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Reaction-Diffusion Fronts in Inhomogeneous Media

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Reaction-Diffusion Fronts in Inhomogeneous Media

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Reaction-Diffusion Fronts in Inhomogeneous Media

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In this thesis, we study the asymptotic behavior of solutions to the reaction-advection-diffusion equation

$$u_t = \Delta_z u + B(z,t) \cdot \nabla_z u + f(u), \quad z \in R^n, \ t > 0$$

under various conditions on the prescribed flow B. Our goal is to characterize, bound, and compute the speed of propagating fronts that develop in the solution u and to describe their dependence on the flow B. We focus mainly on the case when f is the KPP nonlinearity $f(u) = u(1-u)$. In the first section, we consider the case that B is a temporally random field having a spatial shear structure and Gaussian statistics. We show that the solution to the initial value problem develops traveling fronts, almost surely, which are characterized by a deterministic variational principle. In the second section, we use this and other variational principles to derive analytical estimates on the speed of propagating fronts. In the final section, we use the variational principle

to compute the front speed numerically. The mathematical analysis involves perturbation expansions, ergodic theorems, and techniques from the theory of large deviations. We use numerical methods for computing the principal Lyapunov exponents of parabolic operators, which appear in the variational characterization of the front speed.

Table of Contents

List of Tables

List of Figures

Chapter 1

Introduction

1.1 Front Propagation and Reaction-Diffusion Equations

In this thesis, we study the asymptotic behavior of solutions to the semi-linear reaction-advection-diffusion equation

$$u_t = \Delta_z u + B(z,t) \cdot \nabla_z u + f(u), \quad z \in R^n, \ t > 0 \tag{1.1}$$

under various conditions on the prescribed flow B and the nonlinear function $f(u)$. Throughout the thesis, the function $f(u)$ will satisfy $f(0) = f(1) = 0$, and the solution $u(z,t)$ will take values in the interval $[0,1]$. Our goal will be to characterize, bound, and compute the speed of propagating fronts that develop in the solution u as $t \to \infty$ and to describe their dependence on the flow B.

This equation is relevant to models of premixed combustion and spatial ecology, among other applications. In such models, the scalar u may represent a normalized temperature distribution, mass-fraction of burned fuel, or density of a species. The nonlinear function f models either the chemical kinetics or the reproductive behavior of a species. Due to the combination of diffusion and reaction, solutions of (1.1) typically develop propagating fronts that separate regions where u is close to the equilibrium values, $u \approx 0$ and $u \approx 1$. In combustion models, the value $u = 1$ may correspond to the burned state, and the value $u = 0$ may correspond to the unburned state. So, propagating fronts reflect the consumption of fuel as the reaction spreads.

It is of practical interest to understand how inhomogeneities in the flow B effect the asymptotic behavior of propagating fronts. For example, in premixed combustion reactions, turbulent flow can significantly enhance the rate of fuel consumption [21, 73, 75, 76, 91], and many practical power-generating devices critically depend on this enhancement for their efficiency. Intuitively, the reason for this enhancement is that the flow produces a more efficient mixing of the burned and unburned fuel within the reaction zone. Nevertheless, the effect remains difficult to characterize and analyze mathematically.

Motivated by these practical interests, a fundamental mathematical problem is to characterize, bound, and numerically compute the asymptotic speed of traveling fronts that develop in the solutions of equation (1.1). The front speed will depend on the inhomogeneous flow in a nonlinear way, even in the simplest cases. This nonlinear relationship between the flow and the average behavior of the solution is evident in the so-called closure problem – the problem of finding a closed-form equation for the spatial or ensemble averages of the solution u. For example, if $B(z,t)$ is a random field, one might average the equation (1.1) and find that

$$\langle u \rangle_t = \Delta_z \langle u \rangle + \langle B(z,t) \cdot \nabla_z u \rangle + \langle f(u) \rangle.$$

However, the equation for the mean $\langle u \rangle$ cannot be closed, since all moments of the solution are coupled through both the nonlinearity f and the inhomogeneous advection term.

Moreover, it is not possible to separate the effect of the nonlinear reaction from the effect of advection. Notice that by dropping the nonlinear reaction term in (1.1), one obtains the scalar advection-diffusion equation

$$u_t = \Delta_z u + B(z,t) \cdot \nabla_z u, \quad z \in R^n, \ t > 0, \tag{1.2}$$

which has been studied as a simplified model of diffusive transport in a turbulent flow [60]. Under various conditions on B, the rescaled solution $u^\epsilon(x,t) = u(x/\epsilon, t/\epsilon^2)$ converges in some way as $\epsilon \to 0$ to the function $\bar{u}$ that solves an homogenized equation

$$\bar{u}_t = \sum_{i,j} \kappa^*_{i,j} \bar{u}_{z_i z_j}, \tag{1.3}$$

with appropriate initial conditions. In this context, as in the study of (1.1), a fundamental problem is to characterize the effective diffusivity tensor $\kappa^*_{i,j} = \delta_{ij} + \tilde{\kappa}_{i,j}$ in terms of the flow B. It has been shown that if B varies periodically in z and t, the tensor $\tilde{\kappa}$ can be characterized by the cell problem

$$\partial_t \chi_i - \Delta_z \chi_i + B \cdot \nabla \chi_i = -B_i \tag{1.4}$$

defined over the period cell. Then $\tilde{\kappa}$ is given by $\kappa^*_{i,j} = \langle \nabla \chi_i \cdot \nabla \chi_j \rangle$, where $\langle \cdot \rangle$ denotes averaging over the period cell. More general characterizations of κ^* have been obtained for non-period flows, as well [7], [30], [31], [72]. For an extensive discussion of effective diffusivity and diffusive transport of passive scalars, see the survey paper by Majda and Kramer [60].

With this in mind, one might return to the nonlinear problem (1.1) and ask whether the rescaled solution $u^\epsilon(x,t) = u(x/\epsilon, t/\epsilon^2)$ would converge in some sense to a solution of

$$\bar{u}_t = \sum_{i,j} \kappa^*_{i,j} \bar{u}_{z_i z_j} + f(\bar{u}). \tag{1.5}$$

However, this convergence does not hold. In terms of the associated diffusion process defined by (1.1) and (1.2), the effective diffusivity $\kappa^*_{i,j}$ characterizes the mean-squared displacement of the particle trajectories. In other words, as $\epsilon \to 0$ the distribution of particles converges weakly (after rescaling) to a Gaussian distribution with variance $\kappa^*_{i,j}$, and the diffusion process converges weakly to a Brownian motion with diffusivity $\kappa^*_{i,j}$. However, for the nonlinear problem (1.1), the front speed may depend on large deviations of the particle trajectories, not the variance. So, studying the front speed c^* requires more subtle information about the tails of the particle distribution than what is obtained by studying the weak convergence of the distribution via effective diffusivity. As described later, the appropriate scaling for studying the large-scale front motion is $u^\epsilon(x,t) = u(x/\epsilon, t/\epsilon)$, and the limiting equation is a free boundary problem.

1.1.1 Definitions

Before describing analytical approaches to (1.1) and stating the results of this thesis, we make the following definitions that will be used throughout the thesis. First, we will consider the following classes of nonlinearities. In each case, we assume $f(u) = 0$ for all $u \in (-\infty, 0] \cup [1, \infty)$.

Positive nonlinearity: The function f satisfies $f(u) > 0$ for $u \in (0, 1)$. An example of this type of nonlinearity is $f(u) = u^m(1 - u)$ for any power $m \geq 1$.

KPP nonlinearity: The function f is a positive nonlinearity with the additional constraint that $f(u) \leq uf'(0)$. An example of this type of nonlinearity is $f(u) = u(1 - u)$. "KPP" stands for Kolmogorov, Petrovskii, and Piskunov, who were among the first to study solutions to (1.1) for $B \equiv 0$ [54].

Combustion nonlinearity: There is a number $\theta \in (0, 1)$ such that $f(u) = 0$ for $u \in [0, \theta] \cup \{1\}$, and $f(u) > 0$ for $u \in (\theta, 1)$. The parameter θ is sometimes called the ignition temperature, below which no reaction occurs.

Bistable nonlinearity: There is a number $\theta \in (0, 1)$ such that $f(\theta) = f(0) = f(1) = 0$, and $f(u) < 0$ for $u \in (0, \theta)$ and $f(u) > 0$ for $u \in (\theta, 1)$. Also, $f'(0) < 0$, $f'(1) < 0$, and $\int_0^1 f(s)\,ds > 0$. An example of this type of nonlinearity is the cubic function $f(u) = u(1 - u)(u - \theta)$, $\theta < 1/2$.

As we will see, the behavior of $f(u)$ near $u = 0$ has a significant effect on the behavior of solutions to (1.1). For the bistable nonlinearity, both $u = 0$ and $u = 1$ are stable equilibrium solutions of the differential equation $u' = f(u)$. For the positive nonlinearity, however, $u = 0$ is an unstable equilibrium. In the KPP case, the fact that $f(u) \leq uf'(0)$ leads to a linearization of f, and we will see that the minimal speed of propagating fronts is determined completely by $f'(0)$.

Throughout the thesis, the vector field $B(z, t)$ is assumed to be divergence-free, $\nabla_z \cdot B = 0$, for all time. The flow will be either periodic or random in (z, t). In the periodic case, we will generally refer to the period cell in space by $D = [0, L]^n$, or $D_T = D \times [0, T]$ for flows that also are temporally periodic. In other words $B(z + jLe_i, t + kT) = B(z, t)$ for any unit vector $e_i \in R^n$ and integers j, k. In the random case, $B(z, t) = B(z, t, \hat{\omega})$ will be defined over a probability space $(\hat{\Omega}, \mathcal{F}, Q)$, to be specified later. We will often consider the special case of a shear flow: $B(z, t) = (b(z_2, \ldots, z_n, t), 0, \ldots, 0)$. In this case, we will write $z = (x, y)$ where x refers to the one-dimensional variable z_1 and $y \in R^{n-1}$ denotes all the other dimensions $(z_2, \ldots, z_n)$. Hence, we can express the shear flow as $B(z, t) = (b(y, t), \mathbf{0})$. For simplicity, we also write $y \in D \subset R^{n-1}$ to mean the $n - 1$ dimensional cell or domain of dependence for the y variable, since the x dependence is trivial.

Unless otherwise specified, we will consider solutions to the equation (1.1) with nontrivial initial condition $u(z, 0) = u_0(z)$. The function u_0 will generally be smooth and compactly supported with values in the interval $[0, 1]$. Since $u_0 \in [0, 1]$, the properties of f and the maximum principle imply that $u(z, t) \in (0, 1)$ for all $z \in R^n$, $t > 0$.

In addition to considering fronts propagating in R^n, we will consider fronts propagating in an infinite channel, with Neumann boundary conditions imposed on the side of the channel. Specifically, we let D_y be a bounded open subset of R^{n-1} with smooth boundary, and let $D = \{(x, y) \in [0, L] \times D_y\}$ and $D_T = D \times [0, T]$. Furthermore, we assume $B(z, t)$ is periodic in x and t with period L and T, respectively, and we require $B(z_o, t) \cdot \nu = 0$ whenever $y_o \in \partial D_y$ and ν is a vector normal to the exterior of the infinite channel $R \times D_y$ at $z_o = (x_o, y_o) \in R \times \partial D_y$. Under these conditions, we consider fronts propagating through the channel, in the $\pm x$ directions.

1.2 Survey of Recent Results

The study of propagating fronts in solutions to reaction-diffusion equations has a long history. The survey papers of Berestycki [11], Souganidis [80], and Xin [88] contain a thorough review of both classical results and many of the more recent developments described below. Some of the first studies of scalar reaction diffusion equations, were motivated by models of population genetics. For example, see the pioneering works of Fisher [32], Kolmogorov, Petrovsky, and Piskunov [54] who considered solutions to the homogeneous scalar equation

$$u_t = \Delta u + f(u). \tag{1.6}$$

The study of the inhomogeneous problem (1.1) began more recently with Freidlin's [33, 34] probabilistic treatment of (1.1) in the case that $B = B(z)$ is deterministic and periodic in the spatial variables. He also studied one-dimensional fronts in spatially random media. Since this work, there have been multiple analytical approaches to describing front propagation in inhomogeneous flows. In the next chapter, we use probabilistic arguments, like those of Freidlin, to describe front propagation in a temporally random flow. First, we now briefly describe two other major approaches to describing the asymptotic behavior of solutions to (1.1) (the construction of traveling wave solutions and homogenization techniques), and we draw some connections between them.

A traveling wave solution to the equation (1.6) is a solution in the form $u(z,t) = U(z\cdot k - ct)$ for some $c \in R$ and $k \in R^n$, $|k| = 1$, satisfying appropriate boundary conditions for $U(\pm\infty)$. The function U is the profile of the wave in a reference frame moving with constant speed c in the direction k. Inserting this ansatz into (1.6) we see that the profile U must solve the traveling wave equation

$$U'' + cU' + f(U) = 0 \tag{1.7}$$

for both the unknown profile U and the unknown speed c. For each of the nonlinearities described above, such solutions have been shown to exist in the classical sense for all $t \in R$ [88]. For periodically varying media, these notions have been extended by the construction of curved or pulsed traveling waves that oscillate within the moving frame. This technique began with the work of Berestycki, Larrouturou, Lions, and Nirenberg [14, 15], who constructed solutions for the case of a steady shear flow $B(z,t) = (b(y), 0)$. Their curved traveling wave solutions took the form $u(z,t) = U(z\cdot k - ct, y)$, where $k = (1,0)$ and U solves

$$\Delta_{s,y} U + (b(y,t) + c)U_s + f(U) = 0, \;\; U(\pm\infty, y) = 1, 0. \tag{1.8}$$

Pulsating traveling fronts in R^n for more general steady periodic flows were first introduced by Xin [85, 86]. Berestycki and Hamel [12] constructed pulsating traveling waves in more general inhomogeneous domains, including domains with periodically distributed holes. Fréjacques [37, 38], Nolen, Xin, and Rudd [66, 67] have extended the idea of pulsating traveling waves to time-dependent flows. In this case, $B(z) = B(z,t)$ may vary in every direction and in time, so the traveling waves take the form $u(z,t) = U(z \cdot k - ct, z, t)$, and the equation for $U(s,z,t)$ becomes degenerate parabolic:

$$U_t - cU_s = (k\partial_s + \nabla_z)^2 U + B \cdot (k\partial_s + \nabla_z) U + f(U) \,, \tag{1.9}$$

Such solutions have been shown to exist in an appropriate weak sense [66].

Although these traveling wave solutions are special solutions that exist for all time, they have been shown to be asymptotically stable in some situations [57, 63, 85, 90]. For example, in the case of the KPP nonlinearity, it has been shown [63, 66] that solutions to the initial value problem develop fronts that propagate asymptotically at the speed corresponding to the slowest possible traveling wave solution. Thus, studying the effect of B on traveling wave solutions gives insight into the effect of B on solutions to the initial value problem with more general initial data.

The goal of the homogenization approach to studying (1.1) is to obtain an effective or averaged equation that describes the large-scale motion of a front, independent of small-scale inhomogeneities. To do so, one considers the behavior of solutions to (1.1) under the hyperbolic scaling $(z,t) \to (\epsilon^{-1}z, \epsilon^{-1}t)$. This suggests studying the function $u^\epsilon(z,t)$ which solves the equation

$$u^\epsilon_t = \epsilon \Delta u^\epsilon + B(\epsilon^{-1}z, \epsilon^{-1}t) \cdot \nabla u^\epsilon + \epsilon^{-1} f(u^\epsilon) \tag{1.10}$$

with initial data $u^\epsilon(z,0) = u_0(z)$, independent of ϵ. Freidlin [33, 34] studied this equation using probabilistic methods to analyze the trajectories of the related diffusion process. Evans and Souganidis [28] were the first to obtain homogenization results for (1.1) using PDE techniques and the theory of viscosity solutions. For the KPP nonlinearity, Majda and Souganidis [62] extended these arguments to handle more general periodic flows. Although their analysis encompasses more general flows having two separated scales, a particular case of their result is the following theorem:

Theorem 1.2.1. *(Majda and Souganidis [62]) Let f be the KPP nonlinearity and $B(z,t)$ be periodic in both z and t. Let $u^\epsilon(z,t)$ solve (1.10) with initial data $u^\epsilon(z,0) = u_0(z) \in [0,1]$. Let G_0 be the support of $u_0(z)$. Then as $\epsilon \to 0$,*

$$u^\epsilon(z,t) \to \begin{cases} 0 & \text{if } z \in \{\psi < 0\}, \\ 1 & \text{if } z \in \mathit{int}\{\psi = 0\} \end{cases}$$

locally uniformly, where $\psi \in C(R^n \times [0,\infty))$ is the unique viscosity solution to the variational inequality

$$\max(\psi_t - H(\nabla\psi) - f'(0), \psi) = 0, \tag{1.11}$$

with $\psi(z,0) = 0$ for $z \in G_0$ and $\psi(z,0) = -\infty$ for $z \in \bar{G}_0^c$.

Therefore, in the limit $\epsilon \to 0$, the propagating front is defined by the boundary of the set $\{z \in R^n |\ \psi(z,t) < 0\}$ which evolves according to the first-order equation (1.11). The effective Hamiltonian $H(\nabla\psi)$ is the unique constant such that there exists a periodic viscosity solution to the cell problem

$$w_t - \Delta w - |p + \nabla w|^2 - B \cdot (p + \nabla w) = -H(p) \tag{1.12}$$

with $p = \nabla\psi(z,t)$. This cell problem plays a role similar to that of (1.4) in the homogenization of the linear equation (1.2). Similar techniques have also been applied in the case of other nonlinearities [8, 10], systems of multiple equations [9, 34, 35], and spatially random flows [59, 81]. The review paper of Souganidis [80] describes these and many related results in the broader context of viscosity solutions applied to front propagation and interface problems. Other homogenization techniques have been developed recently by Caffarelli, Lee, and Mellet [17]. In this work, the authors scale the nonlinear term $f(u) \mapsto \frac{1}{\delta} f(\frac{u}{\delta})$ and consider the behavior of u in the limit $\delta \to 0$, for a positive nonlinearity f. This scaling leads to a free boundary problem that is different from (1.11).

1.3 Goals and Outline of Thesis

The aim of this thesis is to characterize, bound, and numerically compute the asymptotic speed of traveling fronts that develop in the solutions of equation (1.1) under various conditions on the flow B. This thesis will focus primarily on the case that $f(u)$ is the KPP nonlinearity. When $f(u)$ is the KPP nonlinearity and $B(z,t)$ is periodic in both variables, the analysis of traveling waves, homogenization techniques, and the probabilistic arguments we will use later each lead to a variational representation for the speed of the propagating front (or slowest traveling wave) in a given direction $k \in R^n$. One version of this representation is the formula

$$c^*(k) = \inf_{\lambda > 0} \frac{\mu(\lambda) + f'(0)}{\lambda}. \tag{1.13}$$

The constant $\mu(\lambda)$ is the principal eigenvalue of the operator L_λ defined by

$$L_\lambda \phi = \Delta_z \phi - \phi_t + (B - 2\lambda k) \cdot \nabla_z \phi + (|\lambda^2| - \lambda B \cdot k)\phi = \mu(\lambda)\phi, \tag{1.14}$$

with $\phi > 0$ periodic in (z, t). Freidlin [33] derived this formula using probabilistic techniques. In the context of the theory of traveling waves, this has been derived by Berestycki and Nirenberg [15]; Berestycki and Hamel [12]; Nolen and Xin [67]; and Nolen, Rudd, and Xin [66]. As we will demonstrate, this variational formula is a useful tool for analyzing the relationship between the front speed c^* and the flow B. It is also useful for numerical computation of the front speed.

Notice that if $w(z, t) = \log \phi(z, t)$, then w solves (1.12) with the constant $H(p) = \mu(-p)$, and $p = \lambda k$. So the wave speed could also be expressed in terms of the effective Hamiltonian H. In fact, formula (1.13) can be derived directly from the homogenization theory. Majda and Souganidis [62] have shown that for $B = B(z, t)$, the fronts defined by (1.11) and the boundary of $\{\psi(z, t) < 0\}$ are actually level sets of the solution to a geometric PDE for which the Hamiltonian has an explicit representation in terms of H. Using the Lax representation formula for this solution, one obtains (1.13), as shown in [88].

Most of the results we have described pertain to spatially and temporally periodic flows. Assuming the flow is periodic renders some aspects of the analysis more tractable, and the results give insight into the coupled effect of advection and reaction on the propagating front. However, periodicity may not be an appropriate model of an inhomogeneous flow, since random noise or uncertainty in a flow may effect the asymptotic behavior of the front. Therefore, from the point of view of applications, it is important to develop models for front propagation in random flows and to develop methods for analyzing and approximating the behavior of the fronts.

The first major goal of this thesis is to characterize the propagation of KPP fronts under the influence of shear flows that are periodic in space and random in time. Specifically, the flow $B(z,t)$ is assumed to be Gaussian and unbounded in t, almost surely. In this way, we are adding a temporal randomness into the periodic models that have been studied recently in [12, 15, 62, 66, 67]. Carmona and Xu [19] also have studied such spatially periodic temporally random fields in the context of homogenizing the passive scalar equation (1.2). Although the solution u is a random function on $R^n \times [0,\infty)$, we prove that the solution develops fronts that travel (asymptotically) with a non-random speed.

This introduction of randomness leads to new mathematical difficulties not present with the assumption of periodicity. For example, as we noted in [67], periodicity in t leads to a comparison principle between solutions of the traveling wave equation. Similarly, in [66] we demonstrated how periodicity implies that eigenfunctions of an auxiliary operator can be used to construct sub- and super-solutions to (1.1). When B varies randomly in time, however, it appears that these comparison techniques cannot be extended. Instead, one must rely on ergodic properties of the flow and on large deviations estimates of particles diffusing within the random flow.

Our analysis for fronts in temporally random flows is contained in Chapter 2, and the main results are Theorems 2.1.2 and 2.1.3 which characterize the front speed c^* in terms of an auxiliary linear evolution problem, analogous to the representation (1.13). This work extends techniques employed by Freidlin [33] to analyze KPP fronts in steady periodic flows, and we build on his ideas in order to handle the random temporal dependence of the flows. In Section 2.2, we prove Theorem 2.1.2. In Section 2.3, we adapt Freidlin's method to prove Theorem 2.1.3, as well as many technical estimates needed for the result. In particular, we establish a large deviations principle for the associated diffusion process in the random flow; this is stated in Theorem 2.3.4. We will make frequent use of the sub-additive ergodic theorem [3, 50] and the Borell inequality for Gaussian fields [2]. In Section 2.4, we show that Theorems 2.1.2 and 2.1.3 imply a homogenization result analogous to Theorem 1.2.1.

The second major goal of the thesis is to use (1.13) and related variational principles to analytically estimate the speed c^* when B varies randomly in z or t. This analysis constitutes Chapter 3. In Section 3.1, we derive an asymptotic expansion for c^* when the flow is a small, periodic perturbation of the homogeneous medium. Then, in Section 3.2, we consider an ensemble of fronts propagating in an infinite cylinder under the influence of spatially random flows, and we extend the analysis of Section 3.1 to derive an expansion for the mean speed $E[c^*]$. In Section 3.3, we return to the model studied in Chapter 2, and we analyze the speed of fronts propagating in temporally random shear flows. We use the variational representation derived in Chapter 2 to prove upper and lower bounds on the front speed in terms of statistical properties of the flow. Some of the results of Sections 3.1 and 3.2 have been published already by Nolen and Xin in [69, 70].

The final goal of the thesis is to develop a numerical method for studying c^* based on the variational principle (1.13). A description of this method and the results of our numerical simulations constitute Chapter 4. The main difficulty is computing the constants $\mu(\lambda)$. We demonstrate three techniques for computing $\mu(\lambda)$, depending on the flow structure. In Section 4.1, we simulate fronts propagating in an infinite cylinder under the influence of random shear flows. These simulations correspond to the analytical estimates of Sections 3.1 and 3.2. In Section 4.2, we simulate fronts propagating under the influence of temporally random shear flows, corresponding to the analytical estimates of Section 3.3. Finally, in Section 4.4, we investigate fronts propagating under the influence of temporally and spatially perturbed periodic flows. Some of the results of Section 4.1 have been published already by Nolen and Xin in [70].

Chapter 2

Front Propagation in Temporally Random Flows

2.1 Definitions and Main Results

In this section we describe the propagation of reaction fronts under the influence of prescribed flows that have a statistically homogeneous, or stationary, time dependence. Specifically, we assume that B is a Gaussian random field having the form of a shear flow $B(z,t) = (b(y,t,\hat{\omega}),0)$, where

$$b(y,t,\hat{\omega}) = \sum_{j=1}^{N} b_1^j(y) b_2^j(t,\hat{\omega}) \tag{2.1}$$

for some integer N. The functions $b_1^j(y)$ are assumed to be Lipschitz continuous and periodic with period cell D. The functions $b_2^j(t)$ are assumed to be centered Gaussian random fields over $t \in R$, defined over a probability space $(\hat{\Omega}, \hat{\mathfrak{F}}, Q)$. The covariance function of the field b will be denoted

$$\Gamma(y_1, y_2, t_1, t_2) = E_Q[b(y_1,t_1)b(y_2,t_2)].$$

Furthermore, we assume that b satisfies the following assumptions:

A1: (Stationarity) For each $s \in R^+$ there is a measure preserving transformation $\tau_s : \hat{\Omega} \to \hat{\Omega}$ such that $b_2^j(\cdot + s, \hat{\omega}) = b_2^j(\cdot, \tau_s \hat{\omega})$, for each j. Hence,

$b(y, \cdot + s, \hat{\omega}) = b(y, \cdot, \tau_s \hat{\omega})$, Γ depends only on y_1, y_2 and $|t - s|$, and we can define a global upper bound on the variance of b:

$$\sigma^2 = \sup_{(y,t)} E_Q[b(y,t)^2] = \sup_y E_Q[b(y,0)^2].$$

A2: (Ergodicity) The transformation τ_s is ergodic: if a set $A \in \hat{\mathfrak{F}}$ is invariant under the transformation τ_s, then either $Q(A) = 0$ or $Q(A) = 1$.

A3: (Decay of Temporal Correlations) The function $\hat{\Gamma}(r) = \sup_{y_1, y_2} \Gamma(y_1, y_2, 0, r)$ is integrable over $[0, \infty)$:

$$\int_0^\infty \hat{\Gamma}(r)\, dr = p_1 < \infty \tag{2.2}$$

for some finite constant $p_1 > 0$. This constant will appear later in estimates of the front speed.

We note that due to the assumption of Lipschitz continuity in the spatial variables, there is a finite constant $p_2 > 0$ such that

$$|\Gamma(y_1, y_2, s, t) - \Gamma(y_1, y_3, s, t)| \le p_2 |y_3 - y_2| \hat{\Gamma}(|s - t|).$$

We also note that the processes $b_2^j(t)$ may be almost surely unbounded in time. Consequently, $\sup_{(y,t)} b(y,t) = +\infty$ almost surely with respect to Q. For each fixed y and t, $b(y, t, \hat{\omega})$ is not a member of $L^\infty(\hat{\Omega}, Q)$.

Before stating the main results of this section, let us define the family of diffusion processes associated with the linear part of the operator in (1.1). For a fixed $\hat{\omega} \in \hat{\Omega}$ and for each $z \in R^n$, $t \geq 0$, let $Z^{z,t}(s) = (X^{z,t}(s), Y^{z,t}(s)) \in R^n$ solve the Itô equation:

$$dZ^{z,t}(s) \quad = \quad B(Z^{z,t}(s), t - s)\, ds + dW(s), \quad s \in [0, t]$$

with initial condition $Z^{z,t}(0) = z = (x, y) \in R^n$, where $W(s) = (W_1(s), W_2(s)) \in R^n$ is the n-dimensional Wiener process with $W(0) = 0$, defined on the probability space $(\Omega, \mathcal{F}, P)$. In integral form, this equation is

$$Z^{z,t}(s) \quad = \quad z + \int_0^s B(Z^{z,t}(\tau), t - \tau)\, d\tau + W^0(s) \tag{2.3}$$

where $W^z(s)$ denotes a Wiener process starting at $W^z(0) = z$, P-a.s. Note that the process depends on $\hat{\omega} \in \hat{\Omega}$. So, it can be considered a random walk in a temporally random environment, with $\hat{\omega}$ being an index corresponding to the environment. Due to the assumption of Lipschitz continuity in the spatial variables, there exists a unique strong solution to this stochastic equation. We will use $P^{z,t}$ to denote the corresponding family of measures on $C([0, t]; R^n)$, and we will use $\mathcal{F}^t_{s \leq \tau}$ to denote the σ-algebra of events $\omega \in \Omega$ generated by $Z^{z,t}(s)$ for $s \leq \tau$ (the history of the process up to time τ).

As we will see, the KPP front speed depends on large deviations of the random variable

$$\eta^t_z(\kappa t) = \frac{z - Z^{z,t}(\kappa t)}{\kappa t} \in R^n,$$

which is the average velocity of a trajectory over the time interval $[0, \kappa t]$. The need for the parameter κ results from the time dependence of the field $B(z, t)$ and will become more apparent later. Notice that in the case of a shear flow, equation (2.3) reduces to

$$\begin{aligned} dX^{z,t}(s) &= dW_1(s) + b(W_2(s), t-s)\,ds \\ dY^{z,t}(s) &= dW_2(s). \end{aligned} \tag{2.4}$$

Here we first see the decoupling that results from the shear flow assumption: the component of $Z^{z,t}$ that is orthogonal to the shear direction is just a Brownian motion $Y^{z,t}(s) = y + W_2(s)$.

Now we state the main results. First, the following Proposition will allow us to characterize the speed of propagation:

Proposition 2.1.1. *There is a set* $\hat{\Omega}_0 \subset \hat{\Omega}$ *such that* $Q(\hat{\Omega}_0) = 1$ *and for any* $\hat{\omega} \in \hat{\Omega}_0$ *and any* $\lambda \in R^n$ *the limit*

$$\mu(\lambda, z) = \mu(\lambda) = \lim_{t\to\infty} \frac{1}{t} \log E\left[e^{-\lambda\cdot Z^{z,t}(t)}\right] \tag{2.5}$$

holds uniformly over $z \in D$ *and locally uniformly over* $\lambda \in R^n$*. The limit* $\mu(\lambda)$ *is a finite constant (non-random) for all* $\hat{\omega} \in \hat{\Omega}_0$ *and independent of* $z \in R^n$*. Moreover,* $\mu(\lambda) \geq 0$*, and* $\mu(\lambda)$ *is both convex and super-linear:* $\mu(\lambda)/|\lambda| \to +\infty$ *as* $|\lambda| \to \infty$*.*

Consequently, we can define $S(c)$ to be the Legendre transform of $\mu(\lambda)$

$$S(c) = \mu^*(c) = \sup_{\lambda \in R^n} [c \cdot \lambda - \mu(\lambda)], \tag{2.6}$$

and we find that the speed of propagation can be bounded above in terms of the function S, as stated in the following theorem.

Theorem 2.1.2 (Upper bound on front speed)**.** *Let $u(z,t,\hat{\omega})$ solve (1.1) with initial condition $u(z,0,\hat{\omega}) = u_0(z)$, where $u_0(z) \in [0,1]$ has compact support G_0 and is independent of $\hat{\omega}$. Then, for any closed set $F \subset \{c \in R^n |\ S(c) - f'(0) > 0\}$ and $G' \subset G_0$*

$$\lim_{t\to\infty} \sup_{\substack{c\in F\\ z'\in G'}} u(z'+ct,t,\hat{\omega}) = 0$$

for almost every $\hat{\omega} \in \hat{\Omega}$.

Furthermore, the speed of propagation can be bounded below in terms of the function S:

Theorem 2.1.3 (Lower bound on front speed)**.** *Let $u(x,y,t,\hat{\omega})$ solve (1.1) with initial condition $u(z,0,\hat{\omega}) = u_0(z)$, where $u_0(z) \in [0,1]$ has compact support G_0 and is independent of $\hat{\omega}$. Then, for any compact sets $K \subset \{c \in R^n |\ S(c) - f'(0) < 0\}$ and $G' \subset G_0$,*

$$\lim_{t\to\infty} \sup_{\substack{c\in K\\ z'\in G'}} u(z'+ct,t,\hat{\omega}) = 1 \tag{2.7}$$

for almost every $\hat{\omega} \in \hat{\Omega}$.

Therefore, if for each unit vector $e \in R^n$ we define the constant $c^* = c^*(e) > 0$ by the variational formula

$$c^*(e) = \inf_{\lambda\cdot e>0} \frac{\mu(\lambda) + f'(0)}{\lambda \cdot e}, \tag{2.8}$$

we see from the definition of S that the front spreads asymptotically with speed equal to $c^*(e)$ in the direction of the vector e. Although the solution u

depends on $\hat{\omega} \in \hat{\Omega}$ since B is a random variable over $\hat{\Omega}$, the function $S(c)$ and the speeds $c^*(e)$ are independent of $\hat{\omega}$. They are almost surely constant with respect to $\hat{Q}$, a consequence of the ergodicity assumption A2. Hence, we will refer to the constant $c^*(e)$ as the front speed in direction e. We will frequently suppress the dependence of u on $\hat{\omega}$ for clarity of notation.

As mentioned in the introduction, Theorems 2.1.2 and 2.1.3 extend recent results on KPP front speeds in temporally periodic incompressible flows [62, 66, 67] and an older result of Freidlin (Theorem 7.3.1, p. 494 of [33]) where he treated the case of steady, spatially periodic flows. The following proofs are built on Freidlin's, with additional ingredients to handle both the time-dependence and the stochastic nature of the field B. For example, in the periodic case, $\mu(\lambda)$ is the principal eigenvalue of a periodic-parabolic operator [66, 67], and perturbation theory [47] implies that $\mu(\lambda)$ is differentiable in λ. It then follows that the random variables $\eta_z^t(t)$ satisfy a large deviation principle with convex rate function $S(c)$ given by (2.6) (see Theorem 7.1.1 and Theorem 7.1.2 of [33]). However, if $\mu(\lambda)$ is not known to be differentiable, the large deviation property might not be true. In the present case, $\mu(\lambda)$ is not an eigenvalue of a linear operator, so we cannot readily apply the perturbation theory [47] to get differentiability. Instead, we prove directly that a rate function exists, that it is convex, and that it is characterized by (2.6).

2.2 Proof of Theorem 2.1.2

The proof of Theorem 2.1.2 is based on the assumption that $f(u) \leq f'(0)u$. This allows us to construct super-solutions to equation (1.1) through linearization. Consider the solution to the auxiliary initial value problem

$$\Phi_t = \frac{1}{2}\Delta_z\Phi + (B-\lambda)\cdot\nabla\Phi + (|\lambda|^2/2 + f'(0) - \lambda\cdot B(z,t))\Phi, \Phi(z,0) \equiv 1.$$

where $\Phi = \Phi(z,t) > 0$ is periodic in z. First, we claim that $\mu(\lambda)$ can be expressed in terms of Φ:

$$\mu(\lambda) \quad = \quad -f'(0) + \lim_{t\to\infty}\frac{1}{t}\log\Phi(z,t). \tag{2.9}$$

Using the Feynman-Kac representation for Φ, we see that

$$\Phi = \tilde{E}_z\left[e^{f'(0)t + \frac{|\lambda|^2}{2}t - \int_0^t \lambda\cdot B(\tilde{Z}^{z,t}(s),t-s)ds}\right],$$

where $\tilde{Z}$ solves

$$d\tilde{Z}^{z,t}(s) \quad = \quad \Big(B(\tilde{Z}^{z,t}(s), t-s) - \lambda\Big)\, ds + dW(s).$$

This induces a measure $\tilde{P}^{z,t}$ that is absolutely continuous with respect to $P^{z,t}$. Using the Girsanov theorem [46], one can compute the Radon-Nikodyn derivative of $\tilde{P}$ with respect to P:

$$\frac{d\tilde{P}}{dP} = e^{-\int_0^t \lambda\cdot dW(s) - \frac{1}{2}|\lambda|^2 t}.$$

Hence

$$\Phi = \tilde{E}_z\left[e^{f'(0)t + \frac{|\lambda|^2}{2}t - \int_0^t \lambda\cdot B(\tilde{Z}^{z,t}(s),t-s)ds}\right] = e^{f'(0)t}E_z\left[e^{-\lambda\cdot Z^{z,t}(t)}\right],$$

establishing (2.9).

Now suppose $S(c) - f'(0) > 0$. Then for $\epsilon > 0$ sufficiently small, there exists $\lambda > 0$ such that $\lambda \cdot c > \mu(\lambda) + f'(0) + 2\epsilon$. By Proposition 2.1.1, $\mu(\lambda)$ is finite, and there is a function $R = R(z,t)$ such that $|R| \to 0$ as $t \to \infty$, uniformly in z, and

$$\Phi(y,t) = e^{\mu(\lambda)t + f'(0)t + R(z,t)t}. \tag{2.10}$$

If $\delta > 0$ is sufficiently small, then $\lambda \cdot c' > \mu(\lambda) + f'(0) + \epsilon$, whenever $|c' - c| < \delta$. Then for any $\alpha > 1$, we also have $\lambda \cdot \alpha c' > \mu(\lambda) + f'(0) + \epsilon$. If we define the function $\psi(z,t) = e^{-\lambda \cdot z}\Phi(z,t)$, then ψ solves the equation

$$\begin{aligned} \psi_t = \frac{1}{2}\Delta_z \psi + B \cdot \nabla \psi + f'(0)\psi \\ \psi(z,0) = e^{-\lambda \cdot z}, \end{aligned} \tag{2.11}$$

and ψ is a super solution to the original nonlinear equation (1.1), since $f(u) \le f'(0)u$. Combining (2.10) and (2.11), we see that

$$\begin{aligned} \limsup_{t\to\infty} \psi(\alpha c' t, t) &= \limsup_{t\to\infty} e^{-\lambda \cdot \alpha c' t}\Phi(\alpha c' t, t) \\ &\le \limsup_{t\to\infty} e^{-2\epsilon t + R(\alpha c' t, t)t} \\ &= 0, \end{aligned}$$

since $|R(z,t)| < 2\epsilon$ for t sufficiently large. By definition of $R(z,t)$, the limit is uniform in α and δ, for $\alpha \ge 1$ and δ small. After multiplying ψ by a constant, if necessary, the maximum principle implies that $u(z,t) \le \psi(z,t)$ for all z and t. The function u is therefore trapped below ψ which moves with velocity c'. Now we piece together a collection of such super solutions.

If F is bounded, then it is compact, since it is closed. From the above analysis, we see that we can pick finite sets $\{c_j\} \subset F$ and $\{\lambda_j\}$ such that $F \subset \bigcup_j U_{\delta_j}(c_j) \subset \{S(c) - f'(0) > 0\}$ and $\lambda_j \cdot c' > \mu(\lambda_j) + \epsilon$ whenever $c \in U_{\delta_j}(c_j)$. If we define ψ_j according to (2.11) with $\lambda = \lambda_j$, and set

$$\hat{\psi}(z,t) = \inf_j \psi_j(z,t),$$

we see that

$$\lim_{t\to\infty} \sup_{c\in F, \alpha>1} u(\alpha ct, t) \leq \lim_{t\to\infty} \sup_{c\in F, \alpha>1} \hat{\psi}(\alpha ct, t) = 0$$

Since $\{S(c) - f'(0) \leq 0\}$ is bounded, then general result follow from the fact that the limit is uniform in $\alpha > 1$. This completes the proof of Theorem 2.1.2. $\square$

Note that the function $R(z,t) = R(z,t,\hat{\omega})$ depends on the realization $\hat{\omega} \in \hat{\Omega}$. However, such a function exists almost surely with respect to Q, according to Proposition 2.1.1.

