdot] \tag{5.37}$$

where $\rho_f^1, \ldots, \rho_f^N$ are the irreducible representations of G_f. $H^n[\rho_f^j, \cdot, \cdot]$ can be specified as follows.

1. *If $g \in G_f$, then g acts on $H^n[\rho_f^j, \cdot, \cdot]$ via multiplication by $\chi_j(g)^{-1}$.*
2. *As a $G_{\mathbb{Z}}$ representation,*

$$H^n[\rho_f^j, \cdot, \cdot] \cong \mathbb{C}[[z, z^{-1}]]^{b_j} \oplus \frac{\mathbb{C}[z]}{(z - \lambda_{1j}^{-1})^{p_{1j}}} \oplus \cdots \oplus \frac{\mathbb{C}[z]}{(z - \lambda_{mj}^{-1})^{p_{mj}}} \tag{5.38}$$

where $b^j \geq 0$, $p_{ij} > 0$ and $\lambda_{ij} \in \mathbb{C} - \{0\}$.

Proof. Equation (5.37) is exactly what we proved in section 5.2, and the G_f action was explained in the remark at the end of that section.

Theorem 5.4 says that $H^1 \cong (H_1)^*$. The dual of $\mathbb{C}[z, z^{-1}]^b$ is $\mathbb{C}[[z, z^{-1}]]^b$. This is easy to see if we recall that $\mathbb{C}[z, z^{-1}]^N$ is a direct sum of $\mathbb{C}^N$ while $\mathbb{C}[[z, z^{-1}]]^N$ is a direct product of $\mathbb{C}^N$. It is also straightforward to check that

$$\left(\frac{\mathbb{C}[z]}{(z-\lambda)^p}\right)^* \cong \frac{\mathbb{C}[z]}{(z-\lambda^{-1})^p}. \tag{5.39}$$

So, dualizing equation (5.29) yields the desired result. □

5.4 Major Rank, Exceptional Eigenvalues, Surplus Dimensions: Some convenient shorthands to help us classify cohomology

Before proceeding to the next chapter, we need to introduce some terminology that will help us organize our data. Let A be a $\mathbb{Z}$ representation which can be expressed in the form (5.38). We can write A as

$$A = \mathbf{A_0} \oplus A_1 \oplus \cdots \oplus A_r \tag{5.40}$$

where

$$\mathbf{A_0} \cong \mathbb{C}[[z, z^{-1}]]^b, \tag{5.41}$$

and each A_j is a space of power vectors for some λ_j, with $\lambda_{j'} \neq \lambda_j$ for $j' \neq j$. In other words, there exists q such that

$$(g_{\mathbb{Z}} - \lambda_j)^q A_j = \{0\}. \tag{5.42}$$

Let q_j to be the smallest number q for which 5.42 holds. We call b the *major rank* of A, denoted as majrank(A). We call $\lambda_1, ..., \lambda_r$ the *exceptional eigenvalues* of A. We have a couple of useful ways of organizing the data of A_j. We can write

$$A_j = \hat{A}_j^1 \oplus \cdots \oplus \hat{A}_j^{q_j} \tag{5.43}$$

where

$$\hat{A}_j^l \cong \left[\frac{\mathbb{C}[z]}{(z-\lambda_j)^l}\right]^{m_{jl}}. \tag{5.44}$$

If for any given j we can determine m_{jl} for each l, then we have determined A_j up to isomorphism. Alternatively, we can also write

$$A_j = A_j^{q_j} \supset A_j^{q_j-1} \supset \cdots \supset A_j^2 \supset A_j^1 \tag{5.45}$$

where A_j^t is the space of order-t power vectors in A_j. We may call A_j^t the *t^{th}-order λ_j surplus space*. We call the dimension of A_j^t the *t^{th} order surplus dimension of λ_j in A*. If λ is not an exceptional eigenvalue, the t^{th} order surplus dimension of λ is zero, for all t. We denote the t^{th} order surplus dimension of λ in A by surpdim(A, λ, t), or just surpdim(λ, t) when there can be no confusion over A. If for any given j we can determine surpdim(λ, t), then we have determined A_j up to isomorphism. Let $\mathrm{gdim}(A, \lambda, t)$ (or, again just $\mathrm{gdim}(\lambda, t)$ when there can be no confusion) be the dimension of the entire t^{th} order generalized λ-eigenspace. Notice that $\mathbf{A_0}$ has a bt-dimensional subspace of λ power vectors, and clearly, if $\lambda \neq \lambda_j$, then A_j has no λ power vectors. So we have

$$\mathrm{gdim}(\lambda, t) = bt + \mathrm{surpdim}(\lambda, t). \tag{5.46}$$

Notice that

$$\mathrm{surpdim}(\lambda, t) = \sum_{l=1}^{q_j} \min(m_{jl}, t), \tag{5.47}$$

and it isn't difficult to invert this formula to calculate the m_{jl} from the surplus dimensions. We mention this just to let it be known that the two approaches are equivalent, and it is easy to move back and forth between the two.

In the next chapter, we will be content to work with surplus dimensions and the dimensions of the generalized eigenspaces. In other words, suppose that our goal is to calculate H^n up to isomorphism. We will interpret that goal to mean that for each irreducible representation ρ_f of G_f, we must analyze $H^n[\rho_f, \cdot, \cdot]$ as a $\mathbb{Z}$ representation and calculate (1) its major rank b, (2) the exceptional eigenvalues $\lambda_1, \ldots, \lambda_r$, and (3) $\mathrm{surpdim}(\lambda_j, t)$ for all j and for all t. Equation (5.46) allows us to deduce that $\mathrm{gdim}(\lambda, 1) = b$ for all but finitely many λ, and if $\mathrm{gdim}(\lambda, 1) \neq b$, then λ must be an exceptional eigenvalue. So, if we are able to calculate $\mathrm{gdim}(\lambda, t)$ for all λ and t, we can determine the major rank, exceptional eigenvalues, and their respective surplus dimensions. It really comes down to just calculating the dimensions of the generalized eigenspaces. It will be advantageous to keep this in mind in the next chapter.

Chapter 6

Relating the Rand Cohomology to the Total Cohomology

Let Γ be a CW complex, acted on by the group $G_\Gamma = G_\mathbb{Z} \times G_f$, as in the previous section. Let (V^ρ, ρ) be a finite dimensional representation of the group G_Γ.

As in the last chapter, we have the chain complex

$$\cdots \longleftarrow C_{n-1} \xleftarrow{\partial_n} C_n \xleftarrow{\partial_{n+1}} C_{n+1} \longleftarrow \cdots \tag{6.1}$$

We say that a map $\tau : C_n \to V^\rho$ is a *ρ-invariant n-cochain* if it commutes with the action of G_Γ, i.e. $\tau(gv) = \rho(g)\tau(v)$, and we denote the space of all such maps as C^n_ρ. Since the set of n-cells has only finitely many orbits under the action of G_Γ, C^n_ρ has finite dimension. In the usual way, we can define the cochain complex

$$\cdots \longrightarrow C^{n-1}_\rho \xrightarrow{\delta^{n-1}_\rho} C^n_\rho \xrightarrow{\delta^n_\rho} C^{n+1}_\rho \longrightarrow \cdots \tag{6.2}$$

where $\delta^n_\rho \tau = \tau \circ \partial_{n+1}$. We define

$$H^n_\rho(\Gamma) = \frac{\ker \delta^n_\rho}{\operatorname{Im} \delta^{n-1}_\rho}. \tag{6.3}$$

As in chapter 4, we will refer to $H^n_\rho(\Gamma)$ as the Rand cohomology of Γ.

The goal of this chapter is to establish a method that allows us to determine $H^n(\Gamma)$ *as a G_Γ representation* by knowing the dimension of $H^n_\rho(\Gamma)$ for all indecomposable representations ρ, and vice versa. More explicitly, we want to take the dimensions of each of the H^n_ρ and use that data to calculate the major rank of $H^n[\rho_f, \cdot, \cdot]$, its exceptional eigenvalues, and their respective surplus dimensions. Theorems 6.4 and 6.5, the major results of this chapter if not of the entire paper, show how to do just that, explaining how $H^n(\Gamma)$ and $H^n_\rho(\Gamma)$ are related.

The path to these two theorems is windy. Let us summarize the approach.

First, recall the definition of an *indecomposable* representation. An indecomposable representation is a representation that cannot be expressed as a direct sum of two proper subrepresentations. An *irreducible* representation is a representation with no subrepresentations. Any finite dimensional irreducible representation of an abelian group must be one dimensional, but a group like G_Γ has indecomposable representations of finite dimension greater than one. In this chapter, an irreducible representation will always mean a one dimensional representation, while an indecomposable representation will always mean a finite dimensional indecomposable representation. We need to dedicate some effort just to understanding what indecomposable and irreducible representations of G_Γ look like, to understanding what C^n_ρ looks like when ρ is irreducible or indecomposable, to understanding how C^n_ρ embeds in C^n, and to establishing some good notation. This is the content of section 6.1.

Next, in section 6.2, we will show that $H^n(\Gamma)$ is easily determined if we can calculate the kernels of the coboundary maps δ^n, for each n, and in passing we also note that H^n_ρ is easily determined if we know the kernel of δ^n_ρ. In section 6.3 we will prove the main theorems 6.4 and 6.5 and work through an example that will show us explicity how to use these theorems to calculate H^n from H^n_ρ.

6.1 Irreducible and Indecomposable Representations

A one dimensional irreducible representation (V^ρ, ρ) of $G_\Gamma = G_\mathbb{Z} \times G_f$ is easy to understand. Each element of G_Γ must act by multiplication by some nonzero complex number. Let $G_f = \langle g_1 \rangle \times \cdots \times \langle g_l \rangle$, $G_\mathbb{Z} = \langle g_\mathbb{Z} \rangle$. Then a representation is completely determined by how $g_1, \ldots, g_l$, and $g_\mathbb{Z}$ act, so it is completely determined by scalars $\chi(g_1), \ldots, \chi(g_l)$ and λ where g_j acts as multiplication by $\chi(g_j)$ and $g_\mathbb{Z}$ acts as multiplication by λ. The complex number $\chi(g_j)$ must be a root of unity, so that $(r_j)^{\mathrm{order}(g_j)} = 1$. On the other hand, λ

only needs to be nonzero. Notice that V^ρ is also a one dimensional representation of G_f, determined up to isomorphism by the scalars $\chi(g_1), \dots, \chi(g_l)$.

The situation for a finite dimensional indecomposable representation ρ of G_Γ is subtle enough that it is worth stating as a theorem.

Theorem 6.1. *If (V^ρ, ρ) is a finite dimensional indecomposable representation of $G_\Gamma = G_\mathbb{Z} \times G_f$ of dimension t, then all elements of G_f act as scalar multiplication, while the generator $g_\mathbb{Z}$ of $G_\mathbb{Z}$ acts as a matrix with a single Jordan block of size t.*

Proof. We are aided by the fact that G_Γ is abelian. Suppose V^ρ has dimension t. If $g_1, g_2 \in G_\Gamma$, then $\rho(g_1)$, $\rho(g_2)$ are commuting linear automorphisms of V^ρ. If $\rho(g_1)$ has an eigenvalue λ, then $\rho(g_2)$ maps the λ-eigenspace of $\rho(g_1)$ to itself, because if $\rho(g_1)v = \lambda v$, then

$$\rho(g_1)(\rho(g_2)v) = \rho(g_2)(\rho(g_1)v) = \rho(g_2)(\lambda v) = \lambda(\rho(g_2)v). \tag{6.4}$$

This shows that $\rho(g_2)v$ is an eigenvector of $\rho(g_1)$. If $g \in G_f$, it must be true that $\rho(g)$ is diagonalizable, otherwise $(\rho(g))^n \neq 1$, for all $n \neq 0$. Hence, V^ρ can be decomposed into eigenspaces of $\rho(g)$. Suppose that

$$V^\rho = V_{\lambda_1} \oplus \cdots \oplus V_{\lambda_N}, \tag{6.5}$$

where V_{λ_j} is the λ_j eigenspace of $\rho(g)$ (implicitly, $\lambda_j \neq \lambda_i$ for $j \neq i$). Then, as we just showed, all elements of G_Γ must send V_{λ_j} to itself, which means that each of the V_{λ_j} are representations of G_Γ. But by assumption, V^ρ is indecomposable, hence there can be only one eigenspace. So, we have proven that all elements of G_f act via scalar multiplication.

This also means that *any* subspace of V^ρ is fixed by the action of G_f. Now suppose that the Jordan canonical form of $\rho(g_\mathbb{Z})$ consists of k Jordan blocks. If $W_1, \dots, W_k \subset V^\rho$ are the subspaces corresponding to these Jordan blocks, then

$$V^\rho = W_1 \oplus \cdots \oplus W_k, \tag{6.6}$$

and $\rho(g_\mathbb{Z})$ fixes each W_j as do all elements of G_f, which means that each W_j is a representation of G_Γ. Since V^ρ is indecomposable, this implies that there can be only one Jordan block. $\square$

Theorem 6.1 suggests a succinct notation for referring to indecomposable representations. Since G_f acts via scalar multiplication, ρ is naturally associated to some irreducible representation of G_f, call it ρ_f. When we mean to refer to an indecomposable representation ρ of G_Γ, of dimension t, where $G_{\mathbb{Z}}$ acts as a Jordan block with λ on the diagonal, and where the action of G_f is specified by an irreducible representation ρ_f of G_f, we will write it as

$$\rho = (\rho_f, \lambda, t). \tag{6.7}$$

It is worth pausing to note why we are making the effort to consider the irreducible representations separately from the indecomposable ones. The irreducibles are just a special case of indecomposables, and it turns out that we need the indecomposable representations to tell the whole story of H^n. Thus, as a simple matter of presenting the correct and important results, I could choose to ignore the special case of irreducible representations.

However, such an approach would not give an accurate impression of the situation. The irreducible representations are by far the most important and most common case. As we will see, in order to calculate H^n from H^n_ρ, there is no way to avoid calculating H^n_ρ for all irreducible ρ. The data that we acquire from that calculation will allow us to limit the discussion to only finitely many of the remaining indecomposable ρ. Moreover, in actual calculations, we might be able to make observations that allow us to completely disregard all the higher order indecomposable representations.

In the end, we are forced to consider indecomposable representations. One can concoct cellular complexes Γ and Γ' such that $H^n_\rho(\Gamma) \cong H^n_\rho(\Gamma')$ for all irreducible ρ, yet $H^n(\Gamma)$ and $H^n(\Gamma')$ are not isomorphic. However, the irreducibles merit particular consideration.

For an irreducible representation $\rho = (\rho_f, \lambda, 1)$, it is easy to see how the space of ρ-invariant cochains C^n_ρ embeds in C^n. We can think of $\tau \in C^n$ as being a ρ-invariant cochain if

$$\tau(g\alpha) = \chi(g)\alpha, \tag{6.8a}$$

$$\tau(g_{\mathbb{Z}}\alpha) = \lambda\alpha, \tag{6.8b}$$

for all $g \in G_f$, $\alpha \in C_n$. In our example of the real line with a $\mathbb{Z}$ action, we have

$$C_1 = \bigoplus_{\mathbb{Z}} \mathbb{C} = \mathbb{C}[z, z^{-1}], \qquad C^1 = \prod_{\mathbb{Z}} \mathbb{C} = \mathbb{C}[[z, z^{-1}]]. \tag{6.9}$$

An arbitrary element of C^1 can be represented like this:

$$\tau = (\dots, \tau_{-3}, \tau_{-2}, \tau_{-1}, \tau_0, \tau_1, \tau_2, \tau_3, \dots). \tag{6.10}$$

To evaluate τ on a sequence

$$\alpha = (\dots, \alpha_{-3}, \alpha_{-2}, \alpha_{-1}, \alpha_0, \alpha_1, \alpha_2, \alpha_3, \dots) \in C_1, \tag{6.11}$$

we simply calculate $\sum \tau_j \alpha_j$. Let $\rho = (\rho_0, \lambda, 1)$ (here G_f is trivial so necessarily ρ_0 is the trivial representation). Then we can view τ as ρ-invariant if $\tau_j = \lambda^j \tau_0$:

$$\tau = (\dots, \lambda^{-3}\tau_0, \lambda^{-2}\tau_0, \lambda^{-1}\tau_0, \tau_0, \lambda\tau_0, \lambda^2\tau_0, \lambda^3\tau_0, \dots), \tag{6.12}$$

In the irreducible case, we are aided by the fact that ρ is one dimensional. Elements of both C^n and C^n_ρ are maps that take values in a one-dimensional vector space. So, it is easy to view C^n_ρ as the subspace of C^n which transform in a way prescribed by ρ. If V^ρ is multidimensional, it is a little more difficult to understand which elements of C^n should be viewed as elements of C^n_ρ. At first glance, they seem incompatible. The elements of C^n are maps taking values in a one-dimensional vector space, but elements of C^n_ρ are not.

The key to relating the two is to focus on how the elements of C^n_ρ transform. Notice that if $\rho = (\rho_f, \lambda, 1)$ then every cochain in C^n_ρ is an eigenvector of $\rho(g_\mathbb{Z})$ with eigenvalue λ^{-1}. More generally, if $\rho = (\rho_f, \lambda, t)$ then every cochain in C^n_ρ is a generalized eigenvector (power vector) of $\rho(g_\mathbb{Z})$ of order less than or equal to t with generalized eigenvalue λ.

As an example, consider our toy model with $G_\Gamma = \mathbb{Z}$. Let $\rho = (\rho_0, 1, 2)$. For all $v \in \rho$

$$(\rho(g_\mathbb{Z}) - 1)^2 v = 0. \tag{6.13}$$

Then for all $\tau \in C^n_\rho$ we have

$$(\rho(g_\mathbb{Z}) - 1)^2 \tau = 0. \tag{6.14}$$

The idea is that this equation should exactly define which elements in C^n should be viewed as ρ-invariant. τ satisfies (6.14) if and only if

$$\tau = (\dots, L(-2), L(-1), L(0), L(1), \dots), \tag{6.15}$$

where $L : \mathbb{C} \to \mathbb{C}$ is a linear function. The space of all such τ is clearly two dimensional.

