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Information Structures and their Effects on Consumption Decisions and Prices

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A mi madre, a mi padre
y a mi abuela Concha.

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Information Structures and their Effects on Consumption Decisions and Prices

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This work analyzes the effects that different information structures on the demand side of the market have on consumption decisions and the way prices are determined. We develop three theoretical models to address this issue in a systematic way.

First, we focus our attention on the consumers' awareness, or lack thereof, of substitute products in the market and the strategic interaction between firms competing in prices and costly advertising in such an environment. We find that prior information held by consumers can drastically change the advertising equilibrium predictions. In particular, we provide sufficient conditions for the existence of three types of equilibria, in addition to one previously found in the literature, and provide a necessary condition for a fourth type of equilibrium. Additionally, we show that the effect of the resulting advertising strategies on the expected transaction price is qualitatively significant, although ambiguous when compared to the case of a newly formed market. We can establish, however, that the transaction price is increasing in the size of the smaller firm's captive market.

In the second chapter, we study the optimal timing to buy a durable good with an embedded option to resell it at some point in the future, as well as its reservation price, where the agent faces Knightian uncertainty about the process generating the market prices. The problem is modeled as a stopping problem with multiple priors in continuous time with infinite horizon. We find that the direction of the change in the buyer's reservation price depends on the particular parametrization of the model. Furthermore, the change in the buying threshold due to an increase in ambiguity is greater as the fraction of the market at which the agent can resell the good decreases, and the value of the embedded option is decreasing in the perceived level of ambiguity.

Finally, we introduce Knightian uncertainty to a model of price search by letting the consumers be ambiguous regarding the industry's cost of production. We characterize the equilibria of this game for high and low levels of the search cost and show that firms extract abnormal profits for low realizations of the marginal cost. Furthermore, we show that, as the search cost goes to zero, the equilibrium of the game under the low cost regime does not converge to the Bertrand marginal-cost pricing. Instead firms follow a mixed-strategy that includes all prices between the high and low production costs.

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Chapter 1

Existence Advertising with Partially Informed Consumers

1.1 Introduction

The information held by consumers regarding prices and the availability of goods in the market plays an important role in the pricing decisions of the firms. Not surprisingly, a considerable amount of effort have been devoted to formalize the effect that imperfect information has on the equilibrium pricing strategies and to analyze the firms' transmission of information to consumers. Most notably, Varian (1980) showed that when a fraction of consumers can identify the lowest price (informed consumers) and the rest accept the first price they observe (uninformed consumers) there is no pure strategy equilibrium, but there exist a symmetric equilibrium where firms randomize over a range of prices. However, the information asymmetries in Varian's model are solely determined by whether consumers are informed or not, and there is nothing the firms can do to change those conditions. In their seminal papers Butters (1977) and Grossman and Shapiro (1984) explicitly allow the firms to decide on the intensity at which they provide information to consumers regarding its price and existence, directly influencing the fraction of consumers aware of particular goods in the market.[1] Despite the differences in information and market structures, this models share an important result: unless the information is complete there exist some monopoly power as a result of some of the firms having a captive market at which they can charge higher prices.

This monopoly power creates incentives for the firms to not fully advertise its existence and form costumer bases in order to avoid price competition. This result was formalized by Ireland (1993) and, more recently, Eaton et al. (2010). Ireland analyzes the case of costless information transmission in an oligopoly market with an homogeneous good using a two-stage model in which firms first simultaneously decide on a fraction of consumers to be reach by and advertising strategy, and then set a price given the size of their captive and overlapped markets. The resulting equilibria of this model are characterized by a mixed strategy in the second stage of the game (pricing stage) and an asymmetric pure strategy equilibrium in the first stage (advertising stage) with a dominant firm advertising to all consumers while the rest of the firms advertise to only a fraction of them – half of the consumers in the duopoly case. Eaton et al. extend Ireland analysis by allowing costly advertising and a more general advertising technology. They show that the asymmetric result in advertising holds if the advertising technology generates an overlap in costumer bases with positive probability.

However, most of the work on information transmission assumes a newly created market where no consumer is aware of the existence of any of the prices nor

[1] Butters focuses on an homogeneous market while Grossman and Shapiro analyze the heterogeneous market case. In the latter the firm's information also includes its position in the market.

goods prior to advertising. Here we propose a model that allows for arbitrary starting conditions regarding the prior information in the market and analyze how these conditions affect the advertising decisions of the firms as well as the resulting price distributions in equilibrium. In order to do so we propose a model with two firms producing and homogeneous good and a unit mass of consumers. Similarly to Ireland (1993) and Eaton et al. (2010), firms decide first on the intensity of advertising using a random advertising technology and then set a price in the second stage of the game. Further we provide sufficient conditions for different types of advertising equilibria which differ from the ones previously found in the literature. Additionally, our first-stage results are formulated using a general advertising cost function, as opposed to Eaton et al. who use a particular functional form. We show that different initial configurations of the information held by consumers lead to significantly different results in the advertising levels as well as the distribution of prices.

The initial information structure in a market can be the result of a number of factors such as previous advertising efforts of the firms, consumer search, word of mouth or entry of new firms to an existing market. For example, consider a monopolistic market that, due to a change in regulations, is now open to competition. In this case consumers are aware of the firm that has been serving the market in the past, but might not be initially aware of the entrant firms. In this line of though, Fudenberg and Tirole (1984) present a case where an incumbent firm engages in preemptive advertising to reduce the effect of price competition from an entrant firm. We do not attempt to model the initial conditions in this paper but rather take them as given.

Iyer et al. (2005) and Janssen and Non (2008) include some dependency of the advertising strategies on initial conditions. In the model developed by Iyer et al. an exogenous fraction of consumers have an inherent preference toward one of the goods offered in the market while the rest are comparison shoppers. They show that, when price discrimination is assumed impossible, it is optimal to target to consumers with high preference for its good in order to reduce price competition in the shared market. This initial captive market, however, will buy the good from the firm only if it is reached by advertising unlike our setting where a consumer originally in the captive market will patronize the firm regardless of its advertising effort, provided they offer a price below the reservation price and they are not reached by the other firm advertising message. Janssen and Non allow for consumer search and a fraction of "shoppers" who are aware of all the goods and prices prior to advertising. The rest of the consumers are assumed to have a common search cost. For the case where this search cost is high enough so no search is observed in equilibrium, their model converges to one of the cases here analyzed. Additionally, Janssen and Non's cost function is assumed linear and advertising is model as a binary decision as opposed to our work that builds on a convex cost function and advertising is modeled as a continuous decision.

The problem of informative advertising has also been analyzed with different objectives in mind. For example, Hamilton (2009) and Christou and Vettas (2008) analyze the welfare implications of existence advertising. Among their main findings is that advertising is under-supplied when the products are less differentiated while over-supplied when more differentiated as a result of the possibility to reduce price competition. Additionally, Christou and Vettas find that the effect on prices as the number of firms increases is ambiguous. Robert and Stahl (1993) include a search component to analyze the effect of different information channels in the equilibrium price distributions. They show that as advertising cost decreases the equilibrium converges to perfect competition. However as the consumer's search cost decreases advertising lessens and the equilibrium prices remain above marginal cost.

The rest of the paper is organized as follows. In section 1.2 we set up the model in detail. Later, in Section 1.3, we obtain the pricing equilibrium for the second stage of the game. Section 1.4 provides the sufficient conditions for which distinct types of equilibria arises, and section 1.5 elaborates on the effect that the resulting advertising strategies have on the price distributions. Finally, section 1.6 concludes. The proofs not included in the chapter are collected in Appendix A.

1.2 The Model

Consider a market populated by a unit mass of consumers and two firms. Each consumer has a demand of at most one unit of the product and is only aware of the alternatives available in her consideration set. This set can be originally empty, include only one of the goods, or include both goods in the market. Let s_j denote the fraction of consumers whose consideration set initially includes only the good provided by firm j, s_m be the fraction of consumers whose consideration set includes both products available in the market, and s_0 be the fraction of consumers with a non-empty consideration set. Additionally, consumers are endowed with a common valuation for the good denoted by $r > 0$.

On the production side of the market, each firm produces an homogeneous good at a constant marginal cost assumed to be zero. Initially, both firms decide simultaneously on a fraction of the market to reach through informative advertising, $m_j \in [0,1]$, with a strictly convex cost function $\phi(m)$, and on a price p_j for its good. We assume that the advertising technology does not allow the firms to target specific segments of the market and that each consumer is reached with equal probability.

The timing of the model goes as follows. On the first stage of the game, the firms choose simultaneously their advertising strategies. In a second stage they simultaneously choose a price for their goods. Consumers with both goods in their consideration set will then patronize the firm with the lowest price as long as it is less than the reservation price, and consumers with only one good in their consideration set will buy that good provided the price is less than r.

1.2.1 Advertising Strategies

In our setting, the only way a consumer can expand her consideration set is through advertising from the firms. Clearly, the structure of these consideration sets will play an important role in the pricing equilibrium. In particular, the advertising decisions will affect the effective size of the market, *i.e.* the fraction of consumers with at least one good in their consideration set, as well as the size of the captive markets. In this section we introduce some notation for the shared and captive markets as a fraction of the effective market size that will be useful in our derivation of the equilibrium pricing strategies. Let the effective market size be denoted by $s_1(m_j, m_k)$; specifically, this is the fraction of consumers with a non empty consideration set after advertising.[2] Define $\theta_j(m_j, m_k)$ and $\theta_m(m_j, m_k)$ as the captive share of the market for firm j and the overlap in the costumer bases, respectively. Given the structure of our advertising technology, we can construct this shares as follows:

$$s_1(m_j, m_k) = s_0 + (1 - s_0)(m_j + m_k - m_j m_k) \tag{1.1}$$

$$\theta_j(m_j, m_k) = \left[(1 - s_0)\, m_j(1 - m_k) + s_j(1 - m_k)\right](s_1)^{-1} \tag{1.2}$$

$$\theta_m(m_j, m_k) = \left[(1 - s_0)\, m_j m_k + s_m + s_j m_k + s_k m_j\right](s_1)^{-1} \tag{1.3}$$

Since we assume that each consumer is equally likely to be reached by the firms' advertising efforts, only $(m_j + m_k - m_j m_k)$ fraction of the consumers with empty consideration sets will receive at least one of the messages. Equation (1.1) simply adds these consumers to the initial effective market size. Similarly, in equation (1.2), $m_j(1 - m_k)$ fraction of the consumers will receive only firm j's message so that fraction of the previously unaware consumers will now be part of firm j's captive market while $(1 - m_k)$ fraction of it's previous captive market will still remain in it. Equation (1.3) is obtained using the same logic.

This notation allow us a great flexibility to determine the initial conditions with respect to the awareness of the existence of goos in the market. For example one can set s_j and s_m equal to zero to specify a new market where all consumers have empty consideration sets, $s_j = 0.5$ and $s_m = 0$ would imply a segmented market and no room to grow the captive markets, or one could change the size of the overlap in costumer bases.[3]

[2] Note that $s_1(m_j, m_k)$ is defined with respect to the whole unit mass of consumers.

[3] This is particularly relevant since we do not require the assumption that each captive market is strictly greater that the shared market as in Eaton et al. (2010).

Obtaining a full characterization of all the equilibria for the above described model is quite complicated. However one can focus the attention on the sufficient starting conditions that will led to different sorts of equilibria. In the next section we obtain an equilibrium for the pricing stage of the game.

1.3 The Pricing Equilibrium

As already discussed, m_j, and m_k will determine the effective market size, s_1, as well as the captive and shared markets, θ_j and θ_m. A consumer in firm j's captive market will patronize this firm as long as it charges a price lower than the reservation price, r. A consumer who is aware of the existence of both goods will buy the good from the firm charging the lowest price, in case they both charge the same price she will randomly pick one of the firms. Additionally, the profits must be scaled by the effective size of the market. Therefore the profit function net of advertising costs for firm j is given by:

$$\pi_j(p_j, p_k) = \begin{cases} s_1(\theta_j + \theta_m)p_j & \text{if } p_j < p_k \text{ and } p_j \leq r \\ s_1(\theta_j + \frac{1}{2}\theta_m)p_j & \text{if } p_j = p_k \text{ and } p_j \leq r \\ s_1\theta_j p_j & \text{if } p_j > p_k \text{ and } p_j \leq r \\ 0 & \text{if } p_j > r \end{cases} \tag{1.4}$$

Define $\underline{p}_j$ as the price at which firm j is indifferent between getting its captive and shared fractions of the market, $s_1(\theta_j + \theta_m)\underline{p}_j$, and getting only its captive market at the reservation price, $s_1\theta_j r$, that is,

$$\underline{p}_j = \frac{\theta_j r}{\theta_j + \theta_m}. \tag{1.5}$$

Varian (1980) and Baye et al. (1992) established that, when both firms have captive markets of the same size (uninformed consumers) there exist a unique symmetric equilibrium in mixed strategies with profits equal to $s_1\theta_j r$ equilibrium. Following a strategy similar to Baye el al. we can obtain an equilibrium to this game for an arbitrary size of the captive and shared markets.

Proposition 1.1. *Given the captive market shares, θ_j, and the effective market size s_1 the following statements are satisfied:*

(i) If $\theta_m = 1$ there exist a unique pure strategy equilibrium in prices given by $p_j = 0$ with equilibrium profits $\pi_j^* = 0$.

(ii) If $\theta_m = 0$ there exist a unique pure strategy equilibrium in prices given by $p_j = r$ with equilibrium profits $\pi_j^* = s_1\theta_j r$

(iii) If $0 < \theta_m < 1$ and $\theta_j > 0$ for some j with $\underline{p}_j \geq \underline{p}_k$, there exist a unique mixed strategy equilibrium in prices with price distributions

$$G_j(p) = \begin{cases} 1 - \left[\frac{(\theta_k+\theta_m)\theta_j r}{(\theta_j+\theta_m)\theta_m p} - \frac{\theta_k}{\theta_m}\right] & \text{if } \frac{\theta_j r}{\theta_j+\theta_m} \leq p < r \\ 1 & \text{if } p = r \\ 0 & \text{otherwise} \end{cases} \tag{1.6}$$

$$G_k(p) = \begin{cases} \frac{\theta_j+\theta_m}{\theta_m} - \frac{\theta_j r}{\theta_m p} & \text{if } \frac{\theta_j r}{\theta_j+\theta_m} \leq p \leq r \\ 0 & \text{otherwise} \end{cases} \tag{1.7}$$

and equilibrium profits $\pi_j^* = s_1\theta_j r$ and $\pi_k^* = s_1\frac{(\theta_k+\theta_m)\theta_j r}{(\theta_j+\theta_m)}$.

The first part of Proposition 1.1 refers to the case where no firm has a captive share of the market so, after the advertising decisions have been made, all consumers are aware of the existence of both goods. Therefore the firms play a Bertrand type of game and it is well known that the unique equilibrium is to set both prices equal to the marginal cost yielding zero profits.[4] The second part of our result corresponds to the case where there is no overlap in the costumer bases of the firms, so each of them charges the corresponding monopoly price on its share of the market and obtain the monopoly profits scaled by the effective size of the market.

The third part of Proposition 1.1 corresponds to the more interesting case where at least one of the firms has a captive market and there is some positive overlap in the costumer bases. In this case the firm with the biggest captive market, which corresponds to the firm with the greatest $\underline{p}$ as defined in equation (1.5), will obtain its monopoly profits in expectation while the smaller firm will get a fraction of the large firm profits. Additionally, although both firms share the same support for their equilibrium strategies, the "small" firm assign no mass points throughout this support while the "large" firm puts a mass point in the monopoly price of size $1 - \frac{\theta_k + \theta_m}{\theta_j + \theta_m}$, where θ_j corresponds to the captive market of the largest firm.

Note that, as the shared fraction of the market tends to zero, the lower bound for the equilibrium strategies, $\frac{\theta_j r}{\theta_j + \theta_m}$ goes to r, and the equilibrium mixed strategies converges to a degenerated distribution assigning all the mass to the monopoly price, as in case *(i)*. Similarly as the overlap in costumer bases goes to one, i.e. full overlap, all the mass in the equilibrium mixed strategies collapse to $p = 0$, which is the Bertrand equilibrium described in part *(ii)*.

[4]If in addition the overlap in costumer bases is also zero, $\theta_m = 0$, all consumers have empty consideration set making all combination of prices a trivial equilibrium.

1.4 The Advertising Equilibrium

With the second stage equilibrium profits in hand we can now proceed to analyze the advertising decision of the firms using a standard backwards induction argument. Assume, without loss of generality, $s_j \geq s_k$ and that $s_j > 0$ or $s_m > 0$ (possibly both). The small firm can, potentially, face two problems. For low levels of advertising, relative to m_j, the resulting market shares will make firm k enter the second stage of the game as the small firm and will face profits given by:

$$\pi_k^S = s_1 \frac{(\theta_k + \theta_m)\theta_j r}{(\theta_j + \theta_m)} - \phi(m_k). \tag{1.8}$$

For the rest of the chapter we refer to this case as the *small firm problem.*

For relatively high levels of advertising, firm k will become the large firm in the second stage of the game with profits

$$\pi_k^L = s\theta_k r - \phi(m_k). \tag{1.9}$$

We will call this the *large firm problem.* It is easy to see that both profit functions are strictly concave. For the profit function under the small firm problem:

$$\frac{\partial \pi_k^S}{\partial m_k} = \frac{r\left(m_j\left(1-s_0\right)+s_j\right)\left(1-2(s_0-s_j)-2m_k\left(1-s_0+s_j\right)\right)}{m_j+(1-m_j)(s_j+s_m)} - \phi'(m_k) \tag{1.10}$$

$$\frac{\partial^2 \pi_k^S}{\partial m_k^2} = -\frac{2r\left(1-s_k-s_m\right)\left(s_j+m_j\left(1-s_0\right)\right)}{s_j+s_m+m_j\left(1-s_j-s_m\right)} - \phi''(m_k) < 0, \tag{1.11}$$

and for the profit function under the large firm problem:

$$\frac{\partial \pi_k^L}{\partial m_k} = (1-s_0)(1-m_j)r - \phi'(m_k) \tag{1.12}$$

$$\frac{\partial^2 \pi_k^L}{\partial m_k^2} = -\phi''(m_k) < 0. \tag{1.13}$$

Equations (1.10) and (1.12) provide the first order conditions when firm k faces the small and the large firm problems, respectively. However we should be careful about the existence of corner solutions for each of the choices. Define $\hat{m}_k^{(m_j)}$ as the required level of advertising for the originally small firm to match the size of firm j's captive market by

$$\hat{m}_k^{(m_j)} = \frac{m_j\left(1-(s_j+s_m)\right)+(s_j-s_k)}{1-(s_k+s_m)}. \tag{1.14}$$

Note that $\hat{m}_k^{(m_j)}$ is uniquely defined by $\pi_k^S(m_j, \hat{m}_k^{(m_j)}) = \pi_k^L(m_j, \hat{m}_k^{(m_j)})$.[5]

Therefore, when firm k faces the small firm problem, its strategy space is constrained to $m_k \in [0, \hat{m}_k^{(m_j)}]$, and $m_k \in [\hat{m}_k^{(m_j)}, 1]$ when making a decision as the large firm. The solution to the small firm problem has a corner solution at zero when:

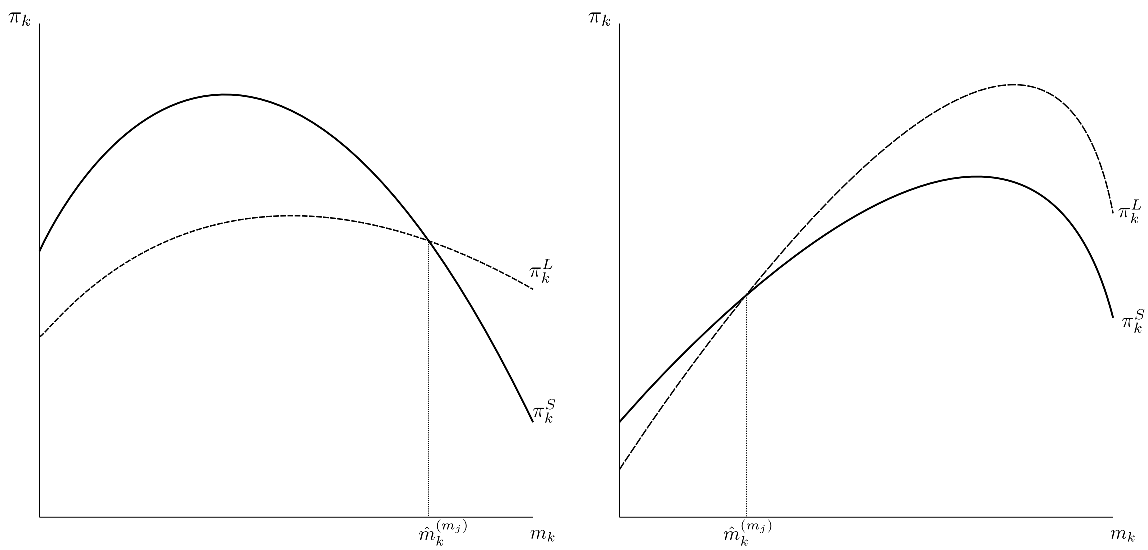
$$\phi'(0) > \frac{r\left(m_j\left(s_0-1\right)-s_j\right)\left(2(s_0-s_j)-1\right)}{m_j+(1-m_j)(s_0-s_k)}, \tag{1.15}$$

which is simply $\partial_{m_k}\pi_k^S|_{m_k=0} > 0$. The solution to the large firm problem has a corner solution at one when:

$$\phi'(1) < (1-s_0)(1-m_j)r, \tag{1.16}$$

or $\partial_{m_k}\pi_k^L|_{m_k=1} > 0$. Constructing the remaining two corner solutions at $\hat{m}_k^{(m_j)}$ can be easily done by substituting equation (1.14) into equation (1.10) and (1.12). The relationship between these two corner solutions is very useful to provide further insight on the sorts of equilibria that can arise in this model and it is summarized in the following Lemma.

