# Even More Cappell-Shaneson Spheres are Standard

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**Abstract**

This thesis examines an important problem in the field of differential topology: the 4-dimensional smooth Poincaré conjecture. More specifically we analyze a class of objects known as Cappell-Shaneson spheres and the question of whether or not they are counterexamples to the conjecture.

The smooth 4-dimensional Poincaré conjecture concerns objects called manifolds, which can intuitively be thought of as geometric surfaces. For example, a (hollow) sphere is a 2-dimensional manifold which sits in 3-dimensional space, and a circle (not filled in) is a one-dimensional manifold which sits in 2-dimensional space, i.e. the in the plane. Although it is harder to visualize, the same mathematical ideas can be used to define manifolds in higher dimensions. The 4-sphere, for example, is the analogue of the sphere or circle, except that it is a 4-dimensional surface which can be thought of as sitting in 5-dimensional space.

Intuitively one can think of two manifolds of the same dimension as being diffeomorphic if one can be deformed into the other in a smooth way, such that reversing the process is also smooth. "Smooth" in this context can be thought of as meaning that there are no tears or sharp movements in the process of deformation. As an example, an egg shape is diffeomorphic to a sphere because it can be "squashed" into one. However, a torus (doughnut shape) is not diffeomorphic to a sphere because it would

have to be torn in order to be deformed into one.

The last important concept we need to introduce is that of homotopy equivalence. Homotopy equivalence is a weaker way of relating manifolds than diffeomorphism. Roughly, two manifolds are homotopy equivalent if one can be mapped onto the other, and then back onto itself in a way that is essentially the same as sending each point to itself on the first manifold. This will be given a formal definition in the section on basic concepts

The smooth Poincaré conjecture states that a manifold (with some conditions) which is homotopy equivalent to a sphere is also diffeomorphic to a sphere. Whether or not this is true has been settled for most dimensions; for example, it is true for 3-dimensional manifolds. However, the case of four dimensions remains unresolved. It has not been proven that a 4-dimensional manifold homotopy equivalent to a sphere must be diffeomorphic to a 4-dimensional sphere, but on the other hand no counterexample to the claim has been found. This is the last unsolved case of the Poincaré conjecture.

As part of the effort to answer the question of whether or not the conjecture is true, Cappell and Shaneson introduced a set of 4-dimensional manifolds which appear to be possible counterexamples to it. That is, they do not seem, intuitively at least, to be diffeomorphic to a sphere, yet they are homotopy equivalent to spheres. These manifolds are called Cappell-Shaneson surfaces, or Cappell-Shaneson homotopy spheres. A Cappell-Shaneson sphere which is in fact diffeomorphic to a 4-dimensional sphere is called "standard". These manifolds are considered to be some of the most promising candidates for counterexamples to the conjecture.

Some Cappell-Shaneson spheres have already been proven to be standard. Gompf [4] recently found a method for proving that even more of them are standard, by relating some Cappell-Shaneson spheres to those that are already known to be standard. It was then conjectured that this method can be used to prove that they are all standard. If true, then this would give strong intuitive support for the Poincaré Conjecture (though would not prove it). This line of reasoning will be the focus of this thesis. We derive several new results to this end, and give an analysis of the current state of the problem.

Our main results, which are summarized in Theorem 3.1, show that even more Cappell-Shaneson spheres can be shown to be standard using the method of Gompf; for example we extend Gompf's bound on the entry $p$ of a Cappell-Shaneson matrix in certain cases. Although it has not yet been shown that they can all be handled this way, some interesting and/or promising observations about the problem are gained from our efforts. In particular, All of the matrices which our standard methods do not solve have either 19 or 37 in the top middle entry. We will explain the attempts at solving this problem that have been made, and discuss what this tells us about how it can be approached from here.

# 1 Basic Concepts

As mentioned above, the concept of a diffeomorphism can roughly be thought of as a smooth deformation of an object. This can be formally defined as follows. First we define a diffeomorphism on n-dimensional space:

Definition 1.1: A diffeomorphism on $\mathbb{R}^n$ is a one-to-one function from $\mathbb{R}^n$ onto itself which is smooth (i.e. each of its component functions is partially differentiable) and which has an inverse function which is also smooth.