Returning to the general case, let ρ_f be an irreducible representation of G_f with character χ, and let $\rho = (\rho_f, \lambda, t)$.

Define $C^n[\rho_f, \lambda, t] \subset C^n$ by $\tau \in C^n[\rho_f, \lambda, t]$ if and only if

$$(g_{\mathbb{Z}} - \lambda^{-1})^t \tau = 0 \text{ and } (g - \chi(g^{-1}))\tau = 0, \tag{6.16}$$

for all $g \in G_f$. Notice that $C^n[\rho_f, \lambda, t]$ is exactly the t^{th}-order λ^{-1} generalized eigenspace of $C^n[\rho_f, \cdot, \cdot]$, and we should think of $C^n[\rho_f, \lambda, t]$ as being exactly the ρ-invariant cochains sitting inside C^n. The following theorem makes this notion rigorous.

Theorem 6.2. *Let $\rho = (\rho_f, \lambda, t)$, for each $n \geq 0$, there exists an injective map*

$$\Phi_n : C^n_\rho \to C^n \tag{6.17}$$

whose image is exactly $C^n[\rho_f, \lambda, t]$. Furthermore, the maps Φ_n commute with the coboundary maps, i.e., $\Phi_n \circ \delta_\rho = \delta \circ \Phi_n$.

Proof. Let

$$L : V^\rho \to \mathbb{C} \tag{6.18}$$

be linear map, and let

$$\begin{aligned} \Phi_n : C^n_\rho &\to C^n \\ \tau &\mapsto L \circ \tau. \end{aligned} \tag{6.19}$$

Let us first check that Φ_n commutes with the action of G_Γ, i.e., for all $g \in G$, $g \circ \Phi_n = \Phi_n \circ g$. Let $\tau \in C^n_\rho$, $g \in G_\Gamma$. Then

$$\begin{aligned} (g \circ \Phi_n)\tau = g(\Phi_n(\tau)) = g(L \circ \tau) = L \circ \tau \circ g^{-1} \\ = \Phi_n(\tau \circ g^{-1}) = \Phi_n(g(\tau)) = (\Phi_n \circ g)\tau. \end{aligned} \tag{6.20}$$

If $\tau \in C^n_\rho$, then $(\rho(g_\mathbb{Z}) - \lambda_Z^{-1})^p\tau = 0$ and $(\rho(g) - \lambda_g^{-1})\tau = 0$ for all $g \in G_f$. So, the fact that Φ_n commutes with the action of G_Γ immediately implies that $(g_\mathbb{Z} - \lambda_Z^{-1})^p\Phi_n\tau = 0$ and $(g - \lambda_g^{-1})\Phi_n\tau = 0$ for all $g \in G_f$. Thus, the image of Φ_n is contained in $C^n[\rho_f, \lambda, t]$.

Now, let us check that the Φ_n maps commute with the coboundary maps.

$$\begin{aligned}(\Phi_{n+1} \circ \delta_\rho)(\tau) &= \Phi_{n+1}(\delta_\rho\tau) = L \circ (\delta_\rho\tau) = L \circ (\tau \circ \sigma) \\ &= (L \circ \tau) \circ \partial = (\Phi_n(\tau)) \circ \partial = (\delta \circ \Phi_n)(\tau). \quad (6.21)\end{aligned}$$

Now, we want to show that we can choose L so that Φ_n injects and surjects onto $C^n[\rho_f, \lambda, t]$.

Claim. *If the kernel of L contains no subspaces fixed by the action of G_Γ (other than $\{0\}$), then Φ_n injects.*

Proof of claim. Suppose that $\Phi_n(\tau) = 0$. Then for all $\sigma \in C^n$, $\Phi_n(\tau)(\sigma) = 0$. Thus $L(\tau(\sigma)) = 0$, so $\tau(\sigma) \in \ker L$, for all $\sigma \in C^n$. Fix a chain σ, then $\rho(g)\tau(\sigma) = \tau(g\sigma) \in \ker L$, for all $g \in G_\Gamma$. But then the span of $\rho(g)\tau(\sigma)$, over all g, is a subspace of L which is fixed by the action of G_Γ, hence $\tau(\sigma) = 0$. Since σ was arbitrary, $\tau = 0$. Thus Φ_n injects. $\square$

Finally, we must show that every cochain in $C^n[\rho_f, \lambda, t]$ is in the image of Φ_n. Suppose $\sigma_1, \ldots, \sigma_s$ are simple cochains representing the distinct orbits $\mathcal{O}^1, \ldots, \mathcal{O}^s$ of $\mathcal{K}_n$ (the collection of n-cells), under the action. So, every chain in C_n is a sum of G_Γ translates of the the σ_j's. Every cochain τ in $C^n[\rho_f, \lambda, t]$ can be expressed as

$$\tau = \tau_1 + \cdots + \tau_n, \tag{6.22}$$

where τ_j vanishes on all translates of σ_k for $k \neq j$. So, if we let $\sigma = \sigma_j$ for some j, it suffices to show that if τ is a cochain that vanishes on all translates of σ_k for $k \neq j$, then τ is in the image of Φ_n.

We need to be more specific about L. Suppose $\dim V^\rho = t$. Let $v^0, \ldots, v^{t-1}$ be vectors spanning V^ρ such that

$$\rho(g_\mathbb{Z})v^k = \begin{cases} \lambda v^0 & \text{if } k = 0, \\ \lambda v^k + v^{k-1} & \text{if } 1 \leq k \leq t-1\ . \end{cases} \tag{6.23}$$

It will be convenient to write $\rho(g_{\mathbb{Z}})$ as $(S+\lambda I)$ where I is the identity map on V^{ρ} and hence

$$S(v^k) = \begin{cases} 0 & \text{if } k=0, \\ v^{k-1} & \text{if } 1 \le k \le t-1 \;. \end{cases} \tag{6.24}$$

Now define

$$L(v^k) = \begin{cases} 1 & \text{if } k=0, \\ 0 & \text{if } 1 \le k \le t-1. \end{cases} \tag{6.25}$$

Note that this implies that

$$L \circ S^l(v^k) = \begin{cases} 1 & \text{if } k=l, \\ 0 & \text{otherwise.} \end{cases} \tag{6.26}$$

Now, let $\tau \in C^n[\rho_f, \lambda, t]$ vanish on all translates of σ_k for $k \neq j$. Notice that if τ' is another such cochain, and if $\tau(g^l_{\mathbb{Z}}\sigma) = \tau'(g^l_{\mathbb{Z}}\sigma)$ for $1 \le l \le t-1$, then $\tau = \tau'$. This is because, by virtue of the fact that both τ and τ' are in $C^n[\rho_f, \lambda, t]$, we know that the functions $f, f' : \mathbb{Z} \to \mathbb{C}$ defined by

$$f(l) = \tau(g^l_{\mathbb{Z}}\sigma), \tag{6.27a}$$

$$f'(l) = \tau'(g^l_{\mathbb{Z}}\sigma) \tag{6.27b}$$

are both the solutions to the same t^{th}-order linear difference equations, with the same initial values, thus for all $l \in \mathbb{Z}$

$$\tau(g^l_{\mathbb{Z}}\sigma) = f(l) = f'(l) = \tau'(g^l_{\mathbb{Z}}\sigma). \tag{6.28}$$

So, for all $l \in \mathbb{Z}$ and $g_f \in G_f$, we have

$$\tau(g_f g^l_{\mathbb{Z}}\sigma) = \chi(g_f)\tau(g^l_{\mathbb{Z}}\sigma) = \chi(g_f)\tau'(g^l_{\mathbb{Z}}\sigma) = \tau'(g_f g^l_{\mathbb{Z}}\sigma). \tag{6.29}$$

With this in mind, we only need to show that we can choose $\bar{\tau} \in C^n_{\rho}$ so that $\tau = \Phi_n(\bar{\tau})$. Per the present discussion, we need only to show that $L \circ \bar{\tau}(g^l\sigma) = \tau(g^l\sigma)$ for $0 \le l \le t-1$. Define $a_0, \ldots, a_{t-1}$ recursively as follows

$$\tau(g^l\sigma) = \sum_{j=0}^{l} \binom{l}{j} \lambda^{j-l} a_l. \tag{6.30}$$

Although this defines a_l implicitly, it is easy to see that there exists a unique list of numbers $a_0, \ldots, a_{t-1}$ satisfying (6.30) because if $a_0 = \tau(\sigma)$, then there is a unique a_1 such that $a_1 + \lambda a_0$ and thus a unique a_2 such that $a_2 + 2\lambda a_1 + \lambda^2 a_0$, et cetera. Now, let

$$\bar{\tau}(\sigma) = \sum_{k=0}^{t-1} a_k v^k, \tag{6.31}$$

and then extend this naturally by letting $\bar{\tau}(g\sigma) = \rho(g)\bar{\tau}(\sigma)$ for all $g \in G_\Gamma$ (and by letting $\bar{\tau}$ vanish on the other orbits). It remains only to check that $\Phi_n(\bar{\tau})(g_{\mathbb{Z}}^l \sigma) = \tau(g_{\mathbb{Z}}^l \sigma)$ for $0 \leq l \leq t-1$.

$$\begin{aligned}
\Phi_n(\bar{\tau})(g_{\mathbb{Z}}^l \sigma) = L(\tau(g_{\mathbb{Z}}^l \sigma)) &= L(g_{\mathbb{Z}}^l(\tau(\sigma))) \\
&= \sum_{k=0}^{t-1} L((S+\lambda)^l a_k v^k) \\
&= \sum_{j=0}^{l} \sum_{k=0}^{t-1} \binom{l}{j} \lambda^{j-l} L \circ S^l (a_k v^k) \\
&= \sum_{j=0}^{l} \binom{l}{j} \lambda^{j-l} a_l = \tau(g_{\mathbb{Z}}^l \sigma).
\end{aligned} \tag{6.32}$$

Hence, $\Phi_n(\bar{\tau}) = \tau$. $\square$

Now, let $\rho = (\rho_f, \lambda, t)$ and let $\bar{\delta}_\rho^n$ be the map δ^n restricted to $C^n[\rho_f, \lambda, t]$. Define

$$K^n[\rho_f, \lambda, t] \equiv \ker \bar{\delta}_\rho^n = \ker \delta^n \cap C^n[\rho_f, \lambda, t]. \tag{6.33}$$

By definition, $K^n[\rho_f, \lambda, t]$ is exactly the t^{th}-order λ^{-1}-generalized eigenspace of $K^n[\rho_f, \cdot, \cdot]$. Define

$$H^n[\rho_f, \lambda, t] \equiv \frac{\ker \bar{\delta}_\rho^n}{\operatorname{Im} \bar{\delta}_\rho^{n-1}}. \tag{6.34}$$

An immediate consequence of theorem 6.2 is that

$$H_\rho^n \cong H^n[\rho_f, \lambda, t]. \tag{6.35}$$

6.2 Kernels of Truth

Observe the following elementary fact: Leaving group actions and representations aside for the moment, suppose that we're calculating the complex valued cohomology of some finite dimensional cochain complex

$$\cdots \longrightarrow C^{n-1} \xrightarrow{\delta^{n-1}} C^n \xrightarrow{\delta^n} C^{n+1} \longrightarrow \cdots \tag{6.36}$$

Then we can determine H^n simply by knowing the dimension of the kernels of all these maps. We have

$$\begin{aligned} \dim H^n &= \dim \ker \delta^n - \dim \operatorname{Im} \delta^{n-1} \\ &= \dim \ker \delta_n - (\dim C^{n-1} - \dim \ker \delta^{n-1}). \end{aligned} \tag{6.37}$$

In other words, kernels are all we really need to calculate, in this situation. A similar statement is true for $\mathbb{Z}$-representation valued cohomology:

Theorem 6.3. *Suppose we have the following complex of $\mathbb{Z}$ representations:*

$$\cdots \longrightarrow C^{n-1} \xrightarrow{\delta^{n-1}} C^n \xrightarrow{\delta^n} C^{n+1} \longrightarrow \cdots \tag{6.38}$$

where all maps commute with the $\mathbb{Z}$ actions and $C^n \cong \prod_{\mathbb{Z}} \mathbb{C}^{l_n}$. Let

$$\ker \delta^n \cong \prod_{\mathbb{Z}} \mathbb{C}^{T_n} \oplus A_{(n)} \tag{6.39}$$

where $A_{(n)}$ is finite dimensional as a $\mathbb{C}$ vector space. Then

$$H^n \cong \prod_{\mathbb{Z}} \mathbb{C}^{T_n-(l_{n-1}-T_{n-1})} \oplus A_{(n)}. \tag{6.40}$$

It will turn out that the key to relating the Rand cohomology to the total cohomology will boil down to understanding the subtle differences between (6.37) and the content of this theorem, but we are getting ahead of ourselves. Our strategy for proving this theorem is to dualize, which moves us into the realm where all modules are finitely generated. The theorem is not difficult to prove but we need to recall some basic facts from module theory. In what follows, assume that all modules are modules over an integral domain R. The *rank* of a module is the maximum number of linearly independent elements. A module M is *free* if $r \in R$, and $m \in M$ and $rm = 0$, then $r = 0$ or $m = 0$. A module is *torsion*[1] if for all $m \in M$, there exists $r \in R$, $r \neq 0$, such that $rm = 0$. We will use the following facts

1. A submodule of a free module is free.

2. A map from a torsion module to a free module is trivial.

3. Whenever a module M can be expressed as $M = F \oplus T$ where F is free and T is torsion (as is always the case for finitely generated modules over principal ideal domains) then the rank of M and the rank of F are the same.

4. Ranks behave the way you expect them to when it comes to quotients. In other words, if P, Q and S are modules of rank p, q and s, and $S \cong Q/P$, then $s = q - p$.

We are now ready to prove the theorem.

Proof. The complex (6.38) has the dual complex

$$\cdots \longleftarrow C_{n-1} \xleftarrow{\partial_n} C_n \xleftarrow{\partial_{n+1}} C_{n+1} \longleftarrow \cdots \tag{6.41}$$

Recall that theorem 5.4 essentially says that cohomology is exactly the dual of homology. Let us apply that theorem to the following to the following chain complex:

$$0 \longleftarrow C_n \xleftarrow{\partial_{n+1}} C_{n+1}. \tag{6.42}$$

Then we have that

$$\frac{C^n}{\operatorname{Im} \partial_{n+1}} \cong (\ker \delta^n)^*. \tag{6.43}$$

Since $\ker \delta^n \cong \prod \mathbb{C}^{T_n} \oplus A_{(n)}$, we have

$$\frac{C_n}{\operatorname{Im} \partial_{n+1}} \cong \bigoplus_{\mathbb{Z}} \mathbb{C}^{T_n} \oplus A^*_{(n)}. \tag{6.44}$$

[1] When the ring R is $\mathbb{Z}$, and M is a finitely generated module, then M is an abelian group, and M is torsion if and only if it is finite. When $R = \mathbb{C}[z, z^{-1}]$, and M is finitely generated, then M must be a finite dimensional complex vector space. For example $\mathbb{C}[z, z^{-1}]/(z - \lambda)^t$ is torsion, since it is annihilated by the ring element $(z - \lambda)^t$.

Similarly,

$$\frac{C_{n-1}}{\operatorname{Im}\partial_n} \cong \bigoplus_{\mathbb{Z}} \mathbb{C}^{T_{n-1}} \oplus A^*_{(n-1)}. \tag{6.45}$$

Since ranks behave nicely with respect to quotients we know that $\operatorname{Im}\partial_n$ has rank $l_{n-1} - T_{n-1}$. Since, $\operatorname{Im}\partial_n$ is a submodule of a free module, it is free, thus

$$\frac{C_n}{\ker\partial_n} \cong \operatorname{Im}\partial_n \cong \bigoplus_{\mathbb{Z}} \mathbb{C}^{l_{n-1}-T_{n-1}}. \tag{6.46}$$

Now, note that if $A \subset B \subset C$ and

$$\varphi : \frac{C}{A} \to \frac{C}{B} \tag{6.47}$$

is the natural map sending $x + A$ to $x + B$, then $\ker\varphi \cong B/A$, and φ surjects, so in the present situation, we have

$$\begin{array}{rccc} \varphi : & \dfrac{C_n}{\operatorname{Im}\partial_{n+1}} & \to & \dfrac{C_n}{\ker\partial_n} \\ & \displaystyle\bigoplus_{\mathbb{Z}} \mathbb{C}^{T_n} \oplus A^*_{(n)} & \to & \mathbb{C}^{l_{n-1}-T_{n-1}}, \end{array} \tag{6.48}$$

and thus $H_n = \ker\varphi$. It remains only calculate $\ker\varphi$ and dualize. Since φ surjects, $\ker\varphi$ must have rank $T_n - (l_{n-1} - T_{n-1})$ (again, by the fact that ranks behave nicely). $A^*_{(n)}$ is torsion, and $\varphi(A^*_{(n)})$ lies in a free module. But we know that any map from a torsion module to a free module must be trivial, thus $\varphi(A^*_{(n)}) = 0$. So, $A^*_{(n)} \subset \ker\varphi$ and since there is no other torsion in $C_n/\operatorname{Im}\partial_{n+1}$, A^* is exactly the torsion of $\ker\varphi$. So, by theorem 5.4 of the previous chapter, we have

$$\begin{aligned} H^n &\cong \left(\frac{\ker\partial_n}{\operatorname{Im}\partial_{n+1}}\right)^* \cong (\ker\varphi)^* \\ &\cong \left(\bigoplus_{\mathbb{Z}} \mathbb{C}^{T_n-(l_{n-1}-T_{n-1})} \oplus A^*_{(n)}\right)^* \\ &\cong \prod_{\mathbb{Z}} \mathbb{C}^{T_n-(l_{n-1}-T_{n-1})} \oplus A_{(n)}. \end{aligned}$$

□

Let us compare and contrast how kernels relate to cohomology in the category of finite dimensional vector spaces, versus how kernels relate to cohomology in the category of $\mathbb{Z}$ representations. In both situations, if you want to calculate H^n up to isomorphism, it suffices to calculate the kernel of δ^n and δ^{n-1}, (assuming, of course, that you also know C^{n-1}). Thus, the title of this section: Kernels give you the whole truth.

