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Longevity Risk Modeling, Securities Pricing and Other Related Issues

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Longevity Risk Modeling, Securities Pricing and Other Related Issues

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Longevity Risk Modeling, Securities Pricing and Other Related Issues

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This dissertation studies the adverse financial implications of "longevity risk" and "mortality risk", which have attracted the growing attention of insurance companies, annuity providers, pension funds, public policy decision-makers, and investment banks. Securitization of longevity/mortality risk provides insurers and pension funds an effective, low-cost approach to transferring the longevity/mortality risk from their balance sheets to capital markets. The modeling and forecasting of the mortality rate is the key point in pricing mortality-linked securities that facilitates the emergence of liquid markets.

First, this dissertation introduces the discrete models proposed in previous literature. The models include: the Lee-Carter Model, the Renshaw Haberman Model, The Currie Model, the Cairns-Blake-Dowd (CBD) Model, the Cox-Lin-Wang (CLW) Model and the Chen-Cox Model. The different models have captured different features of the historical mortality time series and each one has their own advantages.

Second, this dissertation introduces a stochastic diffusion model with a double exponential jump diffusion (DEJD) process for mortality time-series and is the first to capture both asymmetric jump features and cohort effect as the underlying reasons for the mortality trends. The DEJD model has the advantage of easy calibration and mathematical tractability. The form of the DEJD model is neat, concise and practical. The DEJD model fits the actual data better than previous stochastic models with or without jumps. To apply the model, the implied risk premium is calculated based on the Swiss Re mortality bond price. The DEJD model is the first to provide a closed-form solution to price the q-forward, which is the standard financial derivative product contingent on the LifeMetrics index for hedging longevity or mortality risk.

Finally, the DEJD model is applied in modeling and pricing of life settlement products. A life settlement is a financial transaction in which the owner of a life insurance policy sells an unneeded policy to a third party for more than its cash value and less than its face value. The value of the life settlement product is the expected discounted value of the benefit discounted from the time of death. Since the discount function is convex, it follows by Jensen's Inequality that the expected value of the function of the discounted benefit till random time of death is always greater than the benefit discounted by the expected time of death. So, the pricing method based on only the life expectancy has the negative bias for pricing the life settlement products. I apply the DEJD mortality model using the Whole Life Time Distribution Dynamic Pricing (WLTDDP) method. The WLTDDP method generates a complete life table with the whole distribution of life times instead of using only the expected life time (life expectancy). When a life settlement underwriter's gives an expected life time for the insured, information theory can be used to adjust the DEJD mortality table to obtain a distribution that is consistent with the underwriter projected life expectancy that is as close as possible to the DEJD mortality model. The WLTDDP method, incorporating the underwriter information, provides a more accurate projection and evaluation for the life settlement products. Another advantage of WLTDDP is that it incorporates the effect of dynamic longevity risk changes by using an original life table generated from the DEJD mortality model table.

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Chapter 1 Introduction to Longevity Risk

1.1 Background

The adverse financial implications of "longevity risk" and "mortality risk" have attracted the growing attention of insurance companies, annuity providers, pension funds, public policy decision-makers, and investment banks. Longevity risk denotes the adverse financial consequences that ensue when an individual or group live longer than expected (i.e., their mortality rate is lower than what was expected at the time that the financial balancing of assets, set aside for future consumption or future payments, was made). Similarly, mortality risk describes the adverse financial consequences that ensue when an individual or group live a shorter time than expected (their mortality rate is higher than expected in the premium/benefit balancing equation). The International Actuarial Association defines four components of longevity/mortality risk: level, trend, volatility, and catastrophe. The four components can be categorized into two groups, systematic risk and specific risk (Crawford, et al. 2008). Systematic risk is defined as the underestimation or overestimation of the base assumption of mortality rates, including the level component and the trend component. Specific risk is defined as the volatility around the base assumption, including the volatility component and the catastrophe component. According to the Law of Large Numbers specific risks can be reduced by diversifying with a large pool of lives; however, systematic risk cannot be reduced by diversification.

Clearly life insurers are interested in mortality risk because they have to pay death claims earlier than expected, resulting in an unbalanced loss of capital. Annuity providers, defined benefit pension plans and social insurance programs such as Social Security are interested in longevity risk because they have to make financial payments for longer than was originally reserved. In addition to having substantial pension obligations such as social security programs, governments act as residual risk bearers of last resort and are becoming increasingly concerned with the financial consequences of citizens outliving their resources.

The life insurance industry, as well as the annuity and pension industries, functioned reasonably for many decades, according to actuarial estimated projections. These plain vanilla products used standard mortality tables and conservative interest rates. However, situations are changing, and insurers, reinsurers, pension funds, and governmental life security schemes (e.g., social insurance) are changing as well. Illustrative of this on the life (mortality) side is the quote by Richard M. Todd and Neil Wallace in the 1992 Federal Reserve Board-Minneapolis Quarterly Review, "In 1980 the life insurance industry was 150 years old. In 1990 ... [it] was 10 years old." Increased competition by capital markets, increasingly sophisticated financial life insurance products, mutual funds, and new derivative instruments are re-sculpturing the landscape related to the transfer of certain risks in life insurance. Risks can now be transferred using capital markets and are not restricted to just reinsurance market risk transfer. A similar statement could be made in today's environment concerning the status of the annuity/pension/longevity risk market.[1] It is currently undergoing rapid changes caused

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[1] It is interesting to note that annuity market sophistication has progressed substantially since the government of William III of England (November 4, 1650 --March 8, 1702) offered annuities of 14% regardless of whether the annuitant was 30 or 70 years of age (Pearson 1978, p. 134). Indeed, the mindset behind this was that one's death was considered to be an "act of God" which occurred whenever the all powerful God dictated that one's time had come. There was no room for "chance" in this contract. Death

by regulatory pressures, financial innovations, medical advances, pandemic threats, and capital market pressures on firms and governments having substantial pension obligations. The pressure on Pay-As-You-Go social security systems is severe because of the imminent retirement of the baby boom generation, followed by a baby bust generation. The next generation is unable to pick up the needed financial costs associated with longevity. A misestimation of costs incurred by using a life table that does not incorporate longevity changes can also create financial pressure for defined benefit pension plans of private enterprises or annuity providers.

Sources of pressure on longevity risk sensitive entities come from several arenas including advances in medicine, nutrition, and sanitation. In the last several decades, life expectancy in the developed countries has, on average, been increasing by approximately 1.2 months every year. Globally, life expectancy at birth has increased by 4.5 months per year on average over the second half of the 20th century (Gutterman, England, Parikh, and Pokorski, 2002).

Substantial improvements in longevity during the 20th century have posed longevity risk management challenges to pension funds and other entities that originally reserved for expected future costs using what would now be considered incorrectly diminished mortality rates (older mortality tables). A 2006 study of the companies in the U.K.'s FTSE100 index found that many companies had based their estimates of pension

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was deterministic and only appeared random to humans because people were "ignorant of God's will"!The development of rigorous probability theory by Fermat and Pascal (1654) and its subsequent application in 1693 by Edmund Halley (of Halley's comet fame) to create the first mortality table (including an annuity pricing example) revolutionized the annuity and life insurance market then, just as the developments in the studies of the capital markets have revolutionized the insurance industry in more modern times.

liabilities on mortality tables which underestimated expected lifetimes by not recognizing improving longevity, and by not recognizing this underestimation of expected lifetimes would cause the aggregate deficit in pension reserves to more than double from £46 billion to £100 billion (Pension Capital Strategies and Jardine Lloyd Thompson, 2006). In 2010 alone, improved life expectancy added £5 billion to corporate pension obligations in the U.K. (Reuters, 2010). In the U.S., the Internal Revenue Service (IRS) has recently established new mortality assumptions for pension contributions, which according to Watson Wyatt insurance consulting firm, will increase pension liabilities by 5-10%, (Halonen, 2007). Mercer Human Resource Consulting has calculated that the use of up-to-date mortality tables would increase the cost of providing a pension to a male born in 1950 by 8%, Mercer (2006). Additionally, U.S. insurance regulation ignores changes in longevity risk (or mortality risk) in the current Risk Based Capital (RBC) formula for calculation of insurance risk. An imbalanced approach to assets and liabilities can affect the capital structure of insurance or reinsurance companies and increase their default risk. The insurance or reinsurance companies that ignore longevity risk in designing and pricing their insurance products, including their annuity and life insurance products, run a substantial risk of underestimating their ultimate product costs which can cause a liability payment shortage for certain products.

Interaction of the insurance industry with the capital markets (Cummins, 2005) provides a vehicle for mitigating the above mentioned mortality/longevity risk, namely, through financial securitization of life and pension products (c.f., MacMinn, Brockett, and Blake, 2006). Securitization provides an approach to transferring non-diversifiable mortality/longevity risk from the insurer's or pension's balance sheet to the capital market. Moreover, this capital market transfer may provide an attractive alternative to reinsurance because of the size of the liabilities. According to Cummins and Trainer (2009, p.475), "… the traditional reinsurance model begins to break down when risks are correlated, add significantly to the reinsurer asymmetry risk, and are large relative to the reinsurer's equity capital. The cost of capital is also increased by informational asymmetries between reinsurers and capital market and by agency costs and other market frictional costs. Under these conditions, the price of reinsurance may be prohibitively high, and the supply of coverage may be restricted." Longevity risk for pension plans or annuity providers is an example of such correlated risk (mortality improvements or pandemics affect many individuals in the insurer's book of business). In these circumstances securitization can help address the inefficiencies in the reinsurance market such as correlation between risks and counterparty credit risk in the case of large or catastrophic risk (cf., Cummins and Trainer, 2009). Moreover, the capital markets are also significantly larger in capacity than are the reinsurance markets, so spreading the risk among the capital market participants can reduce insolvency risk. Additionally, since the securitized insurance products tend to be uncorrelated (or lowly correlated) with other assets in the economy, these securitized mortality/longevity risk transfer instruments can be attractive to investors wishing to diversify their own risks by putting an essentially zero beta asset into their portfolios. Securitization also enhances the risk capacity of the insurance industry, as illustrated by the catastrophe (CAT) mortality bond and other similar derivatives, whose payment depends on the underlying loss indices and the catastrophic mortality event. On the insurance policy holder side, many investment banks have recently been involved in life-settlement securitization. Investment banks purchased hundreds of thousands of life insurance policies and repackaged them into bonds, then sold bonds to investors such as pension funds. The high return of the life settlement is attractive to the investors. The expected rate of return to an investor in such a bond depends on the projected life expectancy of the members in the pool of life insurance policies (Modu 2009).

Capital market solutions to longevity risk problems have grown increasingly important in recent years, both in academic research and in the Life Markets, the capital markets that trade longevity-linked assets and liabilities. Capital markets can, in principle, provide vehicles to hedge longevity risk effectively. Many new investment products have been created both by the insurance/reinsurance industry and by the capital markets. Mortality catastrophe bonds are an example of a successful insurance-linked security. Some new innovative capital market solutions for transferring longevity risk include longevity (or survivor) bonds, longevity (or survivor) swaps and mortality (or q-) forward contracts.

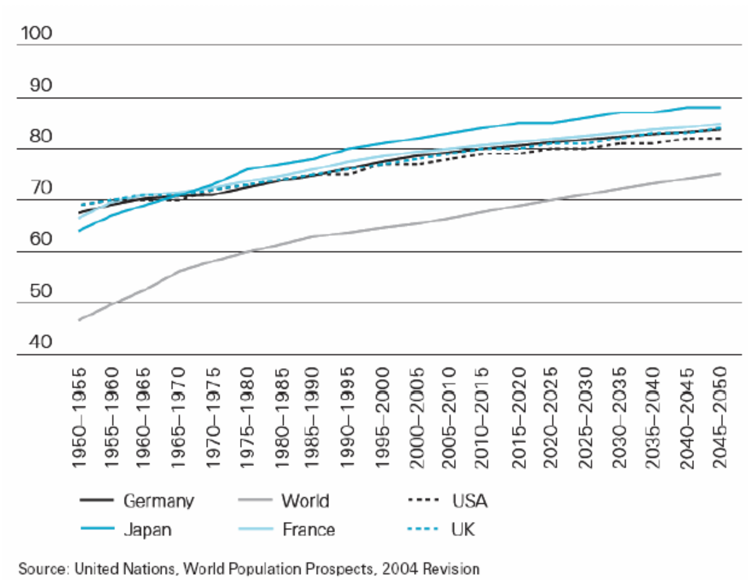
1.2 Trend and Outlook of Retirement Market

A range of previous research has reached the same conclusion: people are living longer than they ever have in the past, or the life expectancy of people has obviously increased. Significant medical progress, improved living standards, healthier lifestyles that include organic food, the absence of global wars and pandemic influenza crises are some of the main environmental reasons for the increase in life expectancy.

In the United States, the number of centenarians (individuals over the age of 100) has increased from 15,000 in 1980 to roughly 72,000 in 2000. The Social Security Advisory Board using U.S. Census Bureau data, predicts the number of centenarians will increase to 4.2 million by 2050, which is approximately 1% of the projected total population (Scotti and Effenberger, 2007).

The life expectancy at birth for people living in several countries that are the members of the Organization for Economic Cooperation and Development (OECD) is shown in Figure 1. In the study of UN World Population prospects, the projections of Japanese lives from 1950 through 2050 indicate that on average life expectancy at birth will increase at a rate of approximately 3.2 months per year for females and 2.7 months per year for males (Scotti and Effenberger, 2007).

Figure1 Life Expectancy at Birth in Different Regions



In the next few decades, most OECD countries are expected to experience what has been called "the demographic time bomb." It is unambiguous that there is a higher life expectancy and a lower birth rate in these countries. The OECD includes Japan, South Korea, etc., which also exhibit very low birth rates and increasing longevity which implies an inversion of the standard age distributions. In 2050, 27% of the European population is expected to be older than 65 years (versus 16% in 2005), and about 10% is projected to be older than 85 (versus 3.5% in 2005) (Scotti and Effenberger, 2007).

Figure 2 Old-Age Dependency Ratio in Selected Countries

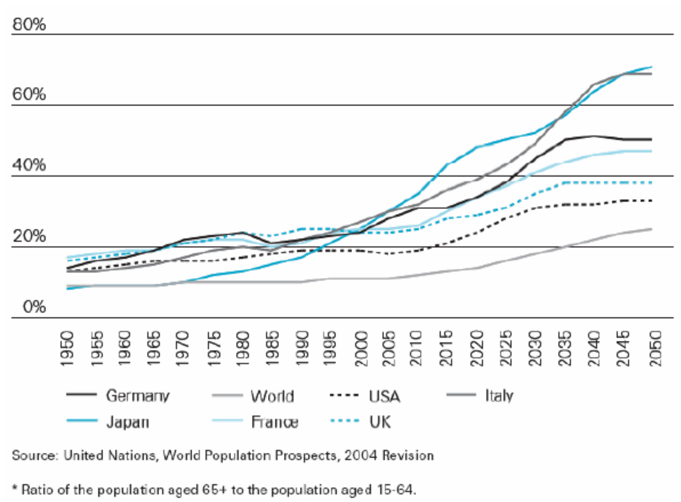
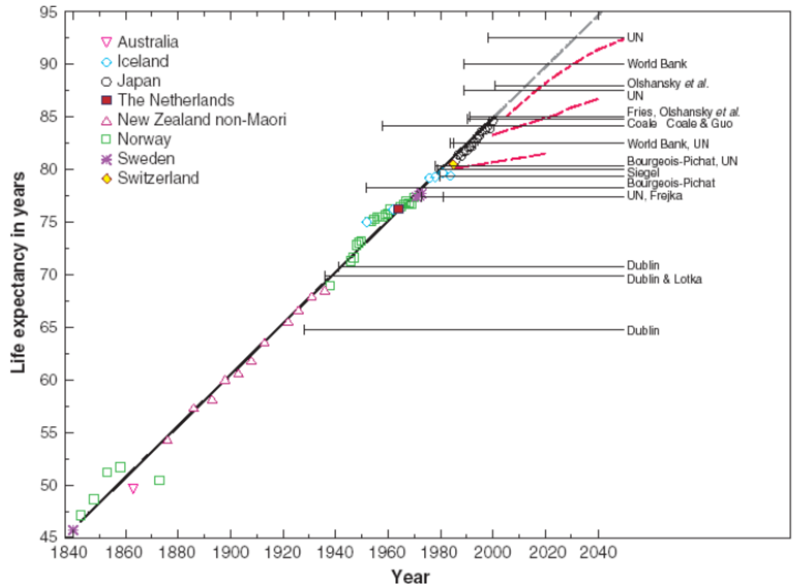


Figure 2 illustrates the old-age dependency ratio (the ratio of the population aged 65 and older to that aged 15 to 64). While today the ratio is around 25% in a typical developed country, in 2050 it is estimated to rise to 70% in countries such as Japan and Italy (Scotti and Effenberger, 2007). Traditionally the over 65 population is considered to be the retired population who are supported by (dependent on) the "working" population aged 15-64. A ratio of 25% means there are four workers supporting (via taxes and other manners) each person over 65. A ratio of 70% means each person over 65 will have to be supported by only approximately 1.4 workers, a burden almost three times as great as that today.

While mortality improvement trends can be observed in the entire population, the specific amount of mortality improvement is different for different age groups and depends on when the individual was born. (See Figure 3) The term "cohort effect" describes anomalies in observed mortality improvement for individuals born during a specific period of time or having specific characteristics in common. Figure 3 shows the longevity in women in various countries and shows that different countries exhibit different effects.

Figure 3 Female Life Expectancy



Source: Oeppen J, et al. Science 2002; 296: 1029-31

In the next section, we shall review the capital market instruments that are proposed to transfer longevity and mortality risk.

Chapter 2 The Longevity Risk Models and Products

2.1 MORTALITY RATE MODELS FOR USE IN CAPITAL MARKET HEDGING

The capital market provides a new way to hedge the longevity risk or mortality risk for pension funds, annuity providers and insurance companies. The pricing of the mortality and longevity risk contingent financial instruments depends on the estimation and projection of the mortality rate in a cohort of lives. This can be cumulative (as in the Swiss Re Mortality Catastrophe Bond) or for a specified birth cohort at a specific age. For any of these there is a need to develop a theoretical model of mortality. We shall discuss several of the currently used mortality models and examine their strengths.

2.2 DISCRETE TIME MODELS

2.2.1 Lee-Carter Model.

Lee and Carter (1992) proposed the first stochastic model for mortality rate.

$$\ln(\mu_{x,t}) = a_x + b_x k_t + e_{x,t}$$

$$k_{t+1} = k_t + m + \sigma Z_{t+1}$$

Here x represents the age of the individual, t represents time (date), and $\mu_{x,t}$ represents the mortality rate of a person aged x in year t. a_x represents the age group shift effect, $\exp(a_x)$ is the general shape across the age of the mortality schedule, and b_x represents the age group's reaction effect to the mortality time-series k_t. The b_x profile tells us which group of mortality rates declines rapidly and which group declines slowly to changes in k_t, and $e_{x,t}$ captures the age group's residual effect not reflected in the model. m is the drift, σ is the variance for the mortality time-series k_t, and Z_{t+1} follows the standard normal distribution.

There is an identification problem in this parameterization. Lee and Carter introduce the normalizing conditions:

$$\sum_t k_t = 0$$

$$\text{and } \sum_x b_x = 1.$$

To estimate the parameters in the model, these constraints resolve an identification problem that would occur in the general model if the constraints were not imposed. The values of age-specific parameters a_x, b_x, and mortality time-series k_t can be generated througth the Singular Value Decomposition (SVD) method with the historic data of $\mu_{x,t}$.