[5]Equations (1.8) and (1.9) can only be equal if $\theta_j = \theta_k$, which is how $\hat{m}_k^{(m_j)}$ is constructed or if $\theta_m = 0$. However, given our advertising technology $\theta_m = 0$ if and only if $s_m = 0$ and $m_j = m_k = 0$ and it cannot be the case that the small firm matches the large firm's captive market with $m_k = 0$.



(a) $\partial_{m_k}\pi_k^L|_{m_k=\hat{m}_k^{(m_j)}} \leq 0$ (b) $\partial_{m_k}\pi_k^S|_{m_k=\hat{m}_k^{(m_j)}} \geq 0$

Figure 1.1: *Profit functions of the small firm with $s_m > 0$.*

Lemma 1.2. *The following two conditions are always satisfied form $s_m > 0$:*

(i) $\partial_{m_k}\pi_k^L|_{m_k=\hat{m}_k^{(m_j)}} \leq 0 \Rightarrow \partial_{m_k}\pi_k^S|_{m_k=\hat{m}_k^{(m_j)}} < 0$

(ii) $\partial_{m_k}\pi_k^S|_{m_k=\hat{m}_k^{(m_j)}} \geq 0 \Rightarrow \partial_{m_k}\pi_k^L|_{m_k=\hat{m}_k^{(m_j)}} > 0.$

The previous lemma essentially states that: *(i)* if the large firm maximization problem has a corner solution at $m_k = \hat{m}_k^{(m_j)}$ then the optimal decision in the small firm problem is always less than $\hat{m}_k^{(m_j)}$ and in fact this is the optimal solution across any advertising level, see Figure 1.1(a), and *(ii)* if the small firm maximization problem has a corner solution at $m_k = \hat{m}_k^{(m_j)}$ then the optimal decision for the large firm across any advertising level is always greater than $\hat{m}_k^{(m_j)}$, see Figure 1.1(b). Our first advertising equilibrium result can be written as a corollary of Lemma 1.2.

Corollary 1.3. *If there is any overlap in the initial costumer bases then there is no equilibrium in advertising where both firms end with captive markets of the same size.*

Proof. An equilibrium where both firms end up with captive markets of the same size requires $m_k = \hat{m}_k^{(m_j)}$ to be a best response for some m_j, which in turn requires that $\partial_{m_k}\pi_k^L \leq 0$ and $\partial_{m_k}\pi_k^S \geq 0$ contradicting Lemma 1.2. ■

Requiring $s_m > 0$ rules out one particular scenario where, for $\hat{m}_k^{(m_j)} = 0$, there exist some values of s_j and s_k where the second condition in Lemma 1.2 can hold with equality. This scenario will show up latter when we propose an equilibrium in advertising where both firms end with the same market size.

1.4.1 Existence of Equilibria

Now we present a list of lemmas that will be useful prove existence of equilibria of many sorts. First we formalize the sufficient conditions for which the small firm find a strictly positive level of advertising to be best response to m_j. For this purpose it is enough to focus our attention in the problem firm k faces as the small firm since a positive level advertising in this case implies that $m_k > 0$ is also a best response for the complete profit function, i.e. including both the small and large firm problem.[6]

[6]Lemma 1.2 rules out cases where a strictly positive m_k in the small firm problem will make firm k's captive market larger than firm j's, but the large firm problem implies $m_k = 0$ since this situation arises only when $\partial_{m_k}\pi_k^L \leq 0$ and $\partial_{m_k}\pi_k^S > 0$ for $\hat{m}_k^{(m_j)} = 0$.

Lemma 1.4 ($m_k^* > 0$)**.**

(i) The smaller firm, denoted by k, will find a strictly positive level of advertising optimal for $m_j = 0$ if

$$s_k + s_m < \frac{1}{2}\left(1 - \frac{\phi'(0)}{r}\left[\frac{s_m + s_j}{s_j}\right]\right).$$

(ii) Additionally if

$$s_k + s_m \leq \frac{s_m}{s_j + s_m},$$

a strict positive level of advertising is optimal for all $m_j \geq 0$.

Proof. Equations (1.10) and (1.15) imply that $m_k^* > 0$ if

$$\frac{r\left(m_j\left(\bar{s}_0 - 1\right) - s_j\right)\left(2(\bar{s}_0 - s_j) - 1\right)}{m_j + (1 - m_j)(\bar{s}_0 - s_k)} > \phi'(0). \tag{1.17}$$

Some simple algebra can show that condition *(i)* is a sufficient and necessary condition for equation (1.17) for $m_j = 0$. Then we verify that *(ii)* ensures that the left hand side of of this equation is increasing in m_j. Therefore the two conditions combined are sufficient for the condition to hold for all $m_j \geq 0$ since the left hand side is constant. We want to show that the derivative of the right hand side of the previous condition is always positive, i.e.

$$\frac{r\left(2(s_k + s_m) - 1\right)\left(s_m\left(s_k + s_m - 1\right) + s_j\left(s_k + s_m\right)\right)}{\left(m_j + (1 - m_j)(s_j + s_m)\right)^2} > 0.$$

Condition *(i)* implies that $(2(s_k + s_m) - 1) < 0$, so it only remains to show that

$$\left(s_m\left(s_k + s_m - 1\right) + s_j\left(s_k + s_m\right)\right) \leq 0,$$

or

$$\begin{aligned} s_j\left(s_k+s_m\right) &\leq s_m\left(1-s_k s_m\right) \\ s_j s_k+s_j s_m &\leq s_m(1-s_0)+s_j s_m \\ s_m+s_k &\leq \frac{s_m}{s_m+s_j}. \end{aligned}$$

which is our condition *(ii)*. ■

We can extract some useful interpretations from the latter result. For example, as the marginal cost at zero increases, condition *(i)* becomes harder to satisfy and the set of initial conditions leading to a positive advertisement of the small firm shrinks. Notice that an additional consumer in the overlapped market will have a greater effect on firm j's advertising decision than an additional consumer in its captive market since an increase in m_j will have both a positive effect on the left hand side and a negative effect of the right hand side of condition *(i)*. Additionally, as the marginal cost at zero decreases, the particular distribution of consumers aware of firm j becomes less important, and only the total fraction is relevant.

Lemma 1.5 (Weak $m_k^* > 0$)**.** *The smaller firm, denoted by k, will find a strict positive level of advertising optimal for $m_j^* = 0$ if $s_m + s_k < s_j - \frac{\phi'(0)}{r}$.*

Proof. We can rewrite equation (1.17) in Lemma 1.4 evaluated at $m_j = 0$ as

$$\begin{aligned} 1-\frac{\phi'(0)}{r} &> 1+\frac{s_j\left(2(s_0-s_j)-1\right)}{(s_0-s_k)} \\ &= s_0+\frac{s_m(1-s_0)+s_j(s_0-2s_j)}{s_j+s_m} \end{aligned}$$

The last term on the previous equation is less than $s_0 + (1 - s_0) + (s_0 - 2s_j) = 1 + (s_0 - 2s_j)$, and equation (1.17) will hold if

$$1 - \frac{\phi'(0)}{r} > 1 + (s_0 - 2s_j),$$

or equivalently if $s_j - \frac{\phi'(0)}{r} > s_m + s_k$. ■

Another useful result of Lemmas 1.4 and 1.5 is summarized in the following Corollary.

Corollary 1.6. *If $s_m + s_k \geq \frac{1}{2}$, the small firm, k, will always find $m_k^* = 0$ to be an optimal solution for the small firm problem for all $m_j \geq 0$.*

Proof. The result follows immediately from equation (1.17) and the fact that $s_0 - s_j = s_m + s_k$. ■

The condition obtained in Lemma 1.5, although weaker than condition *(i)* in Lemma 1.4, is more intuitive. In particular, we can see that the small firm will find a positive level of advertising optimal if the total fraction of consumer aware of the small firm – including the captive and the overlapped markets – is less than the captive share of the large firm, as the marginal cost at zero shrinks. Cor 1.6, on the other hand, establishes that when the total fraction of consumers with firm j in their consideration set is greater than half, not to advertise is a best response when the large firm is not advertising either.

The previous Lemmas establish conditions for which the small firm will find a positive level of advertising profitable. Now we proceed to establish the necessary condition for which firm k will remain smaller than firm j after advertising.

Lemma 1.7 ($m_k^* < \hat{m}_k$)**.** *The smaller firm, k, will find optimal to remain small, i.e $m_k^* < \hat{m}_k$ for all values of m_j, if*

$$s_0 \geq 1 - \frac{\phi'(\hat{m}_k^{(0)})}{r}$$

Proof. The first condition makes equation (1.12) evaluated at $\hat{m}_k^{(0)}$ negative. From Lemma 1.2 this implies that $\partial_{m_k}\pi^S|_{m_k=\hat{m}_k^{(0)})} < 0$ and firm $k's$ best response to $m_j = 0$ must be less than $\hat{m}_k^{(0)}$. Since $\hat{m}_k$ is increasing in m_j and $\phi(m)$ is strictly convex the condition is also satisfied for all $m_j > 0$. ■

1.4.2 Types of Equilibria

In this subsection we provide sufficient conditions for some interesting equilibria. First, there exists an equilibrium where no firm advertises, we refer to it as equilibrium type 1. This equilibrium emerges when: (1) the initial fraction of consumers aware of at least one of the goods in the market, s_0, is large enough, and (2) when the fraction of consumers aware of the smaller firm, $s_k + s_m$, is large enough to make advertising unprofitable for both firms. Notice that the large firm's revenue depends only on the size of its captive market (as shown in Proposition 1.1) which becomes harder to increase as the mass of uninformed consumers, $(1 - s_0)$ decreases. Similarly, firm k's revenue is determined by $(\theta_k+\theta_m)/(\theta_j+\theta_m)$ times firm j's revenue. As $s_k + s_m$ increases, combined with the fact that s_0 is relatively large, increases the cost of adding new consumers to firm k's potential market. The particular thresholds for s_0 and $s_k + s_m$ are provided in the following Proposition.

Proposition 1.8 (Equilibrium type 1)**.** *There exist an advertising equilibrium where* *no advertisement takes place from any of the firms, i.e.* $m_j^* = 0$ *and* $m_k^* = 0$ *if the following initial conditions are satisfied:*

(i) $s_0 \geq (1 - \frac{\phi'(0)}{r})$, *and*

(ii) $s_k + s_m \geq s_j$

Proof. Rewriting equation (1.12) for the large firm j for $m_k = 0$ give us the best response function of the large firm:

$$(1 - s_0)r - \phi'(m_j) = 0 \tag{1.18}$$

Which has a solution equal zero if and only if condition *(i)* is satisfied. Equation (1.10) evaluated at $m_j = 0$ states the necessary condition for $m_k^* = 0$ to be a best response when remaining small to $m_j = 0$; explicitly,

$$\phi'(0) \geq -\frac{rs_j(2(s_0 - s_j) - 1}{s_0 - s_k}.$$

Rearranging terms in the previous expression we obtain:

$$1 - \frac{\phi'(0)}{r} \leq s_0 + \frac{s_m(2(s_k + s_m) + s_j(s_k + s_m - s_j)}{s_0 - s_k}.$$

Clearly if $s_k + s_m - s_j > 0$ the previous expression is satisfied assuming condition *(i)*. We now need to show that firm k's profit at $m_k = 0$ is greater than the profit associated with any $m_k \geq \hat{m}_k(0)$. First, we show that $\pi_k^L(\hat{m}_k(0), 0) < \pi_k^S(0, 0)$, or

$$\begin{aligned} \left(\frac{1 - s_0 + s_k}{1 - s_0 + s_j}\right) s_j r - \phi(\hat{m}_k(0)) &< \left(\frac{1 - s_j}{1 - s_k}\right) s_j r \\ \left(\frac{s_m(s_k - s_j)}{(1 - s_k)(1 - s_k - s_m)}\right) s_j r &< \phi(\hat{m}_k(0)). \end{aligned}$$

By construction $s_k - s_j \leq 0$ so the previous inequality is always satisfied.

Finally, $\delta_{m_k}\pi_k^L(\hat{m}_k(0), 0) \leq 0$ if and only if

$$s_0 \geq 1 - \frac{\phi'(\hat{m}_k(0))}{r},$$

which is satisfied by the convexity of $\phi(m)$ and condition *(i)*. ■

Two cases that satisfy the conditions required for an equilibrium type 1 are of particular interest. First, as the initial information structure converges to a evenly divided market among the two firms with no overlap in the costumer bases, $s_j = s_k$ and $s_m = 0$, a type 1 equilibrium converges to one in which each firm serves its half of the market extracting monopoly profits, provided the size of the captive markets is sufficiently large. Additionally, this is the only case where the perfect equilibrium gives an outcome with both captive markets of the same size and strictly positive. Any other initial information structure with positive overlap will never generate a segmented market as described above from Corollary 1.3. Given the random advertising technology, an initial information structure that allows advertising from any of the firms will generate some overlap in the costumer bases and any advertising strategy that implies captive markets of the same size will contradict Lemma 1.2.

The second case is concerned with the initial information structure as the overlap in costumer bases approaches $1 - \phi'(0)$ and the size of the captive markets approaches zero. In this case a type 1 equilibrium converges to a Bertrand equilibrium with zero profits. Similarly as above, this is the only case where a the perfect equilibrium outcome of the game produces a Bertrand equilibrium since a positive level of advertising from any of the firms will create a disparity in the captive market sizes allowing the large firm to extract some profits above marginal cost. These two cases are formalized in the following Corollaries of Proposition 1.8.

Corollary 1.9. *If* $s_m = 0$*,* $s_j = s_k$*, and* $2s_j \geq 1 - \phi'(0)/r$*, there exist an equilibrium with* $(m_j^*, m_k^*) = (0, 0)$ *with the property that both firms end up the same size after advertising with monopoly profits.*

Proof. $(m_j^*, m_k^*) = (0, 0)$ comes immediately by verifying that the initial conditions satisfy those in Proposition 1.8. Clearly, since $s_m = 0$, $\theta_m = 0$ and monopoly profits follow from part *(ii)* of Proposition 1.1. ■

Corollary 1.10. *If* $s_m \geq 1 - \phi'(0)/r$*, and* $s_j = s_k = 0$*, there exist an equilibrium with* $(m_j^*, m_k^*) = (0, 0)$ *with the property that both firms share the totality of the market with zero profits.*

Proof. $(m_j^*, m_k^*) = (0, 0)$ comes immediately by verifying that the initial conditions satisfy those in Proposition 1.8. Clearly, since $s_j = s_k = 0$, $\theta_m = 1$ and zero profits follow from part *(i)* of Proposition 1.1. ■

Proposition 1.11 presents a second type of equilibrium where the small firm does not advertise, $m_k^* = 0$, and the large firm finds an strictly positive level of advertising optimal, $m_j^* > 0$. We refer to this equilibrium as type 2. Following a similar reasoning as above, if the size of the small firm's costumer base is high enough (half of the market in this case) any advertising will increase the degree of price competition, reducing profits overall. Additionally, if the fraction of consumers with a non empty consideration set, s_0, is too large, increasing the size of firm k's captive market is too costly to try to become the large firm since most of the advertising effort will be wasted on consumers already aware of at least one of the firms. However, firm j will still find some level of advertising optimal as long as s_0 is not too large so it can increase its captive market for it is not concerned of the cost of becoming the large firm.

Proposition 1.11 (Equilibrium type 2)**.** *There exist an advertising equilibrium where the large firm, j, sets $m_j^* > 0$ and the small firm, k, sets $m_k^* = 0$ if the following initial conditions are satisfied:*

(i) $s_k + s_m \geq 1/2$,

(ii) $(1 - \frac{\phi'(\hat{m}_k(0))}{r}) \leq s_0 < (1 - \frac{\phi'(0)}{r})$

Proof. Condition *(i)* guarantees that firm k will choose $m_k = 0$ if remaining small if preferred to become the large firm from Lemma 1.6. The first part of *(ii)* ensure that remaining small is in fact the best response from Lemma 1.7 so $m_k^* = 0$ is a best response for all possible values of m_j. Rewriting equation (1.12) for the large firm j for $m_k = 0$ give us the best response function of the large firm:

$$(1 - s_0)r - \phi'(m_j) = 0$$

With a solution greater than zero if and only if the second part of condition *(ii)* is satisfied. ■

Proposition 1.12 (Equilibrium type 3). *There exist an advertising equilibrium where the large firm, j, sets $m_j^* = 0$ and the small firm, k, sets $0 < m_k^* < \hat{m}_k^{(0)}$ if the following initial conditions are satisfied:*

(i) $s_k + s_m < s_j - \frac{\phi'(0)}{r}$, *and*

(ii) $s_0 \geq 1 - \frac{\phi'(0)}{r}$.

Proof. Condition *(i)* guarantees that firm k will choose $m_k > 0$ if remaining small if preferred to become the large firm from Lemma 1.5. Additionally, firm k will prefer to remain small in response to $m_j = 0$ if $s_0 \geq 1 - \phi'(\hat{m}_k^{(0)})/r$ which is implied by condition *(ii)* so $0 < m_k^* < \hat{m}_k^{(0)}$ is a best response for $m_j = 0$. Finally, rewriting equation (1.12) for j at a given m_k^* we obtain a corner solution at zero if and only if

$$(1 - s_0)(1 - m_k^*)r \leq \phi'(0)$$

The left hand side of the previous equation is decreasing in m_k^* while the right hand side is constant. Therefore if it is satisfies at $m_k^* = 0$ it is satisfied at $m_k^* > 0$ which makes condition *(ii)* a sufficient condition for $m_j^* = 0$ to be a best response to $m_k^* > 0$.

■

The previous Proposition present the type 3 equilibrium characterized by a strictly positive level of advertising from the small firm and no advertising from the large firm, while remaining relatively the same size, i.e. $m_j^* = 0$ and $0 < m_k^* < \hat{m}_k^{(0)}$. This type of equilibrium occurs when the fraction of consumers with an empty consideration set is small enough such that increasing the size of the captive markets is too costly for both firms, so the large firm will not advertise and the small firm will not find optimal to become the large firm. However, firm k's costumer base is small enough to allow for some increase in it without reducing firm k's profits due to tougher price competition.

Proposition 1.13 specifies the sufficient conditions for an equilibrium where both firms find a strictly positive level of advertising optimal while remaining relatively the same size after advertising, named type 4. The same intuition as in the equilibrium type 2 applies for the large firm; while the intuition for the small firm's behavior is similar to the type 3 equilibrium with the addition of one extra condition. In the equilibrium described in Proposition 1.12, the small firm is best responding to $m_j^* = 0$ so condition *(i)* is enough to guarantee a positive level of advertising whereas in this type of equilibrium the small firm needs to accommodate a strictly positive level of advertising form the large firm, requiring condition *(ii)* in the following proposition to ensure that its costumer base is small enough so the profits obtained from a larger market share are not dominated by the the price competition induced from a bigger overlap in the costumer bases as a result of a positive m_j^*.