2.3 Proof of Theorem 2.1.3

Proving Theorem 2.1.3 requires more estimates on the random solutions $u(z,t)$ and the processes $Z^{z,t}(s)$. The first estimate is a lower bound analogous to Lemma 7.3.3 of [33]. It estimates the exponential decay rate of u in terms of the function S:

Lemma 2.3.1. *For any compact set $K \subset \{c \in R^n |\ S(c) - f'(0) > 0\}$,*

$$\liminf_{t\to\infty} \frac{1}{t} \log \inf_{c\in K} u(ct,t) \geq -\max_{c\in K}(S(c) - f'(0)). \qquad (2.12)$$

The second estimate gives a coarse bound on very large excursions of the random process $X^{z,t}$, the first component of the process $Z^{z,t}$:

Lemma 2.3.2. *There are constants $K_1, K_2 > 0$ independent of $\kappa \in (0,1]$ such that, except on a set of Q-measure zero,*

$$\sup_{z\in R^n} P\left(\sup_{s\in[0,\kappa t]} |X^{z,t}(s) - x| \geq \eta t \ \mid W_2^0 \in \Lambda\right) \leq K_1 e^{-K_2\eta^2 t/\kappa} \tag{2.13}$$

for any open set $\Lambda \subset R^n$, for any $\kappa \in (0,1]$, $\eta > 0$, and for t sufficiently large depending on $\hat{\omega}$, κ, and η.

In particular, the lemma holds with $\Lambda = R^n$. Using the fact that the $Y^{z,t}$ component of the process $Z^{z,t}$ is a Wiener process, the lemma implies the following corollary:

Corollary 2.3.3. *There are constants $K_1, K_2 > 0$ independent of $\kappa \in (0,1]$ such that, except on a set of Q-measure zero,*

$$\sup_{z\in R^n} P\left(\sup_{s\in[0,\kappa t]} |Z^{z,t}(s) - z| \geq \eta t\right) \leq K_1 e^{-K_2\eta^2 t/\kappa} \tag{2.14}$$

for any $\kappa \in (0,1]$, $\eta > 0$, and for t sufficiently large depending on $\hat{\omega}$, κ, and η.

For the moment, we delay the proof of these estimates and show how they lead to Theorem 2.1.3. Lemma 2.3.1 is proved in Section 2.3.1; Lemma 2.3.3 is proved in Section 2.3.3.

The proof of Theorem 2.1.3 is based on the observation that when $u < h < 1$, the reaction rate can be bounded below. For each $u \in (0,1]$, define the reaction rate ζ by

$$\zeta(u) = \frac{f(u)}{u}$$

and $\zeta(0) = f'(0)$. Now equation (1.1) can be written

$$u_t = \frac{1}{2}\Delta_z u + B \cdot \nabla u + \zeta(u)u. \tag{2.15}$$

By the properties of $f(u)$ we see that $\zeta(u) > 0$ for $u \in [0,1)$, $\zeta(u)$ is continuous for $u \in [0,1]$, and $\zeta(0) \geq \zeta(u)$ for any $u \in [0,1]$. For example, if $f(u) = f'(0)u(1-u)$, then $\zeta(u) = f'(0)(1-u)$. If $h \in (0,1)$ we define a lower bound on ζ:

$$\zeta_h = \inf_{u \in (0,h)} \zeta(u) > 0.$$

So, in regions where u is bounded away from one, the reaction rate can be bounded below by $\zeta_h > 0$. This prevents quenching of the fronts, a phenomenon that can occur with other nonlinearities that do not satisfy $f(u) \leq f'(0)u$ (for example, see [52]).

For a fixed $\hat{\omega} \in \hat{\Omega}$, we can estimate $u(z,t)$ using the Feynman-Kac formula for the solution of (2.15):

$$u(z,t) = E\left[e^{\int_0^t \zeta(t-s,u(Z^{z,t}(s),t-s))ds} u_0(Z^{z,t}(t))\right], \tag{2.16}$$

where the expectation is with respect to measure $P^{z,t}$. If τ is any stopping time, we also have

$$u(z,t) = E\left[e^{\int_0^{t\wedge\tau} \zeta(t-s,u(Z^{z,t}(s),t-s))ds} u(Z^{z,t}(t-(t\wedge\tau)), t-(t\wedge\tau))\right], \tag{2.17}$$

where $t \wedge \tau = \min(t,\tau)$. Therefore, we can estimate u by carefully choosing stopping times and restricting the expectation to certain sets of paths. The exponential term inside the expectation (2.17) will be large when the path $Z^{z,t}(s)$ passes through regions where u is small and the reaction rate is large. On the other hand, if $u(Z^{z,t}(t-(t\wedge\tau)), t-(t\wedge\tau))$ is too small, then the expectation as a whole may be small.

Now we follow the ideas of Freidlin [33] (see p. 494). For $s \in R$, define the set

$$\Psi(s) = \{c \in R^n |\, S(c) - f'(0) = s\} \quad \text{and} \quad \underline{\Psi}(s) = \{c \in R^n |\, S(c) - f'(0) \le s\}.$$

For any $\delta > 0$ and $T > 1$, define

$$\Gamma_T = \left([\underline{\Psi}(\delta) \times \{1\}] \cup [\bigcup_{1 \le t \le T} (t\Psi(\delta)) \times \{t\}] \right).$$

This defines the boundary of a region that spreads outward in z, linearly in t. Outside this region u may be close to zero, but on the boundary of this region, we have the crucial lower bound from Lemma 2.3.1:

$$u(z,s) \ge e^{-2\delta t} \quad \text{for all} \quad (z,s) \in \Gamma_t \tag{2.18}$$

for t sufficiently large.

Let K be a compact set $K \subset \{c \in R^n |\, S(c) - f'(0) < 0\}$ and $z = ct$ for some $c \in K$. For $h \in (0,1)$, $t, \eta > 0$, define the Markov times

$$\begin{aligned}
\sigma_h(t) &= \min\{s \in [0,t] |\, u(Z^{z,t}(s), t-s) \ge h\}, \\
\sigma_\Gamma(t) &= \min\{s \in [0,t] |\, (Z^{z,t}(s), t-s) \in \Gamma_t\}, \\
\tau_\eta(t) &= \min\{s \in [0,t] |\, |Z^{z,t}(s) - z| > \eta t\}, \\
\hat{\sigma}(t) &= \sigma_h(t) \wedge \sigma_\Gamma(t).
\end{aligned}$$

(We set these variables equal to $+\infty$ if the sets on the right are empty.) Using (2.17) with the stopping time $\hat{\sigma}(t)$ we express $u(z,t)$ as

$$u(z,t) = E[e^{\int_0^{t\wedge\hat{\sigma}} \zeta(t-s,u(Z^{z,t}(s),t-s))ds} \times \\ \times u\left(Z^{z,t}(t-(t\wedge\hat{\sigma})), t-(t\wedge\hat{\sigma})\right)(\chi_{A_1}+\chi_{A_2}+\chi_{A_3})], \tag{2.19}$$

where A_1, A_2, and A_3 are the disjoint sets

$$\begin{aligned} A_1 &= \{\omega|\ \sigma_h(t) \le t\}, \\ A_2 &= \{\omega|\ \sigma_h(t) > t,\ \sigma_\Gamma(t) \ge rt\}, \\ A_3 &= \{\omega|\ \sigma_h(t) > t,\ \sigma_\Gamma(t) < rt\} \end{aligned}$$

for some $r \in (0,1)$ so be chosen. Note that $P(A_1) = P^{z,t}(\sigma_h(t) \le t)$ is the probability that a particle starting at $z = ct$ will encounter the "hot region", $u \ge h$, at or before time t.

Because the sets A_1, A_2, and A_3 are disjoint, the expectation (2.19) splits into three integrals. The first integral, over A_1, can be bounded below by

$$E\left[e^{\int_0^{t\wedge\hat{\sigma}} \zeta(t-s,u(Z^{z,t}(s),t-s))ds} u(Z^{z,t}(t-(t\wedge\hat{\sigma})), t-(t\wedge\hat{\sigma}))\chi_{A_1}\right] \ge \\ \ge hP(A_1). \tag{2.20}$$

since $\zeta \ge 0$. The second integral, over A_2, can be bounded below by

$$E\left[e^{\int_0^{t\wedge\hat{\sigma}} \zeta(t-s,u(Z^{z,t}(s),t-s))ds} u(Z^{z,t}(t-(t\wedge\hat{\sigma})), t-(t\wedge\hat{\sigma}))\chi_{A_2}\right] \ge \\ \ge e^{-2\delta t} e^{\zeta_h rt} P(A_2). \tag{2.21}$$

Combining (2.20) and (2.21) we have

$$u(z,t) \geq hP(A_1) + e^{-2\delta t + \zeta_h rt} P(A_2). \tag{2.22}$$

If we chose δ to be small, depending on h and r, then $-2\delta t + \zeta_h rt > 0$. Since $u(z,t) \leq 1$ for all (z,t), (2.22) then implies that $P(A_2) \to 0$ exponentially fast, if δ is small. Therefore, if we can also show that $P(A_3) \to 0$, then $P(A_1) \to 1$, and (2.22) implies the desired result (2.7), since h can be chosen arbitrarily close to 1.

The compact set K is bounded away from the boundary of $\Psi(0) \subset \Psi(\delta)$, so we can choose η small and then $r \in (0,1)$ sufficiently small so that

$$rt < \sigma_\Gamma(t) \leq t - 1$$

whenever $\tau_\eta(t) > t$. In other words, the trajectory $Z^{z,t}(s)$ stays in the set Γ_t for at least some fixed proportion of the interval $[0,t]$. Therefore,

$$P(A_3) \leq P(\sigma_\Gamma(t) < rt) \leq P(\tau_\eta(t) \leq t).$$

By Corollary 2.3.3,

$$\sup_{z \in R^n} P^{z,t}\left(\tau_\eta(t) < t\right) \to 0 \tag{2.23}$$

as $t \to \infty$, for all $\eta > 0$, except on a set of $Q-$measure zero. Hence $P(A_3) \to 0$, uniformly over $c \in K$. This completes the proof of Theorem 2.1.3. $\square$

We note that the main difficulty in extending Freidlin's argument for the periodic case is the manner in which the estimates (2.18) and (2.23) are obtained. In [33], estimate (2.23) followed from the uniform boundedness of the field B and the independence of B with respect to time, properties that we do not have in the present case.

2.3.1 Proof of Lemma 2.3.1

The main issue in proving the estimate of Lemma 2.3.1 (and thus the lower bound (2.18)) is whether the random variables

$$\eta_z^t(\kappa t) = \frac{z - Z^{z,t}(\kappa t)}{\kappa t} \tag{2.24}$$

satisfy a large deviation principle with a convex rate function $S(c)$, almost surely with respect to Q. The variable $\eta_z^t(\kappa t)$ is the average velocity of a trajectory over time interval $[0, \kappa t]$.

Definition 2.3.1. For fixed $\hat{\omega} \in \hat{\Omega}$, the random variables $\eta_z^t(\kappa t)$ satisfy a large deviation principle with a convex rate function $S(c)$ if there exists a convex function $S(c)$, independent of $z \in R^n$, such that

(i) For each $s \geq 0$, the set $\Phi(s) = \{c \in R^n | \; S(c) \leq s\}$ is compact.

(ii) For any $\delta, h > 0$, there exists $t_0 > 0$ such that for all $t > t_0$

$$P^{z,t}\left(d(\eta_z^t(\kappa t), \Phi(s)) > \delta\right) \leq e^{-\kappa t(s-h)}.$$

(iii) For any $\delta, h > 0$, there exists $t_0 > 0$ such that for all $t > t_0$

$$P^{z,t}\left(\eta_z^t(\kappa t) \in U_\delta(c)\right) \geq e^{-\kappa t(S(c)+h)}. \tag{2.25}$$

If such a function $S(c)$ exists, it might depend on the parameter $\kappa \in (0, 1]$, and it might depend on $\hat{\omega} \in \hat{\Omega}$. However, we will show that

Theorem 2.3.4. *Almost surely with respect to Q, the random variables $\eta_z^t(\kappa t)$ satisfy a large deviation principle (with respect to $P^{z,t}$) with a convex rate function $S(c)$ that is independent of κ and $\hat{\omega} \in \hat{\Omega}$.*

We postpone the proof for the moment while we finish the proof of Lemma 2.3.1. By Proposition 2.1.1, the quantity $\mu(\lambda)$ is well defined and is almost surely constant with respect to Q for $\lambda \in R^n$, independently of κ. Since, by our assumption of Theorem 2.3.4, the variables $\eta_z^t(\kappa t)$ have a convex rate function, it follows (see Section 5.1 of [36]) that the rate function $S(c)$ is the same convex function defined by (2.6):

$$S(c) = \sup_{\lambda \in R^n} [c \cdot \lambda - \mu(\lambda)]. \tag{2.26}$$

Thus, our use of the notation $S(c)$ in Definition 2.3.1, Theorem 2.3.4, and (2.6) anticipates this equivalence. The characterization (2.26) does not hold if the rate function is not convex, in which case the Legendre transform of μ is equal to the convex envelope of the rate function. Let us emphasize that $S(c)$ is independent of $\kappa \in (0, 1]$ and $\hat{\omega} \in \hat{\Omega}$, although the constants t_0 in Definition 2.3.1 may depend on $\kappa, \hat{\omega}$.

Now, by definition of $S(c)$,

$$\liminf_{t\to\infty} \frac{1}{\kappa t} \log \inf_{z \in R^n} P^{z,t} \left\{\eta_z^t(\kappa t) \in U_\delta(c)\right\} \geq -S(c) > -\infty, \tag{2.27}$$

and the lower bound (2.12) of Lemma 2.3.1 is

$$\liminf_{t\to\infty} \frac{1}{t} \log \inf_{c \in K} u(ct, t) \geq f'(0) - \max_{c\in K} S(c). \tag{2.28}$$

To prove the lower bound we now use the Feynman-Kac formula to relate (2.27) to (2.28), as in the arguments of Freidlin in Lemma 7.3.2 in [33]. The compactness of K implies that it suffices to show that given any $\epsilon > 0$, and any c for which $S(c) - f'(0) > 0$,

$$\liminf_{t\to\infty} \left(\frac{1}{t}\log \inf_{\tilde{c}\in U_\delta(c)} u(\tilde{c}t, t)\right) \geq f'(0) - S(c) - \epsilon \tag{2.29}$$

for $\delta > 0$ sufficiently small. Without loss of generality, we assume that the initial data is the characteristic function of a small ball centered at the origin:

$$u_0(z) \geq \chi_{U_\delta(0)}(z) \tag{2.30}$$

for some $\delta > 0$. We define q to be the limit on the left hand side of (2.29):

$$q = \liminf_{t\to\infty} \left(\frac{1}{t}\log \inf_{z\in U_{\delta t}(ct)} u(z, t)\right).$$

We also assume that $S(c) - f'(0) > 0$.

Step 1: The first step is essentially the same as in [33]. Suppose for the moment that we known q is finite. By the representation (2.17) we have for any $\kappa \in (0, 1]$

$$\inf_{\tilde{c}\in U_\delta(c)} u(t\tilde{c}, t) \geq \inf_{\tilde{c}\in U_\delta(c)} E\left[e^{\int_0^{\kappa t}\zeta(t-s,u(Z(s),t-s))ds} u(Z(t-\kappa t), t-\kappa t)\chi_A\right] \tag{2.31}$$

for any set $\mathcal{F}_{s\leq t}$-measurable set A. Recall that when $u \leq h$, the reaction rate $\zeta(u)$ is bounded below by $\zeta_h > 0$. If we choose A to be the set of paths satisfying both

$$Z^{z,t}(\kappa t) \in U_{(1-\kappa)\delta t}((1-\kappa)tc) \tag{2.32}$$

and

$$u(Z^{z,t}(s), t-s) \leq h \ \text{ for all } \ s \in [0, \kappa t], \tag{2.33}$$

then from (2.31) and the assumption that q is finite we have a lower bound

$$q \ \geq \ \zeta_h + \liminf_{t \to \infty} \frac{1}{\kappa t} \log \inf_{\tilde{c} \in U_\delta(c)} P(A), \tag{2.34}$$

provided that the limit on the right also exists and is finite.

Step 2: Now we bound the right hand side of (2.34) and show how it relates to (2.27). Since we have assumed that $S(c) - f'(0) > 0$, Theorem 2.1.2 implies that there is δ sufficiently small so that for any $h \in (0,1)$ there is a constant $t_0 > 0$, depending on h, such that

$$u(c't, t) \leq h \quad \text{for all } \ c' \in U_{6\delta}(c), \ t \geq t_0.$$

Now if $\kappa < 1/2$ and

$$\sup_{s \in [0, \kappa t]} |Z^{z,t}(s) - (t-s)c| \leq 3\delta t, \tag{2.35}$$

then (2.33) is achieved along such paths when $t > 2t_0$. Next, if $\tilde{c} \in U_\delta(c)$ is written $\tilde{c} = c + \delta e_1$ for some $e_1 \in R^n$ with $|e_1| < 1$, then define $\hat{c} = c + 2\delta e_1$. Then for any $|e_2| < 1$

$$\tilde{c}t - \kappa t \hat{c} + \kappa t \delta e_2 \in U_{(1-\kappa)\delta t}((1-\kappa)ct). \tag{2.36}$$

It follows that for each $\tilde{c} \in U_\delta(c)$ there is a $\hat{c} \in U_{2\delta}(c)$ such that (2.32) is achieved whenever $\eta_z^t(\kappa t) \in U_\delta(\hat{c})$, where η is defined by (2.24). This gives us a lower bound on $P(A)$ in terms of the $\eta_z^t(\kappa t)$, the average speed of a trajectory over $[0, \kappa t]$:

$$\inf_{\tilde{c}\in U_\delta(c), z=ct} P(A) \geq \tag{2.37}$$
$$\inf_{\hat{c}\in U_{2\delta}(c), z=\hat{c}t} P\left(\sup_{s\in[0,\kappa t]} |Z^{z,t}(s) - (t-s)c| \leq 3\delta t, \;\; \eta_z^t(\kappa t) \in U_\delta(\hat{c}) \right)$$

For κ sufficiently small, $\kappa < (2\delta)/(3\max(1, |c|))$, we see that

$$\sup_{\hat{c}\in U_{2\delta}(c), z=\hat{c}t} P\left(\sup_{s\in[0,\kappa t]} |Z^{z,t}(s) - (t-s)c| \geq 3\delta t \right) \leq$$
$$\leq \sup_{z\in R^n} P\left(\sup_{s\in[0,\kappa t]} |Z^{z,t}(s) - z| \geq \delta t/3 \right).$$

By Corollary 2.3.3 there are constants $K_1, K_2 > 0$ independent of κ such that (except possibly on a set of Q-measure zero)

$$\sup_{z\in R^n} P\left(\sup_{s\in[0,\kappa t]} |Z^{z,t}(s) - z| \geq \delta t/3 \right) \leq K_1 e^{-K_2\delta^2 t/\kappa} \tag{2.38}$$

for t sufficiently large, depending on $\hat{\omega}$. Therefore, for any $M > 0$, by choosing κ arbitrarily small, we can make $K_2\delta^2/\kappa^2 > M$, so that

$$\limsup_{t\to\infty} \frac{1}{\kappa t} \log(\sup_{z\in R^n} P\left(\sup_{s\in[0,\kappa t]} |Z^{z,t}(s) - (t-s)c| \geq 2\delta t \right)) \leq$$
$$\leq \lim_{t\to\infty} \frac{1}{\kappa t} \log(K_1 e^{-K_2\delta^2 t/\kappa}) \leq -M.$$

Therefore, from (2.34) and (2.37) we now see that for κ sufficiently small,

$$q \;\; \geq \;\; \zeta_h + \liminf_{t\to\infty} \frac{1}{\kappa t} \inf_{\hat{c}\in U_{2\delta}(c), z\in R^n} P\left(\eta_z^t(\kappa t) \in U_\delta(\hat{c})\right) \tag{2.39}$$

provided that the limit on the right is finite and bounded below, independently of κ. However, this follows immediately from Theorem 2.3.4 and the lower bound (2.27), since $S(c)$ is independent of κ. Then (2.29) follows by letting $h \to 0$ so that $\zeta_h \to f'(0)$.

Step 3: It remains to establish the initial claim that $q > -\infty$, almost surely with respect to Q. To see this, define for any $c \in R^n$

$$\hat{q}_\delta(c,t) = \inf_{z \in U_\delta(tc)} P^{z,t}\left(Z^{z,t}(t) \in U_\delta(0)\right), \tag{2.40}$$

a random variable over $\hat{\Omega}$. We will show that for any bounded set $\Lambda \subset R^n$, there is a finite constant $K_1 > 0$ such that the limit

$$\liminf_{t \to \infty} \frac{1}{t} \log \hat{q}_\delta(c,t) \geq -K_1 \tag{2.41}$$

holds uniformly over $c \in \Lambda$. This immediately implies that $q > -\infty$.

For $z \in U_\delta(ct)$, let us write $X^{z,t}(s)$ as

$$X^{z,t}(s) = x + I^{z,t}(s) + W_1^0(s)$$

where $I^{z,t}$ is the first integral term in (2.3) and $W_1^0(0) = 0$. Note that the integral $I^{z,t}$ is independent of W_1, due to the shear structure of the flow. For simplicity of notation, we will write $W_2(t) \in U_\delta(0)$ to mean that $|W_2(t)| < \delta$, even though $U_\delta(0)$ generally denotes an n-dimensional ball. First, we claim that for $z \in U_\delta(tc)$

$$P\left(Z_t^{z,t} \in U_\delta(0) |\, W_2^y(t) \in U_{\delta/2}(0)\right) \geq e^{-(3|c|^2+1)t} \tag{2.42}$$

for t sufficiently large. For $\hat{\omega} \in \Omega$ fixed, let $M > 0$ and define the set

$$A_M = A_M(t) = \{w \in \Omega \mid \sup_{z \in D, s \in [0,t]} |I^{z,t}(s)| \leq Mt\}.$$

Using the fact that $W_1(s)$ and $I^{z,t}(s)$ are independent, we see that for $z \in U_\delta(tc)$,

$$\begin{aligned}
&P(Z_t^{z,t} \in U_\delta(0) |\, W_2^y(t) \in U_{\delta/2}(0)) \geq \\
&\quad \geq P(W_1^0(t) \in U_{\delta/2}(0) - I^{z,t}(t) - x |\, W_2^y(t) \in U_{\delta/2}(0)) \\
&\quad \geq P(W_1^0(t) \in U_{\delta/2}(0) - I^{z,t}(t) - x,\ A_M |\, W_2^y(t) \in U_{\delta/2}(0)) \\
&\quad \geq \inf_{|\hat{e}_1|,|\hat{e}_2| \leq 1} P(W_1^0(t) \in U_{\delta/2}(0) + \hat{e}Mt + ct + \delta\hat{e}_2) P(A_M |\, W_2^y(t) \in U_{\delta/2}(0)) \\
&\quad \geq \frac{\delta}{\sqrt{2\pi t}} e^{-\frac{(Mt+|c|t+\delta)^2}{2t}} P(A_M |\, W_2^y(t) \in U_{\delta/2}(0)). \qquad (2.43)
\end{aligned}$$

By Lemma 2.3.2, $P(A_M |\, W_2^y(t) \in U_{\delta/2}(0)) \geq 1/2$ for t sufficiently large, depending on $\hat{\omega}$ and M. Moreover, if we chose $M = \max(1, |c|)$, then $P(A_M |\, W_2^y(t) \in U_{\delta/2}(0)) \geq 1/2$ for t sufficiently large, independent of c. Using this in (2.43) establishes (2.42), for t sufficiently large. Now, we can bound $\hat{q}_\delta$:

$$\begin{aligned}
\hat{q}_\delta(c,t) &= \inf_{z \in U_\delta(tc)} P\left(Z_t^{z,t} \in U_\delta(0)\right) \\
&\geq \inf_{z \in U_\delta(tc)} P\left(Z_t^{z,t} \in U_\delta(0),\ W_2(t) \in U_{\delta/2}(0)\right) \\
&= \inf_{z \in U_\delta(tc)} E\left[\chi_{W_2(t) \in U_{\delta/2}(0)} P(Z_t^{z,t} \in U_\delta(0) |\, W_2(t) \in U_{\delta/2}(0))\right] \\
&\geq \inf_{z \in U_\delta(tc)} E\left[\chi_{W_2(t) \in U_{\delta/2}(0)} e^{-(3|c|^2+1)t}\right] \\
&\geq e^{-(3|c|^2+1)t} \inf_{z \in U_\delta(tc)} P\left(W_2(t) \in U_{\delta/2}(0)\right) \geq e^{-(4|c|^2+1)t} \qquad (2.44)
\end{aligned}$$

for t sufficiently large. Therefore,

$$\liminf_{t \to \infty} \frac{1}{t} \log \hat{q}_\delta(c,t) \geq -(4|c|^2 + 1) \qquad (2.45)$$

is finite almost surely with respect to Q. For any bounded set $\Lambda \subset R^n$, we can choose K_1 to be

$$K_1 = 1 + \sup_{c\in\Lambda} 4|c|^2 < \infty. \tag{2.46}$$

This establishes the claim (2.41). Having shown that q is finite, we have completed the proof of Lemma 2.3.1. $\square$

For use in the next section, we now show that for all $t > 0$, $\log(\hat{q}_\delta(c,t))$ is integrable with respect to Q. Note that bound (2.44) holds for t sufficiently large, depending on $\hat{\omega}$, so more work is needed in order to establish the integrability of $\log(\hat{q}_\delta(c,t))$.

Lemma 2.3.5. *For each $c \in R^n$,*

$$\sup_{t>1} E_Q[|\frac{1}{t}\log \hat{q}_\delta(c,t)|] < \infty. \tag{2.47}$$

Proof: Using (2.43) we see that

$$\begin{aligned}
\frac{1}{t}\log \hat{q}_\delta(c,t) &\geq \frac{1}{t}\log P\left(Z_t^{z,t} \in U_\delta(0),\ W_2(t) \in U_{\delta/2}(0)\right) \geq \\
&\geq \frac{1}{t}\log\left(P\left(W_2(t) \in U_{\delta/2}(0)\right)\frac{\delta}{\sqrt{2\pi t}}e^{-\frac{((1+2|c|)t+\delta)^2}{t}}P(A_M|\ W_2^y(t) \in U_{\delta/2}(0))\right) \\
&\geq -C_1 + \frac{1}{t}\log\left(e^{-\frac{((1+2|c|)t+2\delta)^2}{t}}P(A_M|\ W_2^y(t) \in U_{\delta/2}(0))\right)
\end{aligned}$$

for a constant $C_1 > 0$ depending only on c and δ, for $t \geq 1$. This constant is uniformly bounded for c in a bounded set and $\delta > 0$ fixed. Let $\hat{g}$ be the term inside the logarithm:

$$\hat{g} = e^{-\frac{(Mt+|c|t+2\delta)^2}{2t}}P(A_M|\ W_2^y(t) \in U_{\delta/2}(0)),$$

a random variable with respect to Q. Then for $\alpha \geq 2C_1$,

$$\begin{aligned} Q\left(\frac{1}{t}\log \hat{q}_\delta(c,t) \leq -\alpha\right) &\leq Q\left(\frac{1}{t}\log \hat{g} \leq -\alpha/2\right) \\ &= Q\left(\hat{g} \leq e^{-\alpha t/2}\right) \qquad (2.48) \\ &= Q\left(P(A_M|\, W_2^y(t) \in U_{\delta/2}(0)) \leq e^{-\alpha t/2} e^{\frac{(Mt+|c|t+2\delta)^2}{2t}}\right). \end{aligned}$$

Also, from Lemma 2.3.2,

$$Q\left(P(A_M|\, W_2^y(t) \in U_{\delta/2}(0)) \leq 1 - e^{-K_2 M^2 t/2}\right) \leq K_1 e^{-K_2 M^2 t/2}. \qquad (2.49)$$

It is easy to see that there exist constants $K_3, K_4 > 0$ independent of t such that whenever $t \geq 1$, $M = K_3\sqrt{\alpha}$, and $\alpha \geq K_4|c|^2$, we have

$$e^{-\alpha t/2} e^{\frac{(Mt+|c|t+2\delta)^2}{2t}} \leq 1/2 \leq 1 - e^{-K_2 M^2 t/2}.$$

By combining (2.48) and (2.49), we now conclude that

$$Q\left(\frac{1}{t}\log \hat{q}_\delta(c,t) \leq -\alpha\right) \leq K_1 e^{-K_2 K_3^2 \alpha t/2} \qquad (2.50)$$

whenever $\alpha \geq K_4|c|^2$ and $t \geq 1$. It follows that for $t \geq 1$

$$\begin{aligned} E_Q[|\frac{1}{t}\log \hat{q}_\delta(c,t)|] &= \int_0^\infty Q\left(|\frac{1}{t}\log \hat{q}_\delta(c,t)| \geq \alpha\right) d\alpha \\ &\leq K_4|c|^2 + \int_{K_4|c|^2}^\infty K_1 e^{-K_2 K_3^2 \alpha t/2}\, d\alpha < \infty. \qquad (2.51) \end{aligned}$$

This is bounded uniformly in t, for $t \geq 1$. □

2.3.2 Proof of Large Deviation Estimates

In this section we prove Theorem 2.3.4. We work first with the case $\kappa = 1$. For $c \in R^n$ and $0 \leq r < s < t$, define the probability

$$q_\delta^z(c,r,t) = P^{z,t}(z - Z^{z,t}(t-r) \in U_{\delta(t-r)}(c(t-r))) = P^{z,t}(\eta_z^t(t-r) \in U_\delta(c))$$

and

$$\begin{aligned} q_\delta^+(c,r,t) &= \sup_{z\in D} q_\delta^z(c,r,t) \\ q_\delta^-(c,r,t) &= \inf_{z\in D} q_\delta^z(c,r,t). \end{aligned}$$

The quantity $q_\delta^z(c,r,t)$ is the probability that a trajectory should have average velocity sufficiently close to c, over a given time interval. This probability depends on the starting point z, so q_δ^+ and q_δ^- are the maximum and minimum possible probabilities. The quantities q_δ^z, q_δ^+, and q_δ^- also depend on $\hat{\omega}\in\hat{\Omega}$, but we will use the sub-additive ergodic theorem to show that $(1/t)\log q_\delta^-(c,0,t)$ converges to a finite constant, $-S_\delta(c)$, almost surely with respect to Q. From these constants we will recover the desired rate function $S(c)$ as the limit of $S_\delta(c)$ as $\delta\to 0$. Then, we will derive a Harnack-type inequality to compare $(1/t)\log q_\delta^-(c,0,t)$ and $(1/t)\log q_\delta^+(c,0,t)$ and show that $S(c)$ satisfies the requirements for Definition 2.3.1. The same analysis will extend to the case of $\kappa<1$.

Define the events

$$\begin{aligned} A &= \left\{\omega\in\Omega \middle|\ z - Z^{z,t}(t-r)\in U_{\delta(t-r)}(c(t-r))\right\} = \left\{\eta_z^t(t-r)\in U_\delta(c)\right\} \\ B &= \left\{\omega\in\Omega \middle|\ z - Z^{z,t}(t-s)\in U_{\delta(t-s)}(c(t-s))\right\} = \left\{\eta_z^t(t-s)\in U_\delta(c)\right\}. \end{aligned}$$

Note that event B is $\mathcal{F}^t_{s'\leq\tau}$ measurable for any $\tau\geq t-s$. Using the Markov property of the Wiener process, we find that $\log(q_\delta^-(c,s,t))$ is super-additive for each $c \in R^n$ since

$$
\begin{aligned}
q_\delta^-(c,r,t) &= \inf_y P(A) \geq \inf_y P(A \cap B) \\
&\geq \inf_z P(\{-Z^{z,t}(t-r) + Z^{0,z,t}(t-s) \in U_{\delta(r-s)}(c(s-r))\} \cap B) \\
&= \inf_z E\left[\chi_B P[\{-Z^{z,t}(t-r) + Z^{z,t}(t-s) \in U_{\delta(s-r)}(c(s-r))\} \mid \mathcal{F}_{t-s}]\right] \\
&= \inf_z E\left[\chi_B P[\{-Z^{z,t}(t-r) + Z^{z,t}(t-s) \in U_{\delta(s-r)}(c(s-r))\} \mid Z^{z,t}(t-s)]\right] \\
&\geq \inf_z E\left[\chi_B \inf_z P[\{z - Z^{z,s}(s-r) \in U_{\delta(s-r)}(c(s-r))\}]\right] \\
&= \inf_z P(\{z - Z^{z,s}(s-r) \in U_{\delta(s-r)}(c(s-r))\}) \inf_z P(B) \\
&= q_\delta^-(c,r,s) q_\delta^-(c,s,t).
\end{aligned}
$$

Also, due to the stationarity of B with respect to t,

$$
\begin{aligned}
\tau_h q_\delta^-(c,r,t) &= \tau_h \inf_y P(z - Z^{z,t}(t-r) \in U_{\delta(t-r)}(c(t-r))) \\
&= \inf_y P(z - Z^{z,t+h}(t-r) \in U_{\delta(t-r)}(c(t-r))) \\
&= q_\delta^-(c, r+h, t+h).
\end{aligned}
$$

For any $\epsilon > 0$, we can bound q below by translating in z and using (2.40):

$$
q_\delta^-(c,r,t) \geq \tau_r \hat{q}_\epsilon(c, t-r) = \inf_{z \in U_\epsilon(ct)} P(Z^{z,t}(t-r) \in U_\epsilon(cr)) \tag{2.52}
$$

if $\epsilon < \delta(t-r)$. Hence, $\log(q_\delta^-(c,r,t))$ is integrable by (2.51). Kingman's ergodic theorem [50] now implies that the limit

$$
\lim_{n\to\infty} \frac{1}{n} \log q_\delta^-(c,0,n) = \sup_{n>0} \frac{1}{n} E_Q[\log q_\delta^-(c,0,n)] = -S_\delta(c) \tag{2.53}
$$

exists and is a finite constant, Q-a.s, because of the ergodicity assumption A2.

To extend the convergence in (2.53) to continuous time, we employ a technique from [3] (see the proof of Theorem 2.5 therein). Let

$$g(\hat{\omega}) = \sup_{\substack{r,t \in [0,2] \\ |r-t| \geq 1}} |\log(q_\delta^-(c,r,t))|.$$

Let $\Upsilon(\hat{\omega}) = \sup_{z \in D, t \in [0,2]} |B(z,t)|$. Then for all $0 \leq r < t \leq 2$, we can bound

$$\sup_{y \in D} \Big| \int_0^{t-r} b(W_2^y(\tau), t-\tau) d\tau \Big| \leq \Upsilon |t-r|$$

independently of the realization of W_2^y. As in (2.43),

$$\begin{aligned} &P(z - Z^{z,t}(t-r) \in U_{\delta(t-r)}(c(t-r))) \\ &\qquad \geq \inf_{|\hat{e}_1|,|\hat{e}_2| \leq 1} P(W_1^0(t-r) \in U_{\delta(t-r)/2}(0) + \hat{e}_1 \Upsilon (t-r) + (c + \delta \hat{e}_2)(t-r)) \times \\ &\qquad\qquad \times P(W_2^0(t-r) \in U_{\delta(t-r)/2}(0)) \\ &\qquad \geq \frac{\delta |t-r|}{\sqrt{2\pi |t-r|}} e^{-\frac{(t-r)^2(\Upsilon + |c| + \delta)^2}{(t-r)}} \frac{\delta |t-r|}{\sqrt{2\pi |t-r|}} e^{-\frac{(t-r)^2(|c|+\delta)^2}{(t-r)}}. \end{aligned}$$

Therefore, since $r, t \in [0,2]$ and $|r-t| \geq 1$ in the definition of $g(\hat{\omega})$,

$$0 \leq g(\hat{\omega}) \leq K_1 + K_2 \Upsilon^2$$

for some constants $K_1, K_2 > 0$ that depend on δ and c. Hence $g(\hat{\omega})$ is integrable with respect to Q, since Υ^2 is integrable by the Borell inequality. By the super-additivity of $\log q_\delta^-(c,r,t)$,

$$\log q_\delta^-(c,0,n-1) - \tau_{n-1} g \leq \log q_\delta^-(c,0,t) \leq \log q_\delta^-(c,0,n+2) + \tau_n g \qquad (2.54)$$

whenever $t \in (n, n+1)$, $n \in Z$. The ergodic theorem implies that

$$\frac{1}{N} \sum_{n=1}^{N} \tau_n g \to E[g] < \infty \qquad (2.55)$$

almost surely. Therefore, $\frac{1}{n}\tau_n g \to 0$ almost surely as $n \to \infty$. It now follows from (2.54) that the limit along continuous time

$$\lim_{t\to\infty} \frac{1}{t} \log q_\delta^-(c, 0, t) = -S_\delta(c) \tag{2.56}$$

holds almost surely with respect to Q.

Now we extend this conclusion to the case of $\kappa < 1$, as well. If $\kappa \in (0, 1)$ and $\delta > 0$, the stationarity of $b(y, t)$ implies that

$$\frac{1}{\kappa t} \log \inf_z P\left(\eta_z^t(\kappa t) \in U_\delta(c)\right) = \frac{1}{\kappa t} \log q_\delta^-((1-\kappa)t, t) \to -S_\delta(c) \tag{2.57}$$

in distribution (with respect to Q) as $t \to \infty$, but this does not immediately imply pointwise, almost-sure convergence. However, the collection of sets $\{[(1-\kappa)t, t]\}_{t\geq 0}$ is a regular family of sets in the sense of [3], since $0 \leq |[0, t]| \leq C|[(1-\kappa)t, t]|$ for all t, with $C = \frac{1}{\kappa}$. It now follows from Theorem 2.8 of [3] that $\lim_{n\to\infty} \frac{1}{n} \log q_\delta^-((1-\kappa)n, n)$ converges almost surely along any rational sequence. Therefore, (2.57) implies that, indeed, this limit is $-S_\delta(c)$

$$\lim_{n\to\infty} \frac{1}{n} \log q_\delta^-((1-\kappa)n, n) = -S_\delta(c), \quad Q - a.s., \tag{2.58}$$

Finally, this convergence can be extended to continuous time, using the same technique as in (2.54).

For each $c \in R^n$, $S_\delta(c)$ can be bounded above independently of $\delta > 0$ using (2.41) and (2.52). From the definition, it is clear that $S_\delta(c) \geq 0$ for all δ, and that

$$S_{\delta_1}(c_1) \leq S_{\delta_2}(c_2) \tag{2.59}$$

whenever $U_{\delta_1}(c_1) \supset U_{\delta_2}(c_2)$. In particular, $S_{\delta_1}(c) \le S_{\delta_2}(c)$ for $\delta_1 > \delta_2$, $c \in R^n$. Therefore, we define for each $c \in R^n$

$$S(c) = \lim_{\delta \to 0} S_\delta(c) = \sup_{\delta > 0} S_\delta(c) \in [0, +\infty).$$

This will be the rate function described in the theorem.

