Acoustic Radiation Force on a Sphere in Tissue

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Abstract. A theory is presented for the acoustic radiation force on a sphere embedded in a soft elastic medium that possesses a shear modulus several orders of magnitude smaller than its bulk modulus. Scattering of both compressional and shear waves is taken into account. There is no restriction on the size of the sphere or, apart from symmetry about an axis passing through the center of the sphere, the form of the incident compressional wave. If the medium is a liquid and the sphere is small in comparison with a wavelength the result reduces to the classical theory developed by Gor'kov [1] and others. The effect of shear elasticity on the radiation force is discussed in the context of parameters relevant to a gas bubble in tissue.

Keywords: acoustic radiation force, gas bubble, tissue
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INTRODUCTION

Theory for acoustic radiation force on a particle in liquid is well established; see, for example, the review by Wang and Lee [2]. Presented here is a theory for the radiation force on a sphere of arbitrary size, embedded in a soft, homogeneous, isotropic elastic medium. By "soft" is meant a medium characterized by a shear modulus that is several orders of magnitude smaller than the bulk modulus, which is the case for soft tissue. The calculation presented here differs from previous analyses for liquids in that Lagrangian coordinates are used. Care must be taken when calculating the radiation force in Lagrangian coordinates to remove elements of the stress tensor that are associated only with volume potential forces. The effect of shear elasticity on the radiation force is evaluated for parameters corresponding to a gas bubble in tissue. For a small sphere in liquid, classical results obtained by Gor'kov [1] and others are recovered.

SOLUTION PROCEDURE

The incident and scattered compressional waves, and the scattered shear wave, are defined by the scalar potential functions ψ and Π, respectively, such that the particle displacement in the elastic medium surrounding the sphere is given by $\mathbf{u} = \nabla\psi + \nabla\times\nabla\times(\mathbf{r}\Pi)$, where $\mathbf{r}$ is position relative to the center of the sphere. The potentials for harmonic waves with angular frequency ω are expressed in spherical coordinates as

$$\psi = \frac{1}{2}e^{-i\omega t}\sum_{n=0}^{\infty} a_n[j_n(kr) + A_n h_n(kr)]P_n(\cos\theta) + \text{c.c.} \tag{1}$$

$$\Pi = \frac{1}{2}e^{-i\omega t}\sum_{n=1}^{\infty} a_n B_n h_n(\kappa r)P_n(\cos\theta) + \text{c.c.} \tag{2}$$

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where j_n and $h_n(\equiv h_n^{(1)})$ are the spherical Bessel and Hankel functions, respectively, P_n the Legendre polynomials, $k = \omega/c_l$ and $\kappa = \omega/c_t$ the wave numbers of the compressional and shear waves having propagation speeds c_l and c_t, respectively, a_n the amplitude coefficients of the incident compressional wave, and A_n and B_n the relative amplitude coefficients of the scattered compressional and shear waves, respectively (a_n has dimensions of length squared, whereas A_n and B_n are dimensionless). Shear viscosity, thermal conductivity, and hysteretic behavior that characterize biological media are not considered here. The sphere itself is elastic with similar potentials, not presented here, describing the field inside.

The starting point for the calculation is the momentum equation expressed in Lagrangian coordinates as $\rho_0 \ddot{u}_k = \partial\sigma_{kl}/\partial x_l$, where the dots indicate time derivatives, ρ_0 is the density of the undeformed medium, and σ_{kl} is the Piola-Kirchhoff pseudostress tensor. In Lagrangian coordinates the surface of the sphere is fixed in the reference frame, and all nonlinearity is thus confined to the stress tensor, which is related to the elastic energy density $\mathscr{E}$ as $\sigma_{kl} = \partial\mathscr{E}/\partial(\partial u_k/\partial u_l)$. The formulation of $\mathscr{E}$ due to Landau and Lifshitz [3] is employed, in which the second-order elastic constants are the bulk modulus K and shear modulus μ, and the third-order elastic constants are A, B, and C.