Proposition 1.13 (Equilibrium type 4)**.** *There exist an advertising equilibrium where the large firm, j, sets $m_j^* > 0$ and the small firm, k, sets $0 < m_k^* < \hat{m}_k^{(0)}$ if the following initial conditions are satisfied:*

(i) $s_k + s_m < s_j - \frac{\phi'(0)}{r}$;

(ii) $s_k + s_m \leq \frac{s_m}{s_m + s_j}$; *and*

(iii) $(1 - \frac{\phi'(\hat{m}_k(0))}{r}) \leq s_0 < (1 - \frac{\phi'(0)}{r})$.

Proof. Conditions *(i)* and *(ii)* guarantee that firm k will choose $m_k^* > 0$ for all m_j when forced to remain small from Lemmas 1.4 and 1.5. As in the proof of Proposition 1.11, the first part of condition *(iii)* makes remaining small form firm k optimal while the second part ensures a positive level of advertising for firm j. ■

The remaining sub-game perfect equilibria require a switch in firm sizes, that is $m_k^* > \hat{m}_k^{(m_j^*)}$. For this equilibria to exist it is necessary for the advertising efforts of the small firm to find enough consumers with and empty consideration set so becoming the large firm is not prohibitively costly. This necessary condition is formalized in the following Proposition.

Proposition 1.14. *There exist an equilibrium in advertising $m_k^* > \hat{m}_k^{(m_j^*)}$ for some m_j^* only if:*

$$s_0 < 1 - \frac{\phi'(\hat{m}_k^{(0)})}{r}$$

Proof. For $(m_j^*, m_k^* > \hat{m}_k^{(m_j^*)})$ to be an equilibrium it must be the case that the small firm finds optimal to become the large firm at some m_j^*, that is

$$(1 - s_0)(1 - m_j^*)r > \phi'(\hat{m}_k^{(m_j^*)}).$$

The left hand side of the previous equation is decreasing in m_j^* while the large hand side is increasing. Therefore, if the equation is going to hold for some m_j^*, then it must be true for $m_j^* = 0$, giving a necessary condition for the existence of such equilibrium.

■

Providing a sufficient condition for the existence of an equilibrium where the relative sizes of the firms change after advertising can be quite complicated without specifying a functional form for the cost function. Clearly when $s_0 = 0$ any asymmetric equilibrium would yield the result.[7] We can, however, specify a sufficient condition for an equilibrium where the originally small firm becomes the large firm after advertising while the initially large firm does not advertise for the case where only one of the firms is endowed with a captive market and there is no initial overlap in the firm's markets. These conditions are particularly relevant since it make the best response function under the small firm problem independent of the other firm's advertising strategy.[8]

Proposition 1.15. *There exists an advertising equilibrium where the small firm, k, sets $m_k^* > \hat{m}_k^{(m_j^*)}$, and the large firm sets $m_j^* = 0$ if $s_m = s_k = 0$, and*

(i) $1 - 2s_j \geq \frac{\phi'(\hat{m}_k^{(0)})}{r}$, *and*

(ii) $(1 - s_j)(1 - 2s_j) \leq \frac{\phi'(0)}{r}$

The first condition in Proposition 1.15 plays a dual role in allowing the small firm to become large after advertising. On one hand, a small s_j reduces the level of advertising required for the small firm to become large, namely $\hat{m}_j^{(0)}$. On the other hand, when $s_m = s_k = 0$, the fraction of consumers aware of at least one good is given by $s_j = s_0$ and so an initially low s_j makes any given level of advertising more

[7] This is in fact the case analyzed by Eaton et al. (2010).

[8] This is property is crucial to identify the overall best response function in Eaton et al. (2010) and Ireland (1993).

effective by reducing the waste on messages received by consumers with a non empty consideration set. At the same time s_j should be sufficiently large so the initially large firm has no incentive to advertise as a response to the initially small firm becoming large, which is stated as condition *(ii)*. A formal proof is presented in Appendix A.

1.5 The Effect on Prices

In this section we analyze the effect that some of the equilibrium types described have on the pricing strategies of the firms. We first calculate the sub-game perfect equilibria for $s_0 = 0$, as a base case for comparison. Throughout the remainder of the chapter we normalize the prices by the reservation price $r = 1$ and parameterize the advertising cost function using

$$\phi(m) = -\gamma \ln(1 - m)$$

with $0 < \gamma < 1$. We chose this particular cost function to obtain the same results as in Eaton et al. (2010).

Proposition 1.16. *If $s_0 = 0$ and the cost function has the form $\phi(m) = -\gamma \ln(1-m)$, then there is an advertising equilibrium in pure strategies where one of the firms, labeled j without loss of generality, advertises twice as much than firm k. In particular*

$$m_k^* = \Delta_1, \text{ and } m_j^* = 2\Delta_1$$

with $\Delta_1 = \frac{3}{4} - \frac{\sqrt{1+8\gamma}}{4}$. Additionally, the proportion of the consumers that buy one of the goods after advertising is $s_1 = (1 - \gamma)$.

A brief proof is included in Appendix A only for the purpose of self-containment as this is the case studied by Eaton, McDonald and Meriluoto. We refer the interested reader to Eaton et al. (2010) for a detailed derivation of the result.

Combining this results with the ones obtained in Proposition 1.1 we can derive the distribution of prices observed in equilibrium. Recall, from equations (1.6) and (1.7), that the equilibrium pricing strategies, G_1 and G_2, have a common support from $[\underline{p}_j, r]$, and firm j's equilibrium distributions has a mass point at r of size $1 - \frac{\theta_k+\theta_m}{\theta_j+\theta_m}$. Plugging the advertising equilibrium strategies we can see that the support of the normalized sub-game perfect equilibrium is

$$\mathrm{Supp}(G_j) = [(1-\Delta_1), 1],$$

and the mass point on firm j's distribution is $\frac{1}{2}$ (see Figure 1.2).

We can extract form these equilibrium distributions some properties useful for comparing this type of equilibrium with other types that arise from different initial conditions. As γ goes to zero the minimum admissible price in the equilibrium distributions goes to $\frac{1}{2}$, giving us the lower bound for the range of observed prices. More importantly, the expected transaction price, p_T, is given by:[9]

$$\mathbb{E}(p_T) = \frac{3(1-\Delta_1)}{3-2\Delta_1}. \tag{1.19}$$

Now let us analyze the equilibrium price distributions that can arise from initial conditions consistent with a type 1 equilibrium, i.e. $(m_j^* = 0, m_k^* = 0)$. Since

[9] $q = \min\{p_j, p_k\}$ has a CDF given by $F(q) = 1-(1-G_j(q))(1-G_k(q))$, and $\mathbb{E}(p_T) = \theta_j\mathbb{E}(p_j) + \theta_k\mathbb{E}(p_k) + \theta_m\mathbb{E}(q)$

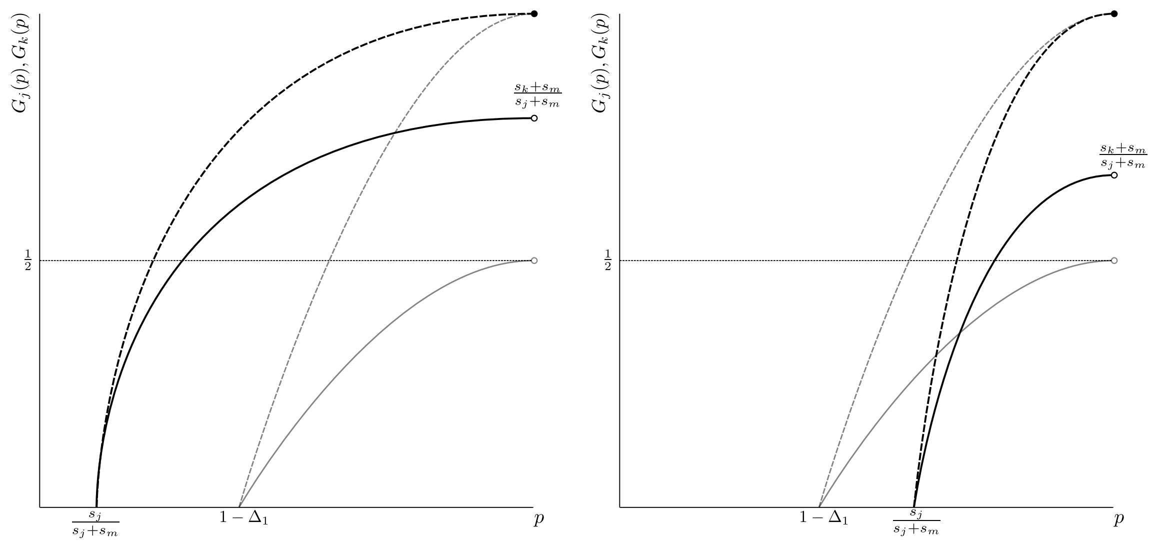
no advertising takes place in equilibrium, it is clear that $\theta_m = s_m/s_0$, $\theta_j = s_j/s_0$, and $\theta_k = s_k/s_0$. The support of the pricing strategies is:

$$\mathrm{Supp}(G_j) = \left[\frac{s_j}{s_j + s_m}, 1\right],$$

with the large firm placing a mass point in its price distribution at one of size $\frac{s_j - s_k}{s_j + s_m}$. The expected transaction price for an equilibrium type 1 is given by:

$$\mathbb{E}(p_T) = \frac{s_j\left(s_j + s_k + 2s_m\right)}{s_0\left(s_j + s_m\right)}. \tag{1.20}$$

As the size of the large firm's captive market decreases, the support of the pricing distributions allows for lower prices. Additionally, if s_j is low enough such that the lower bound of the equilibrium support is smaller under the type one equilibrium than under the base case scenario, the expected prices charged by both firms are lower than the expected prices charged under the base case scenario, resulting in a smaller expected minimum price as well as a smaller expected transaction price. This result comes not only from the fact that the distributions include lower prices, but also because under the common support both firms assign lower probability on higher prices, including the mass point at one from the large firm. On the other hand, if s_j is sufficiently high so the support of the equilibrium strategies under a type 1 equilibrium is smaller than the base case scenario the effect on the expected prices is not that clear. Although the smaller firm will unambiguously charge higher prices (in the stochastic sense) the former is not necessarily true for the large firm as the mass point at the reservation price is still smaller than the one placed under the base case scenario. This result is illustrated in Figure 1.2 and formalized in the following Proposition.



(a) $\frac{s_j}{s_j+s_m} < 1-\Delta_1$ (b) $\frac{s_j}{s_j+s_m} > 1-\Delta_1$

Figure 1.2: *Equilibrium price distributions for firm k (dashed) and firm j (solid).*

Proposition 1.17. *Let G_k^0 and G_j^0 be the equilibrium pricing strategies for the small and large firm for $s_0 = 0$, respectively, and G_k^1 and G_j^1 be the equilibrium pricing strategies for the small and large firm for s_j, s_k and s_m satisfying the conditions in Proposition 1.8. If $\frac{s_j}{s_j+s_m} \leq (1-\Delta_1)$ then $G_k^0 \succeq_{FOSD} G_k^1$ and $G_j^0 \succeq_{FOSD} G_j^1$. If $\frac{s_j}{s_j+s_m} > (1-\Delta_1)$ then $G_k^1 \succeq_{FOSD} G_k^0$.*

Proof. Let $\frac{s_j}{s_j+s_m} \leq (1-\Delta_1)$. For $p \in [(1-\Delta), 1]$, $G_k^0(p) = \frac{1}{\Delta_1}(1 - \frac{1-\Delta_1}{p})$ is increasing in Δ_1 so

$$\begin{aligned} G_k^0(p) &\leq \frac{s_j + s_m}{s_m}\left(1 - \frac{1 - \frac{s_m}{s_j+s_m}}{p}\right) \\ &= \frac{s_j + s_m}{s_m} - \frac{s_j}{s_m p} \\ &= 1 - \frac{(1-p)s_j}{p s_m} \\ &= G_k^1(p) \end{aligned}$$

The last step can be immediately seen by substituting $\theta_m = s_m$, $\theta_j = s_j$ and $\theta_k = s_k$ in $G_k(p)$ as defined in equation (1.7). This same argument works to show that if $\frac{s_j}{s_j+s_m} \geq (1-\Delta_1)$ then $G_k^1(p) \leq G_k^0(p)$. G_j^0 is also increasing in Δ_1 so

$$\begin{aligned} G_j^0(p) &< 1 - \frac{1}{2\frac{s_m}{s_j+s_m}}\left(\frac{1 - \frac{s_m}{s_j+s_m}}{p} - \left(1 - 2\left(\frac{s_m}{s_j+s_m}\right)\right)\right) \\ &= \frac{p s_m - (1-p)s_j}{2 p s_m} \end{aligned}$$

By plugging $\theta_j = s_j$ and $\theta_k = s_k$ in equation (1.6) to obtain G_j^1, we now need to show that

$$\begin{aligned} \frac{p s_m - (1-p)s_j}{2 p s_m} &\leq \frac{(s_k + s_m)(p s_m - (1-p)s_j)}{p s_m (s_j + s_m)} \\ \frac{1}{2} &\leq \frac{s_k + s_m}{s_j + s_m} \\ s_j &\leq 2 s_k + s_m \end{aligned}$$

Which is always satisfied since s_j, s_k and s_m satisfy the conditions in Proposition 1.8.

■

Table 1.1 presents some numerical values for the expected transaction price for different levels of γ and s_k. In order to maintain our results comparable to the

γ	$s_0 = 0$	$s_k = 0$	$s_k = 0.1$	$s_k = 0.2$	$s_k = 0.3$	$s_k = 0.4$	$s_k = 0.499$
0.001	0.75	0.75	0.78	0.81	0.86	0.92	0.99
0.100	0.81	0.75	0.78	0.82	0.87	0.95	
0.200	0.85	0.75	0.79	0.83	0.90	1.00	
0.300	0.88	0.75	0.79	0.85	0.94		
0.400	0.91	0.75	0.80	0.88	1.00		
0.500	0.93	0.75	0.81	0.92			
0.600	0.95	0.75	0.83	1.00			
0.700	0.96	0.75	0.88				
0.800	0.98	0.75	1.00				
0.900	0.99	0.75					
0.999	1.00	0.75					

Table 1.1: *Expected Transaction Price for Equilibrium Type 1*

base case we assigned the smallest s_m and s_j that satisfy the conditions necessary for the equilibrium type 1 to arise. In other words, for a given pair of γ and s_k the initial conditions are chosen such that $s_k + s_m = s_j$ and $s_0 = 1 - \gamma$ so the effective market sizes are the same in the base case scenario and the equilibrium type 1. Additionally, by assigning overlap in costumer bases to its smallest possible value, given γ, we obtain the upper bound for the expected transaction price since it is at this point when price competition is less aggressive. As we can see, the resulting upper bound on the expected transaction price is increasing in s_k since the smaller firm can exercise its monopoly power over a bigger captive market. Additionally, the expected transaction price is increasing in γ. As the advertising cost approaches zero, the overlap in costumer bases needs to be larger to allow for an type 1 equilibrium. Additionally, as the initial captive market of the small firm approaches $\frac{1}{2}$ the overlap in costumer bases is minimized and prices converges to the monopoly prices as shown

γ	$s_0 = 0$	$s_k = 0$	$s_k = 0.1$	$s_k = 0.2$	$s_k = 0.3$	$s_k = 0.4$	$s_k = 0.499$
0.001	0.75	0.75	0.75	0.77	0.80	0.87	0.98
0.100	0.81	0.69	0.71	0.74	0.79	0.89	
0.200	0.85	0.61	0.63	0.67	0.75	0.93	
0.300	0.88	0.49	0.52	0.57	0.68		
0.400	0.91	0.31	0.33	0.39	0.53		
0.500	0.93	0.00					
0.600	0.95						
0.700	0.96						
0.800	0.98						
0.900	0.99						
0.999	1.00						

Table 1.2: *Expected Transaction Price for Equilibrium Type 2*

in Corollary 1.9. Note than, when compare with the case for $s_0 = 0$ the effect of the initial conditions on the expected transaction price is ambiguous under equilibrium type 1. The upper bound on the expected transaction price is smaller than the price observed in the base case for low values of s_k, and larger for higher values of s_k.

Table 1.2 is constructed in a similar way for a type 2 equilibrium, that is s_m and s_j are assigned so $s_k + s_m = 0.5$ and $s_0 = 1 - \phi'(\hat{m}_k^{(0)})$, but in this case these conditions correspond to the lower bound of the expected transaction price. As in the type 1 equilibrium the expected transaction price is increasing in s_k. However $\mathbb{E}(p_T)$ is decreasing in γ. As the cost of advertising increases the required size of the large firm's captive market for an equilibrium type one decreases and the price competition is more aggressive. In particular, when $s_j = 0$ the limiting case converges to the Bertrand equilibrium.[10] As in the case of a type 1 equilibrium the lower bound

[10]Technically, this is not a type 2 equilibrium since at this point $\phi'(\hat{m}_k^{(0)}) = \phi'(0)$ which makes

γ	$s_0 = 0$	$s_k = 0$	$s_k = 0.1$	$s_k = 0.2$	$s_k = 0.3$	$s_k = 0.4$	$s_k = 0.499$
0.001	0.75	0.75	0.78	0.81	0.86	0.92	0.99
0.100	0.81	0.80	0.82	0.85	0.90	0.96	
0.200	0.85	0.84	0.86	0.89	0.94		
0.300	0.88	0.87	0.89	0.94			
0.400	0.91	0.91	0.93	0.99			
0.500	0.93	0.94	0.98				
0.600	0.95	0.98					
0.700	0.96						
0.800	0.98						
0.900	0.99						
0.999	1.00						

Table 1.3: *Expected Transaction Price for Equilibrium Type 3*

for the expected transaction price is smaller than the case when $s_0 = 0$ for low values of s_k, and greater for higher values. The latter is the result of firms engaging in unavoidable price competition since they cannot reduce the overlap in costumer bases, while, when starting with $s_0 = 0$ they can choose an optimal level of overlap.

Finally, Table 1.3 present the expected transaction prices for a type 3 equilibrium with s_m and s_k such that $s_k + s_m < s_j - \gamma$ and $s_0 = (1 - \gamma)$.[11] The values reflect the upper bound for the expected transaction price with similar results as in Table 1.1 with the transaction price being increasing in s_k and in γ. Similarly, when compared to the base case scenario, the effect of the initial conditions are ambiguous yielding expected transaction prices smaller than the $s_0 = 0$ case for small values of s_k and higher transaction prices for high values of s_k.

condition *(ii)* in Proposition 1.11 impossible to hold. However this is the limiting case when we transition from a type 2 to a type 1 equilibrium of the form described in Corollary 1.10.

[11]The first condition is actually implemented as $s_k + s_m = s_j - \gamma - \varepsilon$ with $\varepsilon = 0.01$.

1.6 Conclusion

We have extended Ireland (1993) and Eaton et al. (2010) models of existence advertising by allowing arbitrary initial conditions on the market's information structure. We find that the effect of prior information held by consumers can drastically change the advertising equilibrium predictions. In particular we provided sufficient conditions for four equilibria in the advertising stage of the game. In addition to the two asymmetric pure strategy equilibria previously found we show that: (1) an equilibrium where none of the firms advertise is possible when the fraction of consumers with non empty consideration sets and the smaller firm's costumer base are large enough. Additionally, the monopoly pricing and Bertrand equilibria are limiting cases of this type of equilibrium as the overlap in costumer bases changes. (2) There exist two equilibria where only one of the firms advertises without changing the relative size of the firms, *i.e.* the originally large firm remains large after the advertising decisions are made. (3) An equilibrium where both firms advertises while remaining relatively the same with respect to the initial conditions can also arise. Additionally, we provided a necessary condition for the existence of an equilibrium where firms change their relative size, and a sufficient condition for this kind of equilibrium where only one of the firms start with a captive market and there is no initial overlap in costumer bases.

Last, but not least, we have analyzed the effect that three of these new advertising equilibrium strategies have on prices. We show, in general, that the effect on the expected transaction price is qualitatively significant, although ambiguous when compared to the case of a newly formed market where consumers have no information about any of the firms nor prices. We can establish, however, that the expected transaction price is increasing in the size of the smaller firm's captive market since higher captive markets reduces the initial overlap in costumer bases and softens price competition.