Lemma 2.3.6. *For all $\delta > 0$, the functions $S_\delta(c)$ are continuous and convex in c. Also, $S(c)$ is continuous and convex in c.*

Proof: The continuity and convexity of $S(c)$ follows immediately from the fact that it is the finite, pointwise limit of the continuous, convex functions $S_\delta(c)$. The convexity of $S_\delta(c)$ follows from the Markov property of the process $Z^{z,t}$, as follows.

Let $p \in [0, 1]$ and $c_0 = pc_1 + (1-p)c_2$. Let $t > 0$ and denote $t_1 = pt$, $t_2 = (1-p)t$. Then we see that

$$\begin{aligned}
q_\delta^-(c_0, 0, t) &= \inf_z P\left(z - Z^{z,t}(t) \in U_{\delta t}(c_0 t)\right) \ge \\
&\ge \inf_z P\left(z - Z^{z,t}(t) \in U_{\delta t}(c_0 t)\ , \ \ z - Z^{0,y,t}(t_1) \in U_{\delta t_1}(c_1 t_1)\right) \\
&\ge \inf_z P\left(z - Z^{z,t-t_1}(t_2) \in U_{\delta t_2}(c_2 t_2)\right) \inf_z P\left(z - Z^{z,t}(t_1) \in U_{\delta t_1}(c_1 t_1)\right) \\
&= q_\delta^-(c_2, 0, t_2) q_\delta^-(c_1, t_2, t).
\end{aligned}$$

Hence

$$\begin{aligned}
-\frac{1}{t} \log q_\delta^-(c_0, 0, t) &\le \\
&\le -\frac{1}{t} \log q_\delta^-(c_2, 0, (1-p)t) - \frac{1}{t} \log q_\delta^-(c_1, (1-p)t, t)
\end{aligned} \tag{2.60}$$

By the preceding arguments, both terms on the right converge (Q-a.s.) as $t \to \infty$ to the constants $(1-p)S_\delta(c_2)$ and $pS_\delta(c_1)$. Therefore, we infer that

$$S_\delta(c_0) \leq (1-p)S_\delta(c_2) + pS_\delta(c_1).$$

So, $S_\delta(c)$ is convex and must also be continuous in c, since it is finite for every $c \in R^n$. $\square$

This establishes the existence and convexity of the function $S(c)$. Part (iii) of the Definition 2.3.1 follows from the definition of $S_\delta(c)$ and the fact that $S_\delta(c) \nearrow S(c)$.

To finish the proof of Theorem 2.3.4, we must establish a Harnack-type inequality to relate the probabilities

$$P(z - Z^{z,t}(t) \in U_{\delta t}(ct)) \quad \text{and} \quad P(z' - Z^{z',t}(t) \in U_{\delta t}(ct))$$

corresponding to different starting points $z, z' \in D$. This will allow us to remove the $\inf_z$ in the definition of q and $S(c)$ and to establish parts (i) and (ii) of Definition 2.3.1. Unfortunately, the quantity $\log q_\delta^+$ is not sub-additive or super-additive, so we cannot use the ergodic theorem to show that $\frac{1}{t}\log q_\delta^+ \to -S_\delta(c)$, almost surely, as is the case with $\log q_\delta^-$.

Note that it is not true that two trajectories starting close together will remain close. For example, suppose the flow is $B(z,t) = (\sin(y), 0)$. Then if $z = (0,0)$ and $z' = (0,\pi)$, the X components of the trajectories will satisfy

$$X^{z,t}(t) - X^{z',t}(t) = \int_0^t \sin(W_2(s)) - \sin(\pi + W_2(s))\, ds = 2\int_0^t \sin(W_2(s))\, ds,$$

which we expect will grow like $\sqrt{t}$. Nevertheless, the estimate we need must only relate the distributions of the two processes, not the individual trajectories for fixed realizations of W. We prove the following lemma:

Lemma 2.3.7. *There are constants $K_1, K_2, K_3, K_4 > 0$ such that for all $\kappa \in (0,1]$, $c \in R$, $\epsilon > 0$, and $\delta > 0$,*

$$\inf_z P\left(\eta_z^t(\kappa t) \in U_{(1+\epsilon)\delta}(c)\right) \geq K_4 e^{-K_3\epsilon\delta t} \sup_z P\left(\eta_z^t(\kappa t) \in U_\delta(c)\right) - K_1 e^{-K_3\epsilon^2\delta^2\kappa^2 t^2}$$

Proof of Lemma 2.3.7: For clarity we let $\kappa = 1$. Extension to $\kappa < 1$ is straightforward, as in the proof of Lemma 2.3.3. Because of the shear flow structure, $(Z^{z,t}(t) - z)$ and $\eta_z^t(t)$ are independent of x (where $z = (x, y)$), and the component $Y^{z,t}$ is just a Wiener process. These two facts will enable us to estimate the cost of switching the initial point from z to z'.

For $M > 0$ and $s \in (0, t)$ to be chosen, the Markov property of the process implies that

$$\begin{aligned} P(Z^{z,t}(t) \in U_{\delta t + 2M}(z + ct), |Z^{z,t}(s) - z| \leq M) = & \qquad (2.61) \\ = \int_{|\hat{z}-z|\leq M} \rho(z, t, \hat{z}, s) P^{\hat{z},s}(Z^{\hat{z},s}(t-s) \in U_{\delta t+2M}(z + ct))\, d\hat{z} & \\ \geq \int_{|\hat{z}-z|\leq M} \rho(z, t, \hat{z}, s) P^{\hat{z},s}(Z^{\hat{z},s}(t-s) \in U_{\delta t+M}(\hat{z} + ct))\, d\hat{z} & \end{aligned}$$

where $\rho(z, t, \hat{z}, r)$ denotes the transition density of the Markov process $Z^{z,t}(r)$. The term $P^{\hat{z},s}(Z^{\hat{z},s}(t-s) \in U_{\delta t+M}(\hat{z} + ct))$ inside the integral is independent of $\hat{x}$ (where $\hat{z} = (\hat{x}, \hat{y})$). Using this fact, we will bound the integral in (2.61) by first integrating over $\hat{x}$. Since $Y^{z,t}$ is just a Wiener process, the marginal distribution of ρ with respect to $\hat{y}$ is Gaussian:

$$\int_R \rho(z,t,\hat{z},s)d\hat{x} = \frac{1}{\sqrt{2\pi s}} e^{-\frac{|\hat{y}-y|^2}{2s}} = F(\hat{y}-y,s).$$

Therefore,

$$\int_{|x-\hat{x}|\le M} \rho(z,t,\hat{z},s)d\hat{x} = \frac{1}{\sqrt{2\pi|s|}} F(\hat{y}-y,s) - P^{z,t}(A_M|\, Y^{z,t}(s)=\hat{y})$$

where $A_M = \{\omega|\ |X^{z,t}(s)-x| \ge M\}$. This set will turn out to be very small. For $\epsilon > 0$, let $s = 1$ and $M = \epsilon\delta t/2$ (use $M = \epsilon\delta\kappa t/2$ when $\kappa < 1$). Therefore, integrating only in $\hat{x}$, we have

$$\int_{|\hat{z}-z|\le M} \rho(z,t,\hat{z},s)P^{\hat{z},s}(Z^{\hat{z},s}(t-s)\in U_{\delta t+M}(\hat{z}+ct))\,d\hat{z} \ge \tag{2.62}$$
$$\ge \int_{|\hat{y}-y|\le M/2} F(\hat{y}-y,s)P^{\hat{z},s}(Z^{\hat{z},s}(t-s)\in U_{\delta t+M}(\hat{z}+ct))\,d\hat{y} - G_1$$

where

$$G_1 = M\sup_{\hat{y}} P^{z,t}(A_M|\, Y^{z,t}(s)=\hat{y}).$$

Now we switch the initial point from z to z', such that $|y-y'| \le L$. Then, continuing from (2.62), we have

$$\int_{|\hat{y}-y|\le M/2} F(\hat{y}-y,s)P^{\hat{z},s}(Z^{\hat{z},s}(t-s)\in U_{\delta t+M}(\hat{z}+ct))\,d\hat{y} - G_1 \tag{2.63}$$
$$\ge Ce^{-M/2}\int_{|\hat{y}-y'|\le \frac{M}{2}-L} F(\hat{y}-y',s)P^{\hat{z},s}(Z^{\hat{z},s}(t-s)\in U_{\delta t+M}(\hat{z}+ct))\,d\hat{y} - G_1$$
$$\ge Ce^{-M/2}\int_{|\hat{z}-z'|\le \frac{M}{2}-L} \rho(z',t,\hat{z},s)P^{\hat{z},s}(Z^{\hat{z},s}(t-s)\in U_{\delta t+M}(\hat{z}+ct))\,d\hat{z} - G_1$$
$$\ge Ce^{-M/2}P(Z^{z',t}(t)\in U_{\delta t}(z'+ct), |Z^{z',t}(s)-z'| \le \frac{M}{2}-L) - G_1$$

Combining (2.61) and (2.63) we have

$$\begin{aligned} &P(Z^{z,t}(t) \in U_{\delta t+2M}(z+ct)) \geq && (2.64) \\ &\quad \geq Ce^{-M/2} P(Z^{z',t}(t) \in U_{\delta t}(z'+ct), |Z^{z',t}(s) - z'| \leq \frac{M}{2} - L) - G_1 \\ &\quad \geq Ce^{-M/2} P(Z^{z',t}(t) \in U_{\delta t}(z'+ct)) - Ce^{-M/2} G_2 - G_1 \end{aligned}$$

where

$$G_2 = \sup_{\hat{z}} P^{z,t}(A_{\frac{M}{2}-L}).$$

It follows from Lemma 2.3.2, that

$$G_1, G_2 \leq K_1 e^{-K_2 M^2} = K_1 e^{-K_2 \epsilon^2 \delta^2 t^2/4}$$

for t sufficiently large. Now the lemma follows from (2.64). $\square$

Since $\epsilon > 0$ is arbitrary, the lemma implies that

Corollary 2.3.8. *For all $\epsilon, \delta > 0$, $c \in R^n$, and $\kappa \in (0,1]$,*

$$\begin{aligned} -\lim_{t\to\infty} \frac{1}{\kappa t} \log q_\delta^-(c,(1-\kappa)t,t) &= S_\delta(c) \\ &\geq -\limsup_{t\to\infty} \frac{1}{\kappa t} \log q_\delta^+(c,(1-\kappa)t,t) \\ &\geq S_{(1+\epsilon)\delta}(c) \end{aligned}$$

almost surely with respect to Q.

Now, using this estimate, we can establish parts (i) and (ii) from Definition 2.3.1. From Lemma 2.3.3 there are constants $K_1, K_2 > 0$ such that for t sufficiently large,

$$P(|\eta_z^t(\kappa t)| \geq |c|) \leq K_1 e^{-K_2 |c|^2 t/\kappa}. \tag{2.65}$$

This implies that $\lim_{|c|\to\infty} S(c)/|c| = +\infty$. Hence, $\Phi(s)$ is a bounded set, for each $s \geq 0$. By continuity of $S(c)$, $\Phi(s)$ is compact. Let A be the set

$$A = \{c \in R^n | \, d(c, \Phi(s)) > \delta\}.$$

We must show that for any fixed $\delta > 0$, $h > 0$,

$$P(\eta_z^t(\kappa t) \in A) \leq e^{-\kappa t(s-h)}, \tag{2.66}$$

for $t > 0$ sufficiently large. Because of the bound (2.65), it suffices to show that (2.66) holds with A replaced by any compact subset A' of A (because $K_2|c|^2 > \kappa^2 s$ when $|c|$ is sufficiently large).

We have shown that for $c \in R^n$, there is a set $V(c) \subset \hat{\Omega}$ such that $Q(V) = 0$ and the convergence (2.56) holds for all $\hat{\omega} \in \hat{\Omega} \setminus V(c)$. To obtain convergence on a set independent of c, we define the set $\hat{V} \subset \hat{\Omega}$ by

$$\hat{V} = \bigcup_{c \in \mathbb{Q}^n} V(c), \tag{2.67}$$

which has measure zero. Therefore, for all $c \in \mathbb{Q}^n$, (2.56) holds for all $\hat{\omega} \in \hat{\Omega} \backslash \hat{V}$.

Now, we claim that we can choose $\epsilon > 0$ small enough so that $\epsilon < \delta$ and

$$\inf_{c' \in A'} S_\epsilon(c') > s - \frac{h}{2}. \tag{2.68}$$

If this were not so, then there must be a sequence $\epsilon_k \to 0$ and $\{c_k\} \subset A'$ such that $S_{\epsilon_k}(c_k) \leq s - h/2$. Because A' is compact, there must be a subsequence c_{k_n} converging to some $c_0 \in A'$. Since $c_0 \in A'$, $S_\epsilon(c_0) > s - h/4$ whenever ϵ is less than some $\epsilon_0 > 0$. However, $U_{\epsilon_k}(c_k) \subset U_{\epsilon_0}(c_0)$ for k sufficiently large. It follows from (2.59) that $S_{\epsilon_k}(c_k) > s - h/4$, for k sufficiently large. This is a contradiction, so the claim must hold.

Having chosen ϵ to satisfy (2.68), cover A' with a finite number of balls having size $\frac{\epsilon}{2}$:

$$A' \subset \bigcup_{n=1}^{N} U_{\epsilon/2}(c_n)$$

for some finite set $\{c_n\}_{n=1}^{N} \subset A' \cap \mathbb{Q}^n$. Therefore,

$$\begin{aligned} P(\eta_z^t(\kappa t) \in A') &\leq \sum_{n=1}^{N} P(\eta_z^t(\kappa t) \in U_{\epsilon/2}(c_n)). \\ &\leq \sum_{n=1}^{N} q_{\epsilon/2}^{+}((1-\kappa), t, c_n). \end{aligned}$$

Now we apply Corollary 2.3.8 and inequality (2.68) to conclude that for all $\hat{\omega} \in \hat{\Omega} \setminus \hat{V}$

$$\begin{aligned} &\limsup_{t\to\infty} \frac{1}{\kappa t} \log P(\eta_z^t(\kappa t) \in A') \leq \\ &\quad \leq \limsup_{t\to\infty} \frac{1}{\kappa t} \log \sum_{n=1}^{N} q_{\epsilon/2}^{+}((1-\kappa)t, t, c_n) \\ &\quad \leq -\inf_{c' \in A'} S_{2\epsilon/3}(c') < -(s - \frac{h}{2}) \end{aligned}$$

for t sufficiently large. Thus, (2.66) holds almost surely for t sufficiently large. This proves part (ii) of Definition 2.3.1 for $\kappa \in (0, 1]$ completes the proof of Theorem 2.3.4. $\square$

2.3.3 Estimates on $Z^{z,t}(s)$

In this section we derive some technical estimates on the process $Z^{z,t}(s)$ which follow from our assumptions on the field B and the Borell inequality for

Gaussian fields. In particular, we prove Lemma 2.3.3 which is needed for the bounds (2.23) and (2.38).

Let use first note that by changing variables $r = s - t$, $v = s + t$, it is easy to see from our assumptions on the process B that

$$\begin{aligned} \int_0^T \int_0^T \sup_{y_1, y_2} \Gamma(y_1, y_2, s, t)\, ds\, dt &\le 2 \int_0^{\sqrt{2}T} \int_0^{T/\sqrt{2}} \hat{\Gamma}(r)\, dr\, dv \\ &\le 2\sqrt{2} p_1 T \end{aligned} \tag{2.69}$$

and for $H \in [0, T]$,

$$\begin{aligned} \int_0^T \int_H^T \sup_{y_1, y_2} \Gamma(y_1, y_2, s, t)\, ds\, dt &\le \sqrt{2}|T - H| \int_0^{T/\sqrt{2}} \hat{\Gamma}(r)\, dr \\ &\le \sqrt{2}|T - H| p_1. \end{aligned} \tag{2.70}$$

Let $\rho(s) \in C([0, +\infty), \mathbb{R}^{n-1})$ with $\rho(0) = 0$ be fixed. For $y \in D$, define $\rho_y(s) = y + \rho(s)$. For fixed $t > 0$, the integral

$$f(y, s) = \int_0^s b(\rho_y(\tau), t - \tau)\, d\tau$$

is a Gaussian random field over $M = D \times [0, t]$, with respect to the measure Q. The Borell inequality for Gaussian fields states that if $\|f\| = \sup_{(y,s) \in M} f(y, s)$ is almost surely finite, then $E[\|f\|] < \infty$ and for any $u > 0$,

$$Q\left(\|f\| - E[\|f\|] > u\right) \le e^{-\frac{u^2}{2\sigma_t^2}} \tag{2.71}$$

where $\sigma_t^2 = \sup_{(y,s) \in M} E_Q[f^2]$ (see [2]). By (2.69), $\sigma_t^2 \le 2\sqrt{2} p_1 t$. So, using inequality (2.71), we can control deviations of $\|f\|$, if we bound the growth of $E[\|f\|]$.

Lemma 2.3.9. *There is a finite constant $C > 0$ such that*

$$E[\|f\|] \leq Ct^{1/2}. \tag{2.72}$$

Proof of Lemma 2.3.9: The expectation $E[\|f\|]$ can be bounded by the metric entropy relation

$$E[\|f\|] \leq C \int_0^{\delta} (\log N(\epsilon))^{1/2} \, d\epsilon$$

where $\delta = diam(M)/2$ in the metric

$$d((x,s),(y,z)) = E\left[(f(x,s) - f(y,z))^2\right]^{1/2}$$

and $N(\epsilon)$ is the minimum number of ϵ-balls required to cover M (see [2]). Using (2.69) and (2.70), a straightforward computation shows that

$$E\left[(f(x,s) - f(y,s))^2\right] \leq C|x-y|t$$

and

$$E\left[(f(y,s) - f(y,z))^2\right] \leq C|s-z|$$

for some finite constant C, independent of ρ. Therefore,

$$d((x,s),(y,z)) \leq C_1 \left(|s-z|\right)^{1/2} + C_2(|x-y|t)^{1/2},$$

and there is a constant C_3 independent of t and ϵ such that $d((x,s),(y,z)) \leq \epsilon$ whenever $|s-z| \leq C_3\epsilon^2$ and $|x-y| \leq \frac{C_4\epsilon^2}{t}$. For $\epsilon \in (0, diam(M)/2]$, we have the bound

$$N(\epsilon) \leq \max(C_5\frac{t^2}{\epsilon^4}, 1)$$

and

$$
\begin{aligned}
E[\|f\|] &\leq C \int_0^{C_5^{-1/4} t^{-1/2}} (\log(C_5 \frac{t^2}{\epsilon^4}))^{1/2}\, d\epsilon \\
&= C_6 t^{1/2} \int_0^1 (\log(\frac{1}{\epsilon^4}))^{1/2}, d\epsilon \leq C_7 t^{1/2}. \qquad (2.73)
\end{aligned}
$$

$\square$

Note that the constants depend on the assumed properties of the process b and the size of the domain D, but not on the particular function $\rho(s)$. If $u \geq 2C_7 t^{1/2}$, then by (2.71),

$$
Q\left(\|f\| > u\right) \leq e^{-(u - E\|f\|)^2/2\sigma_t^2} \leq e^{-u^2/8\sigma_t^2} \leq e^{-u^2/8p_1 t}.
$$

It now follows that

Lemma 2.3.10. *For any* $\eta > 0$ *and for* $t \geq t_0 = t_0(\eta) = (2C_7/\eta)^2$*,*

$$
Q\left(\sup_{y \in D,\, s \in [0,t]} \int_0^s b(y + \rho(\tau), t - \tau)\, d\tau > \eta t\right) \leq e^{-\eta^2 t/8p_1}
$$

for any $\rho \in C([0,\infty), R^n)$*,* $\rho(0) = 0$*.*

In applying this lemma, the continuous function ρ will be a realization of the Wiener process $W_2^y(s)$.

Lemma 2.3.11. *For* $\eta > 0$*,* $z \in R^n$*, define the Markov time*

$$
\tau_{\eta,z}(t) = \min\{s \geq 0 |\ |X^{z,t}(s) - x| \geq \eta t\}.
$$

with $\tau_{\eta,z}(t) = +\infty$ *if the set on the right is empty. Then there are constants* K_1, K_2 *such that*

$$
Q\left(P\left(\inf_{z \in R^n} \tau_{\eta,z}(t) \leq t\right) > e^{-K_2 \eta^2 t/2}\right) \leq K_1 e^{-K_2 \eta^2 t/2}
$$

for all $t > 0$.

Proof of Lemma 2.3.11: Note that for the Wiener process $W_1(s)$ with $W_1(0) = 0$

$$\begin{aligned} P\left(\sup_{s\in[0,t]} |W_1(s)| \geq \eta t\right) &\leq 2\sqrt{\frac{2}{\pi}} \int_{\eta\sqrt{t}}^{\infty} e^{-x^2/2}\, dx \\ &\leq K_1 e^{-\eta^2 t/2}. \end{aligned} \tag{2.74}$$

The point of the lemma is that at large times and almost surely with respect to Q, the process $X^{z,t}(s)$ also behaves like a Wiener process in the sense that a bound like (2.74) holds. By definition of $\tau_{\eta,z}(t)$,

$$P\left(\inf_{z\in R^n} \tau_{\eta,z}(t) \leq t\right) = P\left(\sup_{s\in[0,t],z\in R^n} |f_t(y,s) + W_1(s)| \geq \eta t\right).$$

Using Tchebyshev's inequality, (2.74), and Lemma 2.3.10 we see that for any $\eta > 0$, $\alpha > 0$:

$$\begin{aligned} Q\left(P\left(\inf_{z\in R^n} \tau_{\eta,z}(t) \leq t\right) > \alpha\right) &\leq \alpha^{-1} E_Q P\left(\sup_{s\in[0,t],z\in R^n} |f_t(y,s) + W_1(s)| \geq \eta t\right) \\ &= \alpha^{-1} E_P Q\left(\sup_{s\in[0,t],z\in R^n} |f_t(y,s) + W_1(s)| \geq \eta t\right) \\ &\leq \alpha^{-1} E_P Q\left(\sup_{s\in[0,t],z\in R^n} |f_t(y,s)| \geq \eta t/2\right) \\ &\quad + \alpha^{-1} P\left(\sup_{s\in[0,t],z\in R^n} |W_1(s)| \geq \eta t/2\right) \\ &\leq \alpha^{-1}(2e^{-\eta^2 t/32p_1} + K_1 e^{-\eta^2 t/8}) \leq \alpha^{-1} K_1 e^{-K_2\eta^2 t} \end{aligned}$$

for t sufficiently large, for some constants $K_1, K_2 > 0$. The result now follows from a choice of $\alpha = e^{-K_2\eta^2 t/2}$. □

Lemma 2.3.12. *There are constants $K_1, K_2 > 0$ such that, except on a set of Q-measure zero,*

$$\sup_{z \in R^n} P\left(\tau_{\eta,z}(t) \leq t\right) \leq K_1 e^{-K_2 \eta^2 t} \tag{2.75}$$

for t sufficiently large depending on $\hat{\omega}$ and η.

Proof: Note that this implies Lemma 2.3.2 in the case that $\kappa = 1$ and $\Lambda = R^n$. Lemma 2.3.11 and the Borel-Cantelli lemma imply that outside a set of Q-measure zero

$$P\left(\inf_{z \in R^n} \tau_{\eta,z}(k) \leq k\right) \leq e^{-K_2 \eta^2 k/2} \tag{2.76}$$

if $k \in \mathbb{Z}$ is sufficiently large. Now we want to extend this to all real t sufficiently large. Let $t \in [k, k+1]$, $t = k + \tau$, $\tau \in [0, 1]$.

$$\begin{aligned}
& \sup_{z \in R^n} P\left(\sup_{s \in [0,t]} |X^{z,t}(s) - x_0| \geq t\eta\right) \\
\leq\ & \sup_{z \in R^n} P\left(\sup_{s \in [0,\tau]} |X^{z,t}(s) - x_0| \geq t\eta/2\right) + \\
& \quad + \sup_{z \in R^n} P\left(\sup_{s \in [\tau,t]} |X^{z,t}(s) - X^{z,t}(\tau)| \geq t\eta/2\right)
\end{aligned}$$

By the Markov property, this is bounded by

$$\begin{aligned}
\leq\ & \sup_{z \in R^n} P\left(\sup_{s \in [0,\tau]} |X^{z,t}(s) - x_0| \geq t\eta/2\right) + \\
& \quad + \sup_{\bar{z} \in R^n} P\left(\sup_{s \in [0,k]} |X^{\bar{z},k}(s) - \bar{x}_0| \geq t\eta/2\right) \leq \\
\leq\ & P\left(\sup_{z \in R^n, t \in [k,k+1], s \in [0,1]} |X^{z,t}(s) - x_0| \geq k\eta/2\right) + \\
& \quad + P\left(\sup_{\bar{z} \in R^n, s \in [0,k]} |X^{\bar{z},k}(s) - \bar{x}_0| \geq k\eta/2\right)
\end{aligned} \tag{2.77}$$

By (2.76), the second term on the right side of (2.77) is bounded (Q-a.s.) by

$$P\left(\sup_{\bar{z}\in R^n, s\in[0,k]} |X^{\bar{z},k}(s)-\bar{x}_0| \geq k\eta/2\right) \leq e^{-K_3\eta^2 k} \tag{2.78}$$

for $k \in \mathbb{Z}$ sufficiently large. To bound the other term in (2.77), it suffices to show that

$$P\left(\sup_{y\in D, r\in[0,1], s\in[0,1]} |f_k(y,s,r)| \geq k\eta/2\right) \leq e^{-K_4\eta^2 k} \tag{2.79}$$

for $k \in \mathbb{Z}$ sufficiently large, where

$$f_k(y,s,r) = \int_0^s b(W_2^y(\tau), r+k-\tau)\, d\tau.$$

Note that $f_k(y,s,r)$ is a centered Gaussian field (with respect to Q) over $D\times[0,1]\times[0,1]$, and its distribution is invariant with respect to $k>0$, due to the stationarity of $b(y,t)$. For any fixed path $W_2^y(\omega)$, the Borell inequality implies that for k sufficiently large

$$Q\left(\sup_{y\in D, r\in[0,1], s\in[0,1]} |f_k(y,s,r)| \geq k\eta/2\right) \leq K_5 e^{-K_6\eta^2 k^2}$$

for some constants $K_5, K_6 > 0$, independent of k and the realization $W_2^y(\omega)$. Therefore, proceeding as in the proof of Lemma 2.3.10, we see that

$$Q\left(P(\sup_{y\in D, r\in[0,1], s\in[0,1]} |f_k(y,s,r)| \geq k\eta/2) \geq e^{-K_6\eta^2k^2/2}\right) \leq K_7 e^{-K_6\eta^2 k^2/2}.$$

Now (2.79) follows from the Borel-Cantelli lemma. We complete the proof by combining (2.78) and (2.79). $\square$

Proof of Lemma 2.3.2: We have just proved Lemma 2.3.2 in the special case that $\kappa = 1$ and $\Lambda = R^n$. For $\kappa < 1$, modify the preceding bounds for the field

$$f(y, s) = \int_0^s b(\rho_y(\tau), t - \tau)\, d\tau$$

considered over $M_\kappa = D \times [0, \kappa t]$. Now we have $\sigma_t^2 = \sup_{(x,s)\in M} E_Q[f^2] \leq p_1 \kappa t$, so we find that $E[\|f\|] \leq C\sqrt{\kappa t}$ for some constant $C > 0$. Then, just as in Lemma 2.3.11, we have

$$Q\left(P\left(\sup_{s\in[0,\kappa t], z\in R^n} |X^{z,t}(s) - x| \geq \eta t\right) > e^{-K_2\eta^2 t/2\kappa}\right) \leq K_1 e^{-K_2\eta^2 t/2\kappa},$$

and for $\Lambda = R^n$, the rest follows as in the proof of Lemma 2.3.11. To bound the more general conditional probability in Lemma 2.3.2 (with $\Lambda \neq R^n$), observe that whenever $P(W_2^0 \in \Lambda) > 0$

$$\begin{aligned}
&Q\left(P\left(\sup_{s\in[0,\kappa t], z\in R^n} |X^{z,t}(s) - x| \geq \eta t\ |W_2^0 \in \Lambda\right) > e^{-K_2\eta^2 t/2\kappa}\right) = \\
&\quad = Q\left(P\left(\sup_{s\in[0,\kappa t], z\in R^n} |X^{z,t}(s) - x| \geq \eta t, W_2^0 \in \Lambda\right) > P\left(W_2^0 \in \Lambda\right) e^{-K_2\eta^2 t/2\kappa}\right) \\
&\quad \leq \frac{e^{K_2\eta^2 t/2\kappa}}{P\left(W_2^0 \in \Lambda\right)} E_Q P\left(\sup_{s\in[0,\kappa t], z\in R^n} |X^{z,t}(s) - x| \geq \eta t, W_2^0 \in \Lambda\right) \\
&\quad \leq \frac{e^{K_2\eta^2 t/2\kappa}}{P\left(W_2^0 \in \Lambda\right)} E_P\left(\chi_{W_2^0\in\Lambda} Q(\sup_{s\in[0,\kappa t], z\in R^n} |X^{z,t}(s) - x| \geq \eta t)\right)
\end{aligned} \tag{2.80}$$

By Lemma 2.3.10, the probability $Q(\sup_{s\in[0,\kappa t], z\in R^n} |X^{z,t}(s) - x| \geq \eta t)$ is bounded independently of the realization of W_2^0, so the right hand side of (2.80) is bounded by

$$\begin{aligned}
&\frac{e^{K_2\eta^2 t/2\kappa}}{P\left(W_2^0 \in \Lambda\right)} E_P\left(\chi_{W_2^0\in\Lambda} Q(\sup_{s\in[0,\kappa t], z\in R^n} |X^{z,t}(s) - x| \geq \eta t)\right) \\
&\quad \leq \frac{e^{K_2\eta^2 t/2\kappa}}{P\left(W_2^0 \in \Lambda\right)} E_P\left(\chi_{W_2^0\in\Lambda} K_1 e^{-K_2\eta^2 t/\kappa}\right) = K_1 e^{-K_2\eta^2 t/2\kappa}
\end{aligned}$$

Then the rest follows as in Lemma 2.3.11. $\square$

2.3.4 The Lyapunov Exponent $\mu(\lambda)$

In this section we prove Proposition 2.1.1. We study the limit

$$\mu(\lambda, z) = \lim_{t\to\infty} \frac{1}{t} \log E\left[e^{-\lambda\cdot Z^{z,t}(t)}\right] \tag{2.81}$$

As shown in Section 2.2, this is equivalent to the limit

$$\mu(\lambda, z) = \frac{\lambda^2}{2} + \lim_{t\to\infty} \frac{1}{t} \log \phi(z,t)$$

where $\phi(z,t) > 0$ solves that auxiliary initial value problem

$$\phi_t = \frac{1}{2}\Delta_z\phi + (B-\lambda)\cdot\nabla\phi - \lambda\cdot B(z,t)\phi. \tag{2.82}$$

$$\phi(z,0) \equiv 1.$$

Without the drift term, the equation (2.82) is called the parabolic Anderson problem (see [18] and [24]). We will denote by $\rho(\lambda)$ the limit $\lim_{t\to\infty} \frac{1}{t}\log\phi(z,t)$. Therefore, $\mu(\lambda)$ exists independent of z if and only if $\rho(\lambda)$ exists independent of z.

The proof that $\mu(\lambda)$ exists almost surely with respect to Q, independent of z, relies on the sub-additive ergodic theorem and a Harnack-type estimate, provided we assume the necessary decay of temporal correlation of the process $B(y,t)$. Following [24], we define for any continuous path $W \in C([0,t], R^n)$ the exponential

$$\xi(t, W) = e^{-\int_0^t \lambda_1 b(W_2(s)+z,t-s)\,ds - \lambda_1 W_1(t) - \lambda_2 W_2(t)}.$$

which is the exponential term $e^{-\lambda\cdot Z^{z,t}(t)}$ for a fixed realization of the Wiener process. For any fixed path W, $\xi(t,W)$ is lognormal with mean

$$E_Q[\xi(t,W)] = e^{\frac{\lambda^2\hat{\sigma}^2}{2}} e^{-\lambda_1 W_1(t) - \lambda_2 W_2(t)} \tag{2.83}$$

where

$$\hat{\sigma}^2 = \int_0^t \int_0^t \Gamma(X_s, X_r, s, r)\, ds\, dr \le 2\sqrt{2} p_1 t, \tag{2.84}$$

by (2.69). Note that $\hat{\sigma}^2$ is bounded independently of the particular path W.

For $0 \le s < t$, define the random variables

$$\begin{aligned}
q^z(\lambda, s, t) &= E_{z,t}[e^{-\lambda\cdot Z(t-s)}] \\
q_I(\lambda, s, t) &= \inf_{z\in D} q^z(\lambda, s, t) \\
q_S(\lambda, s, t) &= \sup_{z\in D} q^z(\lambda, s, t).
\end{aligned}$$

Using the sub-additive ergodic theorem, we will show that the limits

$$\lim_{t\to\infty} \frac{1}{t} \log q_I(\lambda, 0, t) = \mu_I(\lambda) \tag{2.85}$$

and

$$\lim_{t\to\infty} \frac{1}{t} \log q_S(\lambda, 0, t) = \mu_S(\lambda) \tag{2.86}$$

exist and are finite, almost surely with respect to Q. Then we will show $\mu_I(\lambda) = \mu_S(\lambda)$, and therefore $\mu(\lambda) = \mu_I(\lambda) = \mu_S(\lambda)$ is well-defined, independently of z. By the Markov property of the Wiener process we have for any $s < r < t$:

$$\begin{aligned}
q_I(\lambda, s, t) &= \inf_z E_{z,t}\left[e^{-\lambda\cdot Z^{z,t}(t-r)} E[e^{-\lambda\cdot Z^{a,r}(r-s)} | Z^{z,t}(t-r) = a]\right] \\
&\ge \inf_z E_{z,t}\left[e^{-\lambda\cdot Z^{z,t}(t-r)} \inf_a E_{a,r}[e^{-\lambda\cdot Z^{a,r}(r-s)}]\right] \\
&= q_I(\lambda, s, r) q_I(\lambda, r, t).
\end{aligned}$$

Therefore, $\log(q_I(\lambda, s, t))$ is super-additive:

$$\log(q_I(\lambda, s, t)) \geq \log(q_I(\lambda, s, r)) + \log(q_I(\lambda, r, t))$$

for any $0 \leq s < r < t$. Similarly, the function $\log(q_S(\lambda, s, t))$ is sub-additive. By the stationarity of B, $\tau_r \log(q_I(\lambda, s, t)) = \log(q_I(\lambda, s + r, t + r))$ for any $r \geq 0$.

In order to apply the ergodic theorem, we must show that $\log q_I$ and $\log q_S$ are integrable. First,

$$\begin{aligned}
E_Q[\log(q_I(\lambda, s, t))] &\leq \log E_Q[q_I(\lambda, s, t)] \\
&\leq \log \inf_z E_P E_Q[e^{-\lambda \cdot Z^{z,t}(t-s)}], \qquad (2.87) \\
&= \log \inf_z E_P[e^{\frac{\lambda^2 \hat{\sigma}^2}{2}} e^{-\lambda_1 W_1(t-s) - \lambda_2 W_2(t-s) - \lambda \cdot z}] \\
&\leq C_D|\lambda| + \frac{|\lambda|^2}{2}|t-s| + \log e^{\frac{\lambda^2 \sigma^2}{2}} \\
&\leq C_D|\lambda| + \frac{|\lambda|^2}{2}|t-s| + \sqrt{2}\lambda^2 p_1 |t-s|
\end{aligned}$$

Where the constant $C_D = \sup_{z \in D}|z|$, is independent of λ and $\hat{\omega}$. By Jensen's Inequality

$$\begin{aligned}
E_Q\left[\log(q_I(\lambda, s, t))\right] &\geq E_Q\left[\log(E_P e^{\inf_z -\lambda \cdot Z^{z,t}(t-s)}\right] \\
&\geq E_P E_Q\left[\inf_z -\lambda \cdot Z^{z,t}(t-s)\right] \\
&= -C_D|\lambda| + \frac{|\lambda|^2}{2}|t-s| + \\
&\qquad + E_P E_Q\left[\inf_z - \int_0^t \lambda_1 b(W_2(s) + z, t-s)\, ds\right]
\end{aligned}$$

This last term is finite, by the Borell inequality. Note also that if $M(\hat{\omega}) = \sup_{t\in[0,1],z\in D}|B(z,t)|$, then

$$\begin{aligned}\sup_{s,t\in[0,1]}|\log q_I(\lambda,s,t)| &\leq C_D|\lambda| + |\lambda_1|M(\hat{\omega}) + \sup_{s,t\in[0,1]}|\log E[e^{-\lambda_1 W_1(t-s)-\lambda_2 W_2(t-s)}]| \\ &\leq C_D|\lambda| + |\lambda_1|M(\hat{\omega}) + \frac{|\lambda|^2}{2},\end{aligned}$$

and the latter is integrable with respect to Q. It now follows from the subadditive ergodic theorem (Theorem 2.5 of [3]) and the continuity of $q(\lambda,0,t)$ with respect to t that the limit (2.85) exists almost surely and is finite:

$$\lim_{t\to\infty}\frac{\log(q_I(\lambda,0,t))}{t} = \sup_t \frac{E_Q\left[\log(q_I(\lambda,0,t))\right]}{t} = \mu_I \tag{2.88}$$

Also, by (2.87), $\mu_I \leq \frac{|\lambda|^2}{2} + \sqrt{2}|\lambda|^2 p_1$. Because b is ergodic with respect to translation in t, $\mu_I(\lambda)$ is constant on a set of full measure (with respect to Q).