The focus of the present work is on the acoustic radiation force exerted on a sphere in a tissue-like medium. For soft tissue the ratio μ/K is in the range of 10^{-4} to 10^{-6}, and measurements reported by Catheline et al. [4] reveal that $A = O(\mu)$ and $B = O(K)$. Under these conditions, with the further recognition that $K + B = O(\mu)$ [5], the expression for the stress tensor through quadratic order in the particle displacement reduces to

$$\sigma_{kl} = \mu\left(\frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k}\right) + (K - 2\mu/3)\frac{\partial u_m}{\partial x_m}\delta_{kl} - K\frac{\partial u_m}{\partial x_m}\frac{\partial u_l}{\partial x_k} \tag{3}$$

Omitted from this expression are terms that correspond to the volume potential force produced in the elastic medium by the scattered compressional wave. The volume potential force is not needed in the calculation of the desired surface force on the sphere.

Time averaging eliminates the linear terms, and the remaining term is labeled $\sigma'_{kl} = -K\langle(\partial u_m/\partial x_m)(\partial u_l/\partial x_k)\rangle$, where $\langle\cdots\rangle$ represents time averaging. The quantity σ'_{kl} also contains a component associated with the volume force described above, as well as the volume force due to the scattered shear wave. The former is denoted σ''_{kl}, which can be determined and subtracted from σ'_{kl} when calculating the surface force on the sphere. This volume force is $f_k = \partial\sigma_{kl}/\partial x_l$, which can be expressed as the gradient of a scalar potential function P as $f_k = -\partial P/\partial x_k$, where $P = \frac{1}{2}Kk^2[k^2\psi^2 - (\partial\psi/\partial x_m)^2]$. With $\sigma''_{kl} = -\langle P\rangle\delta_{kl}$, the radiation force in the x_k direction on a sphere of radius R is

$$F_k = \int_S (\sigma'_{kl} - \sigma''_{kl})\frac{x_l}{R}\,dS \tag{4}$$

where S is the surface of the sphere. Because of symmetry, only the axial force F_z is nonzero. Linear theory is used to calculate the quadratic terms in σ'_{kl} and σ''_{kl}.

RESULTS

It is convenient to express the radiation force as $F_z = F_{z1} + F_{z2}$, where F_{z1} involves the scattering of only compressional waves, whereas F_{z2} involves the scattering of both compressional and shear waves. Since for liquids $F_{z2} = 0$, one may anticipate that F_{z1}

provides the dominant contribution in soft tissue. From Eq. (4) is obtained

$$F_{z1} = i\pi K k^2 \sum_{n=0}^{\infty} \frac{n+1}{(2n+1)(2n+3)} a_n^* a_{n+1} (A_n^* + A_{n+1} + 2A_n^* A_{n+1}) + \text{c.c.} \tag{5}$$

$$\begin{aligned} F_{z2} = -\pi K k^2 \sum_{n=1}^{\infty} \frac{(\kappa R) h_n(\kappa R)}{2n+1} [j_n(kR) + A_n^* h_n^*(kR)] \\ \times a_n^* \left[\frac{n(n+1)(n+2)}{2n+3} a_{n+1} B_{n+1} - \frac{(n-1)n(n+1)}{2n-1} a_{n-1} B_{n-1} \right] + \text{c.c.} \end{aligned} \tag{6}$$

These expressions are valid for all kR and κR, and it may be observed that F_{z2} vanishes in an oscillatory fashion as kR and κR tend toward infinity. The incident wave coefficients a_n are prescribed and the scattered wave coefficients A_n and B_n, which are functions of kR and κR, are calculated by solving the linear boundary value problem [6].

Considered first is F_{z1} for a small sphere ($kR \ll 1$). In this case it is sufficient to retain only the first term in the summation in Eq. (5), such that

$$F_{z1} = a_0^* a_1 (A_0^* + A_1 + 2A_0^* A_1) \frac{i\pi}{3} K k^2 + \text{c.c.} \tag{7}$$