However, there is a subtle and important difference between the two categories. In the case of finite dimensional vector spaces, the kernel of δ^{n-1} directly contributes to H^n, because increasing the dimension of the kernel by one, also increases the dimension of the cokernel by one, thus increasing the dimension of H^n by one. The same principle applies in the category of $\mathbb{Z}$ representations, at the level of major rank. Increasing the major rank of the kernel of δ^{n-1} by one increases the major rank of the cokernel by one, thus increasing the major rank of H^n by one. However, the same cannot be said of any additional finite dimensional piece of the kernel. In other words, if $K^{n-1} = \mathbb{C}[[z, z^{-1}]]^{T_{n-1}} \oplus A_{(n-1)}$, where $A_{(n-1)}$ is a finite dimensional $\mathbb{Z}$-representation as above, then $A_{(n-1)}$ does not contribute to H^n in any way.

As a result, H^n and H^n_ρ will detect exceptional behavior differently. For example, let λ be an exceptional eigenvalue of $H^n[\rho_f, \cdot, \cdot]$, and let $\rho = (\rho_f, \lambda, 1)$. Then λ is also an exceptional eigenvalue of $K^n[\rho_f, \cdot, \cdot]$, so the kernel of δ^n_ρ has a higher rank than usual, which implies that both H^n_ρ and H^{n+1}_ρ will have higher dimension than usual. Exceptional behavior in H^n_ρ bleeds over into H^{n+1}_ρ. Exceptional behavior in H^n doesn't bleed over into H^{n+1}, but it does manifest itself in both H^n_ρ and H^{n+1}_ρ. This is the key concept in relating H^n to H^n_ρ, and it is exactly what we saw at the end of chapter 4, when we examined the case in which Γ is the real line with a $\mathbb{Z}$ action.

6.3 H^n and H^n_ρ

We are now in a position to clearly state and prove the major theorems of this section, which relate H^n and H^n_ρ.

Theorem 6.4. *Write*

$$H^n = H^n[\rho_f^1, \cdot, \cdot] \oplus \cdots \oplus H^n[\rho_f^N, \cdot, \cdot] \tag{6.49}$$

where ρ_f^i are the irreducible representations of G_f. Let $\rho = (\rho_f^i, \lambda, t)$, then

$$\begin{aligned}\dim H_\rho^n &= t \times (\mathit{majrank}(H^n[\rho_f^i, \cdot, \cdot])) \\ &\quad + \mathit{surpdim}(H^n[\rho_f^i, \cdot, \cdot], \lambda^{-1}, t) + \mathit{surpdim}(H^{n-1}[\rho_f^i, \cdot, \cdot], \lambda^{-1}, t). \quad (6.50)\end{aligned}$$

Proof. Let

$$C^n[\rho_f^i, \cdot, \cdot] \cong \mathbb{C}[[z, z^{-1}]]^{l_n}, \quad (6.51)$$
$$K^n[\rho_f^i, \cdot, \cdot] \cong \mathbb{C}[[z, z^{-1}]]^{T_n} \oplus A_{(n)}, \quad (6.52)$$

where $A_{(n)}$ is a finite dimensional $\mathbb{Z}$-representation as in theorem 6.3. Let $a^n_{\lambda^{-1},t}$ be the dimension of the t^{th}-order λ^{-1}-generalized eigenspace of $A_{(n)}$. By theorem 6.3

$$H^n[\rho_f^i, \cdot, \cdot] \cong \mathbb{C}[[z, z^{-1}]]^{T_n + (l_{n-1} - T_{n-1})} \oplus A_{(n)}. \quad (6.53)$$

So, the major rank of $H^n[\rho_f^i, \cdot, \cdot]$ is $T_n + (l_n - T_{n-1})$ and the t^{th} order surplus dimension of $H^n[\rho_f^i, \cdot, \cdot]$ is $a^n_{\lambda^{-1},t}$. $C^n[\rho_f^i, \lambda^{-1}, t]$ is exactly the t^{th}-order generalized λ^{-1}-eigenspace of $C^n[\rho_f^i, \cdot, \cdot]$, and $K^n[\rho_f^i, \lambda^{-1}, t]$ is exactly the t^{th}-order generalized λ^{-1}-eigenspace of $K^n(\rho_f^i, \cdot, \cdot)$. So, by equation 5.46, we know that

$$\dim C^n[\rho_f^i, \lambda, t] = l_n t, \quad (6.54)$$
$$\dim K^n[\rho_f^i, \lambda, t] = T_n t + a^n_{\lambda^{-1},t}. \quad (6.55)$$

So we have

$$\begin{aligned}\dim H_\rho^n &= \dim H^n[\rho_f, \lambda, t] \\ &= \dim K^n[\rho_f, \lambda, t] - (\dim C^{n-1}[\rho_f, \lambda, t] - \dim K^{n-1}[\rho_f, \lambda, t]) \\ &= T_n t + a^n_{\lambda^{-1},t} - (l_{n-1} t - (T_{n-1} t + a^{n-1}_{\lambda^{-1},t})) \\ &= t[T_n - (l_{n_1} - T_{n-1})] + a^n_{\lambda^{-1},t} + a^{n-1}_{\lambda^{-1},t} \\ &= t \times (\mathrm{majrank}(H^n[\rho_f^i, \cdot, \cdot])) \\ &+ \mathrm{surpdim}(H^n[\rho_f^i, \cdot, \cdot], \lambda^{-1}, t) + \mathrm{surpdim}(H^{n-1}[\rho_f^i, \cdot, \cdot], \lambda^{-1}, t). \quad \square\end{aligned}$$

Equation 6.50 tells us how to compute H_ρ^n from H^n. The next theorem does the opposite.

Theorem 6.5. *Fix an irreducible representation ρ_f of G_f. For each n, There exists b_n such that for all but finitely many $\lambda \in \mathbb{C} - \{0\}$,*

$$H^n_{(\rho_f,\lambda,1)} = b_n. \tag{6.56}$$

The major rank of $H^n[\rho_f, \cdot, \cdot]$ is b_n. Let $\dim H^n_{(\rho_f,\lambda,t)} = h^n_{\lambda,t}$. Then we have

$$\mathit{surpdim}(H^0[\rho_f, \cdot, \cdot], \lambda^{-1}, t) = h^0_{\lambda,t} - tb_0, \tag{6.57}$$

and for $n > 1$

$$\mathit{surpdim}(H^n[\rho_f, \cdot, \cdot], \lambda^{-1}, t) = h^n_{\lambda,t} - \mathit{surpdim}(H^{n-1}[\rho_f, \cdot, \cdot], \lambda^{-1}, t) - tb_n. \tag{6.58}$$

Proof. There can only be finitely many exceptional eigenvalues of $H^n[\rho^i_f, \cdot, \cdot]$ and $H^{n-1}[\rho^i_f, \cdot, \cdot]$. So for all but finitely many λ, we know that

$$\mathrm{surpdim}(H^n[\rho^i_f, \cdot, \cdot], \lambda, t) = \mathrm{surpdim}(H^{n-1}[\rho^i_f, \cdot, \cdot], \lambda, t) = 0. \tag{6.59}$$

Thus, by equation (6.50), for all but finitely many λ, $\dim H^n_{(\rho_f,\lambda,1)}$ is the major rank of $H^n[\rho_f, \cdot, \cdot]$. With this result in hand, equations (6.57) and (6.58) are trivial consequences of (6.50) as well. □

Notice that once we know the major rank, and if we know $\dim H^n_{(\rho_f,\lambda,t)}$ for all n, then we can use equations (6.57) and (6.58) recursively to calculate $\mathrm{surpdim}(H^n[\rho_f, \lambda, t])$ for all n. More concretely, if b_n is the major rank of $H^n[\rho_f, \cdot, \cdot]$, and $\dim H^n_\rho = h^n_{\lambda,t}$ as in theorem 6.5, we have

$$\mathrm{surpdim}(H^n[\rho_f, \lambda, t]) = (-1)^n \sum_{j=0}^{n} (-1)^j (h^j_{\lambda,t} - tb_j). \tag{6.60}$$

6.3.1 $\mathbb{R}$ with a $\mathbb{Z}$ action, revisited

Let us return to the toy example where Γ is the real line with $\mathbb{Z}$ acting, and let's see if we can use our results to recover the cohomology of $\mathbb{R}$ from its Rand cohomologies. Putting the results of section 4.5.1 in our current notation, we have

$$H^0_{(\rho_0,\lambda,1)}(\mathbb{R}) \cong H^0_{(\rho_0,\lambda,1)}(\mathbb{R}) \cong \begin{cases} \mathbb{C} & \text{if } \lambda = 1, \\ 0 & \text{otherwise }. \end{cases} \tag{6.61}$$

Equation 6.56 tells us that the major rank of both $H^0(\mathbb{R})$ and $H^1(\mathbb{R})$ is zero. 1 is the only possible exceptional eigenvalue. Equation 6.57 tells us that the first order surplus dimension of $\lambda = 1$ in $H^0(\mathbb{R})$ is 1, and subsequently equation 6.58 tells us that first order surplus dimension of $\lambda = 1$ in $H^1(\mathbb{R})$ is 0. Hence, $H^1(\mathbb{R})$ has no exceptional eigenvalues, so we conclude $H^1(\mathbb{R}) \cong 0$.

Now, of course we know that since $\mathbb{R}$ is connected $H^0(\mathbb{R}) \cong \mathbb{C}$, so we have already uncovered all of $H^0(\mathbb{R})$ as well, but let us pretend that we are not privy to such sophisticated topological knowledge, and let us try to determine $H^0(\mathbb{R})$ using Rand cohomology. At this point, all we can say is that $H^0(\mathbb{R})$ is at least one dimensional. We need to calculate the second order surplus dimension of $\lambda = 1$ in $H^0(\mathbb{R})$. Let $\rho = (\rho_0, 1, 2)$. We have

$$\partial([0,1]) = [1] - [0] = g_{\mathbb{Z}}[0] - [0]. \tag{6.62}$$

Let v_1, v_2 be a basis for ρ such that $\rho(g_{\mathbb{Z}})v_1 = v_1$, $\rho(g_{\mathbb{Z}})v_2 = v_1 + v_2$. Let τ_1^j be that element of C_ρ^1 which evaluates to v_j on $[0,1]$, and let τ_0^j be that element of C_ρ^0 which evaluates to v_j on $[0]$. Then we have that

$$\delta_\rho(\tau_0^1)([0,1]) = \tau_0^1(\partial[0,1]) = \tau_0^1(g_{\mathbb{Z}}[0]) - \tau_0^1([0]) = v_1 - v_1 = 0, \tag{6.63a}$$
$$\delta_\rho(\tau_0^2)([0,1]) = \tau_0^2(\partial[0,1]) = \tau_0^2(g_{\mathbb{Z}}[0]) - \tau_0^2([0]) = v_1 + v_2 - v_2 = v_1. \tag{6.63b}$$

Thus

$$\delta_\rho(\tau_0^1) = 0 \qquad\qquad \delta_\rho(\tau_0^2) = \tau_1^1. \tag{6.64}$$

So δ_ρ has a one dimesional kernel. Thus $H_\rho^0(\Gamma) \cong \mathbb{C}$, so the second order surplus dimension of $\lambda = 1$ in $H^0(\mathbb{R})$ is the same as the first order surplus dimension of $\lambda = 1$ in $H^0(\mathbb{R})$. So, we can stop looking for any more zeroeth order cohomology, and we conclude that $H^0(\mathbb{R}) \cong \mathbb{C}$. We have recovered $H^*(\mathbb{R})$.

6.4 Passing to the direct limit

Theorems 6.4 and 6.5 show how to determine H_ρ^n from H^n and vice-versa, but we have been assuming that the underlying space Γ is a CW-complex. At the end of the day, our tiling spaces are not CW-complexes; they are inverse limits of CW-complexes. So, we have to ask ourselves the question: Do theorems 6.4 and 6.5 still hold if we replace H^n with $\varinjlim H^n$ and replace H^n_ρ with $\varinjlim H^n_\rho$? The short answer is: for the pinwheel, yes; in general, no. However, we should be able to make more general statements about how Rand cohomology relates to the total cohomology, once we pass to the direct limit. In this section, we explain why, in the direct limit, the results still work for the pinwheel. We explain what can go wrong in the general case, and how Rand cohomology still relates to total cohomology, in the direct limit. For simplicity, we will set aside the G_f action and assume that $G_\Gamma \cong \mathbb{Z}$, for the rest of this section.

Lemma 6.6. *Let D be a $\mathbb{Z}$ representation such that*

$$D = \prod_{\mathbb{Z}} \mathbb{C}^{b_D} \oplus A_D \tag{6.65}$$

where A_D is a finite dimensional complex vector space. Let $\varphi : D \to D$. There exist s_0 such that for all $s \geq s_0$, $(\varphi)^s D = (\varphi)^{s_0} D$.

Proof. Assume to the contrary that

$$D \supsetneq \varphi D \supsetneq (\varphi)^2 D \supsetneq \cdots . \tag{6.66}$$

Then if we let $\varphi^* : D^* \to D^*$ be the pullback of φ, we have

$$0 \subsetneq \ker \varphi^* \subsetneq \ker (\varphi^*)^2 \subsetneq \cdots \subset D^*. \tag{6.67}$$

This is a contradiction, since D^* is a finitely generated module over a PID, thus Noetherian. $\square$

Now examine the following commutative diagram:

$$\begin{tikzcd}
(\sigma^*)^s C^{n-1}(\Gamma) \arrow[r, "\delta^{n-1}"] \arrow[d, "(\sigma^*)^s"] & (\sigma^*)^s C^n(\Gamma) \arrow[r, "\delta^n"] \arrow[d, "(\sigma^*)^s"] & (\sigma^*)^s C^{n+1}(\Gamma) \arrow[d, "(\sigma^*)^s"] \\
(\sigma^*)^s C^{n-1}(\Gamma) \arrow[r, "\delta^{n-1}"] & (\sigma^*)^s C^n(\Gamma) \arrow[r, "\delta^n"] & (\sigma^*)^s C^{n+1}(\Gamma).
\end{tikzcd} \tag{6.68}$$

Lemma 6.6 says we can choose s large enough so that $(\sigma^*)^{s+1}H^n(\Gamma) = (\sigma^*)^s H^n(\Gamma)$. This means that every element of $\varinjlim H^n(\Gamma)$ ultimately is identified with an element of $(\sigma^*)^s H^n(\Gamma)$. Let $H^n_{ER}(\Gamma) = (\sigma^*)^s H^n(\Gamma)$ (where ER is an abbreviation for "eventual range"), then it is not difficult to show that

$$\varinjlim H^n(\Gamma) \cong \varinjlim H^n_{ER}(\Gamma) \tag{6.69}$$

This is where things can get interesting. If $\sigma^* : H^n_{ER}(\Gamma) \to H^n_{ER}(\Gamma)$ is an isomorphism, then in fact:

$$\varinjlim H^n(\Gamma) \cong H^n_{ER}(\Gamma) \tag{6.70}$$

As we will see in the next chapter, this is exactly what happens for the pinwheel, σ^* *is an isomorphism on its eventual range*. In this case, there is a bit more bookkeeping to be done, but the same arguments that we used to prove theorem 6.4 will apply.

However, let's suppose that $H^n(\Gamma) = \mathbb{C}[[z, z^{-1}]]$, and $\sigma^* : H^n(\Gamma) \to H^n(\Gamma)$ is multiplication by $(z-1)$. In this case, every generalized eigenvector with eigenvalue 1 is eventually identified with 0 in the direct limit. In this case, we have

$$\varinjlim H^n(\Gamma) = \frac{\mathbb{C}[[z, z^{-1}]]}{J_1} \tag{6.71}$$

where J_1 is the space of all generalized eigenvectors with eigenvalue 1, i.e.,

$$J_1 = \{\{p(n)\}_{n=-\infty}^{\infty} | p \text{ is a polynomial}\}. \tag{6.72}$$

This seems a bit daunting, but Rand cohomology can still detect what is going on. For the sake of simplicity, assume $H^{n-1} = 0$. Let ρ_λ be that irreducible representation of $\mathbb{Z} = \langle g_{\mathbb{Z}} \rangle$, in which $g_{\mathbb{Z}}$ acts as multiplication by λ. If were to calculate $H^n_{\rho_\lambda}$ for irreducible ρ_λ, we would find

$$H^n_{\rho_\lambda} = \begin{cases} 0 & \lambda = 1, \\ \mathbb{C} & \lambda \neq 1. \end{cases} \tag{6.73}$$

This is because $\varinjlim H^n(\Gamma)$ has a one-dimensional eigenspace of eigenvectors with eigenvalue λ^{-1}, for all $\lambda \neq 1$, but $\varinjlim H^n(\Gamma)$ has no eigenvectors with eigenvalue 1, all of them having been killed off in the direct limit. As before, Rand cohomology detects the exceptional behavior, and can determine when and how σ^* fails to inject on $H^n_{ER}(\Gamma)$.