Now we show that $(1/t)\log(q_S(\lambda,0,t)$ is integrable. A lower bound on the expectation follows from Jensen's inequality:

$$\begin{aligned}E_Q\left[\log q_S(\lambda,s,t)\right] &\geq \sup_z E_Q E_{z,t}[-\lambda\cdot Z^{z,t}(t-s)] \\ &= \sup_z E_{z,t}E_Q[-\lambda\cdot Z^{z,t}(t-s)] = 0.\end{aligned} \tag{2.89}$$

Now we derive an upper bound. The Borell inequality and Theorem 3.2 of [2] (p. 63, let $\alpha = 1$) imply that there is a finite constant $K_0 > 0$ such that

$$E_Q e^{\sup_z -\int_0^{t_i-t_j}\lambda_1 b(W_2(\tau)+z,t_i-\tau)\,d\tau} < K_0 < \infty \tag{2.90}$$

if $\hat{\sigma}^2 < \frac{1}{2}$, where $\hat{\sigma}^2$ is the variance of the integral $-\int_0^{t_i-t_j}\lambda_1 b(W_2(\tau)+z,t_i-\tau)\,d\tau$ with respect to Q. This variance is small, when $|t_i-t_j|$ is small. Thus by

(2.84), there is a constant $K_1 > 0$ such that (2.90) holds when $|t_i - t_j| \le K_1$. Now for any $s < t$, let N be the smallest integer greater than $|t-s|/K_1$ and $s = t_0 < t_1 < t_2 < \cdots < t_N = t$ with $|t_{i+1} - t_i| = \Delta t = |t-s|/N \le K_1$ for all $i = 0, \ldots, N-1$. Jensen's inequality implies that

$$\begin{aligned}
& E_Q \log(q_S(\lambda, t_i, t_{i+1})) \le \\
& \quad \le \log E_P \left[e^{C_D|\lambda| - \lambda \cdot W(t_{i+1} - t_i)} E_Q [e^{\sup_z -\lambda_1 \int_0^{t_{i+1}-t_i} b(W_2(s)+z, t-s)\, ds}] \right] \\
& \quad \le \log E_P \left[e^{C_D|\lambda| - \lambda \cdot W(t_{i+1} - t_i)} K_0 \right] \\
& \quad = C_D|\lambda| + \frac{|\lambda|^2 (t_{i+1} - t_i)}{2} + \log K_0.
\end{aligned} \tag{2.91}$$

Combining this with the subadditivity of $\log(q_S(\lambda, s, t))$, we derive the upper bound

$$\begin{aligned}
E_Q \log(q_S(\lambda, s, t)) &\le \sum_{i=0}^{N-1} E_Q \log(q_S(\lambda, t_i, t_{i+1})) \\
&\le \frac{|\lambda|^2 (t-s)}{2} + N(C_D|\lambda| + \log K_0) \\
&\le \frac{|\lambda|^2 (t-s)}{2} + (\frac{|t-s|}{K_1} + 1)(C_D|\lambda| + \log K_0).
\end{aligned}$$

The last inequality follows from our definition of N. Moreover,

$$E_P[e^{-C_D|\lambda| - |\lambda|\|B\||t-s| - \lambda \cdot W(t-s)}] \le q_S(\lambda, s, t) \le E_P[e^{C_D|\lambda| + |\lambda|\|B\||t-s| - \lambda \cdot W(t-s)}],$$

so that

$$\sup_{s,t \in [0, K_1]} |\log q_S(\lambda, s, t)| \le K_1 (C_D|\lambda| + |\lambda|\|B\| + \frac{\lambda^2}{2}) \tag{2.92}$$

where $\|b\|$ denotes $\sup_{t \in [0,K_1], z \in D} |B(z,t)|$. The right side of (2.92) is integrable. So, we can apply the sub-additive ergodic theorem to conclude that the limit

$$\lim_{t \to \infty} \frac{\log(q_S(\lambda, 0, t))}{t} = \inf_t \frac{E_Q \left[\log(q_S(\lambda, 0, t))\right]}{t} = \mu_S, \tag{2.93}$$

holds almost surely with μ_S a constant, $\mu_S \in [0, \infty)$. The convergence along continuous time follows from (2.92), the continuity of $q_S(\lambda, 0, t)$, and Theorem 2.5 of [3]. As with μ_I, μ_S is constant on a set of full measure, because of the ergodicity of b with respect to translation in t.

Clearly $\mu_I \leq \mu_S$. To show that $\mu_I = \mu_S$, we will need a kind of Harnack inequality to compare the quantities $q_I(\lambda, 0, t)$ and $q_S(\lambda, 0, t)$. Such a result has been obtained in [24] in the case that $b(y, t)$ is Gaussian in both space and time, with a white-noise temporal dependence. Under the assumptions A1-A3, however, the arguments of [24] imply that the following estimate also holds in the present case.

Theorem 2.3.13. *(Cranston and Mountford [24]) For any fixed $M > 0$, there are positive constants c_1, c_2 such that outside an event of $Q-$probability $e^{-\frac{1}{4}n^{5/6}}$, one has*

$$q_I(\lambda, 0, n) \geq c_1 e^{-c_2 n^{11/12}} \left(q_S(\lambda, 0, n) - e^{-\frac{1}{4}n^{7/6}} \right).$$

From this result it follows immediately that

$$\lim_{n \to \infty} \frac{1}{n} \log E\left[e^{-\lambda \cdot Z^{z,n}(n)}\right] = \mu_I(\lambda) = \mu_S(\lambda),$$

uniformly in z. By (2.88) and (2.93), we see that this extends to continuous time

$$\lim_{t \to \infty} \frac{1}{t} \log E\left[e^{-\lambda \cdot Z^{z,t}(t)}\right] = \mu_I(\lambda) = \mu_S(\lambda) = \mu(\lambda). \tag{2.94}$$

We have now shown that for each $\lambda \in R^n$, $\mu(\lambda)$ is well-defined, independent of $z \in R^n$, almost surely with respect to Q. This means that for each $\lambda \in R^n$, there is a set $\hat{\Omega}_\lambda \subset \hat{\Omega}$ such that $Q(\hat{\Omega}_\lambda) = 1$ and (2.94) holds for all $\hat{\omega} \in \hat{\Omega}_\lambda$. We claim that the set

$$\hat{\Omega}_0 = \bigcap_{\lambda \in R^n} \hat{\Omega}_\lambda \tag{2.95}$$

has full measure: $Q(\hat{\Omega}_0) = 1$. It is clear that the set $\hat{\Omega}_0' = \bigcap_{\lambda \in \mathbb{Q}^n} \hat{\Omega}_\lambda$ has full measure, since $\mathbb{Q}^n$ is a countable set.

Lemma 2.3.14. $\hat{\Omega}_0 = \hat{\Omega}_0'$. *So,* $Q(\hat{\Omega}_0) = 1$.

Proof: Clearly $\hat{\Omega}_0 \subset \hat{\Omega}_0'$. For $\lambda \in R^n$, $t > 0$, $z \in D$, we define the quantities

$$\begin{aligned}
\mu^z(\lambda, t) &= \frac{1}{t} \log E_{z,t}[e^{-\lambda \cdot Z(t)}] = \frac{1}{t} \log q^z(\lambda, 0, t) \\
\mu^+(\lambda, t) &= \inf_{z \in D} \mu^z(\lambda, t) \\
\mu^-(\lambda, t) &= \sup_{z \in D} \mu^z(\lambda, t).
\end{aligned}$$

So, for all $\lambda \in \mathbb{Q}^n$ and $\hat{\omega} \in \hat{\Omega}_0'$, $\mu^+(\lambda, t) \to \mu(\lambda)$ and $\mu^-(\lambda, t) \to \mu(\lambda)$ as $t \to \infty$.

We claim that $\mu^z(\lambda, t)$ and $\mu^+(\lambda, t)$ are convex in λ, for each $t > 0$. Let $r \in [0, 1]$, $\lambda_1, \lambda_2 \in R^n$. By Hölder's inequality,

$$E[e^{-r\lambda_1 \cdot Z^{z,t}(t) - (1-r)\lambda_2 \cdot Z^{z,t}(t)}] \le E[e^{-\lambda_1 \cdot Z^{z,t}(t)}]^r E[e^{-\lambda_2 \cdot Z^{z,t}(t)}]^{1-r}. \tag{2.96}$$

Upon taking a logarithm and dividing by t, this inequality implies that

$$\mu^z(r\lambda_1 + (1-r)\lambda_2, t) \le r\mu^z(\lambda_1, t) + (1-r)\mu^z(\lambda_2, t).$$

Hence, $\mu^z(\lambda, t)$ is convex. Since $\mu^+(\lambda, t)$ is a supremum of convex functions (μ^z), it must also be convex. It follows that the function $\mu(\lambda)$ is continuous in λ, since for each $\lambda \in \Omega_0'$, $\mu(\lambda)$ is the finite, pointwise limit of continuous convex functions $\mu^+(\lambda, t)$ (pointwise for $\lambda \in \mathbb{Q}^n$, a dense subset of R^n). Moreover, $\mu(\lambda)$ must be uniformly continuous on compact sets.

Next, we claim that for each $\hat{\omega} \in \hat{\Omega}_0'$ fixed, $\mu^+(\lambda, t) \to \mu(\lambda)$ and $\mu^-(\lambda, t) \to \mu(\lambda)$ locally uniformly in $\lambda \in R^n$. Let $\Lambda_\delta \subset R^n$ be a closed ball of radius δ. For $\epsilon > 0$, let $k \in \mathbb{Z}$ be large enough so that $|\mu(\lambda_1) - \mu(\lambda_2)| < \epsilon$ whenever $|\lambda_1 - \lambda_2| \leq n2^{-k}$ and $\lambda_1, \lambda_2 \in \Lambda_{2\delta}$. Then, let $t_0 > 0$ be large enough so that $|\mu^+(\lambda, t) - \mu(\lambda)| \leq \epsilon$ for all $\lambda \in \Lambda_{2\delta} \cap 2^{-k}\mathbb{Z}^n$ and $t \geq t_0$. Such a t_0 exists since $\Lambda_{2\delta} \cap 2^{-k}\mathbb{Z}^n$ is a finite subset of $\subset \mathbb{Q}^n$. For k sufficiently large, depending on δ, any $\lambda \in \Lambda_\delta$ can be expressed as a convex combination of points in the set $\{\lambda_j\}_j = \Lambda_{2\delta} \cap 2^{-k}\mathbb{Z}^n$:

$$\lambda = \sum_j w_j \lambda_j \tag{2.97}$$

such that $w_j \geq 0$, $\sum_j w_j = 1$. Moreover, we can require that $w_j = 0$ if $|\lambda_j - \lambda| \geq n2^{-k}$. Therefore, for t sufficiently large,

$$\begin{aligned}
\mu^+(\lambda, t) &= \mu^+(\sum_j w_j \lambda_j, t) \\
&\leq \sum_j w_j \mu^+(\lambda_j, t) \quad \text{(by convexity)} \\
&\leq \epsilon + \sum_j w_j \mu(\lambda_j) \quad \text{(by choice of } t_0\text{)} \\
&\leq \epsilon + \sum_j w_j (\epsilon + \mu(\lambda)) = 2\epsilon + \mu(\lambda).
\end{aligned}$$

This implies that

$$\lim_{t \to \infty} \sup_{\lambda \in \Lambda_\delta} \mu^+(\lambda, t) - \mu(\lambda) \leq 0. \tag{2.98}$$

Now suppose that for some $\epsilon > 0$, there are sequences $z_k \to z_0 \in D$,

$t_k \to \infty$, and $\lambda_k \to \lambda_0 \in \Lambda_\delta$ such that

$$\mu^{z_k}(\lambda_k, t_k) < \mu(\lambda_0) - \epsilon, \quad \text{for } k = 1, 2, 3, \ldots$$

Then for any $\lambda' \in \mathbb{Q}^n$,

$$\frac{\mu^{z_k}(\lambda', t_k) - \mu^{z_k}(\lambda_k, t_k)}{|\lambda_k - \lambda'|} \geq \frac{\mu^{z_k}(\lambda', t_k) - \mu(\lambda_0) + \epsilon}{|\lambda_k - \lambda'|} \tag{2.99}$$

Since $\lambda' \in \mathbb{Q}^n$, $\mu^{z_k}(\lambda', t_k) \to \mu(\lambda')$ as $k \to \infty$. Therefore, by choosing λ' sufficiently close to λ_0, we can make $|\mu^{z_k}(\lambda', t_k) - \mu(\lambda_0)| < \epsilon/2$ for k sufficiently large, since $\mu(\lambda)$ is continuous in λ. Hence, the right hand side of (2.99) can be made arbitrarily large. That is, the slopes of the secant lines through the points $(\lambda', \mu^{z_k}(\lambda', t_k))$ and $(\lambda_k, \mu^{z_k}(\lambda_k, t_k))$ can be made arbitrarily large for k large, since $\lambda_k \to \lambda_0$. However, because the functions $\mu^{z_k}(\lambda, t_k)$ are convex in λ and bounded above by $\mu^+(\lambda, t_k)$, this contradicts (2.98). Hence,

$$\lim_{t\to\infty} \inf_{\lambda \in \Lambda_\delta} \mu(\lambda) - \mu^-(\lambda, t) \leq 0. \tag{2.100}$$

Equations (2.98) and (2.100) imply the claim that $\mu^+(\lambda, t) \to \mu(\lambda)$ and $\mu^-(\lambda, t) \to \mu(\lambda)$ locally uniformly in $\lambda \in R^n$ for each $\hat{\omega} \in \hat{\Omega}_0'$. Therefore, $\hat{\Omega}_0' \subset \hat{\Omega}_0$.

□

To complete the proof of Proposition 2.1.1, we now show that $\mu(\lambda)$ is super-linear in λ. Clearly $\mu(0) = f'(0)$. Let $\lambda = (\lambda_1, 0)$, $\lambda_1 \in R$, so that the nonzero component of λ is in the x-direction. Then ϕ in (2.82) can be chosen to depend only on the y variable: $\phi = \phi(y, t)$. Thus, the problem (2.82) reduces to

$$\phi_t = \frac{1}{2}\Delta_y \phi - \lambda_1 b(y,t)\phi. \tag{2.101}$$
$$\phi(y,0) \equiv 1.$$

Since $b(y,t)$ has the same distribution as $-b(y,t)$, we conclude that $\rho(-(\lambda_1, 0)) = \rho((\lambda_1, 0))$ for all $\lambda_1 \in R$. Hence $\rho((\lambda_1, 0))$ and $\mu((\lambda_1, 0))$ are even functions of λ_1. Using the Feynman-Kac representation for $\rho((\lambda_1, 0))$ as in (2.96), we find that $\rho((\lambda_1, 0))$ is convex in λ_1. Since $\rho(0) = 0$, we conclude that

$$\rho((\lambda_1, 0)) \geq 0 \quad \text{and} \quad \mu((\lambda_1, 0)) \geq \lambda_1^2/2, \quad \forall \lambda_1 \in R. \tag{2.102}$$

If we choose $\lambda = (0, \lambda_2)$, $\lambda_2 \in R$, so that the nonzero component of λ is in the y-direction, then (2.82) ϕ can be chosen to be constant $\phi \equiv 1$, for all time. Hence

$$\rho((0, \lambda_2)) = 0 \quad \text{and} \quad \mu((0, \lambda_2)) = \lambda_2^2/2, \quad \forall \lambda_2 \in R. \tag{2.103}$$

Combining (2.102), (2.103), and the convexity of $\mu(\lambda)$, we conclude that $\mu(\lambda)$ is super-linear. This completes the proof of Proposition 2.1.1.

2.4 Homogenization

We now relate the conclusions of Theorem 2.1.3 and Theorem 2.1.2 to the homogenization results of [62] described in Theorem 1.2.1 of the introduction. The result is the following:

Theorem 2.4.1. *If B is the random field defined by (2.1), then under assumptions A1-A3 the homogenization result of Theorem 1.2.1 holds, almost* *surely, with the effective Hamiltonian defined uniquely by $H(p) = \mu(-p)$ and μ given by (2.5).*

Proof: Let $w(z,t) = \log \Phi(z,t) - f'(0)t - \mu t$, where Φ solves the initial value problem (2.9). Then w is periodic in z and solves the initial value problem

$$w_t - \frac{1}{2}\Delta w - \frac{1}{2}|p + \nabla w|^2 - B \cdot (p + \nabla w) = -H \tag{2.104}$$
$$w(z,0) \equiv 0$$

with $p = -\lambda$ and H being the constant $H(p) = \mu(-\lambda)$. As in the arguments of [62] for temporally periodic flows, the function w can play the role of the corrector function enabling homogenization of (1.1) via the perturbed test function method. In the present case, there can be no cell problem defined on a compact domain, as is possible when the flow is periodic in time. However, because w grows sub-linearly in t, we find that the perturbed test function method can still be used to obtain the homogenization result.

The fact that w grows sub-linearly as $t \to \infty$ follows from the definition of μ. Specifically, $\frac{1}{t}w = -\mu - f'(0) + \frac{1}{t}\log \Phi \to -\mu + \mu = 0$, uniformly in z, so that $w(z,t) = t\hat{w}(z,t)$ where

$$\lim_{t\to\infty} \sup_z |\hat{w}(z,t)| = 0.$$

As a consequence, the scaled function $\epsilon w(\epsilon^{-1}z, \epsilon^{-1}t)$ converges to zero, locally uniformly, as $\epsilon \to 0$, since

$$\begin{aligned} \epsilon w(\epsilon^{-1}z, \epsilon^{-1}t) &= t\frac{1}{\epsilon^{-1}t} w(\epsilon^{-1}z, \epsilon^{-1}t) \\ &= t\hat{w}(\epsilon^{-1}z, \epsilon^{-1}t), \end{aligned} \tag{2.105}$$

and $|\hat{w}(\epsilon^{-1}z, \epsilon^{-1}t)| \to 0$ as $\epsilon \to 0$, locally uniformly in (z,t).

To derive the homogenized equation for (1.1), one follows the method of [62] in defining $\psi^\epsilon(z,t) = \epsilon \log u^\epsilon(z,t)$, where u^ϵ solves (1.10). Then, one determines the equation solved (in the viscosity sense) by the functions

$$\psi^*(z,t) = \limsup_{(y,s,\epsilon)\to(z,t,0)} \psi^\epsilon(y,s) \quad \text{and} \quad \psi_*(z,t) = \liminf_{(y,s,\epsilon)\to(z,t,0)} \psi^\epsilon(y,s).$$

For example, suppose that v is a smooth test function and that $\psi^* - v$ has a strict local max at some point (z_0, t_0) interior to the domain. Then, there must be a sequence $\epsilon_r \to 0$ and points $(z_{\epsilon_r}, t_{\epsilon_r}) \to (z_0, t_0)$ such that

$$\psi^{\epsilon_r}(z,t) - (v(z,t) + \epsilon_r w(\epsilon_r^{-1}z, \epsilon_r^{-1}t)) \tag{2.106}$$

has a local max at $(z_{\epsilon_r}, t_{\epsilon_r})$, and $\psi^{\epsilon_r}(z_{\epsilon_r}, t_{\epsilon_r}) \to \psi^*(z_0, t_0)$. This follows from the crucial fact that $\epsilon w(\epsilon^{-1}z, \epsilon^{-1}t)$ converges to zero, locally uniformly, as demonstrated by (2.105). If w were not sub-linear in t, the perturbation $\epsilon w(\epsilon^{-1}z, \epsilon^{-1}t)$ might stay bounded away from zero. Hence the points $(z_{\epsilon_r}, t_{\epsilon_r})$ where the local maxima of (2.106) are achieved might not converge to (z_0, t_0). Combined with this observation, the arguments of [62] then imply that ψ^* solves

$$\max(\psi^*_t - H(\nabla \psi^*) - f'(0), \psi^*) \le 0. \tag{2.107}$$

where $H(p) = \mu(p)$. Analogous arguments for ψ_*, as in [62], and the comparison principle for viscosity solutions imply that $\psi^\epsilon \to \psi$ locally uniformly, where ψ solves (1.11).

To complete the homogenization result, one must show that $u^\epsilon \to 0$ locally uniformly in the set $\{Z < 0\}$ and that $u^\epsilon \to 1$ locally uniformly in the interior of the set $\{\psi = 0\}$. The convergence $u^\epsilon \to 0$ follows as in [62] (see p. 12). For spatially and temporally periodic flows, Majda and Souganidis [62] have proven the lower bound $u^\epsilon \to 1$ using estimates from [28] and the fact that B is uniformly bounded in all of R^{n+1}. In the present case, however, B is not uniformly bounded in t, so those estimates do not extend. Nevertheless, using Theorem 2.1.2 we can complete the proof.

Proposition 2.4 of [62] implies that the front defined by (1.11) is governed by the geometric equation $\xi_t = F(\nabla\xi)$ where the convex Hamiltonian F is given by $F(p) = \max_{H^*(q)=f'(0)}(p,q)$. Since $H(p) = \mu(-p)$, this is equivalent to

$$F(p) = \max_{S(-q)-f'(0)=0}(p,q).$$

Moreover, in [62] (see p. 19) it is shown that

$$\begin{aligned} \{z|\ Z(z,t) < 0\} &\supseteq \{z|\ \inf_{z'} H^*(\frac{z'-z}{t}) - f'(0) > 0\} \\ &= \{z|\ \inf_{z'} S(\frac{z-z'}{t}) - f'(0) > 0\}. \end{aligned} \tag{2.108}$$

Now we show how this implies the result. For $\delta > 0$, $z' \in G_0$, $T > 0$, define the sets

$$\begin{aligned} \mathcal{C}^0_\delta(z',T) &= \bigcup_{t\in[0,T]} (z' + t\overline{\Psi}(\delta)) \times t \\ \mathcal{C}^1_\delta(z',T) &= \bigcup_{t\in[0,T]} (z' + t\underline{\Psi}(-\delta)) \times t \end{aligned}$$

where

$$\overline{\Psi}(s) = \{c \in R^n|\ S(c) - f'(0) \geq s\} \qquad \underline{\Psi}(s) = \{c \in R^n|\ S(c) - f'(0) \leq s\}.$$

In $R^n \times [0, \infty)$, the set $\mathcal{C}^1_\delta(z', T)$ is an inverted cone with tip at $(z', 0)$, $z' \in G_0$. The set $\mathcal{C}^1_\delta(z', T)$ is contained in the complement of $\mathcal{C}^0_\delta(z', T)$. The continuity and convexity of $S(c)$ imply that $\mathcal{C}^1_\delta(z', T)$ is a convex set. Also, for any $\epsilon > 0$ and $T \in [0, \infty)$, there is a $\delta_0 = \delta_0(\epsilon, T) > 0$ such that $(z, t) \in \mathcal{C}^1_\delta(z', T)$ whenever $\delta < \delta_0$, $t \in [0, T]$, and $|z - z''| > \epsilon$ for all $(z'', t) \in \mathcal{C}^0_\delta(z', T)$. In other words, if z is bounded away from $\mathcal{C}^0_\delta(z', T)$, then z must be contained in $\mathcal{C}^1_\delta(z', T)$, if δ is sufficiently small, since the gap between the two sets goes to zero uniformly as $\delta \to 0$.

Now let $\epsilon > 0$ and suppose K is any compact subset of the interior of $\{(z, t)|\ Z(z, t) = 0\}$ such that $|z - z'| > \epsilon$ whenever $(z', t) \in K$ and $(z, t) \in \{Z(z, t) < 0\}$. It follows from (2.108) and the above observations that there must be a compact set $K' \subset \{c \in R^n|\ S(c) - f'(0) < 0\}$ such that

$$K \subset \{(z' + ct, t)|z' \in G_0, t \in [0, T], c \in K'\}. \tag{2.109}$$

In other words, there is a δ sufficiently small such that K is a compact subset of the interior of

$$\bigcup_{z' \in G_0} \mathcal{C}^1_\delta(z', T). \tag{2.110}$$

The result now follows from the following lemma, which is a consequence of Theorems 2.1.3 and 2.1.2. $\square$

Lemma 2.4.2. *For any $\delta > 0$, and any $T \in [0, \infty)$, $u^\epsilon(z, t) \to 1$ locally uniformly on the interior of the set*

$$\bigcup_{z' \in G_0} \mathcal{C}^1_\delta(z', T). \tag{2.111}$$

Proof: Without loss of generality, suppose $\overline{B_1(0)} \subset int(G_0)$. Then $u_\epsilon(\frac{z}{\epsilon}, \frac{t}{\epsilon}) = u^\epsilon(z,t)$ solves (1.1) with initial data satisfying $u_\epsilon(z,0) = 1$ for $|z| \leq 1/\epsilon$. Let K' be any compact subset of $\{S(c) - f'(0) < 0\}$. Now by Theorem 2.1.2, the periodicity of B, and the maximum principle,

$$\begin{aligned}
\lim_{\epsilon \to 0} \inf_{\substack{c \in K' \\ z' \in \overline{B_1(0)}}} u^\epsilon(z' + ct, t) &= \lim_{\epsilon \to 0} \inf_{\substack{c \in K' \\ |z'| \leq 1/\epsilon}} u_\epsilon(z' + c\frac{t}{\epsilon}, \frac{t}{\epsilon}) \\
&= \lim_{t' \to \infty} \inf_{\substack{c \in K' \\ z' \in D}} u_\epsilon(z' + ct', t') = 1. \qquad (2.112)
\end{aligned}$$

□

2.5 Generalizations

Although the arguments in this chapter rely on the periodicity of $b(y,t)$ in y to provide compactness in the y dimensions, they can be modified to solve the same problem in an infinite cylinder with the zero Neumann boundary condition on the sides of the cylinder. The compactness property remains, and the process $Y^{z,t}(s)$ just needs to be reflected when it hits the boundary $R \times \partial\Omega$.

The proofs also rely on the assumption that the flow has a shear structure; extending the arguments to more general flows will require some modification. Consider a flow having the form

$$B(z,t) = \sum_{j=1}^{N} B_j(z) b_j(t), \qquad (2.113)$$

where the vector fields $B_j \in C^{0,1}(R^n; R^n)$ are periodic in z, and b_j are stationary ergodic random fields over $t \in R$. Under these circumstances, the sub-additive arguments of sections 2.3.2 and 2.3.4 still apply, but the estimates needed to prove the existence of μ and the large deviation principle do not extend. For example, the quantities μ_I (2.85) and μ_S (2.86) would be well defined, almost surely. However, the estimate stated in Theorem 2.3.13 can not be extended in an obvious way. If B has a shear structure then the random variables

$$e^{\int_0^t B(Z^{z_1,t}(s),t-s)ds} \quad \text{and} \quad e^{\int_0^t B(Z^{z_2,t}(s),t-s)ds} \tag{2.114}$$

are both lognormal with respect to the measure Q and can be compared, leading to a comparison between μ_I and μ_S. This follows from the fact that $B(Z^{z_1,t}(s), t-s) = B(z_1 + W(s), t-s)$, and $W(s)$ is independent of the realization $\hat{\omega} \in \hat{\Omega}$. If B has a more general structure, however, then Z depends on the realization of the B, and the random variables in (2.114) have an unknown distribution.

The shear structure also allows us to invoke the Borell inequality to bound large deviations of the random variables

$$\sup_{s\in[0,t]} B(Z^{z,t}(s), t-s) \quad \text{and} \quad \sup_{s\in[0,t]} \int_0^s B(Z^{z,t}(\tau), t-\tau)d\tau, \tag{2.115}$$

since the terms inside the suprema are Gaussian. This technique leads to the estimate in Lemma 2.3.2. If B does not have a shear structure, then $B(Z^{z,t}(s), t-s)$ and $\int_0^s B(Z^{z,t}(\tau), t-\tau)d\tau$ are non-Gaussian. Consequently, the proof of Lemma 2.3.2 cannot be extended without alternative estimates on the distribution of the suprema in (2.115).

Chapter 3

Analysis of Front Speeds via Variational Principles

In this section we use the variational principle for the front speed c^* to describe the dependence of c^* on various properties of the flow B. As mentioned in the introduction, it has been established experimentally that the bulk burning rate in premixed combustion reactions depends on the ambient flow in a nonlinear way that is difficult to characterize mathematically [76]. For the model (1.1), we will analyze three cases:

(i) Temporally and spatially periodic flows,

(ii) Spatially random flows in a bounded cylinder,

(iii) Temporally random shear flows.

We study the asymptotic behavior of c^* when the advection term in (1.1) is relatively strong or weak compared to the diffusion term. To do so, we scale the flow by $B(z,t) \mapsto \delta B(z,t)$ and denote by $c^*(\delta)$ the front speed corresponding to the new flow δB. We will refer to δ as the amplitude or intensity of the flow. We also describe the dependence of c^* on spectral properties of the flow.

Other estimates on the front speed have been obtained by non-variational methods. For example, Constantin, Kiselev, Oberman, and Ryzhik [23] studied wave-like solutions to (1.1) in an infinite cylinder $D = R \times [0, L]$, and they obtained estimates on a quantity called the bulk burning rate, which is defined by

$$V(t) = \frac{1}{t}\int_0^t \left(\int_D u_t(x, y, s)\, \frac{dx\, dy}{L} \right) ds.$$

For KPP fronts propagating in deterministic shear flows, they obtained linear upper and lower bounds on $V(t)$ in the large advection limit $\delta \to \infty$. For cellular flows with bounded streamlines, they showed that $V(t)$ grows sub-linearly as $\delta \to \infty$. More recently, Novikov and Ryzhik [71] have used the KPP variational formula to prove that for periodic cellular flows with bounded streamlines $c^*(\delta) = O(\delta^{1/4})$ in the large advection limit.

3.1 Temporally and Spatially Periodic Flows

In this section we consider the case when $B(z, t)$ is spatially and temporally periodic, but we will not generally require B to have a shear structure or to have zero mean over the period cell. We define $\bar{m} = \frac{1}{D_T}\int_{D_T} B \cdot k\; dz\, dt$ to be the mean flow in the direction k, and $\tilde{B} = B - \bar{m}k$, so that $\int_{D_T} \tilde{B} \cdot k\; dz\, dt = 0$. The following theorems describe the asymptotic behavior of $c^*(\delta)$ when δ is small, corresponding to small perturbations of a homogeneous medium. We let $c_0 = c^*(0)$ denote the speed corresponding to zero advection ($\delta = 0$); in the KPP case, c_0 will denote the minimal speed $c_0 = 2\sqrt{f'(0)}$. We also will use λ_0 to denote the constant $\lambda_0 = \sqrt{f'(0)}$ which is the minimizer of the curve

$\lambda \mapsto (\lambda^2 + f'(0))/\lambda$. For $s \in (0,1)$, we denote by $C^s(D_T)$ the space of s-Hölder continuous functions of D_T.

Theorem 3.1.1. *Let f be the KPP nonlinearity. Let $B \in C^s(D_T)$. Let $\chi = \chi(z,t)$ solve $\chi_t - \Delta_z \chi + 2\lambda_0 k \cdot \nabla \chi = \lambda_0 \tilde{B} \cdot k$, $(z,t) \in D_T$, with periodic or Neumann boundary conditions. Then for δ sufficiently small, depending on B, the minimal speed has the expansion*

$$c^*(\delta) = c_0 - \delta \bar{m} + \frac{\delta^2}{\lambda_0 |D_T|} \int_{D_T} |\nabla \chi|^2 \, dz \, dt + O(\delta^3). \tag{3.1}$$

If $B = B(z)$ is independent of t, then the same result holds assuming only $B \in C^0(D_T)$.

For other nonlinearities, we have a similar result if we assume that $B(z,t) = (b(y), 0, \ldots, 0)$ has a shear structure and $k = (1, 0, \ldots, 0)$. Surprisingly, the leading order behavior of $c^*(\delta)$ is independent of the nonlinearity.

Theorem 3.1.2. *Let f be either the bistable or combustion-type nonlinearity. Let $n = 2$ or 3, and let the shear flow profile $b(y) \in C^s(D)$ for some $s \in (0,1)$ and $D \subset R^{n-1}$ a rectangular region. Then for δ sufficiently small, depending on B,*

$$c(\delta) = c_0 - \delta \bar{m} + \frac{c_0 \delta^2}{2|D|} \int_D |\nabla \chi|^2 \, dy + O(\delta^3). \tag{3.2}$$

where $\Delta \chi = -\tilde{b}(y)$ with Neumann or periodic boundary conditions, and $\tilde{b} = b - \bar{m}$ has zero mean.

We will prove Theorems 3.1.1 and 3.1.2 using variational formulas for c^*. In the next section, we will generalize these results to a class of random flows in the cylinder. For steady flows with zero mean field, Theorem 3.1.1 has also been proven recently by Heinze [42], using the variational formula. The result was also proven by Xin and Nolen [70] for the case of shear flow. Here, we establish the result for more general time-dependent flows and we derive estimates on the higher-order terms in order to extend the results to the case of random flows.

3.1.1 KPP Case

For the KPP nonlinearity, the variational principle for c^* is

$$c^*(k) = \inf_{\lambda>0} \frac{\mu(\lambda) + f'(0)}{\lambda} \tag{3.3}$$

where $\mu(\lambda)$ is the principal eigenvalue to the equation

$$L_\lambda \phi = \Delta_z \phi - \phi_t + (B - 2\lambda k) \cdot \nabla_z \phi + (|\lambda^2| - \lambda B \cdot k)\phi = \mu(\lambda)\phi \tag{3.4}$$

with $\phi > 0$ periodic in (z, t). The eigenvalues in the variational principle can be expressed as

$$\mu(\lambda) = \lambda^2 - \lambda \bar{m} + \rho(\lambda) \tag{3.5}$$

where $\rho(\lambda)$ is the principal eigenvalue for the operator $\tilde{L}_\lambda$:

$$\tilde{L}_\lambda \phi = \Delta \phi - \phi_t + (B - 2\lambda k) \cdot \nabla \phi - \lambda \tilde{B} \cdot k\phi = \rho(\lambda), \phi, \quad \phi > 0 \tag{3.6}$$

and $\tilde{B} = B - \bar{m}k$, has zero mean over D_T. Note that ρ still depends on the mean field $\bar{m}$ since (3.6) contains B on the left hand side. Nevertheless, in the expansion (3.1), the second order term does not depend on the mean field.

In the case of a steady shear flow, the operators L_λ and $\tilde{L}_\lambda$ are self-adjoint, so the eigenvalues can be represented by the Raleigh quotient. For more general flows, including time-dependent flows, we have the following useful representation for the eigenvalues in (3.6).

Theorem 3.1.3. *The following variational formula holds for the principal eigenvalue μ defined by (3.4):*

$$\mu(\lambda) = \inf_{\psi \in E^+} \sup_{(z,t)\in D_T} \frac{L_\lambda \psi}{\psi} = \sup_{\psi \in E^+} \inf_{(z,t)\in D_T} \frac{L_\lambda \psi}{\psi} \tag{3.7}$$

where $E^+ = \{\psi \in C^{2,1}(D_T)\,|\psi > 0\}$.

For spatially-temporally periodic flows, the set D_T denotes the $n+1$ dimensional torus $\mathbb{T}^{n+1}$. By $C^{2,1}(D_T)$ we mean all functions of $\mathbb{T}^{n+1}$ such that the partial derivatives $\partial_{z_i,z_j}\psi$ $(i,j = 1,\ldots,n)$ and $\partial_t \psi$ are continuous. If we impose Neumann boundary conditions on the side of a channel, instead of periodicity, then D_T denotes $D \times \mathbb{T}$, and the infimum is taken over all $\psi \in C^{2,1}(D_T)$ such that $\psi > 0$, and $\frac{\partial \psi}{\partial \nu} = 0$ on ∂D, for all $t \in \mathbb{T}$.

Proof of Theorem 3.1.3: If $\phi^* > 0$ and $\mu^*(\lambda)$ are the principal eigenfunction and eigenvalue of the adjoint operator L^*_λ, then $\mu^*(\lambda) = \mu(\lambda)$. To see this, observe that $\mu^*(\lambda)$ is real and that

$$(L_\lambda \phi, \phi^*) = \mu(\lambda)(\phi, \phi^*) \tag{3.8}$$

and

$$(L_\lambda \phi, \phi^*) = (\phi, L^*_\lambda \phi^*) = \overline{\mu^*(\lambda)}(\phi, \phi^*) = \mu^*(\lambda)(\phi, \phi^*) \tag{3.9}$$

Thus, $\mu = \mu^*$, since $(\phi, \phi^*) > 0$. Now, suppose there is a strict inequality:

$$L_\lambda \psi - \mu(\lambda)\psi = m < 0, \ \ \forall (z,t) \in D_T. \tag{3.10}$$

The Fredholm alternative implies that $(\phi^*, m) = 0$, which is a contradiction since $\phi^* > 0$ and $m < 0$. Hence,

$$\sup_{(z,t)\in D_T} \frac{L_\lambda \psi}{\psi} \geq \mu(\lambda). \tag{3.11}$$

Since $L_\lambda \phi = \mu(\lambda)\phi$, the formula (3.7) follows. The theorem can also be proven by an extension of the arguments in [12], Section 5.2. □

Lemma 3.1.4. *The map $\lambda \mapsto \mu(\lambda)$ is convex in λ.*

Proof of Lemma 3.1.4 This follows by applying Hölder's inequality to the Feynman-Kac representation for $\mu(\lambda)$, as in (2.96). □

Proposition 3.1.5. *For all $\delta > 0$, the front speed $c^*(\delta)$ satisfies $c^*(\delta) \geq c_0 - \delta\bar{m}$, with equality if and only if $\tilde{B} \cdot k \equiv 0$.*

Proof of Proposition 3.1.5: For various special cases, this bound has been observed by others (for example, see [42]). Using (3.6), we see that the function $w = \log \phi$ satisfies

$$\Delta_z w + |\nabla_z w|^2 - w_t + (B - 2\lambda k) \cdot \nabla_z w - \lambda \tilde{B} \cdot k = \rho(\lambda) \tag{3.12}$$

Now we integrate over D_T and conclude that

$$\rho(\lambda) = \frac{1}{|D_T|} \int_{D_T} |\nabla_z w|^2 \, dz \, dt \tag{3.13}$$

If $\tilde{B} \cdot k$ is not identically zero, then w cannot be constant, so the integral on the right is strictly positive. Thus, at the infimum, $\frac{\mu(\lambda)+f'(0)}{\lambda} > \frac{|\lambda_0|^2+f'(0)}{\lambda_0} = c_0$. $\square$

Proof of Theorem 3.1.1 (KPP Case): To estimate $c^*(\delta)$, we bound the principal eigenvalue $\rho(\lambda)$ (and $\mu(\lambda)$) using the representation in Theorem 3.1.3. Motivated by a formal asymptotic expansion of the eigenfunction, we choose a test function $\psi = 1 + \delta\chi + \delta^2 h$, where χ and h are periodic solutions to the equations

$$\begin{aligned} \Delta\chi - \chi_t - 2\lambda k \cdot \nabla\chi &= \lambda\tilde{B} \cdot k, && (3.14) \\ \Delta h - h_t - 2\lambda k \cdot \nabla h &= \gamma(\lambda) - B \cdot \nabla\chi + \lambda\tilde{B} \cdot k\chi. && (3.15) \end{aligned}$$

The constant $\gamma(\lambda)$ is chosen to satisfy the solvability condition for (3.15):

$$\gamma(\lambda) = -\frac{1}{|D_T|}\int_{D_T} \lambda\tilde{B} \cdot k\chi \, dz \, dt = \frac{1}{|D_T|}\int_{D_T} |\nabla\chi|^2 \, dz \, dt. \quad (3.16)$$

Here we have used the fact that $\nabla \cdot B = 0$. We normalize χ and h by setting $\inf_{D_T} \chi = \inf_{D_T} h = 0$. Now let $\psi = 1 + \delta\chi + \delta^2 h$. It follows from parabolic regularity estimates ([58], [55]) that $\psi \in C^{2+s,1+\frac{s}{2}}(D_T)$, so $\psi \in E^+$ and we can use ψ as a test function in the representation (3.7). We find that

$$\frac{\tilde{L}_\lambda\psi}{\psi} = \frac{\delta^2\gamma(\lambda) + \delta^3(B \cdot \nabla h - \lambda\tilde{B} \cdot kh)}{1 + \delta\chi + \delta^2 h},$$

and

$$\begin{aligned} \frac{\tilde{L}_\lambda\psi}{\psi} - \delta^2\gamma(\lambda) &= \frac{\delta^3(B \cdot \nabla h - \lambda\tilde{B} \cdot kh) - \delta^2\gamma(\lambda)(\delta\chi + \delta^2 h)}{1 + \delta\chi + \delta^2 h} \\ &= R_1(\lambda, \delta). && (3.17) \end{aligned}$$

Let us bound this remainder in terms of B, λ, and δ. Regularity estimates for equations (3.14) and (3.15) imply that

$$\|\chi\|_{C^s(D_T)} + \|\nabla\chi\|_{C^s(D_T)} \leq C\lambda\|\tilde{B}\|_s,$$
$$\|h\|_{C^s(D_T)} + \|h\|_{C^s(D_T)} \leq C(1+|\lambda|)\|B\|_s^2. \tag{3.18}$$

Since $\inf_{D_T}\chi = \inf_{D_T} h = 0$, these bounds imply that for $\delta < 1$, and λ bounded

$$|R_1(\lambda,\delta)| \leq C\delta^3\|B\|_s^4.$$

Lemma 3.1.6. *For λ bounded, the constant $\gamma(\lambda)$ is Lipschitz continuous with $|\gamma(\lambda_1) - \gamma(\lambda_2)| \leq C\left(|\lambda_1-\lambda_2||\lambda_2| + |\lambda_1-\lambda_2|^2\right)\|B\|_s^2$, for some constant C independent of λ.*

Proof of Lemma 3.1.6: Suppose the χ_1 and χ_2 solve (3.14) with λ_1 and λ_2, respectively. Then the function $\eta = \chi_1 - \chi_2$ solves

$$\Delta\eta - \eta_t - 2\lambda_1 k\cdot\nabla\eta = (\lambda_1-\lambda_2)\tilde{B}\cdot k + 2(\lambda_1-\lambda_2)k\cdot\nabla\chi_2.$$

Regularity estimates and (3.18) imply that

$$\|\nabla\eta\|_{L^2(D_T)} \leq C|\lambda_1-\lambda_2|\|B\|_s.$$

Therefore, Hölder's inequality implies that

$$\begin{aligned}\left|\int_{D_T}|\nabla\chi_1|^2 - \int_{D_T}|\nabla\chi_2|^2\right| &= \left|\int_{D_T}|\nabla\eta_1|^2 + \int_{D_T}\nabla\chi_2\cdot\nabla\eta\right| \\ &\leq C_1|\lambda_1-\lambda_2|^2\|B\|_s^2 + C_2|\lambda_1-\lambda_2||\lambda_2|\|B\|_s^2.\end{aligned}$$

$\square$

The following lemma allows us to restrict the infimum in the variational formula (3.3) to a bounded set that shrinks like $O(\delta)$ as $\delta\to 0$.