To implement the SVD procedure, first, we need to normalize the condition that sets k_t sums to 0 and b_x sums to 1. Since there are a series of combinations (b_x, k_t) to generate the same result of $b_x k_t$, we choose one group as the standard benchmark with the normalization condition which distributes k_t equally around 0. With the normalization condition, then a_x must equal the average over time of $\ln(\mu_{x,t})$.

$$a_x = \frac{1}{T}\sum_{t=1}^{T}\ln(\mu_{x,t}) \quad (1)$$

Furthermore, k_t must (or almost) equal the sum over age of $(\ln(\mu_{x,t}) - a_x)$, since the sum of b_x has been chosen to be unity. This is not an exact relation, however, since the error terms will not in general sum to 0 for a given age. Then, each b_x can be found by regressing, without a constant term, $(\ln(\mu_{x,t}) - a_x)$ on k_t separately for each age group *x*. See Lee and Carter (1992) for details and further justification and statistical discussions.

Lee-Carter models mortality rate in two steps. First, the logs of the age-specific death rates are modeled as a linear function of an unobserved period-specific intensity index k_t, with parameters depending on a_x and b_x. The model accounts for almost all the variance over time in age-specific death rates as a group. Second, the index k_t is modeled as a time series of random walk with drift. Compared to the traditional hazard models, the Lee-Carter model accounts for the difference of mortality rate in age groups and describes the mortality time-series with a stochastic process. This is the benchmark for a series of extensions.

2.2.2 Renshaw Haberman Model

Renshaw and Haberman (2006) proposed a similar model to the Lee-Carter model but added a term to describe the cohort effect:

$$\ln(\mu_{x,t}) = a_x + b_x k_t + c_x r_{t-x} + e_{x,t}$$

With the normalization condition:

$$\sum_t k_t = 0,$$

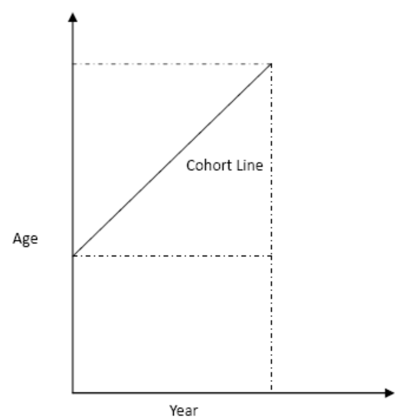
$$\sum_x b_x = 1,$$

$$\sum_{t,x} r_{t-x} = 0,$$

$$\text{and } \sum_x c_x = 1.$$

Here a_x, b_x, k_t, $e_{x,t}$ are defined as in to the Lee-Carter model. The variable r_{t-x} captures the cohort effect and c_x is the parameter corresponding to it. The Lee-Carter model is a special case where c_x and r_{t-x} are set to zero. The model captures the feature that individuals born in the same year have similar environments and healthcare, which affects the mortality rate and life expectancy. The cohort effect is described by r_{t-x}, for example, an individual aged 31 in year 2002 is in the same cohort with the individual aged 35 in year 2006, since they are born in the same year and *t*-*x* has the identical value of 1971. Figure 4 illustrates the cohort line.

Figure 4 The Cohort Line



There is also an identification problem in this parameterization. We need to introduce a normalization using a condition that the sum of r_{t-x} is equal to 0 and the sum of c_x is equal to 1 to reach the unique solution. In this way, the identification problem can be solved.

2.2.3 Currie Model

Currie (2006) introduced the simpler Age-Period-Cohort (APC) model which is a special case of the Renshaw-Haberman model with b_x=1 and c_x=1, namely

$$\ln(\mu_{x,t}) = a_x + k_t + r_{t-x} + e_{x,t}$$

With the normalization condition:

$$\sum_t k_t = 0,$$

$$\text{and} \sum_{t,x} r_{t-x} = 0.$$

Currie (2006) applies P-splines to fit a_x, k_t and r_{t-x} to ensure smoothness.

2.2.4 Cairns-Blake-Dowd (CBD) Model

Cairns, Blake and Dowd (2006) proposed the model for mortality rate:

The forward survival probabilities $p(t, T_0, T_1, x)$ denotes the probability measured at t that an individual aged x at time 0 and still alive at T_0 survives until time $T_1 > T_0$.

The model setup

$$\mu_{x,t} = \frac{e^{A_1(t+1)+A_2(t+1)(x+t)}}{1 + e^{A_1(t+1)+A_2(t+1)(x+t)}}$$

can be simplified in the format in accordance with the models above as:

$$logit\left(\mu_{x,t}\right) = k_t^{(1)} + k_t^{(2)}(x - \bar{x}) + e_{x,t}$$

where $\bar{x} = n_a^{-1} \sum_i x_i$ is the mean age in the sample range. This model has no identification problems.

2.2.5 Cox-Lin-Wang (CLW) Model

Cox, Lin and Wang (2006) proposed a model with permanent jump effects describing the evolution of the mortality factor k_t in the Lee Carter model as follows:

$$k_{t+1} = \begin{cases} k_t + \mu + \sigma Z_{t+1}, & if\ N_{t+1} = 0 \\ k_t + \mu + \sigma Z_{t+1} + Y_{t+1}, & if\ N_{t+1} = 1 \end{cases}$$

where μ and σ are constants, μ denotes the constant increment in the mortality factor k_t, σ denotes the volatility in the mortality factor k_t. N_{t+1} counts the number of the jumps, and Y_{t+1} denotes the jump magnitude. Z_t is a standard normal random variable that is independent of Y_t and N_t. If a jump event occurs in year *t+1*, the magnitude of the jump, Y_{t+1}, is included in the mortality factor k_{t+1}, and this jump effect persists forever.

2.2.6 Chen-Cox Model

Compared to the permanent jump in Cox, Lin and Wang (2006), Chen and Cox (2009) propose their model with transitory jump, that is described with the normal distribution.

Let $\tilde{k}_t$ describe the mortality factor when there is no jump event. They model $\tilde{k}_t$ as a random walk with drift

$$\tilde{k}_{t+1} = \tilde{k}_t + \mu + \sigma Z_{t+1},$$

where μ and σ are constants, μ denotes the constant increment in the mortality factor $\tilde{k}_t$, σ denotes the volatility in the mortality factor $\tilde{k}_t$.

If a jump event occurs in year $t + 1$, then, $N_{t+1} = 1$. The jump Y_{t+1} makes the actual mortality factor k_{t+1} change from $\tilde{k}_{t+1}$ to $\tilde{k}_{t+1}$+Y_{t+1}. Then

$$k_{t+1} = \tilde{k}_{t+1} + Y_{t+1}$$

If there is no jump in year $t + 1$, then, $N_{t+1} = 0$ and

$$k_{t+1} = \tilde{k}_{t+1}$$

Therefore, the dynamics of the mortality factor k_t can be completely expressed as

$$\begin{cases} \tilde{k}_{t+1} = \tilde{k}_t + \mu + \sigma Z_{t+1} \\ k_{t+1} = \tilde{k}_{t+1} + Y_{t+1} N_{t+1} \end{cases}$$

or

$$k_{t+1} = k_t + \mu + \sigma Z_{t+1} + Y_{t+1} N_{t+1} - Y_t N_t$$

where μ and σ are constants, μ denotes the constant increment in the mortality factor k_t, σ denotes the volatility. N_{t+1} counts the number of the jumps, and Z_t is a standard normal random variable that is independent of Y and N.

Table 1 Model Comparison

Model	Description
Lee-Carter Model	The model is a function of age and time which is an improvement to the traditional hazard rate model which only depends on time. The projected value of the mortality rate captures the features that mortality rate increases with age and increases across the time. The model does not account for a cohort effect.
Renshaw -Haberman Model	The model is an extension/generalization of the Lee-Carter Model by adding a cohort effect term, which describes the year of birth. The model captures the feature that individuals born in the same year have similar environment, healthcare etc. experiences and so have similar affects on life expectancy.
Currie Model	The simpler Age-Period-Cohort (APC) model is a special case of the Renshaw-Haberman Model. The P-spline method is applied to fit the data, which assumes that the mortality is a smooth surface.
Cairns-Blake-Dowd (CBD) Model	The model is also a function of age and time, similar to the Lee-Carter Model. The difference is that this model applies the logit transform of mortality rates rather than the natural log of death rates as other models do.
Cox-Lin-Wang (CLW) Model	Extension of the Lee-Carter Model by adding a permanent jump effect term to the mortality factor kt. The severity of the jump follows a standard normal distribution
Chen-Cox Model	Similar to the CLW Model, this model entends the Lee-Carter Model by adding a transitory jump effect term to the mortality factor kt. The severity of the jump follows a standard normal distribution
Brockett-Deng-Macminn (BDM) Model	This model extends the Lee-Carter Model by adding both longevity jump and mortality jump effect terms to the mortality factor kt. The arrival of the jump follows a Poisson process. The severity of the jump follows a double exponential distribution.

2.3 Summary of Products for Individual Longevity Risk Transferring

Defined benefit plans are being changed to defined contribution plans in most countries' retirement markets. More and more individuals have to manage their own personal longevity risk. In the U.S., the baby boomers are entering their retirement age and need to address their own increasing longevity risk and the shortage of retirement plan payments. The retirees need to balance investment and consumption of their accumulated wealth. A key point that needs to be addressed is that extended lifespan may erode their accumulated wealth. Underestimation or overestimation of the personal life expectancy can negatively impact a retiree's life style. If the retiree overestimates his longevity, he will spend less than he could if he has purchased an annuity. If the retiree underestimates his longevity, he will spend aggressively and might outlive the accumulated wealth.

In a survey (SOA, 2006), it was found that more than 40% of both pre-retirees and retirees underestimate average life expectancy by five or more years. Only 33% of retirees and 39% of pre-retirees have bought or plan to buy a product or choose a plan option that will provide them with guaranteed income for life. Similarly, a recent study (Scotti and Effenberger, 2007) in the UK retirement market suggests that individuals underestimate their own mortality by as much as five years on average. A defined benefit plan provides individuals with a guaranteed stream of payments, which decreases the chance of outliving the individuals' assets. The Scotti and Effenberger's study shows that individuals that retire without a pension plan have over an 80% chance of outliving their assets, and individuals with a defined benefit pension plan have only an 18% chance of outliving their assets.

Individuals without defined benefit pension plans choose to self-insure against longevity risk by making their own investment and consumption decisions or choose to do nothing because their income does not allow it. The investment decision is to select an asset allocation strategy among different investment instruments so as to diversify financial risk and minimize the chance of a portfolio shortfall. The consumption decision is to choose the level of withdrawal from the asset pool to reach a satisfactory life style. If individuals who try to self-insure are unsuccessful and run out of money, they will be forced to go through their remaining life without income. This would likely result in relying on children, relatives, or even federal programs to live out their remaining life, which is an undesirable situation for the self-insured individuals. So, a more reliable and scientific approach to manage the longevity risk for individuals is to transfer and diversify longevity risk with annuities and other financial instruments.

Since a major trend in the retirement market is the declining number of defined benefit plans, there is an emerging opportunity for the expansion of the private market solution using financial instruments for individual longevity risk management. For individuals interested in insuring at least a portion of their longevity risk, there are several products that offer lifetime guarantees. These products include:

- Immediate Annuities
    - (e.g., Single Premium Immediate Annuity, SPIA)
    - Impaired Life Annuities

- Deferred Annuities
  - Guaranteed Lifetime Withdrawal Benefits
  - Guaranteed Minimum Income Benefits
- Advanced Life Delayed Annuities
- Corporate Pensions
- Reverse Mortgages
- Structured Settlements
- Life Settlements

A brief description of each of the above is given below.

2.3.1 Immediate Annuities

An immediate annuity, or Single Premium Immediate Annuity (SPIA) is a classical type of annuity product that provides secured payment for life, usually paid for in a lump sum. The term of the contract varies in the frequency and amount of the income payment. They are structured to provide a fixed-level payment, a stream of payments that increase at a pre-specified rate, or a stream of payments that is tied to an underlying equity index (the latter being termed a variable immediate annuity, VIA). Some products provide that the period for the stream of payments continues until the death of the policy holder and some products provide for a certain period, irrespective of the death of the policy holder. Immediate annuities include several types of single-life policies, joint policies, and survivor policies. In the joint policies case, the annuity payments continue while two or more policy holders are alive. In the survivor policies case, the annuity payments continue while at least one of the policy holders is alive, although sometimes with survivor policies the periodic income is reduced and only a percentage of the payment is received by the survivor after the first one dies.

Some longevity risk products also embed options for hedging mortality risk, such as, the participating annuity (available in the U.K. market). Annuitants share in both the investment and longevity mortality gains while benefiting from risk pooling. Individuals can also benefit from other options in the form of minimum investment returns or from insurance benefits such as a minimum death benefit, minimum withdrawal benefit, minimum accumulation benefit, or a minimum income benefit.

The quantification of longevity risk exposure is crucial to the pricing of immediate annuities. In the U.K., for example, regulators have recognized the effect of longevity risk and have adjusted the statutory reserving basis in accordance with this. The changes have been largely based on using revised projections for mortality improvements on a year-of-birth or cohort basis available from the Continuous Mortality Investigation Bureau, CMIB(2002).

Longevity risk exposure can have a serious impact on the pricing of immediate annuities. Pricing risk is higher for companies who assume a larger mortality rate than the actual experience produces. Pricing risk is higher for the products issued to older individuals than to younger individuals. Because annuities for young people are for very long duration, they are more similar to perpetuity which pay forever (and which assumes the infinite lifetime and does not incorporate a mortality component). However, this observation does not mean that there is no exposure to longevity risk, only that the time value of money dominates for younger ages. There is still longevity risk.

Historical mortality rate data corresponding to individual types of products can be used to provide mortality table estimates that can be used in pricing the product. Regardless of the size of the population pool of the product, there can exist large variability in the estimate. This indicates that not only does the central estimate of the mortality rate need to be considered in the pricing, but also a measure of the uncertainty in the mortality rate should be accommodated to produce appropriate pricing margins.

In the U.S., the immediate annuity market is mainly driven by individual sales. However, individual sales can be a problem due to adverse selection, since the product price is identical for individuals with different life expectancies. The individual who knows their life expectancy is lower than average will be more reluctant to purchase the product. As a result, the actual life expectancy in the pool will be higher than insurer expected. This can cause an underestimation of the product price and could produce a loss for the insurer. Adverse selection risk will be lower in the case of mandatory annuitizations, since the individuals with less life expectancy are also required to purchase the annuity. Such mandatory annuitization of accumulated funds occurs in the U.K. and reduces adverse selection effects exhibited by free purchase as in the U.S.

Although the annuities are appropriate products for individuals to transfer longevity risk, the individual annuity market has not developed as fully as researchers expected. One reason is the lack of knowledge of the benefits of annuities and the general lack of knowledge of severity of the longevity risk problem. Another reason is the adverse selection problem which causes the high price of the product and deteriorates the attraction of the product.

2.3.2 Enhanced and Impaired Life Annuities

Enhanced and impaired life annuities offer higher annuity payments to individuals who can prove that they are in poor health or are terminally ill. Enhanced and impaired life annuities can be created to provide longevity risk protection at a reasonable price for individuals who have a life expectancy lower than average. The products also help to solve the adverse selection risk in individual immediate annuities.

Pricing and evaluation of the enhanced and impaired life annuities places a higher weight on exposure to a handful of impairments, particularly cardiovascular disease and conditions related to smoking. Due to the documented worse health condition of the annuitant, higher expected mortality rates are assumed for these policies. Therefore, the effect of longevity risk can be more serious, as there is likely to be less data on the mortality experience of particular subgroups of the population. The relative infrequency of incidence and poor historical reporting of cause of death will cause the estimate of mortality to be biased. The significant deviation in the medical underwriting process is another reason for biased life expectancy (Drinkwater, et al., 2006).

2.3.3 Deferred Annuities

Deferred annuities accumulate tax-deferred savings to distribute later as either an immediate annuity or as a lump sum payment. Fixed deferred annuities, variable deferred annuities, and equity indexed deferred annuities are the three categories.

More recent products, such as Guaranteed Minimum Income Benefit (GMIB) policies, provide the option to annuitize and receive at least a minimum annuity payment at a price dependent on mortality and interest factors. For GMIB policies, the long-term interest rate and longevity risk are the two key risks. If the GMIB policy is based on incorrectly large assumptions for mortality, the cost to the issuer of the guarantee is increased. Moreover, since the interest rate in the capital market has been relatively low recently, the impact of the miscalculation of the interest rate is minor, compared to that of miscalculating the mortality rate (Richards and Jones, 2006). Another similar type of product is the Guaranteed Minimum Withdrawal Benefits policy. The risk level in this policy depends on the form of the guarantee.

2.3.4 Advanced Life Delayed Annuities

Advanced Life Delayed Annuities (ALDAs) are inflation-linked annuities sold to individuals in the early years of their lives that begin paying at age 80, 85, or 90. The cash flow and mortality insurance benefits cannot be exchanged before maturity. ALDAs act as the replacement for the defined benefit pension plan for individuals of advanced age. ALDAs do not include the accumulation phase to the same extent that traditional deferred annuities do. Therefore, they could be considered to be more tailored toward protection against catastrophic longevity (Milevsky, 2004).

Pricing ALDAs depends on the annuitization rate and mortality table. Again, a biased estimation of the mortality rate could result in significant loss for the insurer. Moreover, a long-term risk is involved in the ALDA product. Since the product accumulates the premium for a long period before paying the benefits, the asset liability matching administration systems of most insurers do not support ALDA products very well. Assets to back the lengthy duration are not generally available, which exposes the insurer to reinvestment risk. There is no death benefit offered in the ALDA product, so the policy holder could make years of premium payments and receive no benefit. This potential uncertainty leads to reluctance on the part of insurers to offer this type of product and policyholders to buy it (Milevsky, 2001).

There is a subclass of the ALDA product called "longevity insurance." A lump sum premium is paid today for a benefit payment in the future. The deferral periods are usually in excess of 20 years. The advantage of the product is that it allows individuals to purchase insurance against outliving their assets at a much lower cost than traditional SPIA products (Milevsky, 2001).

2.3.5 Corporate Pensions

Corporate pensions include two categories: defined benefit (DB) and defined contribution (DC) plans. The retirement plans have certain tax advantages, and employers provide for a portion of the employee's contribution. Funds usually cannot be withdrawn without penalty before retirement. With a DB plan, the employer sets up a trust and contributes money annually in amounts sufficient to pay a defined retirement benefit to each employee. The employee receives a fixed income stream after retirement contingent on his/her salary, years of employment, retirement age and other factors. The fund is set up and controlled by the employer and the individual employee accounts are not segregated. With a DC plan, contributions are paid into individual accounts by each employee and the employer may contribute an additional amount. At retirement, a lump sum amount equal to the current account value is available. Defined contribution plans are versatile. The retiree has the options of creating a wide variety of ways to draw down the fund value including simple ad hoc withdrawals or more well-defined withdrawals of a certain percentage of the value per year.

Since DB plans guarantee a fixed stream of cash flows until the death of the policy holder, DB plans have a significant exposure to longevity risk and this risk is born by the employer who has set up the DB plan and administers it. Additionally, the pool of people in the same DB plans usually have similar features, including age, industry, occupation and location. These same risk characteristics can exaggerate the longevity risk presented by DB plans. Additionally, life expectancy is different for different socioeconomic groups (Richards, et al., 2006).

Besides longevity risk, DB plans expose the employer to investment risk. When the investment return does not reach the expected level, the DB pension plans must raise contributions to fund the gap. For new employees they may adjust the formula for determining the pension payments. DB plans are also exposed to inflation risk. If wage inflation outpaces investment returns, pension plans that have benefits linked to final salary will have to increase contributions to the pension fund to account for anticipated salary growth over the employees' work life.