Chapter 7

The Rand Cohomology and Total Cohomology of the Pinwheel Tiling Space

In the last chapter, we saw exactly how to calculate $\check{H}^n$ from H^n_ρ and vice-versa. Now, we will calculate the Rand cohomology of the pinwheel tiling space $H^*_\rho(\Omega)$ for each irreducible representation ρ of the pinwheel group $\mathcal{P}$, and use the results to determine the total cohomology $\check{H}^*(\Omega)$. Recall that the pinwheel group $\mathcal{P}$ is isomorphic to $\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$. Let $\alpha = \tan^{-1}(1/2)$, so that α is the small angle that appears in the triangles of the pinwheel tiling, and let $\beta = 2\alpha$. Then any tiling in Ω rotated by an angle $n\pi/2 + m\beta$ (where $n = 0, 1, 2, 3, m \in \mathbb{Z}$) is again in Ω.

7.1 The Barge-Diamond Technique

In 2008, Marcy Barge and Beverly Diamond [2] invented a beautiful and powerful technique for calculating the cohomology of substitution tiling spaces. They initially applied the technique to one-dimensional tilings. Shortly thereafter, Barge, Diamond, John Hunton, and Lorenzo Sadun [3] showed how to extend the technique to tilings of the plane. Their techniques have made many cohomology calculations much simpler. Some problems that had been computationally intractable are now quite manageable. The pinwheel is perhaps the most shining example of this.

What is the "Barge-Diamond technique"? The linchpin of the Barge-Diamond technique is Barge-Diamond collaring, which is an innovative approximant scheme (i.e., inverse limit structure) one can impose on a substitution tiling space Θ. Like most other useful approximant schemes, the approximants are all the same CW-complex Γ. Unlike previously concocted approximant schemes, the individual two-cells of Γ neatly describe how tilings in Θ piece together: There is one *tile cell* for each tile type, there is one *edge flap* for each way that two different tiles can meet edge-to-edge, and there is one *vertex polygon* for each way that three or more tiles meet at a single point. That is why the technique is beautiful.

The reason why the technique is powerful is that this classification of two-cells into vertex polygons, edge flaps, and tile cells grants us a useful stratification $S_0 \subset S_1 \subset S_2 = \Gamma$ that allows us to employ two tactics that work in tandem to break up the cohomology calculation into manageable pieces. The first tactic is using the snake lemma and relative cohomology. Roughly speaking, this means we only need to calculate the cohomology of the collection of tile cells, the cohomology of the collection of edge flaps, and the cohomology of the collection of vertex polygons, and then put the pieces together. The second tactic is using eventual ranges. The substitution map (which defines the maps between approximants) kills off much of the complex S_0 of vertex polygons. The surviving subcomplex is called the eventual range, and it turns out that we can calculate cohomology by just considering the eventual range. The big payoff comes when we combine these tactics.

We need to flesh out the details. In this section, we define the Barge-Diamond approximant scheme. We will explain how we can use the snake lemma to calculate cohomology from relative cohomology. We will define the eventual range more precisely, and we will explain how these constructs work together harmoniously. In the next section, we will use this machinery to calculate the Rand cohomology, and hence the total cohomology of the pinwheel space.

7.1.1 Barge-Diamond Approximants

Let Θ be a substitution tiling space. In what follows, we will define a series of approximants Γ_k, and maps between the Γ_k which will give us an inverse limit structure on Θ. We will then slightly refine our definition of the Γ_k, which will allow us to sidestep some technical issues.

Let $\epsilon > 0$ be much smaller than the length of any of the sides of the tiles. Define Γ_0 to be the space Θ modulo the relation that two tilings are equivalent if their local patterns agree out to a radius ϵ units. This means that two tilings represent the same equivalence class in Γ_0 if and only if their origin is at the same point in the same type of tile, and the nearby tiles are the same, out to a distance ϵ. Figure 7.1 helps illustrate this.

Now, let μ be the expansion factor of the substitution, and note that any tiling $T \in \Theta$ can be viewed as a tiling of k^{th} order supertiles, for any $k > 0$. With this in mind, for $k > 0$ define Γ_k to be the space Θ modulo the relation that two tilings T and T' are equivalent if their local k^{th} order supertile patterns agree out to a radius $\mu^k\epsilon$. Notice that if the local supertile pattern of two tilings agree out to a radius of r, then clearly their local tile patterns agree to a radius of r, but the converse is not generally true: The condition that the local supertile patterns agree to a radius ϵ is stronger than the condition that the local tile patterns agree to a radius ϵ. More importantly, all the Γ_k are manifestly homeomorphic.

Given a tiling $T \in \Theta$, let $(T)^k$ be the equivalence class of T in Γ_k, and for $k_2 \geq k_1$ define a map

$$\begin{aligned} \sigma_{k_2k_1} : \Gamma_{k_2} &\to \Gamma_{k_1} \\ (T)^{k_2} &\mapsto (T)^{k_1}. \end{aligned} \tag{7.1}$$

This map is well defined because if T and T' are equivalent in Γ_{k_2}, then they are also equivalent in Γ_{k_1} (for $k_2 > k_1$). If two tilings T and T' are equivalent in Γ_k for all k, then they agree with each other out to every radius, which means that they are the same tiling. For this reason, we can identify Θ as the inverse limit of the Γ_k, under the $\sigma_{k_2k_1}$ maps.

Now, we would like to impose a cellular structure on the Γ_k, but it turns out that trying to put a user-friendly CW complex structure on Γ_k introduces some technical issues we'd rather avoid. It is possible to put an easy-to-use cellular structure on a space that is homotopy equivalent to Γ_k, but we can steer clear of this approach by modifying our definition of the Γ_k.

Redefine Γ_k to be the space Θ modulo the relation that two tilings T and T' are equivalent if the following two conditions hold:

1. The local k^{th} order supertile patterns of T and T' agree out to a radius $\mu^k\epsilon$.

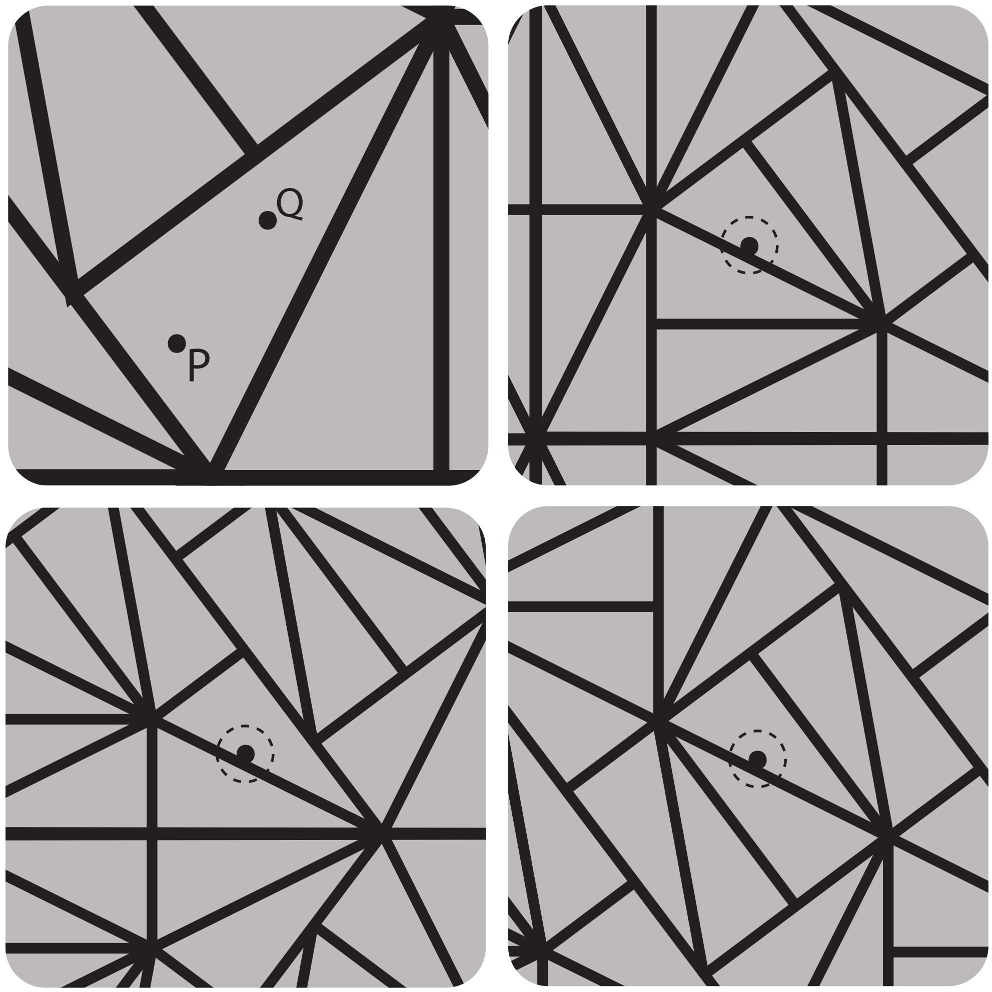

Figure 7.1: Here we see four patches around the origins of various tilings. In the upper left, a tiling with origin at P can never be equivalent in Γ_k to a tiling with origin at Q. If the local patterns of two tilings agree to a radius ϵ, then at the very least, they must be centered at the same point in the same type of tile. In the other three patches, the small circle has radius ϵ. All three tilings are centered at the same point in the same type of tile. However, notice that the local patterns of the upper right and lower right tilings do not agree to radius ϵ. The origin of the upper tiling is within ϵ of a left handed triangle, but the origin of the lower tiling is not. The upper right tiling does agree with the lower left tiling to a radius ϵ, hence the two tilings represent the same point in Γ_0, but the two tilings differ at larger radii, so they do not represent the same point in Γ_3, for example.

2. If condition (1) holds, and the origins of T and T' are within $\mu^k\epsilon$ of two or more edges, then the patterns at the nearby k^{th} order supertile vertex also match.

The value of the second condition is that it allows us to avoid the undesirable situation in which two tilings that are within $\mu^k\epsilon$ of two edges are considered equivalent, despite the fact that the two tilings differ at the nearby supertile vertex. The $\sigma_{k_2k_1}$ maps are still defined the same way, with $(T)^{k_2} \mapsto (T)^{k_1}$.

Let $\Gamma = \Gamma_k$ and $\sigma = \sigma_{k,k-1}$. We can view Γ as a CW complex. We define three different types of two-cells. For each tile type, define a *tile cell* to be the set of tilings whose origin sits in that tile type and is more than a distance ϵ away from any edge. For each edge type, define an *edge flap* to be the set of tilings whose origin is less than a distance ϵ from that edge type, and whose origin is more than ϵ away from other edges. For each vertex type, define a *vertex polygon* to be the set of tilings whose origin is less than a distance ϵ from two or more edges, and whose nearby vertex is of the specified type. This is illustrated in figure 7.2.

The map σ is not a cellular map, but it is easy to see that σ is homotopic to a cellular map $\tilde{\sigma}$. We have

$$\Theta = \varprojlim(\Gamma, \sigma). \tag{7.2}$$

Thus,

$$\check{H}^k(\Theta) = \varinjlim \check{H}^k(\Gamma, \sigma^*) = \varinjlim \check{H}^k(\Gamma, \tilde{\sigma}^*) = \varinjlim H^k_{CW}(\Gamma, \tilde{\sigma}^*). \tag{7.3}$$

Let,

$$\Xi_2 = \varprojlim(\Gamma, \tilde{\sigma}). \tag{7.4}$$

Then we have,

$$\check{H}^k(\Xi_2) = \varinjlim \check{H}^k(\Gamma, \tilde{\sigma}^*) = \check{H}^k(\Theta). \tag{7.5}$$

We calculate $\check{H}^k(\Theta)$ by calculating $\check{H}^k(\Xi_2)$.

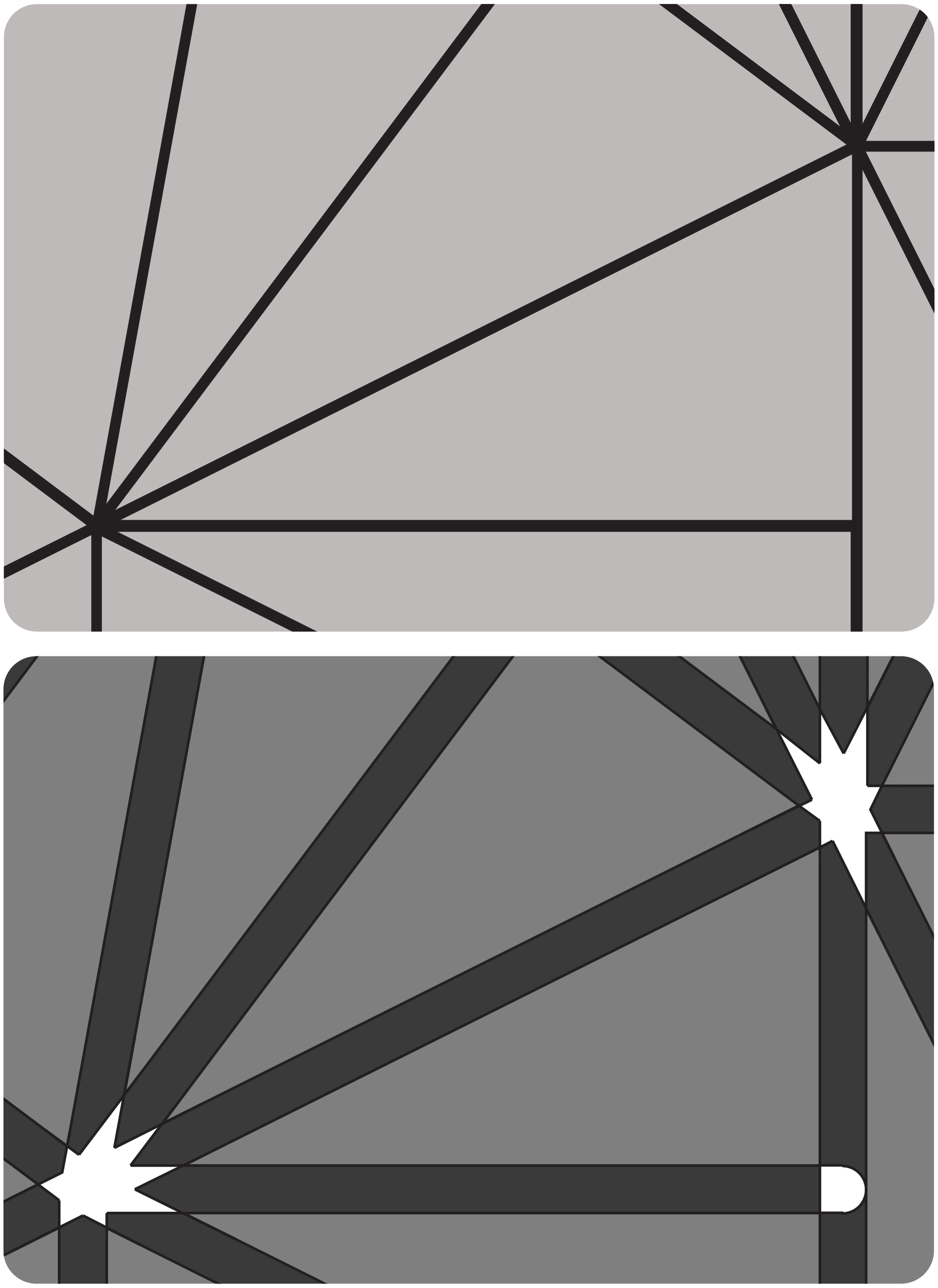

Figure 7.2: The top figure is a patch of tiling, and the bottom figure is the associated cell decomposition. The tile cells are shown in light grey, the edge flaps are shown in dark grey, and the vertex polgons are shown in white.

7.1.2 Using relative cohomology and the snake lemma

In this section, we begin with some elementary homological algebra that can be found in any introductory algebraic topology text (e.g. [8]). Suppose for the moment that we are only concerned with calculating the cohomology of the approximant Γ. Define $S_0 \subset \Gamma$, to be the complex of vertex polygons, define $S_1 \subset \Gamma$ to be the subcomplex of edge flaps and vertex polygons, and define $S_2 = \Gamma$. So, we have a stratification $S_0 \subset S_1 \subset S_2 = \Gamma$. Let $C^k(S_1, S_0)$ be defined as the space of k-cochains defined on $C_k(S_1)$ which vanish on $C_k(S_0)$. We define

$$H^k[S_1, S_0] = \frac{\ker \bar{\delta}^k}{\operatorname{Im} \bar{\delta}^{k-1}}. \tag{7.6}$$

where $\bar{\delta}^k$ is just the coboundary map δ^k restricted to $C^k(S_1, S_0)$. $H^k(S_1, S_0)$ is called the *relative cohomology of the pair* (S_1, S_0).