Now we can define manifolds:

Definition 1.2: An n-dimensional smooth manifold (or n-manifold) is a subset of $\mathbb{R}^m$ (for some integer $m$) such that for each point $x$ of $X$, there is a neighborhood $N$ of $x$ and a map $\phi : \mathbb{R}^n \to N$ which is bijective and smooth, and whose inverse is also smooth. $\phi$ is called a coordinate chart of $X$ on the coordinate neighborhood $N$. $X$ is a closed manifold if it satisfies this condition and is bounded (i.e. it is in the interior of some large enough sphere).

Now we can give a formal definition to the idea of smoothly deforming one manifold into another, as follows:

Definition 1.3: A diffeomorphism between the manifolds $X$ and $Y$ is a bijective (i.e. one-to-one and onto) function $f : X \to Y$ which is smooth such that $f^{-1}$ is also smooth. Here smoothness means for each coordinate neighborhood $N$ and chart $\phi$ of $X$, if $M = f(N)$ and $\psi$ is a coordinate chart on $M$, then $\psi^{-1} \circ f \circ \phi$ is a smooth function from $\phi^{-1}(\mathbb{R}^n)$ to $\psi^{-1}(\mathbb{R}^n)$. Two smooth manifolds are called diffeomorphic if there is a diffeomorphism between them.

In other words, a diffeomorphism is a smooth function from one manifold to another whose inverse is also smooth. The coordinate charts are what allow us to be precise about $f$ being smooth on $X$.

Two more definitions are required to explain the conjecture:

Definition 1.4: Let $f, g : X \to Y$ be smooth functions on manifolds. If there is a function $H : X \times [0, 1] \to Y$ which is smooth such that $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$ for all $x$, then the functions $f$ and $g$ are called homotopic, with $H$ being a homotopy between them.

Definition 1.5: Let $X$ and $Y$ be two manifolds, and let $f : X \to Y$ and $g : Y \to X$. The identity function is the function which sends each point to itself. If $f \circ g$ and $g \circ f$ are homotopic to the identity functions $Y$ and $X$ respectively, then $X$ and $Y$ are said to be homotopy equivalent, and the functions $f$ and $g$ are called homotopy equivalences between them.

With these definitions, the smooth 4-dimensional Poincaré conjecture can be stated easily. The most basic 4-dimensional manifold is the 4-sphere which is defined as $S^4 = \{(x, y, z, u, v) \in \mathbb{R}^5 \mid x^2 + y^2 + z^2 + u^2 + v^2 = 1\}$. It is straightforward to verify from the above definition that this is a smooth 4-manifold. The smooth 4-dimensional Poincaré conjecture states that if $X$ is a closed smooth manifold which is homotopy equivalent to $S^4$, then $X$ is diffeomorphic to $S^4$.

The smooth Poincaré conjecture can also be stated for other dimensions, or with smoothness replaced by a different condition (e.g. only requiring that the maps $\phi$ and $f$ be continuous). whether or not the conjecture is true is fairly well understood for dimensions other than 4. The first known counterexample is dimension 7, and there are known techniques for analyzing the situation in higher dimensions. Another version of the conjecture, in which the definitions above only require continuous functions rather than smooth ones, is known to be true in all dimensions. In the case of 4-dimensional smooth manifolds, however, the conjecture is still unresolved. The behavior of the problem is still not well understood, and the conjecture has not been proven to be true. At the same time, no counterexamples have been found so far.

However, there are potential counterexamples to the conjecture. These are smooth 4-manifolds which are not obviously diffeomorphic to $S^4$, but which are homotopy equivalent to $S^4$. The most well-studied of these are the Cappell-Shaneson homotopy 4-spheres, which were first defined in the 1970s. Since then, some of these manifolds have been proven to be diffeomorphic to $S^4$ (in which case they are called standard spheres) but many are still open as possible counterexamples. In this paper, we will explain their construction, what is known about them so far, and the progress that has been made with them. Then, we will attempt to prove that they are all standard.