DC plans expose the employer to less longevity risk since the plan provides the lump sum to the employees upon retirement, and is not linked to the lifetime of the employees. DC plans also expose the employer to less investment risk, since the participants make their own choice of the funds for investment. The participants, not the employer have to manage longevity risk and investment risk by themselves. The risk is still there, the difference is who must manage it.

Longevity risk exists in DB and DC plans. In DB plans, the employer is at risk. In DC plans, the employee is at risk. Increased life expectancy increases the needed net present value of pension provisions. Because people are living longer, employers or employees have to increase contributions to pension funds, or postpone retirement. The life table and mortality assumptions need to be adjusted for actuarial evaluation and the setup of pension schemes. Future pension plans administration must consider the substantial increases in life expectancy.

2.3.6 Reverse Mortgages

A reverse mortgage or life mortgage is a loan available to seniors, and is used to release the home equity in a property as one lump sum or multiple payments. The homeowner's obligation to repay the loan is deferred until the owner dies, the home is sold. The benefits of the product include: The homeowner does not need to sell the house. The homeowner can use the product to hedge their longevity risk if they choose to receive the loan as a series of annuity payments. The homeowner can make the illiquid housing asset more accessible. From the provider's perspective, longevity risk, real estate price risk and interest rate are the major risks involved in the reverse mortgages. In the U.S. market, Home Equity Conversion Mortgage (HECM) program, Fannie Mae's Home Keeper program, and Financial Freedom's Cash Account Advantage are three major reverse mortgage programs. Collections (pools) of reverse mortgages can be bundled and securitized to produce assets. The structure of securitization for reverse mortgages is similar to that for a collateralized debt obligation (CDO). The valuation of the reverse mortgage, its cash flows and timing, and its structured derivatives securitized asset all depend on using a quantitative model of the mortality rate and life expectancy.

2.3.7 Life Settlements

A life settlement is a financial transaction in which the owner of a life insurance policy sells an unneeded policy to a third party for more than its cash value and less than its face value. A life settlement is an alternative a policyholder has to either surrender or lapse this policy when the owner of the life insurance policy no longer needs or wants the policy. This is also an alternative when the policy is underperforming or the policy owner can no longer afford to pay the premiums. Investment banks have purchased hundreds of thousands of these life insurance policies and repackaged (securitized) them into bonds, then sold bonds to investors such as pension funds. The payment of the bond depends on the life expectancy of the members in the pool of the life insurance policies. "We estimate that life settlements, alone, generate surplus benefits in excess of $240 million annually for life insurance policyholders who have exercised their option to sell their policies at a competitive rate." (Doherty and Singer, 2002). Most purchasers of this type of contract are not in the primary business of trying to make profits from mortality. The cash flows from the life settlements clearly involve longevity risk while the insured lives.

2.4 Summary of Products for Institutional Mortality and Longevity Risk Transferring

2.4.1 Mortality Bonds

The Swiss Reinsurance company issued the first mortality risk contingent securitization in December 2003, the Swiss Re Mortality Catastrophe Bond. When the bond covenants are triggered by a catastrophic evolution of death rates of a certain population, the investors incur a loss in principal and interest. The bond provides the investor higher yield than that in the usual bond market as a compensation for the additional mortality risk the bond purchaser takes. The bond was issued through a special purpose vehicle (SPV) called Vita Capital. This securitization enables the Swiss Re to take extreme catastrophe mortality risk off its balance sheet, such as the Japan tsunami disaster that killed thousands of policy holders. Then Swiss Reinsurance has capital freed up on that bond just when they needed it for claims. Similar products will be of interest to companies that self insure workers compensation.

The bond had a maturity of three years, a principal of \$400m, and a coupon rate of 135 basis points plus the LIBOR. The mortality index, M_t, that was the weighted average of mortality rates over five countries, males and females, and a range of ages. The principal would be repayable in full only if the mortality index does not exceed 1.3 times the 2002 base level during any year of the bond's life, and is otherwise dependent on the realized values of the mortality index. The precise payment schedules are given by the following f_t functions:

$$f_t = \begin{Bmatrix} \text{LIBOR} + \text{spread} & t = 1, \ldots, T-1 \\ \text{LIBOR} + \text{spread} + \max\left\{0{,}100\% - \sum_t L_t\right\} & t = T \end{Bmatrix}$$

$$L_t = \begin{cases} 0\% \\ \left[\frac{M_t - 1.3M_0}{0.2M_0}\right] \times 100\% \\ 100\% \end{cases} \text{if} \begin{cases} M_t < 1.3M_0 \\ 1.3M_0 \le M_t \le 1.5M_0 \\ 1.5M_0 < M_t \end{cases} \text{for all } t \qquad (2)$$

Thus, if the mortality rate does not exceed 130% of the normal mortality defined at time 0 as M_0, then all coupons and principal are given. If mortality M_t does exceed 130% of M_0, then principal repayment is lost; all is lost if the mortality is 150% above the declined level M_0.

2.4.2 Longevity Bonds

The European Investment Bank (EIB) announced a possible issue of the EIB/BNP Paribas Longevity Bond in November 2004, though the structurer/manager BNP Paribas. The bond was issued through a Bermuda-based reinsurer Partner Re. Partner Re contracted to make annual floating rate payments (equal to £50m × S_t to the EIB based on the realized mortality experience of the population of English and Welsh males aged 65 in 2003 (published by the U.K. Office for National Statistics). Partner Re would receive from the EIB annual fixed payments based on a set of mortality forecasts for this cohort. The mortality forecasts used for the first payments were based on the U.K. Government Actuary's Department 2002-based central projections of mortality, adjusted for Partner Re's own internal revisions to these forecasts. This arrangement was then supplemented by a cross currency swap (i.e., fixed-sterling-for-floating-euro) interest-rate swap between the EIB and BNP Paribas, since the EIB also wished to pay a floating rate in Euros. The bond had an initial value of £40 m, an initial coupon of £50 m, and a maturity of 25 years. The floating payment structure is $f_t(S_t) =$ £50m × S_t for $t = 1,2,\dots,T; T = 25$. For lack of investor interest, the longevity bonds were not issued successfully.

2.4.3 Longevity Swaps

There are two types of longevity swaps. Of interest here is a cash-flow swap that indemnifies one party and this type of swap is an over-the-counter transaction. An example of this is the q-forwards proposed by JP-Morgan. These were intended to be simple capital market instruments for transferring longevity risk and mortality risk. The q-forwards enable pension funds to hedge against increasing life expectancy of plan members and also enable life insurers to protect themselves against significant increases in the longevity of policyholders. Similar to other forward swaps, q-forwards are securities involving the exchange of a fixed rate payment for a floating or variable rate payment. In this case, the q-forward variable payment depends on the realized mortality of a population at some future date, whereas the fixed rate is dependent on a fixed mortality rate agreed at inception. The q-forwards form the basic building blocks from which many other complex securities can be constructed. The q-forwards provide a type of standardized contract which can help to create a liquid market. A set of q-forwards that settle based on the LifeMetrics Index could fulfill this role. Since the investors require a risk premium to take on longevity risk, the mortality forward rates at which q-forwards transact will be below the expected, or "best estimate" mortality rates. The other swap type is basis risk and may be over-the-counter or exchange based.

A q-forward contract to hedge the mortality risk of a life insurer is that a life insurer pays fixed mortality rate to JP Morgan and JP Morgan pays realized mortality rate to the life insurer. A q-forward contract to hedge the mortality risk of a pension fund is that a pension fund pays realized mortality rate to JP Morgan and JP Morgan pays the fixed rate to the pension fund. In this way, the pension fund who longs the longevity risk transfers the risk to the life insurer who shorts the longevity risk.

Table 2 Comparison of Products

Date	**Mortality Bonds**	**Longevity Bonds**	**Longevity Swap**
Example	Swiss Re Mortality Catastrophe Bond	EIB/BNP Paribas LB	q-Forwards
Purpose	Hedge Mortality Risk for Issuers	Hedge Longevity Risk for Holders	Hedge Mortality Risk for Fixed Rate Payers Hedge Longevity Risk for Fixed Rate Receiver
Participants	Issuer: Life Insurance Reinsurance	Holder: Pension Annuity Providers	Fixed Rate Payers: Life Insurance or Reinsurance Fixed Rate Receivers: Pensions Annuity Providers
Maturity	Short Term	Long Term	Flexible Term
Coupon	High Yield	Low Yield	N/A
Structure	Complex	Complex	Simple
Underwriting Fees	High	High	Low

Table 3 Publicly Announced Longevity-Swap Transactions

Date	Hedger	Size (m Pounds)	Format	Term	Comments
Jan 2008	Lucida (Insurer)	Not disclosed	Derivative	10 years	Index-based longevity swap First capital markets longevity hedge
July 2008	Canada Life (Insurer)	500	Derivative	40 years	Indemnity longevity swap Distributed to capital markets investors
Feb 2009	Abbey Life (Insurer)	1,500	Insurance	Run-off	Indemnity longevity swap Distributed to reinsurers
Mar 2009	Aviva (Insurer)	475	Derivative	10 years	Collared indemnity swap Distributed to reinsurer and capital markets
Jun 2009	Babcock International (Pension fund)	1,200	Derivative	50 years	Indemnity longevity swap First longevity hedge by a pension scheme
July 2009	RSA (Pension fund)	1,900	Insurance	Run-off	Indemnity longevity swap Distributed to reinsurers
Dec 2009	Royal County of Berkshire (Pension fund)	750	Insurance	Run-off	Indemnity longevity swap First longevity hedge by public sector
Feb 2010	BMW UK (Pension fund)	3,000	Insurance	Run-off	Indemnity longevity swap Distributed to reinsurers
Jul 2010	British Airways (Pension fund)	1,300	Insurance	Run-off	Indemnity longevity swap Distributed to reinsurers

Source: McWilliam, Longevity Risk, 2011

Table 4 Comparison of customized and indexed longevity products

	Customized (indemnity)hedge	**Index (parametric) hedge**
Longevity risk indemnification	Perfect hedge No basis risk Customized for the portfolio of hedged lives	Not perfect hedge With basis risk Expected to hedge around 85% of risk
What is hedged	Hedge of pension plan cash flow	Hedge of liability value over life of swap
Target	For pensioner members	For deferred members
Data requirements	Requires pension plan to provide data over the life of hedge	Pension plan member data only required initially to structure hedge as payout of hedge is based on published index
Other	Bespoke contract tailored to structure of pension plan	Standardized contract More attractive to capital markets investor base Promotes secondary liquidity

Source: Revised from McWilliam, Longevity Risk, 2011

The following section introduces the mortality models currently used for modeling mortality and longevity risk and how the anticipated mortality improvements are incorporated. The next section discusses the q-forward derivative instrument of J.P. Morgan and shows how to price the product in closed form by using a new double exponential jump diffusion model extension of the Lee-Carter mortality formula that allows for cohort and age effects. A final section discusses future research and further application to life settlement securities, reverse mortgages, and longevity sensitive products.

Chapter 3 Double Exponential Jump Diffusion Model

Pricing of the CAT mortality bonds or life-settlement securities depends on the estimation and forecast of mortality rates or life expectancy, which are considerations involving mortality risk and longevity risk. The estimation and forecast of life expectancy or mortality rate also plays a crucial role in longevity/mortality risk management for pension funds or insurers. In this paper, we propose a stochastic model, based on the Brownian Motion process, plus an asymmetric jump diffusion process for the estimation and forecasting of mortality rates and life expectancy.

Longevity jumps and mortality jumps should be incorporated in the modeling and securitization (Cox, Lin and Pedersen 2010), since the jumps are the critical sources of risk which pension funds and insurers should be more cognizant of (Zanjani, 2002). The mortality jumps (such as the 1918 flu) have a short-term intensified effect, while the longevity jumps (caused by the pharmaceutical or medical innovation) have long-term gentle effect. The different frequency and intensity of the mortality jumps and the longevity jumps explain the distribution skewness of the mortality time-series increment, which is important but not considered in previous mortality rate models. Considering the asymmetric jump phenomenon of the mortality time-series, our model adopts a compound Poisson-Double Exponential Jump Diffusion (DEJD) process to capture the longevity jumps and the mortality jumps, respectively.

Very few studies address the modeling of mortality jumps for securitization. Biffis (2005), Bauer, Borger and Russ (2010) apply affine jump-diffusion process to model force of mortality in a continuous-time framework. Our model incorporates the cohort effect, which captures the mortality time-series and adjusts it to fit different age groups. The model with cohort effect captures the feature of the historical mortality time-series more accurately. Chen and Cox (2009) model the jump process with a compound Poisson normal jump diffusion process. Our model makes the contribution of applying the double exponential jump diffusion which differentiates the longevity jumps and the mortality jumps. This captures the distribution skewness of the mortality time-series increment and offers better fitness. Cox, Lin, and Pedersen (2010) propose a model to accommodate both longevity jumps and mortality jumps. Our model has fewer parameters, more concise specification, and can be easily parameterized and applied to securitization.

Our model incorporates the advantages of the Lee-Carter mortality framework, which describes the mortality time-series and considers the cohort effect, making age-specific adjustment for different age groups. This adjustment is critical for the model, since the mortality improvement and extreme positive or negative events (such as influenza pandemic) have different intensity levels in different age groups. In this way, the Lee-Carter model appropriately describes the three dimensional surface of the mortality rate with respect to time horizon and age group horizon. The Lee-Carter framework has been extended by Brouhns, Denuit and Vermunt (2002), Renshaw and Haberman (2003), Denuit, Devolder and Goderniaux (2007), Li and Chan (2007), Chen and Cox (2009), Lin and Cox (2005), Lin and Cox (2008).

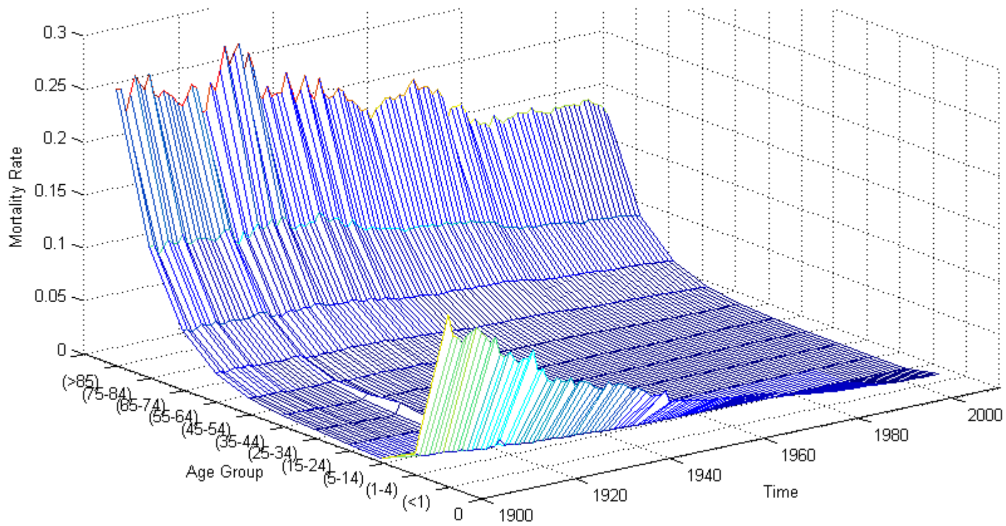
We test our model with historical data and make model fitness comparisons with previous models. The results clearly illustrate the advantage of our model in terms of fit.

In our application, we use the first CAT mortality bond, the Swiss Re Catastrophe Mortality Bond (2003) to calculate the implied risk premium, with two types of changing measure approaches. We implement our model to price the q-forward longevity risk derivative as an example, which illustrates the benefit of our model for providing a closed-form pricing solution for standard structure mortality linked securities. The q-forward derivative, proposed by JP Morgan based on the LifeMetrics index, has the potential to be a future standardized contract which can help to create a liquid market. Beyond the q-forward, our model can provide closed-form pricing solutions to all the mortality-linked securities with cash flows that are linear functions of the mortality rate for each period.

3.1 Data Description

The historical data come from HIST290 National Center for Health Statistics. We chose this because it is the same data used in Chen and Cox (2009). This allows us to compare results and model fit (which we do in section 4). We use the same data in order to facilitate the model fitness comparison in Section 4. The data lists death rates per 100,000 populations for selected causes of death. Death rates are tabulated for age group (<1), (1-4), (5-14), (15-24), then every 10 years thereafter, to (75-84), and finally, the last group is (>85). The data includes both sexes and several race categories. Selected causes for death include major conditions such as heart disease, cancer, and stroke. Figure 5 shows the mortality rate for different age groups and years. Figure 6 shows the comparison of the mortality rate for several sample age groups, including relatively older groups and younger groups.

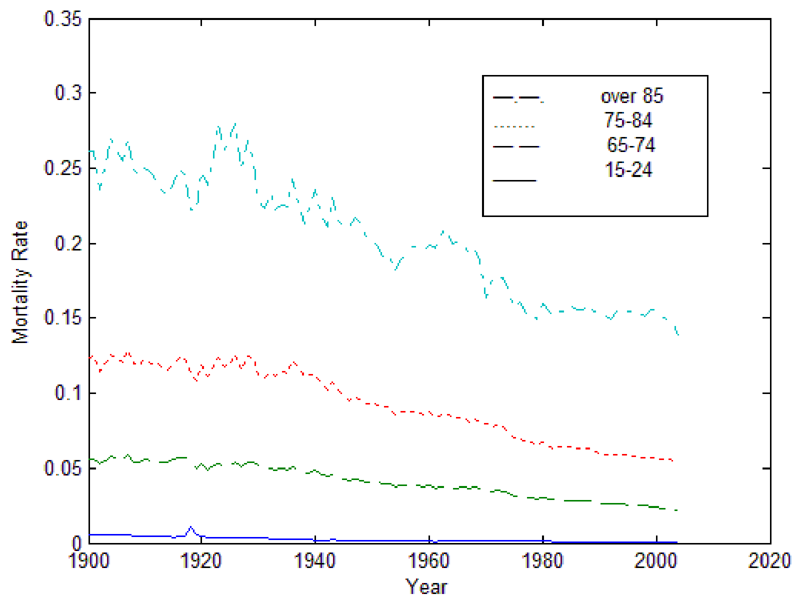
Figure 5 1900-2004 Mortality Rates



We can observe clearly two mortality rate trend properties from Figure 5 and Figure 6. In Figure 5, the downward trend indicates that the mortality rate follows a decreasing trend during 1900-2004, at all ages. For example, in the over 85 age group, the mortality rate decreases from 0.26 to 0.14, while in the younger age groups, such as 15-24, the mortality rate decreases from 0.006 to 0.0008. The decreasing trend shows the improvement of the life times or longevity in all age groups.

In Figure 6, the change in the mortality rate in the older-age groups is more significant than that in the younger-age groups with a steeper downward trend. For example, the mortality rate decreases 0.12 in the age group over 85. During the same time period, the mortality rate decreases only 0.0052 in the age group 15-24. The comparison of the steepness of the mortality surface shows that the improvement in longevity among the older-age group is more significant than that in the younger-age group. We can observe that there is variability and dynamics in the mortality trends. Accordingly we shall use a stochastic model to capture the dynamics features.

Figure 6 Comparison of the Age Group Mortality Rates



3.2 Model Framework and Requirement

The basic requirement of the mortality model is to capture the features described by the data. Various mortality rate models have been provided by previous research. The majority of models are based on the Lee-Carter one-factor model (Lee and Carter, 1992). These models extend the Lee-Carter model to a two-factor model (Blake, Cairns, and Dowd, 2006a), or incorporate a stochastic process in the model (Dahl 2004), or introduce the possibility of a jump process in the stochastic process to accommodate extreme outliers in the mortality time series (Cox, Lin, and Wang 2006; Chen and Cox 2008).