Chapter 2

Consumption of Durable Goods under Ambiguity

2.1 Introduction

We approach the problem of consumption of a durable good with the option to resell it at any moment as an optimal stopping problem where the agent chooses the time of purchase and resale of the good in order to maximize her expected present value at time zero. The idea of modeling economic problems as timing problems is not new. In fact, optimal timing has been used extensively to analyze a wide variety of economically relevant questions such as the firm's entry and exit decisions, and policy implementation of a planner. This stopping time approach, or Real options as introduced by McDonald and Siegel (1986) and Dixit and Pindyck (1994) since it relies on the methods developed to price financial options, can be applied to a broad class of problems sharing three main characteristics: some degree of irreversibility of the decision, ongoing uncertainty, and freedom over the timing of the decision. These characteristics are present in our problem of consumption of durable goods, allowing us to exploit the vast literature on optimal stopping times.

The solution to this type of models takes the form of a reservation value for the state variable, called the exercise threshold, at which the decision maker is indifferent between the termination payoff – received at the moment the option is exercised –

and the continuation payoff – the value associated with the possibility of exercising the option in the future. In our case, the solution is characterized by two exercise thresholds: the *buying threshold*, *i.e.* the exercise threshold for an agent considering to buy the durable good, and the *resale threshold*, *i.e.* the exercise threshold for an agent considering to sell the good, where the state variable is a transformation of the market price. The optimal time to buy (resell) the durable good is the first time the spot price crosses the buyer's (seller's) reservation price.

Furthermore, we develop the model in an environment where the decision maker does not have enough information to assess the probability distribution of future prices precisely. Thus, the agent is subjected to *ambiguity*. The lack of information can be the result of relatively few data available in a new market, unobservable characteristics of the good, etc. The existence of these type of environments was originally argued by Knight (1921), and experimentally motivated latter by Ellsberg (1961). We make a key assumption, however, regarding the structure of ambiguity. In particular, we assume that the agent is ambiguous only when making the buying decision. Once the good is purchased, ambiguity is resolved and the resale decision is made with a well known probability distribution of future prices. This is mostly a simplifying assumption that allows us to first solve the seller's problem and use this result to solve the buyer's problem at a later stage. We do not formally introduce any specific reason for which ambiguity is resolved in this seemingly odd fashion. Nevertheless, one can find economic motivations that justify this assumption. Consider, for example, the case where the price distributions depend on the quality of the good. During the buying stage of the problem quality is unobservable, thus giving room for ambiguity to arise. Once the buying decision is made, the quality of the good is known, both for the agent experiencing the good and the potential buyers in the second hand market if the number of sellers with knowledge about the good is high enough so the information about the quality spreads to the potential buyers, allowing the market to observe the particular probability distribution of prices. We will discuss the reasons for this simplifying assumption in more detail in section 2.3.

This paper is mostly concerned with the effect that ambiguity will have on the buyer's reservation price and the value of the option to purchase the durable good for a risk neutral and ambiguity averse agent. The analysis is done using a continuous-time model with infinite-horizon where prices are assumed to follow a geometric Brownian motion.

In order to model ambiguity averse preferences we use the multiple-prior expected utility representation developed by Gilboa and Schmeidler (1989). In general, under this utility representation, the agent calculates the expectation of future payments using the worst possible probability measure in a set of priors. The set of priors is constructed using the κ-ignorance specification proposed by Chen and Epstein (2002) in their analysis of the multiple-prior representation in continuous time within the context of investment and consumption in an asset pricing model. In particular, this set is built by generating equivalent probability measures perturbed by a bounded stochastic process. The resulting set satisfies the requirements of the *Gilboa-Schmeilder* representation as well as the important rectangularity condition that guarantees time consistency. Additionally, each probability measure in the set corresponds to an underlying Brownian motion with unknown drift term and common variance. For a treatment of this utility representation in discrete time see Epstein and Wang (1994) and Epstein and Schneider (2003).

Developing a theoretical framework for optimal stopping with multiple-priors has been an active research topic in recent years. Riedel (2009) and Miao and Wang (2011) are concerned with a theory of optimal stopping in discrete time while Bayraktar and Yao (2011), Trevino-Aguilar (2012), and Cheng and Riedel (2013) present alternative models in continuous time. In our work we borrow a crucial result from Cheng and Riedel that allow us to find the *worst-case scenario* for the buyer's problem in a fairly simple way.

This paper is related to the work of Miao and Wang (2011), (Nishimura and Ozaki, 2004, 2007), and Schroder (2011) who also examine the effect of ambiguity on the exercise thresholds. Nishimura and Ozaki (2007) and Schroder (2011) use the multiple-priors representation in continuous time to analyze the investment decision of a firm who is subjected to ambiguity regarding the returns on investment. In their results, ambiguity decreases the value of the option to invest and delays investment. In Schroder (2011), the firm uses both the worst and best case scenarios when calculating the optimal stopping time using a generalization of the *Gilboa-Schmeilder* representation (see Ghirardato et al. (2004)). Nishimura and Ozaki (2004) apply the Choquet expected utility model (see Schmeidler (1989)) in a discrete time job search model. They show that ambiguity expedites the exercise of the option, shortening the time the agent engages in search. Miao and Wang (2011) reconcile these seemingly contrasting results using a multiple priors model in discrete time. They analyze the firm's entry and exit decisions and show that ambiguity affects the exercise thresholds differently, depending on whether or not uncertainty is resolved after exercising the option. If uncertainty is not resolved at the moment of exercise, as in the investment problem, the optimal exercise time is delayed. Alternatively, if uncertainty is fully resolved after the option exercise the presence of ambiguity rushes the exercise of the option.

All the applications we have discussed so far focus on simple financial options, *i.e.* American *call* and *put* options, or their equivalent interpretation in the Real Options context. Recent papers, however, have turn their attention into more exotic options. Cheng and Riedel (2013) apply their model to the barrier option and American straddle.[1] They show that ambiguity makes the agent change the probability measure used to evaluate the options. For the barrier option the lowest mean return is used to evaluate the option before it knocks in and the highest mean return is used afterwards. For the American straddle the agent constantly changes the probability measure as the underlying moves in the interval between two exercise thresholds. Chudjakow and Vorbrink (2009) analyze several types of exotic options in discrete time that can be constructed by embedding simple options. As in Cheng and Riedel (2013), the worst-case measure can vary over time as the agent transitions from one simple option into the other.

There are alternative approaches to incorporate ambiguity, of course. Hansen and Sargent (2001) and Barillas et al. (2009), for example, proposed a method based

[1]In a barrier option the contract is only implemented when the underlying hits a predetermined price, otherwise it remains worthless. In the American straddle the contract is implemented over the difference between the underlying and the strike price, regardless of the sign.

on robust control theory. Instead of using a multiple-prior representation, they introduce a zero-sum game where a malevolent player chooses a bounded level of entropy that distorts the decision maker beliefs. As a result, they obtained a Bellman equation with a correction term that accounts for model misspecifications. For a discussion comparing the two approaches see Epstein and Schneider (2003) and Hansen et al. (2006). Ju and Miao (2012) and Klibanoff et al. (2009) use a recursive smooth ambiguity model where the utility function is represented by the composition of two functions, one characterizing the risk preferences and the other the ambiguity preferences.

We follow a procedure similar to Chudjakow and Vorbrink (2009) (for exotic financial options under ambiguity) and Boyarchenko and Levendorskii (2010) (for an firm's entry problem with and embedded option to exit with known drift) and decompose our buying and reselling problem into two simpler embedded options. Our embedded option, however, is more complex than the straddle and barrier options analyzed by Chudjakow and Vorbrink (2009), thus requiring the simplifying assumption discussed above.[2] Specifically, the resale decision is modeled as an option to abandon a stream of payoffs including the utility generated by the consumption of the durable good and the opportunity cost of selling it at a fraction of the spot price. First, we solve this problem for an arbitrary drift parameter, which is revealed to the agent at the moment of purchase, and obtain the value of the resale option by evaluating it at

[2]Consider, for example, the barrier option where a standard American option "kicks in" the first time the price of the underlying hits a predetermined level. In this case, the stopping time at which the option becomes valuable is not an endogenous decision of the agent as it is specified in the contract.

the worst-case scenario, *i.e.* the most unfavorable drift parameter for the seller. Then, we solve the problem for an option to acquire a one time payment equal to the value of the resale option minus the price of the good and find the worst-case scenario to evaluate the option. We find that the agent evaluates the embedded option using two different drifts, the resale value of the good and its optimal resale price is computed using the lowest possible drift in the ignorance interval generated by the κ-ignorance specification, while the optimal buyer's reservation price is computed using the highest possible drift in a similar way to Cheng and Riedel (2013) and Chudjakow and Vorbrink (2009).

Additionally, we examine the effect that an increase in ambiguity has on the buyer's reservation price. This analysis, however, is done numerically since the lack of a close form solution to the optimal buyer's reservation price prevent us to obtain analytic expressions for the derivatives. We show that the direction of the change in the buyer's reservation price depends on the particular parametrization of the model. Our result contrasts with those of Nishimura and Ozaki (2007) and Miao and Wang (2011) who find a definite increase in the optimal exercise thresholds as a response to an increase in ambiguity. The source of the discrepancy is that, as a result of the change in the drift considered to evaluate the embedded options, the positive effect of ambiguity on the buyer's reservation price found by Nishimura and Ozaki (2007) and Miao and Wang (2011) can be dominated by a negative effect product of lower expected resale values. Furthermore, we found that higher levels of perceived ambiguity decrease the value of the embedded option. This last result is consistent with what has been previously found in the literature.

The rests of the chapter is organized as follows. In the next section we formalize the stopping time problem described above and solve for the exercise thresholds and value of the options for the unique prior case. Section 2.3 discusses in detail the construction of the set of priors, elaborates on the solution concept used to obtain the buyer's reservation price and value of the option under ambiguity, and states our main result. Finally, section 2.4 concludes. All the proofs are collected in the appendix.

2.2 The basic model

In this section we analyze the buying decision of a durable good that can be resold at any point in time as an optimal stopping problem. First, we elaborate on the basic setting for the case of a unique prior and introduce some regularity conditions as well as general definitions and previous results that will be used throughout the paper. We then extend the model to allow for multiple priors in section 2.3.

Consider an infinite-horizon optimal stopping problem in continuous time where a risk neutral agent faces the option to buy a durable good which provides a discounted utility at the time of purchase denoted by U. We assume for simplicity that the good does not depreciate over time. Additionally, it is possible to resale the good at a fraction, φ, of the spot market price with $0 < \varphi \leq 1$. The agent discounts future payoff flows using the discount rate $q > 0$ and preferences are represented by a time-additive expected utility.

Let $(\Omega, P, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0})$ be a filtered probability space, and $(X_t)_{t\geq 0}$ be a Brownian motion on $\mathbb{R}$ with respect to P for its filtration $(\mathscr{F}_t)_{t\geq 0}$ – satisfying the usual properties – that solves the stochastic differential equation

$$dX_t = \mu dt + \sigma dW_t;$$

where μ is the drift parameter, σ^2 is the diffusion parameter and dW_t is the increment of a standard Wiener process.[3] We use the stochastic process X_t to characterize the uncertainty about future prices for the durable good with $p_t = e^{x_t}$. Equivalently, we could specify the stochastic process in terms of p_t. In this case p_t follows a geometric Brownian motion with drift $(\mu + \frac{1}{2}\sigma^2)$ and diffusion parameter σ^2.

The agent's problem is to determine both the timing of purchase and resale of the durable good in order to maximize her expected present value at time zero. Moreover, the expected resale value must be taken into account when considering the purchase of the good. Therefore, the agent is presented with two binary choices. The first choice is whether to *stop* and exercise the option to buy the durable good at the current price obtaining the discounted utility U and the embedded option of resale or *continue* for one more period and face the same decision in the future. The second choice corresponds to the problem once the agent has ownership over the durable good; whether to *stop* and resale the good at a fraction φ of the current market price, hence forgoing the utility provided by the durable good, or *continue* and face the same choice in the future.

We start our analysis of the solution to the problem stated above in the following way: First, assuming that the good has been purchased already, we find the

[3] The analysis can be extended to more general Levy processes with some extra assumptions (see Boyarchenko and Levendorskii (2007)).

optimal resale threshold x^*, *i.e.* the value of x at which the agent is indifferent between selling the good and continue to hold it, which allow us to calculate the value of the resale option. Then, we proceed to calculate the optimal buyer's threshold, *i.e.* the value of x at which the agent is indifferent between continue waiting and purchasing the durable good with the embedded resale option, and the value of the embedded option.

2.2.1 The Resale Decision

Once the good has been purchased, at each time t the agent receives a constant stream of payoffs from holding the good equal to qU. However, by holding the durable good, the agent forgoes the payment received from selling it, therefore incurring an opportunity cost equal to the stream of payments whose expected present value is φe^{x_t}. In order to calculate the stream of payoffs let us introduce the *expected present value operator*:[4]

$$\mathcal{E} f(x) = \mathbb{E}_x \left(\int_0^\infty e^{-qt} f(X_t) dt \right).$$

It is possible to obtain the stream of payoffs whose EPV is φe^{x_t} using the previous equation and the moment generating function of the Brownian motion described above. However, it is convenient to use the equality $\mathcal{E} f(x) = (q - L)^{-1} f(x)$, where L is the infinitesimal generator of the Brownian motion defined as:

$$L = \frac{\sigma^2}{2} \partial^2 + \mu \partial.$$

[4]This operator is known in the theory of stochastic processes as the *resolvent* of the process.

Using the EPV-operator we can get the stream of payments, $g(x)$, associated to the resale price by $g(x) = (q - L)(\varphi e^{x_t})$. The profit flow generated from holding the durable good at time t can now be obtained by subtracting $g(x)$ from the consumption payoff received. Let denote this profit flow by:

$$\pi(x) = qU - (q - \psi(1))\varphi e^{x}, \tag{2.1}$$

where $\psi(z) = \frac{\sigma^2}{2} z^2 + \mu z$ is the *Levy exponent* of the process X_t.

In order to guarantee that the solution to our problem is well defined we need to impose some restrictions on the parameters of the model. In particular, we need to guarantee that the function $(e^{\zeta^- x} + e^{\zeta^+ x})^{-1}\pi(x)$ is bounded, *i.e.*, there exist a constant C such that

$$\left|(e^{\zeta^- x} + e^{\zeta^+ x})^{-1}\pi(x)\right| \leq C, \text{ a.e. for all } x. \tag{2.2}$$

for all $z \in [\zeta^-, \zeta^+]$ such that $q - \psi(z) > 0$, for some $\zeta^- \leq 0 \leq \zeta^+$.

The regularity condition stated above assures that the expected value operators used to calculate the optimal stopping time and option value are bounded. In other words, we need $\pi(x)$ not to grow too fast, with respect to the stochastic process, X_t, as x approaches positive or negative infinity. Note that, as x tends to negative infinity, $\pi(x)$ approaches the constant term qU and is always possible to find C satisfying equation (2.2). As x tends to positive infinity the absolute value of $\pi(x)$ grows unbounded. However, it is easy to see that the left hand side of equation (2.2) remains bounded as long as $\zeta^+ \geq 1$. Therefore, we need to restrict the parameters in the model such that $q - \psi(z) > 0$ for all $z \in [1, \zeta^+]$, or in other words, that the positive root of the characteristic equation is greater than one.

Let β^+ and β^- be the positive and negative roots of $q-\psi(z)$, respectively. From our previous discussion we need to set a restriction on the drift and diffusion terms of the Brownian motion such that $\beta^+ > 1$. Using standard algebraic manipulations it can be shown that this condition is satisfied as long as

$$\mu < q - \frac{\sigma^2}{2}. \tag{2.3}$$

For the remainder of the chapter we will assume that the relationship between μ, σ and q stated above holds.

The stopping problem described above is then given by the following equation:

$$V_2(x) = \max_{\tau \geq 0} \mathbb{E}\left(\int_0^\tau e^{-qt}\pi(x)dt\right). \tag{2.4}$$

To reiterate, the problem consists of choosing a random time, $\tau \geq 0$, at which the good will be sold that maximizes the expected present value of the stream of payoffs $\pi(x)$. The solution to equation (2.4) comes in the form of a reservation value for the random variable x, known in the Real Options literature as the *exercise threshold*, which divides the real line into two regions. For values of x above its reservation value it is optimal for the consumer to abandon the stream of payoffs, thus getting a utility of zero thereafter. We refer to this region as the *termination region*. For values of x below the reservation value the agent finds profitable to hold on to the durable good and receives the stream $\pi(x)$ together with the expected value of the option to sell the good further in the future. We refer to this region as the *continuation region*. Note that, as the spot market price for the durable good increases, *i.e.* the state variable x increases, the opportunity cost of holding the good becomes larger, and the continuation payoff becomes negative. Should x rise sufficiently high, it may become optimal to sell the good.

Denote the reservation value at which it is optimal to resale the good by x^*, and the optimal stopping time $\tau_{x^*} = \inf\{t \geq 0 | x_t \geq x^*\}$ as the solution to equation (2.4). So it will be optimal to resale the good the first time x_t crosses x^* from below.

The resale threshold and option value that solves for equation (2.4) can be obtained by applying Theorem 11.6.5 in Boyarchenko and Levendorskii (2007). Below we state a simplified version of the theorem for our particular case and notation. Before applying this theorem, however, we need to verify that our particular profit flow is decreasing in x and has a zero at some x. First, $\partial_x \pi(x) < 0$ since the restriction on the parameters of the model imposed by equation (2.3) is equivalent to $q > \psi(1)$ and $\varphi > 0$ by assumption. Direct inspection of equation (2.1) suffices to see that $\pi(x)$ is negative for sufficiently large values of x and, as x tends to negative infinity, $\pi(x)$ approaches $qU > 0$ thus crossing zero at some point.

Theorem 2.1 (Boyarchenko-Levendorskii)**.** *For $\pi(x)$ as defined in equation* (2.1)*:*

(a) equation $F(x) = 0$ has a unique solution at x^* with

$$F(x) = -\beta^- \int_{-\infty}^{0} e^{-\beta^- y} \pi(x+y) dy;$$

(b) τ_{x^*} is an optimal stopping time; and

(c) the value of the stream of payoffs with the option to abandon it is given by:

$$V_2^*(x) = q^{-1} \beta^+ \int_0^{x^*-x} e^{-\beta^+ y} \left(F(x+y) \right) dy.$$

For a proof of this theorem see Theorem 11.6.5 in Boyarchenko and Levendorskii (2007). Directly applying Theorem 2.1 to our stream of payoffs, $\pi(x)$, give us the optimal resale threshold, x^*, and the value of the option to resell the durable good. The result is presented in the following corollary. Note that equation (2.5) is written in terms of e^{x^*} which directly give us the optimal seller's reservation price.

Corollary 2.2. *For $\pi(x_t) = qU - (q - \psi(1))\varphi e^{x_t}$, the optimal resale threshold, x^*, is given by*

$$e^{x^*} = \frac{qU\,(\beta^- - 1)}{\varphi\beta^-(q - \psi(1))}. \tag{2.5}$$

For $x < x^*$ the value of the stream with the option to resale is

$$V_2^*(x) = U - \varphi e^x + \frac{U}{(\beta^+ - 1)} e^{\beta^+(x - x^*)}, \tag{2.6}$$

and $V_2^*(x) = 0$ for $x \geq x^*$.

2.2.2 The Buying Decision

With the optimal resale threshold and value of the stream of payoffs derived from owning the durable good in hand we now turn to the agent's buying decision. At the moment of purchase, the decision maker is entitled to a stream of payoffs whose value is given by $V_2^*(x)$ as defined in equation (2.6). Therefore, the agent faces the option of acquiring the payoff $V_2^*(x)$ at a price e^x. Define the instantaneous payoff of buying the durable good as $\Pi(x_t) = V_2^*(x_t) - e^x$, or equivalently as

$$\Pi(x_t) = U - (1 + \varphi)e^{x_t} + \frac{U}{(\beta^+ - 1)} e^{\beta^+(x_t - x^*)}. \tag{2.7}$$

Similarly to the previous step we need to verify that $\Pi(x)$ satisfies some regularity conditions for our choice of parameters μ, σ and q. In particular, our solutions are well defined if for all N, there exist C such that

$$\sum_{0 \leq s \leq 2} \left| e^{-\zeta^{-} x} \Pi^{(s)}(x) \right| \leq C \tag{2.8}$$

on the interval $(-\infty, N]$.

Since the instantaneous payoff function $\Pi(x)$ is due when a certain boundary is crossed from above – as it will be optimal to buy the good once the price falls sufficiently low – we need to impose a bound in a neighborhood of negative infinity. However, a simple examination of equation (2.7) suffices to verify that at negative infinity $\Pi(x)$ is bounded by U, and the first and second derivatives are bounded by zero so we do not need extra requirements on the parameters of the model to guarantee that the solution to our problem is well defined.