## 2 Introduction to the Problem

The Cappell-Shaneson 4-spheres are a class of 4-dimensional manifolds which are potential counterexamples to the 4-dimensional smooth Poincaré conjecture. Each of these manifolds is homotopic to a 4-sphere. They are constructed in pairs in such a way that each pair of Cappell-Shaneson spheres is determined by a matrix $A \in SL(3,\mathbb{Z})$ such that $det(A-I)=1$ (These are called Cappell-Shaneson matrices). The explicit construction of these spheres is given by Cappell and Shaneson in [3], with a diffeomorphism $\phi$ of the 3-torus determining the matrix $A$. Due to a choice of framing in the construction, each Cappell-Shaneson matrix gives two manifolds which may not necessarily be diffeomorphic. It is already known [4] that Cappell-Shaneson matrices which are conjugate in $SL_3(\mathbb{Z})$ give identical pairs of Cappell-Shaneson spheres, up to diffeomorphism.

Since conjugate Cappell-Shaneson matrices give the same pair of homotopy 4-spheres, we need only consider conjugacy classes of these matrices in $SL_3(\mathbb{Z})$. Aitchison and Rubinstein [1] show that there are a finite number of these classes for each trace. The "basic" (i.e. best studied) conjugacy for each trace is the class containing

$$A_m = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & m+1 \end{bmatrix}$$

for each $m \in \mathbb{Z}$, corresponding to the trace $m+2$. It has been shown by Akbulut [2] that for each $m$, both Cappell-Shaneson spheres generated by $A_m$ are standard. However, it has proved much more difficult to determine this for the other Cappell-Shaneson matrices. [4] develops an indirect approach which may allow us to show that the other matrices also give standard spheres. This approach relies on another relation, in addition to conjugation, which preserves the diffeomorphism class of the resulting manifold.

In [4] it is shown that if $\phi$ is the diffeomorphism used in constructing a Cappell-Shaneson matrix, then composing $\phi$ with a suitable Dehn twist (on either side) does not change the resulting pair of 4-manifolds. Applying this to the associated matrices in $SL_3(\mathbb{Z})$, it follows from that we may, under certain conditions which will be explained below, multiply a Cappell-Shaneson matrix $A$ (on either side) by the matrix

$$\Delta = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

without changing the resulting Cappell-Shaneson spheres. Because multiplying by $\Delta$ does not preserve conjugacy class, this means that we can use $\Delta$ and conjugation to relate the matrices $A_m$ to other Cappell-Shaneson matrices, showing that those matrices also give standard 4-spheres. [4] gives several examples of this technique, and conjectures that in fact all Cappell-Shaneson matrices can be reduced to some $A_m$. If this conjecture is true, then it would follow that all Cappell-Shaneson spheres are standard, or in other words, that none of them are counterexamples to the smooth 4-dimensional Poincaré conjecture.

## Results of Aitchison and Rubinstein

In [1], Aitchison and Rubinstein show that any Cappell-Shaneson matrix can be conjugated into a matrix of the following form:

$$\begin{bmatrix} 0 & m & n \\ 0 & \lambda & p \\ 1 & x & f \end{bmatrix}.$$

Furthermore, $x$ can be made 0 by another conjugation (subtracting $x$ times column 1 from column 2, while adding $x$ times row 2 to row 1) but this is not necessary for the results of Gompf [4] to apply. [4] shows that a Cappel-Shaneson matrix $A$ can be multiplied by $\Delta$ without changing the resulting homotopy 4-sphere if $A$ is in this form (called the standard form), but multiplying by $\Delta$ is not allowed otherwise. Therefore, when applying Gompf's method, we must use conjugation to ensure that $A$ is in standard form before multiplying by $\Delta$. At the same time, [1] shows that we can assume any Cappell-Shaneson matrix to be in this form (with $x = 0$ if needed).