If the incident field is the travelling compressional plane wave $\psi = \frac{1}{2}\psi_0 e^{i(kz-\omega t)} + \text{c.c.}$, for example, then $a_n = i^n (2n+1)\psi_0$. Regardless of the choice of incident field, for $kR \ll 1$ the coefficients for the monopole (A_0) and dipole (A_1) scattering terms are

$$A_0 = \frac{i}{3}(kR)^3 \left(\frac{K}{K_p} - 1 \right) \left[\left(1 + \frac{4}{3}\frac{\mu}{K_p} \right) - \frac{\omega^2}{\omega_R^2} - \frac{i}{3}(kR)^3 \left(\frac{K}{K_p} - 1 \right) \right]^{-1} \tag{8}$$

$$A_1 = -\frac{i}{3}(kR)^3 \frac{\rho - \rho_p}{\rho + 2\rho_p} \left[1 + \frac{i}{3}(kR)^3 \frac{\rho - \rho_p}{\rho + 2\rho_p} \right]^{-1} \tag{9}$$

where K_p and ρ_p are the bulk modulus and density of the sphere (particle), respectively, and $\omega_R = (3K_p/\rho R^2)^{1/2}$ is the natural frequency of the sphere in the absence of shear forces. For $\mu = 0$, substitution of the expressions for the incident and scattered wave coefficients in Eq. (7) recovers exactly the result obtained by Gor'kov [1].

To evaluate the effect of finite shear modulus it is helpful to simplify Eq. (8) for the case of a gas bubble $(K/K_p \gg 1,\ \rho/\rho_p \gg 1)$ driven below resonance $(\omega^2/\omega_R^2 \ll 1)$ and ignore radiation damping, which is associated with the last term inside the square brackets. With $K_p = \gamma P_0$, where γ is the ratio of specific heats for the gas and P_0 is atmospheric pressure, Eq. (8) reduces to

$$A_0 \simeq \frac{i}{3}(kR)^3 \frac{K}{\gamma P_0} \left(1 + \frac{4}{3}\frac{\mu}{\gamma P_0} \right)^{-1} \tag{10}$$

Since $\rho \gg \rho_p$ the magnitude of A_0/A_1 is of order $K/\gamma P_0 \sim 10^4$, so the contribution from the dipole term is negligible, and F_{z1} is thus proportional to A_0. Reported measurements of shear moduli for soft tissue have values ranging from on the order of 1 kPa on the low end (breast, liver) up to 70 kPa on the high end (cancerous tissue) [7]. Taking $\mu = 70$ kPa, $P_0 = 100$ kPa, and $\gamma = 1.4$ yields $\mu/\gamma P_0 = 0.5$, for which F_{z1} is 40% less than the value predicted for a gas bubble in liquid.

To assess the contribution due to F_{z2} only the lowest-order scattering of the shear wave is considered here. Since there is no monopole scattering of a shear wave by the sphere $(B_0 = 0)$, the lowest-order contribution is dipole scattering (B_1). With $B_n = 0$ for $n \neq 1$ Eq. (6) becomes

$$F_{z2} = a_1 a_2^* B_1 \frac{2\pi}{5} K k^2 (\kappa R) h_2(\kappa R)[j_2(kR) + A_2^* h_2^*(kR)] + \text{c.c.} \tag{11}$$

At leading order F_{z2} thus requires participation of the quadrupole component of the incident wave field (a_2), whereas the highest-order multipole required by F_{z1} is the dipole component (a_1).

SUMMARY

The theory presented here provides the radiation force on an isotropic elastic sphere embedded in a soft elastic medium due to a compressional wave incident on the sphere. The incident field is assumed axisymmetric but is otherwise arbitrary, and there is no restriction on the size of the sphere. The force is separated into two components, one that is independent of the scattered shear wave (F_{z1}), and another that depends on the scattered shear wave as well as the scattered compressional wave (F_{z2}). For small spheres F_{z2} is negligible, and for F_{z1} it is sufficient to use Eqs. (7)–(9), which show explicitly the contribution due to shear elasticity in the medium surrounding the particle. For large spheres both F_{z1} and F_{z2} contribute to the radiation force, and it is important to note a distinction between the two contributions. The removal of the volume potential force associated with the scattered compressional wave results in an expression for F_{z1} that is the desired force, i.e., that which produces finite displacement of the sphere. In contrast, the volume force associated with the scattered shear wave is included in F_{z2}, so while F_{z2} is the true force acting on a stationary sphere, it is not the force that produces finite displacement of the sphere. One possible avenue for future work is to calculate the influence of the volume force on displacement of the sphere.

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