Copyright
by
Chethan Krishnan
2006

The Dissertation Committee for Chethan Krishnan
certifies that this is the approved version of the following dissertation:

Aspects of Cosmology and Quantum Gravity in an Accelerating Universe

Committee:

Willy Fischler, Supervisor

Jacques Distler

Sonia Paban

Steven Weinberg

Lorenzo Sadun

Aspects of Cosmology and Quantum Gravity in an Accelerating Universe

by

Chethan Krishnan, B.Tech.

DISSERTATION

Presented to the Faculty of the Graduate School of

The University of Texas at Austin

in Partial Fulfillment

of the Requirements

for the Degree of

DOCTOR OF PHILOSOPHY

THE UNIVERSITY OF TEXAS AT AUSTIN

August 2006

Acknowledgments

I was extremely fortunate to have Willy as my adviser, and I wish to thank him here for that. His way of focusing on what is ultimately important as opposed to what is cosmetic, has made a crucial impact on my thinking. Doing physics with him and Sonia was an entertaining and enlightening experience. My "theory friends" -Marija, Jeff, Uday, Aron, Marcus, Steve, the Germans, the Italians- need to be mentioned for the many interesting conversations and also for tolerating my loudness. I would also like to thank Hyuk-Jae, who taught me a lot in my beginning years. It is a tribute to human sociology that people *far* less capable than him seem to be able to stay on as "string theorists". Finally, I want to thank Jan Duffy for being a paragon of patience and affection, despite my relentless efforts to test her by missing the registration deadline year after year.

There were some other major influences in my life outside of RLM in these last few years. Resmi, Rao, Anoop, Anand and Vishal were my home away from home. Marianne, Chaturi, Nikki, Shannon, Alessandro and Erik have kept me from going insane more than once, even though they might not have realized it at the time. Finally, I also want to mention the friends, enemies and lovers that I made through my ritualistic expeditions to Sixth street during my unofficial year-and-a-half break from physics. The world outside the ivory

tower is not without its own challenges, attractions, lessons and rewards.

I would not try to belittle the influence that my parents, especially my father, had on me, by thanking them here. Somewhere in the same category is my brother, Nikhil. This thesis is dedicated to the memory of Bryce DeWitt, my late co-advisor, the man who was thinking about quantum gravity long before it became fashionable. It was a real pleasure listening to him talk about physics, politics, the War, and more during our ten-day, one-on-one hike through the Californian Sierra.

Aspects of Cosmology and Quantum Gravity in an Accelerating Universe

Publication No. ________________

Chethan Krishnan, Ph.D.
The University of Texas at Austin, 2006

Supervisor: Willy Fischler

The observation that we are living in a Universe that is expanding at an ever-increasing rate is a major challenge for any fundamental theory. The most obvious explanation for an accelerating Universe is a positive cosmological constant, but we do not really know how to do quantum field theory or string theory in spacetimes that are not asymptotically flat. In this thesis, we address various issues that arise in this general context. The problems we address include the stability and evolution of de Sitter-like compactifications, the possibility of defining a quantum theory in de Sitter space using quantum groups, and finally, the classical evolution of thin shells (boundaries of new phase bubbles) in an inhomogeneous Universe with positive Λ.

Table of Contents

List of Figures

Chapter 1

A Small Positive Cosmological Constant

Over the last few years, significant observational evidence has accumulated showing that our Universe is accelerating. The most natural way to incorporate this phenomenon into classical general relativity is to think of it as arising from a small positive cosmological constant. But if we adopt this point of view, it gives rise to some serious difficulties when we try to put this in the framework of a fundamental theory.

This is the context of this thesis. The rest of this chapter is a rather glib overview of some of these issues, along with an outline of the work.

From a quantum field theory perspective, the cosmological constant arises from the vacuum energy contributions of the fields in the theory. A straightforward calculation shows that the vacuum energies of all the known kinds of particles (in cut-off field theory) gives rise to an answer that is many orders of magnitude different from what is observed. (The precise answer depends on the choice of the cut-off.) We *can* accommodate this discrepancy, but only by fine-tuning the bare cosmological constant to an embarrassing number of decimals. This, is the Cosmological Constant Problem [1].

In string theory, it seems plausible that there are a tremendous number of solutions that look roughly like our Universe, but it also seems forbiddingly hard to construct a specific solution that is *exactly* like ours. This latter problem is expected to be only(!) a technical issue, and we believe in fact that there are a large number of (metastable) string vacua that could be candidates for the vacuum on which our Universe is built. Along with these, there is also a "discretuum" of vacua which gives rise to different values for certain "fundamental" parameters like the cosmological constant. There is a certain democracy between all these huge number of vacua in string theory, and so in this picture, the value of the cosmological constant might just be an anthropically determined parameter. This is the way in which string theory solves the Cosmological Constant Problem: by shooting a million arrows at the bull's eye. Of course this begs the question: is this lack of predictivity[1] (about the values certain fundamental constants) a feature of quantum gravity, or a failure of string theory as a theory of quantum gravity?

This is such an interesting question, that I would like to spend the next few paragraphs presenting the reasons why I believe it is worth doing string theory. The arguments are not very convincing, so the businesslike reader might wish to skip ahead.

To start off, the question of quantizing pure gravity is an unclear one. There is no genuine free theory limit for gravity where one can start off by quantizing just gravity. Even if we could write down an action for a "final

[1]The same lack of predictivity that forces even self-respecting physicists to go anthropic.

theory", the $\sqrt{-g}$ that multiplies every piece of the Lagrangian density means that gravity couples directly to all forms of energy, including its own. So it is not clear how one can decouple *just* matter from the theory by going to some energy scale. This should be contrasted to the case of QED or QCD where there are energy scales at which a free field theory is an acceptable starting point for quantization. So quantizing pure gravity and then adding matter is likely to be a pointless enterprise, even though it might be instructive. This means that to even hope for quantizing gravity along the lines familiar from field theory, and to hope (perhaps) that there might be a non-trivial ultraviolet fixed point where the non-renormalizability is not a problem, we either have to wait until we know all the possible matter content from here to Planck scale (and beyond!), or give up in despair about our ignorance[2].

This of course is not a *reason* to take string theory seriously. If we are stymied by our ignorance about high energy physics in our quest for the quantization of gravity, then that is our own little problem: Nature is not obligated to be kind to us. But string theory does give gravity a satisfying UV completion (instead of the possibility of adding arbitrary stuff at all scales), while still being perfectly capable of producing new physics at higher energies. Also, despite the well-known difficulties of constructing a consistent quantum theory that contains a spin-2 graviton, the situation in string theory is that we cannot *not* have a graviton. There is also the fact that string theories are

[2]It is an interesting question whether we can constrain the matter content allowed in a quantum gravity by stipulating that it *have* a non-Gaussian fixed point.

dual to gauge theories: the holographic reduction in degrees of freedom that one expects in gravity (from horizon-entropy arguments and such) is explicitly realized in AdS/CFT. Then there are the black hole entropy calculations done from the string theory side which agree amazingly well. The bottomline is that in all the situations that we have string theory under computational control, it has exhibited the appropriate features of quantum gravity. It has also revealed very interesting connections between the fundamental mathematical structures that arise in theoretical physics, like (say) gauge theory and gravity. So it seems reasonable that one should pursue string theory, even if it ultimately turns out to be the meta-theory of all possible quantum gravities (whatever that means), and not *the* quantum gravity[3].

Back to the main line of this thesis, and onward with the cosmological constant!

The cosmological constant is also a thorny problem when one tries to define field theory or string theory in terms of an S-matrix. The difficulty here is that an S-matrix is defined in terms of scattering states and these are usually defined in an asymptotically flat spacetime. On the other hand, when there is a positive cosmological constant, the spacetime is (with some qualifications) asymptotically de Sitter, and has a horizon. So, at least naively, defining the S-matrix is a difficult proposition.

The existence of the horizon has also been taken as an indication that

[3]Even the "landscape" and the associated anthropic questions might be a reflection of this problem.

the entropy of de Sitter is finite. One argument for this is based on holography, which is the idea that there is a dimensional reduction in quantum gravity. If we believe that the horizon area/entropy is the entropy of de Sitter, then the finiteness of the horizon area suggests that the de Sitter Hilbert space is finite dimensional. Then we have to confront the problem of defining a unitary quantum theory such that the Hilbert space is a finite dimensional representation of the de Sitter isometry group. This is difficult because $SO(4,1)$, the isometry group of de Sitter, is non-compact and does not allow unitary finite dimensional representations.

Another angle on this problem is along the lines of the recent landscape-type constructions. One allows the possibility that de Sitter is a long-lived resonance or a meta-stable state in quantum gravity. Indeed, in string theory constructions [6], the vacuum energy arises as the (local) minimum for the potential for some scalar field, and such vacua are unstable through tunneling. Just as the Coulomb barrier is enough to give most elements a long lifetime, these constructions can in-principle give cosmologies with long enough lifetimes.

In this thesis, we consider some of these problems. The first issue that we look at is the classical stability of de Sitter-like compactifications. We look at Freund-Rubin-like direct product spacetimes ($dS_p \times S^q$) with fluxes. We do a classical stability analysis and reproduce the results of [3, 4] to determine the instability modes. After determining these modes we try an ansatz that incorporates the radial instability mode, and do a numerical evolution. We see that for some values of the flux, the solution flows to another, stable Freund-Rubin configuration, but for others, the sphere decompactifies.

Next, we turn to the question of defining quantum theory in de Sitter space. As I have already mentioned, it is not clear *a priori* how one could get finite dimensional unitary representations of the $SO(4,1)$ group, because it is non-compact. One possible modification that one could try is to construct a q-deformed version of the de Sitter group, as has been proposed in the literature (Lowe et al.). It is known that for certain values of the parameter q, quantum groups admit finite-dimensional representations, and this was the ultimate philosophy behind pursuing this line of thought. But as a result of our analysis, we find that for the standard one parameter q-deformation, the values of q which give rise to unitary finite dimensional representations, do not lead to an underlying (deformed) de Sitter space. This is a consequence of the interplay between the quantum group, and the underlying space on which it is defined. We have not bothered to extend this result to multi-parametric deformations, instead we merely write down the multi-parametric quantum algebra (the so-called Drinfeld-Jimbo algebra) for the complexified version of $SO(5)$. As far as I know, multi-parametric algebras (which are presumably more useful to the physicist than the corresponding multi-parametric groups, if at all these things ultimately turn out to be useful at all) are not available in the literature. One possible relevance of multi-parametric algebras might be in embedding the static patch of the observer in the full Hilbert space of de Sitter. The technical aspects of this paragraph will be clarified in detail in a later chapter.

One of the questions that could be asked in the context of a landscape picture for the cosmological constant is the possibility of tunneling between vacua. Before elaborating on it, we must mention one major caveat that goes into any of the landscape arguments. The basic strategy in all these constructions is to take a 10-D string theory compactified on a Calabi-Yau orientifold, which gives rise to an $N = 1$, $D = 4$ supergravity as the low energy effective theory. The crucial point about this effective field theory is that it arose as an expansion around flat space. But now, we look for other solutions of this effective theory, which are *not* asymptotically flat. There have been arguments in the literature [7] which indicate that this bootstrapping from the effective field theory vacua to the true vacua of string theory, might not be entirely *kosher*: the potential barriers between vacua could be infinitely high, the vacua might belong to different Hamiltonians, etc. But we will cavalierly ignore those issues. If everything goes perfectly, our aim (which is more bottom-up in flavor), will be to see whether there are observable signatures of the landscape in our Universe. That the tunneling between vacua might leave an observable imprint, perhaps in the CMB spectrum, is not a manifestly impossible scenario.

That is the ultimate goal, but we will not get that far in this work. We will start by setting up the problem of evolution of a bubble in an inhomogeneous universe as a problem in general relativity. As a start, the question we pose is: how is the evolution of the tunneling bubble affected if there are inhomogeneities in the old Universe? This is a question that is of relevance in understanding whether inflation can prevail even in the presence of inhomogeneities. The problem is intractable in the general case, so we restrict it to a case of spherical symmetry. We will be working with the only radially inhomogeneous, isotropic cosmology that I know of - one that was used by Peebles in one of his old papers and which I will refer to as the Peebles metric. With this metric, tracking the bubble evolution using the Israel junction conditions will turn out to be too difficult for analytical methods after a stage. So then, we will resort to numerical techniques. In doing the bubble evolution, we need to make a choice of energy-momentum tensor on the shell, and ideally we would like to derive that on field theory grounds because the bubble is the interface between two phases. But because it is not vacuum on both sides of the shell, this is a difficult thing to do. Instead, for now, we push the formalism all the way and try to get an idea about the solution space for the bubble evolution with certain artificial choices of energy-momentum tensor on the bubble.

Chapter 2

Evolution of Unstable de Sitter Spacetimes

2.1 Introduction

It has been known since the work of Freund and Rubin [2] that when an antisymmetric tensor field of dimension $q-1$ is added to gravity (Einstein-Hilbert action), space-times of dimension $p+q$ can naturally compactify to product spaces of the form $dS_p \times S^q$.

Bousso, DeWolfe and Myers [4] derived analogous results when a positive cosmological constant is added to the gravitational action, and studied the stability of these compactifications under gravitational perturbations. Their analysis showed the stability to depend on the relative value of the flux compared to the cosmological constant as well as on the dimension of the internal space.

The unstable modes, at these compactifications, were identified in [4], generalizing the work of [3]. Here, we try to track the evolution of these instabilities. In doing so we will make the assumption that these remain the only unstable modes throughout the evolution. A similar analysis, in the absence of flux, was recently done by Contaldi, Kofman and Peloso [5].

This paper answers the questions about the fate of these configurations that was posed in the original paper of Bousso, DeWolfe and Myers. It is unclear to what extent the lessons learned here translate into more realistic models, e.g. [6, 8]. In fact, Giddings and Myers [15] have already studied these types of models and argued that positive vacuum energy together with extra dimensions render the four-dimensional space unstable toward decompatification of the extra dimensions. Our study is purely classical and does not incorporate the effect of thermal fluctuations or tunneling.

After this work was completed, we learned that the time evolution of these configurations had already been studied in the past [9, 10]. Our results present a more complete analysis of all the initial conditions and agree with them when these conditions overlap. The goal of these earlier papers was to use the unstable mode as the inflaton mode.

In section 2 we present a summary of the solution described in [4] to establish notation and state the problem. The classical perturbation theory and the resulting mode equations are in an appendix, but the stable parameter values for that system are presented as a table here. This is mostly a rehashing of old work, made more specific for our problem. In section 3, we display a more general solution of the equations of motion that reduces to the solutions found in [4] and hence can interpolate between an unstable initial configuration and a stable final state. In section 4, we will present the time evolution that results from the numerical analysis.

2.2 Product Spacetimes with Flux

We will consider solutions of the following action;

$$S = \frac{1}{2}\int d^{p+q}x\sqrt{-g}(R - 2\Lambda - \frac{1}{2q!}F_{P_1\ldots P_q}F^{P_1\ldots P_q}) \tag{2.1}$$

in units where M_P in $p+q$ dimensions has been set equal to 1. The cosmological constant Λ will be assumed positive. $F_{P_1\ldots P_q}$ is a totally antisymmetric tensor of rank q. Throughout this paper we adopt the notation that upper case Latin indices run from 1 to $p+q$, Greek indices run from 1 to p and lower case Latin indices run from 1 to q. The equations of motion are

$$G_{MN} = \frac{1}{2(q-1)!}F_{MP_1\ldots P_{q-1}}F_N^{P_1\ldots P_{q-1}} - g_{MN}\ (\Lambda + \frac{1}{4q!}F_{P_1\ldots P_q}F^{P_1\ldots P_q}) \tag{2.2}$$

$$\frac{1}{\sqrt{-g}}\partial_M(\sqrt{-g}F^{MP_1\ldots P_{q-1}}) = 0 \tag{2.3}$$

These equations admit a solution that is a product of two spaces, a Lorentzian K_p and a Riemannian S^q:

$$ds^2 = -dt^2 + a(t)^2 d^{p-1}\mathbf{x} + R_0^2 d\Omega_q \tag{2.4}$$

$$F_q = c\,\mathrm{vol}_{S^q} \qquad \oint \mathrm{vol}_{S^q} = R_0^q\,\Omega_q \tag{2.5}$$

To be consistent with the notation of [4] we will parametrize the solutions in terms of

$$\mathcal{F} = \frac{c^2}{4\Lambda}$$

Though $\mathcal{F}$ has been often referred to, in the literature, as the flux, it is clear from (2.5) that the actual flux, with the normalization given above, is also a function of the compactification radius R_0. With a normalization that makes the flux radius independent, $F_q = \frac{c}{R^q(t)} \mathrm{vol}_{S^q}$, there are actually two static solutions, instead of one, for some values of c. The equations of motion for the Lorentzian scale factor, $a(t)$ are:

$$\frac{(p-1)(p-2)}{2}\frac{\dot{a}^2}{a^2} + \frac{(p-1)(p-2)}{2}\frac{k}{a^2} = \Lambda + \Lambda\,\mathcal{F} - \frac{q(q-1)}{2}\frac{1}{R_0^2} \tag{2.6}$$

$k = 1, 0, -1$ corresponds to closed, flat or open spaces. The effective cosmological constant of the p dimensional Lorentzian space, Λ_p, is given by:

$$(p-1)\Lambda_p = \frac{2\Lambda}{p+q-2}\,(1-(q-1)\mathcal{F}) \tag{2.7}$$

Since Λ has been chosen positive, the effective cosmological constant will be positive as long as

$$0 \le \mathcal{F} < \frac{1}{q-1} \tag{2.8}$$

The radius of S^q is given by:

$$\frac{q-1}{R_0^2} = \frac{2\Lambda}{p+q-2}\,(1+(p-1)\mathcal{F}) \tag{2.9}$$

Two other immediate facts that come in handy for the computations in the next section are,

$$R_{\mu\nu} = \frac{2\Lambda}{p+q-2}[1-(q-1)\mathcal{F}]g_{\mu\nu} \tag{2.10}$$

$$R_{\alpha\beta} = \frac{q-1}{R_0^2}g_{\alpha\beta}. \tag{2.11}$$

The early Greek indices are along the internal directions, while the later ones, along K_p. These formulae imply that both subspaces are Einstein spaces.

2.3 Stability Modes of Product Spacetimes with Fluxes

In a later section, we will try an ansatz for the time-evolution of the spacetimes presented above. Our ansatz will be dictated by the structure of the instability modes, so now, we will do a classical small fluctuation analysis of (2.2, 2.3) around the solution (2.4, 2.5). We will closely follow deWolfe et al., except for the slight difference in our case that we have a cosmological constant along with the fluxes. We work with a case that is slightly more general than what we need: we look at an internal manifold M_q that is not necessarily S^q. We start with the direct product spacetime and perturb the corresponding metric-form system. The indices $M, N, ...$ will run over all directions, while $\mu, \nu, ..$ are along the base spacetime and $\alpha, \beta, ...$ are along the internal directions.

2.3.1 Fluctuations

We are interested in studying the stability of linearized fluctuations around the background (2.4, 2.5). We consider the perturbations

$$\delta g_{\mu\nu} = h_{\mu\nu} = H_{\mu\nu} - \frac{1}{p-2} g_{\mu\nu} h^\alpha_\alpha \,, \tag{2.12}$$

$$\delta g_{\mu\alpha} = h_{\mu\alpha} \,, \quad \delta g_{\alpha\beta} = h_{\alpha\beta}, \quad \delta A_{q-1} = a_{q-1} \,, \quad \delta F_q \equiv f_q = da_{q-1} \tag{2.13}$$

where we have defined a standard linearized Weyl shift on $h_{\mu\nu}$ in (2.12), and $F_q = dA_{q-1}$. All of these are functions of all $p+q$ of the spacetime coordinates. Notice that we have allowed (as etiquette demands) for the most general perturbation of the q-form, and now it might have legs in directions outside of the internal manifold. It will be useful to decompose $H_{\mu\nu}$ and $h_{\alpha\beta}$ into trace and traceless parts:

$$H_{\mu\nu} = H_{(\mu\nu)} + \frac{1}{p} g_{\mu\nu} H^{\rho}_{\rho}\,, \qquad h_{\alpha\beta} = h_{(\alpha\beta)} + \frac{1}{q} g_{\alpha\beta} h^{\gamma}_{\gamma}\,, \tag{2.14}$$

where $g^{\mu\nu} H_{(\mu\nu)} = g^{\alpha\beta} h_{(\alpha\beta)} = 0$. To (mostly[1]) fix the internal diffeomorphisms and gauge freedom, we impose the de Donder-type gauge conditions

$$\nabla^{\alpha} h_{(\alpha\beta)} = \nabla^{\alpha} h_{\alpha\mu} = 0\,, \tag{2.15}$$

as well as the Lorentz-type conditions

$$\nabla^{\alpha} a_{\alpha\beta_2\ldots\beta_{q-1}} = \nabla^{\alpha} a_{\alpha\beta_2\ldots\beta_{q-2}\mu} = \cdots = \nabla^{\alpha} a_{\alpha\mu_2\ldots\mu_{q-1}} = 0\,. \tag{2.16}$$

These are consistent choices because we expect to have $p+q$ diffeomorphisms to be fixed by a de Donder condition, and $\frac{(p+q)!}{q!} - \frac{(p+q-1)!}{p!}$ gauge degrees of freedom to be fixed by a generalized Lorentz condition. To see why the latter must be true is a straightforward exercise in combinatorics, the lazy reader is invited to take a look at p.33 of [11].