In the Lee-Carter framework, the mortality rate $\mu_{x,t}$ denotes the mortality rate of the group whose age is *x* during the year *t*. It is decomposed into age-specific parameters a_x, b_x and mortality time-series k_t.

$$\ln(\mu_{x,t}) = a_x + b_x k_t + e_{x,t} \quad (3)$$

$$\mu_{x,t} = \exp(a_x + b_x k_t + e_{x,t}) \quad (4)$$

Here a_x represents the age group shift effect, and $\exp(a_x)$ is the general shape across the age of the mortality schedule, and b_x represents the age group's reaction effect to the mortality time-series k_t. In other words, the b_x profile tells us which group of mortality rates decline rapidly and which group of mortality rates decline slowly in response to temporal changes in mortality k_t, and $e_{x,t}$ captures the age group's residual effect not reflected in the model. Lee and Carter (1992) suggest estimating the parameters in their model using a two-stage singular value decomposition (SVD) based on historical data for $\mu_{x,t}$ to estimate the age-specific parameters a_x, b_x, and to generate the mortality time-series k_t.

To implement the SVD procedure, first, we need to normalize using a condition that the sum of k_t is equal to 0 and the sum of b_x is equal to 1. This enables the sum of b_x to be unity and distribute the k_t equally around 0. Then a_x must equal the average over time of $\ln(\mu_{x,t})$.

$$a_x = \frac{1}{T}\sum_{t=1}^{T} \ln(\mu_{x,t}) \quad (5)$$

Furthermore, by equation (2), k_t must (or almost) equal the sum over age of ($\ln(\mu_{x,t})$-a_x), since the sum of b_x has been chosen to be unity. This is not an exact relation, however, since the error terms will not in general sum to 0 for a given age. Then, each b_x can be found by regressing, without a constant term, ($\ln(\mu_{x,t})$-a_x) on k_t separately for each age group *x*.

In the second stage, we re-estimate the mortality time-series k_t iteratively, given the estimation of a_x and b_x in the first step. This enables the actual sum of death at time *t* (left-hand side) to equal the implied sum of deaths at time *t* (right-hand side).

$$D_t = \sum_x (P_{x,t} \exp{(a_x + b_x k_t)}) \quad (6)$$

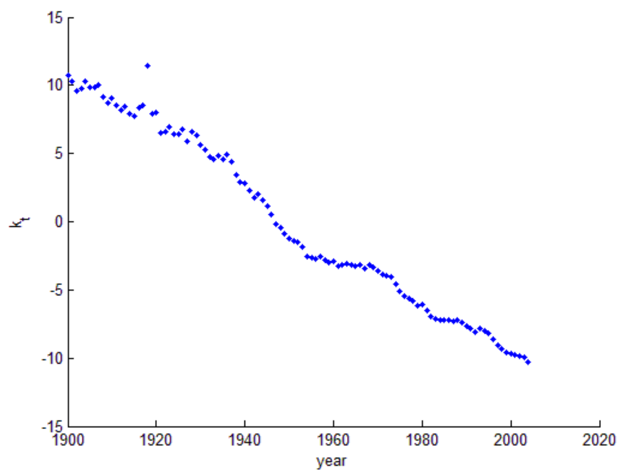
where D_t is the actual sum of deaths at time *t* , and $P_{x,t}$ is the number of members of population in age group *x* at time *t*.

Implementing the SVD two-stage procedure with data on historical U.S. mortality rates during 1900-2004, we obtain the fitted a_x, b_x, in Table 5, and the mortality time-series k_t in Figure 7. The decreasing trend of mortality time-series k_t shows the improvement of mortality along the time as described previously. Figure 7 also shows the big jump in 1918 (which was caused by the flu pandemic) and other jumps around 1920, 1943, etc.

Table 5 Fitted Value for Age-Specific Parameters a_x and b_x during 1900-2004

Age Group	a_x	b_x
<1	-3.4087	0.1455
1-4	-6.2254	0.1960
5-14	-7.1976	0.1492
15-24	-6.2957	0.0994
25-34	-5.9923	0.1044
35-44	-5.4819	0.0855
45-54	-4.7799	0.0608
55-64	-4.0137	0.0468
65-74	-3.2347	0.0426
75-84	-2.4196	0.0409
>85	-1.6119	0.0290

Figure 7 The Mortality Time-Series k_t



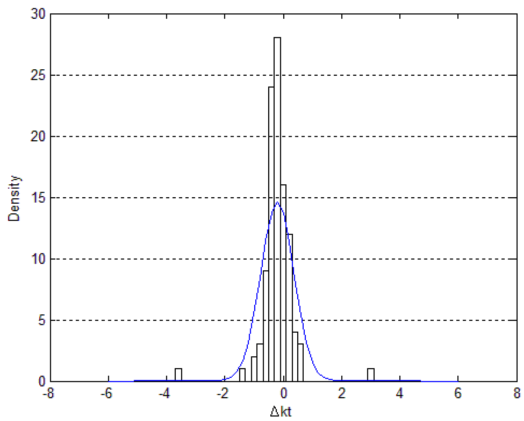
Now, we need a model to capture the features of the shape, the trend and the jumps of the mortality time-series k_t. First, the model should incorporate a stochastic term in the description of the k_t time series, as this has proved to be better than using the model without the stochastic process, (Dahl 2004). Second, as shown in Figure 7, k_t includes both positive and negative values. Since geometric Brownian motion will not generate a negative value from the positive starting value, it does not fit the k_t process by itself.

However, Brownian motion can be selected to fit the k_t time series. Third, we can observe from Figure 7 that the jumps are transient, not permanent. For example, the sudden increase of mortality rate in 1918, caused by the flu, falls back to the normal condition after two years.

Beyond the three points on the model specification listed above, we noted that, the jump phenomenon in mortality rates is two directional. In Figure 7, the positive jumps (the sudden increases in the mortality time-series) occur with higher-severity, while the negative jumps (the suddenly decreases in the mortality time-series) are of lower-severity and higher-frequency. Coughlan et al., 2007 also shows that there is a significant negative autocorrelation in mortality rates, so increase in mortality are often followed by increases in longevity. However, there is a definite downward trend in mortality with the jumps being asymmetric and of unequal frequency. Hence, the model that involves a jump process with a symmetric normal distribution for the size of the jump (Chen and Cox, 2009) may not adequately capture the asymmetry of the mortality jump phenomenon. From biological and demographic perspectives, the positive jumps (mortality jumps) can be explained by sudden catastrophes (e.g., earthquake, tsunami, hurricane) or critical diseases, such as the extreme positive jump caused by flu in 1918. These positive jumps can be transitory (1918 flu) or more lasting (AIDS, antibiotic resistant strains of tuberculosis, etc.). The negative jumps (longevity jumps) are associated with multiple biological and health improvement causes. According to Johnson, Bengtson and Coleman (2005) increases in survival currently reflect a shift in the causes of death from infections to chronic degenerative diseases. Hence, the improvements in mortality due to health or biological reasons affecting chronic diseases appear more frequently and are of a more moderate size over time. The health improvements due to improved medical treatment, for example, may be frequent in occurrence and small in impact, but they have gradually but significantly changed life expectancy. Illustrative of this, since 1960 longevity has increased 1-1.5 % per year due mainly to a 65% reduction in cardiovascular events in the age cohort over age 45 (Iacovino 2009). Again the survival jump effect could be lasting or transitory (e.g., a new drug which loses effectiveness). A large longevity jump could occur in the future if an effective treatment of coronary heart disease or cancer were found, as these two causes of death combined constitute more than half of all deaths among people over the age of 40 (Johnson, Bengtson and Coleman 2005 p. 109). Ultimately, once the jump (positive or negative) occurs, any lasting effect can be subsequently captured in the Gaussian process component of the model. Because the negative jumps are low severity and high frequency they are difficult to discern in the figure except possibly in a more negatively inclined series.

The descriptive statistics of $\Delta k_t = k_{t+1} - k_t$ show leptokurtic features. The skewness of Δk_t equals to -0.451. In other words, the Δk_t distribution is skewed to the left, and has a higher peak and two heavier tails than those of a normal distribution, which we can also observe in Figure 8.

Figure 8 Comparison of Actual Δk_t Distribution and Normal Distribution



In Figure 8, the histogram represents the distribution of actual Δk_t, and, as can be seen, Δk_t cannot be fitted well by a normal distribution. Hence, the Brownian motion process alone cannot be used to describe the mortality time-series k_t.

To incorporate the leptokurtic feature of the Δk_t distribution, the analysis here incorporates a double exponential jump diffusion (DEJD) model to capture both the positive jumps and negative jumps of the k_t process. Compared to Cox, Lin, Pedersen (2010), which also captures the positive jumps and negative jumps (the size of the jumps is normally distributed so there is symmetry in the distribution in Cox and Chen), our model has a concise specification and an easy approach for calibration. What is more, unlike Cox et al. (2008), our model has a closed-form solution for the forecast of the future mortality rate, which facilitates mortality securities pricing.

3.3 THE MODEL SPECIFICATION

To capture the features of the mortality time-series k_t, and to account for the tractability and the calibration of the model, we set the model specification to describe Δk_t in the approximate continuous-time model of dk_t as given below.

The dynamics of the mortality time-series k_t is specified as

$$dk_t = \alpha dt + \sigma dW_t + d(\sum_{i=1}^{N(t)}(V_i - 1)) \qquad (7)$$

where W_t is a standard Brownian motion, $N(t)$ is a Poisson process with rate λ, where λ describes the expected frequency of the jumps. The larger the λ, the more times jumps occur in the mortality time-series. Here V_i is a sequence of independent identically distributed (*iid*) nonnegative random variables, $Y = \log(V)$ has an double exponential distribution with the density

$$f_Y(y) = p\eta_1 e^{-\eta_1 y} 1_{\{y \geq 0\}} + q\eta_2 e^{-\eta_2 y} 1_{\{y<0\}} \qquad (8)$$

$$\eta_1, \eta_2 > 0, \quad p, q \geq 0, \quad p + q = 1.$$

The parameters p and q represent, respectively, the proportion of positive jumps and negative jumps among all jumps. Thus, $p\lambda$ is the expected frequency of positive jumps and $q\lambda$ is the expected frequency of negative jumps. The parameters η_1 and η_2 describe the positive jump severity and the negative jump severity separately. Thus, $Y|Y \geqslant 0$ is exponentially distributed with mean ${\eta_1}^{-1}$, while $-Y|Y \leqslant 0$ is exponentially distributed with mean ${\eta_2}^{-1}$. The larger η_1, the smaller the positive jumps severity. Similarly, the larger η_2 the smaller the negative jump in absolute value. In this way, the positive jumps and negative jumps are captured by similar distributions but with different parameters, based on the asymmetry of jumps in the mortality time-series k_t and the leptokurtic feature of dk_t.

The model specification with the double-exponential distribution has the advantage of mathematical tractability allowing a closed-form formula for the expected future mortality rate to be derived. Because of this closed-form solution, the DEJD model may provide a useful stochastic mortality model for internal company mortality simulation, as well as being useful in the capital market applications we discuss subsequently. The double-exponential distribution has also been widely implemented as a stock price jump-diffusion model, for which closed-form solutions for options and other securities are available (Kou, 2004). The closed-form solution of the expected future mortality rate is presented in equation (20).

3.4 Numerical Solution

3.4.1 Parameter Calibration

The disentangling of jumps from the diffusion components is a serious difficulty in the calibration of the underlying processes. The increments of the underlying process are supposed to be captured by the diffusion process, with a few extreme increments captured by jumps. However, the addition of the jump process may yield the wrong calibrated parameters which leads to high frequency of jumps and small severity of jumps. The double-exponential jump diffusion model for dk_t faces the same problem. A calibration method is needed to generate the right parameters of low frequency and large severity for jumps. Ait-Sahalia and Hansen (2004) demonstrate that maximum likelihood estimation (MLE) has advantages in disentangling jumps from diffusion. Meanwhile, the double-exponential jump diffusion is a linear process with independent increments and an explicit transition density, which fortunately satisfies the requirement of a complete specification of the transition density for using MLE. Therefore, we choose the MLE method to calibrate parameters $\{\lambda, p; \eta_1, \eta_2; \alpha, \sigma\}$ with the dk_t time series.

Let $C = \{k_0, k_1, \dots, k_T\}$ denote the mortality time-series, at equally spaced times $t = 1,2, \dots, T$. By (6), the one period increments $r_i = \Delta k_i = k_i - k_{i-1}$ is independent and identically distributed (*iid*). The unconditional density of one period increment *f*(*r*) is:

$$f(r) = e^{-(\lambda_u+\lambda_d)} f_{0,0}(r) + e^{-\lambda_u} \sum_{n=1}^{\infty} p(n, \lambda_d) f_{0,n}(r)$$

$$+ e^{-\lambda_d} \sum_{m=1}^{\infty} p(m, \lambda_u) f_{m,0}(r) + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} p(n, \lambda_d)\, p(m, \lambda_u) f_{m,n}(r) \qquad (9)$$

where $p(n, \lambda_d) = \frac{e^{-\lambda_d}\lambda_d}{n!}$, $p(m, \lambda_u) = \frac{e^{-\lambda_u}\lambda_u}{m!}$, and $f_{m,n}(r)$ is the conditional density for a one period increment, conditional on the given number of up and down jumps (m, n) in the increment.

$$f_{0,0}(r) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2\sigma^2}(r-u+\frac{1}{2}\sigma^2)^2} \qquad (10)$$

$$f_{0,n}(r) = \frac{\eta_d^n}{(n-1)!\sqrt{2\pi}\sigma} \int_{-\infty}^{0} (-x)^{(n-1)} e^{(\eta_d x - \frac{1}{2\sigma^2}(r-x-\mu+\frac{1}{2}\sigma^2)^2)}\, dx \qquad (11)$$

$$f_{m,0}(r) = \frac{\eta_u^m}{(m-1)!\sqrt{2\pi}\sigma} \int_{0}^{\infty} x^{(m-1)} e^{(-\eta_u x - \frac{1}{2\sigma^2}(r-x-\mu+\frac{1}{2}\sigma^2)^2)}\, dx \qquad (12)$$

$$f_{m,n}(r) = \frac{\eta_u^m \eta_d^n}{(m-1)!(n-1)!\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{0\wedge t} (-x)^{n-1} \; (t-x)^{m-1} \quad e^{(\eta_u+\eta_d)x}\, dx\right)$$

$$\times\, e^{-\eta_u t} e^{-\frac{1}{2\sigma^2}(r-x-\mu+\frac{1}{2}\sigma^2)^2}\, dt \qquad (13)$$

The log-likelihood given T equally spaced increment observations is

$$L(C;\lambda_u,\lambda_d;\eta_1,\eta_2;\alpha,\sigma)=\sum_{i=1}^{T}\ln(f(r_i))$$

where $\lambda=\lambda_u+\lambda_d$, and $p=\frac{\lambda_u}{\lambda}$. Then we can calibrate the parameters $\{\lambda,p;\eta_1,\eta_2;\alpha,\sigma\}$.

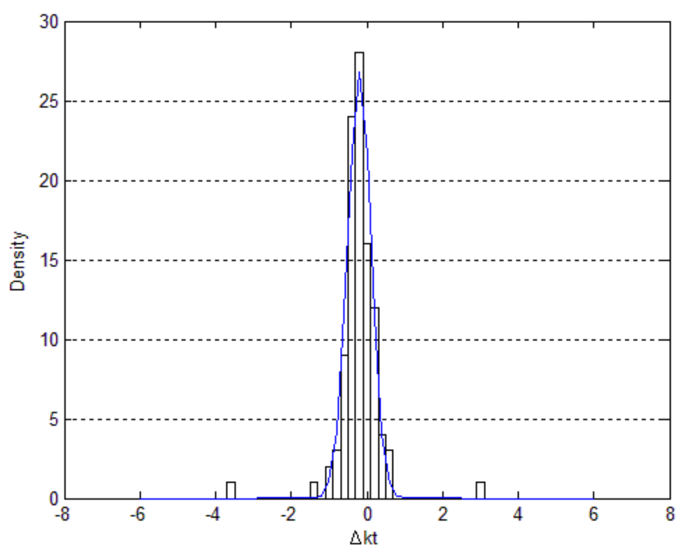
After computation, we get $\{\lambda_u,\lambda_d;\eta_1,\eta_2;\alpha,\sigma\}$ ={0.029,0.035,0.71,0.75;-0.20,0.31}, and $\{\lambda,p;\eta_1,\eta_2;\alpha,\sigma\}$ ={0.064,0.45;0.71,0.75;-0.20,0.31} and maximum likelihood value $L=-49.95$. Here η_1 and η_2 describe the severity of positive jumps (mortality jumps) and negative jumps (longevity jumps), respectively. A larger η represents a smaller severity. The fact that η_1= 0.71< 0.75= η_2 verifies that the severity of positive jumps is larger than the severity of negative jumps, consistent with that observed in Figure 7. Here λ_u and λ_d describes the frequency of positive jumps and negative jumps, respectively. A larger λ represents a larger frequency. Since λ_u= 0.029<0.035= λ_d, this verifies that the frequency of positive jumps is smaller than the frequency of negative jumps, consistent with that observed in Figure 7.

3.4.2 Model Comparison

Figure 9 shows how the DEJD model fits the actual increment of mortality rate Δk_t, by comparing the distribution generated by the DEJD model calibrated to historical data and the actual distribution of Δk_t. Comparing Figure 9 with Figure 8, we observe that the DEJD extension of the Lee-Carter model approximates the distribution of increment of mortality rate Δk_t much better than the commonly used Brownian motion extension. The mean of the distribution of DEJD and the Brownian motion is the same, $\mu_{DEJD}=\mu_{BM}=-0.20$, however the standard deviation of the distribution of

DEJD ($\sigma_{DEJD} = 0.31 < \sigma_{BM} = 0.57$) is much less than that for the Brownian Motion model. This is exactly the reason that the DEJD model is more appropriate to fit the actual distribution, which can be directly observed from the comparison of the two figures. The underlying reason is that the Lee-Carter model includes the outliers in the Brownian motion diffusion process, which causes the calibrated σ_{BM} to be larger. And the Brownian motion diffusion is appropriate to capture the normal distribution shape without a fat tail and a high peak, however this is not the case in this data. In our DEJD model, we include the outliers in the jump diffusion part, which enables the calibrated σ_{DEJD} to be smaller than σ_{BM} and improves the fit to the data.

Figure 9 Comparison of Actual Δk_t Distribution and DEJD Distribution



Next, the DEJD model is compared with both Lee-Carter Brownian motion model and the normal jump diffusion model (Chen and Cox, 2009). For model selection, we adopt the Bayesian Information Criterion (*BIC*) proposed by Schwarz (1978). Unlike the significance test, *BIC* allows comparison of more than two models at the same time and does not require the alternatives to be nested. *BIC* is a "conservative" criterion since it heavily penalizes over parameterization (Ramezani and Zeng, 2007).

Suppose the *k*th model M_k, has parameter vector θ_k, where θ_k consists of n_k independent parameters to be estimated. Denote θ'_k as the MLE of θ_k. Then, *BIC* for Model M_k is defined as:

$$BIC_k = -2\ln f(C|\theta'_k, M_k) + n_k \ln(m), \tag{14}$$

where *m* is the number of observations in data set *C* and $f(C|\theta'_k, M_k)$ is the maximized likelihood function. With the *BIC* criterion, the best "fitting" model is the one with the smallest *BIC* value. Table 6 below gives the *BIC* model fit values for the three competing models. It uses the maximum likelihood function values provided in (Chen and Cox, 2008). It can be observed that the DEJD model fits best[2]

[2] The DEJD model fits the real series of 104 data points better than the other models in spite of being penalized by the BIC criterion for having more parameters. We do get the same ranking of the models by log likelihood fit as by BIC, even if we exclude the 1918 flu year from the data (which we do not think should be done). However, if the 1918 flu year is excluded and the series is made artificially smoother, then using the BIC criterion, the parameter penalty dominates and the ranking is simply according to the number of parameters in the model (Lee-Carter with two parameters, then Chen Cox with 5 parameters, then DEJD model with 6 parameters).