We have the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & C^0(S_1,S_0) \arrow[r, "\iota_0"] \arrow[d, "\bar{\delta}^0"'] & C^0(S_1) \arrow[r, "\eta_0"] \arrow[d, "\delta^0"'] & C^0(S_0) \arrow[r] \arrow[d, "\delta_0"'] & 0 \\
0 \arrow[r] & C^1(S_1,S_0) \arrow[r, "\iota_1"] \arrow[d, "\bar{\delta}^1"'] & C^1(S_1) \arrow[r, "\eta_1"] \arrow[d, "\delta^1"'] & C^1(S_0) \arrow[r] \arrow[d, "\delta^1"'] & 0 \\
0 \arrow[r] & C^2(S_1,S_0) \arrow[r, "\iota_2"] & C^2(S_1) \arrow[r, "\eta_2"] & C^2(S_0) \arrow[r] & 0.
\end{tikzcd} \tag{7.7}$$

where ι_k is the inclusion map, and η_k is the restriction map. The rows are short exact sequences. By the snake lemma, this gives us the following long exact sequence:

$$\begin{tikzcd}
0 \arrow[r] & H^0(S_1,S_0) \arrow[r, "\iota_0^*"] & H^0(S_1) \arrow[r, "\eta_0^*"] & H^0(S_0) \arrow[dll, "(\delta^0)^*"'] \\
 & H^1(S_1,S_0) \arrow[r, "\iota_1^*"] & H^1(S_1) \arrow[r, "\eta_1^*"] & H^1(S_0) \arrow[dll, "(\delta^1)^*"'] \\
 & H^2(S_1,S_0) \arrow[r, "\iota_2^*"] & H^2(S_1) \arrow[r, "\eta_2^*"] & H^2(S_0) \arrow[r] & 0.
\end{tikzcd} \tag{7.8}$$

For the moment we won't worry about the details of ι_k^*, η_k^*, and $(\delta^k)^*$. The point is that if we can calculate $H^k(S_1, S_0)$ and $H^k(S_0)$, we can use this exact sequence to help us calculate $H^k(S_1)$. We then repeat this with the pair (S_1, S_0) replaced by the pair (S_2, S_1) to help us calculate $H^k(\Gamma) = H^k(S_2)$.

Of course, our real goal is to calculate $H^k(\Xi_2)$, not $H^k(S_2)$. Does this homological algebra interact favorably with the limit process? The answer is yes. Earlier, we saw that σ is homotopic to a cellular map $\tilde{\sigma}$. In fact, something much better is true: We can take $\tilde{\sigma}$ to be a map that sends S_j to S_j for $j = 0, 1, 2$. So, we can define $\Xi_0 \subset \Xi_1 \subset \Xi_2$ where

$$\Xi_j = \varprojlim (S_j, \tilde{\sigma}). \tag{7.9}$$

So,

$$\check{H}^k(\Xi_j) = \varinjlim H^k_{CW}(S_j, \tilde{\sigma}^*). \tag{7.10}$$

Furthermore, notice that $\tilde{\sigma}^*$ maps $C^k[S_1, S_0]$ to itself, because if α_0 is a k-chain on S_0, then for all $\tau \in C^k(S_1, S_0)$

$$(\tilde{\sigma}^*\tau)(\alpha_0) = \tau(\tilde{\sigma}^*(\alpha_0)) = 0, \tag{7.11}$$

since $\tilde{\sigma}^*(\alpha_0)$ is a k-chain on S_0. So, it makes sense to speak of the direct limit of the relative cohomologies as well.

For any topological spaces $Y \subset X$ it is possible to define $\check{H}(X, Y)$, the relative Cech cohomology of the pair (X, Y). We have the following theorem (for details, see [18]):

Theorem 7.1. *Let $X = \varprojlim (X_n, f_n)$. Let Y_n be a subcomplex of X_n and suppose $f(Y_n) \subset Y_{n-1}$. Let $Y = \varprojlim (X_n, f_n)$. There is a long exact sequence*

$$\cdots \to \check{H}^k(X, Y) \to \check{H}^k(X) \to \check{H}^k(Y) \to \check{H}^{k+1}(X, Y) \to \cdots \tag{7.12}$$

among the Cech cohomologies of the inverse limit spaces. Furthermore, each term in the long exact sequence, and the maps between them, are direct limits of the corresponding terms in the cohomology long exact sequences of the approximants.

In the present context, this means that the long exact sequence (7.8) of the pair of approximants (S_1, S_0) begets a long exact sequence

$$\begin{tikzcd}
0 \arrow[r] & \check{H}^0(\Xi_1,\Xi_0) \arrow[r, "\iota_0^*"] & \check{H}^0(\Xi_1) \arrow[r, "\eta_0^*"] & \check{H}^0(\Xi_0) \arrow[dll, "d_0^*"] & \\
& \check{H}^1(\Xi_1,\Xi_0) \arrow[r, "\iota_1^*"] & \check{H}^1(\Xi_1) \arrow[r, "\eta_1^*"] & \check{H}^1(\Xi_0) \arrow[dll, "d_1^*"] & \\
& \check{H}^2(\Xi_1,\Xi_0) \arrow[r, "\iota_2^*"] & \check{H}^2_{(}\Xi_1) \arrow[r, "\eta_2^*"] & \check{H}^2(\Xi_0) \arrow[r] & 0.
\end{tikzcd} \tag{7.13}$$

where $\check{H}^k(\Xi_1, \Xi_0)$ is really just the direct limit of the relative cohomology $H^k(S_1, S_0)$ via $\tilde{\sigma}^*$.

So, if we can find a way to calculate the cohomology $H^k(\Xi_0)$ and also the relative cohomology $H^k(\Xi_1, \Xi_0)$, then in principle we can use the exact sequence (7.13) to help us find $H^k(\Xi_1)$, and in turn if we also know $H^k(\Xi_1, \Xi_0)$, we should be able to use that information to determine $H^k(\Xi_2) = H^k(\Theta)$.

7.1.3 Eventual Ranges

Suppose Γ is a finite CW complex, $\mathcal{K}$ is its collection of cells and $\sigma : \Gamma \to \Gamma$ a cellular map. By definition, σ must map the union of the cells in $\mathcal{K}$ into the union of some subcollection of $\mathcal{K}$. So, we have

$$\Gamma = \cup\mathcal{K} \supseteq \sigma(\cup\mathcal{K}) \supseteq \sigma^2(\cup\mathcal{K}) \supseteq \sigma^3(\cup\mathcal{K}) \supseteq \cdots \tag{7.14}$$

If we let $\mathcal{K}^{(n)}$ be that collection of cells such that $\sigma^n(\cup\mathcal{K}) = \cup\mathcal{K}^{(n)}$, then $\mathcal{K} \supset \mathcal{K}^{(1)} \supset \mathcal{K}^{(2)} \ldots$. Since $\mathcal{K}$ is finite, this sequence of inclusions must stabilize, hence there must exist N such that $\sigma^N(\Gamma) = \sigma^{N-1}(\Gamma)$. The subcomplex $\sigma^N(\Gamma) \subset \Gamma$ is called the *eventual range* of Γ under σ, and we denote it Γ_{ER}.

In our case S_0, S_1, and S_2 are not finite complexes, but the same concept applies. Up to rotation, there are only finitely many tile types, finitely many ways that two tiles edges can meet along, and finitely many ways that several tiles can meet at a vertex, which is to say that each S_j has finitely many orbits under the group action. The map $\tilde{\sigma}$ commutes with the action of group, thus it maps the union of cells in any given orbit *onto* the union of cells of some other orbits. Thus, if we let $\mathcal{O}$ be the collection of orbits, then

$$\Gamma = \cup\mathcal{O} \supseteq \sigma(\cup\mathcal{O}) \supseteq \sigma^2(\cup\mathcal{O}) \supseteq \sigma^3(\cup\mathcal{O}) \supseteq \cdots \tag{7.15}$$

(Here, the union symbol means the union over all orbits in the collection and over all cells in each orbit). If we let $\mathcal{O}^{(n)}$ be that collection of orbits such that $\sigma^n(\cup\mathcal{O}) = \cup\mathcal{O}^{(n)}$, then $\mathcal{O} \supset \mathcal{O}^{(1)} \supset \mathcal{O}^{(2)} \ldots$. Since $\mathcal{O}$ is finite, this sequence of inclusions must stabilize, hence there must exist N such that $\sigma^N(\Gamma) = \sigma^{N-1}(\Gamma)$, and again the subcomplex $\sigma^N(\Gamma) \subset \Gamma$ is called the eventual range of Γ.

Intuitively, we might guess that the fact that $\tilde{\sigma}$ ultimately kills all but the eventual range means that we only have to concern ourselves with the eventual range. We have the following theorem:

Theorem 7.2. *Let Γ be a CW complex and σ a cellular map. Let Γ^* be a subcomplex of Γ such that $\sigma(\Gamma^*) \subset \Gamma^*$. Then*

1. $\varprojlim(\Gamma, \sigma) = \varprojlim(\Gamma_{ER}, \sigma)$,
2. $\check{H}^k(\varprojlim(\Gamma, \sigma)) = \varinjlim \check{H}^k(\Gamma_{ER})$,
3. $\check{H}^k(\varprojlim(\Gamma, \sigma), \varprojlim(\Gamma^*, \sigma)) = \varinjlim \check{H}^k(\Gamma_{ER}, (\Gamma^*)_{ER})$,
4. $\check{H}^k(\varprojlim(\Gamma, \sigma), \varprojlim(\Gamma^*, \sigma)) = \varinjlim \check{H}^k(\Gamma_{ER}, \Gamma^* \cap \Gamma_{ER})$.

A proof can be found in [18].

7.1.4 The Barge-Diamond technique in a nutshell

We have provided a partial justification for why the Barge-Diamond technique works (a full justification would require proving the theorems which we have stated without proof). Before proceeding to use this technique to calculate $\check{H}^n_\rho(\Omega)$, we summarize the steps that comprise the Barge-Diamond technique. We assume that the substitution tiling space Θ consists of tilings of the plane, though clearly the steps are generalizable to higher dimensions.

1. Consider the Barge-Diamond approximant Γ of Θ defined above. Determine all the tile cells (one for each tile type), edge flaps (one for each way two tiles can meet along an edge), and vertex polygons. Let S_0 be the complex of vertex polygons, let S_1 be the complex of edge flaps and vertex polygons, and let $S_2 = \Gamma$.

2. Examine the homotoped substitution map $\tilde{\sigma}$ which maps each S_j into itself. Determine $(S_0)_{\mathrm{ER}}$ by applying the substitution map as many times as necessary.

3. Calculate $H^n((S_0)_{\mathrm{ER}}), H^n((S_1)_{ER}, S_0)$, and $H^n((S_2)_{ER}, S_1)$. Take direct limits to calculate $H^n(\Xi_0), H^n(\Xi_1, \Xi_0)$, and $H^n(\Xi_2, \Xi_1)$.

4. Using the long exact sequence of the pair (Ξ_1, Ξ_0) to calculate $H^n(\Xi_1)$, and then use long exact sequence of the pair (Ξ_2, Ξ_1) to calculate $H^n(\Xi_2) = H^n(\Theta)$.

7.2 Calculating $\check{H}^n_\rho(\Omega)$

7.2.1 Calculating the cohomology of the pieces

Our task is the following. For every irreducible representation ρ of $\mathcal{P}$ (and possibly a few other relevant indecomposable representations) we want to fill in this table:

$*$	$H^0_\rho(*)$	$H^1_\rho(*)$	$H^2_\rho(*)$
Ξ_0			
Ξ_1, Ξ_0			
Ξ_2, Ξ_1			

(7.16)

And we use the fact that

$$H^n_\rho(\Xi_0) = \varinjlim H^n_\rho(S_0),$$

$$H^n_\rho(\Xi_1, \Xi_0) = \varinjlim H^n_\rho(S_1, S_0), \tag{7.17}$$

$$H^n_\rho(\Xi_2, \Xi_1) = \varinjlim H^n_\rho(S_2, S_1), \tag{7.18}$$

where the limit is taken via the map $\tilde{\sigma}^*$. Keep in mind that σ is really just the substitution map, so we can see how $\tilde{\sigma}^*$ acts on cochains by examining the substitution.

Consider S_0, the complex of vertex polygons[1]. Applying $\tilde{\sigma}$ to S_0 a few times shows that (up to rotation) only eight vertex polygons survive to $(S_0)_{\mathrm{ER}}$. We define

$$\begin{aligned}
f_1 &= S_R S_L S_R B_R B_L B_R B_L S_L, & f_2 &= S_R S_L B_L B_R B_L B_R S_R S_L,\\
f_3 &= S_R B_R B_L S_L B_L B_R S_R S_L, & f_4 &= S_R S_L B_L B_R S_R B_R B_L S_L,\\
f_5 &= S_R B_R B_L B_R S_R S_L B_L S_L, & f_6 &= S_R B_R S_R S_L B_L B_R B_L S_L,\\
f_7 &= S_R B_R B_L S_L S_R B_R B_L S_L, & f_8 &= S_R S_L B_L B_R S_R S_L B_L B_R.
\end{aligned} \tag{7.19}$$

Here, S_R is the small acute angle of a right handed triangle, B_R is the large acute angle of a right handed triangle, S_L is the small acute angle of a left handed triangle, B_L is the large acute angle of a right handed triangle. Our convention for the orientation is such that the beginning S_R is oriented so that the hypotenuse runs parallel to the x-axis. We then place the subsequent triangles around the first triangle, proceeding in a counterclockwise fashion. This is illustrated in figures 7.4 and 7.3.

Given f_k and a group element $g \in \mathcal{P}$, we will write $(g)f_k$ to refer to the cell that is f_k rotated by g. So, for example f_1 rotated by counterclockwise by the angle β is referred to as $(\beta)f_1$.

What are the one-cells in this complex? We can define a one-cell for every possible adjacency of two angles. Geometrically, these one-cells consist of two line segments sharing a single endpoint, with the two line segments each sitting in a different tile. Up to rotation, we have eight possible one-cells, as

[1]The cellular decomposition I am using is borrowed from [3]. They calculate the Čech cohomology of $\Omega/\mathcal{P}$, and also the Čech cohomology of the pinwheel space equipped with the topology obtained from the full euclidean metric.

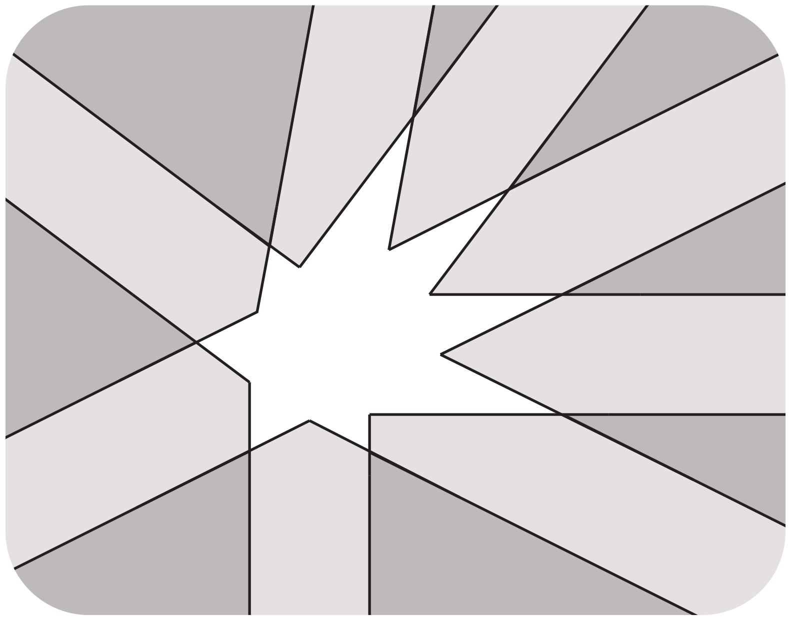

Figure 7.3: The $f_1 = S_R S_L S_R B_R B_L B_R B_L S_L$ vertex polygon. Note that the sixteen geometric edges bounding f_1 constitute eight one-cells in S_0, one for each adjacent pair of tiles that share an edge and meet at the vertex.

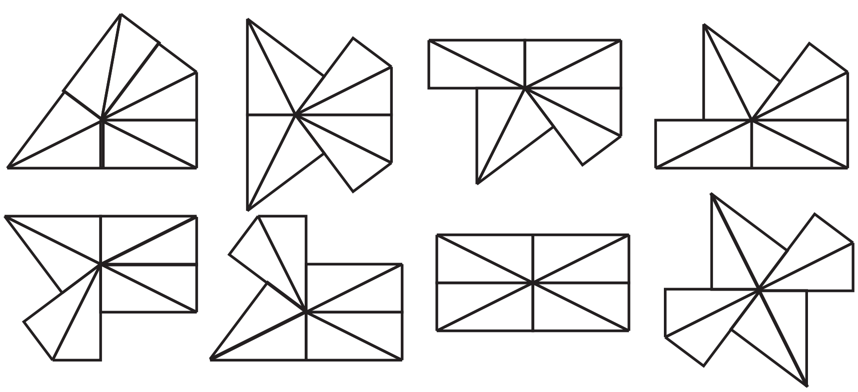

Figure 7.4: The eight vertex types that correspond to vertex polygons in the eventual range of S_0. f_1, f_2, f_3, f_4 are on the top row, and f_5, f_6, f_7, f_8 are on the bottom row.