A generic gauge potential $a_{\alpha_1\ldots\alpha_n\mu_{n+1}\ldots\mu_{q-1}}$, viewed as an n-form on M_q with additional K_p indices, can be expanded as the sum of an exact, a co-exact and a harmonic form on M_q by the Hodge decomposition theorem. The Lorentz conditions (2.16), which state that the form is co-exact, require the

[1] Besides unfixed p-dimensional diffeomorphisms and gauge transformations, extra conformal diffeomorphisms remain on S^q. These are related to the elimination of a $k=1$ mode in the coupled scalar sector.

exact form in the decomposition to vanish, and hence the potentials can be expanded as co-exact forms (curls) and harmonic forms:

$$a_{\beta_1\ldots\beta_n\mu_{n+1}\mu_{q-1}} = \epsilon^{\alpha_1\alpha_2\ldots\alpha_{q-n}}{}_{\beta_1\ldots\beta_n}\nabla_{\alpha_1}b_{\alpha_2\ldots\alpha_{q-n}\mu_{n+1}\ldots\mu_{q-1}} + \beta^{harm}_{\beta_1\ldots\beta_n\mu_{n+1}\mu_{q-1}} \quad (2.17)$$

When the compact space is an S^q there are no nontrivial harmonic forms, but they can appear for other M_q. In a succinct notation, we may write (2.16) and (2.17) as

$$d_q *_q a = 0 \rightarrow a = *_q d_q b + \beta^{harm} \,, \quad (2.18)$$

where d_q and $*_q$ are the exterior derivative and Hodge dual with respect to the M_q space.

With these gauge choices, we may expand the fluctuations in spherical harmonics (M_q is compact) as

$$\begin{aligned}
H_{(\mu\nu)}(x,y) &= \sum_I H^I_{(\mu\nu)}(x)Y^I(y)\,, \quad H^\mu_\mu(x,y) = \sum_I H^I(x)Y^I(y)\,, && (2.19)\\
h_{(\alpha\beta)}(x,y) &= \sum_I \phi^I(x)Y^I_{(\alpha\beta)}(y)\,, \qquad h^\alpha_\alpha(x,y) = \sum_I \pi^I(x)Y^I(y)\,, && (2.20)\\
h_{\mu\alpha}(x,y) &= \sum_I B^I_\mu(x)Y^I_\alpha(y)\,, && (2.21)\\
a_{\beta_1\ldots\beta_{q-1}} &= \sum_I b^I(x)\;\epsilon^\alpha{}_{\beta_1\ldots\beta_{q-1}}\nabla_\alpha Y^I(y)\,, && (2.22)\\
a_{\mu\beta_2\ldots\beta_{q-1}} &= \sum_I b^I_\mu(x)\;\epsilon^{\alpha\beta}{}_{\beta_2\ldots\beta_{q-1}}\nabla_{[\alpha}Y^I_{\beta]}(y) + \\
&\qquad + \sum_h \beta^h_\mu(x)\;\epsilon^{\alpha\beta}{}_{\beta_2\ldots\beta_{q-1}}Y^h_{[\alpha\beta]}\,, && (2.23)\\
&\vdots\\
a_{\mu_1\ldots\mu_{q-1}} &= \sum_I b^I_{\mu_1\ldots\mu_{q-1}}(x)Y^I(y)\,, && (2.24)
\end{aligned}$$

where I in each case is a generic label running over the possible spherical harmonics of the appropriate tensor type, and $h = 1 \dots b^n(M_q)$ runs over the harmonic n-forms on M_q for the gauge field with $(n-1)$ K_p indices. The x coordinates denotes K_p and the y coordinates are on M_q. We have not included a term $\beta(x)$ in (2.22) since compact Riemannian Einstein spaces with positive curvature cannot possess harmonic one-forms. Following deWolfe et al., we will also define

$$b(x,y) \equiv \sum_I b^I(x) Y^I(y) \,, \qquad b_{\mu\alpha}(x,y) \equiv \sum_I b^I_\mu(x) Y^I_\alpha(y) \,. \tag{2.25}$$

2.3.2 Perturbed Einstein Equations and Form Equations

We now consider the Einstein equations to linear order in fluctuations, as well as the form equations that mix with the graviton. The uncoupled form equations are unnecessary for our purposes. We use the following notation: $\Box_x \equiv g^{\mu\nu}\nabla_\mu\nabla_\nu$, $\Box_y \equiv g^{\alpha\beta}\nabla_\alpha\nabla_\beta$, and Max $B_\mu \equiv \Box_x B_\mu - \nabla^\nu\nabla_\mu B_\nu$ is the Maxwell operator acting on vectors on K_p. Additionally, $\Delta_y \equiv -(d_q^\dagger d_q + d_q d_q^\dagger)$ is the Laplacian[2] acting on differential forms on M_p; for vectors, the explicit form is $\Delta_y Y_\alpha \equiv \Box_y Y_\alpha - R_\alpha{}^\beta Y_\beta$. Further, $f \cdot \epsilon \equiv f_{\alpha_1 \cdots \alpha_q}\epsilon^{\alpha_1 \cdots \alpha_q}/q!$.

For convenience, we present the linearized Ricci tensor in our conventions:

$$\begin{aligned} R^{(1)}_{MN} = -\frac{1}{2}[&(\Box_x + \Box_y)h_{MN} + \nabla_M\nabla_N h^P_P - \nabla_M\nabla^P h_{PN} - \nabla_N\nabla^P h_{PM} \\ &- 2R_{MPQN}h^{PQ} - R_M{}^P h_{NP} - R_N{}^P h_{MP}] \,. \end{aligned} \tag{2.26}$$

[2]The negative sign respects the Kaluza-Klein literature.

The sign conventions are that of Misner, Thorne and Wheeler [?]. We employ Einstein's equations in their Ricci form, $R_{MN} = \bar{T}_{MN}$ with $\bar{T}_{MN} \equiv T_{MN} + \frac{1}{2-D}\, g_{MN}\, T^P_P$. For $R_{\mu\nu}$ we find

$$\begin{aligned} R_{\mu\nu}^{(1)} &= -\frac{1}{2}[(\Box_x + \Box_y)(H_{\mu\nu} - \frac{1}{p-2} g_{\mu\nu} h^\gamma_\gamma) + \nabla_\mu \nabla_\nu (H^\rho_\rho - \frac{2}{p-2} h^\gamma_\gamma) - \\ &- \nabla_\mu \nabla^\rho (H_{\rho\nu} - \frac{1}{p-2} g_{\rho\nu} h^\gamma_\gamma) - \nabla_\nu \nabla^\rho (H_{\rho\mu} - \frac{1}{p-2} g_{\rho\mu} h^\gamma_\gamma) - \\ &- R_\mu{}^\rho (H_{\rho\nu} - \frac{1}{p-2} g_{\rho\nu} h^\gamma_\gamma) - R_\nu{}^\rho (H_{\rho\mu} - \frac{1}{p-2} g_{\rho\mu} h^\gamma_\gamma)] - \\ &- 2R_{\mu\rho\sigma\nu} (H^{\rho\sigma} - \frac{1}{p-2} g^{\rho\sigma} h^\gamma_\gamma)\,, \end{aligned} \tag{2.27}$$

which must be equal to

$$\begin{aligned} \bar{T}_{\mu\nu}^{(1)} = -\frac{c^2(q-1)}{2(D-2)} h_{\mu\nu} + h^{\alpha\beta} \frac{q(q-1)c^2}{2(D-2)q!} g_{\mu\nu} \epsilon_{\alpha\gamma_2\cdots\gamma_q} \epsilon_\beta{}^{\gamma_2\cdots\gamma_q} - \\ -\frac{c(q-1)}{(D-2)} g_{\mu\nu}\ (f \cdot \epsilon) + \frac{2\Lambda}{D-2} h_{\mu\nu}\,, \end{aligned} \tag{2.28}$$

resulting in the equation

$$\begin{aligned} -\frac{1}{2}[(\Box_x + \Box_y) H_{\mu\nu} + \nabla_\mu \nabla_\nu H^\rho_\rho - \nabla_\mu \nabla^\rho H_{\rho\nu} - \nabla_\nu \nabla^\rho H_{\rho\mu} - \\ -2R_{\mu\rho\sigma\nu} H^{\rho\sigma} - R_\mu{}^\rho H_{\rho\nu} - R_\nu{}^\rho H_{\rho\mu}] + \frac{1}{2(p-2)} g_{\mu\nu} (\Box_x + \Box_y) h^\gamma_\gamma \\ -\frac{(q-1)^2}{(p-2)R^2} g_{\mu\nu} h^\gamma_\gamma + \frac{(q-1)^2}{(p-1)R^2} H_{\mu\nu} + \frac{q-1}{D-2} g_{\mu\nu} \Box_y cb - \frac{2\Lambda}{D-2} h_{\mu\nu} = 0\,. \end{aligned} \tag{2.29}$$

For linearized $R_{\mu\alpha}$, we find

$$\begin{aligned} R_{\mu\alpha}^{(1)} = -\frac{1}{2}[\Box_x h_{\mu\alpha} - \nabla_\mu \nabla^\nu h_{\nu\alpha} - R_\mu{}^\nu h_{\nu\alpha} + \Box_y h_{\mu\alpha} - R_\alpha{}^\beta h_{\beta\mu} \\ -\nabla_\alpha \nabla^\nu h_{\nu\mu} + \nabla_\mu \nabla_\alpha (H^\rho_\rho - \frac{2}{p-2} h^\gamma_\gamma) - \nabla_\mu \nabla^\beta h_{\beta\alpha}]\,, \end{aligned} \tag{2.30}$$

which is sourced by

$$\bar{T}_{\mu\alpha}^{(1)} = \frac{c}{2(q-1)!} f_{\mu\beta_2\cdots\beta_q}\epsilon_\alpha{}^{\beta_2\cdots\beta_q} - \frac{c^2(q-1)}{2(D-2)}\, h_{\mu\alpha} \tag{2.31}$$

$$= \frac{c}{2}\nabla_\mu\nabla_\alpha b + \frac{c}{2}(\Box_y b_{\mu\alpha} - R_\alpha{}^\beta b_{\mu\beta}) - \frac{c^2(q-1)}{2(D-2)}\, h_{\mu\alpha}\,. \tag{2.32}$$

For $R_{\alpha\beta}$ we have

$$\begin{aligned} R_{\alpha\beta}^{(1)} &= -\frac{1}{2}[(\Box_x + \Box_y)h_{(\alpha\beta)} - 2R_{\alpha\gamma\delta\beta}h^{(\gamma\delta)} - R_\alpha{}^\gamma h_{(\gamma\beta)} - R_\beta{}^\gamma h_{(\gamma\alpha)} \\ &+ \frac{1}{q} g_{\alpha\beta}(\Box_x + \Box_y)h^\gamma_\gamma - \\ &- (\frac{2}{q} + \frac{2}{p-2})\nabla_\alpha\nabla_\beta h^\gamma_\gamma + \nabla_\alpha\nabla_\beta H^\mu_\mu - \nabla_\alpha\nabla^\mu h_{\mu\beta} - \nabla_\beta\nabla^\mu h_{\mu\alpha}]\,, \end{aligned} \tag{2.33}$$

while on the right-hand side, we find

$$\begin{aligned} \bar{T}_{\alpha\beta}^{(1)} &= \frac{c}{2(q-1)!}(f_{\alpha\gamma_2\cdots\gamma_q}\epsilon_\beta{}^{\gamma_2\cdots\gamma_q} + f_{\beta\gamma_2\cdots\gamma_q}\epsilon_\alpha{}^{\gamma_2\cdots\gamma_q}) + \\ &+ \frac{c^2(q-1)}{2(q-1)!}(-h^{\gamma\delta})\epsilon_{\alpha\gamma\theta_3\cdots\theta_q}\epsilon_{\beta\delta}{}^{\theta_3\cdots\theta_q} - \frac{c^2(q-1)}{2(D-2)}(h_{(\alpha\beta)} + \frac{1}{q}g_{\alpha\beta}h^\gamma_\gamma) - \\ &- \frac{c(q-1)}{(D-2)} g_{\alpha\beta}(f\cdot\epsilon) - \frac{q(q-1)c^2}{2(D-2)q!} g_{\alpha\beta}(-h^{\gamma\delta})\epsilon_{\gamma\theta_2\cdots\theta_q}\epsilon_\delta{}^{\theta_2\cdots\theta_q} \\ &= \frac{p-1}{D-2} g_{\alpha\beta}\Box_y cb + \frac{q-1}{R^2} h_{(\alpha\beta)} - \frac{(q-1)^2}{qR^2} g_{\alpha\beta}h^\gamma_\gamma\,, \end{aligned} \tag{2.34}$$

where we have used $(f\cdot\epsilon) = \Box_y b$ and $f_{\alpha_1\cdots\alpha_q} = (f\cdot\epsilon)\epsilon_{\alpha_1\cdots\alpha_q} = \epsilon_{\alpha_1\cdots\alpha_q}\Box_y b$.

We see that the modes of the graviton mix with the form modes b and b_μ. To solve the coupled systems, we must consider certain form equations as well. From the $\nabla^M F_{M\beta_2\ldots\beta_q}$ equation[3], we find the expression

$$\epsilon_\alpha{}^{\beta_2\cdots\beta_q}\big(\nabla^M f_{M\beta_2\cdots\beta_q} - cg^{\mu\nu}\Gamma^{\gamma\,(1)}_{\mu\nu}\epsilon_{\gamma\beta_2\cdots\beta_q} - cg^{\gamma\delta}\Gamma^{\theta\,(1)}_{\gamma\delta}\epsilon_{\theta\beta_2\cdots\beta_q} - \\ -c(q-1)g^{\gamma\delta}\Gamma^{\theta\,(1)}_{\gamma\beta_2}\epsilon_{\delta\theta\beta_3\cdots\beta_q}\big) = 0\,, \qquad (2.35)$$

where we use the linearized Christoffel symbol,

$$\Gamma^{P\,(1)}_{MN} = \frac{1}{2}\left(\nabla_M h^P_N + \nabla_N h^P_M - \nabla^P h_{MN}\right)\,. \qquad (2.36)$$

Contracting with the epsilon tensor on M_q, (2.35) gives

$$\nabla_\alpha[(\Box_x + \Box_y)b + \frac{c}{2}H^\mu_\mu - \frac{c(p-1)}{p-2}h^\gamma_\gamma] + \nabla^\mu[\Box_y b_{\mu\alpha} - R_\alpha{}^\beta b_{\mu\beta} - ch_{\mu\alpha}] = 0 \qquad (2.37)$$

We have used $\nabla^\alpha b_{\mu\alpha} = 0$ in obtaining the last expression. Finally, from the $\nabla^M F_{M\mu\beta_3\ldots\beta_q}$ equation,

$$\nabla^M f_{M\mu\beta_3\cdots\beta_q} - cg^{\gamma\alpha}\Gamma^{\delta\,(1)}_{\gamma\mu}\epsilon_{\alpha\delta\beta_3\cdots\beta_q} = 0\,, \qquad (2.38)$$

which gives rise to

$$\left(\Box_x + \Box_y - \frac{2(q-1)}{R^2}\right)\nabla_{[\alpha}b_{\beta]\mu} - \nabla^\nu\nabla_\mu\nabla_{[\alpha}b_{\beta]\nu} - c\nabla_{[\alpha}B_{\beta]\mu} \qquad (2.39) \\ +2R_\alpha{}^\gamma{}_\beta{}^\delta\nabla_{[\gamma}b_{\delta]\mu} - D_{\beta_3}D^\nu a_{\mu\nu\beta_4\cdots\beta_q}\epsilon_{\alpha\beta}{}^{\beta_3\cdots\beta_q} + (\Box_x\beta_\mu - \nabla^\nu\nabla_\mu\beta_\nu) = 0\,.$$

We now expand these fields in spherical harmonics and collect like terms. Below we present the results, collecting related equations and indicating the origin of each expression as follows: (E1), (E2) and (E3) for the K_p, mixed

[3]One may avoid explicit manipulation of Christoffel symbols by linearizing the equivalent equation $\partial_M\sqrt{-g}F^{MN_2\cdots N_q} = 0$.

and M_q Einstein equations, and (F1) and (F2) for the form equations (2.37) and (2.39), respectively.

Equations for the coupled scalars π^I, b^I and H^I:

$$\text{(E3)}\quad \Big[\big(\Box_x+\Box_y-\tfrac{2(q-1)^2}{R^2}+\tfrac{4\Lambda}{D-2}\big)\pi^I+\Box_y\big(H^I-\tfrac{2(D-2)}{q(p-2)}\pi^I\big)+\tfrac{2q(p-1)}{(D-2)}\Box_y c b^I\Big]Y^I = \\ =0 \tag{2.40}$$

$$\text{(E3)}\qquad \left(H^I-\tfrac{2(D-2)}{q(p-2)}\pi^I\right)\nabla_{(\alpha}\nabla_{\beta)}Y^I=0\,, \tag{2.41}$$

$$\text{(F1)}\qquad \nabla_\alpha\left(\Box_x b^I+\Box_y b^I+\tfrac{c}{2}H^I-\tfrac{c(p-1)}{(p-2)}\pi^I\right)Y^I=0\,, \tag{2.42}$$

Equations for coupled vectors b^I_μ, B^I_μ:

$$\text{(E2)}\quad \left(\text{Max }B^I_\mu+\Delta_y B^I_\mu+\Delta_y c b^I_\mu-\tfrac{2(q-1)^2}{(p-1)R^2}b^I_\mu\right)Y^I_\alpha=0\,, \tag{2.43}$$

$$\text{(F2)}\qquad \nabla_{[\alpha}\left(\text{Max }b^I_\mu+\Delta_y b^I_\mu-cB^I_\mu\right)Y^I_{\beta]}=0\,. \tag{2.44}$$

$$\text{(F1)}\qquad \left(\nabla^\mu b^I_\mu\Delta_y-c\nabla^\mu B^I_\mu\right)Y^I_\alpha=0\,, \tag{2.45}$$

$$\text{(E3)}\qquad (\nabla^\mu B^I_\mu)\nabla_{(\alpha}Y^I_{\beta)}=0\,, \tag{2.46}$$

Equations for symmetric tensors $H^I_{\mu\nu}$:

$$\text{(E1)}\qquad (R^{(1)}_{\mu\nu}(H^I_{\rho\sigma})-\tfrac{1}{2}\Box_y H^I_{\mu\nu}+\tfrac{(q-1)^2}{(p-1)R^2}H^I_{\mu\nu}+ \\ \tfrac{1}{2(p-2)}g_{\mu\nu}(\Box_x+\Box_y)\pi^I-\tfrac{(q-1)^2}{(p-2)R^2}g_{\mu\nu}\pi^I+\tfrac{(q-1)}{(D-2)}g_{\mu\nu}\Box_y c b^I)Y^I=0\,, \tag{2.47}$$

$$\text{(E2)}\qquad \left(-\nabla^\nu H^I_{\nu\mu}+\nabla_\mu H^I-\tfrac{(p+q-2)}{q(p-2)}\nabla_\mu\pi^I+\nabla_\mu c b^I\right)\nabla_\alpha Y^I=0\,, \tag{2.48}$$

Note that in (2.47), $R^{(1)}_{\mu\nu}$ is the linearized Ricci tensor for K_p only, evaluated on the field $H_{\rho\sigma}$. Finally, there remain a few decoupled equations:

$$\text{(E3)} \quad \left[(\Box_x + \Box_y)\delta^\gamma_\alpha \delta^\delta_\beta - 2R_\alpha{}^{\gamma\delta}{}_\beta\right] \phi^I Y^I_{(\gamma\delta)} = 0\,. \tag{2.49}$$

$$\text{(F2)} \quad (\text{Max}\ \beta^h_\mu) Y^h_{[\alpha\beta]} = 0\,, \tag{2.50}$$

$$\text{(F2)} \quad (\nabla^\nu b^I_{\nu\mu}) \nabla_{[\alpha} \nabla_{\beta]} Y^I = 0\,. \tag{2.51}$$

Notice that in passing from (2.39) to (2.44), we commuted the $\Box_y$ through the covariant derivative ∇_α, which not only produced precisely the Laplacian Δ_y acting on vectors, but also canceled all terms in (2.39) involving the Riemann tensor.