## Results of Gompf

We will say that two Cappell-Shaneson matrices are equivalent if one can be transformed into the other by a series of conjugations and multiplications by $\Delta$ while in standard form. Below are the main results of Gompf [4]. The main conjecture of [4] is that these results can be extended (using $\Delta$ and conjugation) to show that all Cappell-Shaneson matrices are equivalent to $A_0$.

Using the techniques described above, the following results are shown in [4] ($A$ is a Cappell-Shaneson matrix in standard form)

1. All matrices $A_m$ are equivalent to $A_0 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$.

2. For a Cappell-Shaneson matrix in standard form, p and either m or x are odd.

3. If $tr(A)$ is congruent to a number in $[-6, 9]$ $(mod\ p)$ then $A$ is equivalent to $A_0$.

4. If $\lambda$ (mod $p$) or $\lambda$ (mod $m + \lambda x$) is in $[-3, 4]$, or if $f \equiv 1$ (mod $p$) or (mod $m + \lambda x$), then $A$ is equivalent to $A_0$.

5. If $A$ is not congruent to $A_0$, then $|p| \geq 17$, $|m + \lambda x| \geq 9$,$\left|\lambda - \frac{1}{2}\right| > 4$ and $\left|\lambda + f - \frac{3}{2}\right| > 8$.

6. If $x = 0$, then $A$ is equivalent to $\begin{bmatrix} 0 & m + \lambda(\lambda - 1) & n + p(\lambda - 1) \\ 0 & \lambda & p \\ 1 & 0 & f + p \end{bmatrix}$ and

$$\begin{bmatrix} 0 & m & * \\ 0 & \lambda + m & * \\ 1 & 0 & f - m \end{bmatrix}$$. (From the proof of Theorem 3.4 in [4]). This is one of our main tools.

7. If $x = 0$, then $n = (f-1)(\lambda - 1)$.

8. The determinant conditions on a Cappell-Shaneson matrix allow $p$ and $n$ to be determined if $m$, $\lambda$ and $f$ are known. Specifically, if $x = 0$, then $mp = 1 + n\lambda = 1 + \lambda(\lambda - 1)(f - 1)$, meaning that there are only finitely many possible value of $m$ and $p$ for given $\lambda$ and $f$. Furthermore, $p$ and $m + x\lambda$ are both odd.

Note that if we set $x = 0$ beforehand by conjugation, then $m + \lambda x$ is replaced by $m$ in these results.

In the next section we give partial results expanding on [4] to show that more Cappell-Shaneson matrices are equivalent to $A_0$.

# 3 Main results

Our main result is the following theorem, which we will prove below:

Theorem 3.1: Let $A = \begin{bmatrix} 0 & m & n \\ 0 & \lambda & p \\ 1 & 0 & 2 \end{bmatrix}$ be a Cappell-Shaneson matrix. If $0 \leq \lambda \leq 94$ and $m \neq 19, 37$, or if $1 \leq p \leq 35$, then $A$ is equivalent to $A_0$.

Furthermore, $A$ is equivalent to the following matrices:

(1) $\begin{bmatrix} 0 & m - 2\lambda + p + 1 & \lambda - p - 1 \\ 0 & \lambda - p & p \\ 1 & 0 & 2 \end{bmatrix}$

$$(2)\begin{bmatrix} 0 & m+\lambda(\lambda-1) & n+p(\lambda-1) \\ 0 & \lambda & p \\ 1 & 0 & 2+p \end{bmatrix} \text{(from [4])}$$

$$(3)\begin{bmatrix} 0 & m & * \\ 0 & \lambda-m & * \\ 1 & 0 & 2+m \end{bmatrix}$$ (from [4]; note that the new values of $p$ and $n$ are determined by the equations for a Cappell-Shaneson matrix)

$$(4)\begin{bmatrix} 0 & 1-m-p-2\lambda & * \\ 0 & 1-m-\lambda & * \\ 1 & 0 & 2 \end{bmatrix}$$

