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Geometric Representations of Quadratic Solutions

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Geometric Representations of Quadratic Solutions

by

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Report

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Abstract

Geometric Representations of Quadratic Solutions

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This report explores several geometric representations of quadratic equations and their solutions. Topics discussed include applications of geometry relating to solving quadratic equations using graphs and constructions as well as deriving compatible pairs of equations from Pythagorean triples. A brief discussion on the inclusion of advanced graphing methods and constructions into a secondary mathematics class is also included.

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Chapter 1: Introduction

A quadratic equation is a polynomial equation of the second degree written in terms of one variable *x* such that $a \neq 0$:

$$ax^2 + bx + c = 0. \qquad (1)$$

The equation has either two real or two complex solutions, or roots, which may or may not be unique. If the leading coefficient is equal to one, then the equation is in monic form. If $a \neq 1$, the equation can be simplified to monic form by dividing by a.

Students in secondary mathematics classes are taught several algebraic methods of finding the solutions of a quadratic equation such as factoring and the quadratic formula, which can be derived from completing the square. Students learn that the graph of a quadratic function is a parabola, and the abscissas, or *x*-coordinates, of the intersection points of the parabola with the *x*-axis are the real solutions to the given equation, but they are seldom exposed to alternative geometric or graphical methods that can be used to find the real roots of a quadratic equation.

Prior to being introduced to the concept of complex numbers, students are taught that the roots of a quadratic equation are given by the *x*-intercepts of its graph, and a parabola that does not intersect the *x*-axis represents an equation with no real solutions. These points of intersection provide a visual proof of the existence of the roots. After being introduced to complex numbers, students are taught that *all* quadratic equations have roots [**3**]. The values of these imaginary solutions can be found algebraically in a similar manner in which real roots are found, but providing a visual interpretation of the solutions can be a trickier endeavor. However, the complex solutions of quadratic equations can actually found by using graphs and added constructions in the real plane.

In addition to using geometry to solve equations, other interesting connections can be observed between geometry and algebra. For instance, the side lengths of a right triangle can be derived from the coefficients in a “compatible” pair of quadratic equations, and vice versa **[6].** In the following, several geometric and graphical methods that can be used to find the real and complex solutions of quadratic equations will be introduced, as well as an interesting relationship between quadratic parameters and Pythagorean triples.

Chapter 2: Real Solutions from Graphs and Constructions

With the availability of graphing calculators and other advanced technology in the classroom, the traditional non-calculator methods of solving quadratic equations still taught in the classroom may seem "ancient" to some [**1**, p. 349]. Given the efficiency and accuracy that such technological methods provide, it is not surprising that centuries-old processes of solving equations geometrically are generally not introduced in the classroom, as they usually require more time, attention to detail, and procedural steps than modern methods. Nevertheless, the following methods present alternatives to the commonly used means of visualizing and calculating the real solutions of quadratic equations and, if for no other reason, these methods are worth discussing because they "illustrate the power and beauty of coordinate geometry" [**2**, p. 362].

Solving by Graphing: Alternatives to the Parabola

In addition to the standard method of graphing a quadratic function and locating the x-intercepts of the parabola, there are several ways in which the solutions of (1) can be found graphically. The roots can be found by graphing the fixed hyperbola $y = 1/x$ and applying a partial substitution of $x = 1/y$ in (1) to obtain the linear function $ax + cy = -b$. The abscissas of the intersection points of the line and the hyperbola are the roots of (1) [**2**, p. 366]. Figure 1 illustrates an example of this method. The solutions of $x^2 - x - 6 = 0$, which are -2 and 3, are found by graphing $y = 1/x$ and $x - 6y = 1$ and identifying the points of intersection.