The interesting aspect of these results is that the properties of M_q enter into almost all these formulas only through the dimension q and the radius R. Consequently we will be able to treat these equations in a completely unified way. The sole exception is the equation (2.49) for the scalars coming from graviton modes on the compact space, which explicitly involves the Riemann tensor on M_q. There is thus no guarantee that the modes ϕ^I will possess the uniform stability properties for different choices of M_q. Indeed, it was shown by deWolfe et al that for $M_q = S^q$ these modes are harmlessly positive mass for all q, while for any product $M_q = M_n \times M_{q-n}$ with $q < 9$ they contain an instability. It should be born in mind here that for these modes, the equations for their case and our case are identical, so the arguments go through without modifications.

2.3.3 Coupled scalars

In this section, we consider the system of modes associated with the coupled scalars π^I, b^I and H^I, equations (2.40), (2.41) and (2.42).

For certain low-lying scalar spherical harmonics Y^I, some or all of their derivatives appearing in the equations of section 2.3.2 may vanish. Let us first treat the generic case where all derivatives of Y^I in (2.40), (2.41) and (2.42) are nonzero and hence the coefficients must vanish. Equation (2.41) then gives us a constraint which may be used to eliminate H^I in favor of π^I. Substituting into equation (2.42), we find

$$\left((\Box_x + \Box_y)\, b^I - c\frac{(q-1)}{q}\pi^I \right) Y^I = 0\,, \tag{2.52}$$

while the second term in parentheses vanishes in (2.40). We obtain from (2.40) and (2.52) the coupled system

$$L^2\Box_x \begin{pmatrix} b^I/c \\ \pi^I \end{pmatrix} = (p-1)^2 \begin{pmatrix} \frac{\lambda^I}{(q-1)^2} & \frac{R^2}{q(q-1)} \\ \frac{4q\lambda^I}{(q-1)R^2} & \frac{\lambda^I}{(q-1)^2} + 2 \end{pmatrix} \begin{pmatrix} b^I/c \\ \pi^I \end{pmatrix}\,, \tag{2.53}$$

where $\Box_y Y^I = -\lambda^I Y^I/R^2$; where $\lambda^I \geq 0$. On diagonalizing this matrix we obtain the mass spectrum

$$m^2L^2 = \frac{(p-1)^2}{(q-1)^2}[\lambda + (q-1)(q-1 \pm \sqrt{4\lambda + (q-1)^2})]\,. \tag{2.54}$$

With the help of these equations, it is possible to map out the parameter regions for stability, and this has been done in [4]. The following table summarizes the results as a function of the free dimensionless parameter $\mathcal{F}$ and the dimension of the Riemannian space q.

	$dS_p \times S^q$		$R^{3,1} \times S^q$	$AdS_p \times S^q$
q	*unstable*	*stable*		
2	$0 \leq \mathcal{F} < \frac{1}{(p-1)}$	$\frac{1}{(p-1)} \leq \mathcal{F} < 1$	$\mathcal{F} = 1$	$1 < \mathcal{F}$
3	$0 \leq \mathcal{F} < \frac{1}{2(p-1)}$	$\frac{1}{2(p-1)} \leq \mathcal{F} < \frac{1}{2}$	$\mathcal{F} = \frac{1}{2}$	$\frac{1}{2} < \mathcal{F}$
4	$0 \leq \mathcal{F} < \frac{1}{3(p-1)}$, $\frac{1}{2(p-1)} < \mathcal{F} < \frac{1}{3}$	$\frac{1}{3(p-1)} \leq \mathcal{F} < \frac{1}{2(p-1)}$	$\mathcal{F} = \frac{1}{3}$	$\frac{1}{3} < \mathcal{F}$
≥ 5	$0 \leq \mathcal{F} < \frac{1}{q-1}$	–	$\mathcal{F} = \frac{1}{q-1}$	$\frac{1}{q-1} < \mathcal{F}$

Not reflected in the previous table is the fact that the number of unstable modes increases as a function of q. For $q = 2,$ and $q = 3$ there is only one unstable mode. It corresponds to the radius of S^q. For $q \geq 4$ other modes became tachyonic as well.

2.4 Evolution of Unstable Spacetimes

There is a broader set of solutions to (2.2),(2.3), that do not correspond to a product spacetime. They are of the form:

$$ds^2 = -dt^2 + a(t)^2 d^{p-1}\mathbf{x} + R(t)^2 d\Omega_q \tag{2.55}$$

$$F_q = \frac{f}{R^q(t)} \mathrm{vol}_{S^q} \qquad \oint \mathrm{vol}_{S^q} = R(t)^q \, \Omega_q \tag{2.56}$$

In this case, the normalization of the differential form F_q is fixed by the equations of motion. The corresponding equations of motion are:

$$\frac{q(q-1)}{2R^2}(1+\dot{R}^2) + \frac{(p-1)(p-2)}{2}\frac{\dot{a}^2}{a^2} + q(p-1)\frac{\dot{R}\,\dot{a}}{R\,a} = \Lambda\left(1 + \frac{\beta}{\Lambda^q R^{2q}}\right) \tag{2.57}$$

$$\frac{q(q-1)}{2R^2}(1+\dot{R}^2) + \frac{(p-2)(p-3)}{2}\frac{\dot{a}^2}{a^2} + q(p-2)\frac{\dot{R}\,\dot{a}}{R\,a} + (p-2)\frac{\ddot{a}}{a} + q\frac{\ddot{R}}{R} = \\ = \Lambda\left(1 + \frac{\beta}{\Lambda^q R^{2q}}\right) \tag{2.58}$$

$$\frac{(q-2)(q-1)}{2R^2}(1+\dot{R}^2)+\frac{(p-1)(p-2)}{2}\frac{\dot{a}^2}{a^2}+(q-1)(p-1)\frac{\dot{R}\,\dot{a}}{R\,a}+$$
$$+(p-1)\frac{\ddot{a}}{a}+(q-1)\frac{\ddot{R}}{R}=\Lambda\left(1-\frac{\beta}{\Lambda^q R^{2q}}\right) \tag{2.59}$$

where $\frac{\beta}{\Lambda^q}=\frac{f^2}{4\Lambda}$. To lighten these already complicated equations we have made the assumption that $k=0$, the generalization to $k=\pm 1$ is straightforward. These equations can be reduced into an equation for the breathing mode of the sphere, often called the radion: $R(t)$

$$\begin{aligned}\ddot{R} \;&= -\frac{dV(R)}{dR}+\frac{q+p-2}{p-2}\frac{\dot{R}^2}{R} \qquad (2.60)\\ \mp\;& \frac{p-1}{p-2}\dot{R}\sqrt{\frac{q(p+q-2)}{p-1}\frac{\dot{R}^2}{R^2}-2\frac{p-2}{p-1}\left[\frac{q(q-1)}{2R^2}-\Lambda\left(1+\frac{\beta}{\Lambda^q R^{2q}}\right)\right]}\end{aligned}$$

where the potential

$$V(R)=-\frac{\Lambda R^2}{p+q-2}+\frac{(p-1)}{(q-1)(p+q-2)}\frac{\beta}{(\Lambda R^2)^{q-1}}+(q-1)\ln(R) \tag{2.61}$$

The evolution of $a(t)$ is given by:

$$\frac{\dot{a}}{a}=-\frac{q}{p-2}\frac{\dot{R}}{R}\pm\frac{1}{p-2}\sqrt{\frac{q(p+q-2)}{p-1}\frac{\dot{R}^2}{R^2}-2\frac{p-2}{p-1}\left[\frac{q(q-1)}{2R^2}-\Lambda(1+\frac{\beta}{\Lambda^q R^{2q}})\right]} \tag{2.62}$$

The initial conditions, i.e. the sign of $\frac{\dot{a}}{a}$ at $\dot{R}=0$, and R equal the radius of the unstable point, decide which branch of the equations (2.60), (2.62) to consider.

As long as the flux is nonvanishing and relatively small, $0<\beta_s$, where

$$\beta_s=\frac{(q-1)^{2q-1}}{p-1}\left(\frac{p+q-2}{2q}\right)^q \tag{2.63}$$

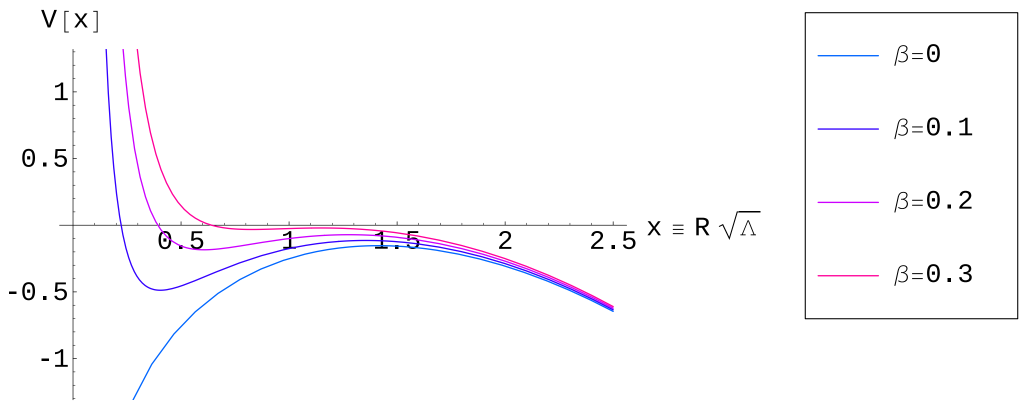

Figure 2.1: Potential for the field $R(t)$ as a function of the flux, β, when $p = 4$, and $q = 2$.

the radion potential will always have two stationary solutions, one stable and one unstable. There is a solution to the equations of motion that corresponds to the field $R(t)$ sitting on the maximum. In this case, the geometry factorizes and corresponds to a $dS_p \times S_q$ space. There are also solutions that correspond to $R(t)$ sitting in the minimum. In these cases, the corresponding geometry also factorizes into the product of the spaces $L_p \times S_q$. The Lorentzian signature space will be a (A)dS or flat Minkowski depending on the value of β. In particular, for $\beta_m < \beta < \beta_s$, where

$$\beta_m = \frac{(q-1)^{2q-1}}{2^q} \tag{2.64}$$

the Lorentzian signature space will also be a de Sitter space.

We are interested in studying the evolution of the unstable solutions away from the maximum. The equations (2.60),(2.62) will capture such an

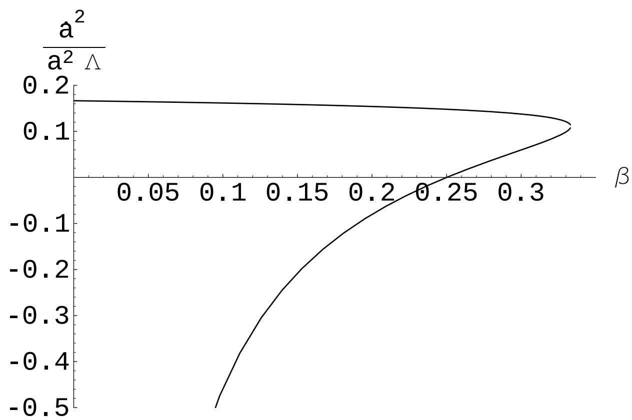

Figure 2.2: Value of $\frac{\dot{a}^2}{a^2}$ at the stationary points as a function of the initial condition β. The upper branch corresponds to the unstable point and the lower branch corresponds to the stable point. (Case $p = 4$, $q = 2$.)

evolution provided that $R(t)$ is indeed the only mode excited along the path away from the maximum. The analysis of [4] proves that this is indeed the only excited mode around the extrema of the potential, provided that $q = 2, 3$, for $q \geq 4$ there will be other unstable modes. In this work we will restrict our attention to the first two cases, and leave the analysis of $q \geq 4$ to future work.

2.5 Results

In this section we report on the numerical solution of the equations (2.60), (2.62). It is useful to distinguish between two cases:

- When the minimum of the potential corresponds to another de Sitter solution ($\beta_m < \beta < \beta_s$)
- When the minimum of the potential corresponds to an Anti de Sitter

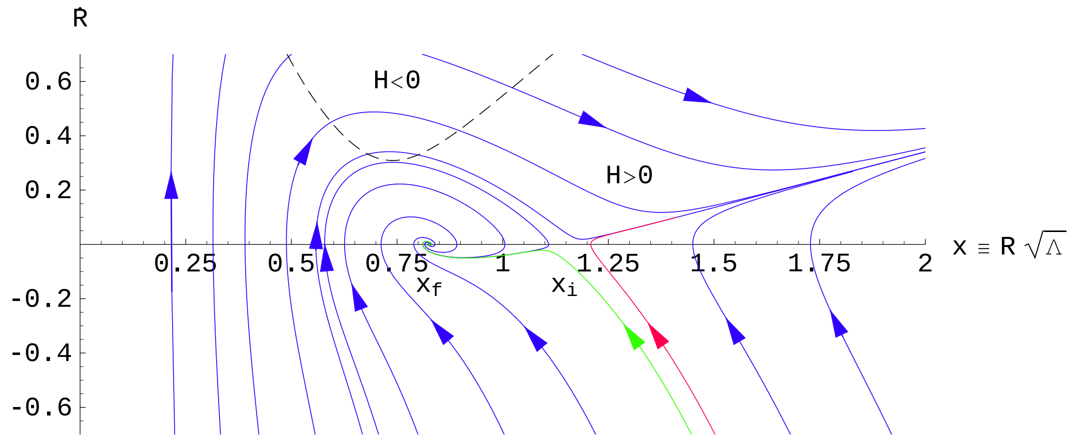

Figure 2.3: Numerical evolution for $p = 4$, $q = 2$, $\beta = 0.3$. The arrows indicate the direction of time evolution.

solution $(0 < \beta < \beta_m)$

2.5.1 $\beta_m < \beta < \beta_s$

Let us choose the initial condition such that $\dot{a}/a > 0$, namely, we start with an expanding de Sitter space. The numerical analysis shows that the evolution from the unstable $dS_p \times S^q$ in this case leads either to a decompactification of the compact dimensions, or to another, stable $dS_p \times S^q$ solution, see Figure 2.3.

In the case of decompactification, the expansion rate of the radius of the sphere S^q asymptotes to:

$$\frac{\dot{R}}{R} = \sqrt{\frac{2\Lambda}{(D-1)(D-2)}} \tag{2.65}$$

where $D = p+q$. It is interesting to note that this expansion rate matches the

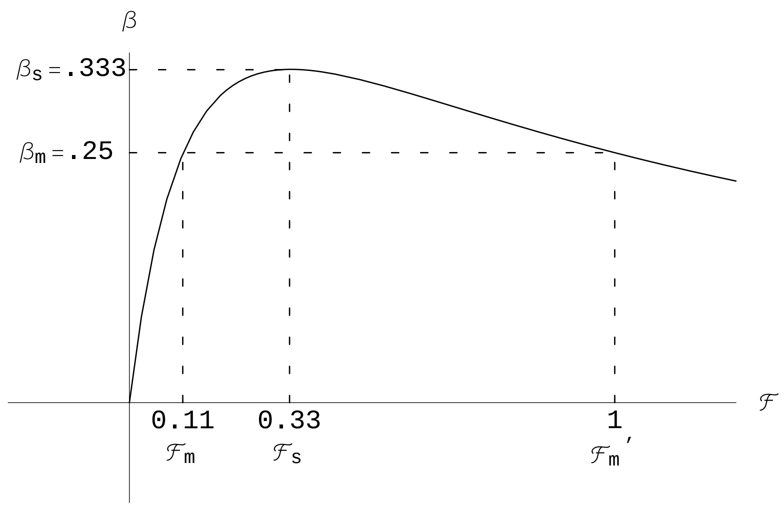

Figure 2.4: Flux β as a function of parameter $\mathcal{F}$, when $p = 4$ and $q = 2$.

asymptotic effective Hubble constant for the de Sitter space, i.e. $\dot{R}/R = \dot{a}/a$ as $R, a \to \infty$. This agrees with the result found in [5], because this fast evolution erases all the information about the initial flux. The other direction of evolution from the unstable $dS_p \times S^q$, with these initial conditions, leads to a stable $dS_p \times S^q$ solution. The two solutions are characterized by the same value of β. Namely, for a static solution we have:

$$\beta/\Lambda^q = \mathcal{F} R^{2q} \tag{2.66}$$

$$R^2 = \frac{(q-1)(D-2)}{2\Lambda[1 + (p-1)\mathcal{F}]} \tag{2.67}$$

For a particular value of β we have two static solutions related by:

$$\frac{\mathcal{F}_f}{\mathcal{F}_i} = \left(\frac{1 + (p-1)\mathcal{F}_f}{1 + (p-1)\mathcal{F}_i} \right)^q \tag{2.68}$$

The stable solution will always have a smaller radius R than the initial, unstable one. The two are related by:

$$R_f^2 = \frac{1+(p-1)\mathcal{F}_i}{1+(p-1)\mathcal{F}_f} R_i^2 \tag{2.69}$$

The effective cosmological constant for the de Sitter is given by (2.7), and it therefore changes:

$$\left(\frac{\dot{a}}{a}\right)_f^2 = \frac{1+(1-q)\mathcal{F}_f}{1+(1-q)\mathcal{F}_i} \left(\frac{\dot{a}}{a}\right)_i^2 \tag{2.70}$$

In particular, for $p = 4$ and $q = 2$ we have:

$$\mathcal{F}_f = \frac{1}{9\mathcal{F}_i} \tag{2.71}$$

$$R_f^2 = \frac{2}{\Lambda} - R_i^2 = 3\mathcal{F}_i R_i^2 \tag{2.72}$$

$$\left(\frac{\dot{a}}{a}\right)_f^2 = \frac{(9\mathcal{F}_i - 1)}{9\mathcal{F}_i(1-\mathcal{F}_i)} \left(\frac{\dot{a}}{a}\right)_i^2 \tag{2.73}$$

$$\Lambda_{4f} = \frac{\Lambda_{4i} - \frac{4}{27}\Lambda}{6\Lambda_{4i} - \Lambda} \Lambda \tag{2.74}$$

Notice, that the final effective cosmological constant can be made very small compared with the initial. An analogous transition is known in the literature in the context of braneworlds. The radius of the internal sphere will remain, however, within the same order of magnitude of the initial one. Let us compare the entropies of the initial and the final solution:

$$\frac{S(dS_4 \times S^2)_f}{S(dS_4 \times S^2)_i} = \frac{A(dS_4)_f \times V(S^2)_f}{A(dS_4)_i \times V(S^2)_i} = \frac{(1-\mathcal{F}_i)(1+3\mathcal{F}_i)}{(1-\mathcal{F}_i)(1+3\mathcal{F}_i)} = 27(\mathcal{F}_i)^2 \frac{1-\mathcal{F}_i}{9\mathcal{F}_i - 1} > 1 \tag{2.75}$$

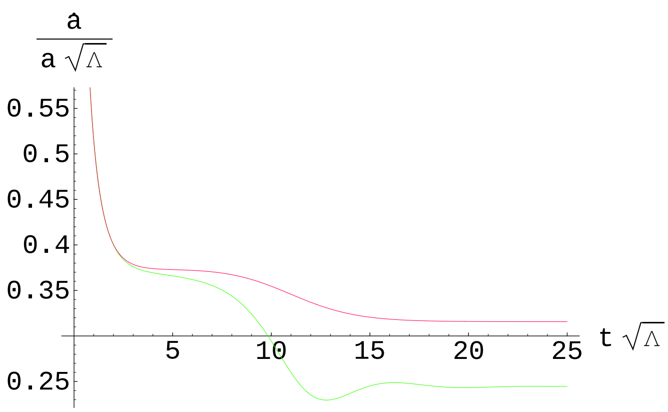

Figure 2.5: The evolution of effective Hubble rate for $p = 4$, $q = 2$. (The colors match the numerical evolution in Figure 2.3)

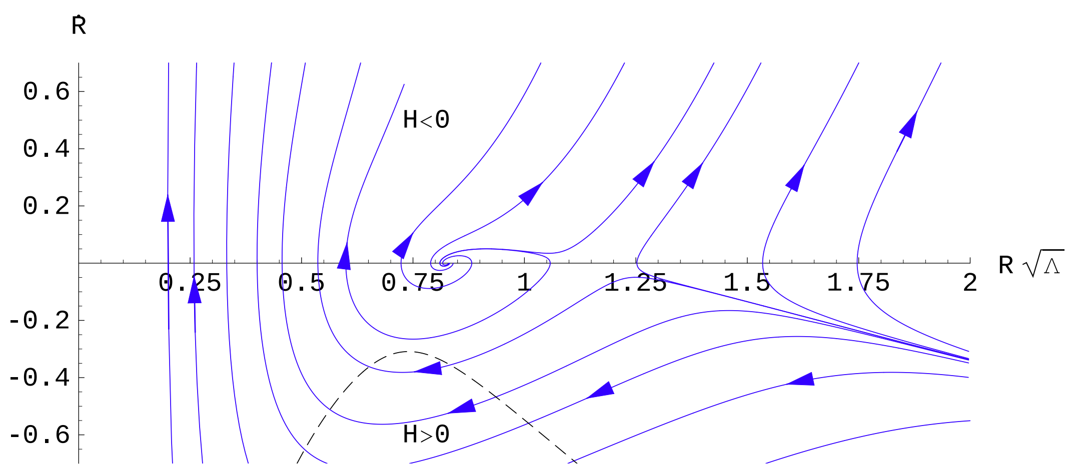

Figure 2.6: Numerical evolution for $p = 4$, $q = 2$, $\beta = 0.3$

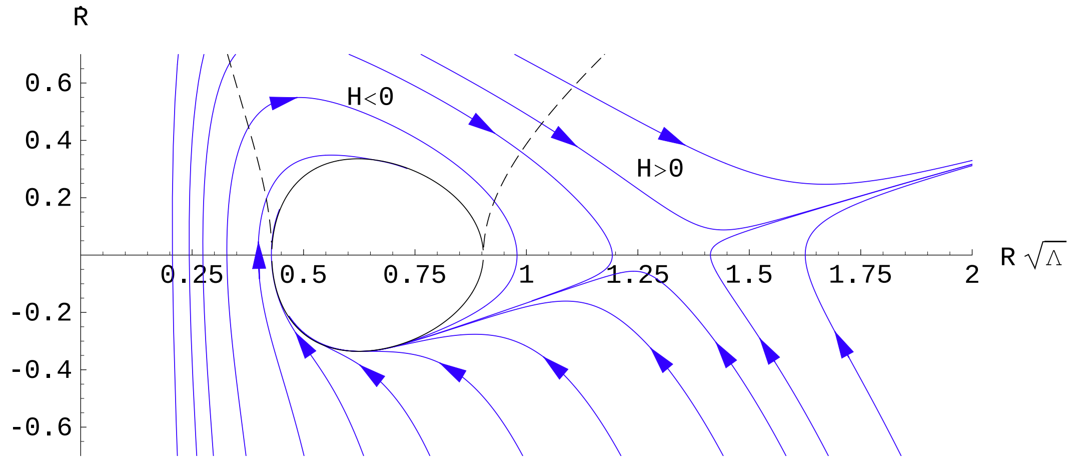

Figure 2.7: Numerical evolution for $p = 4$, $q = 2$, $\beta = 0.15$ or equivalently Anti de Sitter minimum.