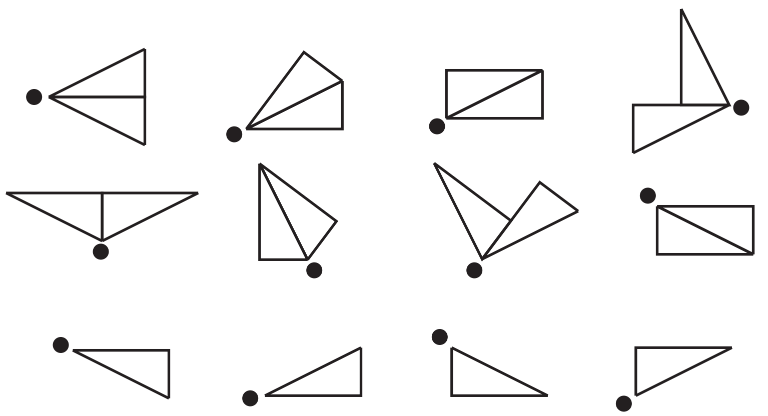

Figure 7.5: The first two rows show the eight possible ways (up to rotation) that two triangles can meet at a vertex whose vertex polygon is in the eventual range of S_0. These define the one-cells $e_1, \dots, e_8$ in $(S_0)_{\mathrm{ER}}$. The last row shows the four possible ways (up to rotation) that a single triangle can touch a vertex whose polygon is in the eventual range of S_0. These define the zero-cells $v_1, \dots, v_4$ in $(S_0)_{\mathrm{ER}}$. Note that this figure is implicitly defines our orientation convention.

shown in figure 7.5, which we will denote $e_1, \dots, e_8$. We have

$$\begin{aligned}
\partial f_1 &= e_1 + (\beta)e_1 + e_2 + (\beta)e_3 + (\pi)e_5 + (\beta)e_5 + (\pi/2)e_6 + e_8, \\
\partial f_2 &= e_1 + e_2 + (-\beta)e_2 + (\pi - \beta)e_4 + (\pi/2)e_5 + e_6 + (\pi - \beta)e_6 + e_7, \\
\partial f_3 &= e_1 + (-\beta)e_2 + e_3 + (\pi - \beta)e_4 + e_5 + (\pi - \beta)e_6 + (\pi - \beta)e_7 + (\pi)e_8, \\
\partial f_4 &= e_1 + e_2 + (\pi)e_3 + e_4 + (\pi)e_5 + e_6 + e_7 + e_8, \\
\partial f_5 &= e_1 + (-\pi/2 - \beta)e_2 + e_3 + (\pi/2 - \beta)e_4 + e_5, \\
&\quad + (\pi/2 - \beta)e_6 + (-\pi/2 - \beta)e_7 + e_8 \\
\partial f_6 &= e_1 + (\pi/2)e_2 + e_3 + (-\pi/2)e_4 + (\pi)e_5 + (\pi/2)e_6 + (\pi/2)e_7 + e_8, \\
\partial f_7 &= e_1 + (\pi)e_1 + e_3 + (\pi)e_3 + e_5 + (\pi)e_5 + e_8 + (\pi)e_8, \\
\partial f_8 &= e_2 + (\pi)e_2 + e_4 + (\pi)e_4 + e_6 + (\pi)e_6 + e_7 + (\pi)e_7.
\end{aligned} \tag{7.20}$$

The zero-cells must then correspond to the end points of the one-cells, and up to rotation, there are only four possibilities. A zero-cell can refer to a point near the small angle of a left-handed triangle, the large angle of a left-handed triangle, the small angle of a right-handed triangle, the large angle of a right-handed triangle.

$$\begin{aligned}
\partial e_1 &= v_2 - v_1, & \partial e_2 &= (\beta)v_1 - v_2,\\
\partial e_3 &= v_4 - v_2 & \partial e_4 &= (\pi)v_2 - (\pi/2)v_4,\\
\partial e_5 &= (\pi)v_3 - v_4, & \partial e_6 &= (\pi/2)v_4 - (\beta+\pi/2)v_3,\\
\partial e_7 &= (\beta+\pi/2)v_3 - (\beta)v_1, & \partial e_8 &= v_1 - v_3.
\end{aligned} \tag{7.21}$$

Since every two- one- and zero- cell is some rotation of some f_k, e_j, or v_m, respectively, any ρ-invariant cochain is defined by the value it takes on the simple chains corresponding to these basic cells. So, C^2_ρ, the space of ρ-invariant two cochains, is eight dimensional, except when $\rho(\pi) = -1$. In that case, any ρ-invariant two cochain must evaluate to zero on both f_7 and f_8, owing to the fact that these two cells are nontrivial rotations of themselves, so $C^2_\rho = \mathbb{C}^6$. C^1_ρ is always eight dimensional and C^0_ρ is always four dimensional. Let $\rho(\beta) = z$ and $\rho(\pi/2) = i^l$, where $l \in 0,1,2,3$. Given the boundary maps above, we can write down the following coboundary maps:

$$\delta^0_\rho = \begin{bmatrix}
-1 & 1 & 0 & 0\\
z & -1 & 0 & 0\\
0 & -1 & 0 & 1\\
0 & (-1)^l & 0 & -i^l\\
0 & 0 & (-1)^l & -1\\
0 & 0 & -i^l z & i^l\\
-z & 0 & i^l z & 0\\
1 & 0 & -1 & 0
\end{bmatrix}. \tag{7.22}$$

$$\delta^1_\rho = \begin{bmatrix}
1+z & 1 & z & 0 & (-1)^l+z & i^l & 0 & 1\\
1 & 1+z^{-1} & 0 & (-1)^l z^{-1} & i^l & 1+(-1)^l z^{-1} & 1 & 0\\
1 & z^{-1} & 1 & (-1)^l z^{-1} & 1 & (-1)^l z^{-1} & (-1)^l z^{-1} & (-1)^l\\
1 & 1 & (-1)^l & 1 & (-1)^l & 1 & 1 & 1\\
1 & i^{-l} z^{-1} & 1 & i^l z^{-1} & 1 & i^l z^{-1} & i^{-l} z^{-1} & 1\\
1 & i^l & 1 & i^{-l} & (-1)^l & i^l & i^l & 1\\
1+(-1)^l & 0 & 1+(-1)^l & 0 & 1+(-1)^l & 0 & 0 & 1+(-1)^l\\
0 & 1+(-1)^l & 0 & 1+(-1)^l & 0 & 1+(-1)^l & 1+(-1)^l & 0
\end{bmatrix}. \tag{7.23}$$

This is not quite an accurate way to summarize δ^1_ρ across all representations. Specifically, when $\rho(\pi) = -1$ (i.e, when l is odd), $C^2_\rho = \mathbb{C}^6$, and δ_2 is actually a 6 by 8 matrix, which is the same as the δ_2 shown above, but with the last two rows removed (which are all zeroes anyway). In all cases, the map $\tilde{\sigma}^*$ is an isomorphism on C^2_ρ. It sends f_1 to a rotation of f_2, and vice versa, and similarly for f_k and f_{k+1}, e_k and e_{k+1}, v_k and v_{k+1}, for k odd. Moreover, $(\tilde{\sigma}^*)^2$ is the identity. So, the cohomology groups are unaffected by the limit process.

In the trivial representation, we have $l = 0$, $z = 1$. In this case, δ^0_ρ has rank 3, but it is of full rank in all other cases. So, as we should expect,

$$H^0_\rho(S_0) = \begin{cases} \mathbb{C} & \rho \text{ trivial,} \\ 0 & \text{otherwise.} \end{cases} \tag{7.24}$$

Also, δ^1_ρ has rank 3 in the trivial representation but has rank 4 in all other representations. So,

$$H^1_\rho(S_0) = \begin{cases} \mathbb{C}^2 & \rho \text{ trivial,} \\ 0 & \text{otherwise.} \end{cases} \tag{7.25}$$

Combining the rank considerations of δ^1_ρ and the fact that $C^2_\rho = \mathbb{C}^8$ if $\rho(\pi) = 1$ while $C^2_\rho = \mathbb{C}^6$ if $\rho(\pi) = -1$, we have

$$H^2_\rho(S_0) = \begin{cases} \mathbb{C}^5 & \rho \text{ trivial,} \\ \mathbb{C}^4 & \rho \text{ non-trivial, } \rho(\pi) = 1, \\ \mathbb{C}^2 & \rho(\pi) = -1. \end{cases} \tag{7.26}$$

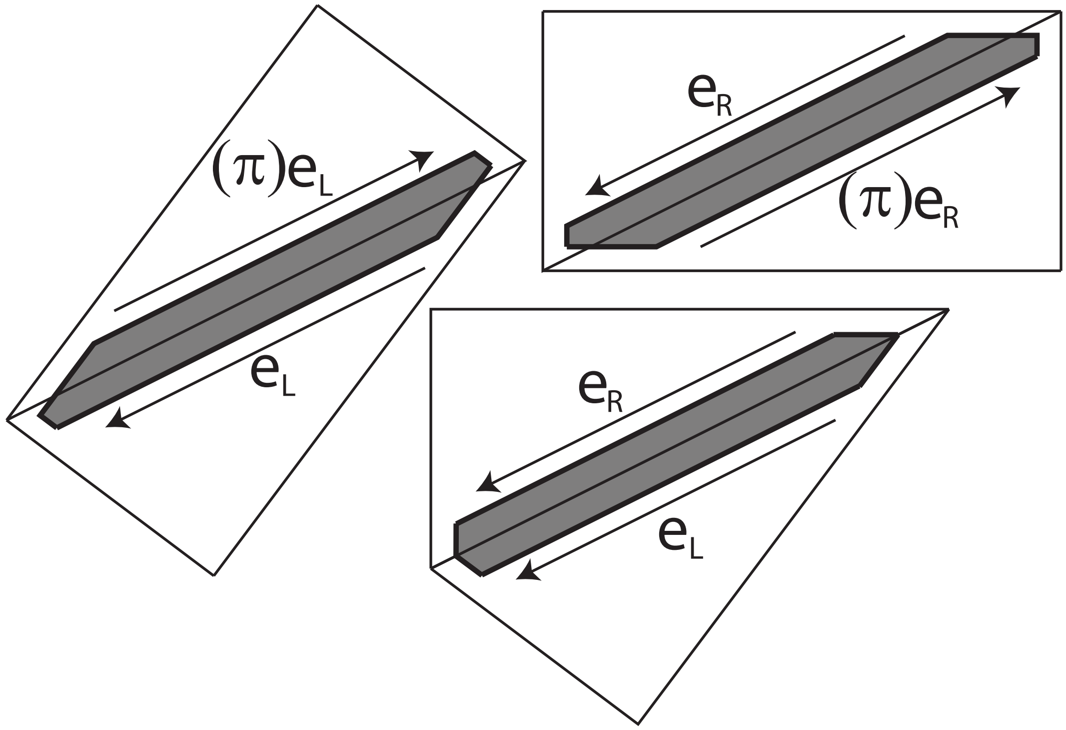

Figure 7.6: The three grey regions are the f_{LL} (left), f_{RR} (upper right), and f_{LR} (lower right) edge flaps, respectively.

At this point, we have calculated the first row of table (7.16). We turn our attention to the second row. S_1 is the complex of edge flaps and vertex polygons. There are two classes of edges in the pinwheel tiling; those edges where hypotenuses of right triangles meet, and those edges where the legs of right triangles meet. $\tilde{\sigma}$ exchanges these two classes, thus $\tilde{\sigma}^2$ maps one class to itself. This convenient fact allows us to choose one of the two classes, calculate the cohomology contribution of that class, take the direct limit under $(\sigma^*)^2$, and then "double" the result, because the contribution of the opposite class will be exactly the same.

We'll work with hypotenuse class. Let $(S_1)_H$ be the complex of hypotenuse edge flaps and vertex polygons. Up to rotation, there are only three ways that two hypotenuses can meet. The hypotenuse of a right-handed triangle can meet the hypotenuse of a left-handed triangle, or the hypotenuses of two right-handed triangles can meet each other, or the hypotenuses of two left-handed triangles can meet each other. As shown in figure 7.6, we define f_{LR} to be a left-right hypotenuse edge flap, and similarly define f_{LL} and f_{RR}. Up to rotation, these are all of the two-cells in $(S_1)_H/S_0$. Define e_L to be that part of the boundary of the edge flaps that lies in the left-hanged triangle, and runs parallel to the hypotenuse. Define e_R similarly. Up to rotation, e_L and e_R are all the one-cells in $(S_1)_H/S_0$. The zero cells would then be the points at the ends of e_L and e_R, but these points are in S_0. Thus, in this cellular decomposition, $C^0_\rho((S_1)_H, S_0) = 0$, which not only trivially implies that $H^0_\rho((S_1)_H, S_0) = 0$, but also allows us to calculate $H^1_\rho((S_1)_H, S_0)$ and $H^2_\rho((S_1)_H, S_0)$ by just determining the kernel and cokernel of δ_2. Regardless of ρ, $C^1_\rho((S_1)_H, S_0) = \mathbb{C}^2$ since no rotation of e_L or e_R maps either of these cells back into itself. However, a rotation by π sends both f_{LL} and f_{RR} back into themselves. As a result, when $\rho(\pi) \neq 1$, any ρ invariant cochain must be zero on both of these cells. So,

$$C^2_\rho((S_1)_H, S_0) = \begin{cases} \mathbb{C} & \rho(\pi) = 1, \\ \mathbb{C}^3 & \rho(\pi) = -1. \end{cases} \tag{7.27}$$

and figure 7.6 shows that

$$\partial f_{LR} = e_R - e_L, \qquad \partial f_{RR} = e_R + (\pi)e_R, \qquad \partial f_{LL} = -e_L - (\pi)e_L. \tag{7.28}$$

So, for $\rho(\pi) = 1$, we have

$$\delta^1_\rho = \begin{bmatrix} 1 & -1 \\ 2 & 0 \\ 0 & -2 \end{bmatrix}. \tag{7.29}$$

This matrix is clearly of rank 2. For $\rho(\pi) = -1$,

$$\delta^1_\rho = \begin{bmatrix} 1 & -1 \end{bmatrix}. \tag{7.30}$$

This matrix has rank 1. Thus,

$$H^1_\rho((S_1)_H, S_0) = \begin{cases} 0 & \rho(\pi) = 1, \\ \mathbb{C} & \rho(\pi) = -1. \end{cases} \tag{7.31}$$

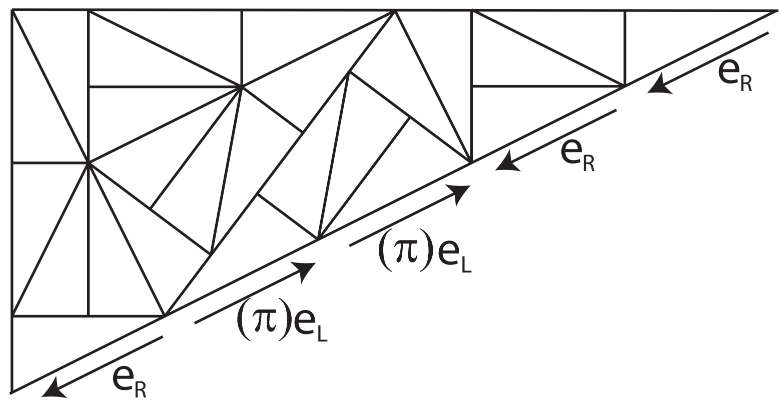

Figure 7.7: σ^2 sends on e_R to $3e_R - 2(\pi)e_L$.

$$H^2_\rho((S_1)_H, S_0) = \begin{cases} \mathbb{C} & \rho(\pi) = 1, \\ 0 & \rho(\pi) = -1. \end{cases} \tag{7.32}$$

To finish calculating $H^*_\rho(\Xi_1, \Xi_0)$ we need to consider $\tilde{\sigma}^2$. Figure 7.7 illustrates how double substitution acts on e_R. We have

$$\sigma^2(e_R) = 3e_R - 2(\pi)e_L, \qquad \sigma^2(e_L) = -2(\pi)e_R + 3e_L. \tag{7.33}$$

For $\rho(\pi) = 1$, $H^1_\rho((S_1)_H, S_0) = 0$, but for $\rho(\pi) = -1$, $H^1_\rho((S_1)_H, S_0) = \mathrm{span}\{e^*_R + e^*_L\}$, and

$$(\sigma^*)^2(e^*_R + e^*_L) = (3e^*_R + 2e^*_L) + (2e^*_R + 3e^*_L) = 5(e^*_R + e^*_L). \tag{7.34}$$

Thus, $(\tilde{\sigma}^*)^2$ acts as multiplication by 5. Now, we also have

$$\begin{aligned} \sigma^2(f_{LR}) &= 3f_{LR} + 2(\pi)f_{LR}, \\ \sigma^2(f_{RR}) &= f_{LR} + (\pi)f_{LR} + 2f_{RR} + f_{LL}, \\ \sigma^2(f_{LL}) &= f_{LR} + (\pi)f_{LR} + f_{RR} + 2f_{LL}. \end{aligned} \tag{7.35}$$

For $\rho(\pi) = -1$, the action of $(\tilde{\sigma}^*)^2$ on two-cochains is moot. For $\rho(\pi) = 1$, we have:

$$(\sigma^*)^2 = \begin{bmatrix} 5 & 0 & 0 \\ 2 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix}. \tag{7.36}$$

note that in cohomology $f^*_{LR} = -2f^*_{RR} = -f^*_{LL}$. So,

$$(\sigma^*)^2(f^*_{LR}) = 5f^*_{LR} + 2f^*_{LL} + 2f^*_{RR} = 3f^*_{LR}. \tag{7.37}$$

So, $(\sigma^*)^2$ acts as multiplication by 3. We have

$$H^0_\rho(\Xi_1, \Xi_0) = 0, \tag{7.38}$$

$$H^1_\rho(\Xi_1, \Xi_0) = \begin{cases} 0 & \rho(\pi) = 1, \\ \mathbb{C}^2 & \rho(\pi) = -1, \end{cases} \tag{7.39}$$

$$H^2_\rho(\Xi_1, \Xi_0) = \begin{cases} \mathbb{C}^2 & \rho(\pi) = 1, \\ 0 & \rho(\pi) = -1. \end{cases} \tag{7.40}$$

We now calculate the last row of table (7.16). S_2/S_1 is complex of tile cells. Up to rotation, there are only two types of two-cells, corresponding to the left handed and right handed triangles. The boundaries of these cells lie in S_1. So, regardless of ρ, we have that $H^0_\rho(\Xi_2, \Xi_1) = H^1_\rho(\Xi_2, \Xi_1) = 0$, while $C^2_\rho(\Xi_2, \Xi_1) = \mathbb{C}^2$, and the only non-trivial consideration is how the substitution map behaves. By inspection of a doubly substituted R or L tile (see, for example, figure 2.5), we see that

$$\begin{aligned} \sigma^2(L) = {} & 5L + 2(\pi/2)L + 2(-\beta)L + 2(\pi - \beta)L \\ & + 2(-\pi/2)L + 2R + 2(\beta)R + 2(\pi + \beta)R \\ & + (-\pi/2)R + 2(\pi)R + (\pi/2 + \beta)R + (-\pi/2 + \beta)R + (\pi/2)R, \end{aligned} \tag{7.41a}$$

$$\begin{aligned} \sigma^2(R) = {} & 5R + 2(-\pi/2)R + 2(\beta)R + 2(\pi + \beta)R \\ & + 2(\pi/2)R + 2L + 2(-\beta)L + 2(\pi - \beta)L \\ & + (\pi/2)L + 2(\pi)R + (-\pi/2 - \beta)L + (\pi/2 - \beta)L + (-\pi/2)L. \end{aligned} \tag{7.41b}$$