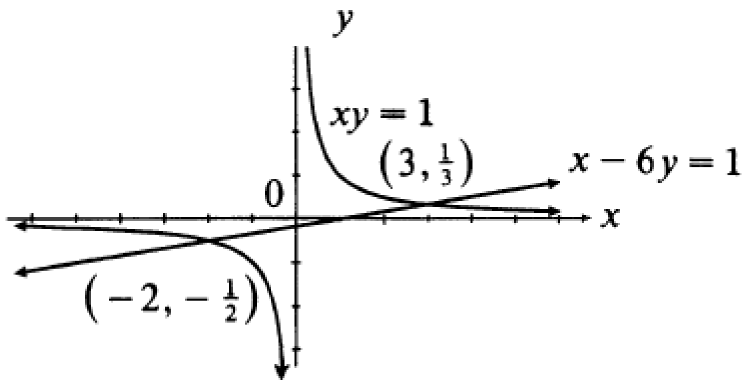

Figure 1. Solving $x^2 - x - 6 = 0$ using a fixed hyperbola [**2**, p. 366].

The next few techniques presented call for the quadratic equation to be in monic form. While this form can be written as $x^2 + px + q$ where $p = b/a$ and $q = c/a$, the explanations instead assume $a = 1$ in (1), so the equation to be solved is

$$x^2 + bx + c = 0. \tag{2.1}$$

The following method was originally proposed by Thomas Carlyle (1795-1881) and can be used to find the real solutions of (2.1) by graphing a circle on a coordinate grid whose diameter has endpoints $(0,1)$ and $(-b, c)$ [**2**, p 363]. If (2.1) has two real solutions x_1 and x_2, the circle will intersect the *x*-axis at $(x_1, 0)$ and $(x_2, 0)$. If the circle does not intersect the *x*-axis, then the equation has no real solutions. Figure 2 shows an example of the featured circle used to determine the solutions of (2.1).

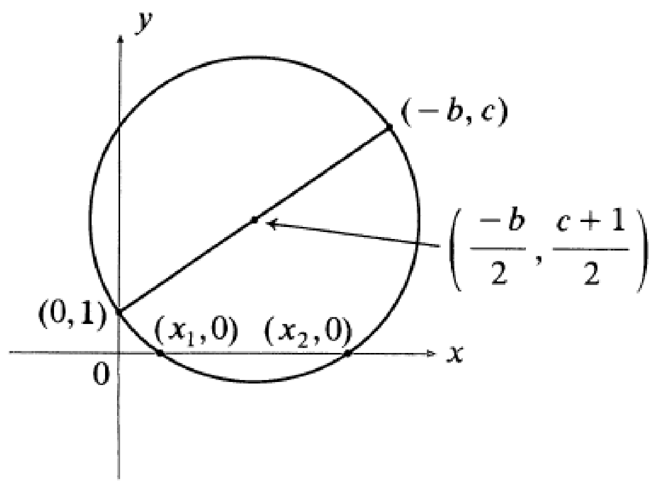

Figure 2. Carlyle's Method [**2**, p. 364].

This result can be verified by considering the equation of the featured circle:

$$x^2 + y^2 + bx - (1 + c)y + c = 0. \tag{2.2}$$

The points at which the circle intersects the *x*-axis can be found by setting $y = 0$, which produces the original equation to be solved (2.1).

The real solutions of (2.1) can also be solved by applying projections on the unit circle as shown in the following method, originally presented by Karl Georg Christian von Staudt (1798 – 1867). The solutions are found by graphing the unit circle with center (0,1) and plotting points $L(\frac{c}{-b}, 0)$, $T(\frac{4}{-b}, 2)$, and $A(0,2)$ (Figure 3). Points *L* and *T* are connected to form a line segment that intersects the circle at points *R* and *S*. The roots of the equation can be found by projecting the points *R* and *S* from *A* onto the *x*-axis to obtain points $(r, 0)$ and $(s, 0)$ where r and s are the solutions to (2.1) [**2**, p. 364]. In Figure 3, von Staudt's method is used to find the solutions of $x^2 - 2x - 8 = 0$, which are $r = -2$ and $s = 4$.