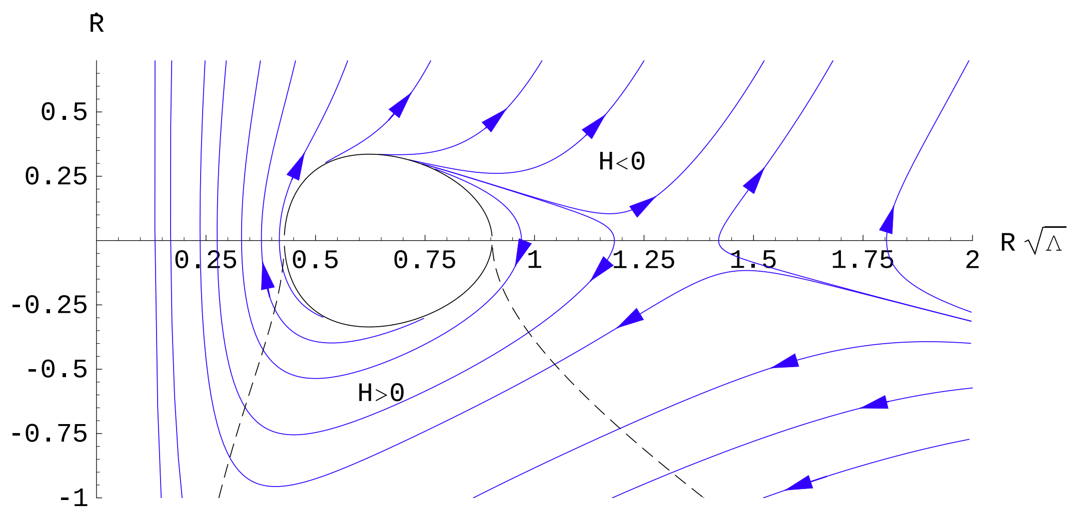

Figure 2.8: Numerical evolution for $p = 4$, $q = 2$, $\beta = 0.15$ or equivalently Anti de Sitter minimum.

as long as, $\frac{1}{9} < \mathcal{F}_i < \frac{1}{3}$, which corresponds to our initial unstable static solution.

Now let's look at the situation where the initial conditions are chosen such that the de Sitter component of the space is initially contracting, $\dot{a}/a < 0$ (See Figure 2.6). In this case, the numerical evolution shows that the internal dimensions necessarily decompactify, while the initial de Sitter dimensions keep contracting. This result may appear surprising, at first, when the initial velocity at the unstable point is taken to be negative, i.e. $\dot{R} < 0$. These initial conditions choose the negative branch in (2.62) and hence the positive branch in (2.60). The friction force in (2.60) is then big enough to overcome the force generated by the potential.

The crunching solution should (2.55) asymptotically become a Kasner type solution. We did not use this ansatz in the numerical analysis that led to Figure 2.6.

2.5.2 $0 < \beta < \beta_m$

When the flux β is very small, only one of the static solutions corresponds to a de Sitter compactification, the other solution corresponds to an anti de Sitter space. In this case the numerical solution in both branches (either initially contracting (Figure 2.8), or expanding (Figure 2.7) de Sitter phase) leads to a decompactification of the inner space, while the initially de Sitter components either expand or contract, depending on the initial conditions and the initial direction of the numerical evolution. This is a sensible result. The classical trajectories studied here cannot connect spaces with different spacial curvatures. Given the ansatz for the metric (2.55) the initial de Sitter configuration can be either flat or closed while the final anti de Sitter configuration has to be open.

Chapter 3

Quantum Groups in de Sitter Space

3.1 Introduction

As mentioned in the introduction, observations indicate that our Universe is currently in a regime of accelerated expansion, and that we might be living in an asymptotically deSitter spacetime[12, 13]. From the perspective of a co-moving observer, deSitter spacetime has a cosmological horizon and an associated finite entropy [12, 17]. Finiteness of entropy suggests finite dimensionality of the Hilbert space, but since the isometry group of deSitter is non-compact and therefore has no finite dimensional unitary representations, we immediately have a problem in our hands [16, 14].

One idea that has been proposed as a way out of this is to look for q-deformations [18, 19, 20, 21] of the isometry group which might admit finite dimensional unitary representations. When the deformation parameter(s) are taken to $\rightarrow 1$, we recover the classical group. It is known that the standard q-deformations of (complexified and therefore non-compact) classical groups have finite dimensional unitary representations when q is a root of unity[22, 23, 24, 25, 26, 27, 29].

In this chapter, we first look at how the geometric structure of the underlying deSitter space is modified when its symmetry group is deformed. It turns out that a deformed symmetry group necessitates a deformed differential calculus on the underlying space in order for the differential structure to be covariant with respect to the new q-symmetry. In section 3.2, we will make use of the work by Zumino et al. [30, 31] to explicitly write down the differential calculus on the quantum Euclidean space underlying the deformed[1] $SO(5;\mathbf{C})$. Quantum groups are defined through their actions[2] on complex vector spaces, and so we need to start with $SO(5;\mathbf{C})$ before restricting to an appropriate real form to obtain $SO(4,1)$.

To obtain this real form, we need to choose a "$*$-structure" (conjugation) on the algebra [32, 35]. We do this for the one-parameter case in section 3.3 and get $SO(4,1)_q$. The definition of conjugation that is necessary for imposing the reality condition on the group elements, induces a conjugation on the underlying quantum space as well. Using this we can choose our coordinates to be real, and by imposing an $SO(4,1)_q$-covariant constraint on the quantum space we get a definition of quantum deSitter space. This is analogous to the imposition of $-(X^0)^2+(X^i)^2=1$ on a five-dimensional Minkowski space (thought of as a normed real vector space) to get the classical deSitter space.

The interesting thing is that the allowed real form of $SO(5;\mathbf{C})_q$ which

[1]We work this out for a multi-parametric deformation, the one-parameter deformation can be obtained in the limit when the parameters are identical.

[2]To be precise we should say co-actions, but we are using the word loosely.

gives rise to $SO(4,1)$ in the $q \to 1$ limit is constrained by the condition that q be real. But the representation theory of standard quantum groups allows finite-dimensional representations only when the deformation parameter is a root of unity[29, 28]. This suggests that to get finite dimensional Hilbert spaces that could possibly be useful for deSitter physics, we might need to look for non-standard deformations. Or it might be an indication that quantum mechanics in deSitter space is too pathological to make sense even after q-deformation.

Later in this chapter, we embark on a different project in the context of quantum groups in de Sitter space: we write down the multi-parametric *algebra* (as opposed to multi-parametric group) for deformed, complexified $SO(5)$. The rationale for doing this is that in the coordinate system of a static observer in deSitter, the manifest isometries are $SO(3)$ and a time-translation (see Appendix A). So one of the questions we need to answer when we quantize in deSitter, is to understand how the static observer and the full de Sitter are related to each other. It is hoped that somehow a multi-parametric algebra might be the first step towards understanding the representation theory of these two Hilbert spaces in relation to each other. In any event, the algebra for multi-parametric q-groups seem to have not been written down in the literature.

The usual one-parameter q-deformation for a Lie Algebra is the Drinfeld-Jimbo Algebra. We will be interested in a construction of this algebra starting with a dual description in terms of R-matrices: using the Faddeev-Reshetikhin-

Takhtadzhyan [32] approach. What we will do is to think of the DJ algebras as defined by the FRT method, and then extend that definition by using a generalized, multi-parametric R-matrix [33, 34]. We will do this explicitly for B_2 and the result will be a multi-parametric generalization of the DJ algebra for B_2 (which is the same as $SO(5)$ at the level of complexified algebras).

3.2 Differential Calculus on the Quantum Euclidean Space

Following [30], we will consider deformations of the differential structure of the underlying space (with co-ordinates x^k, $k = 1, 2, ...N$) by introducing matrices B, C and F (built of numerical coefficients) such that

$$
\begin{aligned}
B^{kl}_{mn} x^m x^n &= 0, && (3.1)\\
\partial_l x^k &= \delta^k_l + C^{km}_{ln} x^n \partial_m, && (3.2)\\
\partial_n \partial_m F^{mn}_{kl} &= 0. && (3.3)
\end{aligned}
$$

In the limit when there is no deformation, these matrices should tend to the limits

$$
\begin{aligned}
B^{kl}_{mn} &\rightarrow (\delta^k_m \delta^l_n - \delta^l_m \delta^k_n), && (3.4)\\
C^{lk}_{nm} &\rightarrow \delta^k_n \delta^m_l, && (3.5)\\
F^{mn}_{kl} &\rightarrow (\delta^n_k \delta^m_l - \delta^n_l \delta^m_k), && (3.6)
\end{aligned}
$$

so that we have the usual algebra of coordinates and their derivatives. We could also deform the commutation relations for 1-forms and exterior differentials, but these follow straightforwardly from the matrices B, C and F upon imposing natural properties like Leibniz rule etc. So we will not concern ourselves with them here.

To construct a calculus on the space that is covariant under the coaction of a quantum group, we will use the R-matrix of the appropriate quantum group to define our matrices B, C and F. To do this, we first look at the matrix[3] $\hat{R}$, which is related to the R-matrix through $\hat{R}^{ij}_{kl} \equiv R^{ji}_{kl}$. This $\hat{R}$ satisfies the quantum Yang-Baxter equation by virtue of the fact that R does:

$$\hat{R}_{12}\hat{R}_{23}\hat{R}_{12} = \hat{R}_{23}\hat{R}_{12}\hat{R}_{23}. \tag{3.7}$$

It has a characteristic equation of the form:

$$(\hat{R} - \mu_1 I)(\hat{R} - \mu_2 I)...(\hat{R} - \mu_m I) = 0. \tag{3.8}$$

It turns out that we can meet all the consistency requirements that B, C and F should satisfy in order for them to define a consistent deformation, if we set

$$C = -\hat{R}/\mu_\alpha, \tag{3.9}$$

$$B = F = \prod_{\beta(\neq\alpha)} (\hat{R} - \mu_\beta I), \tag{3.10}$$

with some choice of the eigenvalue μ_α. With these definitions, the consistency conditions become automatic because $\hat{R}$ satisfies the Yang-Baxter equation.

[3] $\hat{R}$ is also often called the R-matrix, but we will not do so to avoid confusion.

The multi-parametric[33] R-matrix for $SO(2n+1)$ [32, 34] looks like

$$\begin{aligned}
R &= q\sum_{i\neq i'}^{2n+1} E_{ii}\otimes E_{ii} + q^{-1}\sum_{i\neq i'}^{2n+1} E_{ii}\otimes E_{i'i'} + E_{n+1,n+1}\otimes E_{n+1,n+1} + \\
&+ \sum_{i<j,\ i\neq j'}^{2n+1} \frac{q}{r_{ij}} E_{ii}\otimes E_{jj} + \sum_{i>j,\ i\neq j'}^{2n+1} \frac{r_{ij}}{q} E_{ii}\otimes E_{jj} + \\
&+ (q-q^{-1})\Big[\sum_{i>j}^{2n+1} E_{ij}\otimes E_{ji} - \sum_{i>j}^{2n+1} q^{\rho_i-\rho_j} E_{ij}\otimes E_{i'j'}\Big]. \qquad (3.11)
\end{aligned}$$

For $SO(5)$, $n = 2$, i and j run from 1 to 5, and E_{ij} is the 5×5 matrix with 1 in the (i,j)-position and 0 everywhere else. The symbol $\otimes$ stands for tensoring of two matrices. We define $i' = 6 - i$ and $j' = 6 - j$. The deformation parameters are q and r_{ij} and they are not all independent: $r_{ii} = 1$, $r_{ji} = q^2/r_{ij}$ and $r_{ij} = q^2/r_{ij'} = q^2/r_{i'j} = r_{i'j'}$. These relations basically imply that r_{ij} with $i < j \leq n$ determine all the deformation parameters. It should be noted that when all the independent deformation parameters are set equal to each other ($= q$), then the R-matrix reduces to the usual one parametric version. In the case of $SO(5)$, the multi-parametric R-matrix has only two independent parameters, which we will call r and q. Finally, $(\rho_1, \rho_2, ..., \rho_5) = (3/2, 1/2, 0, -1/2, -3/2)$.

The quantum group is defined in terms of matrices $T = (t_{ij})$ so that $RT_1T_2 = T_2T_1R$ where $T_1 = T\otimes I$ and $T_2 = I\otimes T$. For deforming orthogonal groups we also need to specify a norm that is left invariant under the quantum group elements. This is done through the introduction of the matrix $\hat{C}$ (not to be confused with the C introduced earlier) so that $T^t\hat{C}T = T\hat{C}T^t = \hat{C}$ where (for the specific case of $SO(5)$)

$$\hat{C} = \begin{pmatrix} & & & & q^{-3/2} \\ & & & q^{-1/2} & \\ & & 1 & & \\ & q^{1/2} & & & \\ q^{3/2} & & & & \end{pmatrix}. \tag{3.12}$$

We can use an $SO(5)_{q,r}$ invariant constraint of the form $x^t\hat{C}x$ =constant (where $x = \{x^i\}$) to define invariant subspaces of the Euclidean space. With appropriate reality conditions, this can give rise to different signatures in the classical limit.

Using the R-matrix defined above, we can define the $\hat{R}$ matrix for our quantum orthogonal space and when $r \neq q$, it turns out to have five distinct eigenvalues. Of these, $1/q^4, -1/q$ and q are eigenvalues even in the one-parameter case, but $\frac{-1+q^2+\sqrt{(-1+q^2)^2+4r^2}}{2q}$, and $\frac{(-1+q^2)r-\sqrt{4q^4+(-1+q^2)^2r^2}}{2qr}$ exist only when $r \neq q$. By explicit computation using (3.9, 3.10) and (3.1, 3.2, 3.3) we find that the choice of μ_α that gives rise to a non-degenerate deformation of the calculus in the one parameter limit (i.e., when $r \to q$) is $\mu_\alpha = -1/q$. Also (not surprisingly), it turns out that $\mu_\alpha = -1/q$ generates a non-degenerate algebra even in the multi-parameter case. This is the case we will write down in the appendix. The interesting thing is that when $r \neq q$, there are some other choices of μ_α which lead to consistent algebras, which become degenerate[4] in the $r \to q$ limit (i.e, when the multi-parametric deformation reduces to the single-parameter case). In other words, there are some algebras that exist only

[4]i,e., the algebra relations turn into $0 = 0$.

in the multi-parametric case and not in the single-parameter case.

This and the other computations done in this article were implemented using the Mathematica package NCALGEBRA (version 3.7) [39]. The explicit form of the algebra is relegated to an appendix.

3.3 Choice of Real Form for the One-parameter Case

So far we have worked with complexified groups and their deformations. But since we are interested in $SO(4,1)$ which is a specific real form of $SO(5;\mathbf{C})$, we need to impose a reality condition on the q-group elements. For that, we need a definition of conjugation ($*$-structure). The $*$-structure on the quantum group will induce a conjugation on the underlying quantum space, and we want our co-ordinates to be real under this conjugation. In a basis where the co-ordinates and the quantum group elements are real, we can write down the metric $\hat{C}$. If the signature of that metric in the $q \to 1$ limit is $\{-++++\}$, we have the real form that we are looking for. This is the program we will resort to, for writing down $SO(4,1)_q$. In this section, we will be working in the single-parameter context.

Using [32, 35], we define a $*$-structure[5] by the relation

$$T^* \equiv D\hat{C}^t T \hat{C}^t D^{-1} \tag{3.13}$$

[5]A conjugation $*$ on a Hopf algebra A is an algebra anti-automorphism, i.e., $*(\eta a.b) = \bar{\eta}(*(b)).(*(a))$ for all $a, b \in A$ and $\eta \in \mathbf{C}$, that also happens to be a co-algebra automorphism, $\Delta(*) = (* \otimes *)(\Delta)$, $\epsilon(*) = \epsilon$ and an involution, $*^2$ =identity.

where

$$D = \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & -1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix} .$$

The idea here is this: from FRT [32], we know of the conjugation $\star$, which is defined by $T^\star \equiv \hat{C}^t T \hat{C}^t$. Since it is known that $\star$ does not lead to the real form that we are looking for, we use the involution D, to create a new conjugation from $\star$. D can be shown to respect all Hopf algebra structures: it is a Hopf algebra automorphism. Before we go further, it should also be mentioned that in order for the conjugation $\star$ to preserve the RTT equations, the R-matrix should satisfy $\bar{R}^{ij}_{kl} = R^{lk}_{ji}$ which works if $q \in \mathbf{R}$.

Using the fact that quantum groups co-act on the quantum space, we can induce a conjugation on the quantum space which turns out to be $x^* = \hat{C}^t D x$. As per our program, our next step would be to find a linear transformation

$$x \to x' = Mx \tag{3.14}$$

$$T \to T' = MTM^{-1} \tag{3.15}$$

such that x', T' are real under their respective conjugations. Under such a transformation, the metric $\hat{C}$ must go to

$$\hat{C} \to \hat{C}' = (M^{-1})^t \hat{C} M^{-1}. \tag{3.16}$$

One can check that an M that can do this is,

$$M = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & & & & 1 \\ & 1 & & 1 & \\ & & i\sqrt{2} & & \\ & i & & -i & \\ i & & & & -i \end{pmatrix}$$

and under it, the matrix $\hat{C}$ goes to

$$\hat{C}' = \begin{pmatrix} \frac{1}{2q^{3/2}} + \frac{q^{3/2}}{2} & 0 & 0 & 0 & \frac{i}{2q^{3/2}} - \frac{iq^{3/2}}{2} \\ 0 & \frac{1}{2q^{1/2}} + \frac{q^{1/2}}{2} & 0 & \frac{i}{2q^{1/2}} - \frac{iq^{1/2}}{2} & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & -\frac{i}{2q^{1/2}} + \frac{iq^{1/2}}{2} & 0 & \frac{1}{2q^{1/2}} + \frac{q^{1/2}}{2} & 0 \\ -\frac{i}{2q^{3/2}} + \frac{iq^{3/2}}{2} & 0 & 0 & 0 & \frac{1}{2q^{3/2}} + \frac{q^{3/2}}{2} \end{pmatrix}.$$

It is clear that when $q \to 1$, we get the correct signature with one negative and four positive eigenvalues. So we finally have a complete definition of our quantum deSitter space.

We note here that it is not obvious that all the real forms of the classical groups exist after q-deformation. Twietmeyer [36] and Aschieri [35] have classified the possible real forms of $SO(2n+1)_q$, and they find that for real values of q there are 2^n real forms (our deSitter belongs to this category), and for $|q| = 1$ there is only one real form, namely $SO(n, n+1)_q$. Since we expect to get finite dimensional representations of quantum groups only when q is a root of unity[6], for $SO(5)$, these are allowed only for anti-deSitter space[25, 26, 27]: with isometry group $SO(2, 3)$. DeSitter symmetry group can occur only if we choose q to be real, as we saw explicitly.