As usual, let $\rho(\beta) = z$. For $\rho(\pi/2) = 1$,

$$(\sigma^*)^2 = \begin{bmatrix} 9 + 4z & 6 + 6z \\ 6 + 6z^{-1} & 9 + 4z^{-1} \end{bmatrix}. \tag{7.42}$$

For $\rho(\pi/2) = -1$,

$$(\sigma^*)^2 = \begin{bmatrix} 1 + 4z & 2 + 2z \\ 2 + 2z^{-1} & 1 + 4z^{-1} \end{bmatrix}. \tag{7.43}$$

For $\rho(\pi) = -1$,

$$(\bar{\sigma}^*)^2 = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}. \tag{7.44}$$

These matrices have determinants 25, 9, and 25, respectively. In particular, $(\sigma^*)^2$ is an isomorphism, regardless of ρ.
We have now fully determined the table for all irreducible representations. For the trivial representation, we have

$*$	$H^0_\rho(*)$	$H^1_\rho(*)$	$H^2_\rho(*)$
Ξ_0	$\mathbb{C}$	$\mathbb{C}^2$	$\mathbb{C}^5$
Ξ_1, Ξ_0	0	0	$\mathbb{C}^2$
Ξ_2, Ξ_1	0	0	$\mathbb{C}^2$

(7.45)

For any non-trivial representation where $\rho(\pi) = 1$, we have

$*$	$H^0_\rho(*)$	$H^1_\rho(*)$	$H^2_\rho(*)$
Ξ_0	0	0	$\mathbb{C}^4$
Ξ_1, Ξ_0	0	0	$\mathbb{C}^2$
Ξ_2, Ξ_1	0	0	$\mathbb{C}^2$

(7.46)

For any representation where $\rho(\pi) = -1$, we have

$*$	$H^0_\rho(*)$	$H^1_\rho(*)$	$H^2_\rho(*)$
Ξ_0	0	0	$\mathbb{C}^2$
Ξ_1, Ξ_0	0	$\mathbb{C}^2$	0
Ξ_2, Ξ_1	0	0	$\mathbb{C}^2$

(7.47)

7.2.2 Using homological algebra to put the pieces together

In the case of the trivial representation, the exact sequence of the pair (Ξ_1, Ξ_0) is

$$
\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r, "\iota_0"] & H^0_\rho(\Xi_1) \arrow[r, "\eta_0"] & \mathbb{C} \arrow[dll, rounded corners, "(\delta^0_\rho)^*"] & \\
& 0 \arrow[r, "\iota_1"] & H^1_\rho(\Xi_1) \arrow[r, "\eta_1"] & \mathbb{C}^2 \arrow[dll, rounded corners, "(\delta^1_\rho)^*"] & \\
& \mathbb{C}^2 \arrow[r, "\iota_2"] & H^2_\rho(\Xi_1) \arrow[r, "\eta_2"] & \mathbb{C}^5 \arrow[r] & 0.
\end{tikzcd}
\tag{7.48}
$$

The fact that the sequence is exact makes it apparent that $H^0_\rho(\Xi_1) = \mathbb{C}$. On the other hand, in order to determine $H^0_\rho(\Xi_1)$, we need to look more closely at $(\delta^1)^*$. The covectors $(e_3^* - e_8^*), (e_2^* - e_6^*)$ form a basis of $H^1_\rho(S_0) = \mathbb{C}^2$. As we alluded to earlier, $\tilde{\sigma}^2$ is the identity on $C_n(S_0)$, so in particular $\tilde{\sigma}^*$ is an isomorphism, so the cohomology groups are unaffected by the limit process, and it suffices to determine what the map $(\delta^1_\rho)^*(\bar{\tau})$ does to the basis vectors $(e_3^* - e_8^*)$, and $(e_2^* - e_6^*)$. If $\bar{\tau}$ is a cohomology class in $H^1_\rho(S_0)$ with representatives $\tau \in C^1_\rho(S_0)$ then $d_1^*(\bar{\tau})$ is defined by taking τ and walking it back to $C^2[S_1, S_0]$ via

$$
\begin{tikzcd}
& C^1_\rho(S_1) \arrow[r, "\eta_1"] \arrow[d, "\delta^1_\rho"'] & C^1_\rho(S_0). \\
C^2_\rho(S_1, S_0) \arrow[r, "\iota_2"] & C^2_\rho(S_1) &
\end{tikzcd}
\tag{7.49}
$$

Of course, we can view the covectors $(e_3^* - e_8^*)$, and $(e_2^* - e_6^*)$ as elements of $C^1_\rho(S_1)$ as well. Notice that if we examine f_{LR}, f_{RR}, and f_{LL} in $C_n(S_1)$, then we have

$$\partial_1 f_{LR} = e_6 + e_R + (\pi)e_2 - (\pi)e_L, \tag{7.50a}$$

$$\partial_1 f_{RR} = e_3 + e_R + (\pi)e_3 + (\pi)e_R, \tag{7.50b}$$

$$\partial_1 f_{LL} = e_8 - e_L + (\pi)e_8 - (\pi)e_L. \tag{7.50c}$$

So, if we look at δ^1_ρ as a map from $C^1_\rho(S_1)$ to $C^2_\rho(S_1)$ as in (7.49), we have

$$\delta^1_\rho(e_3^*) = 2f_{RR}^* + \delta^1_\rho|_{C^1_\rho(S_0)}(e_3^*), \tag{7.51a}$$

$$\delta^1_\rho(e_8^*) = 2f_{LL}^* + \delta^1_\rho|_{C^1_\rho(S_0)}(e_8^*), \tag{7.51b}$$

$$\delta^1_\rho(e_6^*) = 2f_{LR}^* + \delta^1_\rho|_{C^1_\rho(S_0)}(e_6^*), \tag{7.51c}$$

$$\delta^1_\rho(e_2^*) = 2f_{LR}^* + \delta^1_\rho|_{C^1_\rho(S_0)}(e_2^*). \tag{7.51d}$$

Thus,

$$\delta^1_\rho(e_3^* - e_8^*) = 2(f^*_{RR} - f^*_{LL}) + \delta^1_\rho|_{C^1_\rho(S_0)}(e_3^* - e_8^*) = 2(f^*_{RR} - f^*_{LL}), \tag{7.52a}$$
$$\delta^1_\rho(e_2^* - e_6^*) = \delta^1_\rho|_{C^1_\rho(S_0)}(e_2^* - e_6^*) = 0. \tag{7.52b}$$

Notice that in $H^2(S_1, S_0)$, $2(f^*_{RR} - f^*_{LL}) = \delta^1_\rho(e^*_R + e^*_L)$, so in fact δ^1_ρ annihilates both $(e_3^* - e_8^*)$, and $(e_2^* - e_6^*)$. Thus, δ^1_ρ is trivial. So in (7.48), η_1^* must be an injective map whose image is the kernel of $(\delta^1)^*$ which is $\mathbb{C}^2$, thus $H^1_\rho(\Xi_0) = \mathbb{C}^2$. Also, we now see that ι_2 must inject, thus η_2 is has a two dimensional kernel and a five dimensional image. So, $H^2_\rho(\Xi_0) = \mathbb{C}^7$.

For the next step, we need to find a basis of $H^1_\rho(S_1)$. Our analysis shows that η_1^* is an isomorphism, and η_1^* is just the restriction map pushed down to cohomology. It suffices to find two covectors $\tau_1, \tau_2 \in S_1$ representing cohomology classes in $H^1_\rho(S_1)$ (so they must have vanishing coboundary) such that $\eta_1(\tau_1) = (e_3^* - e_8^*)$, $\eta_1(\tau_2) = (e_2^* - e_6^*)$. We can let $\tau_1 = e_3^* - e_8^* - e_R^* - e_L^*$, and $\tau_2 = e_2^* - e_6^*$.

Now we examine the exact sequence of the pair (Ξ_2, Ξ_1):

$$\begin{array}{l} 0 \longrightarrow 0 \xrightarrow{\iota_0^*} H^0_\rho(\Xi_2) \xrightarrow{\eta_0^*} \mathbb{C} \xrightarrow{(\delta^0_\rho)^*} \\ \phantom{0 \longrightarrow{}} 0 \xrightarrow{\iota_1^*} H^1_\rho(\Xi_2) \xrightarrow{\eta_1^*} \mathbb{C}^2 \xrightarrow{(\delta^1_\rho)^*} \\ \phantom{0 \longrightarrow{}} \mathbb{C}^2 \xrightarrow{\iota_2^*} H^2_\rho(\Xi_2) \xrightarrow{\eta_2^*} \mathbb{C}^7 \longrightarrow 0. \end{array} \tag{7.53}$$

Again, we need to analyze the map $(\delta^1_\rho)^*$. We use the basis covectors τ_1, τ_2 and walk them back to $C^2(S_2, S_1)$ via

$$\begin{tikzcd} & C^1_\rho(S_2) \arrow[r, "\eta_1"] \arrow[d, "d^1_\rho"'] & C^1_\rho(S_1). \\ C^2_\rho(S_2, S_1) \arrow[r, "\iota_2"] & C^2_\rho(S_2) & \end{tikzcd} \tag{7.54}$$

Consider e_2, e_3, e_6, e_8, e_R, and e_L as edges in $C^1(S_2)$. Notice that equations 7.50 still apply here, but now we also have

$$\partial_1 R = e_R + \text{(other edges corresponding to the legs of the triangle)}, \quad (7.55a)$$
$$\partial_1 L = -e_L + \text{(other edges corresponding to the legs of the triangle)}, \quad (7.55b)$$

which gives us

$$\delta^1_\rho e^*_R = 2f^*_{RR} + f^*_{LR} + R^*, \quad (7.56)$$
$$\delta^1_\rho e^*_L = -2f^*_{LL} - f^*_{LR} - L^*. \quad (7.57)$$

So,

$$\delta^1_\rho(e^*_3 - e^*_8 - e^*_R - e^*_L) = 2(f^*_{RR} - f^*_{LL}) = L^* - R^*. \quad (7.58)$$

and we still have that $\delta^1_\rho(e^*_2 - e^*_6) = 0$. Thus $(\delta^1_\rho)^*$ has a one dimensional kernel and image, which gives us that

$$H^0_\rho(\Gamma) = H^0_\rho(\Xi_2) = \mathbb{C}, \quad (7.59a)$$
$$H^1_\rho(\Gamma) = H^1_\rho(\Xi_2) = \mathbb{C}, \quad (7.59b)$$
$$H^2_\rho(\Gamma) = H^2_\rho(\Xi_2) = \mathbb{C}^8, \quad (7.59c)$$

when ρ is the trivial representation.

When ρ is an irreducible nontrivial representation for which $\rho(\pi) = 1$, the fact that the first two columns of table (7.46) vanish makes the analysis of the long exact sequences trivial, and it is easy to see that

$$H^0_\rho(\Gamma) = H^0_\rho(\Xi_2) = 0, \quad (7.60a)$$
$$H^1_\rho(\Gamma) = H^1_\rho(\Xi_2) = 0, \quad (7.60b)$$
$$H^2_\rho(\Gamma) = H^2_\rho(\Xi_2) = \mathbb{C}^8. \quad (7.60c)$$

Let ρ be an irreducible representation for which $\rho(\pi) = -1$, so that $\rho(\pi/2) = \pm i$, $\rho(\beta) = \lambda \in \mathbb{C}$. The analysis of the long exact sequence of the pair (Ξ_1, Ξ_0) is easy and we get

$$H^0_\rho(\Xi_1) = 0, \quad (7.61)$$
$$H^1_\rho(\Xi_1) = \mathbb{C}^2, \quad (7.62)$$
$$H^2_\rho(\Xi_1) = \mathbb{C}^2. \quad (7.63)$$

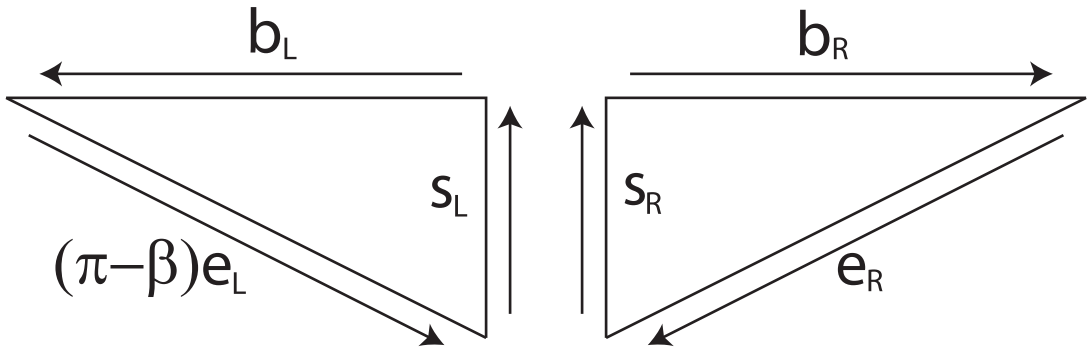

Figure 7.8: An illustration of $\partial\bar{R}$ and $\partial\bar{L}$.

Note that $H^1_\rho(S_1)$ comes entirely from the pair $H^2_\rho(S_1, S_0)$, since $H^1_\rho(S_0) = 0$. Now, we have that the long exact sequence of the pair (Ξ_2, Ξ_1) is

$$\begin{tikzcd} 0 \arrow[r] & 0 \arrow[r, "\iota_0"] & H^0_\rho(\Xi_2) \arrow[r, "\eta_0"] & 0 \arrow[dll, "(\delta^0_\rho)^*"] & \\ & 0 \arrow[r, "\iota_1"] & H^1_\rho(\Xi_2) \arrow[r, "\eta_1"] & \mathbb{C}^2 \arrow[dll, "(\delta^1_\rho)^*"] & \\ & \mathbb{C}^2 \arrow[r, "\iota_2"] & H^2_\rho(\Xi_2) \arrow[r, "\eta_2"] & \mathbb{C}^2 \arrow[r] & 0. \end{tikzcd} \tag{7.64}$$

Earlier we saw how $C^1_\rho(S_1, S_0)$ has two edge classes, the hypotenuse class and the leg class. Each edge class accounts for one of the two dimensions of $H^1_\rho(S_1) = H^1_\rho(S_1, S_0)$, and $e^*_R + e^*_L$ represents a cohomology class, and is from the hypotenuse edge class. If $\tau_1 = e^*_R + e^*_L$ and τ_2 is cohomology class from the leg class, then τ_1 and τ_2 form a basis of $H^1_\rho(S_1)$. Examine figure 7.8. The edges e_L and e_R are s_L, s_R, b_L and b_R are edges corresponding to the other legs of the triangles, as shown. Let $\tau_2(s_L) = 1$. The cochain τ_2 must evaluate to zero on the boundary of any edge flap. Clearly, $s_L - s_R$ bounds an edge flap, thus $\tau_2(s_R) = 1$. Also, $b_L - (\pi/2)s_R + (-\pi/2)s_L$ bounds an edge flap, and since $\rho(\pi/2) = \pm i$, we have that $\tau_2(b_L) = \pm 2i$. Finally, $b_L - (\pi)b_R$ bounds an edge flap, thus $\tau_2(b_R) = \mp 2i$. So, we have

$$\tau_1 = e_R^* + e_L^*, \tag{7.65a}$$
$$\tau_2 = s_L^* + s_R^* \pm 2ib_L^* \mp 2ib_R^*. \tag{7.65b}$$

Let $\bar{R} = (\pi)R$, the tile cell corresponding to the right handed triangle in figure 7.8, and similarly $\bar{L} = (\pi)L$. We have

$$\tau_1(\partial\bar{R}) = 1, \qquad \tau_1(\partial\bar{L}) = \lambda^{-1}, \tag{7.66a}$$
$$\tau_2(\partial\bar{R}) = 1 \mp 2i, \qquad \tau_2(\partial\bar{L}) = -1 \mp 2i. \tag{7.66b}$$

So,

$$(\delta_\rho^1)^*(\tau_1) = \bar{R}^* + \lambda^{-1}\bar{L}^*, \tag{7.67a}$$
$$(\delta_\rho^1)^*(\tau_2) = (1 \mp 2i)\bar{R}^* + (-1 \mp 2i)\bar{L}^*. \tag{7.67b}$$

$\bar{R}^*$ and $\bar{L}^*$ span $H^2(S_2, S_1)$. By equations 7.67, $(\delta_\rho^1)^*$ is an isomorphism *except* when

$$\lambda = \frac{-1 \pm 2i}{1 \pm 2i} = \frac{3 \pm 4i}{5}. \tag{7.68}$$

So in this case, $H_\rho^1(\Xi_2) = \mathbb{C}$, $H_\rho^1(\Xi_2) = \mathbb{C}^3$. Otherwise, $H_\rho^1(\Xi_2) = 0$, $H_\rho^1(\Xi_2) = \mathbb{C}^2$.

We have calculated the Rand cohomology of the pinwheel tiling space for all irreducible representations. This is important enough to state as a theorem.