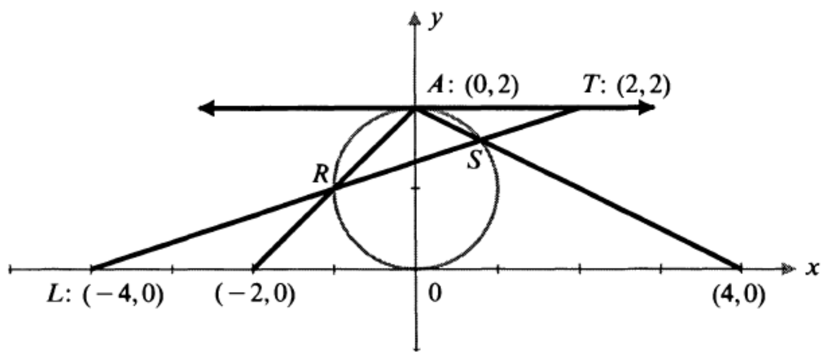

Figure 3. von Staudt's method: solving $x^2 - 2x - 8 = 0$ [**2**, p. 365].

This method can be verified by first deriving the following equations:

$$\text{Circle: } x^2 + y(y-2) = 0 \tag{2.3}$$

$$\text{Line AR: } 2x + r(y-2) = 0 \tag{2.4}$$

$$\text{Line AS: } 2x + s(y-2) = 0. \tag{2.5}$$

Multiplying the left hand side of equations (2.4) and (2.5) and setting this product equal to a factor of (2.5) produces an equation of a graph that passes through points *A*, *R*, and *S*:

$$4x^2 + 2(r+s)(y-2)x + (rs)(y-2)^2 = 4x^2 + 4y(y-2), \quad (2.6)$$

which simplifies to

$$(y-2)[2x(r+s) + rs(y-2) - 4y] = 0. \quad (2.7)$$

It follows that points *A*, *R*, and *S* must lie on the graphs of $y = 2$ or

$$2x(r+s) + rs(y-2) - 4y = 0. \quad (2.8)$$

Neither *R* nor *S* lie on $y = 2$, therefore (2.8) must be the equation for line *RS*.

Recall that the points plotted at the beginning of the construction, *L* and *T*, occur on line *RS* when $y = 0$ and $y = 2$. The intersection points of line *RS* with $y = 0$ and $y = 2$ are $(\frac{rs}{r+S}, 0)$ and $(\frac{4}{r+s}, 2)$, respectively. Note that if *r* and *s* are solutions to (2.1), then the following equivalency holds:

$$x^2 + bx + c = (x-r)(x-s) = x^2 - (r+s)x + rs. \quad (2.9)$$

It follows that $-b = r + s$ and $c = rs$. By substitution, the coordinates of *L* and *T* are $(\frac{c}{-b}, 0)$ and $(\frac{4}{-b}, 0)$, thus the validity of the method is established.

The methods presented in this section are just a few of the many ways in which the real roots of a quadratic function can be found graphically. In addition to graphing, there are many other ways in which these roots can be derived geometrically.

Solving Using Constructions

Euclidean constructions provide another geometric process that can be used to find the real solutions of (2.1), [**2**, p. 363]. Since b and c represent the lengths of segments in the featured construction, both values must be positive, so the equation is written as four distinct cases:

$$x^2 - bx + c = 0 \tag{2.10}$$

$$x^2 + bx + c = 0 \tag{2.11}$$

$$x^2 - bx - c = 0 \tag{2.12}$$

$$x^2 + bx - c = 0. \tag{2.13}$$

Figure 4 illustrates a construction that can be used to derive the solutions for the first two cases. The following construction begins with forming two circles with diameters $(c + 1)$ and b units long where A and B are the endpoints of the second circle's diameter. An inscribed right triangle with height $\sqrt{c}$ is drawn in the circle with diameter $\overline{AB}$ and C is the intersection of the altitude and $\overline{AB}$. The values of the solutions are expressed as lengths of line segments in the featured construction. The solutions of (2.10) are given by $x_1 = AB$ and $x_2 = CB$, and the solutions of (2.11) are $x_1 = -AB$ and $x_2 = -CB$.