## The Case $f = 2$

Here we will prove the above result and analyze the problem in more detail. It is useful to focus on the case $f = 2$, as it is the simplest which has not been shown to give standard spheres (the $f = 1$ case is resolved in [4]). We will also be assuming that $x = 0$, which as Aitchison and Rubinstein [1] show can be assumed without changing $f$. In this case, $n = (f-1)(\lambda-1) = \lambda-1$. We will assume that $\lambda > 0$; the case where $\lambda$ is negative can be dealt with symmetrically. In addition to the transformations derived by Gompf in (6) above, the case $f = 2$ allows another simple transformation using $\Delta$. This is done by multiplying by $\Delta^{-1}$ on the right, then adding the first column to the second while subtracting the second row from the first. If we start with $A = \begin{bmatrix} 0 & m & \lambda-1 \\ 0 & \lambda & p \\ 1 & 0 & 2 \end{bmatrix}$, then the result is $A' = \begin{bmatrix} 0 & m-2\lambda+p+1 & \lambda-p-1 \\ 0 & \lambda-p & p \\ 1 & 0 & 2 \end{bmatrix}$. For many matrices, this move can be repeated until $m < 9$ or until (6) can be used to reduce to one of the cases already resolved in [4]. Using this method, it is straightforward to check that for $\lambda \leq 94$, $A$ is equivalent to $A_0$ except for the matrices for which $\lambda = 46, 50, 65, 69, 80, 84, 85$ and 88, though the $\lambda = 46$case is handled below. The others remain unsolved.

An interesting pattern arises from this analysis for small values of $\lambda$ : in all of the exceptions just listed, the value of $m$ is 19, except when $\lambda = 85$, in which case $m = 37$. Furthermore, the only case where $m = 19$ so far and the method does work is the matrix $\begin{bmatrix} 0 & 19 & 26 \\ 0 & 27 & 37 \\ 1 & 0 & 2 \end{bmatrix}$. So far most attempts to use other methods to reduce one of the matrices with $m = 19$ have resulted in matrices with very large entries with no clear way to reduce them to $A_0$, though we will describe one promising method below. It is not yet clear what structure, if any, links these matrices, if they are in fact counterexamples. The $m = 37$ case may be an example of a similar pattern, but so far only one such matrix has been found (except for $\begin{bmatrix} 0 & 37 & 26 \\ 0 & 27 & 19 \\ 1 & 0 & 2 \end{bmatrix}$ which is handled in the same way as the previous matrix).

We can find some general results for when the new move described above reduces $A$ to $A_0$. As shown above, we can transform $m$ to $m' = m - 2\lambda + p + 1$. Repeating this move $k$ times gives $m' = m - 2k\lambda + k + k^2p = pk^2 + (1-2\lambda)k + m$, which is a quadratic expression in $k$. The discriminant of this expression is $-3$, so that if $m > 0$, then $m' > 0$. Therefore, the minimum possible value of $\mid m' \mid$ will occur when $k$ is the integer closest to $x_{min}$, where $x_{min}$ minimizes $px^2 + (1-2\lambda)x + m$. Using the formula for the minimum of a quadratic function, $x_{min} = \frac{2\lambda - 1}{2p}$, at which $px^2 + (1-2\lambda)x + m = \frac{3}{4p}$. Because $\mid k_{min} - x_{min} \mid \leq \frac{1}{2}$, we can substitute $x_{min} \pm \frac{1}{2}$ into the expression for $m'$ to conclude that the minimum value of $m'$ is no greater than $f(x_{min} \pm \frac{1}{2})$ where $f(x) = px^2 + (1 - 2\lambda)x + m$. Simplifying the resulting expression gives $m'_{min} \leq \frac{3+p^2}{4p}$. If $0 < p \leq 35$, then this expression is less than 9. Therefore, using the results of [4], it follows that $A$ is standard if $f = 2$ and $1 \leq p \leq 35$. Because $p$ is required to be odd, this effectively doubles the bound on $p$ from [4] in the case that $f = 2$.