[6]At other values of q, the representation theory of quantum groups is "pretty much isomorphic" to the representation theory of classical groups.

As already stressed, it might be possible to skirt this issue by working with more generic (multi-parametric or otherwise) deformations and corresponding $*$-structures. But we will not be pursuing those lines here. Maybe deSitter should only be looked at as a resonance or a metastable state in some fundamental theory like string theory.

3.4 Multiparametric Deformation of de Sitter Isometry

Now we turn to a new problem, namely the construction of a multiparametric ("Drinfeld-Jimbo") algebra for the complexified de Sitter group. The motivation for looking at this problem was discussed in the introduction to this chapter, so we turn to the problem straightaway.

3.4.1 One-parameter DJ Algebra and its Dual Description

Drinfeld-Jimbo algebra is a deformation of the universal enveloping algebra of the Lie algebra of a classical group. A universal enveloping algebra is the algebra spanned by polynomials in the generators, modulo the commutation relations. When we deform it, we mod out by a set of deformed relations, instead of the usual commutation relations. These relations are what define the DJ algebra. When the deformation parameter tends to the limit unity, the algebra reduces to the universal enveloping algebra of the usual Lie algebra.

We will write down the algebra relations in the so-called Chevalley-Cartan-Weyl basis. The rest of the generators of the Lie Algebra can be generated through commutations between these. The Drinfeld-Jimbo algebra is constructed as a deformation of the relations between the Chevalley generators. So without any further ado, lets write down the form of the DJ algebra [29] for a generic semi-simple Lie algebra $\mathbf{g}$ of rank l and Cartan matrix (a_{ij}). In what follows, q is a fixed non-zero complex number (the deformation parameter), and $q_i = q^{d_i}$, with $d_i = (\alpha_i, \alpha_i)$ where α_i are the simple roots of the Lie algebra. The norm used in the definition of d_i is the norm defined in the dual space of the Cartan sub-algebra, through the Killing form. These are all defined in standard references [38, 29]. The indices run from 1 to l.

With these at hand we can define the Drinfeld-Jimbo algebra $U_q(\mathbf{g})$ as the algebra generated by E_i, F_i, K_i, K_i^{-1}, $1 \leq i \leq l$, and the defining relations,

$$K_i K_j = K_j K_i, \;\; K_i K_i^{-1} = K_i^{-1} K_i, \tag{3.17}$$

$$K_i E_j = {q_i}^{a_{ij}} E_j K_i, \;\; K_i F_j = {q_i}^{-a_{ij}} F_j K_i, \tag{3.18}$$

$$E_i F_j - F_j E_i = \delta_{ij} \frac{K_i - {K_i}^{-1}}{q_i - q_i^{-1}}, \tag{3.19}$$

$$\sum_{r=0}^{1-a_{ij}} (-1)^r [[1-a_{ij}; r]]_{q_i} E_i^{1-a_{ij}-r} E_j E_i^r = 0, \;\; i \neq j, \tag{3.20}$$

$$\sum_{r=0}^{1-a_{ij}} (-1)^r [[1-a_{ij}; r]]_{q_i} F_i^{1-a_{ij}-r} F_j F_i^r = 0, \;\; i \neq j, \tag{3.21}$$

with

$$\big[[n; r]\big]_q = \frac{[n]_q!}{[r]_q! [n-r]_q!} \tag{3.22}$$

and

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}, \;\; [n]_q! = [1]_q [2]_q ... [n]_q, \;\; [0]_q \equiv 1. \tag{3.23}$$

The relations containing only the Es or the Fs are called Serre relations and they should be thought of as the price that we have to pay in order to write the algebra relations entirely in terms of the Chevalley generators. Sometimes, it is useful to write K_i as $q_i^{H_i}$. In the limit of $q \to 1$, the DJ algebra relations reduce to the Lie Algebra relations written in the Chevalley basis, with H_i's the generators in the Cartan subalgebra and the E_i's and F_i's the raising and lowering operators.

We will be interested in the specific case of $SO(5)$ (Cartan's B_2), and we will rewrite the DJ algebra $U_{q^{1/2}}(\mathbf{so(5)})$ for that case in a slightly different form for later convenience:

$$k_1 k_2 = k_2 k_1,\ k_1^{-1} = q^{H_1 + H_2/2},\ k_2^{-1} = q^{H_2/2} \tag{3.24}$$

$$k_1 E_1 = q^{-1} E_1 k_1,\ k_2 E_1 = q E_1 k_2, \tag{3.25}$$

$$k_1 E_2 = E_2 k_1,\ k_2 E_2 = q^{-1} E_2 k_2, \tag{3.26}$$

$$k_1 F_1 = q F_1 k_1,\ k_2 F_1 = q^{-1} F_1 k_2, \tag{3.27}$$

$$k_1 F_2 = F_2 k_1,\ k_2 F_2 = q F_2 k_2, \tag{3.28}$$

$$[E_1, F_1] = \frac{k_2 k_1^{-1} - k_2^{-1} k_1}{q - q^{-1}},\ [E_2, F_2] = \frac{k_2^{-1} - k_2}{q^{1/2} - q^{-1/2}}. \tag{3.29}$$

The Serre relations take the form:

$$E_1^2 E_2 - (q + q^{-1}) E_1 E_2 E_1 + E_2 E_1^2 = 0 \tag{3.30}$$

$$E_1 E_2^3 - (q + q^{-1} + 1) E_2 E_1 E_2^2 + (q + q^{-1} + 1) E_2^2 E_1 E_2 - E_2^3 E_1 = 0 \tag{3.31}$$

with analogous expressions for the Fs.

Drinfeld-Jimbo algebra is one way to describe a “quantum group”. Another way to do this is to work with the groups directly and deform the group structure using the R-matrices, rather than to deform the universal envelope of the Lie algebra. It turns out that both these approaches are dual to each other, and one can obtain the DJ algebra by starting with R-matrices. Faddeev-Reshetikhin-Takhtadzhyan have constructed a formalism for working with the R-matrices, and to construct the DJ algebra starting from the dual approach. So, a natural place to look for when trying to generalize the DJ algebra of $SO(5)$ is to look at this dual construction and try to see whether it admits any generalizations.

In the rest of this section, we will review the construction of the DJ algebra starting with the R-matrices. In the next section, we will start with a multi-parametric generalization of the R-matrix for $SO(5)$ and follow an analogous procedure to obtain the multi-parametric $SO(5)$ DJ algebra.

As already mentioned, the deformation of the group structure is done in the dual picture through the introduction of the R-matrix. The duality between the two approaches is manifested through the so-called L-functionals [29]. If one defines the L-functionals as certain matrices constructed from the DJ algebra generators, then the R-matrix and the L-functionals would together satisfy certain relations (which we will call the duality relations), as a consequence of the fact that the generators satisfy the DJ algebra. Conversely, we could start with L-functionals thought of as matrices with previously unconstrained matrix elements, and then the duality relations would be the statement that the matrix elements should satisfy the DJ algebra. Thus, the L-functionals, together with the duality relations is equivalent to the DJ algebra.

For any R-matrix, we can define an algebra $A(R)$, with $N(N+1)$ generators l^+_{ij}, l^-_{ij}, $i \leq j, j = 1, 2, ..., N$, and defining relations

$$L^\pm_1 L^\pm_2 R = R L^\pm_2 L^\pm_1,\ L^-_1 L^+_2 R = R L^+_2 L^-_1 \tag{3.32}$$

$$l^+_{ii} l^-_{ii} = l^-_{ii} l^+_{ii} = 1,\ i = 1, 2, ..., N \tag{3.33}$$

where the matrices $L^\pm \equiv (l^\pm_{ij})$ and $l^+_{ij} = 0 = l^-_{ji}$, for $i > j$ (that is, they are upper or lower triangular). The subscripts 1 and 2 have the following meaning: L^+_1 stands for L^+ tensored with the $N \times N$ identity matrix, and L^+_2 stands for the $N \times N$ identity matrix tensored with L^+. So the matrix multiplication with R is well-defined because the R-matrix is an $N^2 \times N^2$ matrix. The above relations will be referred to as the duality relations. It turns out that this algebra has a Hopf algebra structure with comultiplication $\Delta(l^\pm_{ij}) = \sum_k l^\pm_{ik} \otimes l^\pm_{kj}$, counit $\epsilon(l^\pm_{ij}) = \delta_{ij}$, and antipode $S(L^\pm) = (L^\pm)^{-1}$.

Now, lets choose the R-matrix in the above case to be the one-parameter R-matrix for $SO(N)$, with $N = 2n+1$.

$$\begin{aligned} R &= q \sum_{i \neq i'}^{2n} E_{ii} \otimes E_{ii} + q^{-1} \sum_{i \neq i'}^{2n} E_{ii} \otimes E_{i'i'} + E_{n+1,n+1} \otimes E_{n+1,n+1} + \\ &+ \sum_{i \neq j,j'}^{2n} E_{ii} \otimes E_{jj} + \\ &+ (q - q^{-1}) \Big[\sum_{i>j}^{2n} E_{ij} \otimes E_{ji} - \sum_{i>j}^{2n} q^{\rho_i - \rho_j} E_{ij} \otimes E_{i'j'} \Big]. \end{aligned} \tag{3.34}$$

Most of the notation should be clear from the earlier sections, but we repeat the explanations here for convenience. E_{ij} is the $2n \times 2n$ matrix with 1 in the (i, j)-position and 0 everywhere else, and the symbol $\otimes$ stands for tensoring of two matrices. $i' = 2n + 2 - i$ and similarly for j'. The deformation parameter is q. Finally, $(\rho_1, \rho_2, ..., \rho_{2n}) = (n - 1/2, n - 3/2, ..., 1/2, 0, -1/2, ..., -n + 1/2)$.

Let $I(so(N))$ be the two-sided ideal in $A(R)$ generated by

$$L^{\pm} C^t (L^{\pm})^t (C^{-1})^t = I = C^t (L^{\pm})^t (C^{-1})^t L^{\pm} \tag{3.35}$$

where I is the identity matrix, and the metric C defines a length in the vector space where the quantum matrices are acting. C provides the constraint arising from the fact that the underlying classical group is an orthogonal group: $TC^{-1}T^tC = I = C^{-1}T^tCT$ for quantum matrices T (see [34]). For $SO(N)$,

$$C = (C^i_j), \; C^i_j = \delta_{ij'} q^{-\rho_i} \tag{3.36}$$

with j' and ρ_i are as defined above.

Now, $I(so(N))$ is a Hopf ideal of $A(R)$ [29], so the quotient $A(R)/I(so(N))$ is also a Hopf algebra which we will call $U_q^L(\mathbf{so(N)})$. Now, there is a theorem which says that $U_q^L(\mathbf{so(N)})$ is isomorphic to $U_{q^{1/2}}(\mathbf{so(2n+1)})$, which is the DJ algebra for $SO(2n+1)$ with deformation parameter $q^{1/2}$. Explicitly, this isomorphism can be written down as,

$$\begin{aligned}
l^+_{ii} = q^{-H'_i}, \; l^+_{i'i'} = q^{H'_i} \quad &, \quad l^+_{n+1,n+1} = l^-_{n+1,n+1} = 1, \\
l^+_{k,k+1} = (q - q^{-1}) q^{-H'_k} E_k \quad &, \quad l^+_{2n-k+1,2n-k+2} = -(q - q^{-1}) q^{H'_{k+1}} E_k, \\
l^-_{k+1,k} = -(q - q^{-1}) F_k q^{H'_k} \quad &, \quad l^-_{2n-k+2,2n-k+1} = (q - q^{-1}) F_k q^{-H'_{k+1}},
\end{aligned}$$

$$
\begin{aligned}
l^+_{n,n+1} &= (q^{1/2}+q^{-1/2})^{1/2}(q^{1/2}-q^{-1/2})q^{-H'_n}E_n, \\
l^+_{n+1,n+2} &= -q^{-1/2}(q^{1/2}+q^{-1/2})^{1/2}(q^{1/2}-q^{-1/2})E_n, \\
l^-_{n+1,n} &= -(q^{1/2}+q^{-1/2})^{1/2}(q^{1/2}-q^{-1/2})F_n q^{H'_n}, \\
l^-_{n+2,n+1} &= q^{1/2}(q^{1/2}+q^{-1/2})^{1/2}(q^{1/2}-q^{-1/2})F_n
\end{aligned}
$$

Here, $i = 1, 2, ..., n$ as always, and $1 \leq k \leq n-1$. $H'_i = H_i + H_{i+1} + ... + H_{n-1} + H_n/2$. The above relations (which we will call the isomorphism relations) define the relations between elements of the L matrices and the Chevalley-Cartan-Weyl generators. Sometimes it will be convenient to call $q^{-H'_i}$ as k_i because it makes comparison with $SO(5)$ DJ algebra (written earlier) more direct.

3.4.2 The Multi-parametric Algebra

Our procedure for constructing the multi-parametric algebra is straightforward. Instead of using the usual one-parametric R-matrices in the duality relations, we use the multi-parametric R-matrices that Schirrmacher has written down [34]. We keep the isomorphism relations to be the same as above and use the duality relations to define the new multi-parametric algebra.

In principle this procedure could be done for all the multi-parametric R-matrices of all the different Cartan groups using their associated isomorphism relations. We have endeavored to do this procedure for only the case of $SO(5)$, but at least for the smaller Cartan groups, the exact same procedure can be performed on a computer using the appropriate R-matrices. To write down the form of the multi-parametric DJ algebra for a generic semisimple Lie algebra is an interesting problem which we have not attempted to tackle here.

The multi-parametric R-matrix for $SO(2n+1)$ (which for our purposes is the same thing as B_n) was written down earlier, but we repeat it here.

$$\begin{aligned} R &= r\sum_{i\neq i'}^{2n} E_{ii}\otimes E_{ii} + r^{-1}\sum_{i\neq i'}^{2n} E_{ii}\otimes E_{i'i'} + E_{n+1,n+1}\otimes E_{n+1,n+1} + \\ &+ \sum_{i<j,\ i\neq j'}^{2n} \frac{r}{q_{ij}} E_{ii}\otimes E_{jj} + \sum_{i>j,\ i\neq j'}^{2n} \frac{q_{ij}}{r} E_{ii}\otimes E_{jj} + \\ &+ (r-r^{-1})\Big[\sum_{i>j}^{2n} E_{ij}\otimes E_{ji} - \sum_{i>j}^{2n} r^{\rho_i-\rho_j} E_{ij}\otimes E_{i'j'}\Big]. \end{aligned} \tag{3.37}$$

The deformation parameters are r and q_{ij} and they are not all independent: $q_{ii}=1$, $q_{ji}=r^2/q_{ij}$ and $q_{ij}=r^2/q_{ij'}=r^2/q_{i'j}=q_{i'j'}$. These relations basically imply that q_{ij} with $i<j\leq n$ determine all the deformation parameters. It should be noted that when all the independent deformation parameters are set equal to each other ($=q$), then the R-matrix reduces to the usual one parametric version. In the case of $SO(5)$, the multi-parametric R-matrix has only two independent parameters, which we will call r and q.

The Mathematica package NCALGEBRA (version 3.7)[39] was used extensively to do the computations, since the matrix elements (being generators of an algebra) are not commuting objects. The first task is to obtain the duality relations between the matrix elements explicitly. The L matrices are chosen to be upper and lower triangular. The task is straightforward but tedious because the duality relations are 25 by 25 matrix relations for the case of $SO(5)$. So one has to scan through the resulting output to filter out the relations that are dual to the relations between the Chevalley-Cartan-Weyl generators. Doing the calculation for the single-parameter case will give a hint about which relations are relevant in writing down the algebra.

The first line of the isomorphism relations (for the specific case of $SO(5)$) implies that we can use $k_1, k_2, 1, k_2^{-1}$ and k_1^{-1} instead of $l_{11}, l_{22}, l_{33}, l_{44}$ and l_{55} respectively. With this caveat, the algebra looks like the following in terms of the relevant L matrix elements:

$$\begin{aligned}
k_1 k_2 \quad &= \quad k_2 k_1, \\
k_1 l_{12}^+ = \frac{r}{q^2} l_{12}^+ k_1 \quad &, \quad k_2 l_{12}^+ = \frac{q^2}{r} l_{12}^+ k_2, \\
k_1 l_{23}^+ = \frac{q}{r} l_{23}^+ k_1 \quad &, \quad k_2 l_{23}^+ = q^{-1} l_{23}^+ k_2, \\
k_1 l_{21}^- = r l_{21}^- k_1 \quad &, \quad k_2 l_{21}^- = \frac{r}{q^2} l_{21}^- k_2, \\
k_1 l_{32}^- = \frac{q}{r} l_{32}^- k_1 \quad &, \quad k_2 l_{32}^- = q l_{32}^- k_2, \\
\left[l_{45}^+, l_{21}^-\right] \quad &= \quad (q - q^{-1})(k_1^{-2} - k_2^{-2}), \\
\left[l_{23}^+, l_{32}^-\right] \quad &= \quad (q - q^{-1})(k_2 - k_2^{-1}), \\
{l_{12}^+}^2 l_{23}^+ - \left(\frac{q}{r} \quad + \right. &\left. \quad \frac{r}{q^3}\right) l_{12}^+ l_{23}^+ l_{12}^+ + \frac{1}{q^2} l_{23}^+ {l_{12}^+}^2 = 0, \\
\frac{q^2}{r^2} {l_{23}^+}^3 - \left(\frac{q^2}{r} + \frac{q^5}{r^3} + \frac{q}{r}\right) {l_{23}^+}^2 l_{12}^+ l_{23}^+ \quad &+ \quad \left(q + \frac{q^4}{r^2} + \frac{q^5}{r^2}\right) l_{23}^+ l_{12}^+ {l_{23}^+}^2 - \frac{q^4}{r} l_{12}^+ {l_{23}^+}^3 = 0.
\end{aligned}$$

The last two equations correspond to the Serre relations. (We write them down only for the L^+ matrix elements.). As an example of the general procedure for obtaining these algebra relations from the duality relations (i.e., the Mathematica output), we will demonstrate the derivation of the first Serre relation. The relevant expressions that one gets from Mathematica are:

$$
\begin{aligned}
l^+_{12} l^+_{23} - \frac{q}{r} l^+_{23} l^+_{12} &= -(q - q^{-1}) l^+_{13} k_2, && (3.38) \\
l^+_{12} l^+_{13} &= \frac{1}{q} l^+_{13} l^+_{12}. && (3.39)
\end{aligned}
$$

Solving for l^+_{13} from the first equation by multiplying by k_2^{-1} on the right, plugging it back into the second equation and using the commutation rules for k_2, we get our Serre relation. This kind of manipulation is fairly typical in the derivation of the above algebra.

As a next step, we use the isomorphism relations defined at the end of the last section to rewrite the above algebra in terms of the Chevalley-Cartan-Weyl type generators. The result is

$$
\begin{aligned}
k_1 k_2 &= k_2 k_1, \\
k_1 E_1 = \frac{r}{q^2} E_1 k_1 \quad &, \quad k_2 E_1 = \frac{q^2}{r} E_1 k_2, \\
k_1 E_2 = \frac{q}{r} E_2 k_1 \quad &, \quad k_2 E_2 = \frac{1}{q} E_2 k_2, \\
k_1 F_1 = r F_1 k_1 \quad &, \quad k_2 F_1 = \frac{r}{q^2} F_1 k_2, \\
k_1 F_2 = \frac{q}{r} F_2 k_1 \quad &, \quad k_2 F_2 = q F_2 k_2, \\
\frac{q}{r} E_1 F_1 - \frac{r}{q} F_1 E_1 &= \frac{k_2 k_1^{-1} - k_2^{-1} k_1}{q - q^{-1}}, \\
[E_2, F_2] &= \frac{k_2^{-1} - k_2}{q^{1/2} - q^{-1/2}}, \\
E_1^2 E_2 - \Big(\frac{q^2}{r} &+ \frac{r}{q^2}\Big) E_1 E_2 E_1 + E_2 E_1^2 = 0, \\
E_2^3 E_1 - \Big(\frac{r}{q} + \frac{q^2}{r} + \frac{r}{q^2}\Big) E_2^2 E_1 E_2 &+ \Big(\frac{r^2}{q^3} + 1 + q\Big) E_2 E_2 E_2^2 - \frac{r}{q} E_1 E_2^3 = 0.
\end{aligned}
$$

This, is our final form for the multi-parametric version of $SO(5)$ Drinfeld-Jimbo algebra. Notice that this reduce to the one-parameter DJ algebra of $SO(5)$ in the limit of $r \to q$.