Lemma 3.1.7. *There is a constant β satisfying $0 < \beta \leq C\|B\|_s$ such that whenever δ is sufficiently small,*

$$\inf_{\lambda>0} \frac{\mu(\lambda) + f'(0)}{\lambda} = \inf_{|\lambda-\lambda_0|\leq\beta\delta} \frac{\mu(\lambda) + f'(0)}{\lambda} \tag{3.19}$$

where $\lambda_0 = \sqrt{f'(0)}$, the point where $\frac{\lambda^2+f'(0)}{\lambda}$ achieves its minimum.

Proof of Lemma 3.1.7: Will write $\mu = \mu(\lambda, \delta)$ to emphasize the dependence of μ on both λ and δ. First, by applying the maximum principle to equation (3.4), it is clear that

$$|\mu(\lambda, \delta) - \mu(\lambda, 0)| \leq \lambda\delta\|B\|_\infty.$$

Therefore, the curve $\frac{\mu(\lambda,\delta)+f'(0)}{\lambda}$ is trapped between the curves $g^+(\lambda) = \frac{\mu(\lambda,0)+f'(0)}{\lambda} + \delta\|B\|_\infty$ and $g^-(\lambda) = \frac{\mu(\lambda,0)+f'(0)}{\lambda} - \delta\|B\|_\infty$. Since $\partial_{\lambda\lambda}g^+(\lambda) = \partial_{\lambda\lambda}g^-(\lambda) > 0$ for λ near λ_0, there is there is a universal constant C_1 such that for any $\epsilon > 0$ and $\delta\|B\|_\infty \leq C_1\epsilon^2$, the infimum of $\frac{\mu(\lambda,\delta)+f'(0)}{\lambda}$ is attained at a point $\lambda \in (\lambda_0 - \epsilon, \lambda_0 + \epsilon)$.

Now set $\epsilon = \lambda_0/2$ and suppose $\delta\|B\|_\infty \leq C_1\epsilon^2$. Fix another constant $C_2 > 0$ and suppose that $\delta\|B\|_s^2 < C_2$. From (3.17) we see that

$$\mu(\lambda, \delta) = \mu(\lambda, 0) - \delta\lambda\bar{m} + \delta^2\gamma(\lambda) + R_1(\lambda, \delta)$$

Since $\lambda > \lambda_0/2$,

$$\begin{aligned}
\frac{\mu(\lambda,\delta)+f'(0)}{\lambda} &= \frac{\mu(\lambda,0)+f'(0)}{\lambda} - \delta\bar{m} + \delta^2\frac{\gamma(\lambda)}{\lambda} + \frac{R_1(\lambda,\delta)}{\lambda} \\
&\le \frac{\mu(\lambda,0)+f'(0)}{\lambda} - \delta\bar{m} + \delta^2\frac{\gamma(\lambda_0)}{\lambda} + \\
&\quad + \delta^2\frac{(|\lambda-\lambda_0||\lambda_0| + |\lambda-\lambda_0|^2)\,\|B\|_s^2}{\lambda} + \frac{R_1(\lambda,\delta)}{\lambda} \\
&\le \frac{\mu(\lambda,0)+f'(0)}{\lambda} - \delta\bar{m} + \delta^2 C_3\|B\|_s^2 + \delta^3 C_4\|B\|_s^4 \\
&\le \frac{\mu(\lambda,0)+f'(0)}{\lambda} - \delta\bar{m} + \delta^2 C_5\|B\|_s^2 \qquad (3.20)
\end{aligned}$$

with C_3, C_4, C_5 bounded, independently of δ and B and λ. Combining this with a similar lower bound shows that the curve $\frac{\mu(\lambda)+f'(0)}{\lambda}$ satisfies

$$\begin{aligned}
-\delta\bar{m} - \delta^2 C_5\|B\|_s^2 &\le \frac{\mu(\lambda,\delta)+f'(0)}{\lambda} - \frac{\mu(\lambda,0)+f'(0)}{\lambda} \\
&\le -\delta\bar{m} + \delta^2 C_5\|B\|_s^2 \qquad (3.21)
\end{aligned}$$

Now, using the fact that $\frac{\mu(\lambda,0)+f'(0)}{\lambda} = (\lambda^2 + f'(0))/\lambda$ attains its infimum at λ_0 and satisfies $\partial_{\lambda\lambda}\frac{\mu(\lambda,0)+f'(0)}{\lambda} > 0$ for λ near λ_0, it is easy to see that (3.21) implies the lemma with $\beta \le C_6\|B\|_s$, for some constant C_6 independent of δ and B. $\square$

Returning to the variational formula, we apply Theorem 3.1.3, (3.17), and (3.19) to conclude that for δ sufficiently small

$$\begin{aligned}
c^*(\delta) &= \inf_{|\lambda-\lambda_0|\le\beta\delta} \frac{\mu(\lambda)+f'(0)}{\lambda} \\
&\ge \inf_{|\lambda-\lambda_0|\le\beta\delta} \frac{|\lambda|^2 + f'(0) - \lambda\delta\bar{m} + \delta^2\gamma(\lambda) - |R_1|}{\lambda} \qquad (3.22) \\
&= \inf_{|\lambda-\lambda_0|\le\beta\delta} \frac{|\lambda|^2 + f'(0) - \lambda\delta\bar{m} + \delta^2\gamma(\lambda_0) - \delta(\gamma(\lambda_0)-\gamma(\lambda)) - |R_1|}{\lambda}
\end{aligned}$$

where $\beta \leq C\|B\|_s$. Applying Lemma 3.1.6, we see that this is bounded by

$$
\begin{aligned}
c^*(\delta) &\geq \inf_{|\lambda-\lambda_0|\leq\beta\delta^2} \frac{|\lambda|^2 + f'(0) - \lambda\delta\bar{m} + \delta^2\gamma(\lambda_0) - C\delta^3\|B\|_\infty^3 - |R_1|}{\lambda} \\
&\geq \inf_{|\lambda-\lambda_0|\leq\beta\delta^2} \frac{|\lambda|^2 + f'(0) - \lambda\delta\bar{m} + \delta^2\gamma(\lambda_0)}{\lambda} - C(\delta^3\|B\|_s^3 - |R_1|) \\
&= 2\sqrt{f'(0) + \delta^2\gamma(\lambda_0)} - \delta\bar{m} - C(\delta^3\|B\|_s^3 - |R_1|) \\
&= c_0 - \delta\bar{m} + \frac{1}{\lambda_0}\delta^2\gamma(\lambda_0) - |R_2|, \qquad (3.23)
\end{aligned}
$$

where $|R_2| \leq C\delta^3(1+\|B\|_s^4)$ for $\delta < 1$. The last equality follows from the Taylor expansion of $\sqrt{f'(0) + \delta^2\gamma(\lambda_0)}$ and the fact that $c_0 = 2\sqrt{f'(0)}$. Similarly, we derive an upper bound on $c^*(\delta)$ using Theorem 3.1.3:

$$
\begin{aligned}
c^*(\delta) &= \inf_{|\lambda-\lambda_0|\leq\beta\delta^2} \frac{\mu(\lambda) + f'(0)}{\lambda} \\
&\leq \inf_{|\lambda-\lambda_0|\leq\beta\delta^2} \frac{|\lambda|^2 + f'(0) - \lambda\delta\bar{m} + \delta^2\gamma(\lambda) + |R_1|}{\lambda} \qquad (3.24) \\
&\leq c_0 - \delta\bar{m} + \frac{1}{\lambda_0}\delta^2\gamma(\lambda_0) + |R_2|, \qquad (3.25)
\end{aligned}
$$

Thus, we have proven that

$$
|c^*(\delta) - c_0 - \delta\bar{m} - \frac{1}{\lambda_0}\delta^2\gamma(\lambda_0)| \leq C\delta^3(1 + \|B\|_s^4) \qquad (3.26)
$$

for δ sufficiently small.

In the case that $B = B(z)$ is independent of t, then $W^{2,p}$ estimates (see [40]) applied to (3.14) and (3.15) imply that

$$
\|\chi\|_{W^{2,p}(D_T)} \leq C\|\lambda\tilde{B}\cdot k\|_{L^p(D_T)} \leq C|\lambda|\|B\|_\infty \qquad (3.27)
$$

for any $p > 1$ and some constant C independent of λ. We can choose $p > 1$ sufficiently large, depending on n, such that $\|\chi\|_\infty, \|\nabla\chi\|_\infty \leq C(p)\|\chi\|_{W^{2,p}(D_T)}$.

Therefore, (3.18) may be replaced by

$$\|\chi\|_\infty,\ \|\nabla\chi\|_\infty \le C|\lambda|\|B\|_\infty. \tag{3.28}$$

Similarly, all of the preceding estimates expressed in terms of $\|B\|_s$ can be expressed in terms of $\|B\|_\infty$, instead. This does not work in the case that B depends on t, since $W^{2,1,p}(D_T)$ does not embed continuously in $C^{1,s}(D_T)$. Hence, $\|\nabla\chi\|_\infty$ might not be controlled by $\|B\|_\infty$.

□

3.1.2 Bistable and Combustion Cases

When $f(u)$ does not satisfy the conditions for the KPP-type nonlinearity, the variational principle (3.3) does not hold. However, when f is either the bistable or combustion nonlinearity, the front speed c^* satisfies a min-max variational characterization derived by Hamel [41], Heinze, Papanicoloaou, and Stevens [44]. Following the notation of [44], define the functional

$$\psi(v) = \psi(v(z)) \equiv \frac{L\,v + f(v)}{\partial_x v} \equiv \frac{\Delta v + \delta b(y)\partial_x v + f(v)}{\partial_x v}. \tag{3.29}$$

The strong form of the min-max variational formula of [44] for $c(\delta)$ is:

$$\sup_{v\in K}\inf_{z\in D_R} \psi(v(z)) = c(\delta) = \inf_{v\in K}\sup_{z\in D_R} \psi(v(z)) \tag{3.30}$$

where $z = (x, y) \in D_R = R \times D$, and K is a set of admissible functions:

$$K = \{v \in C^2(D_R)|\ \partial_x\, v > 0,\ 0 < v(z) < 1, v \in I_s\}.$$

If one can construct an admissible test function $v \in K$ that approximates well the exact traveling front solution, then the variational principle (3.30) is a useful tool for getting tight bounds on c. This is the case when δ is small and the test function is a perturbation of the known traveling front solution when $\delta = 0$. In [44], a test function is proposed for deterministic shear flow, and the quadratic speed enhancement law of Theorem 3.1.2 is recovered if δ is small enough. To handle an ensemble of shear flows in the random case, as in the next section, one must know which norm of b controls the smallness of δ by analyzing an admissible test function v and the functional $\psi(v)$. However, in the analysis of [44] there is a delicate divergence problem when perturbing the traveling front at $\delta = 0$, making it necessary to use a new form of multi-scale test function to obtain Theorem 3.1.2. In this section we prove Theorem 3.1.2 using this new test function. These results have been published already by Nolen and Xin in [69]. The analysis follows the ideas sketched in [44] with the addition of estimates on the remainder terms in order to generalize to the random case in the next section.

Proof of Theorem 3.1.2:

Suppose that b is a given Hölder continuous deterministic function, and assume that $\bar{m} \equiv \frac{1}{|D|} \int_D b(y)\, dy = 0$. Otherwise, the integral mean contributes a linear term $\delta \bar{m}$ to the front speed. Because of the shear structure, the mean $\bar{m}$ has no other higher order effect on the speed, as may be the case for non-shear flows. Let $U = U(x + c_0 t)$ be the traveling front when $\delta = 0$, satisfying the equation:

$$U'' - c_0 U' + f(U) = 0,$$

$U(-\infty) = 0$, $U(+\infty) = 1$, $U' > 0$, and U approaches zero and one at exponential rates.

In the small δ regime, define the test function

$$v(z) = U(\xi) + \delta^2 \tilde{u}(\xi, y), \tag{3.31}$$

where the variable ξ is

$$\xi = (1 + \alpha\delta^2)x + \delta\chi, \tag{3.32}$$

with α a constant to be determined and $\chi = \chi(y)$ solution of:

$$-\Delta_y \chi = \tilde{b} = b, \quad y \in D,$$

subject to the appropriate boundary conditions, periodic or Neumann. We normalize χ by setting $\int_D \chi \, dy = 0$. The test function (3.31) reduces to the one in [44] if $\alpha = 0$. However, we shall see that $\alpha \neq 0$ is essential in suppressing certain divergence arising in the evaluation of $\psi(v)$; if $\alpha = 0$, the function v is not admissible.

For Hölder continuous $b(y)$, χ is $C^{2+p}(D)$, for some $p \in (0, 1)$. Assume that $\tilde{u}$ is C^2, and decays sufficiently fast as $|\xi| \to \infty$. A straightforward computation shows that:

$$\begin{aligned}
Lv + f(v) &= U'' + f(U) \\
&\quad + \delta^2 \left(2\,\alpha\,U'' + \Delta_{\xi,y}\tilde{u} + U''|\nabla\chi|^2 + f'(U)\tilde{u}\right) \\
&\quad + \delta^3 (b\,\alpha\,U' + 2\nabla_y \tilde{u}_\xi \cdot \nabla\chi) \\
&\quad + \delta^4 (\alpha^2\,U'' + 2\,\alpha\,\tilde{u}_{\xi\xi} + \Delta_\xi\,\tilde{u}\,|\nabla\chi|^2) \\
&\quad + f(v) - f(U) - f'(U)\,\delta^2\,\tilde{u} \\
&\quad + \delta^5\,\alpha\,b\,\tilde{u}_\xi + \delta^6\,\alpha^2\,\tilde{u}_{\xi\xi}, \qquad (3.33)
\end{aligned}$$

where $f(v) - f(U) - f'(U)\delta^2\tilde{u} = O(\delta^4\tilde{u}^2)$ and the $O(\delta)$ terms cancel by choice of $-\Delta\chi = b$. On the other hand, up to an undetermined constant γ, we have:

$$\begin{aligned}
(c_0 + \delta^2\gamma)\,\partial_x\,v &= c_0 U' + \delta^2 (c_0 \alpha U' + c_0 \tilde{u}_\xi + \gamma U') \\
&\quad + \delta^4 (\gamma\alpha U' + c_0 \alpha \tilde{u}_\xi + \gamma \tilde{u}_\xi) \\
&\quad + \delta^6 (\alpha\gamma\tilde{u}_\xi). \qquad (3.34)
\end{aligned}$$

All terms in (3.33) and (3.34) are evaluated at (ξ, y). Let us choose $\tilde{u}$ to solve the equation:

$$\bar{L}\,\tilde{u} \;=\; \Delta_{\xi,y}\,\tilde{u} - c_0\,\tilde{u}_\xi + f'(U)\,\tilde{u} = -|\nabla\chi|^2\,U'' - 2\,\alpha\,U'' + \gamma\,U' + c_0\,\alpha\,U', \quad (3.35)$$

subject to periodic or Neumann boundary conditions at ∂D, and exponential decay as $|\xi| \to \infty$. The solvability of (3.35) will be discussed later. Then we would have

$$Lv + f(v) = (\partial_x v)(c_0 + \delta^2\gamma) + R_1. \qquad (3.36)$$

The remainder R_1 is:

$$R_1 = \delta^3 A + \delta^4 B + \delta^5 D + \delta^6 E + O(\delta^4 \tilde{u}^2), \tag{3.37}$$

where

$$\begin{aligned}
A &= b\,\alpha U' + 2\nabla_y \tilde{u}_\xi \cdot \nabla\chi \\
B &= \alpha^2 U'' + 2\alpha\,\tilde{u}_{\xi\xi} + \Delta_\xi\,\tilde{u}\,|\nabla\chi|^2 - \gamma\,\alpha\,U' - c_0\,\alpha\,\tilde{u}_\xi - \gamma\,\tilde{u}_\xi \\
D &= \alpha b\,\tilde{u}_\xi \\
E &= \alpha^2\,\tilde{u}_{\xi\xi} - \alpha\,\gamma\,\tilde{u}_\xi.
\end{aligned} \tag{3.38}$$

The constant γ is determined by a solvability condition for (3.35). The right hand side must be orthogonal to the function $U'(\xi)\,e^{-c_0\xi}$ which spans the kernel of the adjoint operator $\bar{L}^* v = \Delta_{\xi,y} v + c_0 v_\xi + f'(U)v$. Thus,

$$\begin{aligned}
\gamma \int_R (U')^2 e^{-c_0\xi}\,d\xi &= \frac{1}{|D|}\int_{D_R} \left((|\nabla\chi|^2 + 2\alpha)U''U' - c_0\alpha(U')^2\right) e^{-c_0\xi} d\xi\,dy \\
&= \frac{c_0}{2}\langle|\nabla\chi|^2 + 2\alpha\rangle \int_R (U')^2 e^{-c_0\xi}\,d\xi - c_0\alpha \int_R (U')^2 e^{-c_0\xi}\,d\xi
\end{aligned}$$

where the bracket denotes the integral average over D. We have performed integration by parts once, the boundary terms decay to zero, and the integral $\int_R (U')^2 e^{-c_0\xi}\,d\xi$ converges for both bistable and combustion type nonlinearities. It follows that equation (3.35) is solvable when

$$\gamma = \frac{c_0}{2|D|}\int_D |\nabla\chi|^2\,dy = \frac{c_0}{2}\langle|\nabla\chi|^2\rangle, \tag{3.39}$$

regardless of the choice of α. For such γ, we may choose $\alpha = -\frac{1}{2}\langle|\nabla\chi|^2\rangle$, so $\gamma + c_0\,\alpha = 0$ and the $\tilde{u}$ equation becomes:

$$\bar{L}\tilde{u} = \Delta_{\xi,y}\tilde{u} - c_0\tilde{u}_\xi + f'(U)\tilde{u} = \left(\langle|\nabla\chi|^2\rangle - |\nabla\chi|^2\right) U''. \tag{3.40}$$

Note that the right hand side of (3.40) has zero integral average over D.

Now we show that (3.40) is solvable, and we obtain estimates on the remainder in order to show that $\frac{R_1}{\partial_x v}$ is $O(\delta^3)$. As suggested in [44], the solution $\tilde{u}$ of (3.40) can be expanded in eigenfunctions of the Laplacian Δ_y:

$$\begin{aligned}\tilde{u}(\xi, y) &= \sum_{j \in Z_+^{n-1} \cup \{0\}} u_j(\xi)\phi_j(y), \\ |\nabla_y \chi|^2 &= \sum_{j \in Z_+^{n-1} \cup \{0\}} a_j \phi_j(y),\end{aligned}$$

where Z_+^{n-1} denotes nonnegative integer vectors in R^{n-1} with at least one positive component; $-\Delta_y \phi_j = \lambda_j \phi_j$ with periodic or Neumann boundary conditions on ∂D. The eigenfunctions ϕ_j are normalized so that $\|\phi_j\|_{L^\infty} = 1$, and the eigenvalues $\lambda_j = O(|j|^2)$ for large $|j|$. Clearly, $\phi_0 = 1$ and $\lambda_0 = 0$, $a_0 = \langle |\nabla \chi|^2 \rangle$. The functions u_j solve the equations

$$L_j u_j = u_j'' - \lambda_j u_j - c_0 u_j' + f'(U) u_j = -(a_j - \delta_{j,0}\, a_0)\, U''. \tag{3.41}$$

The $j = 0$ equation has zero right hand side, and we take $u_0 \equiv 0$. If α were zero (as in [44]), equation (3.35) implies that the right hand side of the $j = 0$ equation would be $-a_0\, U'' + \gamma\, U'$. The general solution (zeroth-mode) for u_0 is $-\frac{a_0}{2} \xi\, U'(\xi) + C\, U'(\xi)$, for any constant C. The linear factor ξ would render $v(x)$ unable to stay in the set K due to divergence at large ξ (and x).

Now consider the equations for $j \neq 0$. As shown in [88], the operators L_j are invertible in part due to $\lambda_j > 0$ for $j \neq 0$. As in Lemma 2.3 of [88], we have the regularity estimate:

$$\|u_j\|_{C^2(R)} \leq C_1 |a_j|, \quad j \neq 0, \tag{3.42}$$

where the constant C_1 is independent of j and depends only on U and its derivatives.

Next we show that for $j \neq 0$, the ratio $\frac{u_j}{U'}$ is bounded by $C_2 \, |a_j|$, where C_2 is a positive constant independent of j. It suffices to prove a uniform bound if the a_j factor is one. Consider the function $w_j = \beta U' - u_j$ for some positive constant β. For some $r_0 > 0$ sufficiently large, $f'(U) \leq 0$ whenever $|\xi| \geq r_0$. Choose β as

$$\beta \geq \beta_1 \equiv \frac{C_1}{\min_{|\xi| \leq r_0} U'(\xi)}, \tag{3.43}$$

so that by (3.42), $w_j > 0$ in the region $|\xi| \leq r_0$. By equation (3.41) for u_j and the fact that U' solves $L_j U' = -\lambda_j U' < 0$, we have

$$L_j w_j = -\beta \lambda_j U' + U''. \tag{3.44}$$

Using the fact that U'' is bounded by a constant multiple of U', and that $\inf_{j \neq 0} \lambda_j > 0$, we may further increase β if necessary to ensure that $L_j w_j < 0$ in the region $|\xi| > r_0$. The maximum principle implies that $w > 0$ for all $\xi \in R$. Repeating the argument for $-u_j$ and taking into account the a_j factor gives

$$\frac{|u_j|}{U'} \leq C_3 |a_j|, \tag{3.45}$$

where the constant C_3 depends only on U, and (3.45) holds uniformly in $j \neq 0$. Inequality (3.45) says that there is no resonance when inverting the operator L_j to find u_j, for $j \neq 0$. In other words, u_j decays the same as U' as $\xi \to \pm\infty$.

To improve the estimate above, we move the term $f'(U) \, u_j$ to the right hand side of (3.41) which is then bounded by $C_3' |a_j| U'$, for C_3' independent of

j. The left hand side operator becomes $u_j'' - \lambda_j\, u_j - c_0\, u_j'$. An explicit formula can be written for u_j, and the asymptotics of λ_j imply that

$$\frac{|u_j^{(i)}(\xi,y)|}{U'(\xi)} \leq C_4 \frac{|a_j|}{(1+|j|^2)}, \tag{3.46}$$

for a constant C_4 depending only on U, where $i = 0,1,2$ denotes the order of derivatives with respect to ξ.

If b is p-Hölder continuous, Schauder-type estimates [40] imply that

$$\| |\nabla\chi|^2 \|_{C^{1+p}(D)} \leq C\, \|b\|^2_{C^p(D)}, \tag{3.47}$$

For a rectangular cross section D (dimension $n-1$), the ϕ_j's are trigonometric functions. Then (3.47) implies that

$$|a_j| \leq C_5\, \|b\|^2_{C^p}\, (1+|j|)^{-(1+p)}, \;\; p \in (0,1), \tag{3.48}$$

for some constant C_5 (see [93] regarding the relationship between regularity and the decay of Fourier coefficients). Combining (3.46) and (3.48), we see that the eigenfunction expansion of $\tilde{u}$

$$\tilde{u}(\xi,y) = \sum_{j\in Z_+^{n-1}} u_j(\xi)\, \phi_j(y) \tag{3.49}$$

converges uniformly in (ξ, y). Moreover,

$$\begin{aligned} \frac{|\tilde{u}_\xi^{(i)}(\xi,y)|}{U'(\xi)} &\leq \sum_{j\in Z_+^{n-1}} |u_j^{(i)}|(\xi)\, |\phi_j|(y)/U' \\ &\leq C_4\, C_5\, \|b\|^2_{C^p} \sum_{j\in Z_+^{n-1}} (1+|j|)^{-(3+p)} = C_6\, \|b\|^2_{C^p}, \end{aligned} \tag{3.50}$$

if $n = 2, 3$. The mixed derivative term is bounded as:

$$\begin{aligned} \frac{|\nabla_y \tilde{u}_\xi(\xi, y)|}{U'(\xi)} &\leq \sum_{j \in Z_+^{n-1}} \frac{|u_j^{(1)}|(\xi)}{U'} |j| \, |\phi_j|(y) \\ &\leq C_5 \, \|b\|_{C^p}^2 \sum_{j \in Z_+^{n-1}} (1 + |j|)^{-(2+p)} = C_7 \, \|b\|_{C^p}^2, \end{aligned} \tag{3.51}$$

if $n = 2, 3$. In case $n = 2$, Ω is an interval, and $b \in L^\infty(\Omega)$ suffices. This is because $\nabla_y \chi$ is Lipschitz, so $|a_j| \leq O(\|b\|_\infty^2 |j|^{-p})$ for some $p \in (0, 1)$ which replaces estimate (3.48). The exponent in (3.51) decreases to $1 + p$, which is still enough for convergence of the sum over $j \in Z_+^1$. As a result, for $n = 2$, $i = 0, 1, 2$, the estimates:

$$\frac{|\tilde{u}_\xi^{(i)}(\xi, y)|}{U'(\xi)} \leq C_8 \, \|b\|_\infty^2, \quad \frac{|\nabla_y \tilde{u}_\xi(\xi, y)|}{U'(\xi)} \leq C_9 \, \|b\|_\infty^2, \tag{3.52}$$

hold.

With these estimates, we can verify the admissibility of test function v. Clearly, v approaches zero (one) exponentially as $x \to -\infty$ $(+\infty)$. Because

$$v_x = (1 + \delta^2 \alpha) U' + \delta^2 (1 + \delta^2 \alpha) \tilde{u}_\xi,$$

we have for $n = 3$:

$$v_x \geq \frac{1}{2} U' - \frac{1}{2} \delta^2 |\tilde{u}_\xi| \geq \left[\frac{1}{2} - \delta^2 \frac{C_6}{2} \, \|b\|_{C^p}^2 \right] U' \geq \frac{1}{4} \, U' > 0, \tag{3.53}$$

if

$$\delta^2 \leq \min \left(\frac{1}{2|\alpha|}, \frac{1}{2} \, (C_6 \, \|b\|_{C^p}^2)^{-1} \right). \tag{3.54}$$

For $n = 2$, in view of (3.52), $v_x \geq \frac{1}{4} U'$ if

$$\delta^2 \leq \min \left(\frac{1}{2|\alpha|}, \frac{1}{2} \, (C_8 \, \|b\|_\infty^2)^{-1} \right). \tag{3.55}$$

It follows from the monotonicity of v in x and its behavior as $x \to \pm\infty$ that $0 < v < 1$ for all x, y.

With v being an admissible test function, we see from (3.36) that

$$\psi(v(z)) = \frac{Lv + f(v)}{\partial_x v} = c_0 + \delta^2\gamma + \frac{R_1}{\partial_x v}, \tag{3.56}$$

with R_1 defined by (3.37). By (3.54) or (3.55), $\partial_x v \geq \frac{1}{4}U'$. Also, each term in R_1 is bounded by a multiple of U'. It follows that for δ satisfying (3.54) or (3.55), $|\frac{R_1}{\partial_x v}| \leq |4\frac{R_1}{U'}| \leq C_{10}\,\delta^3$, where $C_{10} = C_{10}(U, \|b\|_{C^p}), p \in (0,1)$, if $n = 3$, $C_{10} = C_{10}(U, \|b\|_\infty)$ if $n = 2$. This completes the proof of Theorem 3.1.2. $\square$

3.1.3 Fourier Analysis of the Quadratic Enhancement Term

Before extending our analysis to random flows, we conclude this section by computing the quadratic enhancement term

$$\gamma(\lambda_0) = \frac{1}{D_T}\int_{D_T} |\nabla\chi|^2$$

in terms of the Fourier coefficients of B. Let $D_T = [0, L]^2 \times [0, T]$,

$$\chi = \sum_{(m,w)\in\mathbb{Z}^3_+} a_{m,w} e^{-i(m2\pi z/L + w2\pi t/T)},$$

and

$$\tilde{B} = \sum_{(m,w)\in\mathbb{Z}^3_+} \bar{\beta}_{m,w} e^{-i(m2\pi z/L + w2\pi t/T)}, \tag{3.57}$$

where $\mathbb{Z}^3_+ = \mathbb{Z}^3 \setminus (0,0,0)$, since χ and $\tilde{B}$ have zero mean. Then plugging into (3.14) we see that

$$a_{m,w} = \frac{\lambda_0 \bar{\beta}_{m,w} \cdot k}{\left(-\frac{4\pi^2|m|^2}{L^2} + i(w2\pi/T + 4\pi\lambda_0\frac{k\cdot m}{L})\right)}. \tag{3.58}$$

Thus $\gamma(\lambda_0) = \frac{4\pi^2}{L^2}\sum_{(m,w)\in\mathbb{Z}^3_+}|m|^2|a_{m,w}|^2$ is given by

$$\gamma(\lambda_0) = \frac{\lambda_0^2 4\pi^2}{L^2}\sum_{(m,w)\in\mathbb{Z}^3_+}\frac{|m|^2|\bar{\beta}_{m,w}\cdot k|^2}{\frac{16\pi^4}{L^4}|m|^4 + (w2\pi/T + 4\pi\lambda_0\frac{k\cdot m}{L})^2}. \tag{3.59}$$

Formula (3.58) shows that $|a_{m,w}| \to 0$ as $T \to 0$, whenever $w \neq 0$. This means that all modes that depend nontrivially on time will vanish as the frequency of temporal oscillations increases to ∞. As a result, $\gamma(\lambda_0)$ tends to zero as $T \to 0$ whenever B has a product structure

$$B(z,t) = \sum_j B_1^j(z)b^j(t)$$

where $\{B_1^j\}$ are a family of periodic, divergence free velocity fields, and $\{b^j(t)\}$ are all periodic with zero mean over $[0,T]$. Hence, faster temporal oscillations in the flow tend to diminish the enhancing effect of spatial variation in the flow. If $B(z,t) = (b(y,t),0)$ is a shear flow, then (3.58) reduces to

$$\gamma(\lambda_0) = \frac{4\pi^2}{L}\sum_{(m,w)\in\mathbb{Z}^2_+}\frac{|m|^2|\beta_{m,w}|^2}{\frac{16\pi^4}{L^4}|m|^4 + (w2\pi/T)^2} \tag{3.60}$$

since $k\cdot m = 0$ whenever $\beta_{m,w} \neq 0$, due to the shear structure. In this case, the coefficients $|a_{m,w}|$ decrease to zero monotonically in T. In the general case, however, there may be a kind of resonance between the temporal and spatial oscillations. This is seen in the fact that (3.59) may have a local maximum with respect to T.

From (3.59) we also see that as $T \to \infty$ (very slow temporal oscillations), $\gamma(\lambda_0)$ has a nontrivial limit

$$\lim_{T\to\infty}\gamma(\lambda_0) \quad = \quad \frac{4\pi^2}{L^2}\sum_{m\in\mathbb{Z}^2_+}\frac{|m|^2|\beta'_m|^2}{\frac{16\pi^4}{L^4}|m|^4 + (4\pi\lambda_0\frac{k\cdot m}{L})^2} \tag{3.61}$$

where $\beta'_m = \sum_{w \in Z} |\bar{\beta}_{m,w} \cdot k|^2$.

As an illustrative example, which we will study numerically in the next chapter, consider the periodically perturbed flow

$$B(z,t) = B_c(z + \epsilon k(\sin(\frac{2\pi}{T}t)),$$

where $B_c = B_c(x,y)$ is independent of t and $k = (1,0)$. If $\{\bar{\beta}_m\}$ are the Fourier coefficients of B_c, then $B(z,t)$ has the representation

$$B(z,t) = \sum_{m \in \mathbb{Z}^2_+} \bar{\beta}_m e^{-i(2\pi m \cdot z/L)} e^{-i(\frac{m \cdot k 2\pi\epsilon}{L} \sin(\frac{2\pi}{T}t))}.$$

If $\frac{\epsilon}{L}$ is sufficiently small, then we can approximate

$$e^{-i(\frac{m \cdot k 2\pi\epsilon}{L} \sin(\frac{2\pi}{T}t))} \approx 1 - i\frac{m \cdot k 2\pi\epsilon}{L} \sin(\frac{2\pi}{T}t). \tag{3.62}$$

Now suppose that $B_c = \nabla^\perp H$ is defined by the stream function $H(x,y) = \sin(2\pi x/L)\sin(2\pi y/L)$. With the approximation (3.62), the nonzero Fourier coefficients of B correspond to $(m,w) = (\pm 1, \pm 1, 0)$ and $(m,w) = (\pm 1, \pm 1, \pm 1)$, and the enhancement term γ can be explicitly computed. In Figure 3.1, we plot a computation of $\tilde{c}^*/\tilde{c}^{**}$ where $\tilde{c}^*$ is the leading order approximation of c^* given by (3.1), and $\tilde{c}^{**}$ is the leading order approximation of c^{**}, the minimal speed when there is no temporal perturbation ($\epsilon = 0$). We observe a resonance effect when $T = L/c_0$. That is, the front speed enhancement is greatest when the time scale of the perturbation matches that of the unperturbed flow. As the temporal frequency $\omega = \frac{1}{T} \to \infty$, we observe that $c^* \to c^{**}$. However, as $\omega \to 0$, c^* converges to a limit that is strictly greater than c^{**}.

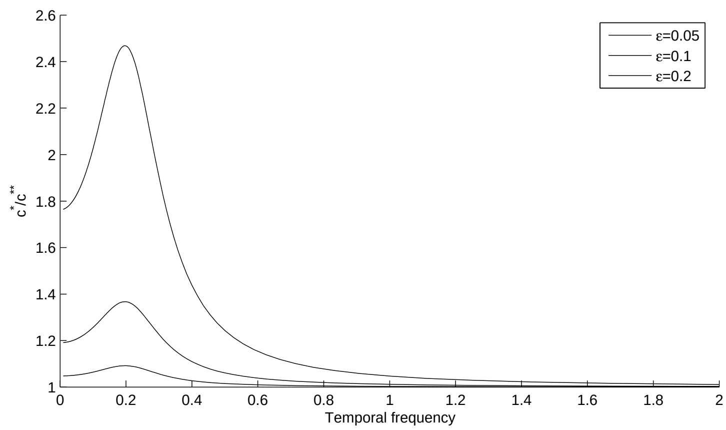

Figure 3.1: Enhancement of c^* as a function of temporal frequency $\omega = 1/T$.

3.2 Spatially Random Flows in a Cylinder

In this section, we assume channel conditions on the domain and the flow B. We consider the case when the flow B is perturbed by a random shear flow

$$B(z) = B_1(z) + b(y, \omega) \tag{3.63}$$

where $B_1(z)$ satisfies the channel conditions, and $b(y, \omega)$ is a random field. Since D is a channel, we consider only propagation in the x-direction: $k = (1, 0)$. As before, we scale the flow $B \mapsto \delta B$. In this case, the corresponding minimal speed $c^*(\delta) = c^*(\delta, \omega)$ is a random variable for each δ, and we consider how the expectation $E[c^*(\delta)]$ scales with the parameter δ. For a fixed realization, Theorems 3.1.1 and 3.1.2 imply that $c^*(\delta)$ scales like $O(\delta^2)$ when δ is sufficiently small, depending on the realization. We now show that the same quadratic scaling law remains valid for enhancement of the averaged front speeds under suitable moment conditions on the flow. As before, we define $\bar{m} = \bar{m}(\omega) = \frac{1}{|D|}\int_D B \cdot k\, dz$.

Theorem 3.2.1. *Let f be the KPP-type nonlinearity. Let $B(z)$ be defined by (3.63), where $b(y,\omega)$ is a stationary random process in R^{n-1} so that sample paths are almost surely continuous and*

$$E[\|b\|_\infty^8] < +\infty. \tag{3.64}$$

Then for δ small, the expectation $E[c^*(\delta)]$ has the expansion

$$E[c^*(\delta)] = c_0 - \delta E[\bar{m}] + \frac{\delta^2}{\lambda_0 |D|}\int_D E[|\nabla \chi|^2]\, dz + O(\delta^3), \tag{3.65}$$

where $B(z,\omega) = \bar{m}(\omega)k + \tilde{B}(z,\omega)$; and $\chi = \chi(z,\omega)$ solves $-\Delta_z \chi + 2\lambda_0 k \cdot \nabla \chi = \lambda_0 \tilde{B} \cdot k,\ z \in D$, subject to zero Neumann boundary conditions in y.

If we assume $B_1 \equiv 0$, we have a similar result for the bistable and combustion case, in dimension $n = 2$ or $n = 3$ and with slightly stronger conditions on the flow regularity:

Theorem 3.2.2. *Let f be either the bistable or combustion-type nonlinearity. Let $D = [0,L]^2$, and b be a random field satisfying*

$$E[\|b\|_s^4] < \infty. \tag{3.66}$$

Then the expectation $E[c(\delta)]$ has the expansion

$$E[c(\delta)] = c_0 + \delta^2 \frac{c_0}{2|D|}\int_D E[|\nabla \chi|^2]\, dy + O(\delta^3).$$

where χ satisfies: $-\Delta_{\tilde{x}}\, \chi = b_1$, subject to zero Neumann boundary condition at ∂D.