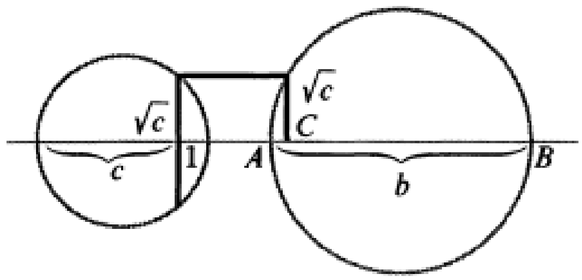

Figure 4. Solving using Euclidean constructions [**2**, p. 363].

This result can be verified by using similar right triangle ratios to observe that $\frac{AC}{\sqrt{c}} = \frac{\sqrt{c}}{CB}$, therefore c$= AC \cdot CB$. Consider the solutions to (2.10): If $x_1 = AC$ then $CB = b - x_1$ (Figure 4). By substitution, $c = x_1(b - x_1)$, therefore $x_1^2 - bx_1 + c = 0$ which is equivalent to (2.10). The same method can be used to confirm that $x_2 = CB$ for (2.10) and that $-AC$ and $-CB$ are the solutions to (2.11).

Chapter 3: Complex Roots in the Real Plane

When using algebraic methods such as the quadratic formula, the procedure used to solve an equation with real solutions can also be used to solve an equation with complex solutions. However, the graphical methods used to solve a quadratic equation with real solutions cannot always be used to solve an equation with complex solutions. The following methods show how complex solutions can be derived geometrically from the graph of a quadratic equation in the real plane.

For the first procedure, consider a quadratic function of the form

$$y = a(x-h)^2 + k \tag{3.1}$$

where (h, k) is the vertex of the parabola. This form can be easily derived from an equation in standard form by letting $h = \frac{-b}{2a}$ and $= c - \frac{b^2}{4a}$. For the featured example, let a and k be positive to produce a concave up parabola that does not intersect the *x*-axis, and let the roots of (3.1) be given by $\alpha \pm i\beta$.

The values of α and β can be found by plotting the parabola and the line $y = 2k$ on the same grid (Figure 5). The value of α is equal to the *x*-coordinate of the graph's vertex, so $\alpha = h$. The value of β is equal to half the length of the chord formed by the intersection of $y = 2k$ and the parabola [**4**, p. 284]. Since $y = 2k$ intersects the parabola at points $\left(\left(h + \sqrt{k/a}\right), 2k\right)$ and $\left(\left(h - \sqrt{k/a}\right), 2k\right)$, the length of the chord is $2\sqrt{k/a}$ thus $\beta = \sqrt{k/a}$.

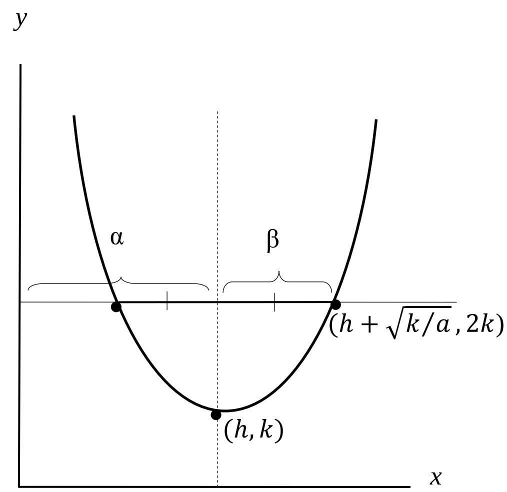

Figure 5: Graph of (3.1) with roots $\alpha \pm i\beta = h \pm i\sqrt{k/a}$.

This result can be verified by setting $y = 0$ in (3.1) and using the quadratic formula to solve for *x*, so

$$x = \frac{2ah}{2a} \pm \frac{\sqrt{-4ak}}{2a} \tag{3.2}$$

which simplifies to

$$x = h \pm i\sqrt{k/a}. \tag{3.3}$$

By substitution we have that $x = \alpha \pm i\beta$ when $y = 0$, which confirms the result shown in Figure 5.