Table 6 Comparison of Model Fitness

	Number of Parameters	*ln* (likelihood)	BIC
Double Exponential Jump Diffusion Model (DEJD Model)	6	-49.95	127.76
Normal Jump Diffusion Model (Chen-Cox Model)	5	-62.52	148.26
No Jump Diffusion Model (Lee-Carter Model)	2	-94.27	197.82

The underlying reasons that our DEJD model fits the data better are as follows. First, the outliers in the mortality time-series cause there to be fat tails and a high peak in the increment Δk_t distribution which rules out the normal distribution. The Lee-Carter model treats outliers the same as other points in the mortality time-series evolution process. As a result, the outliers enhance the variability of the process and cause overestimation of the standard deviation σ. Our DEJD model applies a compound Poisson double exponential jump diffusion process separately from the Brownian motion diffusion process. This avoids the problem of mismatching the fat tail and high peak with the normal distribution and hence provides a better fit.

Second, the Chen-Cox model applies the normal jump diffusion model which is composed of a Brownian motion diffusion process and a normal jump diffusion process. This model also treats the outlier with a normal distribution. Actually, the positive outlier (or jump) and the negative outlier (or jump) are due to different biological and technical reasons. The positive outliers (for example, that caused by the 1918 flu pandemic) have short-term intensified effects, while the negative outliers (caused by pharmaceutical or medical innovation) have long-term gentle effects. The different frequency and intensity of the positive outliers and the negative outliers explain the skewness of the mortality time-series increment, which is not appropriately described by the symmetric normal jump diffusion.

The *BIC* comparison above shows the model with asymmetric jumps fits the mortality time-series k_t better than the Lee-Carter model or the Chen-Cox model. An important feature of the data is the 1918 flu epidemic. In our view, the flu is in the data, might recur, and should therefore be included in any model, especially one aimed toward financially hedging against such events in the future. In an alternative view, Lee Carter treated it as an outlier, essentially regarding it as a one of a kind event whose inclusion would result in misleading average forecasts. Additionally, Chen Cox (2010) and Li and Chan (2005, 2007), in order to reveal a smoother mortality trend, performed a systematic time-series outlier analysis for the mortality data in United States and Canada, and fit the adjusted outlier-free mortality series to the Lee--Carter model. In their data, U.S., 1900 to 2000, they find seven outliers, which occurred in years 1916, 1918, 1921, 1928, 1936, 1954, and 1975, respectively. Most of these outliers result from influenza epidemics according to their explanations, except for the data in 1954 and 1975. To explore the effect of such deletions on our model, we did another analysis where we deleted the outliers found by Li and Chan from the original mortality data, and re-estimated the mortality factor k_t. The parameters obtained were {0.01,0.05,0.92,0.85,-0.23,0.39}, and the *ln*(likelihood) value was {-47.62} with a *BIC* value of 123.1063. The maximum likelihood value is similar as the result treated by Chen Cox model and Lee Carter model. (Chen Cox 2010). When the jumps are treated as outliers and deleted from the trend, the parameters describing the arrival of the positive jump or negative jump converge to 0.

The parameters describing the Gaussian component (drift and volatility) converge to that in the Lee Carter model.

However, almost surely catastrophic mortality event will occur again, the question is only when, and how can vulnerable entities (life insurers, pension funds, etc.) prepare for it and mitigate their financial exposure to this risk. For the purpose of this paper, the rationale for the interest in mortality and longevity risk is precisely to be able to handle such outliers as the 1918 flu pandemic. If, on the other hand, the purpose were to model mortality rates alone for aggregate smooth life table models, such as the goal of Lee-Carter, then their removal prior to parameter fitting might be justified. In our case if there were no possibility of jumps, especially large potentially bankrupting jumps in mortality or longevity rate, then there would be no interest in longevity or mortality derivatives to be introduced into the capital markets. It is precisely because of such jumps that the problem arises and hence we strongly feel that the inclusion of these outliers should be maintained. Having said that, without the 1918 outlier, as expected, the *BIC* improves since one is fitting a smoother series. Still, we feel that the outlier point needs to be included as a matter of principle.

3.4.3 Implied Market Price of Risk

Like other insurance products, such as annuities, the longevity risk contingent securities are priced in an incomplete capital market. Hence, the risk premium should be considered in pricing the issues, since it represents the price that pension funds or insurers are willing to pay to transfer longevity or mortality risk. In previous research papers, (Blake, Cairns, Dowd, MacMinn 2006), (Chen, Cox 2009), the Swiss Re mortality catastrophe bond has been used to calculate the implied mortality risk premium, given its payment structure and issue price.

The Swiss Re Mortality Catastrophe Bond

The Swiss Reinsurance company issued the first mortality risk contingent securitization in December 2003. If the bond is triggered by a catastrophic evolution of death rates of a certain population, the investors incur a loss in principal and interest. The bond provides the investors higher yield as compensation for the mortality risk they take. The bond was issued through a special purpose vehicle (SPV) called Vita Capital, which enabled Swiss Re to remove extreme catastrophic risk from its balance sheet.

The bond had a maturity of three years, a principal of $400m, the coupon rate of 135 basis points plus LIBOR rate. The mortality index, M_t, was a weighted average of mortality rates over five countries, males and females, and a range of ages. The principal was repayable in full only if the mortality index did not exceed 1.3 times the 2002 base level during any year of the bond's life. If mortality did exceed this threshold, the payment was dependent on the realized values of the mortality index. The precise payment schedules were given by the following f_t functions:

$$f_t = \begin{Bmatrix} \text{LIBOR} + \text{spread} & t = 1, \dots, T-1 \\ \text{LIBOR} + \text{spread} + \max\left\{0{,}100\% - \sum_t L_t\right\} & t = T \end{Bmatrix}$$

where the function L_t specifies the amount of payment that is lost due to mortality experience, namely

$$L_t = \begin{cases} 0\% \\ \left[\frac{M_t - 1.3M_0}{0.2M_0}\right] \times 100\% \\ 100\% \end{cases} \quad \text{if} \begin{cases} M_t < 1.3M_0 \\ 1.3M_0 \leq M_t \leq 1.5M_0 \\ 1.5M_0 < M_t \end{cases} \text{for all } t \qquad (15)$$

Risk-Neutral Pricing

Risk-neutral pricing was used by Milevsky and Promislow (2001) and by Blake, Cairns, and Dowd (2006a). The method is derived from financial economic theory that is applicable even in an incomplete market. If the overall market has no arbitrage, then there exists at least one risk-neutral measure Q which can be used for calculating fair prices. We apply the approach in Blake, Cairns, and Dowd (2006b), which assumes the market price of mortality risk is constant and estimates it from the Swiss Mortality Bond. As a more liquid mortality linked securities market develops, a more accurate market price of risk can be calculated based on the adequacy of deals and data.

The payoff of the longevity derivative instrument that we consider in this paper is dependent on the difference between the experienced and expected mortality (or equivalently, survival) rate. However, the mortality rate in our model for a fixed age is itself linearly dependent on the time series k_t, that we modeled using the DEJD. In an incomplete market, the risk neutral pricing will allow pricing of the derivative. Kou and Wang (2004) have discussed derivative pricing using the DEJD model for security prices, and they have derived the risk neutral measure for this stochastic process. Using their results, the DEJD model in the physical measure has a risk neutral DEJD model with parameters given below where the asterisk symbol * denotes that the parameter corresponds to the risk neutral measure:

$$dk_t = (\alpha^* - \lambda^*\gamma^*)dt + \sigma^* dW^* + d(\sum_{i=1}^{N^*(t)}(V_i^* - 1)) \tag{16}$$

$$\gamma^* = E^*[V^*] - 1 = \frac{p^*\eta_1^*}{\eta_1^*-1} + \frac{q^*\eta_2^*}{\eta_2^*+1} - 1 \tag{17}$$

We may integrate (16) to obtain

$$k_s = k_0 + \left(\alpha^* - \frac{1}{2}\sigma^{*2} - \lambda^*\gamma^*\right)s + \sigma^* W_s^* + \sum_{i=1}^{N^*(s)} Y_i^* \tag{18}$$

According to Kou and Wang (2004) the characteristic function is

$$\Phi^*(\theta) = E^*\left[e^{\theta(k(s)-k_0)}\right] = \exp\,[G(\theta)s] \qquad \text{or}$$

$$E^*\left[e^{\theta k(s)}\right] = \exp\,(\theta k_0)\exp\,[G(\theta)s] \tag{19}$$

where

$$G(\theta) = \theta\left(\alpha^* - \frac{1}{2}\sigma^{*2} - \lambda^*\gamma^*\right) + \frac{1}{2}\theta^2\sigma^{*2} + \lambda^*\left(\frac{p^*\eta_1^*}{\eta_1^*-1} + \frac{q^*\eta_2^*}{\eta_2^*+1} - 1\right) \tag{20}$$

Using b_x for θ in (19), the closed-form expression for the expected future mortality rate $\mu_{x,t}$ is derived as

$$E^*\left[\mu_{x,t}\right] = \exp(a_x) \times E^*[\exp(b_x k_t)] = \exp\,(a_x) \times \Phi^*(b_x)$$

$$= \exp\,(a_x + b_x k_0 + b_x t\left(\alpha^* - \frac{1}{2}\sigma^{*2} - \lambda^*\gamma^*\right) + \frac{1}{2}b_x^2\sigma^{*2}t + \lambda^* t(\frac{p^*\eta_1^*}{\eta_1^*-1} + \frac{q^*\eta_2^*}{\eta_2^*+1} - 1)) \tag{21}$$

Formula (20) can be used to calculate the expected future mortality rate directly with parameters $\{\lambda^*, p^*; \eta_1^*, \eta_2^*; \alpha^*, \sigma^*\}$, which is much faster and more convenient than using simulation to project and average the paths of future mortality rates. This model is especially suitable for pricing the mortality linked securities whose cash flow each period is a linear function of the mortality rate (e.g., the q-forward). We will discuss the example of q-forward pricing in section 4.4.

We can derive the implied market price of risk ζ based on the actively traded mortality linked securities on the market whose fair price is already known, and then apply the same ζ to price the unknown mortality linked securities. In the previous research, the annuity price (Cox and Lin, 2006) or the mortality bond price (Chen and Cox, 2009), was used as the known traded price. In this paper, we use the Swiss Re mortality catastrophe bond to determine a known market price of mortality risk to enable us to calculate ζ, and then use this in our DEJD model to price the q-forward incorporating ζ as an implementation example of our DEJD model. For comparison purposes, we consider three possible models corresponding to the Brownian motion, the positive jump severity and the negative jump severity, along with $\{\alpha^*, \eta_1^*, \eta_2^*\}$. Since the mortality linked securities are priced in an existing incomplete market, the value of ζ or the risk-neutral measure Q is not unique. Practically speaking we have only one mortality linked security to use (the Swiss Re Bond) but need to calculate three market prices of risk. Accordingly we use the process suggested by Blake, Cairns, and Dowd, (2006b), to estimate the set of ζ by sequentially changing one and fixing the rest.

Consider the market prices of risk set $\zeta = \{\zeta_1, \zeta_{2,}\zeta_3\}$ with $\alpha^* = \alpha + \zeta_1$; $\eta_1^* = \eta_1 + \zeta_2$; and $\eta_2^* = \eta_2 + \zeta_3$. We can then estimate the components of ζ by changing only one and fixing the rest. The algorithm below is similar to the traditional procedure for calculating the market price of risk with Wang transform approach (Chen and Cox, 2009):

Step 1. Based on the known 2003 mortality time-series, simulate 10,000 times the future mortality time-series $k(t)$ for 2004-2006, using the DEJD model (7) with the calibrated parameter set $\{\lambda, p; \eta_1, \eta_2; \alpha, \sigma\}$={0.064, 0.45; 0.71, 0.75; -0.20, 0.31}, and the initial assumed set $\zeta = \{\zeta_1, \zeta_{2,}\zeta_3\} = \{0,0,0\}$, with the risk-neutral transform function.

Step 2. Calculate the mortality rate $\mu_{x,t}$ by the formula (4) and calculate the average μ_t based on year 2000 standard population and corresponding weights. The year 2000 standard population and corresponding weights is based on the technique notes of the NCHS report GMWK293R. The weights are 0.013818 for the (<1) age group, 0.055317 for the (1--4) age group, 0.145565 for the (5--14) age group, 0.138646 for the (15--24) age group, 0.135573 for the (25--34) age group, 0.162613 for the(35--44) age group, 0.134834 for the (45--54) age group, 0.087247 for the (55--64) age group, 0.066037 for the (65--74) age group, 0.044842 for the (75--84) age group, and 0.015508 for the (>85) age group. These are tabulated in Table 10.

Step 3. Calculate the expected value of the principal payment in every period T by the formula, $E_T^*[payment] = \$400{,}000{,}000 \times [\max(1 - \sum_{t=2004}^{2006} L_t, 0)]$, where L_t is given by (15). The coupon payment in every period is calculated based on the par spread plus 135 basis points, the latter of which was obtained by reference to the risk premium of the Swiss Re Mortality Bond Contract.

Step 4. Iteratively adjust the market price of risk set ζ, and repeat step.1-step.3 until the discounted expected value of the coupon payment in 2004-2006 plus the principal payment in 2006 equals the face value of the mortality bond $400,000,000.

Table 7 Implied Market Prices of Risk by Risk-Neutral Approach

ζ_1	ζ_2	ζ_3
6.15	0	0
0	0.21	0
0	0.18	0.18

Wang Transform:

We do not have efficiently traded underlying mortality index to create a replicating portfolio for pricing. In such an incomplete market situation, Wang (2002) develops a method for pricing risks which combines financial and insurance pricing theories. Wang's distorted method transforms the underlying distribution to enable the securities price to exactly equal the discounted expected values. Wang's transform is intuitive finance theory since it is in accordance with the capital asset pricing model (CAPM) for underlying assets and the Black-Scholes formula for options. Wang's transform is practical and can be easily applied in calculation.

Given a random payment X and cumulative density function $F_X(x)$ under the measure P, then the Wang Transform is defined as that the "distorted" or transformed distribution $F_X^*(x)$ is determined by the market price of risk δ according to the equation

$$F_X^*(x) = \Phi[\Phi^{-1}\big(F_X(x)\big) - \delta] \tag{22}$$

where $\Phi(x)$ is the standard normal cdf, and δ is the implied market price of risk which reflects the level of market systematic unhedgeable risk. After the transform, the fair price of X, or the expectation of X under $F_X^*(x)$ should be the discounted expected value using the transformed distribution.

We can derive the implied market price of risk δ based on the actively traded mortality linked securities on the market whose fair price is already known, and then apply the same δ to price the unknown mortality linked securities. In the previous research, annuity (Cox and Lin, 2006) or the mortality bond (Chen and Cox, 2009), is applied as the known traded price. In this paper, we use Swiss Re mortality catastrophe bond as the known price to calculate δ, then to price the q-forward as an implementation example of our DEJD model. The set of implied market price of risk $\delta = \{\delta_1, \delta_2, \delta_3\}$ is in correspondance to the Brownian motion, the positve jump scale and the negative jump scale, $\{\alpha^*, \eta_1^*, \eta_2^*\}$. Since the mortality linked securities are priced in the incomplete market, the value of δ or the risk-neutral measure Q is not unique. We have only one mortality linked security but need to calculate four market prices of risk. Following (Cairns, Blake, and Dowd, 2006b), we can estimate set of δ by changing one and fixing the rest.

Following is the traditional procedure for calculating the market price of risk (Chen and Cox, 2009):

Step 1. Based on the known 2003 mortality indicator, simulate 10,000 times the future mortality indicator $k(t)$ for 2004-2006, using the DEJD model (7) with the calibrated parameter set $\{\lambda, p; \eta_1, \eta_2; \alpha, \sigma\}$={0.064, 0.45; 0.71, 0.75; -0.20, 0.31}, and the initial assumed set $\delta = \{\delta_1, \delta_{2,} \delta_3\}$, with Wang Transform function.

Step 2. Calculate the mortality rate $\mu_{x,t}$ by the formula (4) and calculate the average μ_t based on year 2000 standard population and corresponding weights.

Step 3. Calculate expected value of the principal payment in every period T by the formula, $E_T^*[payment] = \$400{,}000{,}000 \times [\max(1 - \sum_{t=2004}^{2006} L_t, 0)]$, L_t follows (15). The coupon payment in every period is calculated based on the par spread plus 1.35% risk premium.

Step 4. Adjust the market price of risk set δ, and repeat step.1-step.3 until the discounted expected value of the coupon payment in 2004-2006 plus the principal payment in 2006 equal the face value of the mortality bond $400,000,000.

Table 8 Implied Market Prices of Risk by Wang Transform

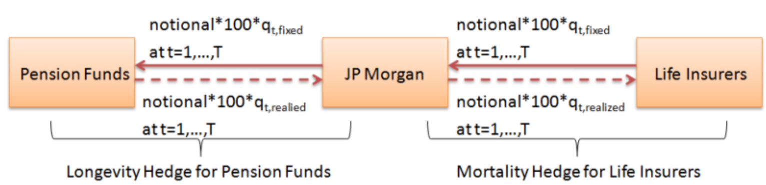
$\boldsymbol{\delta_1}$	$\boldsymbol{\delta_2}$	$\boldsymbol{\delta_3}$
6.15	0	0
0	1.82	0
0	1.79	1.79

The value of ζ_1 and δ_1 are the same for the risk neutral approach and the wang transform, which are correspondance to the drift of the Brownian Motion. The value of (ζ_2, δ_2) (ζ_3, δ_3) are different since the approaches for the implied market price of risk are different.

3.5 Example: q-forward Pricing

JP Morgan proposed q-forwards derivative contracts as simple capital markets instrument for transferring longevity and mortality risk. q-forwards enable pension funds to hedge against increasing life expectancy of plan members and life insurers to protect themselves against significant increases in the mortality of policyholders. Similar in structure to other forwards, q-forwards are securities involving the exchange of the realized mortality of a population at some future date, in return for a fixed mortality rate agreed at inception. Theses q-forwards can provide the basic building blocks from which many other complex mortality/longevity risk securities can be constructed. A q-forward provides a type of standardized contract which could help to create a liquid longevity risk capital market. A set of q-forwards that settle based on the LifeMetrics Mortality Index (JP Morgan, 2007) could fulfill this role. Since the counterparty who is not exposed to the longevity risk requires a risk premium to take on longevity risk, the mortality forward rates at which q-forwards transact will be below the expected, or "best estimate" mortality rates The standard actuarial notation uses q for mortality rate, and this is how q-forward derivative is named. In our notation, q is denoted by μ.

Figure 10 Longevity Risk Hedge and Mortality Risk Hedge



A q-forward type contract that can be used by a life insurer to hedge mortality risk occurs when the life insurer pays the fixed mortality rate to the investment bank and the investment bank pays the realized mortality rate to the life insurer. The life insurer receives the payment $(Notional\ Amount) \times (q_{realized} - q_{fixed}) \times 100$, as shown in Table 9. When the realized mortality rate increases, the investment bank pays more (the realized rate minus the fixed rate) to the life insurer which offsets the loss incurred by the life insurer due to the experienced increase in mortality. A q-forward type contract can also be used by a pension fund or by Social Security to hedge longevity risk. The pension fund pays the realized mortality rate to the investment bank and the investment bank pays the fixed rate to the pension fund. The pension fund receives the payment $(Notional\ Amount) \times \left(q_{fixed} - q_{realized}\right) \times 100$. When the mortality decreases (longevity increases), the investment bank pays more (fixed rate minus realized rate) to the pension fund which can then use these funds to cover the loss incurred by the pension fund due to the experienced increase in longevity. In this way, pension funds that are long on longevity risk can transfer the risk to investors who are willing to take on this extra risk for the increased return they receive. Similarly, life insurers who are long on mortality risk can transfer the risk to investors who want to short the mortality risk[3]. Of course the pension fund and the life insurer are natural counterparties, and the investment bank can serve as a financial intermediary facilitating their mutual hedging as in Figure 10.