The buying problem is formally characterize by the equation:

$$V_1(x) = \max_{\tau \geq 0} \mathbb{E} \left(e^{-q\tau} \Pi(x_\tau) \right). \tag{2.9}$$

For the buying decision, in the termination region the agent acquires a instantaneous payoff, $\Pi(x)$, which includes the stream of utilities derived from the consumption of the durable good and its embedded resale value. In the continuation region the agent receives no payment, obtaining the expected value of waiting to exercise the option to buy the good. Additionally, as the spot market price for the durable good decreases, *i.e.* the state variable x decreases, the instantaneous payoff increases. Intuitively, a lower value of x reduces the price that the agent needs to pay in order to acquire the durable good. Furthermore, a reduction in x implies a lower expected resale value, reducing $V_2^*(x)$ as well. This result is formalize in the following lemma. Therefore, should x fall sufficiently low it may become optimal to acquire the instantaneous payoff $\Pi(x)$. Denote the threshold at which it is optimal to exercise this option by x_*, and its corresponding stopping time $\tau_{x_*} = \inf\{t \geq 0 | x_t \leq x_*\}$. Note that it is never optimal to buy the good for $x \geq x^*$ since in this interval the acquired value is zero, *i.e.* $V_2^*(x|x \geq x^*) = 0$, as stated in Corollary 2.2. Thus, it must be the case that $x_* < x^*$, and the value of the durable good with the option to resale is given by $\Pi(x)$ for all $x < x_*$.

Lemma 2.3. $\Pi(x)$ *as defined in equation* (2.7) *is a decreasing function of* x *for all* $x \leq x_*$.

In order to find the optimal buying threshold and option value that solves the problem defined by equation (2.9) we make use of Theorem 11.5.6 in Boyarchenko and Levendorskii (2007). Similarly as above, we present a simplified version of the Theorem relevant to our particular case and notation.

Theorem 2.4 (Boyarchenko-Levendorskii)**.** *Assume that there exists* x_* *such that*

(i) $\Pi(x) - (\beta^-)^{-1}\Pi'(x) > 0$ *for all* $x < x_*$, *and*

(ii) $\Pi(x) - (\beta^-)^{-1}\Pi'(x) < 0$ *for all* $x > x_*$.

Then

(a) x_* *is an optimal threshold;*

(b) τ_{x_*} *is an optimal stopping time; and*

(c) the value of the option with payoff $\Pi(x)$ *is given by*

$$V_1^*(x) = -\beta^- \int_{-\infty}^{x_*-x} e^{-\beta^- y}(\Pi(x+y) - (\beta^-)^{-1}\Pi'(x+y))dy.$$

A proof is provided by Boyarchenko and Levendorskii (2007) Theorem 11.5.6. Unlike the case of resale, the previous Theorem does not guarantee the existence of the optimal threshold, x_*, so we need to verify that conditions *(i)* and *(ii)* are satisfied for our function $\Pi(x)$.

Lemma 2.5. *Let* $G(x) = \Pi(x) - (\beta^-)^{-1}\Pi'(x)$, *with* $\Pi(x)$ *defined as in equation* (2.7). *Then,* $G(x)$ *has a zero at some* $x_* < x^*$. *Furthermore,* x_* *is unique and* $G(x)$ *changes sign as* x *passes* x_* *with* $G(x) > 0$ *for all* $x < x_*$.

Applying Theorem 2.4 and using the result in Lemma 2.5 we can obtain the optimal buying threshold and value of the option to buy the durable good with embedded resale option.

Corollary 2.6. *For* $\Pi(x_t) = V_2^*(x_t) - e^{x_t}$, *the optimal buying threshold,* x_*, *is given implicitly by*

$$U - \frac{e^{x_*}(1+\varphi)(\beta^- - 1)}{\beta^-} + \frac{U(\beta^- - \beta^+)}{(\beta^+ - 1)\beta^-} e^{\beta^+(x_* - x^*)} = 0. \tag{2.10}$$

For $x > x_*$ *the value of the option to buy the durable good is*

$$V_1^*(x) = \Pi(x_*)e^{\beta^-(x - x_*)}, \tag{2.11}$$

and $V_1^*(x) = \Pi(x)$ *for* $x \le x_*$.

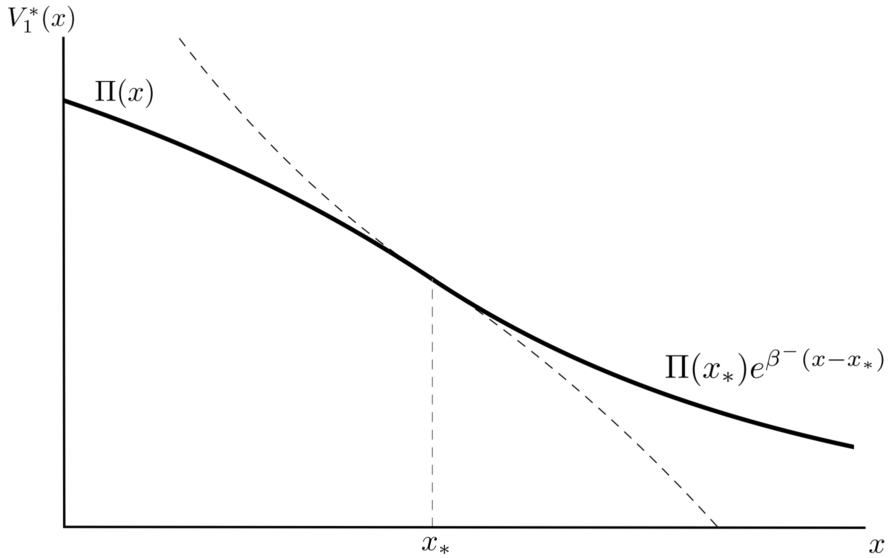


Figure 2.1: *Value of the option to buy a durable good,* $V_1^*(x)$

Proof. Equation (2.10) is obtained using Lemma 2.5 and $G(x)$ defined therein evaluated at x_*. Equation (2.11) follows from directly calculating the integral in Theorem 2.4

$$V_1^*(x) = \left[U - e^{x_*}(1+\varphi) + \frac{U}{(\beta^+ - 1)} e^{\beta^+(x_* - x^*)} \right] e^{\beta^-(x-x_*)},$$

and recognizing the term in brackets as $\Pi(x_*)$ in equation 2.7. ∎

The buyer's reservation price, e^{x_*}, can be calculated numerically from equation (2.10). Figure 2.1 shows the value of the option to buy the durable good and the optimal buying threshold. This concludes our analysis of the basic setup. We will come back latter to the results obtained in this section and compare them with those of the multiple priors case, which is the main focus of our work.

2.3 The multiple-priors model

In this section we extend the previous model by adding ambiguity about the drift of the underlying Brownian motion. First, we describe the structure of ambiguity in the model and discuss the importance of our simplifying assumption. Next, we construct a suitable set of probability measures to account for the uncertainty about the drift. We then elaborate on our solution concept and present our main result.

In the previous section we assumed that the agent knows (at least subjectively) the probability measure underlying the stochastic process X_t. Alternatively, we now allow the agent to be uncertain about the particular measure governing the state space and considering, instead, a set of probability measures denoted by $\mathscr{P}$. It can be argued that this type of uncertainty is more common in realistic decision problems. We incorporate ambiguity to the model by using the multiple-priors utility representation proposed by Gilboa and Schmeidler (1989). Under the *Gilboa-Schmeilder* representation of preferences, and ambiguity averse agent evaluates the expectation of future payments using the *worst case scenario* measure. Formally,

$$\mathbb{E}f(x_t) = \inf_{P \in \mathscr{P}} \left(\mathbb{E}^P f(x_t) \right).$$

for a suitable set of priors, $\mathscr{P}$. In the following subsection we discuss in detail the structure of $\mathscr{P}$ and the the technical reasons for our simplifying assumption of ambiguity resolving once the durable good is bought.

2.3.1 The set of priors

The set of priors, $\mathscr{P}$, is constructed using the κ-*ignorance* specification proposed by Chen and Epstein (2002). In this type of models the agent is assumed to be uncertain about the drift term of the underlying Brownian motion while the variance term is observable. In order to account for ambiguity, the agent considers a family of Brownian motions whose drift parameters take any value in the interval $[\nu - \sigma\kappa, \nu + \sigma\kappa]$, thus ambiguity is parameterized by a constant $\kappa > 0$ that generates an interval for the drift parameters centered at an arbitrary value, ν, hence the name.

In particular, let $(\Omega, P^{\nu}, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0})$ be a filtered probability space, and $(X_t)_{t\geq 0}$ be a Brownian motion defined on $(\Omega, \mathscr{F}, P^{\nu})$ where $(\mathscr{F}_t)_{t\geq 0}$ is the filtration generated by X_t satisfying the usual conditions augmented by the P^{ν}-null sets of $\mathscr{F}$. P^{ν} is only used as a reference measure to generate a set equivalent probability measures with respect to it (and each other) and should not be interpreted as the true probability measure. We generate $\mathscr{P}$ as the set of probability measures Q *mutually absolutely continuous* with respect to P^{ν}, which is the probability measure characterizing a Brownian motion with drift ν and variance σ^2. Define $\mathscr{D}_{\kappa}$ as the set of all real-valued processes $\theta = (\theta_t)_{0\geq t\geq T}$ with $|\theta_t| \leq \kappa$. Let B_t be a Brownian motion under P^{ν}. Now define,

$$z_t = \exp\left\{\int_0^t \theta_s dB_s ds - \frac{1}{2}\int_0^t \theta_s^2 ds\right\}$$

for some $\theta \in \mathscr{D}_{\kappa}$. For each $T > 0$, $Q_{\theta}(F) = \mathbb{E}_P(z_T \mathbf{1}_F)$ defines a probability measure *a.c.* with respect to P^{ν}. From Girasanov's theorem

$$\tilde{B}_t = B_t + \int_0^t \theta_t ds,$$

where $\tilde{B}_t$ is a Brownian motion under Q_θ with instantaneous drift $(\nu - \sigma\theta_t)$ and variance σ^2. The set of priors $\mathscr{P}$ is constructed as the set of Q_θ for all $\theta \in \mathscr{D}_\kappa$.

In this way we can model uncertainty over the drift of the underlying Brownian motion, and therefore the agent needs to consider any drift in the interval $M = [\nu - \sigma\kappa, \nu + \sigma\kappa]$ at all times, we will refer to this interval as the *ignorance interval.* In order to guarantee that our solutions under ambiguity are well defined we need to impose a restriction on the parameters in the ignorance interval similar to the one stated in equation (2.3). Namely, the highest drift under κ-*ignorance* should satisfy

$$\nu + \sigma\kappa < q - \frac{\sigma^2}{2}. \tag{2.12}$$

We can interpret κ as a measure of the degree of ambiguity. As κ tends to zero, the interval of drifts considered by the agent shrinks around ν, and the drift used to evaluate the expectations gets closer the the drift of the actual Brownian motion generating the state process. As κ increases the set of drifts gets larger and the worst-case scenario drift can be, in general farther away from the actual drift. Finally, Chen and Epstein (2002) showed that a set of priors constructed in this manner satisfies the rectangularity condition, which is sufficient for the dynamic consistency of the problem.

For rectangularity to hold, however, it is important for the instantaneous drift of the Brownian motion under Q_θ to be time varying and stochastic whenever ambiguity is present. It is because of this requirement that we need to restrict the time at which the agent is subjected to ambiguity in order to be able to embed our two decisions and solve the problem in a way similar to the previous section. In particular, we assume that ambiguity is only present at the time when the decision maker is considering to buy the durable good, and once the good is purchased the actual probability measure, denoted by $Q \in \mathscr{P}$, is revealed thus resolving ambiguity. When facing the resale decision, the agent still experiences uncertainty over future prices, but this uncertainty can be assessed using the now observed measure. We restrict Q even further by requiring it to be the probability measure underlying a Brownian motion with constant drift.

To see why resolving ambiguity in this manner is so crucial to obtain a solution to the model let $\Pi(x|\mu)$ be the *ex post* instantaneous payoff as defined in equation (2.7) for a particular realization of the (constant) drift for the Brownian motion generated by Q at the point of purchase, x. The *ex ante* value of the instantaneous payoff received at the moment of buying the durable good can then be obtained by evaluating $\Pi(x|\mu)$ at each μ in the ignorance interval and selecting the drift that minimizes it. That is

$$\tilde{\Pi}(x) = \min_{\mu \in M} \Pi(x|\mu). \tag{2.13}$$

Let $V(x)$ be the the value of the embedded stopping time problem. Under the *Gilboa-Schmeilder* representation and assuming ambiguity fully resolves after the good is purchased we have:

$$V(x) = \max_{\tau \geq 0} \min_{P \in \mathscr{P}} \mathbb{E}_0^P \left(e^{-q\tau_P} \tilde{\Pi}(X_{\tau_P}^P) \right), \tag{2.14}$$

Time consistency of the model is preserved since at the moment of purchase ambiguity is resolved, and the resale problem of the embedded option reduces to the standard case with single priors discussed in subsection 2.2.1. Therefore, the buying decision can be modeled as the multiple priors version of acquiring an instantaneous payoff $\tilde{\Pi}(x)$.

2.3.2 The solution concept

Assuming that the agent observes the actual drift of the underlying Brownian motion generating the process X_t at the moment of purchase we can obtain the optimal value of the stream $\pi(x|\mu)$ and the optimal resale threshold, $x^*_{(\mu)}$, as in subsection 2.2.1. In order to make the dependence of our solutions on the *ex post* value of the drift term evident, denote $\beta^+_{(\mu)}$ and $\beta^-_{(\mu)}$ as the positive and negative roots of the characteristic equation $q - \psi_{(\mu)}(z)$. By means of Corollary 2.2, the optimal resale threshold, $x^*_{(\mu)}$, is characterized by:

$$e^{x^*_{(\mu)}} = \frac{qU\left(\beta^-_{(\mu)} - 1\right)}{\varphi\beta^-_{(\mu)}(q - \psi_{(\mu)}(1))}, \tag{2.15}$$

and the value of the stream with the option to resale in the continuation region $x < x^*_{(\mu)}$ is given by:

$$V_2^*(x|\mu) = U - \varphi e^x + \frac{U}{\left(\beta^+_{(\mu)} - 1\right)} e^{\beta^+_{(\mu)}\left(x - x^*_{(\mu)}\right)}. \tag{2.16}$$

As in the previous section, the *ex post* instantaneous payoff of buying a durable good with an option to resell it is simply given by:

$$\Pi(x|\mu) = V_2^*(x|\mu) - e^x. \tag{2.17}$$

The *ex ante* version of equation (2.17) is then constructed as the worst-case scenario for $\Pi(x|\mu)$ over all possible vaues of μ in the ignorance interval as denoted in equation (2.13).

Lemma 2.7. $\Pi(x|\mu)$ *as defined in equation* (2.17) *is a strictly increasing function of* μ *for all* $\mu \in [\nu - \sigma\kappa, \nu + \sigma\kappa]$.

Therefore, the problem stated in equation (2.13) has a corner solution at $\underline{\mu} = \nu - \sigma\kappa$, and the *ex ante* value of the instantaneous payoff received at the moment of buying the durable good is given by:

$$\tilde{\Pi}(x) = U - (1+\varphi)e^x + \frac{U}{\left(\beta^+_{(\underline{\mu})} - 1\right)} e^{\beta^+_{(\underline{\mu})}\left(x - x^*_{(\underline{\mu})}\right)}. \tag{2.18}$$

Now let us revisit the option to acquire an instantaneous payoff equal to $\tilde{\Pi}(x)$ under ambiguity. Using the *Gilboa-Schmeilder* representation of preferences, the optimal time to buy a durable good with the possibility of reselling it is given by the stopping time τ_* that solves for

$$\mathscr{V}_1(x) = \max_{\tau} \min_{P \in \mathscr{P}} \mathbb{E}^P \left(e^{-q\tau}\tilde{\Pi}(x)\right). \tag{2.19}$$

Cheng and Riedel (2013) produced a result that can be applied to solve the problem in equation (2.19), provided $\tilde{\Pi}(x)$ is monotone in the state variable. To put it in terms of the discussion in Cheng and Riedel, what we have in equation (2.19) is a simple real option problem with a put-like payoff. Below we present an abbreviated version of their result for reference, but first let us establish a couple of properties of $\tilde{\Pi}(x)$ that allow us to use this result in our model.

Clearly, $\tilde{\Pi}(x)$ is a continuous function of x. Additionally, the restriction on the parameter values imposed by equation (2.12) guarantees that $\tilde{\Pi}(x)$ is bounded in the sense of equation (2.8) for all μ in the ignorance interval and in particular for

$\underline{\mu}$. This restriction on the parameter values is, in fact, stronger than the "growing condition" used by Cheng and Riedel. Lemma 2.3 established that $\Pi(x)$ is decreasing for all $x < x_*$, independently of the value of μ, thus $\tilde{\Pi}(x)$ is also decreasing in x for said interval.

Theorem 2.8 (Cheng-Riedel)**.** *Denote by v^κ the value function of the standard optimal stopping problem with payoff function $\tilde{\Pi}(x)$ under the measure P^κ. Then the optimal stopping time under κ-ignorance has value function $V_t = v^\kappa(t, X_t)$ where v^κ is the value function of the classical optimal stopping problem under the measure P^κ with the least favorable drift of the underlying process X_t.*[5]

For a proof see Theorem 4.1 in Cheng and Riedel (2013). The previous theorem states that, for the case of the κ-ignorance specification, the solution to the optimal stopping problem under ambiguity is the standard solution for a single prior evaluated at the worst possible drift in the ignorance interval. Therefore, we can characterize the optimal buying threshold and option value as in equations (2.10) and (2.11) evaluated at the drift that minimizes $V_1^*(x)$.

Denote $\lambda^+_{(\eta)}$ and $\lambda^-_{(\eta)}$ as the positive and negative roots of the characteristic equation $q - \xi(z)$ where $\xi(z) = \frac{\sigma^2}{2} + \eta z$ and η is the drift of the Brownian motion characterized by P^κ, and define y_* as the optimal buying threshold under ambiguity. Using Corollary 2.6 we can directly establish that, for $x > y_*$ the value of the option

[5]The original Theorem 4.1 in Cheng and Riedel (2013) is stated in terms of an increasing payoff function. Here, we present the clearly analogous result for a decreasing function that fits better for our purposes.

to buy the durable good is

$$\mathscr{V}_1^*(x) = \tilde{\Pi}(y_{*(\eta)})e^{\lambda^-_{(\eta)}(x-y_{*(\eta)})}, \tag{2.20}$$

with $y_{*(\eta)}$ given implicitly by

$$U - \frac{e^{y_{*(\eta)}}(1+\varphi)\left(\lambda^-_{(\eta)} - 1\right)}{\lambda^-_{(\eta)}} + \frac{U\left(\lambda^-_{(\eta)} - \beta^+_{(\underline{\mu})}\right)}{\left(\beta^+_{(\underline{\mu})} - 1\right)\lambda^-_{(\eta)}} e^{\beta^+_{(\underline{\mu})}\left(y_{*(\eta)} - x^*_{(\underline{\mu})}\right)} = 0. \tag{2.21}$$

The only remaining step to have a complete characterization of the value of the option and optimal buying threshold under ambiguity is to find η associated to the measure P^κ that minimizes $\mathscr{V}_1^*(x)$. From an economic point of view, it seems natural to infer that the highest possible drift is the most unfavorable since higher expected increments of the market price are clearly not aligned with the interests of an agent considering to buy the durable good. Our intuition is, in fact, correct and the result is formalized in the following Lemma.

Lemma 2.9. *$\mathscr{V}_1^*(x)$ as defined in equation* (2.20) *is a decreasing function of the drift term, η, for all $x > y_*$.*

Consequently, the value for the option to buy the durable good and the optimal buying threshold under ambiguity are given by equations (2.20) and (2.21) with $\bar{\eta} = \nu + \sigma\kappa$, *i.e.* the highest drift in the ignorance interval.

We now summarize the main result of the section and harmonize some notation to make the dependence of the *ex ante* option value and optimal buying threshold on the level of perceived ambiguity clear. Define $\tilde{\beta}^\pm = \beta^\pm_{(\underline{\mu})}$ with $\underline{\mu} = \nu - \sigma\kappa$, and $\tilde{\lambda}^\pm = \lambda^\pm_{(\bar{\eta})}$ with $\eta = \nu + \sigma\kappa$. Similarly, let $\tilde{y}_* = y_{*(\bar{\eta})}$ and $\tilde{x}^* = x^*_{(\underline{\mu})}$.