Chapter 4

Evolution of Tunneling Bubbles in an Inhomogeneous Cosmology

4.1 Introduction

Tunneling of a new Universe into an old one is an idea has that been known for a long time, but it becomes a particularly interesting scenario in the context of the landcape. One of the questions that has not been adequately (if at all) discussed in the literature is the evolution of such a bubble when the ambient Universe is inhomogeneous. This is an interesting problem on two counts. First, there is the possibility that as the bubble evolves through various inhomogeneous regions, it leaves a signature on the new Universe. The bubble could be thought of as a Casimir cavity with moving boundaries and the resulting radiation could leave detectable signatures on the cosmic microwave background. Second, there is the question of inflation. The question of evolution of the vacuum bubble through inhomogeneities is of direct relevance in understanding how robust inflation is, in smoothing out inhomogeneities.

In this chapter, we will try to formulate the problem by considering the bubble evolution through a specific radially inhomogeneous cosmology, one which we will refer to as the Peebles cosmology. The new phase bubble, we will assume to be a Minkowski spacetime, even though the generalization to de Sitter does not complicate most of the calculations that follow. It is the evolution of the interface that we are interested in. For that, we will use the Israel junction conditions, which will be the equations of motion for the shell. Most of this chapter will be devoted to the construction of this machinery.

To fully describe the evolution of the bubble, we need to prescribe an energy-momentum tensor on the shell hypersurface[1]. Ideally, we would like to derive that from a field theory interpolation between the two phases on either side, and taking the thin-wall limit. But because the metric and the stress tensor on the Peebles side are not known as explicit analytic functions (as we will see later), this is not the route we will take. Following the oldest rule of progress in science, namely that *doing something is better than doing nothing*, we will arbitrarily make the choice that the density on the shell be constant. This choice will be (almost) purely dictated by the criterion that it lend itself to analysis without creating too many complications and that it consistently satisfy Einstein's equations. So what we are doing is not totally wrong, but its real significance is only as a starting point for the numerics before going on to more realistic models. In particular, it is not clear with our choice that the pressure satisfies the positivity conditions throughout the evolution for all choices of the density.

[1] In fact, the symmetries of the problem reduce the number of unknown functions in the shell tensor to just two (density and pressure), and even these have to satisfy the conservation law.

It should be mentioned that we *have* made some progress with a much better choice which satisfies the energy conditions at all stages, namely, a perfect-fluid on the shell, but since our purpose here is to present the formalism and the numerics, I will refrain from presenting those incomplete results.

4.2 The Background Spacetime: Peebles Cosmology

We are interested in the evolution of a Minkowski bubble into a radially inhomogeneous ambient cosmology. The only radially inhomogeneous cosmology that I know of is one (probably) discovered by Lemaitre, but brought to our attention by the work of Peebles. Peebles spacetime, as originally written down, is for the case with no cosmological constant, so we will generalize that to include a positive Λ. We take the (4-dimensional) metric in the form

$$ds^2 = dt^2 - \frac{[(a(t,x)x)_{,x}]^2}{1 - x^2/R(x)^2} dx^2 - (a(t,x)x)^2 d\Omega^2. \tag{4.1}$$

Here x is a radial coordinate, and R can be thought of as a position-dependent spatial curvature. One can immediately see that this form is a direct generalization of Friedmann-Robertson-Walker models. The Einstein equations for this metric can be written as,

$$\frac{1}{x^2 a^2 (ax)_{,x}} \frac{\partial}{\partial x}\left(\frac{ax^3}{R^2} + (\partial_t a)^2 a x^3\right) = \rho + \Lambda \tag{4.2}$$

$$\frac{1}{a^2}\left(\frac{1}{R^2} + (\partial_t a)^2 + 2a\partial_{tt} a\right) = \Lambda \tag{4.3}$$

These come from the G_{tt} and G_{xx} equations: the $G_{\theta\theta}$ equation is just a derivative of the G_{xx} equation, so contains no extra information. The right hand sides contain factors of $8\pi G$, which we have set to 1. Multiplying the second equation throughout by $a^2\dot{a}$, we can rewrite it as,

$$\partial_t\Big[\frac{a}{R^2}+(\partial_t a)^2 a-\frac{\Lambda a^3}{3}\Big]=0,\quad \frac{a}{R^2}+(\partial_t a)^2 a-\frac{\Lambda a^3}{3}=F(x). \tag{4.4}$$

The choice of $F(x)$ is a choice of coordinates: we can make it a constant, call it A, and then use it in the first Einstein equation. The final result is that the system is completely specified by the two equations:

$$(\partial_t a)^2-\frac{A}{a}-\frac{\Lambda a^2}{3}+\frac{1}{R^2} \;=\; 0 \tag{4.5}$$

$$\rho a^2 (ax)_{,x} \;=\; 3A. \tag{4.6}$$

By choosing different profiles for $R(x)$, we can solve for a from the first equation (with an arbitrary integration "constant" $C(x)$), and then plug it back into the second equation to solve for ρ as a function of t and x. Thus the system is completely specified by the choice of $R(x)$ and $C(x)$, both of which are purely functions of x. In fact, we will always set $C(x)$ to zero to keep things simple. There is one caveat though: we need to make sure that the ρ that we calculate with a given choice of $R(x)$, is always positive.

It is possible to make plots of a as a function of t and x for different choices of $R(x)$. Since our plan is to push the technology all the way before getting caught up in the fancier aspects, we will focus almost exclusively on the choice $R(x)=\exp(x)$. Notice that the spacetime is spatially flat at spatial infinity with this choice. It is interesting to look at more complicated $R(x)$'s (which translate to more interesting inhomogeneity profiles) and see how a

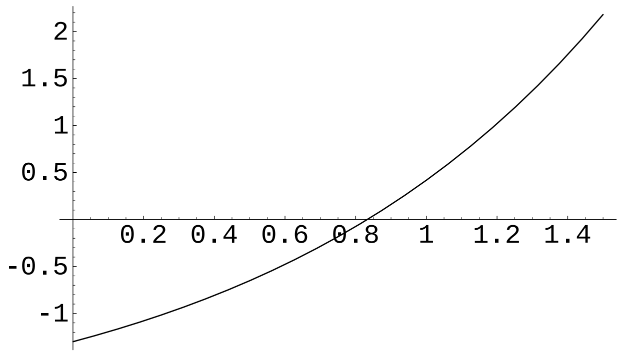

Figure 4.1: Plot of $\left[R(x) - \left(\frac{4}{9\Lambda A^2}\right)^{1/6}\right]$ for $R(x) \equiv R_0(x) = \exp(x)$.

bubble might evolve: this question is currently being looked into. The next few pages contain plots of a couple of choices for $R(x)$ and the corresponding plots of $a(t, x)$.

One thing about the generalized Friedmann equation (4.5), is that for each different value of x, we can integrate for a and find out the time-dependence of the scale factor at that point in the coordinate grid. Notice that x, like in FRW, is just a grid location here, but unlike in FRW, the scale factor $a(t, x)$ is not uniform across the grid, but depends on x. Thus, if we set our initial conditions so that we start out with a Big-Bang at all points on the grid[2], then the question of whether there is a Big-Crunch singularity is a point-dependent question. By tuning $R(x)$, we can control which are the regions where Big-Crunch happens. To be more precise, we can think of (4.5), as an energy equation for a (like that of the point particle in classical me-

[2]this was essentially the idea in setting the integration "constant" $C(x)$ to zero.

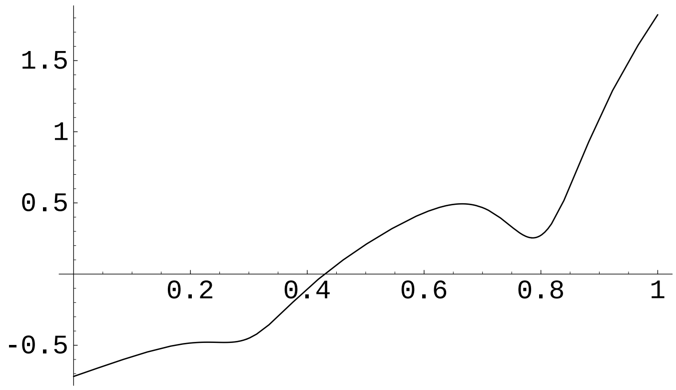

Figure 4.2: Plot of $\left[R(x) - \left(\frac{4}{9\Lambda A^2}\right)^{1/6}\right]$ for $R(x) \equiv R_1(x) = 0.7\left[\left(\frac{4}{9\Lambda A^2}\right)^{1/6} - \frac{0.01}{\pi}\left(\frac{1}{0.1^2+(x-0.3)^2}\right) - \frac{0.02}{\pi}\left(\frac{1}{0.1^2+(x-0.8)^2}\right)\right]\exp(x)$

chanics) at zero total energy. The crucial point is that for each x, we have a different equation because $R(x)$ is not a constant. For those values of x for which the maximum of $V(a) \equiv -\frac{A}{a} - \frac{\Lambda a^2}{3} + \frac{1}{R^2}$ is bigger than zero, if we launch a at zero (Big-Bang), it will reach a maximum and then recollapse. It is easy to check that this happens for all those grid points for which $R(x)$ is less than $(4/9\Lambda A^2)^{1/6}$.

With these explanations, it is easy to understand the Mathematica plots of a as a function of t and x. The collapse happens at those points for which (with the given choice of the function $R(x)$) the above condition is satisfied. Before we move on, lets emphasize a trivial point: the evolution of $a(t, x)$ is not to be confused with the evolution of the bubble through the spacetime, to be introduced later. So far we have only described the ambient spacetime, and by specifying (perhaps numerically) the time-dependence of $a(t, x)$ at each

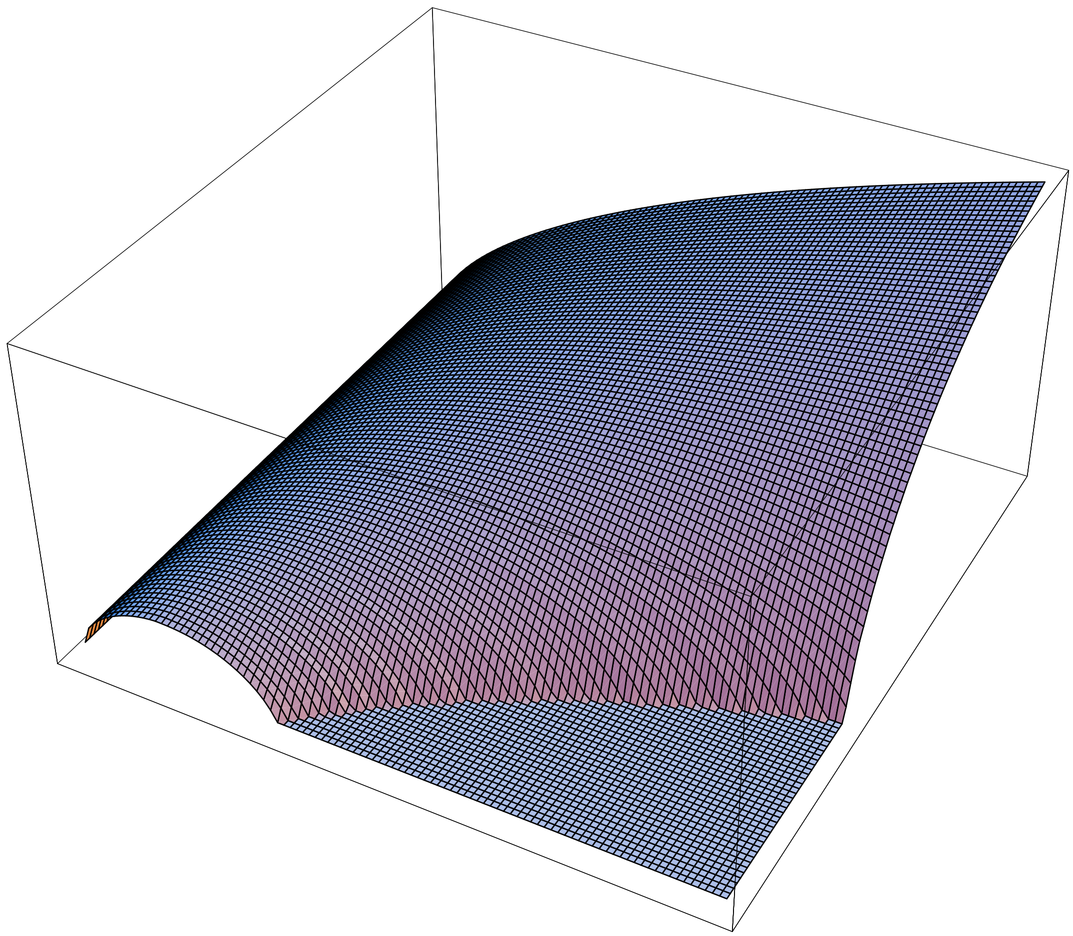

Figure 4.3: Plot of $a(t,x)$ for $R(x) = R_0(x)$. With right-hand thumb rule, the thumb (vertical axis) denotes $a(t,x)$, the index finger denotes time t, and the middle-finger, the grid location x.

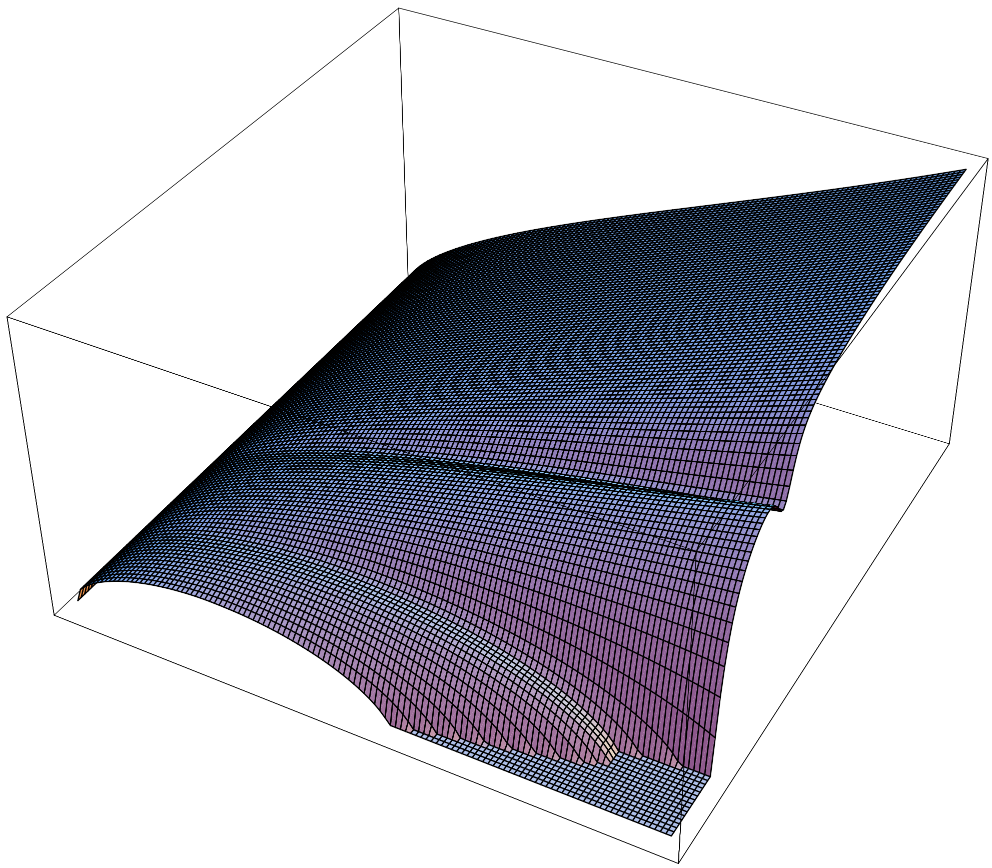

Figure 4.4: Plot of $a(t, x)$ for $R(x) = R_1(x)$. The axes are as before.

x, we can complete that task for any choice of $R(x)$. But the evolution of a Minkowski bubble that can form inside this spacetime is an entirely new game in itself, to which we now turn.

4.3 Israel's Thin Shell Approach

Because the machinery of thin shells is usually presented from a formal perspective in most references, we will attempt to give a practical introduction to the subject here based on Israel's original work.

4.3.1 The Setup

The situation we have is this: a hypersurface Σ separates spacetime into two regions V^+ and V^-. For us, these regions will be the Peebles side and the Minkowski side respectively. We will take $x^\pm_\alpha$ and $g^\pm_{\alpha\beta}$ to stand for the coordinates and the metric on the two sides. When we match the two spacetimes at Σ, we want to find out what are the conditions under which the metrics on the two sides together give rise to a consistent solution of Einstein's equations. We are willing to think of the interface as a limit of progressively sharper transitions, so we are willing to think of Einstein's equations in terms of distributions, but that is as far as we are willing to go.

The problem is non-trivial because of the diffeomorphism invariance of gravity: comparing the spacetimes on either side is not straightforward because $x^\pm$ might be different sets of coordinates, so the metrics are not directly comparable. Israel's strategy for overcoming this issue is to formulate the problem in terms of the surface Σ and the quantities defined on the surface (three-tensors). These quantities can be defined on the surface by projecting four-dimensional quantities from either side: once they are defined on the surface in terms of the three-dimensional geometry, it is alright to compare[3] them. The resulting conditions for a smooth matching will be called the *Israel junction conditions*, which are essentially Einstein's equations in disguise, written in terms of three-dimensional quantities.

As a first step towards this program, we first lay down coordinates x^α in an open subset of $V^+ \cup V^-$ that contains a patch of the hypersurface Σ. The importance of these coordinates is that unlike $x^\pm$, these coordinates extend from V^- to V^+. We are allowed to define these coordinates, because we are looking not at discontinuities in the spacetime manifold itself, but only on some functions (perhaps curvature, stress tensor etc.) defined *on* this manifold. These coordinates obviously overlap with $x^\pm$ (where they are defined) and have the corresponding transition functions. It is crucial to notice here that x^α are defined only temporarily, and we will only need them as a crutch to get to the junction conditions: in particular, we need only the fact that such a coordinate system exists, we will not explicitly calculate the metric (for example) in these coordinates.

We also lay down coordinates y^a which are defined on the shell. It is conventional in the literature to reserve Latin indices for coordinates on the

[3]the details of what it means to "compare", "match" etc. will be clear in a minute.

hypersurface and Greek indices for spacetime coordinates, and we follow that convention.

Next, we need to define a normal to the hypersurface, because it is along the normal direction that we want to understand the discontinuities, matching etc. The easiest way to do this is to note that in terms of the coordinates x^α, the hypersurface is defined by an equation

$$\Phi(x^\alpha) = 0 \tag{4.7}$$

for some function Φ of the coordinates. So we can define the normal to the hypersurface according to

$$n_\alpha = \frac{-\Phi,_\alpha}{|g^{\mu\nu}\Phi,_\mu \Phi,_\nu|^{1/2}} \equiv -\partial_\alpha n. \tag{4.8}$$

We will take Φ to be increasing from V^- to V^+, and the surface Σ to be time-like (i.e, $n^\alpha n_\alpha = -1$), so the normal is outward directed ($n^\alpha \Phi,_\alpha > 0$). The negative sign on (4.8) is to accommodate these choices, it is in fact the same negative sign from the time-likeness of the hypersurface. Note that n, as defined, is a measure of the separation from Σ. It effectively is a coordinate along the normal direction.

4.3.2 The First Junction Condition

In terms of distributions, we can write

$$g_{\alpha\beta} = \theta(n) g^+_{\alpha\beta} + \theta(-n) g^-_{\alpha\beta} \tag{4.9}$$

where θ is the step function and the metrics on the right hand side are thought of as written in the x^α coordinates[4]. Our task now is straightforward. We just need to make sure that with this distributional definition of the metric tensor, when we calculate the Einstein equations, we can make sense of the results as distributions.

To calculate the Einstein equations, we need to take two derivatives of the metric, and after the first derivative, we find an expression that looks like

$$g_{\alpha\beta,\gamma} = \theta(n) g^{+}_{\alpha\beta,\gamma} + \theta(-n) g^{-}_{\alpha\beta,\gamma} - \delta(n)[g_{\alpha\beta}]n_\gamma, \tag{4.10}$$

where $\delta(n)n_\gamma = \partial_\gamma \theta(n)$, and we use the standard notation:

$$[X] \equiv X|_{\Sigma^+} - X|_{\Sigma^-}, \tag{4.11}$$

i.e., $[X]$ stands for the jump of the quantity X across the hypersurface.

As already advertised, we want the Einstein equation to make sense as a distribution. But if we allow the last term in the above equation to be non-zero, then there will be mixed terms containing $\theta(n)\delta(n)$ (which does not make sense as a distribution: it cannot be integrated) in the Christoffel symbols. So we come to the conclusion that $[g_{\alpha\beta}] = 0$ if we want the matching to work. This is the first junction condition. We can project this onto the 3-surface, by using the $e^\alpha_a = \frac{\partial x^\alpha}{\partial y^a}$. These e^α_a are continuous across the 3-surface (because x^α is continuous and y^a is a coordinate on the shell and so it is the same from

[4]Again, these coordinates are mere crutches used to derive our results, so we will never need to do these coordinate transformations explicitly.

either side.), so they can be taken inside the square brackets. Thus we end up with the intrinsic form of the first junction condition:

$$[h_{ab}] \equiv [g_{\alpha,\beta} e^\alpha_a e^\beta_b] = 0, \tag{4.12}$$

where h_{ab} is, by definition, the intrinsic metric tensor on the shell. The first junction condition is natural because we would expect the metric on the surface to be the same, as projected from either side. Notice that in this form, the equation is independent of our choice of coordinates, as promised - this is an intrinsic formulation.