Theorem 7.3. *The Rand cohomology of the pinwheel tiling space is summarized in the following table*

ρ	$H_\rho^0(\Omega)$	$H_\rho^1(\Omega)$	$H_\rho^2(\Omega)$
trivial	$\mathbb{C}$	$\mathbb{C}$	$\mathbb{C}^8$
non-trivial, $\rho(\pi) = 1$	*0*	*0*	$\mathbb{C}^8$
$\rho(\pi/2) = \pm i, \rho(\beta) = (3 + 4\rho(\beta))/5$	*0*	$\mathbb{C}^1$	$\mathbb{C}^3$
$\rho(\pi/2) = \pm i, \rho(\beta) \neq (3 + 4\rho(\beta))/5$	*0*	*0*	$\mathbb{C}^2$

(7.69)

7.2.3 Going from $H_\rho(\Omega)$ to $\check{H}_\Omega$

We have just a couple more things to check before we can confidently write down the total cohomology of Ω. Let ρ_i be that representation of $G_f = \mathbb{Z}/4\mathbb{Z}$ in which $\pi/2$ acts as multiplication by i. Table (7.69) shows that $H^1[\rho_i;\cdot;\cdot]$ has major rank zero, and that its only exceptional eigenvalue is $\lambda = (3+4i)/5$. However, table (7.69) only allows us to conclude that λ has surplus dimension at least one. We can show that λ has surplus dimension exactly one. To do so, we will follow the spirit of chapter 6, and calculate $H^1_{(\rho_i,\lambda,2)}$.

Above, we showed that when $\rho(\pi/2) = \pm i$, we have that $H^1_\rho(\Xi_2,\Xi_1) = 0$, $H^1_\rho(\Xi_2) = H^2_\rho(\Xi_2,\Xi_1) = H^2_\rho(\Xi_1) = \mathbb{C}^2$. This is true regardless of how $\rho(\beta)$ acts. So, we also know that the major rank of $H^1[\rho_i,\cdot,\cdot](\Xi_2,\Xi_1)$ is zero, the major ranks of $H^2[\rho_i,\cdot,\cdot](\Xi_1)$, $H^2[\rho_i,\cdot,\cdot](\Xi_2,\Xi_1)$, and $H^2[\rho_i,\cdot,\cdot](\Xi_1)$ are all equal to two, and that none of these spaces have any exceptional eigenvalues. Hence, we know that if $\rho = (\rho_i,\lambda,2)$, we have $H^1_\rho(\Xi_2,\Xi_1) = 0$, $H^1_\rho(\Xi_2) = H^2_\rho(\Xi_2,\Xi_1) = H^2_\rho(\Xi_1) = \mathbb{C}^4$.

$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r, "\iota_1"] & H^1_\rho(\Xi_2) \arrow[r, "\eta_1"] & \mathbb{C}^4 \arrow[dll, "(\delta^1_\rho)^*"] & \\
& \mathbb{C}^4 \arrow[r, "\iota_2"] & H^2_\rho(\Xi_2) \arrow[r, "\eta_2"] & \mathbb{C}^4 \arrow[r] & 0.
\end{tikzcd} \tag{7.70}$$

Let w_1, w_2 be a basis of the representation ρ with $\rho(\beta)w_1 = \lambda w_1$ and $\rho(\beta)w_2 = \lambda w_2 + w_1$. For any cell K, let K_j^* be the ρ-invariant cochain which evaluates to w_j on K, and zero on the other orbits. Using the same analysis that led us to equations (7.65), we can write down the following basis of $H^1_\rho(\Xi_2)$:

$$\tau_{1,1} = (e_R)_1^* + (e_L)_1^*, \tag{7.71a}$$
$$\tau_{1,2} = (e_R)_2^* + (e_L)_2^*, \tag{7.71b}$$
$$\tau_{2,1} = (s_L)_1^* + (s_R)_1^* \pm 2i(b_L)_1^* \mp 2i(b_R)_1^*, \tag{7.71c}$$
$$\tau_{2,1} = (s_L)_2^* + (s_R)_2^* \pm 2i(b_L)_2^* \mp 2i(b_R)_2^*, \tag{7.71d}$$

which gives us

$$(\delta^1_\rho)^*(\tau_{1,1}) = \bar{R}^*_1 + \lambda^{-1}\bar{L}^*_1, \tag{7.72a}$$

$$(\delta^1_\rho)^*(\tau_{1,2}) = \bar{R}^*_2 + \lambda^{-1}\bar{L}^*_2 + L^*_2\lambda^{-2}\bar{L}^*_1, \tag{7.72b}$$

$$(\delta^1_\rho)^*(\tau_{2,1}) = (1 \mp 2i)\bar{R}^*_1 + (-1 \mp 2i)\bar{L}^*_1, \tag{7.72c}$$

$$(\delta^1_\rho)^*(\tau_{2,2}) = (1 \mp 2i)\bar{R}^*_2 + (-1 \mp 2i)\bar{L}^*_2. \tag{7.72d}$$

The image of $(\delta^1_\rho)^*$ is three-dimensional, as we can see that $(\delta^1_\rho)^*(\tau_{1,1})$ is a multiple of $(\delta^1_\rho)^*(\tau_{2,1})$, but $(\delta^1_\rho)^*(\tau_{1,2})$, $(\delta^1_\rho)^*(\tau_{2,1})$ and $(\delta^1_\rho)^*(\tau_{2,2})$ are linearly independent. So the exact sequence (7.70), tells us that $H^1_\rho(\Xi_2) = \mathbb{C}$, and hence the surplus dimension of λ in $H^1[\rho_i; \cdot; \cdot]$ is exactly one. Similarly, the surplus dimension of $\lambda^{-1} = (3-4i)/5$ in $H^1[\rho_{-i}; \cdot; \cdot]$ is exactly one.

We also see that $H^0[1, \cdot, \cdot]$ has exceptional eigenvalue 1, with surplus dimension at least one. We could perform a similar analysis, but it is easier to appeal to the fact that Ω is connected, which implies that $H^0(\Omega)$ must have dimension one. Thus the eigenvalue 1 has surplus dimension at most one in $H^1[1, \cdot, \cdot]$.

There are no other exceptional eigenvalues, so we can deduce $\check{H}^n(\Omega)$.

Theorem 7.4. *As a $\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$ representation, $\check{H}^0(\Omega)$, $\check{H}^1(\Omega)$, and $\check{H}^n(\Omega)$ are as follows. In what follows β always acts as multiplication by z.*

$$\check{H}^0(\Omega) = \frac{\mathbb{C}[z]}{(z-1)} \tag{7.73}$$

where $\pi/2$ (and also β) act trivially.

$$\check{H}^1(\Omega) = \frac{\mathbb{C}[z]}{(z-\lambda)} \oplus \frac{\mathbb{C}[z]}{(z-\lambda^{-1})} \tag{7.74}$$

where $\lambda = (3+4i)/5$. In the first summand $\pi/2$ acts as multiplication by i, and in the second summand $\pi/2$ acts as multiplication by $-i$. More to the point, $\check{H}^1(\Omega) = A \oplus \bar{A}$, where A and $\bar{A}$ are one dimensional and the angle θ acts as multiplication by $e^{i\theta}$ on A, and as multiplication by $e^{-i\theta}$ on $\bar{A}$.

Finally,

$$\check{H}^2(\Omega) = \mathbb{C}[[z, z^{-1}]]^8 \oplus \mathbb{C}[[z, z^{-1}]]^8 \oplus \mathbb{C}[[z, z^{-1}]]^2 \oplus \mathbb{C}[[z, z^{-1}]]^2 \tag{7.75}$$

where $(\pi/2)$ acts as $1, -1, i$ and $-i$ on the first, second, third, and fourth summands, respectively.

7.2.4 A brief discussion of the meaning of $\check{H}^*(\Omega)$

We originally defined pattern equivariant cohomology of a tiling of the plane as the deRham cohomology restricted to pattern equivariant differential forms. With that in mind, certain cohomology classes are apparent. Any constant function is a closed, non-exact zero-form, and a constant function is rotation invariant, so the space of constant functions are a ρ-invariant when ρ is the trivial representation. This exactly accounts for our $\check{H}^0(\Omega)$.

Next, we see that dx and dy are closed, non-exact pattern equivariant one forms. The span of dx and dy is mapped to itself by any rotation. Rotation by θ acts on a one form $a_1dx + a_2dy$ as follows

$$\begin{aligned}\theta(a_1dx + a_2dy) &= (\theta^{-1})^*(a_1dx + a_2dy) \\ &= (a_1\cos\theta - a_2\sin\theta)dx + (a_1\sin\theta + a_2\cos\theta)dy. \quad (7.76)\end{aligned}$$

Using dx, dy as a basis, we can just write

$$\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}. \quad (7.77)$$

In terms of representations, this means that span of dx and dy is a space of ρ-equivariant one forms, where ρ is the two dimensional representation in which the angle $\theta \in \mathcal{P}$ acts via the matrix (7.77). But we can diagonalize this matrix, which shows that this representation can be decomposed into two subrepresentations A and $\bar{A}$. A is the span of $dx - idy$, and θ acts as multiplication by $e^{i\theta}$. $\bar{A}$ is the span of $dx + idy$, and θ acts as multiplication by $e^{-i\theta}$. In other words, on A, $\pi/2$ acts as multiplication i, and β acts as multiplication by $(3 + 4i)/5$, and on $\bar{A}$, $\pi/2$ acts as multiplication $-i$, and β acts as multiplication by $(3 - 4i)/5$. This is exactly $\check{H}^2(\Omega)$. Notably, and perhaps disappointingly, this accounts for *all* of $\check{H}^1(\Omega)$.

Finally, the area form $dxdy$ is rotation invariant one-form, and thus it shows up in the trivial representation. Of course, there is so much top cohomology, that the area form seems like just another face in a crowd, but it is there.

Chapter 8

Conclusion and Suggestions for Further Research

In this thesis, I answered the question of how the Rand cohomology of a tiling T relates to its total pattern equivariant cohomology, in the case where the rotation group is an abelian group of free rank 1. Informally speaking, when we go to calculate the Rand cohomology $H^k_\rho(T)$ for all irreducible representations ρ, we will find finitely many representations for which $H^k_\rho(T)$ is exceptional (i.e. is of higher dimension than usual). This extra cohomology will be reflected in the total cohomology $H^k(T)$ when it comes from the kernel of the coboundary map δ^k_ρ, but not if it comes from the cokernel of δ^{k-1}_ρ. Theorems 6.4 and 6.5 state the results precisely. In chapter 7, we calculated the Rand cohomology of the pinwheel tiling space and used our results to calculate the total cohomology.

I can see several possible avenues of further research that might deepen and extend the results and concepts from this thesis. As I see it, the ideas for further research fall into two different areas.

A first area of ideas is to extend the results algebraically. One way to do this is to attempt to work over the integers (or other well chosen finite extensions of the integers, like $\mathbb{Z}[1/p]$), instead of the complex numbers. When we calculated the cohomology of the equithirds tiling, we saw factors like $\mathbb{Z}[1/3]$ occurring in the cohomology. Finite groups like $\mathbb{Z}/2\mathbb{Z}$ can also show up in the integer cohomology of tiling spaces. These groups always tell us something interesting about the underlying tilings. By working over $\mathbb{C}$, we were able to appeal to some convenient algebraic facts (e.g. irreducible representations are always one dimensional, $\mathbb{C}[z, z^{-1}]$ is a PID). The price that we pay for this approach is that divisibility factors like $\mathbb{Z}[1/3]$ become just $\mathbb{C}$, and torsion factors like $\mathbb{Z}/2\mathbb{Z}$ vanish, so some interesting information is lost.

Another way to extend the results algebraically is to work with more general rotation groups. Why limit ourselves to tilings whose rotation group is an abelian group of free rank 1? Radin and Conway [6] invented a three-dimensional aperiodic tilings with infinite, non-abelian rotation group, called the "quaquaversal" tiling. Dirk Frettloeh showed me an example of a two-dimensional tiling, which he dubbed "Uber-pinwheel", whose rotation group is $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$. In this thesis, we were able to view the chain complex of the Barge-Diamond complex as a finitely generated module over the PID $\mathbb{C}[z, z^{-1}]$, the theory of which is easily accessible. In these other examples, we cannot appeal to such an easily accessible theory.

In chapter 3, we briefly mentioned that some of the tilings (one of which was illustrated in figure 3.1) in Ω do not have tiles in all rotations allowed by the the pinwheel group. Letting T' be such a tiling, the tiling space $X_{\mathbb{R}^2}^{T'}$ does not admit an action by the pinwheel group $\mathcal{P}$. $X_{\mathbb{R}^2}^{T'}$ *does* admit an action by the semigroup $\mathbb{N} \times (\mathbb{Z}/4\mathbb{Z})$. How can we generalize our results to handle the situation in which we have a rotation *semigroup*, instead of a rotation group?

This last example is a partial segue into a second area of ideas for further research, which is admittedly somewhat vague at this point. As we discussed in chapter three, the pinwheel tiling T is rotationally FLC, but not translationally FLC. Finite local complexity is a nice property. Among other things, it means that the corresponding tiling space is compact. So, $\Omega = X_{\mathbb{R}^2}^{T}$, the translational orbit closure of a pinwheel tiling, is not compact, while $X_{E(2)}^{T}$, the euclidean orbit closure of a pinwheel tiling, is compact. For this reason, tiling space theorists have often preferred to study $X_{E(2)}^{T}$.

However, in order to answer the question, "What does Rand cohomology tell us about a tiling with infinite rotation group like the pinwheel," we must study $X_{\mathbb{R}^2}^{T}$, the translational orbit closure. Otherwise, the action of the pinwheel group $\mathcal{P}$ is nasty (i.e. nowhere properly discontinuous). So, I openly embrace the non-compact pinwheel tiling space because I value an easy-to-use group action more than I value the compactness of the underlying space. This idea of trading compactness for added algebraic structure bears some resemblance to the approach of associating a non-commutative C^*-algebra to a compact tiling space ([14]).

The open-ended question is: Can this approach bear fruit for different types of non-FLC tilings? Infinite rotation groups are not the only way to break finite local complexity. Certain substitutions can create tilings in which the tiles do not meet full edge to full edge, and in fact a tile of type A can meet a tile of type B in infinitely many different ways (see [13]), which obviously means that the tilings are not FLC. The principle strategy for studying these tilings has been to introduce a new topology, in which the associated tiling space is compact. This thesis points suggests an alternative approach: If T is one of these tilings, we can study the usual translational orbit closure of $X^T_{\mathbb{R}^2}$ (which is noncompact) while trying to exploit the natural algebraic structures that act on $X^T_{\mathbb{R}^2}$. Having illustrated that Rand cohomology can yield information about non-FLC tilings, I believe it would be worthwhile to attempt to construct something analogous to Rand cohomology in this context, and to use it to gain information about $X^T_{\mathbb{R}^2}$ and T.

In summary, the success of Rand cohomology in helping us understand the noncompact pinwheel tiling space Ω should motivate several avenues of further research.

Bibliography

[1] Jared E. Anderson and Ian F. Putnam. Topological invariants for substitution tilings and their associated C^*-algebras. *Ergodic Theory Dynam. Systems*, 18(3):509–537, 1998.

[2] Marcy Barge and Beverly Diamond. Cohomology in one-dimensional substitution tiling spaces. *Proc. Amer. Math. Soc.*, 136(6):2183–2191, 2008.

[3] Marcy Barge, Beverly Diamond, John Hunton, and Lorenzo Sadun. Cohomology of substitution tiling spaces. *Ergodic Theory and Dynamical Systems*, First View:1–21, 2009.

[4] Daniel Berend and Charles Radin. Are there chaotic tilings? *Comm. Math. Phys.*, 152(2):215–219, 1993.

[5] Raoul Bott and Loring W. Tu. *Differential forms in algebraic topology*, volume 82 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1982.

[6] John H. Conway and Charles Radin. Quaquaversal tilings and rotations. *Invent. Math.*, 132(1):179–188, 1998.

[7] David S. Dummit and Richard M. Foote. *Abstract algebra*. John Wiley & Sons Inc., Hoboken, NJ, third edition, 2004.

[8] Allen Hatcher. *Algebraic topology*. Cambridge University Press, Cambridge, 2002.

[9] Johannes Kellendonk. Noncommutative geometry of tilings and gap-labelling. *Rev. Math. Phys.*, 7:1133–1180, 1995.

[10] Johannes Kellendonk. Pattern-equivariant functions and cohomology. *J. Phys. A*, 36(21):5765–5772, 2003.

[11] Johannes Kellendonk and Ian F. Putnam. The Ruelle-Sullivan map for actions of $\mathbb{R}^n$. *Math. Ann.*, 334(3):693–711, 2006.

[12] B. Mossé. Puissances de mots et reconnaissabilité des point fixes d'une substitution. *Theor. Comput. Sci.*, 99:327–334, 1992.

[13] Natalie Priebe Frank and Lorenzo Sadun. Topology of some tiling spaces without finite local complexity. *Discrete Contin. Dyn. Syst.*, 23(3):847–865, 2009.

[14] Ian Putnam and Johannes Kellendonk. *Tilings, C*-algebras and K-theory*. CRM Monograph Series, American Mathematical Society, Providence, RI, 2000.

[15] Charles Radin. The pinwheel tilings of the plane. *Ann. of Math. (2)*, 139(3):661–702, 1994.

[16] Betseygail Rand. *Pattern equivariant cohomology of tilings spaces with rotations*. PhD thesis, University of Texas at Austin, 2006.

[17] Lorenzo Sadun. Pattern-equivariant cohomology with integer coefficients. *Ergodic Theory Dynam. Systems*, 27(6):1991–1998, 2007.

[18] Lorenzo Sadun. *Topology of tiling spaces*, volume 46 of *University Lecture Series*. American Mathematical Society, Providence, RI, 2008.

[19] Boris Solomyak. Nonperiodicity implies unique decomposition for self-similar translationally finite tilings. *Discrete and Computational Geometry*, 20:265–279, 1998.