[3] Of course investors can take either side of the transaction, they do not need to go long or short as described above, only take the side they view as advantageous.

Table 9 Example of q-Forward Structure

Notional Amount	**$ 50,000,000**
Trade Date	31 Dec. 2006
Effective Date	31 Dec. 2006
Maturity Date	31 Dec. 2016
Reference Year	2015
Fixed Rate q_{fixed}	0.8765% (or 87.65 basis points)
Fixed Rate Payer	XYZ Investment Bank
Fixed Amount	Notional Amount $\times$ Fixed Rate $\times$ 100
Reference Rate	LifeMetrics Index
Floating Amount Payer	XYZ Pension or Annuity Provider
Floating Amount	Notional Amount $\times$ Reference Rate $\times$ 100
Settlement	Net settlement = Fixed amount - Floating amount

Based on our DEJD model and the q-forwards product structure above, the fixed rate can be calculated with the closed-form formula (21) directly.

$$E^*[\mu_t] = \sum_x W_x \times \{\exp(a_x) \times E^*[\exp(b_x k_t)]\}$$

$$= \sum_x W_x \times \{\exp\,(a_x + b_x k_0 + b_x t\left(\alpha^* - \frac{1}{2}\sigma^{*2} - \lambda^*\gamma^*\right)$$

$$+\frac{1}{2}b_x^2\sigma^{*2}t + \lambda^* t(\frac{p^*\eta_1^*}{\eta_1^* - 1} + \frac{q^*\eta_2^*}{\eta_2^* + 1} - 1))\}$$

In our context, the Fixed Rate q_{fixed} is represented as $E^*[\mu_{x,t}]$, the expected future mortality rate in the risk neutral measure with W_x being a weight associated with the age category. These weights are given in Table 10.

Table 10 Parameters for the Closed-Form Solution of the q-Forward[4]

Age-specific Parameters	W_x	a_x	b_x	Other Parameters	Value
<1	0.013818	-3.4087	0.1455	k_0	-10.302
1-4	0.055317	-6.2254	0.1960	t	10
5-14	0.145565	-7.1976	0.1492	α^*	-0.20
15-24	0.138646	-6.2957	0.0994	σ^*	0.31
25-34	0.135573	-5.9923	0.1044	λ^*	0.029
35-44	0.162613	-5.4819	0.0855	γ^*	-1.25
45-54	0.134834	-4.7799	0.0608	p^*	0.035
55-64	0.087247	-4.0137	0.0468	${\eta_1}^*$	0.89
65-74	0.066037	-3.2347	0.0426	q^*	0.065
75-84	0.044842	-2.4196	0.0409	${\eta_2}^*$	0.93
>85	0.015508	-1.6119	0.0290		

The fixed rate or equivalently the mortality forward rate quoted by an investment bank would be formed using a combination of (i). best estimate mortality projection, (ii). a risk premium, and (iii). mid-to-bid spread (half of the ask-bid spread).

The best estimate of mortality will depend on the model used, e.g., the Lee-Carter model, and so may be biased if the model is not the most appropriate one. The risk premium must also be appropriate for the market and based on transactions there. Unfortunately to date, there are very few transactions involving mortality based derivatives and so the risk premium calculation in this market is currently problematic.

The calculation presented here uses the well known risk neutral valuation approach with adjustments for mortality and longevity jumps. The jump processes play a role in fitting the data and in estimating the risk premium. The mortality forward rate

[4] We apply the risk-neutral measure change on the positive jump severity ${\eta_1}^*$ and negative jump severity ${\eta_2}^*$.

flows from the closed-form solution. For the U.S. data used here, the fixed rate 10 year q-forward contract is priced at 0.8765% (or 87.65 basis points). This DEJD pricing may differ from that of the Lee-Carter or Chen-Cox models.

The Lee-Carter model, the Chen-Cox model and our DEJD model will naturally yield different values for the best estimate mortality projection. Not incorporating the asymmetric jump effect will make a difference in the calculated expected mortality rate that enters into the determination of the fixed rate payment component of the swap in Figure 10. Using the same data, whole age groups, and both sexes in the reference year for the U.S. national population, the result for the best estimate mortality projection for q-forward fixed rate is 0.8583% using the Lee-Carter model, 0.8594% using the Chen-Cox model, and 0.8765% using the DEJD model. Thus, the DEJD model will yield a larger expected fixed payment affecting the size of the experienced swap.

The Lee-Carter model captures the significant positive jump (1918 flu) and moderate negative jump in the baseline Brownian motion process, which caused a larger standard deviation and a positively biased mean for the baseline Brownian motion process. If we set the Lee-Carter model as the benchmark and compare the Chen-Cox model and the DEJD models, we conclude the following.

Compared to the Lee-Carter model, the Chen-Cox model considers the significant positive jump and moderate negative jump in the normal jump process, not in the baseline Brownian motion process. This reduces the standard deviation and the mean in the baseline Brownian motion process. However, in the projection of future mortality rates, the jumps are assumed to occur symmetrically in the jump diffusion model of the

Chen-Cox model, so the overall jump effects offset each other (positive and negative). The jump diffusion doesn't affect the mean of the Brownian motion process by very much, so the pricing of the fixed rate is not much different from that obtained with the Lee-Carter model.

Compared to the Lee-Carter model, our DEJD model considers the possibility of both positive jumps and negative jumps in the jump diffusion process, not in the baseline Brownian motion process. This also reduces the standard deviation and the mean in comparison with the baseline Brownian motion process. The difference of the DEJD model from the Chen-Cox model is that, in the projection of the future mortality rate, the jumps in the DEJD model are assumed to occur asymmetrically following the historical rule of infrequent significant positive jumps and more frequent moderate negative jumps. In this way, the overall jump effect adds a positive increment to the mean of Brownian motion process, which causes the pricing of the fixed rate to be higher than that obtained with the Lee-Carter model.

Some final comments about the q-forward prices developed in this paper are in order. A pricing difference in the q-forward also can arise from the use of a different implied market price of risk. Due to the lack of a sufficient number of mortality securities in the market, the implied market price of risk used in this article in conjunction with our model was estimated using only one product, i.e., the Swiss Re mortality catastrophe bond. Applying a different estimated implied market price of risk in the formula will cause a difference in the quoted q-forward fixed rate. Also, in realistic deals, the mid-bid spread (defined as the half of the bid-ask spread), and other factors including judgment calls can also cause a difference in the quoted fixed rate of q-forward.

3.6 CONCLUSION

This paper proposed a quantitative model to price mortality-linked securities, and provided a possible approach to measuring and managing longevity/mortality risk. Marked improvement in life expectancy has attracted public attention to the financial consequences of longevity risk on pension plans, long term care insurance, and Social Security solvency. Longevity risk can also seriously affect the asset and liability balance of the pension fund and annuity providers. On the other side of the longevity/mortality market, life insurers have begun to show increased concern about increased mortality risk caused, for example, by sudden influenza or other natural or man-made catastrophic sources. A series of mortality linked securities, e.g. longevity bond, mortality bond, and other types of securities have been issued to manage and transfer the risk. Additionally, the recent market involving life settlement securitization whose pricing depends on modeling the life expectancy of insurance policy holders has boomed. Hence, modeling and pricing mortality linked securities is crucial to risk management, product innovation and formation of a liquid intermediate market.

This paper proposed a stochastic mortality model to capture the observed feature of the historical mortality rate process, and used this model to price mortality linked securities. The baseline component of our model incorporates the advantages of the Lee-Carter framework, which describes the main trend and regular dynamics of historical mortality rate and is able to adjust for the cohort effect. The jump diffusion component in our model applies the compound Poisson-double exponential jump diffusion to extend the Lee Carter model so as to describe longevity jumps (negative jumps) and mortality jumps (positive jumps) separately. Our model accommodates the different features of longevity jumps and mortality jumps. Hence the model fits the mortality time-series increment distribution much better than the previous models and explains the distribution skewness effect. In addition, our proposed model has an advantage of mathematical tractability and the ability to obtain a closed-form solution for the standard securities, like the q-forward contract, whose price depends on the expected future mortality rate. Since the DEJD model has the concise specification and closed-form density function, the likelihood function for the parameters can be easily expressed. In this way, the small number of parameters and concise likelihood function facilitates the calibration and application of the model in practice.

We calibrated our model with the historical mortality rate data 1900-2004 from National Center for Health Statistics and we compared the prices of the q-forward fixed rate, calculated by the Lee-Carter model, the Chen-Cox model and our DEJD model. We found that our mortality model fits this data better and that the q-forward fixed rate in the contracted swap used for hedging longevity risk is higher from our model.

As a caveat, however, should note that we use only 104 years of recent data, and the last 104 years may not be the same as the next 104 year (technological innovations are occurring more rapidly now than in the past). Moreover, 104 years may not be adequate for modeling certain low frequency events. Can a 1918 flu epidemic scale event be expected once every 100 years or once every 200 years? Also, the fact that large longevity jumps have not occurred within this data set does not mean they have not occurred in the past (the acceptance of the germ theory of disease prior to this data set, for example) or that they will not occur in the future. Cox, Lin and Pedersen (2010) note that some experts conjecture that we may experience extreme events of both types in the future. As with all actuarial modeling, have to accept that there is a real risk that the coefficients we estimate may prove to be incorrect if the past data is not reflective of the future. Model risk is always an actuarial issue, and our mortality model is no exception. For the creation of capital market financial instruments to hedge longevity risk such as in Figure 10, however, the model risk does not affect the hedger (the pension fund's hedge still works and the life insurer's hedge still works), and the investment bank intermediary is protected as long as they did not take a long or short position. Thus, while model risk is an important concern as far as mortality modeling is concerned, it is of less concern in the dual hedging context of this paper.

Chapter 4 Longevity Risk in Life Settlement Products Pricing

4.1 INTRODUCTION

While the effect of longevity risk is traditionally thought of in terms of its impact on pensions, social security systems and corporate defined benefit plan solvency, there is another market that is vulnerable to longevity risk, perhaps more than the above, namely the life settlement (and securitization) market. A life settlement is a financial arrangement whereby the third party (or investor) purchases a life insurance policy from the person who originally purchased the life insurance policy. This third party pays the insured an amount greater than the cash surrender value of the policy -- in effect, the trade-in value of the policy as determined by the originating insurance company[5] -- but less than the face value (or the death benefit). They do this in exchange for the right to collect the death benefit upon the death of the insured. The investor also agrees to make the future life insurance premium payments until the death of the insured. It can be a win-win situation, as the investor can obtain a return on their initial investment and premium payments once the death benefit becomes payable (assuming the insured does not live too much longer than expected when setting the purchase price) and the owner of the policy obtains more money than they would if they had surrendered the policy for its cash value or allowed it to lapse. The win-win situation is for the contract participants (the insured and the life settlement investor). Other third parties may have losses due to this transaction, however. The insurer, for example, loses the ability to recapture the policy

[5] The cash value of the policy is also known as the non-forfeiture value since this is the least amount the insurer can pay to a surrendering policy holder. Formula for calculating the cash value can be found in Bowers et al (1997), Actuarial Mathematics.

value upon lapse by the policy owner. The losses of recaptured lapse value by the insurer may also cause the insurer to have to raise premiums to future customers if the recaptured lapse value becomes significant due to a significant growth in life settlements (unless the insurer itself enters the life settlement market to capture the otherwise lost profit, or to use a life settlement portfolio as a natural hedge against longevity risk for their own life insurance book of business). This market has grown. According to Annin, DeMars, and Morrow (2010 p. 1); "It is estimated that in the past five years alone, more than $40 billion of the face value has been sold in the life settlement market."

In this market, there is a vulnerability to longevity risk as increased longevity implies longer periods of investors paying premiums prior to collecting their money, and hence the potential for losing money or going bankruptcy. The rise and fall of the viatical settlement market, where the life settlement market arose, illustrates these dangers and susceptibility to increases in longevity.

The practice of buying and selling of "viatical settlements" began in the late 1980s when the AIDS epidemic presented a devastating medical shock to thousands of previously healthy Americans (Stone and Zissu 2006). Due to the extremely high medical costs associated with treatments for this disease and the difficulty for too-ill-to-work HIV positive individuals to sustain an active income, many AIDS patients and their families became financially vulnerable. Thus, a market developed to relieve some of the monetary stress of AIDS victims.

Seen as a new opportunity, investors (predominately entrepreneurs) stepped in and offered to buy AIDS patients' life insurance policies for a price less than their face value. The investors would take over the premium payments---also a burden the terminally ill found too heavy to carry with deteriorating health---and become the beneficiary of the policy when a certain "waiting period" had passed. Then, once the insured died, the investor would obtain the proceeds of the life insurance policy. Since these patients were given very little time left to live (typically two to three years), the investor would not have to pay many premiums, and after subtracting the initial payment to the insured and subsequent premiums from the final payout of the life insurance policy, the investor would theoretically end up with a large profit (Quinn 2008).

As the success of viatical settlement investments went public, the market for such investments grew and companies were created that specialized in fulfilling investors' desires for viatical settlements. It was not much later, however, that this new market collapsed due to a change in the longevity risk. At the 1996 International AIDS Conference in Vancouver, papers were presented that gave evidence of a new drug capable of substantially reducing, perhaps even to zero, the level of HIV in its infectees. This research had a twofold impact: On one hand, it offered new hope for increased life to the AIDS infected community but, on the other hand, this sudden increase in longevity pronounced a death sentence to firms that had been surviving off profits from the sale of viatical settlements. The second effect is evident in the collapsed value of the viatical settlement firm, Dignity Partner, and the significant decrease in prices being offered to AIDS victims for their insurance policies. As evidence grew that policies might take a substantially longer time to mature, their value plummeted (Stone and Zissu 2006).

As the viatical settlements market collapsed, investment companies, in order to keep the life settlement backed securities market alive, expanded their life insurance purchases to those belonging to the elderly. Companies selected elderly people with estimated low life expectancies because a low life expectancy meant a greater possibility of profiting sooner from the purchase of life insurance policies. Today, this life settlement market has a growing potential as the baby boomers are now entering old age. As the population ages, funding retirement over the last few years of life becomes an escalating concern. Life settlements for senior citizens have become popular, partly due to the extensive marketing pursued by life settlement companies. The senior market now comprises about 80% of the entire viatical and life settlements industry (ViaticalWeb 2011). Moreover this market may continue to grow. Due to gradual increases in technology and beneficial medical treatment in the United States, the number of centenarians (individuals over the age of 100) has increased from 15,000 in 1980 to roughly 72,000 in 2000 and the number is predicted by the Social Security Advisory Board to reach to 4.2 million, (or approximately 1% of the projected total population) by 2050 (Scotti and Effenberger, 2007).

The life settlement market has developed at a rapid pace in its early years. Two recent surveys estimate that the available market size will grow from $13 billion in 2004 to $161 billion over the next few decades through a combination of population aging, increasing life expectancy and increasing market penetration. Life settlement has attracted a broad range of attention, including dominant investment banks and major reinsurance companies as intermediaries, the Securities and Exchange Commission

(SEC), the National Association of Insurance Commissioners (NAIC) and National Conference of Insurance Legislators (NCOIL), as well as, state regulators and other rating agents and life expectancy underwriters.

However, just as advancements in treating AIDS led the viatical settlement market to succumb to longevity risk, the substantial increases in longevity during the 20th and 21st centuries can pose substantial risks to the current life settlement market. A large longevity jump could occur in the future if an effective treatment of coronary heart disease or cancer is found, as these two causes of death combined constitute more than half of all deaths among people over the age of 40 (Johnson, Bengtson and Coleman 2005 p. 109). Thus, the modeling of longevity risk is of potentially more important in the life settlements market than it is in the pension market because the life settlement market is based (and funded) on shorter horizons.

Very little literature has discussed the issue of determining a pricing model for life settlement portfolios subject to jump discontinuities between mortality and longevity. The main factor in the life settlement securities pricing currently is the estimation of the life expectancy of the insured. Incorporating longevity risk into the pricing structure when evaluating life settlement securities turns out to be more complicated.

In this paper, I propose a Whole Life Time Distribution Dynamic Pricing (WLTDDP) method to evaluate the life settlement products. The method determines the life expectancy and generates a life table for different birth year cohorts with potential jump discontinuities in both mortality and in longevity rates. The method incorporates the updated information on the individual insured’s life expectancy (obtained from an expert medical underwriter), which is a critical factor that enters into the evaluation of the life settlement products.

In past life settlement transactions, the life expectancy of the insured was considered as the most critical (often the only) variable used in determining the sales price of the policy as this represents the expected life length of the insured when he or she sells the life insurance policy to the third party as a life settlement (the time to payment for the investor). Essentially, if T represents the (random) future life of the insured, then the life expectancy is $\mu = E[T]$. This life expectancy is computed using an appropriate life table, or in the case of life settlements, is usually given by a medical expert based on their examination of the current medical record of the insured. If the discount factor is $v = 1/(1+r)$ where r denotes the required rate of return, then the current value of the pay off of a life insurance policy with a benefit of $B is traditionally calculated as Bv^{μ}. However, this is a systematically biased assessment of the value of the payoff and leads to a systematically inaccurate evaluation of the value of the life settlement product.

According to Jensen's inequality, if f is any convex function, and X is any random variable, then $E[f(X)] \geq f(E[X])$. In the current situation we $f(X) = v^x = e^{-\delta_x}$)) with $\delta > 0,$ to observe $f'' > 0$, so f is convex. According to Jensen's inequality, if $f(X)$ is convex, and $X = T$ denotes the life length of the insured, then $E[v^T] \geq v^{E[T]}$ which means the expected value of the discounted benefit is always greater than the benefit discounted by the expected time to death. Thus the traditional evaluation of the present value of the life insurance payoff using the life expectation alone as in $Bv^{E[T]}$

systematically underestimates the true expected payoff of $E[Bv^T]$. This results in a negatively biased price to the insured policyholder. Note that this bias may be intentionally used as a mechanism to increase profitability by the purchaser or to hedge against adverse selection or longevity risk. While this bias in always present regardless of the probability distribution used for T, the appropriate distribution is also necessary for a correct assessment of the actual expected net present value of the benefit payment in the life settlement, namely $E[Bv^T]$.

In the past, life expectancy was considered in the pricing as a solo variable representing the expected life time of the insured when he sells his life insurance policy to the third party as a life settlement. However, this is an inaccurate approach to evaluate the life settlement product. According to Jensen's inequality, the value of life settlement products are contingent on the expected value of the functions of life time, which is always larger than the value of the function of the expected life time (or life expectancy). So, the previous pricing method based on life expectancy has a negative bias for pricing the life settlement products. In order to solve this problem, we need to generate the whole distribution for the life time.

The advantage of Whole Life Time Distribution Dynamic Pricing (WLTDDP) method is that it generates a complete life table with the whole distribution of life time instead of the expected life time (life expectancy). In this way, the method provides a more accurate projection and evaluation for the life settlement products, through incorporating more statistical information of the insured's future life time. The statistical methodology is based on information theory for adjusting mortality tables to obtain exactly some known individual characteristics while obtaining a table that is as close as possible to a standard one.