Another graphical method for which the complex solutions of a monic quadratic can be found is shown in Schultze's text published in 1908 [**5**]. This method can be used to solve (2.1) by splitting the quadratic into two separate functions: $y = x^2$ and $y = -bx - c$. If the graphs of these equations intersect, then the equation has real solutions which are given by the abscissas of the intersection points. However, if the graphs do not intersect, then the equation has two complex solutions [**2**, p. 366]. Note that if there are no real solutions, one can assume that $b^2 - 4c < 0$. Since $\sqrt{b^2 - 4c}$ is equal to $\sqrt{(-1)(4c - b^2)}$, the complex solutions obtained from the quadratic formula are given by

$$x = \frac{-b}{2} \pm \frac{i\sqrt{4c - b^2}}{2}. \tag{3.4}$$

The values of $\frac{-b}{2}$ and $\frac{\sqrt{4c-b^2}}{2}$ can be derived separately by adding constructions to the non-intersecting graphs. Figures 6 and 7 show the case where $b < 0$ and > 0 . To begin by finding the real component $\frac{-b}{2}$ of each solution, let the graph of $y = -bx - c$ be line *L* and construct a chord $\overline{AB}$ on the parabola parallel to *L* with a midpoint *M* . Extend a line from *M* perpendicular to the *x*-axis that intersects the parabola at point *P*, the axis at *V*, and line *L* at *N* (Figure 6). The coordinate of *V* is given by $(-\frac{b}{2}, 0)$, thus the real component of each solution is given by the abscissa of *V*.

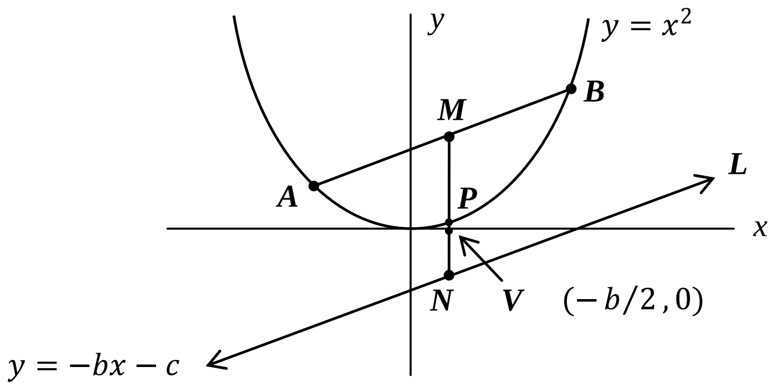

Figure 6. Real component $\frac{-b}{2}$ of complex solution is given by *V*.

To verify that the *x*-coordinate of *V* is $\frac{-b}{2}$, it is sufficient to show that the *x*-coordinate of *M* is $\frac{-b}{2}$. Let the coordinates of *A* and *B* be given by (x_1, x_1^2) and(x_2, x_2^2). Therefore, the slope of $\overline{AB}$ is $(x_1 + x_2)$ and the *x*-coordinate of point *M* is $\left(\frac{x_1+x_2}{2}\right)$. Also, since the slope of *L* is $-b$, the slope of $\overline{AB}$ is also $-b$. In summary, the slope of $\overline{AB}$ is equal to $x_1 + x_2$ and $-b$, thus

$$\frac{x_1 + x_2}{2} = \frac{-b}{2}. \tag{3.5}$$

Therefore, the *x*-coordinate of point *M* (and *V*) is $\frac{-b}{2}$.

To find the imaginary part of each solution, continue the construction by adding a point Q on $\overline{MN}$ such that $PQ = PN$ (Figure 7). Construct a chord $\overline{ST}$ through point Q parallel to line *L*, then extend a line from *T* perpendicular to the *x*-axis that intersects the axis at W. The magnitude of the imaginary part of each solution $\frac{\sqrt{4c-b^2}}{2}$ is represented by the length of $\overline{VW}$.