Result 2.10. *The optimal buying threshold under ambiguity, $\tilde{y}_*$, is given implicitly by*

$$U - \frac{(1+\varphi)\left(\tilde{\lambda}^- - 1\right)}{\tilde{\lambda}^-} e^{\tilde{y}_*} + \frac{U\left(\tilde{\lambda}^- - \tilde{\beta}^+\right)}{\left(\tilde{\beta}^+ - 1\right)\tilde{\lambda}^-} e^{\tilde{\beta}^+(\tilde{y}_* - \tilde{x}^*)} = 0. \tag{2.22}$$

For $x > \tilde{y}_*$ the value of the option to buy the durable good is

$$\mathscr{V}_1^*(x) = \tilde{\Pi}(\tilde{y}_*) e^{\tilde{\lambda}^-(x-\tilde{y}_*)}, \tag{2.23}$$

and $\mathscr{V}_1^*(x) = \tilde{\Pi}(x)$ for $x \leq \tilde{y}_*$.

Intuitively, at time zero the pessimistic agent presumes that the *ex post* mean increments of the resale price will be at its lowest. This is consistent with ambiguity aversion since the utility received from consuming the good is constant and the only source of uncertainty after purchase is the resale value of the durable good. Before purchasing the good, however, the most unfavorable process for the agent is the one with the highest mean price increments, regardless of the realized process at the time of purchase. In this sense, at time zero – or at any time before the good is bought – the agent uses two drifts to evaluate the embedded option of buying a durable good with the possibility of reselling it at some point in the future: (1) the highest possible drift when evaluation the option to buy the good, and (2) the lowest possible drift to compute the resale value as well as the optimal resale threshold. Chudjakow and Vorbrink (2009) and Cheng and Riedel (2013) find a similar change in the drift used to evaluate their exotic options as the problem transitions from one simple option to another.

2.3.3 Changes in the level of ambiguity

Now we analyze the effect that an increase in the level of ambiguity has on the optimal buyer's and seller's reservation price, as well as on the value of the option to buy the durable good. We start our analysis by determining the effect that a change on $\underline{\mu}$ have on the buyer's and seller's optimal reservation prices.

Proposition 2.11. *The optimal resale threshold, $\tilde{x}^*$, is an increasing function of $\underline{\mu}$.*

Intuitively, as $\underline{\mu}$ increases the agent considers the option to resale the good using a process with higher expected price increments. This has no effect on the termination payoff and increases the value of the option to resale in its continuation region, thus increasing the value of waiting to exercise the option. This in turn will increase the optimal resale threshold. Since the durable good is going to be sold the moment the market price crosses the reservation resale price from below, a higher resale threshold implies a longer waiting time (in expectation) to resale the durable good. From our Result 2.10 we have that, under ambiguity, the *ex ante* drift used to evaluate the option to resale the good is given by $\underline{\mu} = \nu - \sigma\kappa$. Therefore, an increase in the level of ambiguity, κ, will reduce $\underline{\mu}$ and the seller's reservation price, $e^{\tilde{x}^*}$. Recall from our discussion in subsection 2.2.2 that $x^* < \tilde{y}_*$. Proposition 2.11 guarantees that, once the *ex post* value of μ is revealed, the agent does not find herself immediately in the termination region of the option to resale the good since $\tilde{x}^* \leq x^*$ as the *ex post* value of μ can only be greater than $\underline{\mu}$.

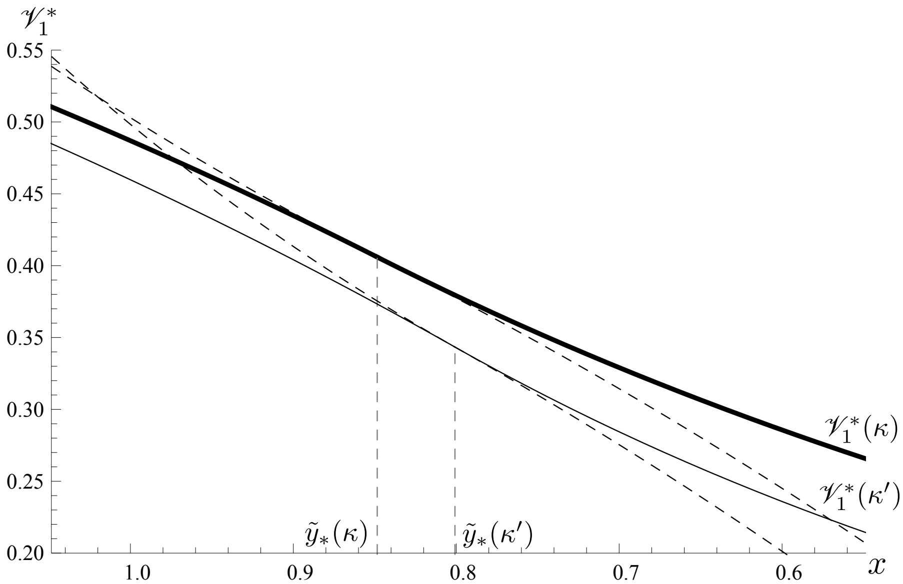
Let us now turn to the effect that an increase in κ has on the buyer's optimal reservation price, $e^{\tilde{y}_*}$, and the value of the option to buy the durable good, $\mathscr{V}_1^*(x)$ in its continuation region. Note that $\tilde{y}_*$ is a function of both $\bar{\eta}$ and $\underline{\mu}$. Thus, in order to evaluate the total effect that a change in ambiguity has on the buyer's optimal threshold we first need to establish the partial effects of $\bar{\eta}$ and $\underline{\mu}$ on $\tilde{y}_*$.

Proposition 2.12. *The optimal buying threshold, $\tilde{y}_*$, is an increasing function of both $\bar{\eta}$ and $\underline{\mu}$.*

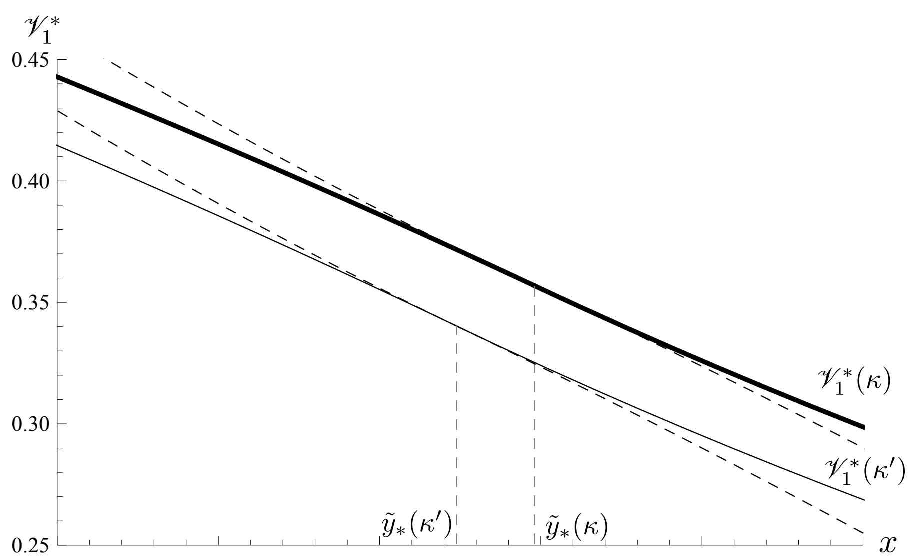
In order to see the intuition behind Proposition 2.12 notice that for higher values of $\bar{\eta}$ the agent uses the process with greater expected price increments to evaluate the decision of buying the durable good. This perceived environment of increasing prices, relative to the price increments for lower ambiguity levels, makes optimal to increase the reservation price for the good in question, thus increasing $\tilde{y}_*$. On the other hand, a decrease in $\underline{\mu}$ reduces the value of the option in its termination region by lowering the prospect of a high resale price for the durable good. The optimal reservation price will be lower due to the lower (expected) resale value of the good, therefore reducing the optimal buying threshold, $\tilde{y}_*$. An increase in ambiguity will then have two contrasting partial effects: (1) an increase in the optimal exercise threshold due to changes in $\bar{\eta}$, and (2) a decrease in $\tilde{y}_*$ through changes in $\bar{\mu}$.

As opposed to previous results in the literature (see Nishimura and Ozaki (2007) and Miao and Wang (2011)), the direction of the change in the buyer's optimal reservation price as a response to an increase in ambiguity depends on the particular parametrization of the model. That is, the partial positive effect of an increase in $\bar{\eta}$ on $\tilde{y}_*$ due to a change in ambiguity dominates the negative effect of an increase in $\underline{\mu}$ only for a subset of parameters. Figure 2.2 presents two particular parametrizations for which an increase in ambiguity has opposite effects on the optimal buying threshold.

Characterizing the regions for the parameter values that will guarantee a particular direction of the change in the optimal buyer's reservation price is quite complicated due to the lack of a close form solution for $\tilde{y}_*$. However, we can identify some regularities by computing the value of the optimal buying threshold numerically. In particular, we conjecture that $\tilde{y}_*$ is increasing in κ when $\underline{\mu}$ is "low enough" with respect to the initial level of ambiguity. Figure 2.3 presents the region at which $\tilde{y}_*$ is decreasing in κ for a particular set of parameter values. In Appendix C, we vary the values of σ, φ, and q to examine the result at "extreme" values of these parameters.



(a) $\tilde{y}_*$ increasing in κ ($\nu = 0$, $\kappa = 0$, $\kappa' = 0.4$)



(b) $\tilde{y}_*$ decreasing in κ ($\nu = 0.3$, $\kappa = 0$, $\kappa' = 0.1$)

Figure 2.2: *Change in $\tilde{y}_*$ as a response to an increase in κ ($\sigma = 1$, $q = 1$, $\varphi = 0.5$).*

From an economic point of view, the total benefit of owning the durable good has two sources: the utility generated from the use of it and its resale value. When $\underline{\mu}$ is low, the resale motive plays a smaller role in the *ex ante* purchase decision as the agent considers a relatively pessimistic process to calculate the seller's reservation price. Therefore, further reductions to the originally low mean increments of the resale price will be dominated by higher expected price increments used to evaluate the buying decision. This intuition is similar to the case when the fraction of the spot market price at which the good can be resold, φ, is close to zero. In this case, the effect of an increase of ambiguity in the termination region gets smaller, and the decreasing pressure on $\tilde{y}_*$ through changes in $\bar{\mu}$ becomes less relevant. At the extreme case, when $\varphi = 0$, our model collapses to that of Nishimura and Ozaki (2007) and Miao and Wang (2011) where, in the absence of a resale option, the agent faces a simple perpetual American put-like option with instantaneous payoff $U - e^{x_t}$. Similarly, for a given $\underline{\mu}$, a higher initial κ implies a relatively higher $\bar{\eta}$. If this $\bar{\eta}$ is "too high", further increases will be dominated by the lower expected price increments used to compute the *ex ante* resale value of the good, making $\tilde{y}_*$ decreasing in κ.

To finalize this section, we examine the effects of an increase in ambiguity on the value of the option to buy the durable good both in its termination and continuation regions. First, in the termination region the value of the option is given by $\tilde{\Pi}(x)$, which is independent of $\bar{\eta}$ by assumption. As shown in Lemma 2.7, $\tilde{\Pi}(x)$ is increasing in $\underline{\mu}$, thus an increase in ambiguity will reduce the value of the option in its termination region since an increase in κ reduces the value of $\underline{\mu} = \nu - \sigma\kappa$.[6]

[6]The result in Lemma 2.7 holds for all μ in the ignorance interval, and in particular for $\underline{\mu}$.

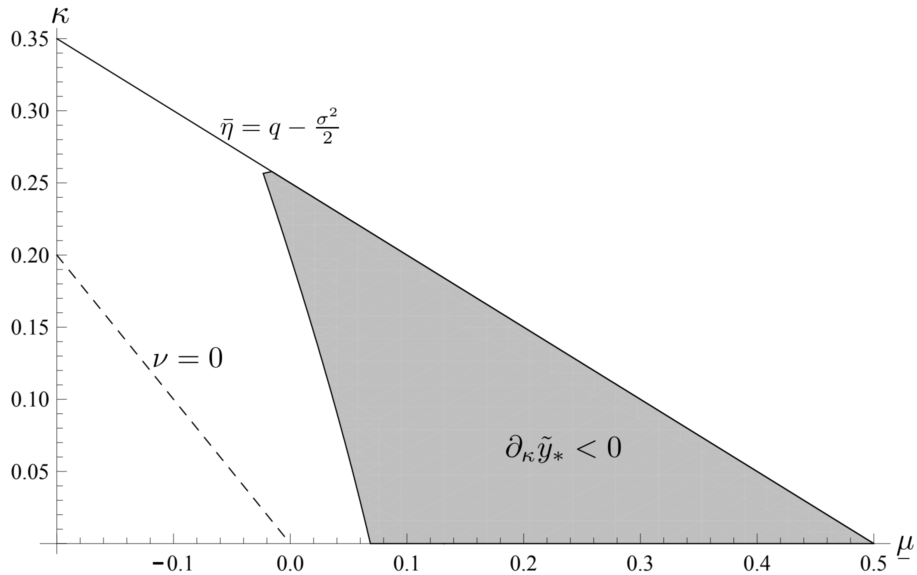


Figure 2.3: *Region Plot for $\partial_\kappa \tilde{y}^* < 0$ ($\sigma = 1$, $q = 1$, $\varphi = 0.5$).*

In order to analyze the effect of an increase in κ in the continuation region we first need to establish the following result.

Proposition 2.13. *The value of the option to buy a durable good, $\mathscr{V}_1^*(x)$, is an increasing function of $\underline{\mu}$ for all x in the continuation region, $x > \tilde{y}_*$.*

Using our results from Lemma 2.9 and Proposition 2.13 we can see that the partial effects that a change in the level of ambiguity has on the value of the option in the continuation region coincide. That is, $\mathscr{V}_1^*$ decreases as a result of increases in κ, since it is decreasing in $\bar{\eta}$ and increasing in $\underline{\mu}$. As the level of perceived ambiguity increases the agent uses a process with higher expected price increments. This reduces the value of the option to buy the good as the agent expects to pay more for a good whose utility derived from its consumption is independent of the market price. The *ex* *ante* resale decision is made using lower expected price increments which negatively affect the resale value of the good.

2.4 Conclusion

We have developed a model to analyze the option to buy a durable good with an embedded option to resell it at any point in time at a fraction of the spot market price where the agent is ambiguous regarding the drift of the Brownian motion characterizing the process of the state variable before the moment of purchase. We assume that ambiguity is resolved after the good is bought, and the resale decision is made with a well known probability distribution of future prices. We find that while the agent is considering to buy the durable good she uses two drifts to evaluate the embedded option: (1) the highest possible drift to compute the value of the option to buy the good and the optimal buying threshold, and (2) the lowest possible drift to calculate the *ex ante* resale value and optimal resale threshold. In addition we use computational methods to analyze the effect that an increase in the perceived level ambiguity has on the buyer's reservation price. We show that the direction of the change in the buyer's reservation price depends on the particular parametrization of the model. Furthermore, the change in the buying threshold due to an increase in ambiguity is greater as the fraction of the market at which the agent can resell the good decreases, and when this fraction gets closer to zero our problem gets closer to the perpetual American put-like option. As the last result, an increase in ambiguity reduces the value of the option in both its continuation and termination regions.

The results in Cheng and Riedel (2013) and Boyarchenko and Levendorskii

(2010) that we used in this work can be applied to more general Levy processes, thus there is room to generalize our model in that direction. Finally, another natural extension of the model is to remove the assumption that ambiguity vanishes at the moment of purchase or to formally incorporate it into the model by allowing the utility derived from the consumption of the durable good to be a function of the state variable or even the drift of the Brownian motion directly.

Chapter 3

Price Search and Ambiguity

3.1 Introduction

The interaction between firms an consumers in different market settings is one of the most researched topics in microeconomics. In particular, a significant effort has been devoted to understand the effects that information acquisition by consumers have on the firms' pricing behavior. The main focus of this chapter is to extend the models of price search to accommodate consumers' ambiguity regarding the production cost. Loosely speaking, and agent is said to be subjected to ambiguity if his information is too vague to reduce uncertainty to a single probability measure governing the state of the world.

Although the seminal work on price search can be traced back to Stigler (1961), it was Diamond (1971) who first developed a model where both sides of the market act optimally when making their consumption and pricing decisions. In his work Diamond showed that when all consumer have strictly positive search cost, the unique equilibrium of the game is the monopoly price, thus ruling out the possibility of price dispersion. This result is known in the literature as *Diamond Pricing*. Further efforts then focused on developing theoretical models that support price dispersion in equilibrium; being the most influential of such models Varian (1980).[1] Varian demonstrated that price dispersion arises when we allow an exogenous fraction of consumers to be fully informed about the prices set by the firms, while the rest of the consumers randomly chose a store to buy from. A closely related line of research concentrated its attention on costly price search (e.g. Braverman (1980); Salop and Stiglitz (1977); and Stahl (1996)). In a nutshell, it has been shown that a wide set of equilibria is attainable, ranging from the Bertrand's marginal cost pricing to the Diamond pricing, as we allow for different consumers' search cost structures.

More recently, this sort of models have been used to explain the phenomenon of asymmetric price adjustment, known as *rockets and feathers*, where prices react faster to increases in the production costs while lower costs lead to slow falling prices (see Yang and Ye (2008); Tappata (2009); and Cabral and Fishman (2011)). These models have two important characteristics in common. First, the cost faced by the firms are assumed to follow a Markov process, and for the case of Tappata (2009) and Cabral and Fishman (2011) the realized state is artificially revealed to the consumers at some point in time. Second, search is made non-sequentially and, more important, all search decisions are made prior to observe any price in the market, i.e. at the beginning of the period consumers from a belief regarding the production cost level without observing any of the actual prices, and then decide to either buy from a random store or become fully informed about the prices and buy from the firm offering the lowest price.[2] This paper uses a similar setting to the one presented by Yang and

[1] In fact Varian's simple arguments and proofs are still an important building block of current research on price search and our work is not the exception.

[2] In Cabral and Fishman's model there are only two types of firms so there is no distinction

Ye, and Tappata with two crucial distinctions: (1) search is partially sequential in the sense that consumers observe one price in the market and use that information to make the search decisions, and if a consumer engage in search he will observe all the remaining prices in the market; and (2) the process generating the production costs is assumed to be irreducible to a single prior from the perspective of the consumers, thus leading to an ambiguous setting.

We show that, under the presence of ambiguity, the prices fully reflect the high cost structure and any reduction on expected prices due to the possibility of a lower production cost vanishes. However, if the realization of the production cost is low, firms are still able to extract extra profits from consumers as costly search makes consumption at higher prices optimal and prices does not adjust completely to the pricing strategies expected under perfect information. Additionally we characterize a unique pricing equilibrium with continuous support for sufficiently high search costs and show that, as the search cost goes to zero, the resulting equilibrium under ambiguity does not converges the Bertrand pricing for low realizations of the production cost. In fact, firms price its goods at high marginal costs with positive probability while the remaining probability is assigned at search-inducing price levels.

The remainder of the chapter is organized as follows. In section 3.2 we set up the model in detail. In Section 3.3 we characterize a pricing equilibrium with connected support for sufficiently high values of the search cost. Section 3.4 elaborates on the existence of an equilibrium with disconnected support for low values of the

between sequential and non-sequential search.

search cost. Finally, section 3.5 concludes. All the proofs for this chapter are collected in Appendix D.

3.2 The Model

Consider a market with $N \geq 2$ stores selling an homogeneous good with a common marginal cost $c \in \{c_l, c_h\}$ identically and independently distributed at any point in time with $c_l < c_h$. To simplify the model we assume no firms enter nor exit the market.

The demand side is composed by a unit mass of consumers with unit demand and a common reservation price, $v > c_h$. At the beginning of each period, consumers observe only one (randomly assigned) price in the market. Before the consumption decisions are made, consumers can engage in non-sequential search for a price lower than the one offered to them such that if a buyer decides to search they will pay a search cost and observe all the prices available in the market. We assume that a fraction $\lambda > 0$ of consumers have zero search cost and therefore will always search for lower prices as it is costless. This type of consumers is closely related to Varian (1980) informed consumers in the context of an environment without price search and Stahl (1996) *shoppers* when search is allowed from whom we borrow the term. The remaining $(1 - \lambda)$ fraction of the market have a common search cost denoted by $s > 0$. If a consumer with strictly positive search cost decides to search, we say that he undertakes *costly search* to distinguish this endogenous search decision from the exogenously imposed search from the *shoppers*. The expected fraction of consumers that engages in *costly search* is denoted by $\mu \in [0, (1 - \lambda)]$. The expected fraction of consumers searching in the model, including endogenous and exogenous search, is then given by $(\lambda + \mu) \in [\lambda, 1]$ and we refer to it as the search intensity of the market.