Proof of Theorem 3.2.1: The proofs of Lemma 3.1.7 and Theorem 3.1.1 show that there are universal constants C_1 and C_2 such that if $\delta\|B\|_\infty \leq C_1$ and $\delta\|B\|_\infty^2 \leq C_2$, then (3.26) holds. Let $A = \{w| \ \delta\|B\|_\infty \leq C_1, \ \ \delta\|B\|^2 \leq C_2\}$. Then

$$\begin{aligned} &E[|c^*(\delta) - c_0 + \delta\bar{m} - \frac{1}{\lambda_0}\delta^2\gamma(\lambda_0)|] = \\ &\quad = E[|c^*(\delta) - c_0 + \delta\bar{m} - \frac{1}{\lambda_0}\delta^2\gamma(\lambda_0)|\chi_A] + \\ &\quad\quad + E[|c^*(\delta) - c_0 + \delta\bar{m} - \frac{1}{\lambda_0}\delta^2\gamma(\lambda_0)|\chi_{A^C}] \qquad (3.67) \\ &\quad \leq C\delta^3(1 + E[\|B\|_\infty^4]) + E[|c^*(\delta) - c_0 + \delta\bar{m} - \frac{1}{\lambda_0}\delta^2\gamma(\lambda_0)|\chi_{A^C}] \end{aligned}$$

By Hölder's inequality,

$$\begin{aligned} &E[|c^*(\delta) - c_0 + \delta\bar{m} - \frac{1}{\lambda_0}\delta^2\gamma(\lambda_0)|\chi_{A^C}] \leq \\ &\leq E[|c^*(\delta) - c_0 + \delta\bar{m} - \frac{1}{\lambda_0}\delta^2\gamma(\lambda_0)|^2]^{1/2}P(A^C)^{1/2} \\ &\leq \delta E[\|B\|_\infty^4]^{1/2}P(A^C)^{1/2} \qquad (3.68) \end{aligned}$$

We can bound $P(A^C)^{1/2}$ by Tchebychev's inequality,

$$\begin{aligned} P(A^C) &\leq P(\|B\|_\infty \geq \frac{C_1}{\delta}) + P(\|B\|_\infty^2 \geq \frac{C_2}{\delta}) \\ &\leq C(\delta^4 E[\|B\|_\infty^4 + \delta^4 E[\|B\|_\infty^8]) = O(\delta^4) \qquad (3.69) \end{aligned}$$

Combining (3.67), (3.68), and (3.69) we see that

$$E[|c^*(\delta) - c_0 + \delta\bar{m} - \frac{1}{\lambda_0}\delta^2\gamma(\lambda_0)|] = O(\delta^3) \qquad (3.70)$$

□

Proof of Theorem 3.2.2: From (3.54) and (3.55), there is a constant C_{17}, independent of δ and b, such that the test function v is admissible if

$$\|b\|_{C^p} \leq \frac{C_{17}}{\delta}. \tag{3.71}$$

In this case, $c(\delta)$ has the expansion

$$c(\delta) = c_0 - \delta\bar{b} + \delta^2\gamma + R. \tag{3.72}$$

Using (3.37), (3.38), and (3.52), we find that $|R| \leq C\delta^3\|b\|_p^4$. Thus, the remainder satisfies the bound

$$E[|R|] \leq C\delta^3 E[\|b\|_p^4] = O(\delta^3).$$

Let A_δ be the set

$$A_\delta = \left\{ w| \ \ \|b(\cdot,\omega)\|_p \leq \frac{C_1}{\delta} \right\}$$

Then we have

$$\begin{aligned} E\left[|c(\delta) - c_0 - \delta^2\gamma|\right] &= \\ &\leq E\left[|(c(\delta) - c_0 + \delta^2\gamma)\chi_{A_\delta}|\right] \\ &\quad +E\left[|(c(\delta) - c_0 + \delta^2\gamma)\chi_{A_\delta^c}|\right] \\ &\leq E[|R|] + E\left[|(c(\delta) - c_0 + \delta^2\gamma)\chi_{A_\delta^c}|\right]. \end{aligned}$$

Using Hölder's inequality and the fact that $|c(\delta)-c_0-\delta^2\gamma| \leq \max(\delta^2|\gamma|, \delta\|b\|_\infty) \leq C\delta\|b\|_p^2$ when $\delta < 1$, this reduces to

$$E\left[|c(\delta) - c_0 - \delta^2\gamma|\right] \leq E[|R|] + C\delta E[\|b\|_p^4]^{1/2} P(A_\delta^c)^{1/2}. \tag{3.73}$$

To bound $P(A_\delta^c)$, we use Tchebychev's inequality:

$$P\left(\|b(\cdot,\omega)\|_p \geq \frac{C_1}{\delta}\right) \leq C_1\delta^4 E[\|b\|_p^4] = O(\delta^4) \tag{3.74}$$

for some constant C_1. Consequently, $C\delta E[\|b\|_p^4]^{1/2} P(A_\delta^c)^{1/2} = O(\delta^3)$. Equation (3.73) now implies that

$$E[c(\delta)] \;\; = \;\; c_0 + \delta^2 E[\gamma] + O(\delta^3).$$

□

Lemma 3.2.3. *Let $b(y,\omega)$ be mean zero, stationary Gaussian process on D such that the function*

$$p(u) = \max_{\|s-t\|\leq|u|\sqrt{2}} [E|b(s)-b(t)|^2]^{1/2}.$$

is Hölder continuous. Then b satisfies the hypotheses of Theorem 3.2.1 and Theorem 3.2.2.

Proof of Lemma: To prove this, we will need probabilistic estimates of the Hölder norm of b, which we obtain by using Garsia's Theorem and the Karhunen-Loève expansion. For a mean zero Gaussian field, this expansion is related to the structure of the covariance function $R(t,s) = E[b(t)b(s)]$, $t,\ s \in \Omega$. If the covariance function is continuous, positive and non-negative definite, there exists a Gaussian process with this covariance [2]. The symmetric integral operator: $\phi \to \int_D R(t,s)\,\phi(t)\,dt$ generates a complete set of orthonormal eigenfunctions ϕ_j on $L^2(D)$ with nonnegative eigenvalues λ_j, $j = 1, 2, \cdots$. Define:

$$p(u) = \max_{\|s-t\|\leq|u|\sqrt{2}} [E|b(s)-b(t)|^2]^{1/2}.$$

Consider the partial sum (m a positive integer):

$$X^{(m)}(t,\omega) = \sum_{j=1}^{m} \sqrt{\lambda_j}\, \phi_j(t)\theta_j(\omega), \tag{3.75}$$

where θ_j's are independent unit Gaussian random variables. The convergence of the partial sum to b is given by Garsia's Theorem (Theorem 3.3.2, p. 52, [1]). It says that if $\int_0^1 (-\log u)^{1/2}\, dp(u) < \infty$, then with probability one, the functions $X^{(m)}(t)$ are almost surely equicontinuous and converge uniformly on Ω. The resulting infinite series is the Karhunen-Loève expansion. Moreover, the following estimate holds for all m:

$$\begin{aligned} |X^{(m)}(s) - X^{(m)}(t)| \;&\leq\; 16\sqrt{2}[\log H]^{1/2}\, p(\|s-t\|) \\ &\quad +32\sqrt{2}\int_0^{\|s-t\|} (-\log u)^{1/2}\, dp(u), \end{aligned} \tag{3.76}$$

where $H = H(\omega)$ is a positive random variable, $E[H^2] \leq 32$.

Suppose that $p(u)$ is Hölder continuous with exponent $s \in (0,1)$, then b is almost surely Hölder continuous with exponent s, and the Hölder norm of b is bounded by $\alpha_1[\log H]^{1/2} + \alpha_2$, where α_1, α_2 are two positive deterministic constants. Tchebychev's inequality gives:

$$\text{Prob}\left([\log H]^{1/2} \geq \lambda\right) = \text{Prob}\left(H \geq e^{\lambda^2}\right) \leq E(H^2)/e^{2\lambda^2} \leq 32\, e^{-2\lambda^2}. \tag{3.77}$$

This implies that

$$\text{Prob}\left(\|b\|_s > \lambda\right) \leq 32 e^{-2\left(\frac{\lambda-\alpha_2}{\alpha_1}\right)^2} \leq C_{15} e^{-2\lambda^2} \tag{3.78}$$

Therefore, all moments of the random variables $\|b\|_s$ and $\|b\|_\infty$ are finite. □

For the stationary Ornstein-Uhlenbeck (O-U) process with $E[b] = 0$, we can derive an explicit formula for the leading order enhancement.

Corollary 3.2.4 (Explicit Average Speed Formula)**.** *Consider the O-U process $b(y, \omega)$ as solution of the Itô equation:*

$$dX(y) = -a\, X(y)\, dy + r\, dW(y), \quad y \in [0, L], \tag{3.79}$$

where $W(y, \omega)$ is the standard Wiener process, $X(0, \omega) = X_0(\omega)$ is a Gaussian random variable with mean zero, and variance $\rho = r^2/(2a)$. Then $X(y, \omega)$ satisfies the moment conditions in Theorem 2.1. The averaged KPP front speed in the channel $R \times [0, L]$ is given by

$$E[c^*(\delta)] = c_0 + \frac{c_0 \delta^2}{2} \mathrm{enh} + O(\delta^3), \quad \delta \ll 1, \tag{3.80}$$

where:

$$\mathrm{enh} = \frac{r^2}{2a} \left(e^{-aL} \left(\frac{4}{L^2 a^4} - \frac{1}{3a^2} \right) + \frac{L}{3a} - \frac{4}{L^2 a^4} - \frac{5}{3a^2} + \frac{4}{La^3} \right). \tag{3.81}$$

Proof: The O-U process is stationary and Markov. Its sample paths are almost surely Hölder continuous though nowhere differentiable. The process can be written as

$$b(y, \omega) = e^{-ay} b(0, \omega) + r \int_0^y e^{-a(y-s)} dW_s(\omega). \tag{3.82}$$

The covariance function of this process is $E[b(y)b(s)] = \rho e^{-a|y-s|}$, and therefore $p(u)$ is Hölder continuous.

To compute the quadratic enhancement term in (3.80), notice that

$$(\chi_x(x))^2 = \int_0^x \int_0^x b_1(s) b_1(y)\, ds\, dy,$$

and

$$E[(\chi_x(x))^2] = \int_0^x \int_0^x E[b_1(s) b_1(y)]\, ds\, dy.$$

We can calculate $E[b_1(s)b_1(y)]$ in terms of the covariance function $E[b(s)b(y)]$, since

$$E[b_1(y)b_1(s)] = E[b(s)b(y)] - E[b(s)\bar{b}] - E[b(y)\bar{b}] + E[\bar{b}^2],$$

$$E[b(s)\bar{b}] = \frac{1}{L}\int_0^L E[b(y)b(s)]\, dy,$$

$$E[\bar{b}^2] = \frac{1}{L^2}\int_0^L \int_0^L E[b(s)b(y)]\, dy\, ds.$$

After computing these integrals, we find that

$$\begin{aligned} E[\langle|\chi_x|^2\rangle] &= \frac{r^2}{2a}\left(\frac{2L}{3a} + \frac{2}{3a^2}(e^{-aL} - 1)\right) \qquad (3.83) \\ &\quad -\frac{r^2}{2a}\frac{2}{L}\int_0^L \frac{2x^2}{La} + \frac{x}{La^2}(e^{-ax} - 1) + \frac{x}{La^2}(e^{-aL} - e^{-a(L-x)})\, dx \\ &\quad +\frac{r^2}{2a}\frac{1}{L}\int_0^L \frac{2x}{a} + \frac{2}{a^2}(e^{-ax} - 1)\, dx \\ &= \frac{r^2}{2a}\left(e^{-aL}\left(\frac{4}{L^2a^4} - \frac{1}{3a^2}\right) + \frac{L}{3a} - \frac{4}{L^2a^4} - \frac{5}{3a^2} + \frac{4}{La^3}\right). \end{aligned}$$

□

The above results imply that the mean front speed grows quadratically with δ, when δ is small. However, in the limit of large advection, the mean front speed grows linearly. The following theorem follows from a recent result of S. Heinze [42]:

Theorem 3.2.5. *If the stationary shear process $b(y,\omega)$ has almost surely continuous sample paths and satisfies $E[\|b\|_\infty] < \infty$, then the amplified shear field $\delta\, b(y,\omega)$ generates an average front speed that satisfies*

$$\lim_{\delta\to\infty} E[|c^*(\delta,\omega)|]/\delta = E[d^*] < \infty$$

where d^* is the random variable

$$d^*(\omega) = \sup_{\psi\in D_1} \int_D b(y,\omega)\, \psi^2(y)\, dy,$$

where:

$$D_1 = \{\psi \in H^1(D) : \|\nabla\psi\|^2_{L^2(D)} \leq f'(0), \|\psi\|_{L^2(D)} = 1\}.$$

Proof: By Theorem 1.3 of [42], $\frac{c*(\delta,\omega)}{\delta} \to d^*(\omega)$ as $\delta \to \infty$, for each realization. Because of the upper bound $|c^*(\delta,\omega)| \leq |c_0| + \delta\|b\|_\infty$, we also have $\frac{|c^*(\delta,\omega)|}{\delta} \leq 1 + \|b\|_\infty \equiv Y$, when $\delta > |c_0|$. By assumption, $E(Y) < \infty$, so the dominated convergence theorem implies the limit:

$$E[\frac{|c^*(\delta,\omega)|}{\delta}] \to E[d^*(\omega)] \leq E(Y).$$

□

If a realization of b were to have a flat piece in D, $d^*(\omega)$ would be equal to $\sup_D b(y,\omega)$. However, this happens with zero probability for a nontrivial stationary Gaussian random field. It appears that the distribution of $d^*(\omega)$ is analytically unknown. In fact, even the distribution of $\sup_D b(y)$ is known only in a few special cases [1], not including the O-U process. For this reason, an efficient numerical technique like the one described in the following section is very useful for the analysis of the statistical behavior of $c^*(\delta)$ as $\delta \to \infty$.

3.3 Temporally Random Shear Flows

In this section we derive bounds on the speed c^* when the flow is temporally random and spatially periodic. We will assume that the shear $b(y,t)$ has the form

$$b(y,t) = \sum_{j=1}^{N} b_1^j(y) b_2^j(t) \tag{3.84}$$

where $b_1^j(y)$ are Lipschitz continuous and periodic in y, and $b_2^j(t)$ are stationary centered Gaussian fields such that the Assumptions A1-A3 of Section 2.1 are satisfied. We have shown that if $b(y,t)$ is periodic in both space and time then $c^*(\delta) = c^*(0) + O(\delta^2)$ for δ small and $c^*(\delta) = c^*(0) + O(\delta)$ for δ as $\delta \to \infty$. For temporally random shear flows, the following theorem gives some analytical bounds consistent with the asymptotic behavior in the periodic case. Throughout this section we consider front propagation in the direction $k = (1, \mathbf{0})$, which is aligned with the direction of the shear.

Theorem 3.3.1 (Bounds on c^*)**.** *For all $\delta \geq 0$, $c^*(\delta)$ satisfies the bounds*

(i) $c^*(\delta) \geq c^*(0)$.

(ii) $c^*(\delta) = c^*(0)$ *if* $b(y,t) = b(t)$.

(iii) $c^*(\delta) \leq c^*(0) + \delta \sum_{j=1}^{N} \|b_1^j\|_\infty E_Q[|b_2^j|]$.

(iv) $c^*(\delta) \leq c^*(0)\sqrt{1 + \delta^2 p_1}$.

From (iv), we also have

$$c^*(\delta) \leq c^*(0)(1 + \frac{\delta^2 p_1}{2}) + O(\delta^3)$$

when δ is small.

We also have a linear lower bound on the growth of $c^*(\delta)$ as $\delta \to \infty$.

Theorem 3.3.2 (Linear growth of c^*)**.** *The non-random constant $\bar{C} \in [0, +\infty)$ defined by*

$$\liminf_{\delta \to \infty} \frac{c^*(\delta)}{\delta} = \bar{C} \tag{3.85}$$

is equal to zero if and only if $b(y,t) \equiv b(t)$.

As shown in the preceding chapter, the variational formula for c^* holds if we replace the eigenvalue with the principal Lyapunov exponent $\mu(\lambda)$ defined by

$$\mu(\lambda) = |\lambda|^2 + \lim_{t \to \infty} \frac{1}{t} \log \phi(z,t), \tag{3.86}$$

where ϕ solves the auxiliary initial value problem (2.82). Our proof of these result relies on growth estimates for the principal Lyapunov exponents. Recall that $\rho(\lambda) = \mu(\lambda) - |\lambda|^2$. For the fronts propagating in the direction $k = (1, \mathbf{0})$, (2.82) reduces to

$$\phi_t = \frac{1}{2}\Delta_y \phi - \lambda b(y,t)\phi, \quad \phi(y,0) \equiv 1.$$

Proof of Theorem 3.3.1: The first bound (i) follows from (2.102) and the formula

$$c^*(\delta) = \inf_{\lambda > 0} \frac{\mu(\lambda) + f'(0)}{\lambda} \geq \inf_{\lambda > 0} \frac{\lambda}{2} + \frac{f'(0)}{\lambda} = c^*(0).$$

The function $\psi = \log(\phi)$ satisfies

$$\begin{aligned} \psi_t &= \frac{1}{2}\Delta\psi + \frac{1}{2}|\nabla_y \psi|^2 - \lambda b(y,t) \\ \psi(y,0) &\equiv 0. \end{aligned} \tag{3.87}$$

Integrating (3.87) over $D \times [0,t]$, we have

$$\frac{1}{t}\int_D \psi(y,t)\,dy = \frac{1}{2t}\int_0^t\int_D |\nabla_y\psi|^2\,dy\,dt - \frac{\lambda}{t}\int_0^t\int_D b(y,t)\,dy\,dt. \tag{3.88}$$

Now let $t \to \infty$:

$$\begin{aligned}\rho(\lambda) = \lim_{t\to\infty}\frac{1}{t}\int_D \psi(y,t)\,dy &\geq -\lambda \lim_{t\to\infty}\frac{1}{t}\int_0^t\int_D b(y,t)\,dy\,dt \\ &= -\lambda E_Q\left[\int_D b(y,t)\,dy\right] = 0,\end{aligned}$$

almost surely with respect to Q. If $b(y,t) = b(t)$, then the first integral on the right hand side of (3.88) vanishes since $|\nabla_y\psi|^2 \equiv 0$. Then taking the limit as $t \to \infty$, we have equality:

$$\rho(\lambda) = E_Q\left[\int_D \delta b(t)\,dy\right] = 0.$$

Hence $c^*(\delta) = c^*(0)$. This proves part (ii).

For the linear upper bound (iii), note that

$$\begin{aligned}\frac{1}{t}\log E[e^{\lambda\delta\int_0^t b(W(s),t-s)\,ds}] &\leq \frac{1}{t}\log E[e^{\lambda\delta\sum_j \|b_1^j\|_\infty \int_0^t |b_2^j(t-s)|\,ds}] \\ &= |\lambda|\delta\sum_{j=1}^N \|b_1^j\|_\infty \frac{1}{t}\int_0^t |b_2^j(s)|\,ds.\end{aligned}$$

As $t \to \infty$, this last term converges almost surely to $|\lambda|\delta\sum_{j=1}^N \|b_1^j\| E_Q[|b_2^j|]$. Therefore, $c^*(\delta)$ always satisfies the linear upper bound

$$\begin{aligned}c^*(\delta) = \inf_{\lambda>0}\frac{\mu(\lambda)+f'(0)}{\lambda} &\leq \inf_{\lambda>0}\frac{\lambda}{2} + \frac{f'(0)}{\lambda} + \delta\sum_{j=1}^N \|b_1^j\| E_Q[|b_2^j|] \\ &= c^*(0) + \delta\sum_{j=1}^N \|b_1^j\| E_Q[|b_2^j|].\end{aligned} \tag{3.89}$$

Finally, for upper bound (iv), observe that under the scaling $b \mapsto \lambda\delta b$, the constant p_1 defined in assumption A3 can be replaced by $p_1 \mapsto \lambda^2\delta^2 p_1$. Then by (2.87),

$$\rho(\lambda) \leq \sqrt{2}\lambda^2\delta^2 p_1$$

and

$$\begin{aligned} c^*(\delta) = \inf_{\lambda>0} \frac{\mu(\lambda) + f'(0)}{\lambda} &\leq \inf_{\lambda>0} \frac{\lambda}{2} + \frac{f'(0)}{\lambda} + \frac{\lambda^2\delta^2 p_1}{2} = c^*(0) \\ &= 2\sqrt{(1+\delta^2 p_1) f'(0)/2} \\ &= c^*(0)\sqrt{(1+\delta^2 p_1)} \qquad (3.90) \\ &= c^*(0)(1 + \frac{\delta^2 p_1}{2}) + O(\delta^3). \end{aligned}$$

□

To prove Theorem 3.3.2, we will need the following growth estimate on the principal Lyapunov exponents.

Proposition 3.3.3. *There is a constant $K > 0$ such that for λ sufficiently large, $\rho(\lambda) \geq K\lambda$.*

Proof of Proposition 3.3.3: In the case that $b(y,t)$ is a Gaussian field with white-noise time dependence, the authors of [24] studied the behavior of $\rho(\kappa)$ as $\kappa \to 0$, where $\kappa > 0$ is a diffusion constant (replace Δ with $\kappa\Delta$ in (1.1)). Here we modify their strategy in order to treat the large advection limit when b has the form (3.84).

For clarity of notation, we assume $y \in D = [-L/2, L/2]$. The argument generalizes to multiple dimensions in a straightforward way. For $0 \leq s < t <$

∞, let $A^k_{s,t}$ be the set of functions $g \in C^{0,1}([s,t];R)$ such that $g(s) = g(t) = 0$, and $\|g'\|_\infty \leq k$. Define the random variable

$$I^k(s,t) = \sup_{f \in A^k_{s,t}} \int_0^{t-s} b(f(\tau), t-\tau)\, d\tau.$$

The variable $I^k(s,t)$ is super-additive, and $\tau_h I^k(s,t,\hat{\omega}) = I^k(s+h, t+h, \hat{\omega})$. Then, the sub-additive ergodic theorem implies that the limit

$$\lim_{t\to\infty} \frac{1}{t} I^k(0,t) = c(k)$$

exists Q-almost surely, and that $c(k)$ is a non-random constant given by the formula

$$c(k) = \sup_{t>0} \frac{1}{t} E_Q[I^k(0,t)]. \tag{3.91}$$

We claim that $c(k) > 0$. Therefore, given $\epsilon \in (0,1)$

$$Q\left(\sup_{f \in A^k_{0,t}} \int_0^t b(f(s), t-s)\, ds \geq (c(k) - \epsilon)t \right) \geq 1 - \epsilon \tag{3.92}$$

if t is sufficiently large. That is, for any ϵ small there is a set of probability at least $(1-\epsilon)$ such that we can find $f(s) = f(s,\hat{\omega}) \in A^k_{0,t}$ satisfying

$$\int_0^t b(f(s,\hat{\omega}), t-s, \hat{\omega})\, ds \geq c(k)(1-\epsilon)t, \tag{3.93}$$

and we expect that Brownian paths staying close to this f will make a significant contribution to the exponential in the definition of $\rho(\lambda)$. For a constant $\gamma > 0$ to be determined and $f \in A^k_{0,t}$, we let $B_t(f,\gamma)$ be the γ-neighborhood of f in $C([0,t];D)$:

$$B_t(f,\gamma) = \{X \in C([0,t],D) \mid \ \|X - f\|_{C^0} < \gamma\}.$$

Using the Girsanov transformation, one can show that there are constants K_1, K_2 independent of λ, t, and $f \in A^k_{0,t}$ such that

$$P(B_t(f,\gamma)) \geq K_1 e^{-K_2(k^2+1/\gamma^2)t}$$

for $t > 1$. Because the $\{b^j_1(y)\}$ are assumed to be Lipschitz continuous, we see that for any path $X \in B_t(f,\gamma)$,

$$|\int_0^t b(X(s), t-s)\, ds - \int_0^t b(f(s), t-s)\, ds| < \gamma M \sum_{j=1}^{N} \int_0^t |b^j_2(s)|\, ds \qquad (3.94)$$

where M is the maximum of the Lipschitz constants for the functions $\{b^j_1(y)\}_{j=1}^N$. By (3.93) and (3.94) with $\epsilon > 0$ sufficiently small, there is a set of Q-probability at least $(1-\epsilon)$ such that

$$\begin{aligned} E_P\left[e^{\lambda \int_0^t b(W^y_s(s), t-s)\, ds}\right] &\geq E_P\left[e^{\lambda \int_0^t b(W^y_s(s), t-s)\, ds} \chi_{B_t(f,\gamma)}\right] \\ &= e^{\lambda c(k)(1-\epsilon)t} e^{-\lambda\gamma M V} P(B_t(f,\gamma)) \\ &\geq e^{\lambda c(k)(1-\epsilon)t} e^{-\lambda\gamma M V} K_1 e^{-K_2(k^2+1/\gamma^2)t}, \qquad (3.95) \end{aligned}$$

where $V = \sum_{j=1}^N \int_0^t |b^j_2(s)|\, ds$ and $f \in A^k_{0,t}$ is chosen to satisfy (3.93). For t large, independently of λ, k, and γ, V can be bounded by

$$V \leq t(\sum_{j=1}^{N} E[|b^j_2(0|] + 1)$$

except on a set of probability less than ϵ. Therefore, we can choose γ small so that

$$\gamma \leq \frac{\epsilon c(k)}{M(\sum_{j=1}^N E[|b^j_2(0|] + 1)}.$$

Hence $e^{\lambda c(k)(1-\epsilon)t} e^{-\lambda\gamma MV} \geq e^{\lambda c(k)(1-2\epsilon)t}$ for t sufficiently large. Then by choosing λ large, $\lambda \geq \frac{K_2(k_1^2/\gamma)}{c(k)\epsilon}$, and we obtain from (3.95)

$$E_P\left[e^{\lambda \int_0^t b(W_s^y(s),t-s)\,ds}\right] \geq e^{\lambda(c(k)-3\epsilon)t}$$

with Q-probability at least $(1-2\epsilon)$, for t sufficiently large, independently of λ. Since the limit defining $\rho(\lambda)$ exists Q-almost surely, this establishes the lemma with $K = c(k)(1-3\epsilon)$, for any $\epsilon \in (0,1)$, $k > 0$.

It remains to establish the claim that $c(k) > 0$. Note that for all $k \geq 0$, $E_Q[I^k(0,t)] \geq \sup_{f\in A_{0,t}^k} E_Q[\int_0^t b(f(s),t-s)ds] = 0$. Also, $E_Q[I^{k_2}(0,t)] \geq E_Q[I^{k_1}(0,t)]$ whenever $k_2 > k_1$, since $A_{0,t}^{k_2} \supset A_{0,t}^{k_1}$. Without loss of generality, suppose that there is an $\epsilon > 0$ such that for all $j = 1,\ldots,N$ we have $|b_1^j(y) - b_1^j(0)| \neq 0$ if $|y| < \epsilon$ and $y \neq 0$. This means that the $b^j(y)$ do not have a flat spot touching $y = 0$. Define the set $G = \{\hat{\omega}|\ b_1^j(y)b_2^j(s) > b_1^j(0)b_2^j(s), \forall s \in [0,1],\ y \in (0,\epsilon), j = 1,\ldots,N\}$. Then $Q(G) > 0$. For $k > 0$, let $\tilde{f} \in A_{0,1}^k$ such that $\tilde{f}(s) \in (0,\epsilon)$ for $s \in (0,1)$. Then we have

$$\begin{aligned}
E_Q[I^k(0,1)] &= E_Q[\sup_{f\in A_{s,t}^k} \int_0^1 b(f(s),1-s)ds\chi_G] + \\
&\qquad + E_Q[\sup_{f\in A_{s,t}^k} \int_0^1 b(f(s),1-s)ds\chi_{G^c}] \\
&\geq E_Q[\int_0^1 b(\tilde{f}(s),1-s)ds\chi_G] + E_Q[\int_0^1 b(0,1-s)ds\chi_{G^c}] \\
&> E_Q[\int_0^1 b(0,1-s)ds\chi_G] + E_Q[\int_0^1 b(0,1-s)ds\chi_{G^c}] \\
&= E_Q[\int_0^1 b(0,1-s)ds] = 0.
\end{aligned}$$

Combining this with (3.91) establishes the claim that $c(k) > 0$ for all $k > 0$.

$\square$

In proving Theorem 3.3.2, we also will make use of the following property of the curve $\lambda \mapsto \mu(\lambda)/\lambda$:

Lemma 3.3.4. *Let* $\lambda_0 = \sqrt{2f'(0)}$. *There is an interval* $[\lambda_1^*, \lambda_2^*] \subset (0, \lambda_0]$ *such that* $\frac{\mu(\lambda)+f'(0)}{\lambda}$ *is constant on* $[\lambda_1^*, \lambda_2^*]$ *and* $\frac{\mu(\lambda^*)+f'(0)}{\lambda^*} \leq \frac{\mu(\lambda)+f'(0)}{\lambda}$ *for all* $\lambda^* \in [\lambda_1^*, \lambda_2^*]$ *and* $\lambda \in (0, \infty)$. *Outside of this interval the curve* $\frac{\mu(\lambda)}{\lambda}$ *has no other minima.*

Note that if $\mu(\lambda)$ is strictly convex, as in the case of temporally periodic media, then $\lambda_1^* = \lambda_2^*$, so that the infimum of $\frac{\mu(\lambda)}{\lambda}$ is attained at a unique point.

Proof of Lemma 3.3.4: This follows from the fact that $\mu(\lambda)$ is convex, since $\mu(\lambda) = \lambda^2/2 + f'(0) + \rho(\lambda)$ with ρ being convex in λ and $\rho(0) = 0$ (see discussion leading to (2.102)). The point λ_0 is the value of λ where the infimum of the curve $\lambda/2 + f'(0)/\lambda$ is attained. $\square$

Proof of Theorem 3.3.2: The fact that $\bar{C} \in [0, +\infty)$ follows from Theorem 3.3.1. Also, if $b(y,t) \equiv b(t)$ then $\bar{C} = 0$ since $c^*(\delta) = c^*(0)$ for all $\delta > 0$. By Lemma 3.3.4 there is a unique $\lambda = \lambda_\delta \in (0, \lambda_0]$ such that

$$c^*(\delta) = \inf_{\lambda > 0} \frac{\mu(\lambda) + f'(0)}{\lambda} = \frac{\mu(\lambda_\delta) + f'(0)}{\lambda_\delta}.$$

Let $\delta_j \to \infty$ as $j \to \infty$ and suppose that $\limsup_{j\to\infty}(\lambda_{\delta_j}\delta_j) \leq M$. This implies that

$$\liminf_{j\to\infty} \frac{c^*(\delta_j)}{\delta_j} = \liminf_{j\to\infty} \frac{\mu(\lambda_{\delta_j}) + f'(0)}{\lambda_{\delta_j}\delta_j} \geq \liminf_{j\to\infty} \frac{f'(0)}{\lambda_{\delta_j}\delta_j} \geq \frac{f'(0)}{M} > 0.$$

So, in this case the result holds with $\bar{C} = f'(0)/M$.

Now suppose $\lambda_{\delta_j}\delta_j$ is unbounded as $j \to \infty$. By Proposition 3.3.3, there is a positive constant K such that

$$\rho(\lambda_{\delta_j}\delta_j) \geq K\lambda_{\delta_j}\delta_j > 0$$

for j sufficiently large. Note that Proposition 3.3.3 treats the case of $\delta = 1$; this is why we use $\rho(\lambda_{\delta_j}\delta_j)$ instead of $\rho(\lambda_{\delta_j})$. Therefore,

$$\liminf_{j\to\infty} \frac{c^*(\delta_j)}{\delta_j} = \liminf_{j\to\infty} \frac{\lambda_{\delta_j}}{2\delta_j} + \frac{f'(0)}{\lambda_{\delta_j}\delta_j} + \frac{\rho(\lambda_{\delta_j}\delta_j)}{\lambda_{\delta_j}\delta_j} \geq K > 0$$

since $\lambda_{\delta_j} \in (0, \lambda^*]$ and $\lambda_{\delta_j}\delta_j \to \infty$. Hence $\bar{C} \geq K > 0$. □

Theorems 3.3.1 and 3.3.2 give a linear upper and lower bounds on the enhancement of c^* as the flow intensity increases. However, experiments with premixed flames have shown that increasing turbulence intensity does not lead to unlimited linear enhancement of the turbulent burning rate. Denet [25] has proposed that this "bending" of the turbulent burning velocity in high-intensity flows can be explained by a rapid temporal decorrelation of the flow (see also Ashurst [5]). For the present model, the following upper bound confirms the hypothesis that rapid temporal decorrelation leads to sub-linear enhancement of the front speed. Notice that the derivation uses no information about the spatial structure of the flow other than the maximum value ($\|b_1^j\|_\infty$). As a result, it is likely that the actual speed may grow more slowly than $\delta^{1/2}$, or that c^* eventually decreases with δ, as suggested by the numerical experiments of [25] and [5] for temporally periodic flows.

Corollary 3.3.5. *For $\delta > 0$, let $\{b_2^j(t)\}_{j=1}^N$ be a family of stationary Gaussian fields on $[0,\infty)$ satisfying $E_Q[b_2^j(s)b_2^k(t)] \le C_1 e^{-\alpha_{j,k}|t-s|}$, where $\alpha_{j,k} > 0$ and $C_1 > 0$. Then for the scaled flow $b^\delta(y,t) = \sum_{j=1}^N \delta b_1^j(y) b_2^j(\delta t)$*

$$\limsup_{\delta\to\infty} \frac{c^*(\delta)}{\sqrt{\delta}} < +\infty. \tag{3.96}$$

Proof: For the flow $\sum_{j=1}^N b_1^j(y) b_2^j(\delta t)$,

$$\begin{aligned} \hat{\Gamma}(r) = \sup_{y_1,y_2} \Gamma(y_1,y_2,0,r) &\le \sum_{j,k} \|b_1^j\|_\infty \|b_1^k\|_\infty E_Q[b_2^j(0) b_2^k(\delta r)] \\ &\le \sum_{j,k} \|b_1^j\|_\infty \|b_1^k\|_\infty C_1 e^{-\alpha_{j,k}\delta|r|} \end{aligned} \tag{3.97}$$

Then from (2.2) we have

$$\int_0^\infty \hat{\Gamma}(r)\, dr \le p_1 = C_1 \sum_{j,k} \frac{\|b_1^j\|_\infty \|b_1^k\|_\infty}{\alpha_{j,k}\delta}$$

The result now follow from part (iv) of Theorem 3.3.1.

□

Chapter 4

Numerical Simulation of Front Speeds via Variational Principles

In this chapter, we use the KPP variational formula (3.3) to compute the propagation speed c^* and study its dependence on properties of the flow B. In each case considered, one might approximate the speed directly by solving the original semi-linear equation (1.1) in a truncated domain, evolving the solution for a sufficiently long time, and tracking the front location. Although this approach has been successful (for example, see [82]), the variational principle offers some advantages. First, in the cases considered here, the variational principle reduces computation to a finite domain, and the reduced domain may also have fewer dimensions than the original domain, depending on the flow structure. Second, the nonlinearity is removed from the equation. Therefore, one can compute c^* using methods for linear eigenvalue problems or linear evolution problems, as we will describe. Furthermore, randomness in the flow may greatly distort the front, making it difficult to track the front for a long time within a truncated domain.

The idea of using a variational characterization to compute c^* has also been used by Embid, Majda, Souganidis [27], Khouider, Bourlioux, and Ma-

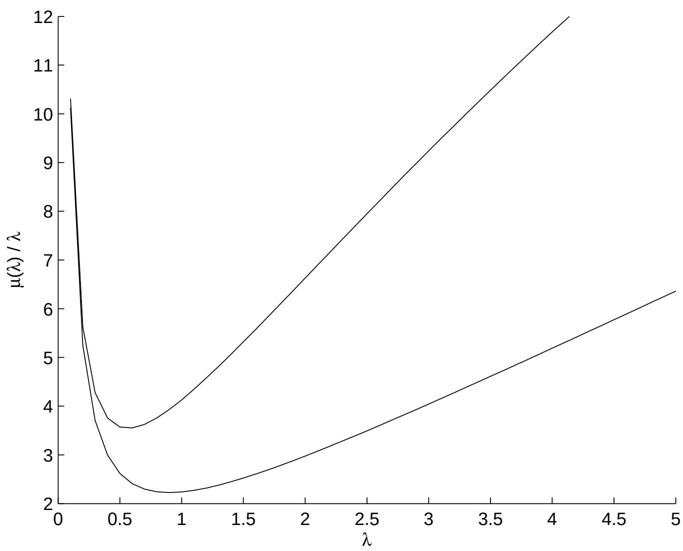

Figure 4.1: Two computed curves $\lambda \mapsto \frac{\mu(\lambda)}{\lambda}$ corresponding to steady shear flow.

jda [49] to study the front speed in deterministic, periodic shear flows. In this setting, the authors of [27] have shown that non-rigorous approaches to computing the effective front speed may give values of c^* that are an order of magnitude below the correct value given by the variational formula.

The trade-off in using the variational formula to compute c^* is that one must now solve a nonlinear minimization problem in λ. However, as we have proven analytically, the curve $\lambda \mapsto \frac{\mu(\lambda)+f'(0)}{\lambda}$ has a unique global minimum and no other local minima, so that $\frac{\mu(\lambda)+f'(0)}{\lambda}$ can be optimized easily with a standard algorithm [16] using only a few evaluations of $\mu(\lambda)$. Thus, the most challenging part of computing with the variational formula is the approximation of the eigenvalues or Lyapunov exponents $\mu(\lambda)$.

We compute μ using one of two different approaches. For spatially and temporally periodic media, μ is truly an eigenvalue, given by (3.4). So, one might approximate μ by the solution to an algebraic eigenvalue problem, $A_\lambda \bar{\phi} = \mu(\lambda)\bar{\phi}$, that follows from a suitable discretization of the operator L_λ. This approach works well if B is a steady shear flow. In this case, the operator L_λ is self-adjoint, the matrix A_λ is symmetric, and the algebraic eigenvalue problem can be solved very quickly. However, if B is not a steady shear flow, then L_λ will not be self-adjoint, so that the resulting nonsymmetric eigenvalue problem is more difficult to solve numerically.

The second method of computing μ follows from the definition of μ as a Lyapunov exponent, or growth rate of a solution to an initial value problem. That is,

$$\mu(\lambda) = \lambda^2 + \lim_{t\to\infty} \frac{1}{t} \log \phi(z,t) \tag{4.1}$$

where ϕ solves

$$\begin{aligned} \phi_t = \Delta\phi + (B - 2\lambda k)\cdot\nabla\phi - \lambda B\cdot k\phi \\ \phi(z,0) \equiv 1. \end{aligned} \tag{4.2}$$

Note that (4.1) holds when B is constant in time, periodic in time, or random in time, uniformly in z. Therefore, to compute $\mu(\lambda)$ numerically, we may approximate

$$\mu(\lambda) \approx \hat{\mu}(\lambda,t) = \lambda^2 + \frac{1}{t}\log\|\phi(z,t)\|_1 \tag{4.3}$$

for sufficiently large t. We will call this approach to computing μ the time-evolution approach.

Although $\frac{1}{t}\log\|\phi(z,t)\|_1$ in (4.3) will converge almost surely, the convergence may be rather slow. For example, suppose $B(z,t)$ is periodic in space and time. If $\Phi(z,t) > 0$ is the eigenfunction, then $L_\lambda\Phi = \mu(\lambda)\Phi$, and the function $\psi(x,t) = \Phi(x,t)e^{\mu t}$ solves (4.2) with initial data $\psi(x,0) = \Phi(x,0)$. Since, $\Phi > 0$, the maximum principle implies that

$$M_1\Phi(x,t)e^{\mu t} \leq \phi \leq M_2\Phi(x)e^{\mu t} \quad \forall \;\; t \geq 0 \tag{4.4}$$

when $M_1 \leq \frac{1}{\sup_x \Phi(x,t)}$ and $M_2 \geq \frac{1}{\inf_x \Phi(x,t)}$. This bound and the approximation (4.3) imply that

$$\mu(\lambda) + \frac{M_1 \inf_x \Phi(x,t)}{t} \leq \hat{\mu}(\lambda,t) \leq \mu(\lambda) + \frac{M_2 \sup_x \Phi(x,t)}{t}. \tag{4.5}$$

So, in the worst case, we expect convergence on the order $O(1/t)$. Even if we started with the exact eigenfunction $\phi(z,0) = \Phi(z,0)$, instead of initial data $\phi(z,0) \equiv 1$, the convergence of $\hat{\mu}(\lambda,t)$ would still be of the order $O(1/t)$.