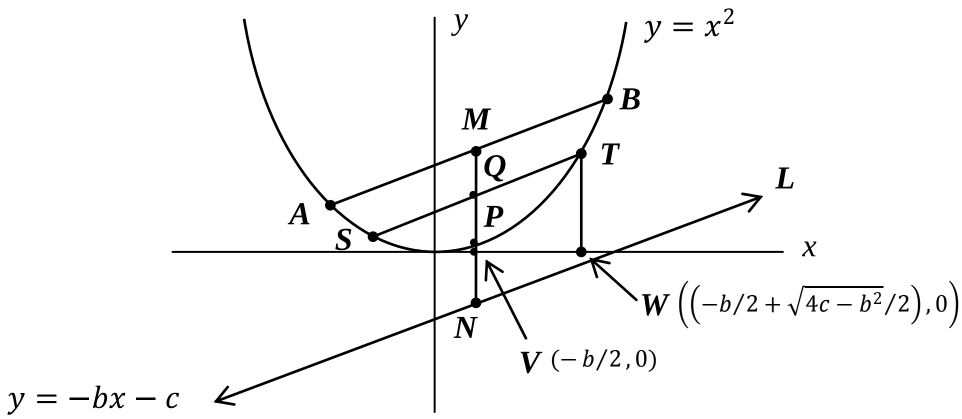

Figure 7. Imaginary component of complex solution $\frac{\sqrt{4c-b^2}}{2}$ is given by VW

To prove that $VW = \frac{\sqrt{4c-b^2}}{2}$ the *x*-coordinate of point *W* must be found. Since $PN = PQ$, the *y*-coordinate of Q is c, so the equation of $\overline{ST}$ is given by

$$y - c = -b\left(x + \frac{b^2}{2}\right). \tag{3.6}$$

Point T lies on $y = x^2$, and the intersection of this parabola with (3.6) produces the equation

$$x^2 + bx + \left(-c + \frac{b}{2}\right) = 0 \tag{3.7}$$

which can be solved for *x*:

$$x = \frac{-b}{2} \pm \frac{\sqrt{4c - b^2}}{2}. \tag{3.8}$$

The *x*-coordinate of *T* is given by the positive solution to (3.8), which is equivalent to the *x*-coordinate of *W*. It follows that

$$VW = \left(\frac{-b}{2} + \frac{\sqrt{4c - b^2}}{2}\right) - \left(\frac{-b}{2}\right) = \frac{\sqrt{4c - b^2}}{2} \tag{3.9}$$

thus verifying the result.

Chapter 4: Integer Roots and Pythagorean Triples

Instead of outlining another way in which solutions of quadratic equations can be found geometrically, this chapter presents an interesting connection between coefficients in a *compatible* pair of quadratic polynomials and the side lengths of a right triangle. If both $x^2 - bx + c$ and $x^2 - bx - c$ are factorable over the integers where $b, c > 0$, they form what is called a compatible pair [**6**, p. 273]. For example, $x^2 - 5x \pm 6$ is a compatible pair since it can be factored as $(x - 3)(x - 2)$ and $(x - 6)(x + 1)$.

The coefficients in a compatible pair of quadratic equations can be used to generate Pythagorean triples, or sets of integers *x*, *y*, and *z* that represent the side lengths of a right triangle satisfying the Pythagorean Theorem $x^2 + y^2 = z^2$, and vice versa. Compatible pairs can be derived from the values in Pythagorean triples.

By the quadratic formula, the solutions to $x^2 - bx \pm c = 0$ are given by

$\frac{b}{2} \pm \frac{\sqrt{b^2+4c}}{2}$ and $\frac{b}{2} \pm \frac{\sqrt{b^2-4c}}{2}$. If the solutions are integers, then $b^2 + 4c$ and $b^2 - 4c$ are perfect squares, so there must exist integers *k* and *l* such that $k^2 = b^2 + 4c$ and $l^2 = b^2 - 4c$. From this the following equations can be derived:

$$k^2 + l^2 = 2b^2 \tag{4.1}$$

$$c = \frac{k^2 - l^2}{8}. \tag{4.2}$$

Exploring possible values of *k* and *l* that yield compatible pairs is algebraically similar to finding Pythagorean triples. Given integers *k* and *l* that satisfy (4.1) where

$x^2 - bx \pm c$ is a compatible pair and given any Pythagorean triple (x, y, z), the relationship shown in Figure 8 exists. The value of b in $x^2 - bx \pm c$ the is equal to the length of the hypotenuse, and the lengths of the legs in the triangle can be derived from k and l if $x = \frac{k-l}{2}$ and $y = \frac{k+l}{2}$.