Next, further analysis of the case m=19 reveals that at least one of these matrices is in fact equivalent to $A_0$. Consider the Cappell-Shaneson matrix $\begin{bmatrix} 0 & 19 & 45 \\ 0 & 46 & 109 \\ 1 & 0 & 2 \end{bmatrix}$. We can first apply the second move in (6) to reduce $\lambda$ to the smallest possible absolute value, then apply $\Delta$ to the rows of the new matrix, then finally perform the second move of (6) enough times to reduce the absolute value of $f$ as much as possible. This results in the matrix $\begin{bmatrix} 0 & -37 & -134 \\ 0 & -66 & 239 \\ 1 & 0 & -1 \end{bmatrix}$. We can then apply $\Delta$ to the columns of this matrix, and following that use conjugation to reset to second entry of the bottom row to 0. Doing this gives us $\begin{bmatrix} 0 & 175 & 134 \\ 0 & 173 & -239 \\ 1 & 0 & -1 \end{bmatrix}$. Finally, the second move of (6) will reduce the value of $\lambda$ to $-2$, which by the results of Gompf shows that the original matrix is equivalent to $A_0$.

In this way, we can see that one of the matrices that was not covered by tour original method can in fact be transformed into $A_0$ by a different method. At this point, it seems intuitive that this may always be the case, although it is not yet clear how to prove this for a general matrix. We can, however, hope to understand the problem better by analyzing some of the special cases that have not yet been solved. For example, an interesting pattern emerges if we apply the method just used to the other matrices with $m = 19$. Applying the first three steps to any of the matrices with $m = 19$ (that have been discovered so far) results in a matrix with $f = -1$. For example, starting with the matrix for which $\lambda = 50, m = 19$ gives $\begin{bmatrix} 0 & -37 & 238 \\ 0 & -118 & 759 \\ 1 & 0 & -1 \end{bmatrix}$. However, this matrix cannot be reduced to $A_0$ in the same way as the previous one, since $\lambda \pmod{m}$ is not reduced to a very small value as it was before. Even so, it is interesting that all of the matrices which we have not been able to solve so far should have the property that $f$ can be changed to $-1$ by the same process. This might hint at some deeper structure which can be used to resolve all or most of these matrices. As a result, we recommend investigating this pattern further in the hopes of finding a general result for the case $f = 2$.

There is another aspect of the matrices with $f = 2$ which, while it has not yet been applied to reducing any specific matrix to $A_0$, nonetheless appears promising. Consider a Cappell-Shaneson matrix of the form $A = \begin{bmatrix} 0 & m & \lambda - 1 \\ 0 & \lambda & p \\ 1 & 0 & 2 \end{bmatrix}$. We can transform it by the following process. Apply $\Delta$ to the columns of $A$ (i.e. multiply $A$ by $\Delta$ on the right), then use an elementary conjugation to interchange the first and second rows (and columns). Now add the second row to the first while subtracting the first column from the second. Finally, use elementary conjugations to reduce the first and second elements of the first column to 0. Call the resulting matrix $A' = \begin{bmatrix} 0 & m' & \lambda' - 1 \\ 0 & \lambda' & p' \\ 1 & 0 & f' \end{bmatrix}$. Carrying out these calculations explicitly shows that $f' = f = 2$, $\lambda' = 1 - m - \lambda$, and $m' = 1 - m - p - 2\lambda$. The value of $p'$ is $\lambda - 1 - (1 - \lambda - m)^2 = -\lambda' - (\lambda')^2 - m$. One reason why this operation seems interesting is because it changes the relationship between $m$ and $\lambda$ in a simple way; it is easy to verify that $\lambda' - m' = \lambda + p$. In other words,

$\lambda' \equiv \lambda + p \pmod{m'}$, a relation which may prove useful in trying to reduce $\lambda \pmod{m}$, and hence $\lambda$. It might be useful to explore this possibility in future research.