Additionally, consumers are assumed to be ambiguous with respect to the probability measure generating the firms' marginal cost. That is, it is impossible for a consumer to form a unique prior regarding the probability of the marginal cost taking any of its particular values. In turn, we assume consumers have a set of priors, Φ, which includes the true probability of $c = c_h$.

The timing of the model goes as follows. First, the stores observe the realization of the marginal cost, c, and set prices simultaneously according to a price distribution $F(p; c)$. Throughout the paper we focus our attention exclusively on symmetric Nash equilibrium (NE) strategies. Then, each consumer observes a random price $p_i \in \boldsymbol{p}$, where $\boldsymbol{p}$ is a vector size N including all quoted prices. After observing one free sample from $\boldsymbol{p}$, consumers form beliefs about the level of the marginal cost and decide whether or not incur the cost s and observe all N prices in the market, while the *shoppers* observe the remaining prices, effectively determining μ as a function of the observed price and the equilibrium pricing strategies, $F(p; c)$. Once all costly search decisions are made, consumers buy from the store offering the lowest observed price at no additional cost, provided that it is lower than the reservation price, v. If the consumer does not incur the search cost the lowest price observed is simply the initial free quote. In case that more than one firm charges the lowest price, one is randomly selected.

For $F(p; c)$ to be an equilibrium, the firms must take into account the consumers' optimal search rule; which, in turn, is a function of the equilibrium pricing strategies. Therefore any NE of the game must be consistent with the search intensity generated by $F(p;c)$ and vice versa.

3.2.1 Belief formation under ambiguity

We assume consumers form their expectations about c according to the *maxmin* expected utility model introduced by Gilboa and Schmeidler (1989). Let $\phi_i(p_i, F)$ denote the belief that $c = c_h$ given the observed quoted price, p_i, and the pricing strategies of the firms, $F(p;c)$. Also, let $\Phi = [0,1]$ be the set including all possible probabilities of $c = c_h$.

The utility derived from making a consumption decision without searching for a consumer originally observing p_i is given by:

$$U_1(p_i; v) = \max\{v - p_i, 0\}.$$

Alternatively, the utility of making a consumption decision after undertaking costly search can be written as:

$$U_2(p_i; v) = \max\{v - \min\{\boldsymbol{p}\}, 0\}.$$

The gains of search, net of search costs, is given by $GS(p_i) = U_2 - U_1$. Note that when the realized minimum of the $N-1$ prices observed after search is greater than than p_i, consumers will buy at price p_i, yielding zero gains from search. The expected gains from search under ambiguity for a consumer with price p_i in hand can then be constructed as:

$$\mathbb{E}GS(p_i) = \min_{\phi' \in \Phi} \mathbb{E}_{\phi'} \left[\max\{p_i - \min\{p_{-i}\}, 0\}\right]. \tag{3.1}$$

That is, consumers use the *worst-case-scenario* to evaluate the expectation over $GS(p_i)$.

Given this behavior, we can potentially have three types of consumers distinguished by their beliefs: (1) consumers observing an initial quoted price only compatible with an equilibrium price distribution for high marginal costs will be certain that the cost is in fact high; (2) consumers whose initial quoted price could come from an equilibrium distribution consistent with the two cost regimes will use equation (3.1) to determine their beliefs; and (3) consumers observing p_i incompatible with the equilibrium price distribution for a high marginal cost will be certain that the cost is low. Formally,

$$\phi_i(p_i, F) = \begin{cases} 1 & \text{for } p_i \in \operatorname{supp} F(p, c_h) \setminus \operatorname{supp} F(p, c_l) \\ \arg\min_{\phi' \in \Phi} \mathbb{E}GS(p_i) & \text{for } p_i \in \operatorname{supp} F(p; c_h) \bigcap \operatorname{supp} F(p; c_l) \\ 0 & \text{otherwise.} \end{cases}$$

The previous specification implicitly assigns *off-equilibrium* beliefs to the low marginal cost regime. In other words, if a consumer observes a price outside the support of any equilibrium distribution under the two cost regimes, we say that such consumers will set $\phi_i = 0$.

By the linearity of the expected value operator, the $\arg\min_{\phi' \in \Phi} \mathbb{E}GS$ has a corner solution at either $\phi_i = 1$ or $\phi_i = 0$. It is natural to expect that, in equilibrium, consumers will be worse-off when the marginal cost is in fact higher. To formally prove this statement, however, we first need to characterize the firms' equilibrium pricing strategies, but to give a characterization of the equilibrium pricing strategies we need some structure on the belief formation, so we need to assume a starting point if we want to solve for an equilibrium in this model. A reasonable place to start is to make this fairly uncontroversial conjecture and, once the equilibrium pricing strategies are characterized, come back and verify that our conjecture is actually true. If the reader is nevertheless uncomfortable with our starting point, the following conjecture can also be stated in terms of "bad" and "good" cost regimes from the perspective of the consumer and trivially labelling the "bad" cost regime as the *wosrt-case-scenario* and we will later verify that the "bad" cost regime is the case of a high marginal cost.

Conjecture 3.1. *If $F(p;c)$ is a symmetric NE for all $c \in \{c_l, c_h\}$, then* $\arg\min_{\phi' \in \Phi} \mathbb{E}GS(p_i) = 1$.

Using our Conjecture regarding the *wosrt-case-scenario*, we can simplify the function determining the consumers' beliefs to

$$\phi_i(p_i, F) = \begin{cases} 1 & \text{for } p_i \in \operatorname{supp} F(p, c_h) \\ 0 & \text{otherwise.} \end{cases} \tag{3.2}$$

That is, either a consumer observes p_i incompatible with the high cost regime, in which case he will be certain that the cost is low, or they will be unable to distinguish between the two costs, in which case he will behave as if the cost were high with probability one; leaving us with at most two different beliefs for any realization of the cost and random vector $\boldsymbol{p}$. Furthermore, when $c = c_h$, all initial quoted prices are generated from $F(p; c_h)$ and all consumers will have $\phi_i(p_i, F) = 1$ which leave us with a common belief across consumers.

3.2.2 Consumers search rule

We now focus our attention of the search rule for consumers with strictly positive search cost, s. Using the equilibrium pricing strategies of the firms under the two cost regimes we can calculate the expected gains of search in equation (3.1) for each ϕ_i.

$$\mathbb{E}GS(p_i, \phi_i) = \begin{cases} \mathbb{E}\left[\max\{p_i - \min\{p_{-i}\}, 0\} | F(p; c_h)\right] & \text{if } \phi_i = 1 \\ \mathbb{E}\left[\max\{p_i - \min\{p_{-i}\}, 0\} | F(p; c_l)\right] & \text{if } \phi_i = 0 \end{cases}. \tag{3.3}$$

A consumer will find optimal to search if the expected gains are greater than the search cost, *i.e.* $\mathbb{E}GS(p_i, \phi_i) > s$, and we can identify the reservation price, $r_i(\phi_i)$, at which the consumer will be indifferent between searching and making a consumption decision immediately as the solution to

$$\mathbb{E}GS\left[r_i(\phi_i), F\right] = s. \tag{3.4}$$

Therefore, we can specify the optimal searching rule for the consumers as a reservation price rule: search if the observed price is greater than $r_i(\phi_i)$; and do not search if the observed price is less or equal to $r_i(\phi_i)$. Combining this search rule with the fact that consumers will never buy at a price greater than their valuation for the good we can summarize the consumers' search and consumption decisions by:

(i) do not search if $p_i \leq r_i(\phi_i)$ and buy from the first store only if $p_i \leq v$; and

(ii) search if $p_i > r_i(\phi_i)$ and buy from the firm offering the lowest price only if $\min\{\boldsymbol{p}\} \leq v$.

Finally, we assume that consumers with zero search cost, *shoppers*, always engage in search independently of p_i and F.[3]

[3]This assumption technically specifies a different way of resolving the indifference case for the shoppers, as they search even if $p_i = r_i(\phi_i)$. For the case of zero search costs, however, the reservation price is simply the lowest possible price under $F(p; c)$, making the search decision irrelevant as they will continue to buy from the firm offering the lowest price.

3.2.3 Firms Expected Profits

Let $\boldsymbol{r}$ denote the collection of distinct reservation prices satisfying equation (3.4). As mentioned in Section 3.2.1, when $c = c_h$ all consumers share the same belief $\phi_i = 1$ and $\boldsymbol{r}$ includes only one element, *i.e.* $\boldsymbol{r} = \{r(1)\}$. For the case when $c = c_l$, there are two potential beliefs with their corresponding two reservation prices and $\boldsymbol{r} = \{r(0), r(1)\}$. Also, let $\mu(\boldsymbol{r}, \boldsymbol{p})$ be the expected fraction of consumers engaging in costly search as a function of the reservation prices and initial quoted prices. In order to analyze the pricing decision of an arbitrary firm, let $\mu_j(r_j, p_j)$ be the costly search induced only by firm j and $\mu_{-j}(\boldsymbol{r}_{-j}, \boldsymbol{p}_{-j})$ be the costly search induced by the all other firms, such that $\mu_j + \mu_{-j} = \mu$

In our model firms can directly affect the reservation price of consumers initially observing it's price and, therefore, the expected share of consumers patronizing the firm. In particular, a firm setting price p_j below its corresponding reservation price, $r_j(\phi_j(p_j, F))$, can be certain that the fraction of consumers observing p_j will stop and buy from it at that price which implies $\mu_j = 0$. Additionally firm j will serve the remaining fraction of consumers searching for the lowest price, $\lambda + \mu_{-j}$, if p_j is the lowest price in the market. On the other hand, setting $p_j > r_j(\phi_j(p_j))$ will induce costly search on consumers observing p_j, and $\mu_j = (1 - \lambda)/N$. In this case the demand from firm j is given by $\lambda + \mu$ only if p_j is the lowest price in the market and zero otherwise. The expected profits of a store charging price p_j, when the symmetric NE distribution is $F(p; c)$ are given by:

$$\mathbb{E}\pi(p_j;c) = \begin{cases} \left((\lambda+\mu_{-j})[1-F(p;c)]^{N-1}+\frac{1-\lambda}{N}\right)(p_j-c) & \text{if } p_j \leq \min\{r_j,v\} \\ \left((\lambda+\mu_{-j}+\frac{1-\lambda}{N})[1-F(p;c)]^{N-1}\right)(p_j-c) & \text{if } p_j \in (r_j,v] \\ 0 & \text{otherwise.} \end{cases} \tag{3.5}$$

This is an important distinction from previous work on the literature where differences on the reservation prices are consequence of different consumer search costs. In this sort of models the stores' expected market share always include a fraction of the *non-searchers* as they are visited by consumers with different search costs. In our case, however, the initial price quote observed by consumers determines the reservation price and all non-shoppers visiting the same store share the same reservation price.

3.3 Pricing Equilibrium with Connected Support

Before characterizing the equilibrium pricing strategies it is convenient to formulate some necessary conditions on the support of $F(p;c)$. These conditions emanate from the requirement of the expected profits to be equal throughout the support of the equilibrium distribution. Later in the section we use these necessary conditions to characterize the equilibrium for each of the cost regimes.

3.3.1 Necessary conditions of a symmetric NE

A common result for a wide class of search models with a mass of consumers with zero search cost is that any symmetric Nash Equilibrium is atomless throughout its support (see Varian (1980), Tappata (2009), Stahl (1996)). In our setting, however, this sort of equilibria arises only for a subset of the parameter space, namely for sufficiently high search cost. This difference originates from the decision faced by the firms of revealing information about the cost level to consumers by setting a price outside the high cost regime equilibrium support, potentially increasing the search intensity in the market. In such a case, it might not be possible to increase profits discretely by a marginal decrease in price, which is the fundamental argument behind a continuous price distribution. Nonetheless, we can evoke the atomless property at least at the lower bound of any equilibrium distribution, as marginally undercutting any mass point at such price will eliminate the probability of a tie, increasing the expected market share discretely. In this case, increasing the search intensity of the market will have no effect on the deviating firm's profits since this firm will be charging the lowest price with probability one and any direct increase in the fraction of consumers searching will continue to patronize the deviating firm. This first result is formalized in the following Lemma.

Lemma 3.2. *Let $\alpha(\underline{p})$ be the size of a mass point placed at the lower bound of the support of a pricing strategy $F(p;c)$. If $F(p;c)$ is a symmetric NE strategy, then* $\alpha(\underline{p}) = 0$.

Furthermore, if we concentrate our attention on equilibrium distributions with connected support such that every price $p \in [\underline{p}, \bar{p}]$ is also in the support of F, then the equilibrium pricing strategies must have no mass points throughout the support of F.[4] Formally,

[4]The support of a cdf, $\operatorname{supp} F \subset \mathbb{R}$, is *path connected* if for every $p, p' \in \operatorname{supp} F$ there is a continuous function $f : [0,1] \to \operatorname{supp} F$ such that $f(0) = p$ and $f(1) = p'$. The support of a cdf, $\operatorname{supp} F \subset \mathbb{R}$ is *connected* if and only if it is path connected.

Proposition 3.3. *If $F(p;c)$ is a symmetric NE with connected support, then $F(p;c)$ is atomless for all $p \in supp\, F(p;c)$.*

The assumption of connectedness on the support of a pricing equilibrium does not come for free. In fact, such an assumption imposes an extra requirement on the reservation prices formed by consumers as a response to the pricing strategy. By assuming that every price between the upper and lower bounds on the support of F is also part of the pricing equilibrium we are demanding that the expected profits are the same throughout the interval $[\underline{p}, \bar{p}]$. This is only possible when marginally undercutting a price will not result in a discrete fall in market share due to a increase in the search intensity of the model. In other words, if there there exist a price in the support of F that makes consumers to engage in costly search then an equilibrium strategy, if exist, must have a disconnected support. Therefore, if we are concerned with equilibrium strategies with connected support it must be the case that $r(\phi(p)) \geq p$ for all p in the support of F. This argument is formally presented in Lemma D.2 as part of the proof of Proposition 3.3 (see Appendix D).

An important consequence of Proposition 3.3 is that no endogenous search will occur in an equilibrium with connected support as the reservation price triggering search is above all the prices set in equilibrium. Also, since a pure strategy equilibrium is a degenerated cdf with a point of mass one, and a point in the real line is trivially connected, an immediate result form Proposition 3.3 is:

Corollary 3.4. *There are no pure-strategy symmetric NE in prices.*

In addition to the previous results, it is convenient to provide a necessary condition for the upper bound of the support for a mixed-strategy equilibrium.

Lemma 3.5. *Let $\bar{p}$ be the upper bound of a symmetric NE with connected support, then* $\bar{p} = \min\{r(\phi(\bar{p})), v\}$.

Remember that $r(\phi(\bar{p}))$ is the reservation price formed by a consumer observing $\bar{p}$. The previous Lemma states that for any symmetric NE, the stores never set a price above such reservation price nor the valuation for the good. The latter is clear since pricing above the consumer's valuation yields zero profits. To see the former note that by setting a price at the upper bound of the equilibrium distribution, the store forgoes all consumers engaging in search with probability one. Increasing the price will then have no effect on the firm's market share as long as $\bar{p} < r(\phi(\bar{p}))$ so it is possible to increase profits; violating the requirements of an equilibrium. From our discussion in Section 3.2.1, we know that there are at most two distinct reservation prices, namely $r(0)$ and $r(1)$, and there is a unique reservation price for the high cost regime given by $r(1)$, so Lemma 3.5 identifies the upper bound for $F(p; c_h)$ and provides a useful necessary condition for $F(p; c_l)$.

Before proceeding to characterize the symmetric Nash Equilibria for each of the possible marginal costs we need to rule out the possibility of an equilibrium where firms convey no information regarding the level of the cost. In particular we are concerned with the possibility that firms find optimal to price its good as if the cost where high, regardless of the actual realization.

Lemma 3.6. *Let $F(p; c_k)$ be the firms' pricing equilibrium distribution for $c = c_k$. Then $supp\, F(p; c_h) \neq supp\, F(p; c_l)$.*

Lemma 3.6 rules out equilibria where the two cost regimes are impossible to distinguish for all the prices in the equilibrium strategies. Intuitively, the existence of a mass of consumers with zero search cost introduces the incentive to reduce the price offered to get those consumers, albeit facing lower per-costumer profits. A pricing equilibrium will then impose a particular relationship between the bounds of the equilibrium distribution such that the incentive for undercutting the price below the equilibrium support is eliminated. When the cost is lower, such relationship does not hold anymore and the equilibrium support must change in order to eliminate the incentive to undercut prices and get a higher market share.

3.3.2 The case of high marginal costs

As discussed above, when the realization of the marginal cost is high, consumers have a common reservation price given by $r(1)$. From Lemma 3.5, it must then be the case that $\bar{p}_h = \min\{r(1), v\}$, where $\bar{p}_h$ denotes the upper bound of an equilibrium distribution under the high cost regime. We established in Proposition 3.3 that, if we restrict our attention to equilibria with connected supports, there are no mass points in the equilibrium cdf and $F(p; c_h)$ is a continuous function with pdf $f(p) = dF(p; c)$ almost everywhere. The symmetric NE for $c = c_h$ for a given $\bar{p}_h$ is then characterized by the following Proposition.

Proposition 3.7. *For the case when $c = c_h$, the unique symmetric NE with connected* *support is a mixed strategy with*

$$F(p; c_h) = 1 - \left[\frac{1-\lambda}{\lambda N}\left(\frac{\bar{p}_h - p}{p - c_h}\right)\right]^{\frac{1}{N-1}},$$

and support $\operatorname{supp} F = \left[\underline{p}_h, \bar{p}_h\right]$*; where*

$$\underline{p}_h = \left[\frac{1-\lambda}{1+\lambda(N-1)}\right](\bar{p}_h - c_h) + c_h.$$

and expected equilibrium profits

$$\mathbb{E}\pi^* = \frac{(1-\lambda)}{N}(\bar{p}_h - c_h).$$

No endogenous search will occur in equilibrium, that is, only the shoppers will find optimal to search. More importantly, the additional requirement on the reservation prices, $r(\phi(p)) \geq p$, is always satisfied. This is a direct consequence of Lemma 3.5, as there is a unique reservations price and $\bar{p}_h = \min\{r(1), v\}$ implying that for all prices set in equilibrium, consumers with non-zero search cost will stop and buy from the first store.

To fully characterize the equilibrium, the reservation price $r(1)$ must be consistent with the pricing strategies of the stores. Let us relabel the reservation price for $\phi = 1$ as $r_h \equiv r(1)$ to properly identify it as the reservation price held by consumers with beliefs that the cost is high. We can obtain a close form solution for r_h by using $F(p; c_h)$ as in Proposition 3.7 and solving for equation (3.3) with $\phi_i = 1$ for all i. Note that the minimum price observed after search has a cdf given by:

$$G(p; c_h) = 1 - [1 - F(p; c_h)]^{N-1},$$

and can be represented by a pdf, $dG(p; c_h)$. We can, therefore rewrite the gains of search as:

$$\begin{aligned}
\mathbb{E}GS(r_h, F(p; c_h)) &= \int_{\underline{p}_h}^{r_h} (r_h - p)\, dG(p; c_h) \\
&= r_h - \left[c_h + \frac{(r_h - c_h)(1-\lambda)}{N\lambda} \log\left(\frac{1+\lambda(N-1)}{1-\lambda}\right)\right].
\end{aligned}$$

To simplify notation, define

$$\gamma(\lambda) \equiv \frac{1-\lambda}{N\lambda} \log\left(\frac{1+\lambda(N-1)}{1-\lambda}\right).$$

We can now use the previous expression to obtain the optimal reservation price r_h that solves equation (3.4).

$$r_h = \frac{s}{1-\gamma(\lambda)} + c_h;$$

and the upper bound of the symmetric pricing NE is given by:

$$\bar{p}_h = \min\left\{\frac{s}{1-\gamma(\lambda)} + c_h, v\right\}, \tag{3.6}$$

fully characterizing the equilibrium under the high cost regime.