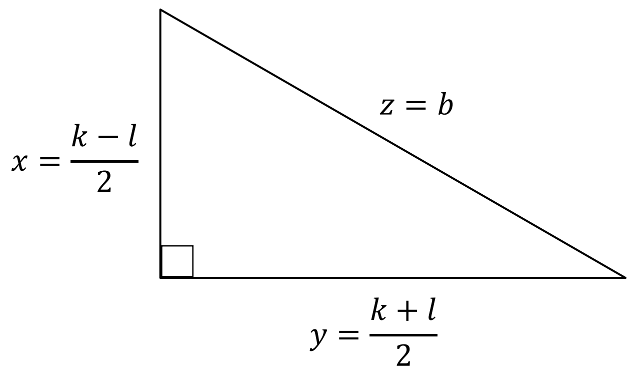

Figure 8. Pythagorean triple in terms of k, l, and b.

To verify this relationship, let k and l be integers that yield a compatible pair. Substituting equation (4.1) into the following equivalency

$$(k-l)^2 + (k+l)^2 = 2(k^2 + l^2). \tag{4.3}$$

produces the equation

$$(k-l)^2 + (k+l)^2 = 4b^2. \tag{4.4}$$

It follows that

$$\left(\frac{k-l}{2}\right)^2 + \left(\frac{k+l}{2}\right)^2 = b^2. \tag{4.5}$$

In summary, if k and l are integers that yield a compatible pair, then $\left(\frac{k-l}{2}, \frac{k+l}{2}, b\right)$ is a Pythagorean triple. As an example, consider the compatible pair $x^2 - 5x \pm 6$ where $b = 5$ and $c = 6$. By (4.1), there must exist two perfect squares whose sum is $2b^2$ or $2(5)^2$. Since $50 = (7)^2 + (1)^2$, one has that $k = 7$ and $l = 1$. Using the result from Figure 8, confirm that $\left(\frac{7-1}{2}, \frac{7+1}{2}, 5\right)$, or $(3, 4, 5)$, is a Pythagorean triple.

Conversely, the values of *x*, *y*, and *z* in a Pythagorean triple can generate integers *k* and *l* that produce a compatible pair. Solving for *k* and *l* using the relationship shown in Figure 8 gives the following equivalencies: $k = x + y$ and $l = y - x$. Use the Pythagorean triple $(3, 4, 5)$ from previous example to justify this result. Thus, $b = 5$ and $k = 3 + 4$ and $l = 4 - 3$. Substituting *k* and *l* in (18.2) yields $c = 6$, which produces the given compatible pair $x^2 - 5x \pm 6$.

Chapter 5: Conclusion

While the featured geometric techniques used to find and represent quadratic solutions may not provide the most efficient path to an answer, they do display fascinating relationships between algebra and geometry. Students in secondary mathematics classes have become increasingly comfortable with solving quadratic and other polynomial equations using graphing calculators as well as using the routine algebraic methods still taught in the classroom. It is less common for students to explore alternate graphical and geometric ways of finding and representing solutions to quadratic equations. However, the methods discussed in this report for solving quadratic equations could easily be incorporated into a Geometry or Algebra II lesson.

The National Council of Teachers in Mathematics secondary standards suggests that students "write equivalent forms of equations, inequalities, and systems of equations and solve them with fluency" as well as "use geometric models to represent and explain numerical and algebraic relationships" [**7**]. As an Algebra II exercise, students could be instructed to find the real roots of a quadratic equation using the fixed hyperbola method or Carlyle's Method. Then, they could verify that finding the intersections points of these graphs is algebraically equivalent to finding the points of intersection of a parabola with the x-axis. In a geometry class, students could be guided through the Euclidean construction method of solving quadratics. This activity would provide a review of similar right triangle relationships as well as present an interesting relationship between geometric and algebraic problem solving. Though the methods for solving quadratics presented in the report may have become obsolete in the in the current high school mathematics curriculum, “these methods do indeed exist and were once studied in the classroom. And what was beautiful and interesting then is still so today” [**2**, p. 368**].**

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Vita

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