In applying the time-evolution approach to compute μ, we use a computational trick to average out temporal oscillations in the approximation (4.3). After evaluating $\log\|\phi\|_1$ at several regularly spaced points in time, we take $\hat{\mu}$ to be the slope of the best-fit line through these points, determined by the least-squares algorithm. We initialize this sampling after some brief interval so that transient behavior does not significantly effect the approximation. This strategy averages out the temporal oscillations in $\log\|\phi\|_1$, as seen in the following simple example. Consider the function $f(t) = t\mu + \sin(t)$. Then $\frac{1}{t}f(t)$ converges as $t \to \infty$ to μ with the error being $O(1/t)$. However, the slope of the best fit line through regularly sampled points $(t, f(t))$ will converge to μ much more rapidly. Our numerical experiments also show this to be the case for the approximation of μ in (4.3).

4.1 Spatially Random Flows in a Cylinder

We first consider the case of a spatially random shear flow restricted to an unbounded, two-dimensional cylinder. For the shear process, we let $b(y, \omega)$ be the Ornstein-Uhlenbeck process defined by the stochastic ODE (3.79). Because of the shear structure, the variational principal reduces computation of the front speed to a one dimensional problem on a bounded domain $[0, L]$. For a given $\lambda > 0$, we must compute the principal eigenvalue $\mu(\lambda)$ with corresponding eigenfunction $\phi = \phi(y) > 0$, $y \in [0, L]$, by solving:

$$\begin{aligned} \phi_{yy} + [\lambda^2 + \lambda\, b(y)]\phi = \mu(\lambda)\phi, \quad & y \in (0, L), \\ \frac{\partial \phi}{\partial y} = 0, \quad & y = 0,\ L. \end{aligned} \tag{4.6}$$

Here we suppress the random parameter ω, as computation is done realization by realization. We use a standard second order finite-difference method. Denote the uniform partition of the domain by points $\{y_i\}_{i=1}^m$, and the numerical solution by $\bar{\phi} = \left\{\bar{\phi}_i\right\}_{i=1}^m$, where $h = L/(m-1)$, $y_i = (i-1)h$, and $\bar{\phi}_i \approx \phi(y_i)$. The discretized system is

$$\frac{1}{h^2}\bar{\phi}_{i-1} + (\lambda^2 + \lambda b_i - \frac{2}{h^2})\bar{\phi}_i + \frac{1}{h^2}\bar{\phi}_{i+1} = \mu(\lambda)\bar{\phi}_i \quad i = 2, \ldots, m-1,$$

with second order approximation of the Neumann boundary conditions. This reduces the problem to finding the principal eigenvalue of a symmetric tridi-

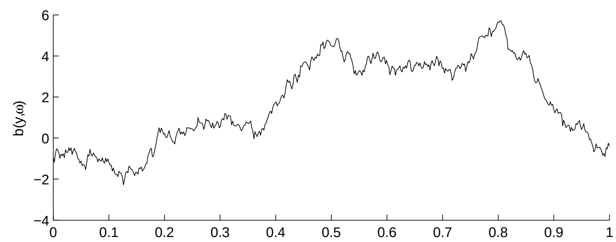

Figure 4.2: One sample path of the Ornstein-Uhlenbeck process $b(y, \omega)$.

agonal matrix, easily accomplished with double precision LAPACK routines [4].

To generate realizations of $b(y, \omega)$, we numerically evaluate (3.79) with the Milstein scheme (see [53]). Although this scheme is first order, we use a discrete spacing $\bar{h} \leq h^2$, where h is the discrete grid spacing for the eigenvalue problem, so that the method is still second order accurate in the parameter h. Figure 4.2 shows a sample path of the shear.

To approximate the expectation $E[c^*(\delta)]$ we generate N independent realizations (indexed by $i = 1, \ldots, N$) of the shear and compute the corresponding minimal speeds $\{c_i^*\}$ for each δ. Then we compute the average

$$E[c^*(\delta)] \approx \bar{E}(\delta) = c_0^* + \frac{1}{N} \sum_{i=1}^{N} M_i(\delta), \tag{4.7}$$

where $M_i(\delta) = c_i^*(\delta) - c_0^* - \delta \bar{b}_i$. That is, we subtract the linear part due to the mean of the shear being nonzero. To compute the distributions of $M(\delta)$, we partition the range of values into Q disjoint intervals: $\{[x_j, x_{j+1})\}_{j=1}^{Q}$. Then, we define the approximate probability density function by

$$pdf(x) = \frac{1}{N}\sum_{i=1}^{N} \frac{\chi_j(M_i(\delta))}{(x_{j+1}-x_j)} \quad \text{if } \ x \in [x_j, x_{j+1})$$

where $\chi_j(x)$ is the characteristic function of the interval $[x_j, x_{j+1})$.

4.1.1 Numerical Results

In Figure 4.3, we show computed probability distributions of $M(\delta)$ for fixed amplitudes $\delta = 1$ and $\delta = 14$. These distributions were computed with $N = 100,000$ samples, and $Q = 300$ intervals. The values $M(\delta)$ are the enhancement of the minimal speeds due to the variation of the shear, after the effect of the mean field has been subtracted off. Since a mean zero shear always enhances the minimal speeds, we should expect $M(\delta) > 0$ for all δ, for all realizations. Moreover, since b is random and Gaussian, we should expect the distribution of M to have unbounded support. Figure 4.4 shows good convergence of the distributions when using $N = 100,000$ samples.

4.1.1.1 Scaling with Shear Amplitude δ

In Figure 4.5 and Figure 4.26, we show the results of a simulation using $N = 100,000$ realizations of a shear with small and large root mean square (rms) amplitudes, respectively. The covariance function of the process is $E[b(y)b(s)] = 2e^{-4|y-s|}$. In each plot, we show multiple curves, corresponding to various domain sizes. In Figure 4.5, corresponding to small δ, the solid curves are the numerically computed values; the dashed curves are given by the approximation formula (3.80). With the averages $\bar{E}(\delta)$ for each δ, we

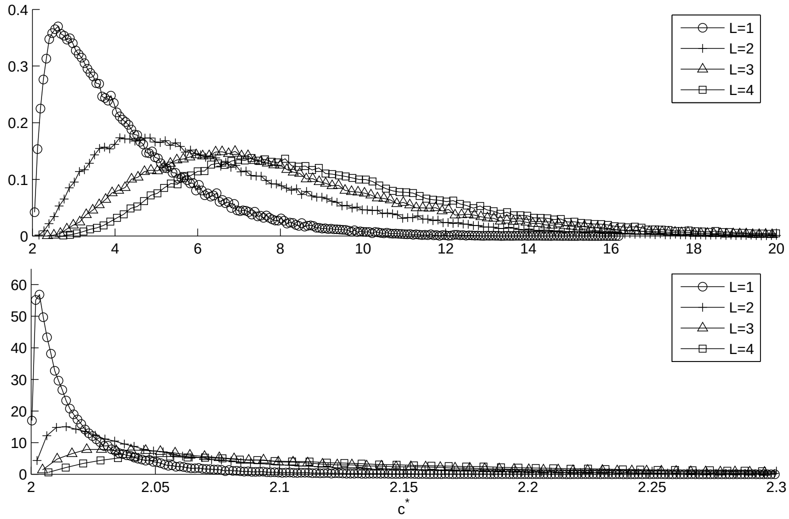

Figure 4.3: Computed probability density functions (pdf's) for the enhancement $M(\delta)$, $\delta = 1.0$ (bottom), $\delta = 14.0$ (top).

compute the scaling exponents p using the least squares method to fit a line to a log-log plot of speed versus amplitude. That is, the exponent p is the slope of the best-fit line through the data points $(\log(\delta), \log(\bar{E}[c^*(\delta)] - c^*(0)))$ for each shear amplitude δ. We find that the enhancement of the minimal speed scales quadratically for small amplitudes and linearly for large amplitudes, which is consistent with the predictions of Theorem 3.1.1 and Theorem 3.2.5. The computed exponents are shown in Table 4.1.

Table 4.1: Computed scaling exponents p for $E[c^*(\delta)] = c^*_0 + O(\delta^p)$.

	$L = 1.0$	$L = 2.0$	$L = 3.0$	$L = 4.0$
$\delta \ll 1$	2.00	1.98	1.96	1.93
$\delta \gg 1$	1.09	1.05	1.04	1.03

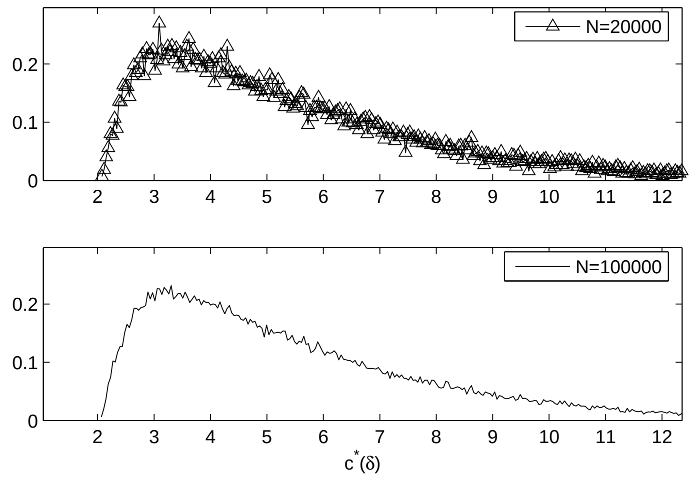

Figure 4.4: Convergence of speed enhancement distribution at $\delta = 14.0$.

For deterministic shear flows, Theorem 2 of [43] shows the upper bound

$$c^*(\delta, \omega) \leq \min(g_1(\omega), g_2(\omega)) \tag{4.8}$$

where $g_1 = c_0 + \delta\|b\|_\infty$, $g_2 = c_0\sqrt{1 + \frac{\delta^2}{\kappa^2}\|\nabla\chi\|_\infty}$, and κ is the diffusion constant (equal to 1 in equation (1.1)). In the present case, g_1 and g_2 are random variables, so whether g_1 or g_2 is the tighter bound depends on b. In Figure 4.7, we compare the distribution of $c^*(\delta)$ with the distributions of $g_1(\omega)$ and $g_2(\omega)$, for $\delta = 50$ and $\kappa = 0.01$. Based on our remarks following Theorem 3.2.5, the bound $c^*(\delta) \leq g_1$ is almost surely not a tight bound. This fact is reflected in Figure 4.7, where we see that the distribution of g_1 has a larger tail than the distribution of c^*.

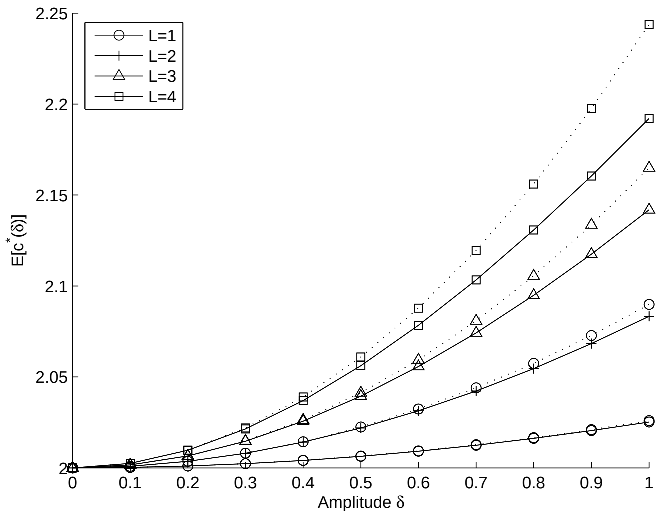

Figure 4.5: Enhancement of $E[c^*]$ in small amplitude shear flow. The solid curves are the numerically computed values; the dashed curves are given by the approximation formula (3.80).

4.1.1.2 Dependence on the Spatial Correlation Length

Next, we consider the relationship between the minimal speed and the spatial correlation of the shear process. The covariance $E[b(y)b(s)]$ is a function of $|t| = |s - y|$, so we will write $V(t) = E[b(y)b(s)]$. By choosing $r = \sqrt{2}\alpha^{3/4}$, we constructed O-U processes with covariances given by

$$V(t) = \sqrt{\alpha}e^{-\alpha\,|t|} \quad . \tag{4.9}$$

The correlation length is $1/\alpha$. By this choice of r, the L^2 norm of $V(t)$ remains constant as α changes, so that the total energy in the power spectrum of b remains constant. Since $\frac{r^2}{2a} = \sqrt{a}$, we see from equation (3.81) that for fixed

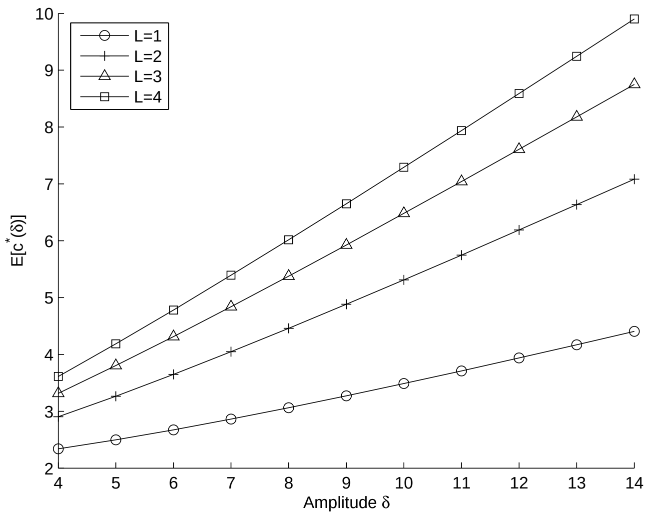

Figure 4.6: Enhancement of $E[c^*]$ in large amplitude shear flow.

L,

$$\lim_{\alpha\to+\infty} E[\langle|\chi_x|^2\rangle] = \lim_{\alpha\to 0^+} E[\langle|\chi_x|^2\rangle] = 0 \tag{4.10}$$

and that $E[\langle|\chi_x|^2\rangle]$ achieves a maximum for some finite value of $\alpha \in (0,\infty)$. This suggests that there is some optimal correlation length $1/\alpha$, depending on the domain size L, at which the enhancement of $E[c^*(\delta)]$ is maximized.

Fixing the grid spacing $dx = 0.002$, we computed the expected value $E[c^*(\delta)]$ for a range of α and for $L = 1.0, 2.0, 3.0, 4.0$. Note that for each α, we must choose the initial points b_0 to have variance $E[b_0^2] = \sqrt{\alpha}$ so that the process remains stationary for each α. Varying the covariance does not effect the order of the scaling in δ. That is, in each case the enhancement scales like $O(\delta^2)$ for small δ and $O(\delta)$ for large δ, as in the preceding simulation.

Figure 4.8 shows the enhancement $E[c^*(\delta)]$ for a fixed $\delta = 1.0$ and a

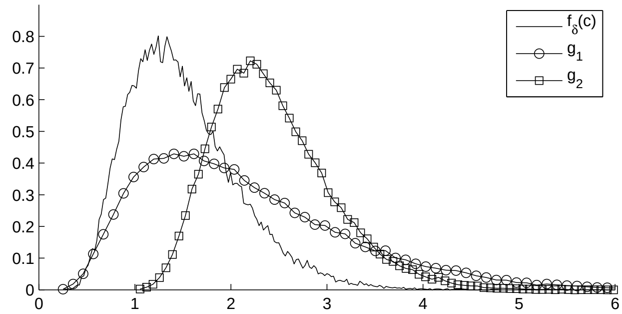

Figure 4.7: Probability density functions for $E[c^*]$ and analytical bounds at $\delta = 50$.

range of α. Figure 4.9 shows the results of the same computation for $\delta = 15.0$, corresponding to the large δ regime. In this case, formula (3.3) is no longer valid. Nevertheless, we see the same effect as in the small amplitude regime: the existence of an optimal correlation length $1/\alpha$ that grows with the channel width L.

This effect can be interpreted in terms of $V(t)$ and its Fourier transform or power spectrum:

$$\hat{V}(w) = \sqrt{\frac{2}{\pi\alpha}}\left(1 + \left(\frac{w}{\alpha}\right)^2\right)^{-1}. \tag{4.11}$$

As $\alpha \to 0$, $\hat{V}(w)$ concentrates at the origin, and so the energy of the shear process is concentrated more in the large scale spatial modes. The domain D, to which the process is restricted, is bounded, and variations over a length scale that is much greater than the diameter of D have little effect on the average enhancement of the front. As a result, $E[c^*]$ decreases as $\alpha \to 0^+$. In the other limit $\alpha \to \infty$, $\|V\|_{L^1} \to 0$ so that $\|\hat{V}\|_\infty \to 0$. That is, $\hat{V}$ spreads

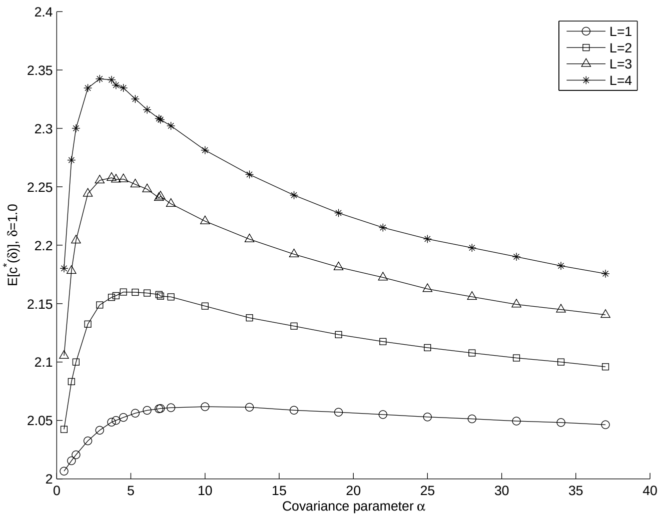

Figure 4.8: Dependence of front speed on covariance parameter α, for $\delta = 1.0$.

out so that the energy over any finite band of frequencies goes to zero, causing $E[c^*]$ to decrease as well. Note that as $\alpha \to 0$, the family of processes does not converge to white noise, whose covariance function equals the Dirac delta function.

4.1.1.3 Dependence on the Domain Size

It is interesting to consider the behavior of c^* as the channel width increases. If D is unbounded, then the front speed is not well-defined in general. Note that $E[\langle|\nabla\chi|^2\rangle]$ given by (3.81) diverges as $L \to +\infty$, which suggests that c^* also diverges as $L \to \infty$. The work of Xin [89] implies that if the channel $R \times [0, L]$ is replaced by R^2 then the front velocities diverge logarithmically in time due to the almost sure growth of the running maximum of the process $b(y, \omega)$. Majda and Souganidis [61] also studied KPP fronts

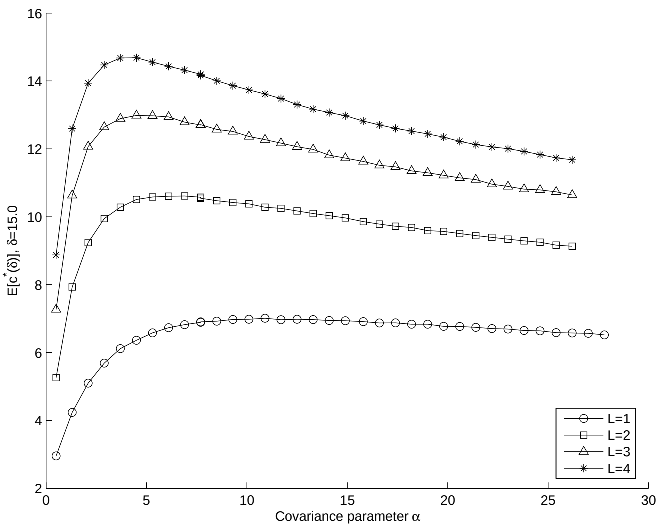

Figure 4.9: Dependence of front speed on covariance parameter α, for $\delta = 15.0$.

propagating under the influence of random shears in R^2. For certain scalings of the spectral density of the shear process, they found that the homogenized fronts can diverge like $c^* \sim t^{1+H}$ where $H \in (0,1)$ is the Hurst exponent corresponding to the process.

In our simulation, we leave the spectral density fixed with respect to L. We observe that as the width L of the channel increases (with δ fixed), the expectation $E[c^*(\delta)]$ increases monotonically and diverges sub-linearly, while the variance $Var[c^*(\delta)]$ decreases. These behaviors are shown in Figure 4.10 and Figure 4.11. Figure 4.12 illustrates the change in the distribution of $c^*(\delta)$ as $L = 4, 16, 30$. For each of these figures, the amplitude is fixed at $\delta = 50$, and the diffusion constant is $\kappa = 0.01$ (set to 1 in equation (1.1)). Also, the covariance parameter α has been modified (in comparison to the α used for Figure 4.5 and Figure 4.26) to give optimal enhancement for the larger

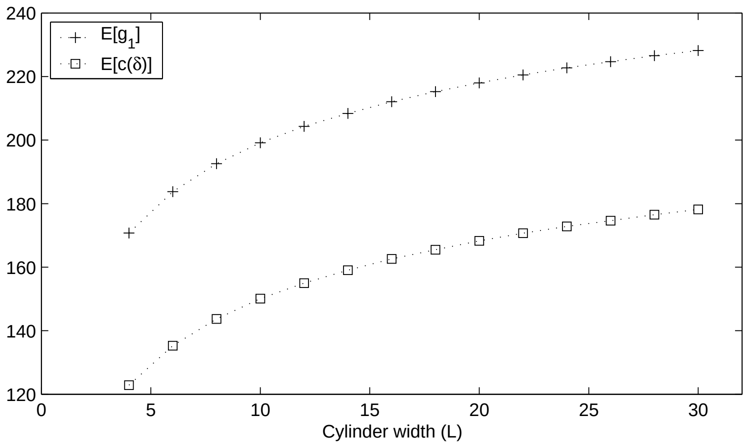

Figure 4.10: Sub-linear growth of $E[c^*]$ as domain width increases, $\delta = 50$, $\kappa = 0.01$.

amplitude and larger domain widths.

We observe that there is a close correlation between the front speed and the global maximum of the shear, which determines the random variable $g_1(L) = c_0^* + \delta \sup_{y\in[0,L]} b(y)$. Indeed, Figure 4.10 and Figure 4.11 show that the mean and variance of $c^*(\delta)$ mimic the mean and variance of $g_1(L) = c_0^* + \delta \sup_{y\in[0,L]} b(y)$ as L increases, even though the distributions of these two quantities were observed to be qualitatively different (Figure 4.7). The average speed obeys the upper bound

$$E[c^*(\delta)] \leq E[g_1] = c_0^* + \delta E[\sup_{y\in[0,L]} b(y)]. \tag{4.12}$$

It is also known [6, 13] that for deterministic shear flows, if the diffusion coefficient κ is small enough, the ratio $c^*(\delta)/\delta$ is close to $\sup_{y\in[0,L]} b(y)$ when δ is sufficiently large. Therefore, it may be that $E[c^*(\delta)/\delta]/E[\sup b] \to k$ as $L \to \infty$ for some positive constant $k \in (0, 1]$.

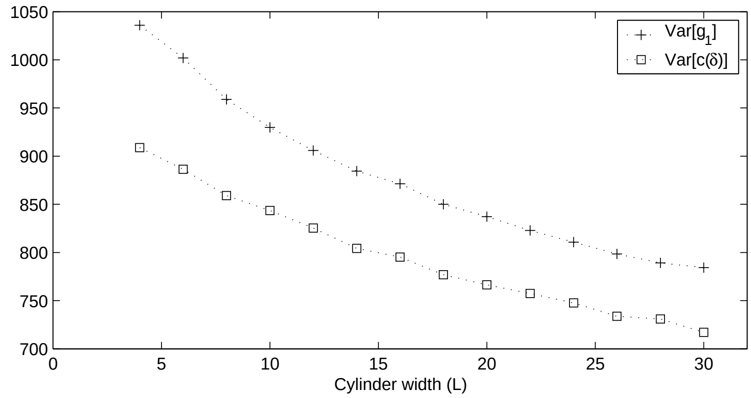

Figure 4.11: Decay of the speed variance as domain width increases, $\delta = 50$, $\kappa = 0.01$.

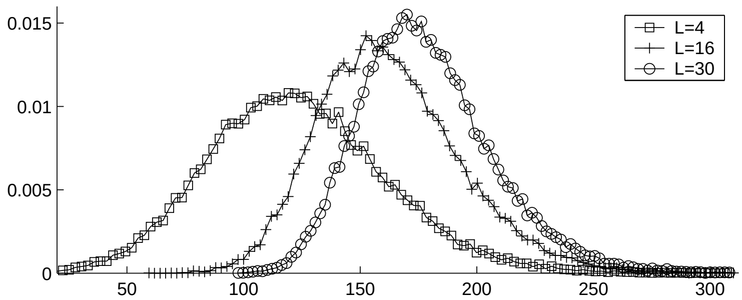

Figure 4.12: Probability density functions for c^* in the large rms shear amplitude regime with varying domain widths, $\delta = 50$, $\kappa = 0.01$

From the theory of extremal distributions for stationary Gaussian fields ([1], Chapter 6), we have as $L \to \infty$:

$$E[\sup_{y\in[0,L]} b(y)] \equiv A(L) \sim O(\sqrt{\log(L)}),$$
$$\mathrm{Var}\,[\sup_{y\in[0,L]} b(y)] \equiv B(L) \sim O(\frac{1}{\sqrt{\log(L)}}),$$
$$\mathrm{Prob}\,(B^{-1}(L)(\sup_{y\in[0,L]} b(y) - A(L)) < u) \to \exp\{-\exp\{-u\}\}. \quad (4.13)$$

In view of (4.12), this implies that the growth of $E[c^*(\delta)]$ with respect to L can be no more than $O(\sqrt{\log L})$. Moreover, the correlation between the moments of c^* and the moments of g_1 suggests that as $L \to \infty$ the random front speed c^* satisfies a central limit theorem, with a limiting distribution that may be non-Gaussian, as in (4.13). However, because of the very slow convergence of these quantities as $L \to \infty$, additional computation as $L \to \infty$ does not yield more insight into the subtle behavior of $c^*(\delta)$ in this limit.

4.1.1.4 Comparison with Direct Simulation

When comparing the results of the preceding sections with the results of a direct simulation, we find that computing the minimal speeds using the variational formula offers significant advantages. As described in [68], we first computed the quantities $E[c^*(\delta)]$ via direct simulation of the original equation (1.1) on a truncated domain using an upwind finite difference scheme. For each realization of the shear, we evolved the solution of (1.1) for a sufficiently long time until the front attained a more or less constant speed (see [68] for details of the method). Then, we repeated the process for a large ensemble of shears to compute the expectations $E[c^*(\delta)]$.

Because the shears are random and may greatly distort the wave front, one major challenge in accurately approximating the front speeds is tracking the widely varying front region over a long time. In the region of the front, gradients are relatively large, so accurately tracking the front requires either a very fine uniform grid spanning a large domain or some kind of adaptive-mesh scheme. In either case, accurate direct simulation is computationally more expensive compared to the simple variational method, which allowed us to compute a much larger number of samples in a fraction of the time and avoid the inaccuracies resulting from truncation of the channel domain.

Nevertheless, the direct simulation approach applies to more general nonlinear terms; we considered the KPP nonlinearity, the combustion nonlinearity, and the bistable nonlinearity. Our results show that, for small amplitudes, $E[c^*(\delta)]$ scales like $O(\delta^2)$ and that the distributions of $\frac{M(\delta)}{c_0^*}$ (defined by (4.7)) are very similar. Figure 4.13 shows the distributions of the computed values $\frac{M(\delta)}{c_0^*}$ for the combustion and bistable nonlinearities, respectively, for $L = 1$.

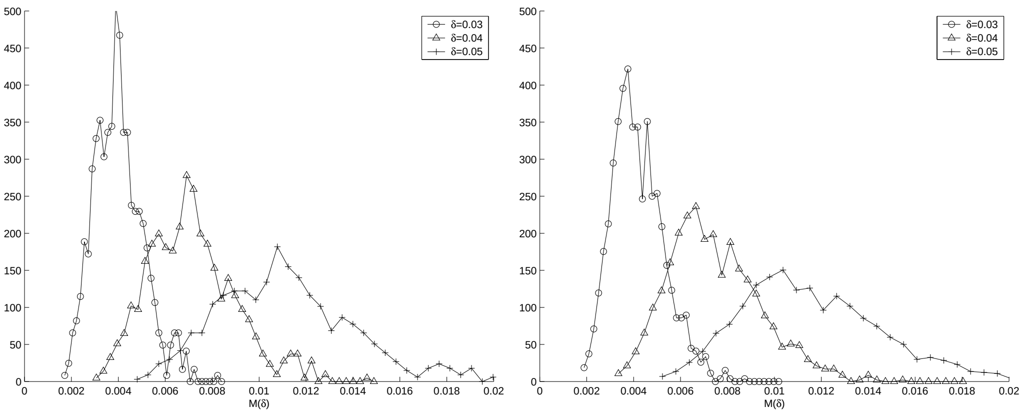

Figure 4.13: Probability density functions for c^* computed via direct simulation, combustion and bistable nonlinearity.

4.2 Temporally Random Shear Flows

Next, we use the variational formula to compute the propagation speed c^* in the case that B is a temporally random shear flow having the structure

(2.1). We also compare the numerical results with the analytical bounds on c^* derived in Section 3.3. In our numerical simulations we consider the specific case of $b(y,t) = b_1(y)b_2(t)$, $y \in R^1$, where $b_1(y)$ is a smooth periodic function of y. For $b_2(t)$, we use the Ornstein-Uhlenbeck process defined by the Itô equation (3.79) (replace y with t). This is a centered, stationary Gaussian process with covariance function

$$E_Q[b_2(t)b_2(s)] = \Gamma(|t-s|) = \frac{r^2}{2a}e^{-a|t-s|}.$$

As in the previous section, we will choose $r = \sqrt{2}\alpha^{3/4}$ so that the covariance function is

$$E_Q[b_2(t)b_2(s)] = \sqrt{\alpha}e^{-\alpha\,|t-s|} = V(|t-s|), \tag{4.14}$$

and the L^2 norm of $V(z)$ remains constant as α changes, so that the total energy in the power spectrum is invariant. Because $b_1(y)$ is periodic and because of the rapid decay of the covariance function V, $b(y,t) = b_1(y)b_2(t)$ satisfies assumptions A2-A3 of Section 2.1.

We approximate $\mu(\lambda) = \lambda^2/2+\rho(\lambda)$, using the time-evolution approach. We discretize the auxiliary initial value problem

$$\begin{aligned} \phi_t &= \frac{1}{2}\Delta\phi - \lambda b_1(y)b_2(t)\phi, \\ \phi(y,0) &\equiv 1 \end{aligned} \tag{4.15}$$

using the Crank-Nicholson scheme:

$$\frac{\phi_m^{n+1} - \phi_m^n}{\Delta t} = \frac{1}{2}(D^2\phi_m^{n+1} + D^2\phi_m^n) + \frac{1}{2}(F_m^{n+1}\phi_m^{n+1} + F_m^n\phi_m^n), \tag{4.16}$$

where $D^2\phi_m^n$ denotes the standard second order discretization of the Laplacian (or $\frac{1}{2}\Delta_y$) centered at the discrete point (y_m, t_n). The term F_m^n corresponds to the reaction term $\lambda b_1(y)b_2(t)$ evaluated at discrete points (y_m, t_n). The Crank-Nicholson scheme is implicit, second-order in both time and space. In all simulations we use $b_1(y) = \delta \sin(6\pi y)$, where $\delta > 0$ is a scaling parameter, and we compute on the domain $y \in [0, 1]$ with discrete grid spacing $\Delta y = 0.01$. We use an adaptive time step, since the implicit treatment of the reaction term requires that $\Delta t < 2/(F^{n+1})$, and F^{n+1} may grow very large with time. To generate realizations of $b_2(t)$ we integrate the Itô equation (3.79) using an implicit order 2.0 strong Taylor scheme (see [53]) with a discrete spacing $\Delta t_{b_2} \leq 0.1(\Delta t)$, where Δt is the adapting time step for (4.16).

To approximate $\mu(\lambda)$, we iterate (4.16) for a long time $t = T_f$ and approximate

$$\mu(\lambda) \approx \mu_t(\lambda) = \lambda^2/2 + \frac{1}{t}\log(\|\phi\|_1). \tag{4.17}$$

As we have shown analytically, μ_t converges to μ almost surely, so we need only generate one realization of the process. Alternatively, we could generate N realizations of $b_2(t)$, evolve (4.16), and approximate

$$\begin{aligned}\mu(\lambda) \approx E_Q[\mu_t(\lambda, \hat{\omega})] &= \lambda^2/2 + E_Q[\frac{1}{t}\log(\|\phi\|_1)] \\ &= \lambda^2/2 + \frac{1}{N}\sum_{i=1}^{N}\frac{1}{t}\log(\|\phi_i\|_1). \qquad (4.18)\end{aligned}$$

In practice, we observe that T_f can be chosen much smaller when using (4.18) instead of (4.17), since the mean converges much faster than an individual sample. In Figure 4.14, we show one realization of the approximation $\mu_t(\lambda)$ com-

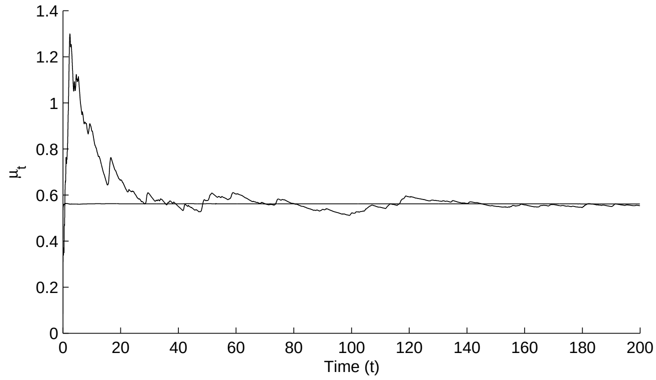

Figure 4.14: One realization of $\mu_t = \frac{1}{t}\log(\|\phi(y,t)\|_1) = \rho_t + \lambda^2/2$. The nearly flat curve shows the sample mean, $N = 40,000$ realizations.

pared with the ensemble mean (4.18). After a very short time, the mean shows relatively little fluctuation compared to the individual realization $\mu_t(\lambda, \hat{\omega})$. The variance of $\mu_t(\lambda)$, shown in Figure 4.15, decays like $O(1/t)$. However, the need to compute a large number of realizations in (4.18) makes this approach no less computationally expensive than evolving only one sample for a very large time. So, we generally use (4.17) to approximate $\mu(\lambda)$. Typically we use a final time up to $T_f = 30,000$. Figure 4.16 shows the convergence of μ computed by (4.17). In Figure 4.17 we show the distribution of μ_t at different points in time. For this simulation we generated $N = 40,000$ realizations.

4.2.1 Numerical Results

The results of our numerical computations suggest that $c^*(\delta)$ is a monotone increasing function of δ, and they confirm both the quadratic and linear growth of $c^*(\delta)$ for small and large δ, respectively. In Figure 4.18, we plot $c^*(\delta) - c^*(0)$

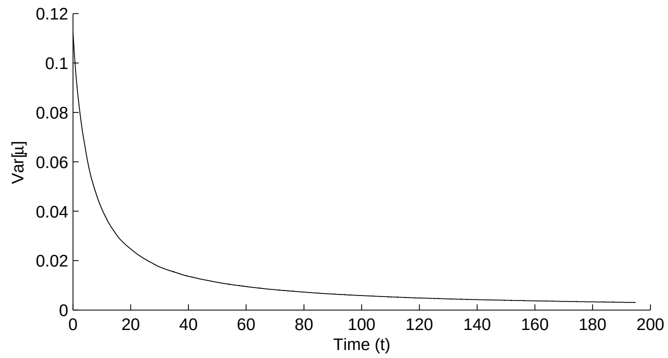

Figure 4.15: Variance of μ_t, $N = 40,000$ realizations.

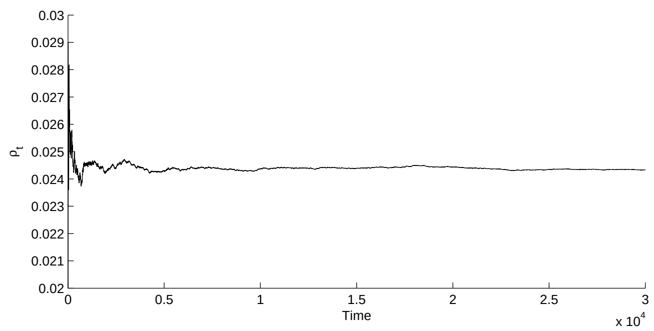

Figure 4.16: Convergence of $\rho_t = \frac{1}{t}\log(\|\phi(y,t)\|_1) = \mu_t - \lambda^2/2$. $T_f = 30,000$.

on a log-log scale for a few values of the covariance parameter α. We observe a transition from quadratic scaling in the small δ regime to linear scaling in the large δ regime. We also plot the upper bounds g_1 and g_2 given by (ii) and (iii) of Theorem 3.3.1 using $\alpha = 16.0$. Of the two upper bounds, g_2 lies closer to the numerically computed values and mimics the transition from quadratic to linear scaling. Indeed, we should expect g_1 to be a course bound, since it contains no information about the temporal correlation of the process.