Another advantage of WLTDDP is that it incorporates the effect of the dynamic longevity risk through the original life table which is generated from the Double Exponential Jump Diffusion model (DEJD) (Deng, Brockett and MacMinn 2010). The DEJD model incorporates the longevity jump (caused by medical improvement, etc), mortality jump (caused by pandemic influenza, etc) and dynamic main trend of the mortality rate which provide a better explanation and fit to the historical mortality rate data. The longevity jump or mortality jump can seriously distort the features of the mortality rate trend and affect the evaluation of the life settlement products which is contingent on the mortality rate and life expectancy. This model is an extension of the Lee-Carter Mortality model which differentiates age and cohort effects. It incorporates a Brownian motion process for smooth mortality changes plus an asymmetric jump diffusion process allowing for jumps up or down in the mortality rate in the critical temporal mortality process. This model is used for the estimation and forecasting of mortality rates and life expectancy and ultimately the pricing of the individual life settlement. The life table projected by the DEJD model incorporates the features of the historical mortality trend and the longevity risk and it provides a more accurate base for pricing life settlement products.

The life settlements literature does not allow jumps in longevity (such as those that destroyed the viatical settlements market), so this paper will be the first to allow for such jumps while still having uncertainty in the possibility of such jumps occurring.

Additionally, jumps in mortality (as opposed to longevity) may also occur (such as an infectious disease which differentially impacts vulnerable elderly populations) and this is allowed when using the DEJD model for pricing. Increased mortality would cause an increase in the value of the life settlement to the investor. Currently, jump changes that increase or decrease the expected mortality rate are not incorporated in previous models. Jumps are important sources of return uncertainty in life settlement investments, which is the reason they are used in my model. Finally, as population longevity increases, especially among the very old, the usefulness to individuals of using life settlements to obtain additional money in their old age could attract more individuals and proper pricing using the DEJD model may have a social benefit, as well as, the more rational pricing developed here.

4.2 Description of the Life Settlement Market

The life settlement market developed rapidly in its early years. The face amount of life insurance settled was estimated at $10 billion at 2005, and this continued to grow to $12 billion in 2007. Similar to other financial product markets, the life settlement market experienced a contraction during 2008 and the face amount was estimated at $11.7 billion in 2008. (Conning Research, Oct. 8, 2008, Life settlements: New challenges to growth.) Two recent surveys have estimated that the available market size will grow from $13 billion in 2004 to $161 billion over the next few decades, through a combination of population aging, increasing life expectancy and increasing market penetration of life settlement options available to policy holders (Bernstein Research 2005, 2006). Life settlement as an investment asset class has attracted a broad range of attention, including dominant investment banks and major reinsurance companies as intermediaries, the Securities and Exchange Commission (SEC), National Association of Insurance Commissioners (NAIC) and National Conference of Insurance Legislators (NCOIL) as regulators, and other rating agents and life expectancy underwriters as participant information suppliers.

Before the life settlement market emerged, if a policy owner no longer wanted, needed, or could not afford to pay the premiums for a life insurance policy they had limited choices. They could cash out a policy by surrendering the policy to the insurance company to receive the surrender value or they could simply stop making premium payments and allow the policy to lapse. In most cases, the policy would be worth considerably more than the surrender value making it an unattractive option. The surrender value is based on the commissioner's standard ordinary (CSO) mortality tables, in force at the time the policy was issued and usually many years prior to the decision to surrender. These are smooth mortality tables used for conservative non-forfeiture value calculations and do not anticipate health changes in individuals but only in the aggregate group and are the basis for the table construction. Later on, after health changes, another mortality table may more accurately reflect the anticipated mortality probabilities of an individual insured. The cash value calculation is incorporated as part of the insurance contract and is not negotiable. Lapsing the policy forfeits (or slowly runs out) the cash value in most cases. Under either choice scenario, the extra value in an unwanted or unneeded policy was relinquished back to the life insurance company who issued the policy and not captured by the insured.

A life settlement provides a secondary financial market for this contingent claims contract producing an alternative option to the policy holder other than the surrender or lapse of a policy. In this way, the policy holder can gain the extra value inherent in the policy instead of giving it back to the insurer. When the owner of a life insurance policy no longer needs or wants the policy, the policy is underperforming or the he can no longer afford to pay the premiums, he should have the right to resell the policy to the third party for the highest payment.

Several market intermediaries play a role in the accomplishment of the life settlement, including insured individuals (or policy owners), producers (financial advisors or insurance agents), settlement brokers (insurance agents), life expectancy underwriters (who evaluate the life expectancy of the underlying insured life at the time of sale), providers (parties acquiring the policy and paying the insured for the right to claim the life insurance benefits), and investors (who either bundle collections of life settlements and securitize them for resale, or keep them for investment purposes as a new asset class in their own portfolio). The majority of investors in today's life settlement market are large institutional investors seeking to acquire large pools of policies. Retail investors also participate in the life settlement market, generally by purchasing fractional interests in settled policies. To the investor, the life settlement portfolio provides an essentially zero-beta asset which can help diversify a larger portfolio of financial market sensitive assets6.

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[6] It can also be used as a zero beta asset for valuation of portfolios in a Black type Capital Asset Portfolio Model instead of the Market portfolio which has well known identifiably problems since Roll's criticism of the CAPM. (Black, et al., 1972),(Roll, R., 1977).

Most insured policy holders participating in the life settlement market are seniors with a life expectancy of more than two years. The process or procedures involved in the life settlement transaction transpire as follows:

1. Insured individuals or policy owners initiate the process to contact a producer (usually financial advisors or insurance agents). Sometimes the producer contacts the insured because they know the insured has a need for the sale of their life insurance policy.

2. The producer contacts one (or usually more than one) life settlement broker with a license to do business in life settlements in the policy holders' state of residence (insurance is a regulated industry).

3. The settlement broker(s) collects the medical information concerning the current health status of the policy holder and "settles" the policy by contacting life expectancy underwriters.

4. The contacted life expectancy underwriters are responsible for preparing a life expectancy assessment and evaluating the mortality risk of the insured based on the current health information provided by the settlement broker.

5. Providers review the data on policy terms, life expectancy, premium amounts, and bid on the amount they would be willing to pay the insured policy owner to take over premium payments in return for collecting the ultimate life insurance benefit upon the death of the insured. This bid is based on supplied information and settlement applications prepared by settlement brokers.

6. The existing insured elects to either hold (not sell on the secondary market) or to sell their policy. This can then be held in a portfolio or resold into a life settlement

securitization issue which expands the asset class to the broader class of investors with interests in life settlements.

Pricing of the life settlement securities depends on the estimation and forecast of mortality rates or life expectancy, which are considerations involving mortality risk and longevity risk. In this paper, I apply the DEJD stochastic mortality model (Deng, et al, 2010) to model mortality underlying the life settlement securities. The model is based on the Brownian motion process, plus an asymmetric jump diffusion process for the estimation and forecasting of mortality rates and life expectancy.

4.3 The Stochastic Mortality Model Specification

To capture the features of the mortality time-series k_t and to account for the tractability and the calibration of the model, we set the model specification to describe Δk_t in the approximate continuous-time model of dk_t as given below.

The dynamics of the mortality time-series k_t is specified as:

$$dk_t = adt + \sigma dW_t + d(\sum_{i=1}^{N(t)}(V_i - 1))$$

where W_t is a standard Brownian motion, $N(t)$ is a Poisson process with rate λ, where λ describes the expected frequency of the jumps. The larger the λ, the more times jumps occur in the mortality time-series. Here V_i is a sequence of independent identically distributed (iid) nonnegative random variables, $Y = \log(V)$ has a double exponential distribution with the density:

$$f_Y(y) = p\eta_1 e^{-\eta_1 y} 1_{\{y \geq 0\}} + q\eta_2 e^{\eta_2 y} 1_{\{y<0\}}$$

$$\eta_1, \eta_2 > 0,\ \ p, q \geq 0,\ \ p + q = 1.$$

The parameters p and q represent respectively, the proportion of positive jumps and negative jumps among all jumps. Thus, $p\lambda$ is the expected frequency of positive jumps and $q\lambda$ is the expected frequency of negative jumps. The parameters η_1 and η_2 describe the positive jump size or severity and the negative jump size or severity respectively. Thus, $\mathrm{Y}|\mathrm{Y} > 0$ is exponentially distributed with mean ${\eta_1}^{-1}$, while $-\mathrm{Y}|\mathrm{Y} \leqslant 0$ is exponentially distributed with mean ${\eta_2}^{-1}$. The larger η_1, the smaller the positive jump severity. Similarly, the larger η_2 the smaller the negative jump in absolute value. In this way, the positive jumps and negative jumps are captured by similar distributions but with different parameters based on the asymmetry of jumps in the mortality time-series k_t and the leptokurtic feature of dk_t.

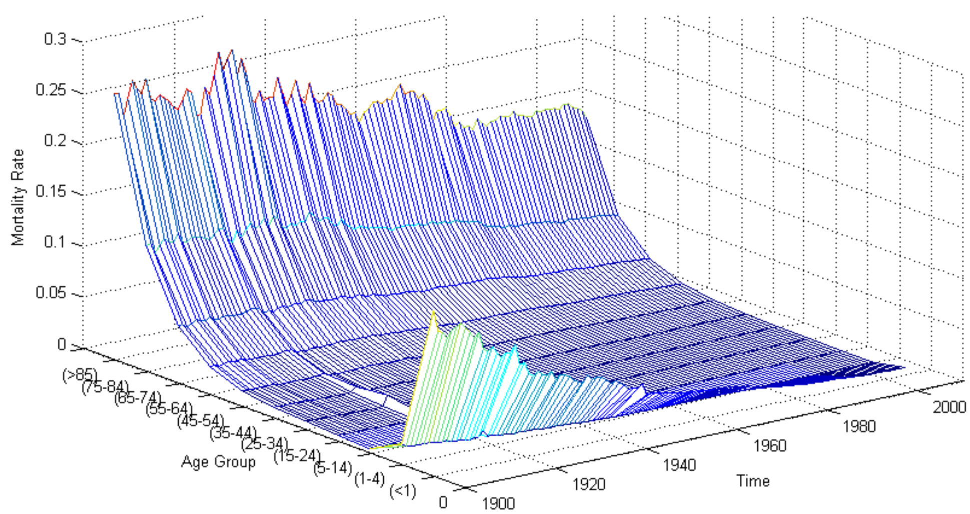
The model specification with the double-exponential distribution has the advantage of mathematical tractability allowing a closed-form formula for the expected future mortality rate to be derived. Because of this closed form solution, the DEJD model may provide a useful stochastic mortality model for internal company mortality simulation, as well as, being useful in the capital market applications I discuss subsequently. The double-exponential distribution also has been widely implemented as a stock price jump-diffusion model, for which closed-form solutions for options and other securities are available (Kou, 2004) (Deng, Brockett and MacMinn 2010). In these papers, the closed-form solution for the expected future mortality rate is presented in the equation below:

$$E^*\left[\mu_{x,t}\right] = \exp(a_x) \times E^*[\exp(b_x k_t)]$$

$$= \exp\left(a_x + b_x k_0 + b_x t\left(\alpha^* - \frac{1}{2}\sigma^{*2} - \lambda^*\gamma^*\right) + \frac{1}{2}{b_x}^2\sigma^{*2}t + \lambda^* t\left(\frac{p^*\eta_1^*}{\eta_1^* - b_x} + \frac{q^*\eta_2^*}{\eta_2^* + b_x} - 1\right)\right)$$

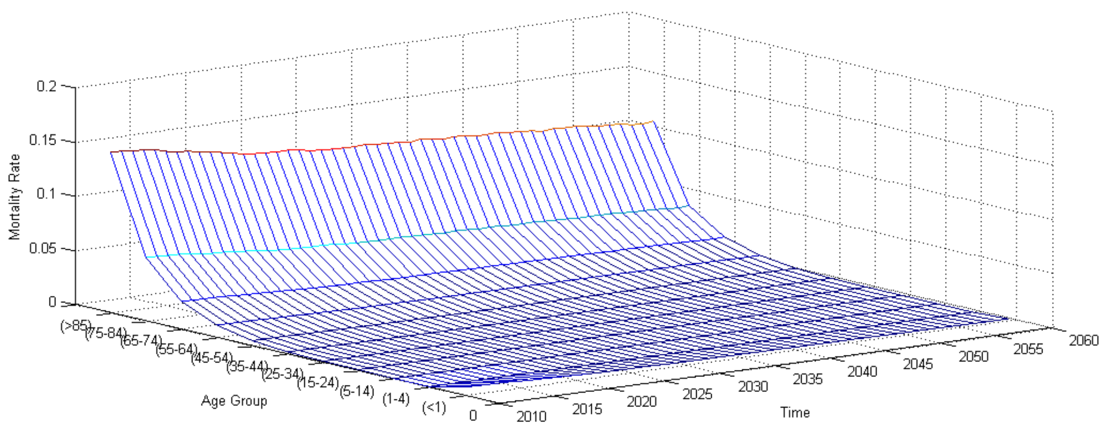
We can observe clearly the two properties of the mortality rate trend from Figure 11. First, in Figure 11, the downward trend indicates that the mortality rate follows a decreasing trend during 1900-2004 at all ages. For example, in the over 85 age group, the mortality rate decreases from 0.26 to 0.14, while in the younger age groups, such as 15-24, the mortality rate decreases from 0.006 to 0.0008. The decreasing trend shows the improvement of the life times or longevity in all age groups. Second, the change in the mortality rate in the older-age groups is more significant with a steeper downward trend than that in the younger-age groups.

Figure 11 1900-2004 Historical Mortality Rates



In Figure 12, we apply the DEJD model to generate the mortality rate for all the age groups from 2010 to 2060 with the following equation and the parameters in Table 10.

Figure 12 2010-2060 Projected Mortality Rates



The projected mortality rate also presents the downward trend which indicates that the mortality rate follows a decreasing trend during 2010-2060, at all ages. The trend keeps the feature of longevity in all age groups. Similar to the historical trend, the improvement of the mortality rate in the older-age groups is more significant with a steeper downward trend than that in the younger age groups.

A life settlement usually depends on the following characteristics of the policy being settled:

1. The insurance carrier

2. The face value (or benefit)

3. The age of the insured person

4. The gender of the insured person

5. The premium

6. The issue date

7. The estimated life expectancy of the insured

8. The primary diagnosis of the insured's illnesses if the insured is in impaired health

9. The bidding or asking price of the policyholder.

Among the attributes that affect the price of the life settlement, life expectancy is the key factor which is difficult to forecast and to measure. Valuation of the life insurance is initially based on a stable life table selected at the time of policy issue, usually a version of the Commission Standard Ordinary Mortality Table (CSO). These tables do not accommodate the improvement trends and dynamics of the mortality rate. That is, they are smoothed (graduated) to produce aggregate fits to data and smoothed progressions of premiums from age to age. The overestimation of the mortality rate and the underestimation of the life expectancy in stable historical life tables will cause the underestimation of the life insurance price at future points, and will have limited permissible variability across individuals due to jumps. To correctly estimate the price of the life insurance policies at future points in time, to effect a life settlement, I adopt the Double Exponential Jump Diffusion model in Deng, Brockett and MacMinn (2010) to project the future mortality rate and the life expectancy. Given the life expectancy, we can estimate the price of the life settlement incorporating both expected and unexpected mortality changes.

To explain the pricing method we use, an illustrative life insurance contract example is set up and a life settlement contract is derived from it. The illustrative contract is State Farm's whole life policy as presented in Baranoff, Brockett and Kahane (2009).

Assume a female purchased a whole life level premium insurance contract at the age of 50 in 1986. The schedule of benefits and the schedule of the premiums are shown in Table 11 and Table 12. The face value (or the benefit payable at death) is $50,000. The annual premium is $565.50. Assume that a third party wants to acquire the life insurance policy when the person is 70 in 2006. In the following, we provide a model to price the life settlement when there is information from a doctor or an underwriting expert concerning the life expectancy of the individual at age 70.

While the CSO table was used to calculate the cash value and reserves for the policy, the most appropriate mortality table for life settlement calculations may no longer be this table due to various changes in longevity and the current health state of the individual.

4.4 Application of Information Theory

Based on information theory, I will present a statistical methodology for adjusting mortality tables by incorporating known individual characteristics, and the adjusted table is as close as possible to the original one (Brockett and Cox, 1984).

The use of known information about an individual to adjust a standard mortality table to reflect the individual's underwriting characteristics is a common problem in actuarial science concerns. The actuary can price life contingent financial instruments more accurately with the adjusted table. I present a statistical approach to mortality table adjustment that simultaneously adjusts survival probabilities at all ages in a consistent, logical manner. I obtain a life table that includes the characteristics of the standard table and the add-in information for adjustment. One can start with the life table and then systematically adjust for the particular individual characteristics that reflect expected life length or a 50 percent confidence interval on the life length.

The problem is summarized as follows: In testifying as an expert witness about the distribution of life time and mortality rate, an actuary is asked to adjust a standard mortality table to obtain a table appropriate for a particular individual given the updated information for the life expectancy. As an expert witness on the same side as the actuary, a physician testifies, that the expected remaining life of the decedent at the date of an untimely death was m years. The actuary needs to value a life settlement based on the distribution of the life time which depends on the life expectancy testified by the expert witness (medical report). In order to do so in a manner consistent with the physician's testimony, the actuary must construct a mortality table which has $E[x] = m$, where x is the years remaining of the decedent under normal circumstances. If the standard table satisfies this condition, then there is no problem. However, since this is usually not the case, I suppose that for the standard table, $E[x] \neq m$.

Here we show explicitly how to obtain an adjusted table that is as indistinguishable as possible from the standard table and that satisfies the physician's constraint $E[x] = m$.

The method I use is based on the principle of minimum discrimination information explained in the next section. The prototype problem of life table adjustment is carried to a numerical conclusion. Since our prototype example is discrete in character, I phrase all the formulas for the discrete case.

4.4.1 Minimum Discrimination Information Estimation

Consider the problem of distinguishing between two probability densities f and g after observing the value t of the random variable under study. In the application considered here, f and g will correspond to potential densities for the survival time of the individual. The technique presented here, however, is applicable to other problems of interest to the actuary (Brockett and Cox, 1984), (Brockett, 1991), (Brockett and Song, 1995), (Brockett, Golden and Zimmer, 1990), (Brockett, Cooper, Learner and Phillips, 1995).

For distinguishing between two densities f and g, the statistic $ln[f(t)/g(t)]$ is a sufficient statistic and represents the log odds ratio in favor of the observation having come from f. It can be thought of as the amount of information contained in the particular observation t for discriminating in favor of f over g (Kullback, 1959). In a long sequence of observations from f, the long-run average log odds ratio is:

$$E_t = \left(ln\frac{f(t)}{g(t)}\right) = \sum_l f(t_l)\, ln\frac{f(t_l)}{g(t_l)}, \qquad (23)$$

The equation reflects the expected amount of information in an observation for discriminating between f and g. In the statistics and engineering literature this quantity is called the *divergence* between the densities f and g and is denoted by $\mathrm{I}(f|g)$. It is not difficult to show that $\mathrm{I}(f|g) \geq 0$, with $\mathrm{I}(f|g) = 0$ if and only if $f = g$. Thus, the size of $\mathrm{I}(f|g)$ is a measure of the closeness of the densities f and g. Such a global measure of loss of densities will be very useful for adjusting mortality tables.

Suppose that we are given a density function g, and we wish to find another density f that is as close as possible to g, and that satisfies certain moment constraints, such as:

$$
\begin{aligned}
1 = \theta_0 &= \sum f_l, \\
\theta_1 &= \sum a_1(t_l) f_l, \\
&\cdots \\
&\cdots \qquad\qquad (24) \\
&\cdots \\
\theta_k &= \sum a_k(t_l) f_l,
\end{aligned}
$$

For example, if $a_1(t) = t$, then the first constraint says that the mean for f is known to be θ_1. Similarly, by taking $a_1(t)$ to be unity on a certain interval and zero off the interval, we arrive at a constraint on the probability for that interval. This would be useful, for example, if one wanted to use a medical study that gives decennial survival probabilities, however, yearly (or more frequent) survival probabilities are required. One would then find a survival density that was as close as possible to a standard mortality table and that reflected the decennial survival rates quoted by the medical study.