It might seem that we are losing some information in going from $[g_{\alpha\beta}] = 0$, with ten components, to $h_{ab} = 0$, with six. But this is not the case because the extra four are in fact automatically satisfied. To see this, we can project the $[g_{\alpha\beta}] = 0$ onto the normal direction (which is the only remaining direction). But now, since Φ and x^α are both continuous across Σ, so is $n_\alpha \sim -\Phi_{,\alpha}$ and thus it can be freely taken inside the square brackets. Since the metric is continuous (by the first junction condition), the normal with upper indices is also continuous. So when we project, we get $0 = [g_{\alpha\beta}]n^\beta = [n_\alpha]$, which is a four-component identity. So, we are not missing anything by not looking at the metric components along the normal direction.

4.3.3 The Second Junction Condition

With one derivative, we got the first junction condition, but we need the Einstein equations which contain two derivatives, so we press on. From what we have from the previous subsection, we calculate the Christoffel symbols to be

$$\Gamma^{\alpha}{}_{\beta\gamma} = \theta(n)\Gamma^{+\alpha}_{\ \ \beta\gamma} + \theta(-n)\Gamma^{-\alpha}_{\ \ \beta\gamma}. \tag{4.13}$$

To construct the curvature tensors, we first take the derivative of this:

$$\Gamma^{\alpha}{}_{\beta\gamma,\delta} = \theta(n)\Gamma^{+\alpha}_{\ \ \beta\gamma,\delta} + \theta(-n)\Gamma^{-\alpha}_{\ \ \beta\gamma,\delta} - \delta(n)[\Gamma^{\alpha}{}_{\beta\gamma}]n_{\delta}, \tag{4.14}$$

and using these the Riemann tensor becomes,

$$R^{\alpha}{}_{\beta\gamma\delta} = \theta(n)R^{+\alpha}_{\ \ \beta\gamma\delta} + \theta(-n)R^{-\alpha}_{\ \ \beta\gamma\delta} - \delta(n)[\Gamma^{\alpha}{}_{\beta\delta}n_{\gamma} - \Gamma^{\alpha}{}_{\beta\gamma}n_{\delta}]. \tag{4.15}$$

Once we know the delta-function part of the Riemann tensor we can calculate the delta-function piece in the Einstein tensor by taking traces. The result is,

$$G_{\alpha\beta} = \theta(n)G^{+}_{\alpha\beta} + \theta(-n)G^{-}_{\alpha\beta} - \delta(n)\Big[\Gamma^{\alpha}{}_{\beta\delta}n_{\alpha} - \Gamma^{\alpha}{}_{\beta\alpha}n_{\delta} - \frac{1}{2}g_{\beta\delta}\big(g^{\mu\nu}\Gamma^{\alpha}{}_{\mu\nu}n_{\alpha} - \Gamma^{\alpha}{}_{\mu\alpha}n^{\mu}\big)\Big]. \tag{4.16}$$

If the stress-tensor outside/inside is $T^{\pm}_{\alpha\beta}$, then the above Einstein tensor will satisfy the Einstein equation if the total stress tensor is

$$T_{\alpha\beta} = \theta(n)T^{+}_{\alpha\beta} + \theta(-n)T^{-}_{\alpha\beta} + \delta(n)S_{\alpha\beta}, \tag{4.17}$$

with

$$S_{\alpha\beta} = -\Big[\Gamma^{\alpha}{}_{\beta\delta}n_{\alpha} - \Gamma^{\alpha}{}_{\beta\alpha}n_{\delta} - \frac{1}{2}g_{\beta\delta}\big(g^{\mu\nu}\Gamma^{\alpha}{}_{\mu\nu}n_{\alpha} - \Gamma^{\alpha}{}_{\mu\alpha}n^{\mu}\big)\Big]. \tag{4.18}$$

So we see that the delta-function in curvature has an interpretation as the energy-momentum tensor on the shell.

This is a perfectly correct result, but it would be more useful to write it in terms of intrinsic 3-surface quantities. For that, first we note by explicit calculation that $S_{\alpha\beta}n^{\beta} = 0$. So, we can project $S_{\alpha\beta}$ on the shell and still retain all the information. Thus we define

$$S_{ab} = S_{\alpha\beta}e^{\alpha}_{a}e^{\beta}_{b}. \tag{4.19}$$

It can be shown that this S_{ab} can be written in terms of the extrinsic curvature of the shell. The extrinsic curvature is defined as

$$K_{ab} = n_{\alpha;\beta}e^{\alpha}_{a}e^{\beta}_{b}, \tag{4.20}$$

and after some algebra, we can show that

$$S_{ab} = [K_{ab}] - [K]h_{ab}, \tag{4.21}$$

where $K = K_{ab}h^{ab}$. The only inputs for this calculation are the definitions of the relevant quantities, the formula connecting covariant derivatives and Christoffel symbols, and the continuity conditions for the various quantities across the 3-surface that we have already written down. This equation is the second junction condition.

So the upshot is that a delta-function curvature discontinuity can be interpreted as an energy-momentum tensor on the shell, and we can still have a consistent interpretation of Einstein's equations.

4.4 The Minkowski-Peebles Junction

Now we use these conditions for the specific case of a spherically symmetric match between a Minkowski bubble and an ambient Peebles spacetime. The outside spacetime is given by the Peebles metric (4.1), and the inside metric is Minkowski, which we will take in the form

$$ds^2 = dT^2 - dr^2 - r^2 d\Omega^2. \tag{4.22}$$

These two spacetimes are matched on a spherical 2-D shell whose evolution is unknown to begin with. Since we are interested in finding out the evolution of a shell that separates two phases, ideally, we would like to have a field theoretic model for the stress tensor on either side of the shell-wall. We would like to calculate the shell stress tensor by interpolation between the two phases in the field theory and taking the thin wall limit. But since the Peebles metric is fairly complex, we do not know the explicit analytic form of the metric or the energy tensor there. We resorted to numerical evolution to make plots of $a(t, x)$ in an earlier section. Without having an explicit form of the energy-momentum tensor, it is hard to come up with a field theoretic model for it. Coupled to this is the fact that we do not know the shell evolution and the hypersurface that it traces out, to begin with. All these problems force us to start out with a simplified model for the energy-momentum tensor without worrying too much about an underlying field-theoretic derivation.

In any event, the assumption of spherical symmetry restricts the form of the intrinsic metric on the shell to the form,

$$ds_3{}^2 = d\tau^2 - \rho(\tau)^2 d\Omega^2. \tag{4.23}$$

We are using the same symbol ρ for the proper radius of the shell, as well as the matter density in the Peebles spacetime (defined earlier), but the context should clarify the meaning. On the shell, the Peebles metric and the Minkowski metric must reduce to the shell 3-D metric. Spherical symmetry implies that there is only one independent coordinate on the shell, the shell proper time[5]. So looked at from the Peebles side, on the shell, we can parametrize the coordinates as $x(\tau)$ and $t(\tau)$, and from the Minkowski side, $T(\tau)$ and $r(\tau)$. Since the metric from either side on the shell should agree with the 3-metric on the shell (the first junction condition), we get two conditions for the Minkowski side,

$$r(\tau) = \rho(\tau), \ \left(\frac{dT}{d\tau}\right)^2 = \left(\frac{dr}{d\tau}\right)^2 + 1, \tag{4.24}$$

and two for the Peebles side,

$$a\big(t(\tau), x(\tau)\big)x(\tau) = \rho(\tau), \ \left(\frac{dt}{d\tau}\right)^2 = \frac{\big(a(t,x)x\big)_{,x}{}^2}{1 - x^2/R(x)^2} + 1, \tag{4.25}$$

with everything thought of as functions of τ.

Now we turn to the second junction conditions. Since we have different sets of coordinates in each region, and since not all of these coordinates can be written in terms of the others, we will write the matching conditions in terms

[5]apart from the angular coordinates on which nothing depends.

of the Peebles coordinates[6]. The second junction conditions are essentially evolution equations for the shell if the shell stress tensor is the input data, so once we figure out that evolution numerically, we can in principle convert to other coordinates if we want.

We will use the second junction condition with one index up and the other low. Because of the spherical symmetry and the form of the three metric, K_a^b has components K_τ^τ and $K_\theta^\theta = K_\phi^\phi$, while S_a^b contains S_τ^τ and $S_\theta^\theta = S_\phi^\phi$. So the second junction conditions become

$$-\frac{1}{2}S_\tau^\tau = [K_\theta^\theta] \tag{4.26}$$

$$-S_\theta^\theta = [K_\tau^\tau] + [K_\theta^\theta]. \tag{4.27}$$

We will call S_τ^τ as σ and S_θ^θ as p from now on for convenience. These are the only two components of the shell energy-momentum tensor and they are both purely functions of τ by spherical symmetry. As already stressed, since we don't know how exactly to determine these functions from a field theory, we will for the moment assume (as a toy model for developing the numerics) that σ is a constant. From the first equation, this then will fix the evolution of the shell (i.e, we will know what the hypersurface looks like) and then we can plug that information back into the second equation and use that as a *definition* of p. We will not worry about the fact that the p obtained this way might not always satisfy the various energy positivity conditions. This approach essentially gets

[6]Since Peebles metric is known only numerically, there is no hope of writing it in terms of other coordinates.

rid of the second equation since if σ is a constant, it is just a definition. This affords considerable simplification in understanding the solution space of the evolution curves for the bubbles. In fact, the understanding gained by this enormous simplification turns out to be very useful in getting a handle on the more realistic case where instead of a constant σ, we assume a linear equation of state[7] between σ and p.

So lets write down the first equation for the matching for our case. We start with

$$K_\theta^\theta = h^{\theta\theta} K_{\theta\theta} = \frac{1}{\rho^2} n_{\alpha;\beta} e_\theta^\alpha e_\theta^\beta = \frac{1}{\rho^2} n_{\theta;\theta} e_\theta^\theta e_\theta^\theta = \frac{1}{\rho^2} n_{\theta;\theta}, \tag{4.28}$$

which is true on either side of the shell. We have used spherical symmetry to note that some of the components of the projectors are zero. To proceed from this point on, and thereby determine the K_θ^θ on either side, we will have to write down the normal vector in the corresponding coordinates. We will start with the Peebles side where the coordinates are $x^\alpha = (t, x, \theta, \phi)$. First, lets write down the projectors:

$$u_\alpha \equiv e_\tau^\alpha = \Big(\frac{dt}{d\tau}, \frac{dx}{d\tau}, 0, 0\Big) \tag{4.29}$$

$$e_\theta^\alpha = \big(0, 0, 1, 0\big) \tag{4.30}$$

$$e_\phi^\alpha = \big(0, 0, 0, 1\big) \tag{4.31}$$

Since u^α is the 4-velocity of the bubble, the normal n_β will be determined (upto a sign) by the two conditions $u^\alpha n_\alpha = 0$ and $n^\alpha n_\alpha = -1$, where raising

[7]We do not report on these perfect fluid solutions here, though.

and lowering are done with the Peebles metric. A simple calculation yields,

$$n_\alpha = \left(\frac{-(ax)_{,x}\,\dot{x}}{\sqrt{(1-x^2/R^2)\dot{t}^2 - {(ax)_{,x}}^2\dot{x}^2}}, \frac{(ax)_{,x}\,\dot{t}}{\sqrt{(1-x^2/R^2)\dot{t}^2 - {(ax)_{,x}}^2\dot{x}^2}}, 0, 0 \right) \tag{4.32}$$

where everything is thought of as a function of τ (and the dots are with respect to τ as well). A little thought, and the discussion around (4.8) reveals that the overall sign of n_α has been chosen so that this is the outward normal. With this, we can now calculate the K^θ_θ on the Peebles side to be,

$$\begin{aligned}
K^\theta_{\theta+} &= \frac{1}{\rho^2}\Big(n_{\theta,\theta} - \Gamma^\alpha_{\theta\theta} n_\alpha\Big) \\
&= \frac{1}{\rho^2}\Big(- \Gamma^t_{\theta\theta} n_t - \Gamma^x_{\theta\theta} n_x\Big) \qquad (4.33)\\
&= \frac{1}{2\rho^2} g^{tt} g_{\theta\theta,t} n_t + \frac{1}{2\rho^2} g^{xx} g_{\theta\theta,x} n_x \qquad (4.34)\\
&= \frac{(ax)_{,t}\,(ax)_{,x}\,\dot{x} + (1-x^2/R^2)\dot{t}}{\rho\sqrt{(1-x^2/R^2)\dot{t}^2 - {(ax)_{,x}}^2\dot{x}^2}}. \qquad (4.35)
\end{aligned}$$

We have used (4.25) to do some of the simplifications. This equation is written in a mixed notation containing coordinates intrinsic to the shell as well as the Peebles coordinates, we will fix that shortly and write it purely in terms of Peebles. Before doing that, we will repeat the above calculation for K^θ_θ on the Minkowski side, with coordinates (T, r, θ, ϕ). Proceeding as before, we have

$$n_\alpha = \left(\frac{-\dot{r}}{\sqrt{\dot{T}^2 - \dot{r}^2}}, \frac{\dot{T}}{\sqrt{\dot{T}^2 - \dot{r}^2}}, 0, 0 \right) \tag{4.36}$$

and so,

$$K^\theta_{\theta-} = \frac{\dot{T}}{r\sqrt{\dot{T}^2 - \dot{r}^2}} \tag{4.37}$$

With these the first junction condition becomes,

$$\frac{(ax)_{,t}\,(ax)_{,x}\,\dot{x} + (1 - x^2/R^2)\dot{t}}{\rho\sqrt{(1 - x^2/R^2)\dot{t}^2 - (ax)_{,x}{}^2\dot{x}^2}} - \frac{\dot{T}}{r\sqrt{\dot{T}^2 - \dot{r}^2}} = -\sigma/2. \tag{4.38}$$

We came to this equation through a route that is fairly specific to our problem, but we can rewrite this in a standard form by using (4.24) and (4.25). The second term on the LHS (the Minkowski piece) becomes, $-\sqrt{\dot{\rho}^2+1}/\rho$. Less trivial, but still straightforward, is the demonstration that the first term (Peebles) can written as $\sqrt{\dot{\rho}^2 + 1 - x^2/R^2 - (ax)_{,t}{}^2}$. This demonstration is easier if we start from the final expression, expand $\dot{\rho}$ as $(ax)_{,t}\,\dot{t} + (ax)_{,x}\,\dot{x}$ and use the relation

$$\left(1 - \frac{x^2}{R^2}\right)(\dot{t}^2 - 1) = (ax)_{,x}{}^2\dot{x}^2 \tag{4.39}$$

once, assemble a perfect square from the pieces, and then use (4.39) again. The relation (4.39) is just a rewriting of (4.25). So finally we get the standard form:

$$\sqrt{\dot{\rho}^2 - \Delta_+} - \sqrt{\dot{\rho}^2 - \Delta_-} = -\frac{\sigma\rho}{2}, \tag{4.40}$$

where $\Delta_+ = -(1 - x^2/R^2) + (ax)_{,t}{}^2$, and $\Delta_- = -1$. We can do some more massaging to bring it into a slightly more convenient form. First, use the Peebles equation of motion (the analog of the Friedmann equation) to substitute for $\dot{a}$ and rewrite Δ_+ as,

$$\Delta_+ = -1 + \left(\frac{A}{a^3} + \frac{\Lambda}{3}\right)\rho^2. \tag{4.41}$$

Next, we square (4.40) twice to write it in the form,

$$\dot{\rho}^2 = \frac{\sigma^2\rho^2}{16} + \frac{\Delta_+ + \Delta_-}{2} + \frac{(\Delta_+ - \Delta_-)^2}{\sigma^2\rho^2}. \tag{4.42}$$

Now, we take up the task of writing these equations purely in the Peebles coordinates, for otherwise, despite being correct, they will be of no practical value in calculating anything. Using (4.41), we can write (4.42) as

$$\dot{\rho}^2 = \rho^2 B^2 - 1, \tag{4.43}$$

where

$$B^2 = \frac{1}{9\sigma^2}\Big(\frac{3\sigma^2}{4} + \Lambda + \frac{3A}{a^3}\Big). \tag{4.44}$$

The advantage of this form is that everything except $\dot{\rho}$ and ρ are in Peebles coordinates, so we just have to focus on these. Since $\rho = ax$, the task really is to look express $\dot{\rho}$ in terms of x and t. The idea is that instead of choosing $\rho(\tau)$ as the evolution curve of the bubble-evolution[8], we want to parametrize it as $x(t)$. (Notice that x and t are dependent variables *on* the shell.). So once we make this choice, the parameter on the shell is t, not τ. Being careful about the partial derivatives, we can rewrite $\dot{\rho}$ as

$$\dot{\rho} = \frac{\Big(x\partial_t a + (ax)_{,x}\frac{dx}{dt}\Big)}{\Big(1 - \frac{(ax)_{,x}{}^2}{1-x^2/R^2}\Big(\frac{dx}{dt}\Big)^2\Big)}. \tag{4.45}$$

Using this is (4.43), expanding, and using (4.1) for $\partial_t a$, we end up with,

$$\frac{dx}{dt} = \frac{\left\{\begin{aligned} &-x(1-\tfrac{x^2}{R^2})\Big(\tfrac{A}{a} + \tfrac{\Lambda a^2}{3} - \tfrac{1}{R^2}\Big)^{1/2} \pm \\ &\qquad \pm x\Big\{(1-\tfrac{x^2}{R^2})(a^2x^2B^2 - 1)\Big(a^2B^2 - \tfrac{A}{a} - \tfrac{\Lambda a^2}{3}\Big)\Big\}^{1/2}\end{aligned}\right\}}{(ax)_{,x}\,(a^2x^2B^2 - x^2/R^2)} \tag{4.46}$$

[8]actually each point on the evolution-"curve" is a sphere because the (θ, ϕ) coordinates go along for the ride.

This is the shell evolution equation in a form that is directly applicable for numerical simulations, which is what we will do in the next section. It should be mentioned here that we have only used one (4.26) of the two equations from the second junction condition for deriving this. The second equation, as we stated earlier, is being thought of as a definition for p, so we don't need it to determine the evolution. If we were to allow the possibility that σ was not constant -with p determined through an ideal fluid equation of state- then we would have to solve a set of simultaneous equations, the one for x the same as above (but with σ time-dependent), and the other a differential equation for σ coming from (4.27)[9].

4.5 Shell Evolution

We plot the results of the shell evolution using Mathematica and plot them in this section. The plots of x and ρ as functions of t are plotted below, for the choice $R(x) = \exp(x)$. Notice that to do the evolution, we first have to numerically evaluate $a(t, x)$ as well. The Mathematica code for doing these is provided in an appendix.

The general structure of the curves is easy to understand. The evolution equation is quadratic in $\frac{dx}{dt}$, so there is a critical curve on the $x - t$ plane on

[9] The second equation (4.27) in the second junction condition, actually contains two time derivatives. But we can use a derivative of the first equation (4.26) to rewrite it as a first order equation. This is analogous to the use of covariant energy conservation instead of an Einstein equation. In our attempts to extend the present work to the case of a perfect fluid, this is what we do, and that is what brings the system down to a set of first order differential equations.