By varying the parameter α in (4.14), we consider the effect of temporal

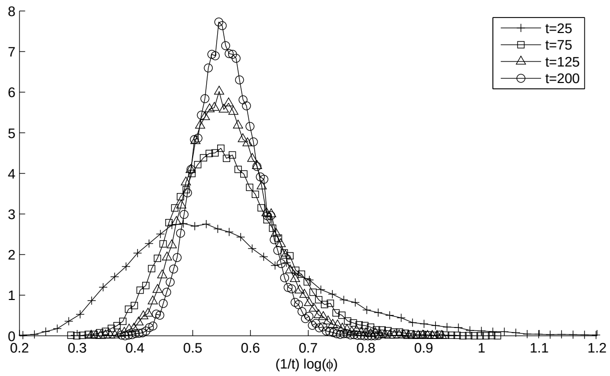

Figure 4.17: Probability density function for $\frac{1}{t}\log(\|\phi(y,t)\|_1)$ at different times.

correlations on the enhancement of the speed c^*. Because of the simple structure of the field $b_1(y)b_2(t)$ we can compute explicitly the constant p_1 appearing in Assumption A3 and Theorem 3.3.1. Using (4.14),

$$\sup_{y_1,y_2} \Gamma(y_1, y_2, 0, r) = \|b_1\|_\infty^2 E_Q[b_2(0)b_2(r)] = \sqrt{\alpha}\|b_1\|_\infty^2 e^{-\alpha r}.$$

Hence $p_1 = \alpha^{-1/2}\|b_1\|_\infty^2$. Therefore, from (iv) of Theorem 3.3.1,

$$c^* \leq c^*(0)\sqrt{(1 + \delta^2\alpha^{-1/2}\|b_1\|_\infty^2)} \tag{4.19}$$

So, as $\alpha \to \infty$, $c^* \to c^*(0)$. This limit corresponds to the correlation time $1/\alpha$ becoming very small and is consistent with the case of periodic time dependence: faster temporal oscillation of the shear tends to decrease the enhancement of the front speed, as suggested by (3.60) (see also [49, 67, 68]). As $\alpha \to 0$, the bound (4.19) blows up. From (iii) of Theorem 3.3.1, however,

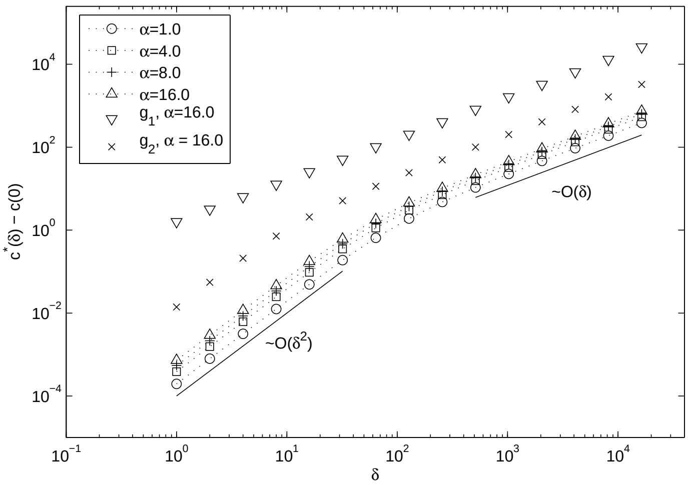

Figure 4.18: Log-log plot of front speed $c^*(\delta)$ versus δ. For comparison, the solid lines on the bottom have slope $p = 2.0$ and $p = 1.0$. The data sets g_1 and g_2 represent the upper bounds (iii) and (iv), respectively, contained in Theorem 3.3.1. Both bounds were computed for $\alpha = 16.0$.

we also have

$$\begin{aligned} c^* &\leq c^*(0) + \delta \|b_1\| E_Q[|b_2|] \\ &= c^*(0) + \delta \alpha^{1/4} \|b_1\| E_Q[|Z|] \end{aligned} \tag{4.20}$$

where Z is a normally distributed random variable. This implies that $c^* \to c^*(0)$ as $\alpha \to 0$, as well. So, we should expect to observe an optimal correlation time $1/\alpha \in (0, \infty)$ at which enhancement is maximal.

For $\alpha \in [1/10, 2^{16}/10]$, we computed the expected speed $c^*(\delta)$ for fixed amplitudes $\delta = 2.0$ and $\delta = 40$. Note that for each α, we must choose the initial points $b_2(0)$ to have variance $E[b_2(0)^2] = \sqrt{\alpha}$ so that the process remains stationary for each α. Also, as α becomes large, we adjust the PDE time

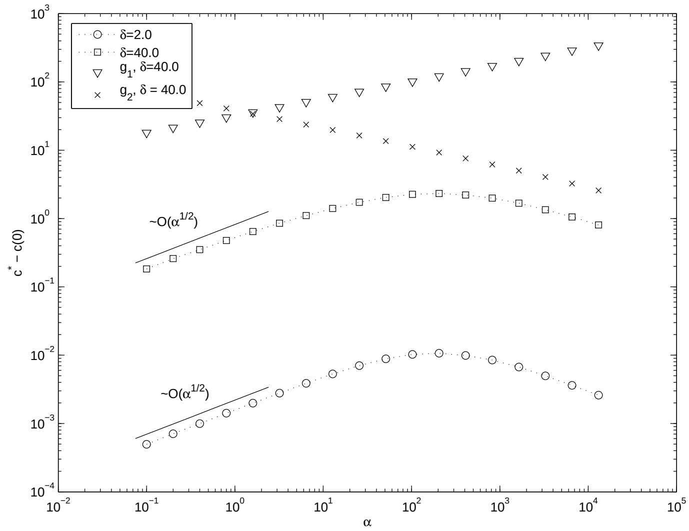

Figure 4.19: Log-log plot of c^* vs. covariance parameter α ($\delta = 2.0, 40.0$). For comparison, the solid lines have slope $p = 1/2$. The data sets g_1 and g_2 represent the upper bounds (iii) and (iv), respectively, contained in Theorem 3.3.1. Both bounds were computed for $\delta = 40.0$.

step so that $\Delta t \leq 0.5/\alpha$, in addition to other restrictions already mentioned. Otherwise, the numerical method cannot resolve the fast oscillations of the shear process, and we observe that the speeds diverge as α grows, violating the known upper bound (4.19).

In Figure 4.19, we plot c^* versus α on a log-log scale. We also plot the upper bounds g_1 and g_2 given by (4.20) and (4.19) for $\delta = 40$. For small values of α, both bounds are rather coarse, an order of magnitude larger than the numerically computed enhancement. As $\alpha \to \infty$, however, the bound g_2 lies relatively close to the data. As $\alpha \to 0$, g_1 is qualitatively better, predicting that $c^* \to c^*(0)$, although the scaling of the bound is different from the scaling observed in the data. By computing the slope of a best-fit line through the points on the log-log plot, we find that for $\delta = 2.0$, c^* scales according to $c^* = c^*(0) + O(\alpha^{0.50})$ and for $\delta = 40$, $c^* = c^*(0) + O(\alpha^{0.44})$. So in both cases, $|c^* - c^*(0)| \to 0$ faster than the $O(\alpha^{0.25})$ convergence predicted by the bound g_1 in (4.20). This scaling behavior can be understood by analogy with the periodic case. Note that $\delta b_1(y) b_2(t) = \alpha^{1/4} \delta b_1(y) \hat{b}_2(t)$ where $\hat{b}_2$ has unit variance and correlation length $1/\alpha$. If $b(y,t) = b_1(y)\hat{b}_2(t)$ and $\hat{b}_2$ is periodic with very long wavelength, then the enhancement is approximately equal to the enhancement caused by b_1 only (a steady shear). The very slow oscillations in the shear field do not significantly slow the front. In the random case, when α is small the correlation time is very large. So by analogy, we expect that in the random case, when α is very small, c^* will behave as if the shear were just $\alpha^{1/4}\delta b_1$. In the small amplitude regime, c^* scales quadratically with amplitude. Hence $c^* \approx c^*(0) + O((\alpha^{1/4}\delta)^2) = c^*(0) + O(\alpha^{1/2}\delta^2)$, which is consistent with our numerical computations for α small. Figure 4.20 shows the data for $\delta = 40.0$ in terms of correlation time $1/\alpha$.

4.3 Cellular Flows

In this section we consider general spatially periodic, divergence-free flows in two dimensions. In this case, the flow can be defined by a stream function $H(x, y, t)$:

$$B = \nabla^{\perp} H(x, y, t) = (-\partial_y H, \partial_x H). \tag{4.21}$$

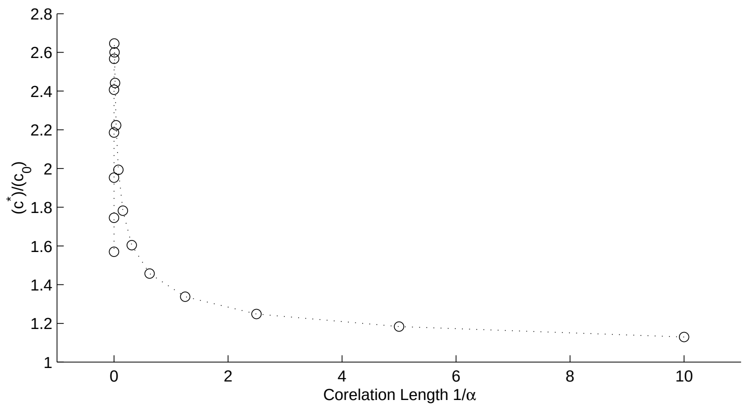

Figure 4.20: Dependence of c^* on correlation time $1/\alpha$ ($\delta = 40.0$).

For shear flows $H = H(y,t)$. When the flow has a more complicated structure, the eigenvalue problem cannot be reduced to a one-dimensional problem, and the evaluation of $\mu(\lambda)$ is more challenging. Therefore, we use the time-evolution method of computing $\mu(\lambda)$, which eliminates the need to solve a nonsymmetric eigenvalue problem.

Using essentially non-oscillatory (ENO) schemes for Hamilton-Jacobi equations, Bourlioux and Khouider [48] have developed a method for computing the effective Hamiltonian defined by the cell problem (1.12) without the Δw term. The absence of the Δw term results from homogenizing the equation with a different scaling than what is used in (1.10) (see [62]). Their idea for computing H is to differentiate the cell problem to obtain an equation for Dw, the gradient of the eigenfunction. The equation for Dw is then solved by marching in time until a steady state is achieved. At each step, the condition that Dw be a discrete gradient is enforced to ensure convergence to a solution.

In the present case, the equation for $\log \phi$ is exactly the cell problem (1.12). Nevertheless, the presence of a diffusive term renders the gradient preserving scheme unnecessary.

The equation (4.2) has the form

$$\phi_t = \Delta\phi + B \cdot \nabla\phi + F\phi \tag{4.22}$$

with $\nabla \cdot B = 0$. When then scaling parameter δ is large, the advection is very strong relative to the diffusion, and the reaction term is stiff. To evolve the system in time, we use the so-called Strang splitting technique:

$$\phi^{n+2} = e^{\Delta t A_1} e^{2\Delta t A_2} e^{\Delta t A_1} \phi^n \tag{4.23}$$

where Δt denotes the discrete time step, A_1 denotes the operator $A_1\phi = \Delta\phi$, and A_2 corresponds to the advection and reaction terms, $A_2\phi = B \cdot \nabla\phi + F\phi$. This time splitting introduces errors of the order $O((\Delta t)^3)$ since the operators A_1 and A_2 do not commute [56].

For the diffusive step, we use the implicit Crank-Nicholson scheme, which is unconditionally stable, and we use the Fast Fourier Transform to approximate $\Delta\phi^n$ and $\Delta\phi^{n+1}$ with spectral accuracy.

For the advective step, we compute $e^{\Delta t A_2}\phi$ using a semi-Lagrangian scheme, as in [29], [79]. Suppose the solution ϕ^n is known at time $t_n = n\Delta t$. Let $Z^{z,t}(s)$ solve the characteristic equation moving backwards in time from the point (z, t):

$$\frac{dZ^{z,t}}{ds} = -B(Z^{z,t}(s), t - s), \quad Z^{z,t}(0) = z, \tag{4.24}$$

Now if $\phi^{n+1} = e^{\Delta t A_2}\phi^n$, then we compute $\phi^{n+1}(z)$ by tracing backwards along the characteristics emanating from the point (z, t_{n+1}):

$$\phi^{n+1}(z) = \phi^n(Z^{z,t_{n+1}}(\Delta t))e^{\int_0^{\Delta t} F(Z^{z,t_{n+1}}(s),t_{n+1}-s)\,ds}. \tag{4.25}$$

Formula (4.25) is exact, if the integral term and the characteristics are evaluated exactly.

Our approximate solution is denoted $\hat{\phi}$, and the values of $\hat{\phi}$ at the discrete grid points will be denoted by $\phi_{i,j}^n = \hat{\phi}(i\Delta x, j\Delta y, n\Delta t)$. At each time step, we compute $e^{\Delta t A_2}\phi^n$ by applying (4.25) starting from $z = (i\Delta x, j\Delta y)$ at time $t_{n+1} = (n+1)\Delta t$. The characteristics are approximated by a second-order Runge-Kutta scheme with a time step $(\Delta t)_{RK}$ that is typically $(\Delta t)_{RK} = \Delta t/30$. The points $Z^{z,t_{n+1}}(\Delta t)$ where the characteristics terminate may not coincide with one of the discrete grid points. So, we evaluate $\phi^n(Z^{z,t_{n+1}}(\Delta t))$ by bilinear interpolation of the values $\phi_{i,j}^n$ at the grid points closest to $Z^{z,t_{n+1}}(\Delta t)$. Using bilinear interpolation to evaluate $\phi^n(Z^{z,t_{n+1}}(\Delta t))$ makes the scheme monotone so that numerical oscillations are avoided. For greater accuracy in the interpolation step, however, one might also apply a higher order monotone interpolation scheme as described in [29], [74], or [78].

The main advantage of this semi-Lagrangian approach is that it permits time steps much larger than those allowed by the CFL constraint on Δt. With any Godunov-type scheme, the CFL constraint makes computation very slow when the flow amplitude δ is very large. In practice, we may take Δt to be as large as $30T_c$ where T_c is the maximum time step determined by the CFL condition. Larger time steps reduce the numerical diffusion that arises from the averaging or interpolation at each step.

4.3.1 Numerical Results

4.3.1.1 Steady Cellular Flow with Shear Perturbation

In our first simulations we demonstrate the sensitive dependence of c^* on the spatial structure of the flow. We use a stream function defined by

$$H(x, y) = \sin(2\pi x)\sin(2\pi y) + \epsilon\cos(\omega_1 y), \tag{4.26}$$

so that the corresponding flow is $B = B_c + \epsilon B_s$, where $B_c(x, y)$ is a rotating cellular flow, and $B_s(y)$ is a shear flow. When $\epsilon = 0$, the flow is a pure cellular flow with bounded streamlines, except for a separatrix that spans all of R^2. For this case, Novikov and Ryzhik [71] have proven recently that the front speed should scale like $c^*(\delta) = O(\delta^{1/4})$ in the large advection limit $\delta \to \infty$. When δ is small, the expansion (3.25) predicts that $c^*(\delta)$ should scale quadratically with δ. As shown in Figure 4.21, we observe both of these scalings in our simulations. The plot shows the transition from $O(\delta^2)$ to $O(\delta^{1/4})$ scaling as δ increases. The different data series correspond to different values of $\tau = \kappa/f'(0)$ used in the simulation (κ is the diffusion constant). Figure 4.22 also show the $O(\delta^{1/4})$ scaling for larger amplitudes in the lower most plot corresponding to $\epsilon = 0.0, 0.2, 0.4$.

When $\epsilon \neq 0$, the streamlines are perturbed, and the new flow may have infinite channels in the direction of the front, depending on the geometry of B_s and the magnitude of ϵ. The behavior of $c^*(\delta)$ as $\delta \to \infty$ is closely related to

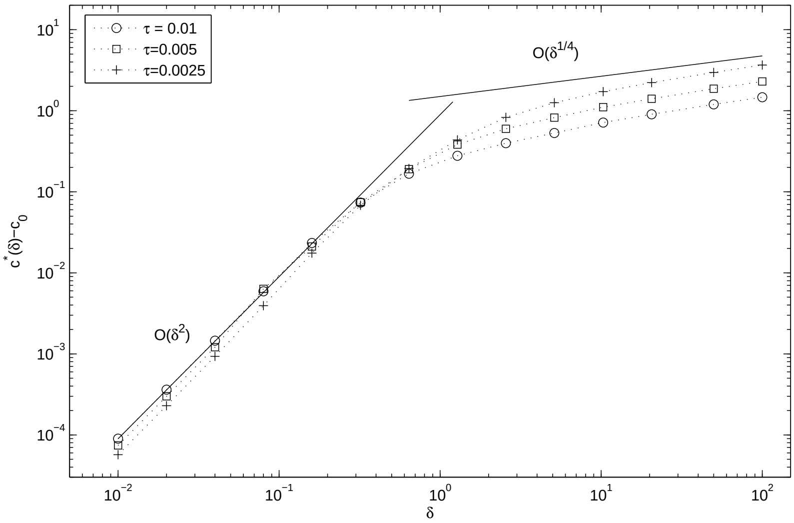

Figure 4.21: Log-log plot of $c^*(\delta)$ showing the transition from quadratic to sub-linear scaling in δ. Here, $\tau = \kappa/f'(0)$ and $\epsilon = 0$. For comparison, the two solid lines have slopes $p = 2$ and $p = 1/4$. The line with slope $p = 2$ is determined by the leading order approximation (3.1) when $\tau = 0.01$.

the properties of first-integrals of B. A first integral is a function $v \in H_1(D)$ that satisfies $B \cdot \nabla v = 0$ almost everywhere and $v \neq 0$. Berestycki, Hamel, and Nadirashvili [13] have shown that if the condition

$$\int_D (B \cdot k) v^2 > 0 \tag{4.27}$$

holds for some first integral, then $c^*(\delta) = O(\delta)$ as $\delta \to \infty$. One can see that (4.27) can be satisfied if $\epsilon > 0$ is sufficiently large, depending on B_s. So, for ϵ sufficiently large, $c^*(\delta)$ should scale linearly with δ. However, B_s may be such that (4.27) cannot be satisfied for small ϵ, so that $c^*(\delta)$ grows sub-linearly.

Our numerical simulations reveal a sharp, nonlinear transition from

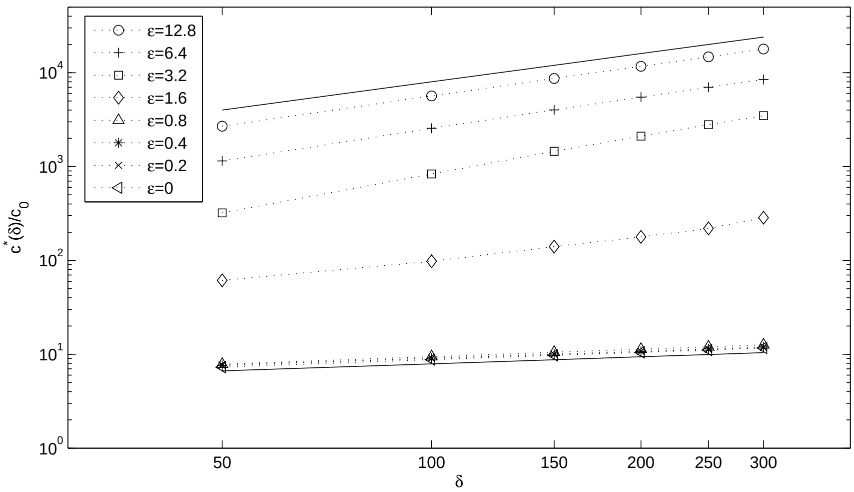

Figure 4.22: Log-log plot of scaling of $c^*(\delta)$ with respect to flow amplitude δ. For comparison, the two solid lines have slopes $p = 1/4$ and $p = 1$.

the sub-linear regime $c^*(\delta) \sim O(\delta^{1/4})$ to the linear regime $c^*(\delta) \sim O(\delta)$ as ϵ increases. This is seen in Figure 4.22, where we plot $c^*(\delta)$ versus δ for various values of ϵ fixed. When ϵ is small, c^* grows like $O(\delta^{1/4})$, and there is very little difference between the curves. This corresponds to the situation where the streamlines are still closed, bounded by an infinite separatrix spanning R^2. However, once ϵ reaches a critical level ϵ^* at which infinite channels are created, then $c^*(\delta)$ grows linearly with δ. This sharp transition is also shown in Figure 4.23, where we plot c^* as a function of ϵ, for δ fixed. There we see that when ϵ is below the critical level ($\epsilon^* \approx 10^0$), c^* changes very little with ϵ and does not vary monotonically with ϵ. However, when ϵ is sufficiently above the critical level, c^* increases linearly with both ϵ and δ.

In Figures 4.24 - 4.26, we show approximations of the eigenfunctions Φ

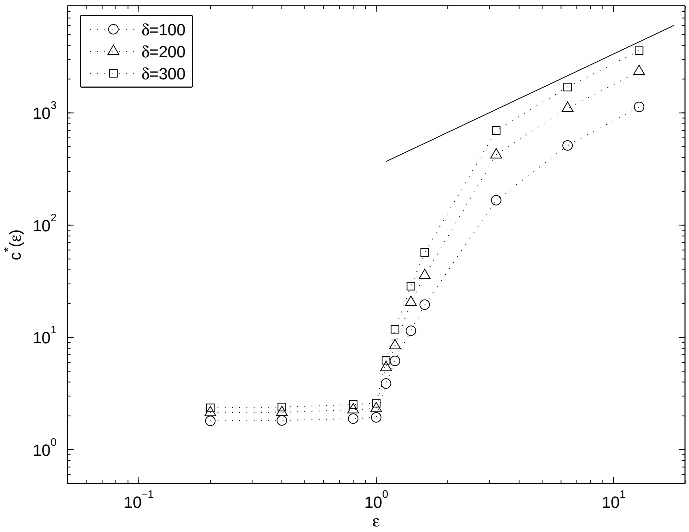

Figure 4.23: Front speed as a function of perturbation amplitude ϵ, for fixed δ. For comparison, the solid line has slope=1.

defining $c^* = (\mu(\lambda^*) + f'(0))/\lambda^*$, for fixed $\delta = 300$. Actually, we have plotted the functions $\phi(z,t)$ at the final simulation time $t = T_f$, but ϕ converges to the eigenfunction (up to scaling by an exponential) in the case that B is independent of time, so ϕ should give a good approximation to Φ. We observe that the structure of the eigenfunctions changes dramatically as ϵ increases beyond the critical value. In Figures 4.24(a), 4.24(b), and 4.25(a), ϵ is below the critical value and the structure of the eigenfunctions reflect the boundedness of the streamlines. In Figures 4.25(b), 4.26(a), and 4.26(b), the structure of the eigenfunctions changes as unbounded channels develop and widen. As $\epsilon \to \infty$, the eigenfunctions become relatively constant in the direction of the flow, as in the case of a shear flow.

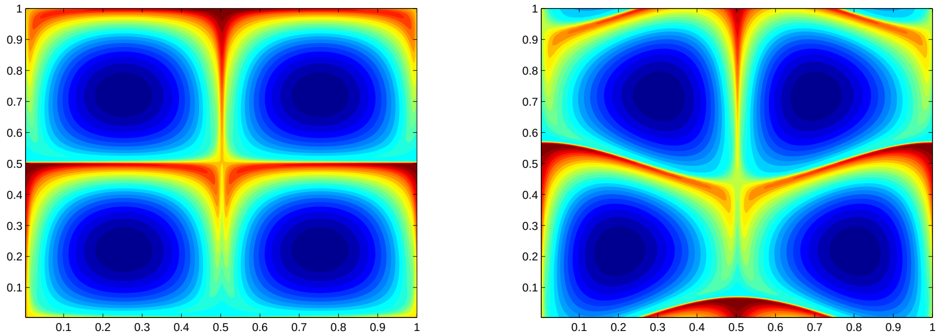

Figure 4.24: Normalized eigenfunctions corresponding to perturbation amplitudes $\epsilon = 0.0, 0.4$.

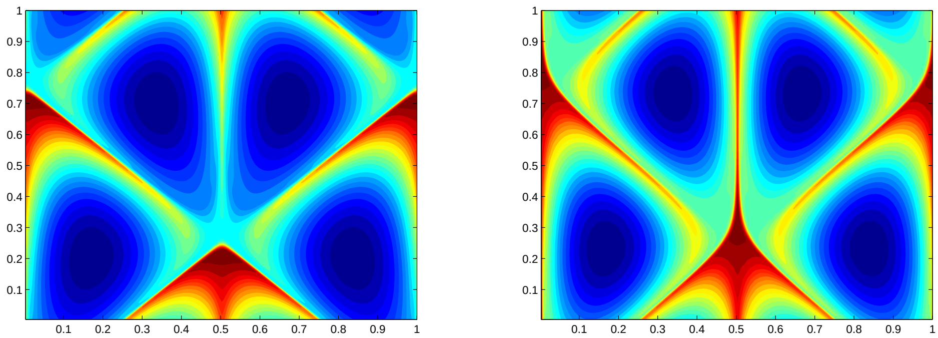

Figure 4.25: Normalized eigenfunctions corresponding to perturbation amplitudes $\epsilon = 1.0, 1.1$. Sufficiently large perturbation of the cellular flow creates unbounded channels, as reflected in the eigenfunction on the right.

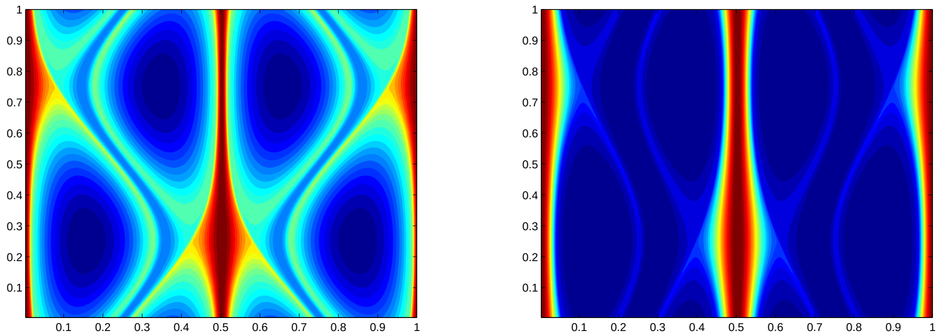

Figure 4.26: Normalized eigenfunctions corresponding to perturbation amplitudes $\epsilon = 1.6, 3.2$.

4.3.1.2 Unsteady Cellular Flows

Next, we investigate the manner in which temporal perturbations of the flow effect the speed c^*. We consider cellular flows that are "shaken", so that the positions of the saddle points oscillate periodically or randomly in time:

$$B(z,t) = B_c(z + \gamma(t)), \tag{4.28}$$

where B_c is the cellular flow defined by (4.26) (with $\epsilon = 0$), and $\gamma(t)$ is periodic or random in t. If $Z'(t) = B_c(Z+\gamma(t))$ then $\zeta(t) = Z+\gamma(t)$ solves the equation

$$\zeta(t) = \zeta(0) + \int_0^t B_c(\zeta(s))\, ds + \gamma(t). \tag{4.29}$$

When $\gamma(t)$ is random, this is a stochastic equation, and $\gamma(t)$ may be unbounded and nowhere differentiable. When $\gamma(t)$ is the periodic perturbation $\gamma(t) = (\epsilon \cos(\omega t), 0)$, the equation for ζ is

$$\zeta'(t) = B_c(\zeta(t)) + \gamma'(t) = B_c(\zeta(t)) - \epsilon\omega k \sin(\omega t), \quad k = (1,0). \tag{4.30}$$

For a fixed time t_0, the streamlines of the flow $B(z, t_0)$ are bounded, as in the previous section with $\epsilon = 0$. Nevertheless, the dynamical system associated with (4.30) may exhibit chaos [45]. This is reflected in the spatial and temporal complexity of the functions ϕ shown in Figures 4.27 compared to the steady case shown in Figure 4.24a.

Using dynamical systems techniques, the authors of [45] have shown that nearly ballistic orbits result from the creation of oscillating, unbounded channels within the shaken reference frame. The temporal perturbation γ allows particles to hop from one bounded cell to another, traveling a distance of order ω^{-3} in a time scale of order $\omega^{-3} \log \omega$, where ω is the frequency of the temporal oscillations. This behavior suggests that the speed of a front moving through such a flow might scale very differently from the speed of a front in the unperturbed flow, which scales like $O(\delta^{1/4})$ as $\delta \to \infty$.

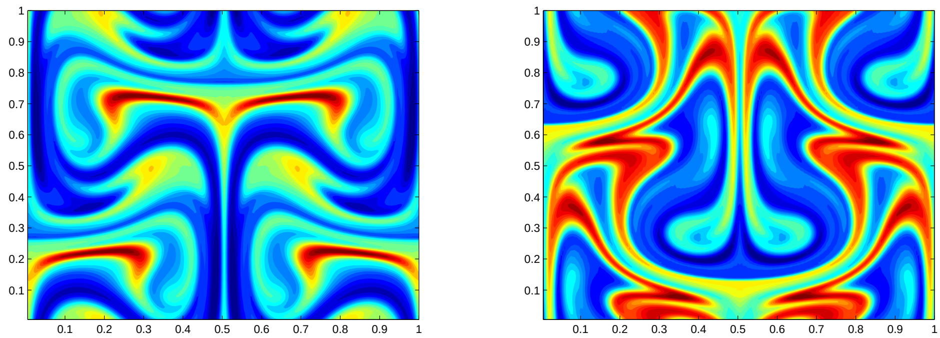

Figure 4.27: Functions $\phi(z, t)$ at final simulation time with random temporal shaking.

Cencini and collaborators [20] studied the case of (4.28) when γ is the periodic perturbation $\gamma(t) = (\epsilon \cos(\omega t), 0)$ and the flow amplitude δ is relatively small. They did this using numerical techniques for the so-called G-equation:

$$G_t + B \cdot \nabla G = c_0 |\nabla G|.$$

The level sets of this function G are assumed to approximate the behavior of the front under the condition that the reaction time-scale is much faster than the diffusive and advective time-scales. By assuming that the front is periodic in space and time, they argue that the speed must be given by the ratio of the temporal and spatial periods, which must be an integer multiple of the spatial and temporal periods of the underlying flow. This suggests that the front speed should exhibit a frequency-locking phenomenon. In particular, c^* is not a monotone function of the shaking frequency ω and may be above or below the front speed corresponding to the unshaken flow. Their numerical simulations with the G-equation support these arguments.

Our simulations using the variational principle for c^* also reveal some resonance between the flow frequency and the amplitude, for certain ranges of the flow amplitude δ. In Figure 4.28a, we plot c^* as a function of ω for fixed flow amplitude $\delta = 0.10$ and perturbation amplitude $\epsilon = 0.02$. We observe only one strong resonance peak, which agrees with the analysis of Section 3.1.3 and the computation shown in Figure 3.1. As δ increases, the small oscillations visible in the Figure 4.28a enlarge. Figure 4.28b shows the same simulation at $\delta = 80$, where we observe multiple resonance peaks, as in the work of [20]. The peaks appear to be more pronounced for larger perturbation amplitudes.

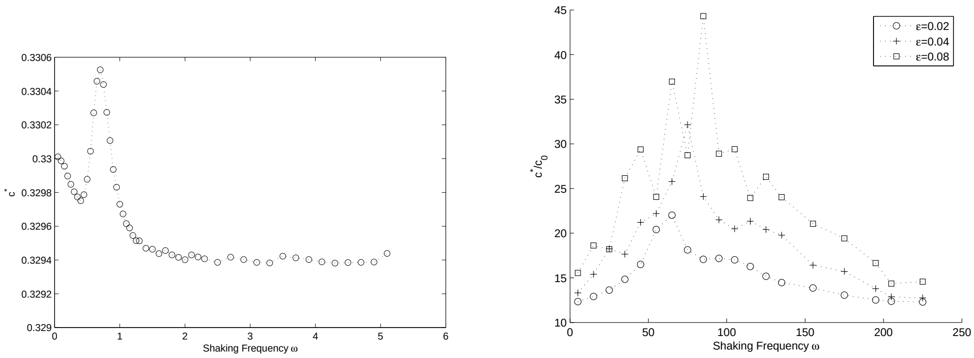

Figure 4.28: Front speed as a function of shaking frequency ω. The amplitude δ is fixed at $\delta = 0.1$ (left) and $\delta = 80$ (right).

The behavior of c^* for small δ and small perturbation amplitude ϵ is qualitatively different from what is predicted in the work of [20]. Even for shear flows with no temporal perturbation, it has been shown [27] that the G-equation leads to inaccurate prediction of the front speed. The G-equation approximation of the front is based on the assumption that the reaction layer is very thin. So, when δ is small relative to the diffusion constant, as in Figure 4.28a, we should expect the G-equation approximation to be very inaccurate.

When δ is large relative to the flow frequency, we observe that $c^*(\delta)$ scales sub-linearly, as shown in Figures 4.29(a,b). Both figures show that the front speed may not be a monotone function of the flow amplitude δ. Figure 4.29b corresponds to a relatively small perturbation amplitude ϵ and it shows that the scaling in δ is proportional to $O(\delta^{1/4})$, as in the unperturbed case. This suggests that, although the temporal perturbation produces nearly ballistic orbits, the effect of these orbits diminishes when the flow is very strong. To

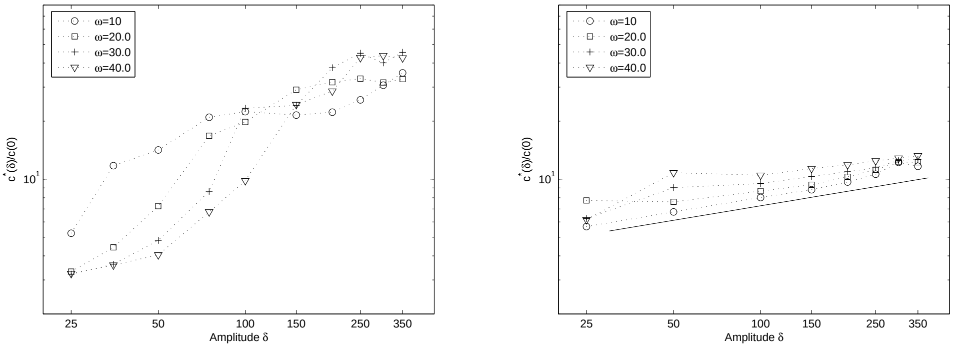

Figure 4.29: Log-log plot of c^* versus flow amplitude for various shaking frequencies. For comparison, the solid line has slope $p = 0.25$. The perturbation amplitude is fixed at $\epsilon = 2.0$ (left) and $\epsilon = 0.02$ (right).

see this, consider the flow at a frozen time t_0, as suggested in [45]. Then in the oscillating reference frame defined by ζ and (4.30), there are open channels, of width $O(1/\delta)$ when δ is large. Since $\epsilon\gamma'(t)$ is bounded independently of δ, this means that as $\delta \to \infty$, the channels will shrink like $O(1/\delta)$. So the probability that a particle will travel very far becomes very small.

When $\gamma(t)$ is a random process, the flow B comprises a continuum of temporal scales, so we should not expect to observe the frequency locking phenomenon that characterizes the periodic case. To simulate this scenario, we let $\gamma(t)$ be the Ornstein-Uhlenbeck process defined by (3.79). As before, we integrate (3.79) using an implicit second-order strong Taylor scheme. Figure 4.30 shows c^* as a function of flow amplitude for various fixed correlation times. As in the periodic case, we observe that c^* scales sub-linearly with δ. In the random case, however, we observe that c^* increases monotonically with δ, without local resonance peaks.

Figure (4.31) shows c^* as a function of correlation time with $\delta = 50$ and $\delta = 125$ fixed. We observe that there is an optimal correlation time leading to maximal speed enhancement. As the correlation time goes to zero, the front speed converges to the speed corresponding to the unperturbed flow. When the correlation time is greater than the optimal value, the front speed c^* decreases with increasing correlation time. This behavior qualitatively resembles the behavior predicted by the approximation argument in Section 3.1.3. However, in the random case we observe this behavior when δ varies over a much larger range, not just when δ is sufficiently small. Moreover, we do not observe multiple resonance peaks, as in the periodic case.

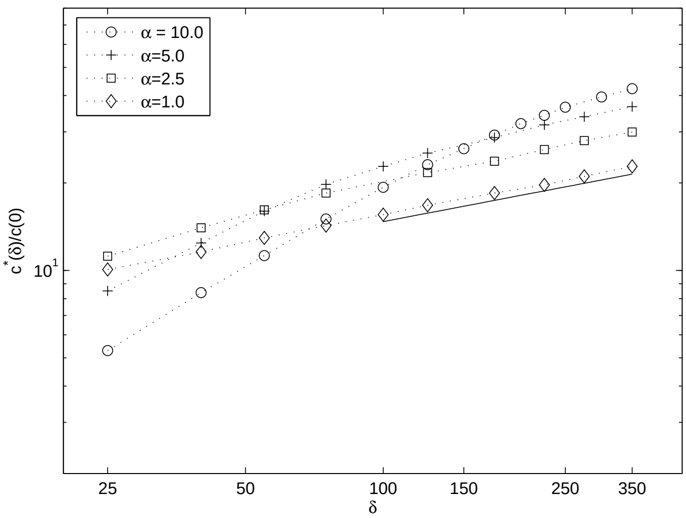

Figure 4.30: Log-log plot of c^* versus flow amplitude for various correlation times $1/\alpha$. For comparison, the solid line has slope $p = 0.3$. The perturbation amplitude is fixed at $\epsilon = 2.0$.

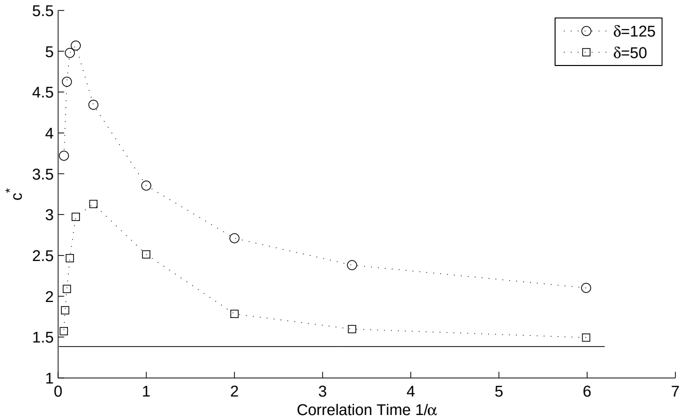

Figure 4.31: Front speed c^* as a function of correlation time for fixed amplitude δ. The perturbation amplitude is fixed at $\epsilon = 2.0$. The solid line corresponds to the front speed in the unperturbed flow ($\epsilon = 0$) for $\delta = 50$.

Chapter 5

Conclusion and Discussion

In this thesis, we have characterized KPP front propagation in prescribed flows that have a spatially periodic and temporally random shear structure. This makes it possible to study the front speed c^* using a variational principle, as in the temporally periodic case. Using variational principles, we derived estimates on the front speed for both periodic and random flows, including temporally random flows. We proved an asymptotic expansion for the front speed when the flow amplitude is small. In the case of temporally random flows, we derived upper and lower bounds on the front speed in the large amplitude regime, as well.

In the last chapter, we used the variational principle to numerically compute the front speed under various assumptions on the prescribed flow. Using this technique, we described the dependence of the front speed on the domain and on statistical properties of the flow. We found that the variational formula offers some significant advantages when compared to direct numerical simulation of the propagating fronts, especially when the flow has a shear structure.

There are many related problems, both analytical and computational, that this thesis did not address. As discussed in Section 2.5, it is not known whether the results of Chapter 2 extend to the case of temporally random flows that have a more general spatial structure. New analytical techniques are needed to prove the existence of the Lyapunov exponents and to establish a large deviation principle for the related diffusion process. If the results of Chapter 2 do extend, then another challenge will be to develop techniques for analytically estimating the front speed, as we have done in Section 3.3 for flows with a shear structure. Moreover, it would be interesting to know whether the fronts described by Theorems 2.1.2 and 2.1.3 obey a central limit theorem, as is the case for random waves governed by Burgers' equation [83].

The recent work of Lions and Souganidis [59] shows that the homogenization results of Theorem 1.2.1 and Theorem 2.4.1 extend to the case of steady, spatially random flows that are uniformly bounded in space. In order to use these results to study the dependence of c^* on the flow, one must develop techniques for analytically estimating or numerically computing the effective Hamiltonian. Analytically, this will be challenging since the flow geometry is random and much more complex than in the periodic cases that have been analyzed (for example, the case of shear flows described here, or the case of periodic cellular flows analyzed in [71]). Computationally, this presents a very challenging problem, because the effective Hamiltonian cannot be defined by a cell problem on a compact domain.

Another interesting question is whether the front speed can be predicted reasonably by first homogenizing the flow field without the reaction term. In other words, is the front speed that results from equation (1.5) a good estimate of the front speed corresponding to equation (1.1)? The analysis of this thesis suggests that the main issue is whether the tails of the associated particle distribution are controlled by the variance. If the distribution is Gaussian, the probability of large deviations is completely determined by the variance. In general, however, this is not the case. Some partial answers to this question in the case of steady periodic flows have been obtained recently by Heinze [42].

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