## 4 Further Research

Our analysis of matrices with $f = 2$ has revealed interesting patterns that may point toward a solution of the general problem. In particular, it seems likely that our new bound on $p$ can be improved. As mentioned above, if $k_{min}$ is close to $x_{min}$, then it may be possible for the method to still work even for quite large values of $p$. If this direction is pursued further, we may be able to find conditions on $\lambda$, $m$ and $p$ which generalize Theorem 3.1. All matrices for which $\lambda > p$ which we have analyzed so far can be reduced to a small enough value of $m$ or $\lambda$ using only (1) in Theorem 3.1, and we conjecture that this is true in general. Generalizing Theorem 3.1 may allow us to determine this. Lastly, the proof of Theorem 3.1 does not take into account the possibility of combining (1) with other moves; presumably this would give even better results.

Furthermore, there are clear patterns in all of the matrices which we have not yet been able to solve. As was already mentioned, all except one of them satisfy $m = 19$, and all of these can the transformed into a matrix with $f = -1$ by the same process. Only one of these matrices is shown to be equivalent to $A_0$ by that process, but nonetheless there seems to be a structural connection which we might be able to exploit. Investigating this structure further might lead to a method for reducing more matrices to $A_0$. Furthermore, more matrices with $f = 2$ (i.e. higher values of $\lambda$) should be examined to see if the matrix with $m = 37$ is part of a similar pattern to the $m = 19$ case.

Finally, it seems very possible at this point that other methods of reducing matrices to $A_0$ using the allowed move may exist. If these could be combined with the methods of Theorem 2.1, then it may be possible to derive a more powerful result. Further experimentation with matrices for which $f = 2$ may help reach this goal.

In our proof that $A$ is equivalent to $A_0$ if $f = 2$ and $1 \leq p \leq 35$, we used the fact that the distance between $k_{min}$ and $x_{min}$ cannot be greater than $\frac{1}{2}$. Since $k_{min}$ may be much closer than $\frac{1}{2}$ to $x_{min}$, this result can most likely be significantly improved with some conditions on $k$. In addition, the proof above does not take into account the possibility of supplementing our new move with the moves of [4] such as in (6), which allowed many of the above matrices to be reduced. Also, the result may apply to matrices with negative values of $p$ and/or $\lambda$; this may be an interesting topic for further research. Therefore, it seems that there is a promising possibility of better results if this line of reasoning is pursued further.

Finally, in this thesis, we have focused on analyzing the case $f = 2$; the case $f = 1$ was resolved in [4]. It might prove to be just as instructive to examine other special cases, or even to look at the general case with no restrictions on $f$. We have not yet analyzed matrices with $f \neq 2$, but there does not seem to be a reason why they could not be just as instructive as the results obtained here. Therefore, while we have found several areas of interest in the case $f = 2$ to explore further, there are certainly other possible directions in which the problem can be taken.

# 5 Conclusions

The general question of whether or not all Cappell-Shaneson matrices are equivalent to $A_0$ remains unanswered. However, the results which we have found so far can be seen as providing intuitive evidence that they are in fact equivalent to $A_0$. At first, the difficulty of making any progress on the matrices with $m = 19$

made it seem that there are counterexamples. However, the discovery of a way to resolve the case $A = \begin{bmatrix} 0 & 19 & 45 \\ 0 & 46 & 109 \\ 1 & 0 & 2 \end{bmatrix}$ leads us to believe that these apparent counterexamples can be resolved. In addition, above-mentioned (apparent) connection between this and the other matrices with $m = 19$ is promising. Finally, it seems very possible at this point that other methods of reducing matrices to $A_0$ using the allowed move may exist. If these could be combined with the methods of Theorem 3.1, then it may be possible to derive a more powerful result. Further experimentation with matrices for which $f = 2$ may help reach this goal. Therefore we can say that so far the pattern seems to be that every matrix can somehow be reduced to $A_0$. If true, then it would follow that all Cappell-Shaneson spheres are standard. We recommend pursuing the areas of interest described above with the goal of ultimately proving this.

# 6 References


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