To phrase the problem mathematically, I desire to find a vector of probabilities $f = (f_1, f_2, \ldots)$ that solves the problem:

$$\min \mathrm{I}(f|g) \qquad (25)$$

subject to the constraints (24). Here $g = (g_1, g_2, \ldots)$ is the vector of probabilities corresponding to the standard probability distribution. Brockett, Charnes and Cooper (1980) show that the problem (25) has a unique solution, which is:

$$f_l = g_l \exp\left[-(\beta_0 + 1) - \beta_1 a_1(t_l) - \cdots - \beta_k a_k(t_l)\right], \qquad (26)$$

where the β_i's are constant parameters selected in such a way that the constraints (24) are all satisfied. They further show that the parameters β_i can be obtained easily as the dual variables in an unconstrained convex programming problem:

$$\min_\beta \sum \{g_i \exp[-(\beta_0 + 1) - \beta_1 a_1(t_l) - \cdots - \beta_k a_k(t_l)] - (\beta_0 + \theta_1\beta_1 + \cdots + \theta_k\beta_k)\} \qquad (27)$$

The solution to (27) can be obtained easily by any number of efficient nonlinear programming codes. In the following section I use the Newton-Raphson technique.

4.4.2 Information Theoretic Life Table Adjustments

The study of life contingencies is intrinsically a study of biostatistics. For example, the life expectation is the expected value of a random variable K that equals the integral number of years a person now aged x will live. We have $K = 0$ with probability q_x, $K = 1$ with probability $p_x q_{x+1}$, etc.

According to the standard mortality table, the distribution of the random variable K is given by the $(\omega - x + 1)$ dimensional probability vector $g = (g_0, g_1, \ldots, g_{\omega-x})$, where $g_k = {}_k p_x q_{x+k}$ for $k = 0, 1, \ldots, \omega - x - 1$.

Consider now the problem of finding the mortality table that is as close as possible to the standard table and that satisfies certain given constraints. This translates into finding a probability distribution $f = (f_0, f_1, \ldots, f_{\omega-x})$ for the random variable K that satisifies the desired constraints. If, for example, the desired constraints involve the expectation of functions such as those given above, then the density (26) is the least distinguishable density from g among the class of all densities satisfying the constraints.

The physician has testified that the expectation of life for the decedent is m more years. Thus, the constraint set is:

$$1 = \sum f_l, \qquad m = \sum k f_k \tag{28}$$

(all sums are over {0, 1, 2 . . . . . $\omega - x$ }). Appealing to the principle of minimum discrimination information, we select the density f to satisfy:

$$\min \mathrm{I}(f|g) = \min \sum f_l \ln (f_l|g_l)$$

subject to the constraints (28).

We could now of course apply the result (26) directly; however, it is perhaps more instructive to show how to obtain the desired density directly by standard methods in this simple situation. Let $n = \omega - x$. The probability distributions that we are considering can be viewed as $n+1$ vectors $f = (f_0,\ f_1, . . . \ , f_n)$ that satisfy $f_k \geq 0,\ \sum f_k = 1$, and $\sum k f_k = m$. Letting β_0 and β_1 denote the Lagrangian multipliers for the equality constraints (28) allows us to replace the original problem and minimize the function:

$$\mathrm{L}(f, \beta) = \sum f_k\, ln(f_k|g_k) - \beta_0 \left(1 - \sum f_k\right) - \beta_1 (m - \sum k f_k)$$

subject to $f_k \geq 0,\ k = 1, . . . \ , n$. The $\mathrm{n} + 3$ first-order conditions found by differentiating with respect to $f_{0,}\ f_1,\ . . . ,\ f_n,\ \beta_0$ and β_1 are as follows:

$$ln(f_k|g_k) + 1 + \beta_0 + k\beta_1 = 0, \qquad k = 0, \ . \ . \ . \ , \ n;$$

$$-1 + \sum f_k = 0;$$

$$-m + \sum kf_k = 0$$

The first $n + 1$ equation give $f_k = g_k exp\,(-1 - \beta_0 - k\beta_1)$ for $k = 0, \ . \ . \ . \ , n.$ The last two equalities allow me to find the parameters $1 + \beta_0$ and β_1. Consider the function $\Phi(\beta_1) = \sum g_k e^{-k\beta}$. Since $\sum f_k = 1$, we have $1 = \sum g_k e^{-1-\beta_0-k\beta_1} = e^{-1-\beta_0}\Phi(\beta_1)$. Therefore, $1 + \beta_0 = \ln\Phi(\beta_1)$. Because $\Phi'(\beta) = -\sum g_k k e^{-k\beta}$, we obtain:

$$\Phi'(\beta_1) = -\sum kg_k e^{-k\beta_1} = -e^{(1+\beta_0)}\sum kg_k e^{-1-\beta_0-k\beta_1} =$$

$$-e^{(1+\beta_0)}\sum kf_k = -e^{(1+\beta_0)}m = -\Phi(\beta_1)m$$

Thus, in order to find the precise numerical value for β_1, we solve:

$$\Phi'^{(\beta_1)} = -\Phi(\beta_1)m$$

or equivalently:

$$\frac{d}{d\beta}\ln[\Phi(\beta)] = -\,m$$

For $\beta = \beta_1$. We then may obtain the other parameter β_0 through the equation: $1 + \beta_0 = \ln\Phi(\beta_1)$. After we have the two parameters β_0 and β_1, we easily calculate the desired density $f_k = g_k e^{-(1+\beta_0+\beta_1 k)}$.

We used Newton's method to solve $\frac{d[ln\Phi(\beta)]}{d\beta} = -m$ for β_1. Recall that to solve the equation $F(\beta) = 0$ by Newton's method, one uses the recursion relation:

$$\beta^{j+1} = \beta^j - \mathrm{F}(\beta^j)/\mathrm{F}'(\beta^j)$$

In my case $\mathrm{F}(\beta) = \frac{d[ln\Phi(\beta)]}{d\beta} + m,$ and this reduces to:

$$\beta^{j+1} = \beta^j - \frac{\frac{\Phi'(\beta)}{\Phi(\beta)} + m}{\{\Phi''(\beta)\Phi(\beta) - [\Phi'^{(\beta)}]^2\}/\Phi(\beta)^2}|_{\beta=\beta^j}$$

$$= \beta^j - \frac{\Phi'(\beta^j)\Phi(\beta^j) + m\Phi(\beta^j)^2}{\{\Phi''(\beta^j)\Phi(\beta^j) - [\Phi'(\beta^j)]^2\}}$$

where $\Phi(\beta) = \sum g_k e^{-k\beta}$, $\Phi'(\beta) = \sum k g_k e^{-k\beta}$, and$\Phi''(\beta) = \sum k^2 g_k e^{-k\beta}$

For illustrative purposes, I shall do a numerical example that is a special case of the above. Assume an insured purchased the life insurance policy at age 40 in 1986 and is age 70 in year 2006. The medical report shows the remaining expected life time is 2 years. The insurance policy for the insured is worth \$50,000 face value (death benefit). Details of the policy are listed in Table 11 and Table 12.

First, I project the standard table rate of mortality q_x with the Double Exponential Jump Diffusion model (Deng, Brockett and MacMinn 2010). Mortality rate q_x denotes the probability that a person who is alive at age x will die before age $x+1$, or for a pool of people, q_x = (the number of people die between age x and age $x+1$)/ (the number of people alive at age x). We use the DEJD mortality table instead of the standard CSO table upon where the cash value was derived in 1986. This is more appropriate for this purpose since the DEJD model incorporates the dynamic and asymmetric longevity jump and mortality jump to describe the trend of mortality rate. In this way, the DEJD model considers the longevity risk in generating the morality rate and life table which provides more accurate projection and better fit

Second, the standard table probability g_k is deduced from q_x.. For a pool of people, g_k = (the number of people die between the time k and $k+1$)/(the total number of people at time 0). $g_k = (1 - q_{x-1}) * q_{x-1}$.

Third, the standard table survival function is calculated by $l_k = 1 - g_k$.

Fourth, the adjusted table rate f_k is deduced from g_k by information theory.

Fifth, the adjusted table rate $q_x{}'$ is deduced from f_k. $q_x{}'$ has the same definition as q_x except that $q_x{}'$ is calculated based on the updated information.

Sixth, the adjusted table survival function $l_k' = 1 - f_{k-1}$. l_k' has the same definition as l_k except that l_k' is calculated based on the updated information.

I simulate on each age group using the Double Exponential Jump Diffusion model for the life time.

Table 11 Schedule of Benefits

Form	**Description**	**Initial Amount**	**Benefit Period Ends**	**Annual Premium**	**Premiums Payable**
07000	Basic Plan	$50,000	With Life	$565.50	To 2047

Table 12 Parts of Schedule of Insurance and Values

Insurance Amount		**Guaranteed Values**	
		End of Policy Year	Cash Value Dollars
50,000	1996	Age 60	3,100.00
50,000	1998	Age 62	4,038.50
50,000	2001	Age 65	5,567.00
50,000	2006	Age 70	8,438.50

Table 13 Adjusted and Standard Mortality Table for Age 70 with 2 Years Remaining Life Time

Year	Time	Age	Standard Table Rate	Standard Table Probability	Standard Table Survival Function	Adjusted Table Rate	Adjusted Table Probability	Adjusted Table Survival Function
k	k'	x	q_x	g_k	l_k	q'_x	f_k	l'_k
2006	0	70	0.02539	0.02539	1.00000	0.30926	0.30926	1.00000
2007	1	71	0.02664	0.02596	0.97461	0.30696	0.21203	0.69074
2008	2	72	0.02796	0.02652	0.94865	0.30328	0.14518	0.47871
2009	3	73	0.02934	0.02706	0.92213	0.29769	0.09929	0.33353
2010	4	74	0.03084	0.02757	0.89507	0.28948	0.06781	0.23424
2011	5	75	0.03233	0.02805	0.8675	0.27785	0.04624	0.16643
2012	6	76	0.03394	0.02849	0.83945	0.26199	0.03149	0.12019
2013	7	77	0.03563	0.0289	0.81096	0.24135	0.02141	0.08870
2014	8	78	0.03741	0.02926	0.78206	0.21592	0.01453	0.06729
2015	9	79	0.03928	0.02957	0.7528	0.18657	0.00984	0.05276
2016	10	80	0.04125	0.02983	0.72323	0.15510	0.00666	0.04292
2017	11	81	0.04385	0.03042	0.69343	0.12540	0.00455	0.03626
2018	12	82	0.04663	0.03092	0.66299	0.09773	0.00310	0.03172
2019	13	83	0.04962	0.03136	0.63208	0.07365	0.00211	0.02862
2020	14	84	0.05282	0.03173	0.60072	0.05392	0.00143	0.02651
2021	15	85	0.05625	0.03201	0.56899	0.03854	0.00097	0.02508
2022	16	86	0.05995	0.03219	0.53698	0.02702	0.00065	0.02411
2023	17	87	0.06391	0.03226	0.50479	0.01866	0.00044	0.02346
2024	18	88	0.06817	0.03221	0.47253	0.01273	0.00029	0.02302
2025	19	89	0.07275	0.03203	0.44032	0.00859	0.00020	0.02273
2026	20	90	0.07768	0.03172	0.40828	0.00575	0.00013	0.02253
2027	21	91	0.08298	0.03125	0.37657	0.00357	0.00008	0.02240
2028	22	92	0.08869	0.03063	0.34532	0.00224	0.00005	0.02232
2029	23	93	0.09485	0.02985	0.31469	0.00135	0.00003	0.02227
2030	24	94	0.10147	0.02893	0.28484	0.00090	0.00002	0.02224
2031	25	95	0.10862	0.02786	0.25594	0.00045	0.00001	0.02222
2032	26	96	0.11633	0.02654	0.22814	0.00000	0.00000	0.00000
2033	27	97	0.12465	0.02513	0.20162	0.00000	0.00000	0.00000
2034	28	98	0.13363	0.02358	0.17647	0.00000	0.00000	0.00000
2035	29	99	0.14333	0.02191	0.15289	0.00000	0.00000	0.00000
2036	30	100	0.15381	0.02015	0.13097	0.00000	0.00000	0.00000
2037	31	101	0.16515	0.01833	0.11083	0.00000	0.00000	0.00000
2038	32	102	0.17746	0.01641	0.09253	0.00000	0.00000	0.00000
2039	33	103	0.19067	0.01451	0.07611	0.00000	0.00000	0.00000
2040	34	104	0.20503	0.01263	0.06169	0.00000	0.00000	0.00000
2041	35	105	0.22058	0.01088	0.04897	0.00000	0.00000	0.00000
2042	36	106	0.23743	0.00906	0.03817	0.00000	0.00000	0.00000
2043	37	107	0.25571	0.00744	0.02911	0.00000	0.00000	0.00000
2044	38	108	0.27552	0.00597	0.02166	0.00000	0.00000	0.00000
2045	39	109	0.29703	0.00466	0.01569	0.00000	0.00000	0.00000
2046	40	110	0.31926	0.00352	0.01103	0.00000	0.00000	0.00000
2047	41	111	1.00000	0.00751	0.00751	0.00000	0.00000	0.00000
Total				1.00000			1.00000	

4.5 LIFE SETTLEMENT PRICING

For life settlement pricing, we treat life settlements as zero-coupon bonds with random maturity whose Maturity is equal to the Life Expectancy (T) of the individual whose policy is being settled. The Par Value is equal to the Net Death Benefit (NDB), whose Initial Price is equal to the Purchase Price (M), and where the Yield to Maturity is the same as that for a zero coupon bond of this character, as shown in Figure 13.

Basic Definitions:

B: Net death benefit (or face value) of the life policy

P: Premium for each year

i: Yield to maturity

V: Discount rate, $V = \frac{1}{1+i}$, i is the yield to maturity

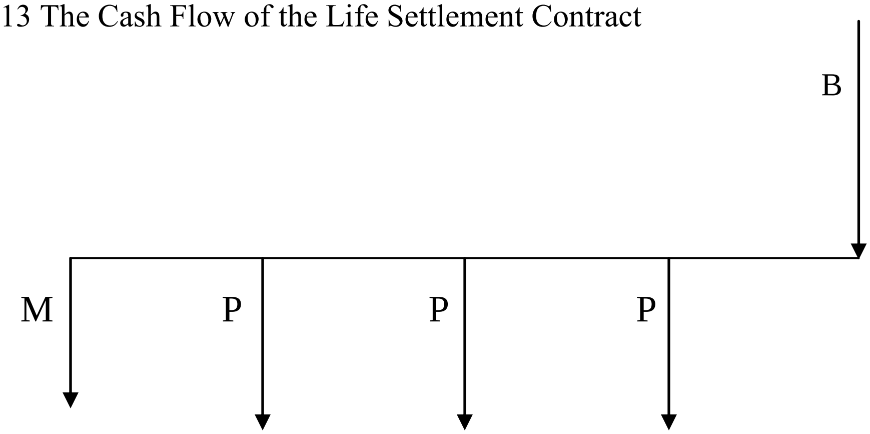
M: Purchase Price for the life settlement

T: Random variable of life time

Consider a hypothetical life settlement contract with a life expectancy of 4 years. Label this contract as contract j. Assume that only 3 premium payments are made and the death benefit kicks in after the three premium payments, as shown in Figure 13.

The relevant cash flows from buying this contract are:

Figure 13 The Cash Flow of the Life Settlement Contract



The formula for calculating the Purchase Pricing M is:

$$M = \mathrm{E}[BV^T - P\left(\frac{1 - V^T}{iV}\right)]$$

$BV^T = \frac{B}{(1+i)^T}$ measures the present value of the net death benefit. This is the positive cash flow for the life settlement purchaser.

$P\left(\frac{1-V^T}{iV}\right) = \sum_{t=1}^{T-1} \frac{P}{(1+i)^t}$ measures the present value of the premiums the purchaser has to continue to pay until the death of the policy holder. This is the negative cash flow for the life settlement purchaser.

M is the purchase price at the date of purchasing.

Table 14 Life Settlement Prices for the Different Yield to Maturity

Yield to Maturity *i*	**Life Settlement Price M ($)**
5.00%	41190.75
6.00%	40071.93
7.00%	39010.62
8.00%	38002.51
9.00%	37043.73
10.00%	36130.74
11.00%	35260.35
12.00%	34429.67
13.00%	33636.03
14.00%	32877.02
15.00%	32150.44
16.00%	31454.24
17.00%	30786.58
18.00%	30145.72

Table 14 illustrates the third party purchase price for the different yields to maturity. The higher yield to maturity the third party requests, the lower the purchase price. But even the lowest purchase price $30,145.72 when the yield to maturity is 18% at the highest, is still higher than the cash value $8,438.50 in Table 12. This means the insured can gain more by selling the policy to the third party for the price of at least of $30,145.72 than to the insurance company at the surrender value $8,438.50. This is a win-win situation for both the insured and the third party (investor), which explains the potential huge profit and investment opportunity in the life settlement market.

Table 15 Life Settlement Price Sensitivity for different Yield to Maturity

Yield to Maturity i	**Life Settlement Price Sensitivity** $\Delta M/\Delta i$
5.00%	-111882
6.00%	-106131
7.00%	-100811
8.00%	-95878
9.00%	-91299
10.00%	-87039
11.00%	-83068
12.00%	-79364
13.00%	-75901
14.00%	-72658
15.00%	-69620
16.00%	-66766
17.00%	-64086
18.00%	N/A

Table 15 illustrates the purchase price sensitivity for the different yield to maturity. There is a decreasing trend for the sensitivity. The sensitivity for the low yield is larger than that for the high yield.

4.6 CONCLUSION

In this paper, I investigate the methodology for pricing life settlement products. The life settlement products provide a way for a third party to purchase the life insurance policy for a price greater than the cash surrender value and less than the face value (or death benefit) in exchange for the right to collect the death benefit. Since the main element for pricing life settlement products is the estimation of the life expectancy of the insured, I adopt the Double Exponential Jump Diffusion (DEJD) model as the kernel for the projection of the mortality rate and the life expectancy. The DEJD model incorporates the longevity jump (caused by medical improvement,), the mortality jump (caused by pandemics) and the dynamic main trend of the mortality rate which provides better explanation and fit to the historical mortality rate data. Based on the DEJD model, I propose a Whole Life Time Distribution Dynamic Pricing (WLTDDP) method to evaluate life settlement products. The method incorporates the updated information of life expectancy, a critical factor, into the evaluation of life settlement products. According to Jensen's inequality, the value of the life settlement product is contingent on the expected value of the function of life time, which is always larger than the value of the function of the expected life time (or life expectancy). The WLTDDP method has the advantage of fixing the negative bias for pricing life settlement products in the previous method by generating the whole distribution for the life time instead of the solo life expectancy variable. The method incorporates more statistical information of the insured's future life time. The statistical methodology is based upon information theory for adjusting mortality tables to obtain exactly some known individual characteristics while obtaining a table that is as close as possible to the standard one.

I use an example of an insured who purchased a life insurance policy at age 40 in 1986 and is age 70 in year 2006. The medical report shows the remaining expected life time is 2 years. The WLTDDP method generates the complete mortality table with the updated information of 2 years remaining life for the insured as illustrated in Table 13. Based on the complete life table, I can generate life settlement prices for different yields to maturity and the sensitivity. The results illustrate that the insured can gain more by selling the policy to a third party than to the insurance company at the surrender value. This is proof for the win-win situation for both the insured and the third party (investor) in the life settlement market. The results explain the potential huge profit and investment opportunity in the life settlement market.

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