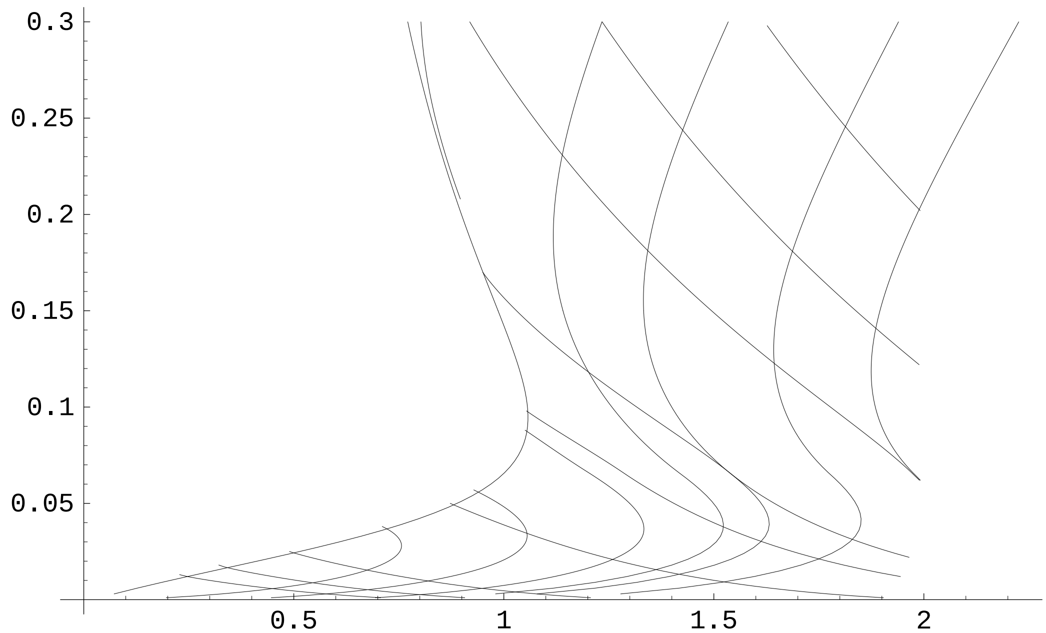

Figure 4.5: Plot of x vs. t shell evolution.

one side of which there are no solutions. This corresponds to the region where the discriminant of the quadratic equation is negative. On the other side of the critical line, we expect that through each point, there would be two curves - because the degree of this equation is two, even though its order is one. This is precisely what we see. The shape off the critical curve depends on the inhomogeneity profile $R(x)$. I am referring to $R(x)$ as the inhomogeneity profile even though its really a profile of spatial curvature, because it directly affects the determination of $\rho(t, x)$.

In the case that we have worked out here, the $\exp(x)$ gives rise to a central region of the Peebles Universe where there is a collapse. The outer regions do not collapse. We see from the graph that the evolution of the bubble depends on where we launch it. There are some regions of initial conditions where the bubble ultimately recollapses on to the critical line, along both the curves through that point. This is vaguely analogous to the case of black holes where both the outgoing and ingoing light-fronts get refocused, and we suspect that this is a symptom of the fact that the Peebles universe in our case has a collapsing central region where we are launching the bubble. For some initial conditions, one would expect that a bubble launched in a collapsing region would not be able to escape that region, whether it starts off on the expanding or collapsing branch of the quadratic equation. There are of course some other choices of initial conditions where the bubble seems to get out of the collapsing region in the coordinate grid.

These plots are a first step towards a better understanding of bubble evolution in an inhomogeneous Universe. There are a few directions that we could work on from here. The first is, trying more complicated inhomogeneity profiles $R(x)$. It seems not inconceivable that with other choices of $R(x)$ there might be more complicated critical curves, for instance, there might be islands of criticality in the $x - t$ plane. These would be useful in understanding the generic evolution patterns of bubbles.

Second, the real meaning of the critical line needs to be clarified further in the context of tunneling. In the case of a de Sitter-de Sitter bubble, the function B that was defined for the shell equation of motion is a constant, and there the critical proper radius of the bubble is given by $B = 1/\rho_c$, which is a constant. So there, the critical line is a straight line p to the ρ axis. This is the minimum proper radius for a tunneling bubble to nucleate so that it can grow. But in our case we see that the critical radius is not a constant and depends on where you launch the bubble. And in fact a classically nucleated bubble can sometimes recollapse back to criticality later on. This needs interpretation from the tunneling perspective.

Finally, we need to make a better choice of the energy-momentum tensor on the shell. The perfect fluid is currently on its way. The difference there, as already stressed, is that we need to keep track of the second of the two equations that arises from the second junction condition as well. This second equation can be taken in the form of a covariant energy conservation equation[10] (instead of the original Israel formula), and that considerably simplifies things by bringing down the order of the differential equation by one. The crucial difference between the constant-σ case is that now we have a system of two differential equations, so the plots should be thought of as curves in (x, σ, t)-space. So their projections on the $x - t$ plane can be significantly more complicated. Also, the critical curve now becomes a critical surface.

[10]Even for the constant σ case that we have considered here, we can use this equation to see how p evolves. It is an interesting problem to see whether there is any connection between the nature of bubble evolution curves and the values of p at which positivity conditions of energy-momentum is violated.

Chapter 5

The Future

We looked at some problems that arise when we try to understand fundamental physics in a Universe with a positive cosmological constant. The cosmological constant is one of the most obstinate problems one faces in any theory of quantum gravity, and therefore, as Polchinski has commented, perhaps one of the best clues we have. It was as much of a problem in string theory as it was in field theory, but maybe through the recent landscape proposals now we have a new (perhaps philosophically unsatisfactory) paradigm for its resolution.

Instead of taking stock of what was done in the last few chapters, I will speculate on possible future directions that extend this work. The most important place where I see the need for extension is in the evolution of tunneling bubbles: it would be nice to have a realistic energy-momentum tensor on the shell. The next best thing would be to try a perfect fluid on the shell, and that is currently under investigation. Once we have that in place, we could try to understand the radiation inside the shell as it moves through the various inhomogeneous regions. This is the four-dimensional analogue of to the moving mirror in 2-dimensions. Realistically, it seems unlikely that this might leave an observable imprint on the CMB spectrum, but at least we might be able to rule out certain scenarios or potentials in the landcape.

On a broader scale than the narrow problems addressed in this thesis, there is always the question of whether we are on the right track in following string theory. Despite the nice things that happen everywhere in string theory, on some days it is hard to escape the feeling that one might just be playing with an elaborately constructed mathematical tautology which is so general that it contains (almost) everything imaginable, and therefore perhaps contains no physical information. This is of course, not to say that the "other" candidate quantum gravity theories in town are somehow better. The arguments presented in the introduction, make the author think that the ultimate "final theory" (or quantum gravity), will definitely be a string theory. But the real task is to figure out why *our* Universe with *our* familiar particle physics and gravity is the one that arises in string theory in some natural way, and not some qualitatively similar other Universe. And it would be nice if we could do this without having to go anthropic. Despite the present excitement over the landscape, I think it would be nice if we could

a) construct (from string theory) *a* vacuum that looks precisely like our Universe: with the cosmological constant and the standard model - whether that vacuum is unique or not. But there are some no-go "theorems" in the literature that say that this problem is probably impossible in a practical, computational complexity sense.

or b) find some non-perturbative breakthrough which gives us a grasp of the vacuum selection mechanisms in string theory. At the very least, with conclusive results along these lines, we can be confident that going anthropic is a necessity in string theory and not a result of the frustration of a generation from battling hard problems for too long.

These are obviously *very* difficult problems (at least until they are solved), but non-negligible progress in theses directions will be a wonderful thing for the morale of the troops.

Appendices

Appendix A

Time Translation of the de Sitter Observer

In this appendix we want to show that the boosts in $SO(4,1)$ correspond to time translations for the static observer. The metric for the static patch is,

$$ds^2 = -(1-r^2)dt^2 + \frac{dr^2}{1-r^2} + r^2 d\Omega_2^2 \tag{A.1}$$

We take $\Lambda/3 = 1$, where Λ is the Cosmological constant.

The easiest way to think about the DeSitter isometry group ($SO(4,1)$) is to think of it as being embedded in a 5D Minkowski space. In terms of these Minkowski coordinates, the static patch can be written as,

$$\begin{aligned} X^0 &= \sqrt{1-r^2}\sinh t, &\text{(A.2)}\\ X^3 &= r\cos\theta, &\text{(A.3)}\\ X^1 &= r\sin\theta\cos\phi, &\text{(A.4)}\\ X^2 &= r\sin\theta\sin\phi, &\text{(A.5)}\\ X^4 &= \sqrt{1-r^2}\cosh t. &\text{(A.6)} \end{aligned}$$

Its easy to check that $-(X^0)^2 + (X^i)^2 = 1$ and that $-dX^{0^2} + dX^{i^2}$ is equal to the metric on the static patch. Boosts in $SO(4,1)$ look like,

$$\begin{pmatrix} X^{0\prime} \\ X^{4\prime} \end{pmatrix} = \begin{pmatrix} \cosh\beta & \sinh\beta \\ \sinh\beta & \cosh\beta \end{pmatrix} \begin{pmatrix} X^0 \\ X^4 \end{pmatrix}. \tag{A.7}$$

Plugging in the expressions for X^0 and X^4 in terms of r and t, multiplying out the matrices and simplifying, we end up with,

$$\begin{pmatrix} X^{0\prime} \\ X^{4\prime} \end{pmatrix} = \begin{pmatrix} \sqrt{1-r^2}\sinh(t+\beta) \\ \sqrt{1-r^2}\cosh(t+\beta) \end{pmatrix}, \tag{A.8}$$

which is just the time-translated version of the original expressions.

Appendix B

$SO(5)_{q,r}$-covariant Differential Calculus

In this appendix we explicitly write down the form of the $SO(5)_{q,r}$-covariant differential calculus with $\mu_\alpha = -1/q$ as our chosen eigenvalue. If we set $q = r$, we will get the single parameter limit. The only choice of the $\hat{R}$ eigenvalue which has a non-degenerate $r \to q$ limit is $\mu_\alpha = -1/q$. For $r \neq q$ there are other choices, and they are no harder to compute, but since the explicit forms of these algebras are rather long and cumbersome, we do not write them down here.

We first write down the commutators between the coordinates x^i where i goes from 1 to 5. We define $\gamma_1 = -\frac{(-1+q^2)r-\sqrt{4q^4+(-1+q^2)^2r^2}}{2q^2}$ and $\gamma_2 = -\frac{(-1+q^2)-\sqrt{(-1+q^2)^2+4r^2}}{2r}$.

$$x^1x^2 = \gamma_1 x^2x^1, \ x^2x^5 = \gamma_1 x^5x^2$$

$$x^1x^4 = \gamma_2 x^4x^1, \ x^4x^5 = \gamma_2 x^4x^5$$

$$x^1x^3 = qx^3x^1, \ x^2x^3 = qx^3x^2, \ x^3x^4 = qx^4x^3, \ x^3x^5 = qx^5x^3$$

$$q(x^1x^5 - x^5x^1) + (q-1)(x^2x^4 + qx^4x^2) + q^{1/2}(q-1)x^3x^3 = 0$$

$$q(q-1)(x^1x^5 - x^5x^1) + (1-q+q^3)x^2x^4 - q^2x^4x^2 + q^{5/2}(q-1)x^3x^3 = 0$$

$$q^{3/2}(x^1x^5 - x^5x^1) + q^{1/2}(q^2x^2x^4 - x^4x^2) + (q^3 - q^2 + q - 1)x^3x^3 = 0$$

$$q^{3/2}(q-1)(x^1x^5 - x^5x^1) - q^{3/2}x^2x^4 + q^{1/2}(1-q^2+q^3)x^4x^2 - (q-1)x^3x^3 = 0$$

Some of the relations above are redundant. Also, these relations should be thought of in conjunction with the condition that $x^t\hat{C}x = \frac{3}{\Lambda}$ which (with the appropriate reality condition) is the embedding corresponding to deSitter space. Here Λ could be interpreted as the cosmological constant. One can rewrite the relations above by making use of this constraint and eliminating x^3x^3. For instance in the one-parameter case, after some algebra the second last equation above becomes

$$\left(\frac{1}{q^2}x^1x^5 - q^2x^1x^5\right) + \left(\frac{1}{q}x^2x^4 - qx^4x^2\right) = \frac{\Lambda(1-q^4)}{3q^{1/2}(1+q^3)}. \tag{B.1}$$

It should be noted that these deformed commutators constructed a 'la Zumino, are the same as the ones written down by [32] and [37] in the $r = q$ limit. Their prescription was to split the R-matrix into projection operators so that

$$\hat{R} \equiv qP_S - q^{-1}P_A + q^{-4}P_1, \tag{B.2}$$

and use that to define the deformations according to $P_A(\mathbf{x} \otimes \mathbf{x}) = 0$. The projectors are onto the eigen-subspaces, so everything is consistent. It is an interesting question (that we have not thought about) what happens to the symmetric, anti-symmetric and singlet subspaces under the multi-parametric deformation.

The algebra of the partial derivatives (which is controlled by the matrix F) can be obtained from the co-ordinate algebra above if we substitute $x^i \rightarrow \partial_{x^{i'}}$, where $i' = 6 - i$.

To complete the definition of the deformed calculus, we need to spell out the algebra that deforms the commutators between the co-ordinates and derivatives. It turns out that they are, for x^i and ∂_{x^j} with $i = j$,

$$\begin{aligned}
\partial_{x^1} x^1 &= 1 + q^2 x^1 \partial_{x^1} + (q^2-1)(x^2\partial_{x^2} + x^3\partial_{x^3} + x^4\partial_{x^4}) + \left(1 - \frac{1}{q^3}\right)(q^2-1) x^5 \partial_{x^5} \\
\partial_{x^2} x^2 &= 1 + q^2 x^2 \partial_{x^2} + (q^2-1)(x^3\partial_{x^3} + x^5\partial_{x^5}) + \left(1 - \frac{1}{q}\right)(q^2-1) x^4 \partial_{x^4} \\
\partial_{x^3} x^3 &= 1 + q x^3 \partial_{x^3} + (q^2-1)(x^4\partial_{x^4} + x^5\partial_{x^5}) \\
\partial_{x^4} x^4 &= 1 + q^2 x^4 \partial_{x^4} + (q^2-1) x^5 \partial_{x^5} \\
\partial_{x^5} x^5 &= 1 + q^2 x^5 \partial_{x^5}
\end{aligned}$$

and for $i \neq j$,

$$\begin{aligned}
\partial_{x^1} x^2 = \frac{q^2}{r} x^2 \partial_{x^1} + \left(\frac{1}{q^2} - 1\right) x^5 \partial_{x^4} \quad &, \quad \partial_{x^2} x^1 = \frac{q^2}{r} x^1 \partial_{x^2} + \left(\frac{1}{q^2} - 1\right) x^4 \partial_{x^5} \\
\partial_{x^1} x^3 = q x^3 \partial_{x^1} + \frac{1-q^2}{q^{3/2}} x^5 \partial_{x^3} \quad &, \quad \partial_{x^3} x^1 = q x^1 \partial_{x^3} + \frac{1-q^2}{q^{3/2}} x^3 \partial_{x^5} \\
\partial_{x^1} x^4 = r x^4 \partial_{x^1} + \frac{1-q^2}{q} x^5 \partial_{x^2} \quad &, \quad \partial_{x^4} x^1 = r x^1 \partial_{x^4} + \frac{1-q^2}{q} x^2 \partial_{x^5} \\
\partial_{x^2} x^3 = q x^3 \partial_{x^2} + \frac{1-q^2}{q^{1/2}} x^4 \partial_{x^3} \quad &, \quad \partial_{x^3} x^2 = q x^2 \partial_{x^3} + \frac{1-q^2}{q^{1/2}} x^3 \partial_{x^4} \\
\partial_{x^2} x^5 = \frac{q^2}{r} x^5 \partial_{x^2} \quad &, \quad \partial_{x^5} x^2 = \frac{q^2}{r} x^2 \partial_{x^5} \\
\partial_{x^3} x^4 = q x^4 \partial_{x^3},\ \partial_{x^3} x^5 = q x^5 \partial_{x^3} \quad &, \quad \partial_{x^4} x^3 = q x^3 \partial_{x^4},\ \partial_{x^5} x^3 = q x^3 \partial_{x^5}, \\
\partial_{x^4} x^5 = r x^5 \partial_{x^4} \quad &, \quad \partial_{x^5} x^4 = r x^4 \partial_{x^5} \\
\partial_{x^1} x^5 = x^5 \partial_{x^1},\ \partial_{x^5} x^1 = x^1 \partial_{x^5} \quad &, \quad \partial_{x^2} x^4 = x^4 \partial_{x^2},\ \partial_{x^4} x^2 = x^2 \partial_{x^4}.
\end{aligned}$$

This completes the definition of the $SO(5)_{q,r}$-covariant calculus on the quantum space.

Bibliography

[1] S. Weinberg, "The Cosmological Constant Problem," Reviews of Modern Physics, vol.61, No.1, January 1989.

[2] P. G. O. Freund and M. A. Rubin, Phys. Lett. B **97**, 233 (1980).

[3] O. DeWolfe, D. Z. Freedman, S. Gubser, G. T. Horowitz, I. Mitra, arXiv:hep-th/0105047.

[4] R. Bousso, O. DeWolfe and R. C. Myers, Found. Phys. **33**, 297 (2003) [arXiv:hep-th/0205080].

[5] C. R. Contaldi, L. Kofman and M. Peloso, JCAP **0408**, 007 (2004) [arXiv:hep-th/0403270].

[6] S. Kachru, R. Kallosh, A. Linde and S. P. Trivedi, Phys. Rev. D **68**, 046005 (2003) [arXiv:hep-th/0301240].

[7] T. Banks, M. Dine, E. Gorbatov, arXiv:hep-th/0403270.

[8] A. Saltman and E. Silverstein, arXiv:hep-th/0411271.

[9] Y. Okada, Phys. Lett. B **150**, 103 (1985).

[10] E. W. Kolb, FERMILAB-PUB-86-138-A

[11] J. Gomis, J. Paris, S. Samuel, arXiv:hep-th/9412228.

[12] W. Fischler, "Taking deSitter Seriously," Talk given at The Role of Scaling Laws in Physics and Biology (Celebrating the 60th birthday of Geoffrey West), Santa Fe, Dec. 2000.

[13] W. Fischler, A. Kashani-Poor, R. McNees, S. Paban, JHEP 0107 (2001) 003, arXiv:hep-th/0104181.

[14] T. Banks, W. Fischler, S. Paban, JHEP 0212 (2002) 062, arXiv:hep-th/0210160.

[15] S. B. Giddings, R. C. Myers, arXiv:hep-th/hep-th/0404220

[16] T. Banks, arXiv:hep-th/0007146.

[17] R. Bousso, JHEP **0011**, 038 (2000) arXiv:hep-th/0411271.

[18] P. Pouliot, Class. Quantum. Grav. **21**, 145, 2004, arXiv:hep-th/0306261.

[19] A. Guijosa, D. Lowe, Phy. Rev. D **69**, 106008 (2004), arXiv:hep-th/0312282.

[20] D. Lowe, Phys. Rev. D **70**, 104002 (2004), arXiv:hep-th/0407188.

[21] C. Krishnan, E. di Napoli, arXiv:hep-th/0511147.

[22] V. K. Dobrev, P. J. Molyan, Phys. Lett. **315B** (1993) 292-298.

[23] V. K. Dobrev, R. Floreanini, J. Phys. A: Math. Gen. **27** (1994) 4831-4840.

[24] V. K. Dobrev, "Positive energy representations of noncompact quantum algebras", Invited talk at the Workshop on Generalized Symmetries in Physics (Clausthal, July 1993), Proceedings, eds. H.-D. Doebner et al, (World Sci, Singapore, 1994, ISBN 981-02-1771-4)), pp. 90-110.

[25] H. Steinacker, Rev. Math. Phys. 13 (2001) 1035, arXiv:math.QA/9907021.

[26] H. Steinacker, arXiv:hep-th/9705211.

[27] H. Steinacker, Commun. Math. Phys. 192 (1998) 687, arXiv:q-alg/9611009.

[28] M. Rosso, Commun. Math. Phys. 117 (1988), 581.

[29] A. Klimyk, K. Schmudgen, "Quantum Groups and their Representations," Springer (1997).

[30] B. Zumino, arXiv:hep-th/9212093.

[31] J. Wess, B. Zumino, Nucl. Phys. (Proc. Suppl.) 18B (1990) 302.

[32] N. Yu. Reshetikhin, L. A. Takhtadzhyan, L. D. Faddeev, Leningrad. Math. J. **1**, 193 (1990).

[33] N. Reshetikhin, Lett. Math. Phys. **20**, 331 (1990).

[34] A. Schirrmacher, J. Phys. A: Math. Gen. **24**, L1249 (1991).

[35] P. Aschieri, Lett. Math. Phys. **49** (1999) 1, arXiv:math.QA/9805120.

[36] E. Twietmeyer, Lett. Math. Phys. **24** (1992) 49.

[37] U. Carow-Watamura, M. Schlieker, S. Watamura, Z. Phys. C **49**, (1991) 439.

[38] R. N. Cahn, "Semi-Simple Lie Algebras and their Representations," Benjamin/Cummings, 1984.

[39] J. W. Helton, R. L. Miller, NCAlgebra version 3.7, // http://www.math.ucsd.edu/ ncalg/

[40] S. J. Rey, hep-th/0207108; S. J. Rey and J. T. Yee, hep-th/0303235.

Vita

Chethan Krishnan was born in Tiruvalla, Kerala, India on 24 January 1978, the son on K. P. Krishnankutty Nair and V. R. Vijayalakshmi. He received the Bachelor of Technology degree in Electrical Engineering from the Indian Institute of Technology, Madras in 1999. Upon receiving his Ph.D., he plans to join as a Research Fellow at the University of Brussels.

Permanent address: 3543 Greystone Drive, Apt# 1.2040
Austin, Texas 78731

This dissertation was typeset with LaTeX[†] by the author.

[†]LaTeX is a document preparation system developed by Leslie Lamport as a special version of Donald Knuth's TeX Program.