3.3.3 The case of low marginal costs

Now we turn to the case when the realization of the marginal cost is low. In this situation the reservation price is not necessarily the same across consumers. Instead consumers can potentially have two distinct reservation prices, depending on the particular initial price observed, *i.e.* $\boldsymbol{r} = \{r(0), r(1)\}$. We can, however specify a condition for the upper bound of a pricing equilibrium distribution stronger than the one stated in Lemma 3.5 for the case of a low marginal cost together with a necessary condition for the lower bound.

Lemma 3.8. *If $\bar{p}_l$ and $\underline{p}_l$ are the upper and lower bounds of a symmetric NE with connected support, respectively; then (a) $\bar{p}_l = \bar{p}_h$, and (b) $\underline{p}_l < \underline{p}_h$.*

This necessary condition simply states that the two upper bounds of the equilibrium distributions for the two cost regimes must coincide. Intuitively, setting a price at $\bar{p}_l$ implies that no *shoppers* will buy from the store, as they will find a lower price at which they can buy from, leaving the firm only with the fraction of consumers with positive search cost observing price $\bar{p}_l$. However, it is always possible to manipulate the beliefs of those consumers by increasing the price to $\bar{p}_h$ as they will then act as if the cost where high and continue to buy from the firm without searching. Additionally, the lower bound of any equilibrium distribution under the low cost regime must be less than the lower bound under the high cost regime as lower costs makes profitable to undercut prices in the support of the high cost distribution.

Furthermore, any symmetric NE with connected support must be atomless, and no endogenous search will arise in equilibrium (see Lemma D.2 in the Appendix D). Therefore, we can obtain the symmetric equilibrium price density using the same procedure as in Proposition 3.7 using

$$\mathbb{E}\pi(p;c_l) = \int_{\underline{p}_l}^{\bar{p}_h} \left(\left(\lambda[1 - F(p;c_l)]^{N-1} + \frac{1-\lambda}{N} \right) (p - c_l) \right) dF(p;c_l)$$

instead of equation (D.6) in the proof of Proposition 3.7 to calculate the expected profits, where the integration limits are given by Lemma 3.8. The symmetric NE under the low cost regime is then given by:

Proposition 3.9. *For the case when $c = c_l$, the unique symmetric NE with connected support is a mixed strategy with*

$$F(p; c_l) = 1 - \left[\frac{1-\lambda}{\lambda N}\left(\frac{\bar{p}_h - p}{p - c_l}\right)\right]^{\frac{1}{N-1}},$$

and support $\operatorname{supp} F = \left[\underline{p}_l, \bar{p}_h\right]$*; where*

$$\underline{p}_l = \left[\frac{1-\lambda}{1+\lambda(N-1)}\right](\bar{p}_h - c_l) + c_l.$$

and expected equilibrium profits

$$\mathbb{E}\pi^* = \frac{(1-\lambda)}{N}(\bar{p}_h - c_l).$$

In this case the previous result fully characterizes the pricing equilibrium as the upper bound is the same as in equation (3.6). The proof of Proposition 3.9 is omitted as it follows closely that of Proposition 3.7.

Figure 3.1 depicts the symmetric NE cdf's for the two cost regimes. Note that, for the case when firms observe low marginal costs, it is necessary that $r(0) \geq \underline{p}_h$ for an equilibrium with connected support to exist. In this way, consumers observing a price between $\underline{p}_l$ and $\bar{p}_h$ will form a belief consistent only with a low cost distribution and their reservation price will be given by $r(0)$. Only prices above $r(0)$ will trigger costly search, but $r(0) \geq \underline{p}_h$ guarantees that consumers observing a price in that range will form a belief consistent with a high marginal cost and no costly search will occur in equilibrium. Now we need to verify that $r(0) \geq \underline{p}_h$ is in fact consistent with the reservation prices formed under $F(p; c_l)$. Parallel to the high cost case, let us relabel the reservation price for $\phi = 0$ as $r_l \equiv r(0)$. Solving the integral implied by equation (3.3) with $\phi_i = 0$ for all i we have:

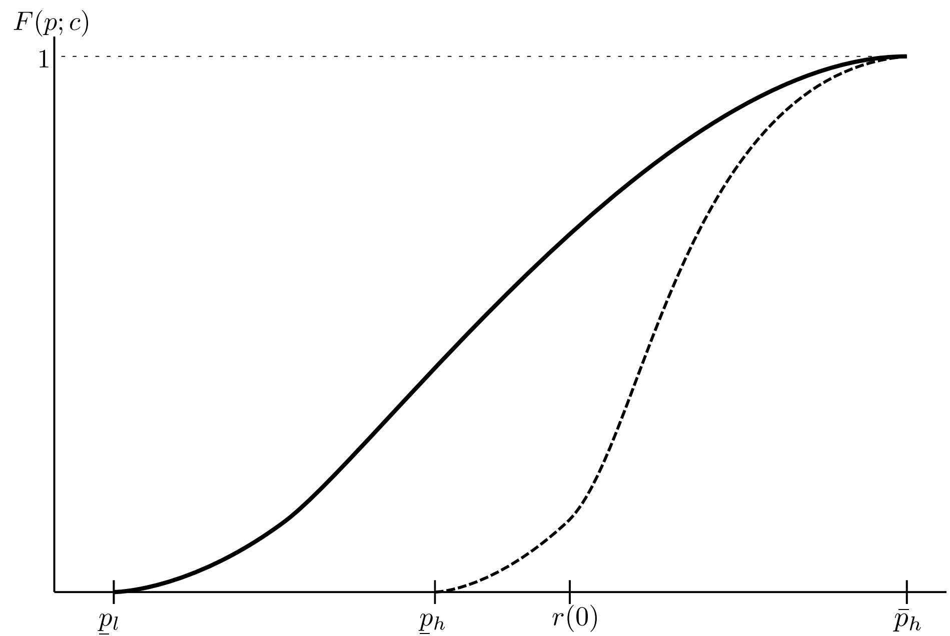


Figure 3.1: *Pricing equilibrium cdf's for $c = c_h$ (dashed) and $c = c_l$ (solid).*

$$\begin{aligned}
\mathbb{E}GS(r_l, F(p;c_l)) &= \int_{\underline{p}_l}^{r_l} (r_l - p)\, dG(p;c_l) \\
&= r_l - \frac{(1-\lambda)}{N\lambda}\left[(\bar{p}_h - r_l) + (\bar{p}_h - c_l)\log\left[\left(\frac{r_l - c_l}{\bar{p}_h - c_l}\right)\left(\frac{1+\lambda(N-1)}{1-\lambda}\right)\right]\right].
\end{aligned}$$

Unlike the high marginal cost case, there is no closed form solution to

$$\mathbb{E}GS(r_l, F(p;c_l)) = s. \tag{3.7}$$

Nevertheless the following two Lemmas show that there exist a unique solution to the previous equation and provide conditions on the parameters of the model such that the solution lies above the lower bound of the equilibrium distribution for the high

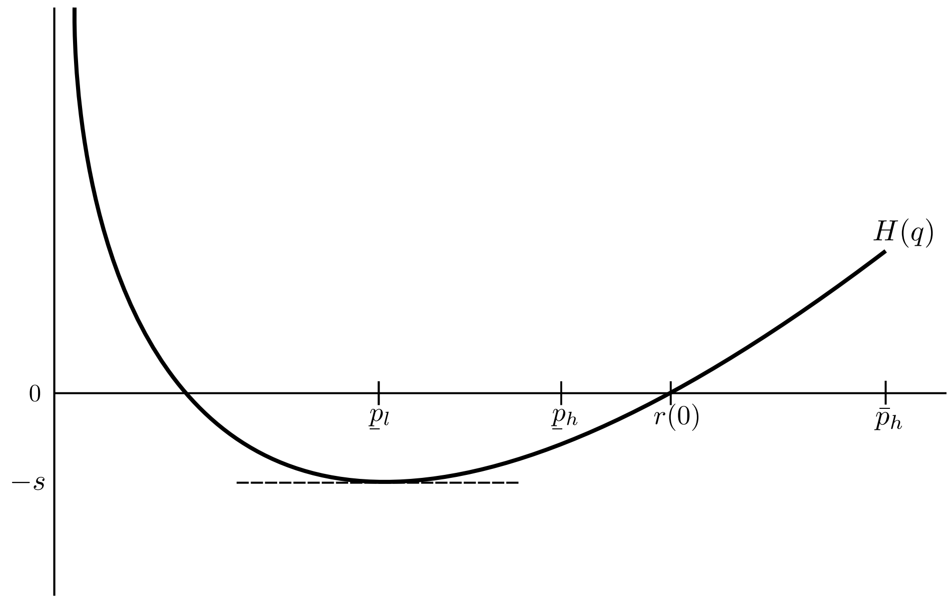


Figure 3.2: *$H(p)$ for parameter values satisfying equation* (3.9)

cost regime. Define

$$H(p) = \mathbb{E}GS(p, F(p; c_l)) - s. \tag{3.8}$$

Now the problem reduces to show that $H(p)$ has a unique root at $r_l > \underline{p}_h$. From our definition of $\mathbb{E}GS$, $H(p)$ is continuous for all $p > c_l$ since the term inside the logarithmic function is well defined and the algebraic sum of continuous functions is continuous. Therefore, if we can find two values at which $H(p)$ changes sign we can guarantee the existence of a root between those two values by the intermediate value theorem. Lemma 3.10 provides such values.

Lemma 3.10. *$H(p)$ satisfies the following properties:*

(i) $H(\bar{p}_h) > 0$*; and*

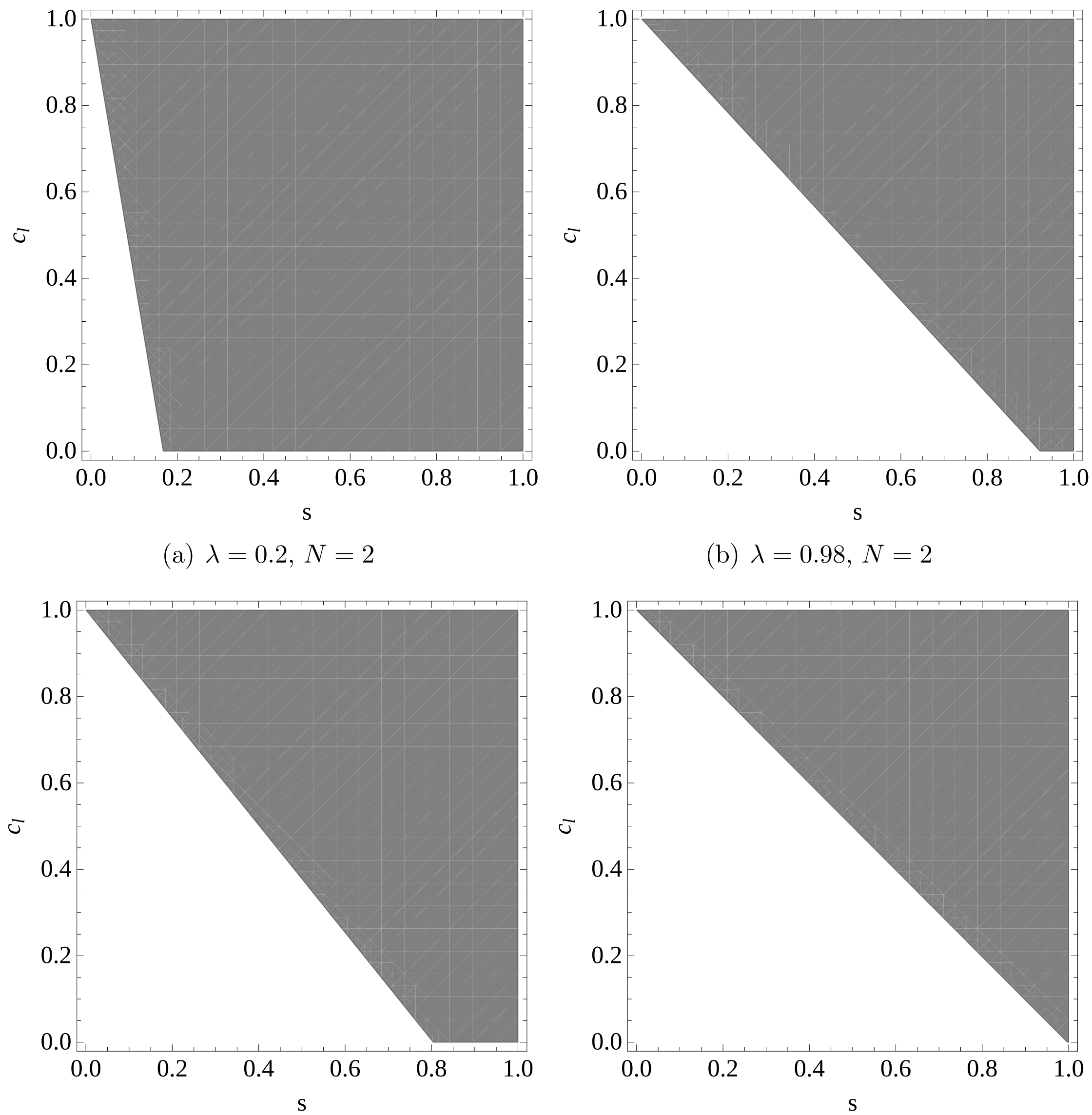
(ii) $H(\underline{p}_h) < 0$ *if and only if*

$$c_h - c_l < s - (\bar{p}_h - c_l)\left(\frac{1-\lambda}{N\lambda}\log\left(\frac{\bar{p}_h - c_l}{\underline{p}_h - c_l}\right) - \gamma(\lambda)\right). \tag{3.9}$$

The inequality in the previous Lemma is intimidating, to say the least. In order to bring some insight on the set of parameters for which this restriction is satisfied we can evaluate equation (3.9) at different parameter values. Figure 3.3 show these regions for different values of λ and N, after normalizing $c_h = 1$. For low values of λ and N, the restriction presented in equation (3.9) is fairly loose, but as the number of firms in the market and the fraction of shoppers increases it becomes harder to satisfy, and the pricing cdf presented in Proposition 3.9 ceases to be an equilibrium. Furthermore any equilibrium, if exist, must have a disconnected support from Lemma D.2 (see Appendix D). It is worth mentioning that, for all parameter values, there is always a level of s low enough such that restriction (3.9) is not satisfied, so as s goes to zero there is no connected equilibrium for the game. In the next section we will formalize this statement and characterize an equilibrium with such a property.

Lemma 3.11. *$H(p)$ is strictly increasing for all $p > \underline{p}_l$.*

The proof for Lemma 3.11 consist of two parts. First, we show that $H(p)$ is strictly convex, so $dH(p)$ is increasing in p. Then, we verify that $dH(p) = 0$ at $\underline{p}_l$, so it must be the case that $dH(p) > 0$ for $p > \underline{p}_l$, and therefore strictly increasing for values greater than $\underline{p}_h$, which implies that the root found in Lemma 3.10 must be unique. Figure 3.2 presents $H(p)$ satisfying the discussed properties. Therefore we can conclude that, if the condition in equation (3.9) is satisfied, $r(\phi(p)) \geq p$ for



(a) $\lambda = 0.2,\ N = 2$

(b) $\lambda = 0.98,\ N = 2$

(c) $\lambda = 0.2,\ N = 100$

(d) $\lambda = 0.98,\ N = 100$

Figure 3.3: (s, c_l) *satisfying equation* (3.9) *for selected values of* λ *and* N*.*

all p in the support; and the reservation prices formed as a response to the pricing equilibrium strategies are consistent with a continuous cdf.

3.3.4 Verification of the worst-case-scenario.

In Section 3.2.1 we formulated a Conjecture regarding the worst-case-scenario from the perspective of the consumers. In particular we hypothesized that, in equilibrium, consumers will be worse-off under the high cost regime. Now we proceed to verify this conjecture to finalize our characterization of the symmetric pricing equilibrium of the firms.

Lemma 3.12. *If $F(p; c_h)$ and $F(p; c_l)$ are symmetric NE with connected supports, then $F(p; c_h)$ first order stochastically dominates $F(p; c_l)$.*

Therefore, the distribution governing the minimum price observed under the high cost regime will also first order stochastic dominate the density under the low cost regime, which directly implies that, for all p_i the expected minimum price observed after search will be higher for a high marginal cost and equation (3.1) will achieve its minimum level at $\phi = 1$, thus verifying our initial conjecture.

3.4 Pricing Equilibrium with Disconnected Support

In this section we characterize an equilibrium for the case when restriction (3.9) is not satisfied. This characterization will prove to be easier than it seems as most of the heavy work is already covered in the previous section. A direct result from lemma 3.10 is that such an equilibrium cannot have a connected support, *i.e.*

some prices in the range $(\underline{p}_l, \bar{p}_h)$ will not be part of the equilibrium pricing strategies. Furthermore, we can easily see that the prices that are never charged in equilibrium must belong to the set of prices that trigger costly search. Assume, to the contrary, that there is a price $p' \in [\underline{p}_h, \bar{p}_h)$ outside the support of the equilibrium strategy, then any firm charging the next lower price in the support of F can safely increase it to p' without risking to lose any consumer engaging in search (costly or non-costly) as the probability of being the firm charging the smallest price remains the same. The same argument holds for any $p' \in (\underline{p}_l, r(0)]$. By definition of an equilibrium, all firms must be indifferent between charging $\underline{p}_l$ and $\bar{p}_h$, thus the relationship between these two prices is still determined by Lemma (7) and, as there is no costly search in a neighborhood around those two values its precise definition is still given by its corresponding values in the previous section.[5]

As $\underline{p}_h > r(0)$, some mass must be placed between these two prices, otherwise the equilibrium cdf will not be properly defined. Consider undercutting $\underline{p}_h$; then consumers with positive search costs observing $p = \underline{p}_h - \varepsilon$ will engage in costly search. These type of deviation will be dominated by $\underline{p}_h$ as no costly search will occur unless there is a mass point that increases market share by breaking a possible tie. Now consider setting a price just above $r(0)$. In this case, the increase in profits due to a higher price is negligible and the previously captive consumers will now engage in costly search. In order to overcome the potential loss in market share there must be a discrete increase in price above $r(0)$, defined by a. Therefore we know

[5]It is for this reason we do not include extra notation to distinguish between the lower and upper bound of the equilibrium strategy.

that any potential equilibrium cannot include prices in the interval $(r(0), a)$. Finally, all prices in $[a, \underline{p}_h)$ must be charged with zero probability, since any mass point can be undercut without increasing costly search beyond the level already implied by a tie at $p \in [a, \underline{p}_h)$. Figure 3.4 and Proposition 3.13 summarizes this description of a disconnected equilibrium.

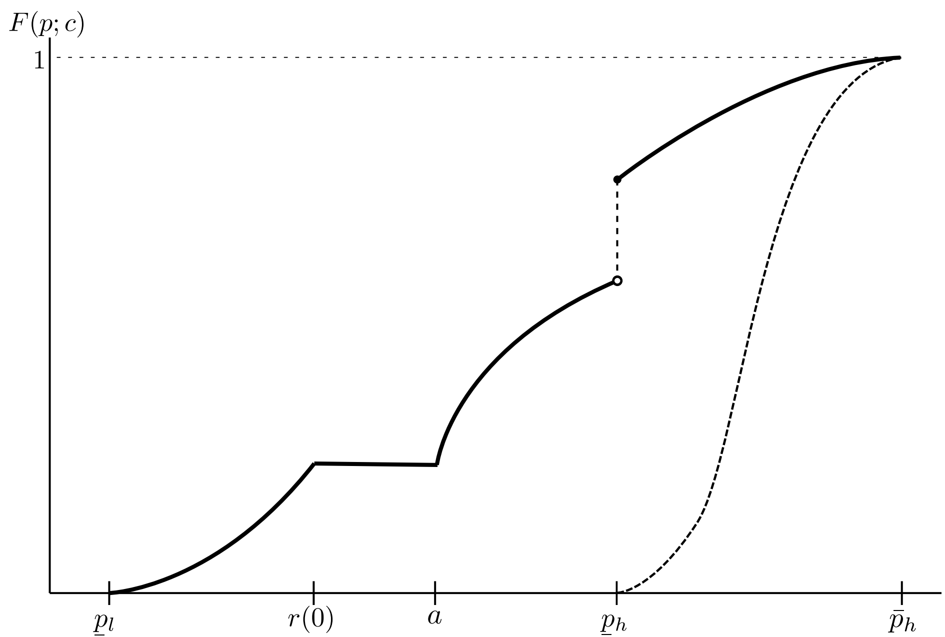


Figure 3.4: *Disconnected Pricing equilibrium cdf's for $c = c_h$ and $c = c_l$.*

Proposition 3.13. *Let $\hat{F}(p)$ be the segment of $F(p)$ in the interval $[\underline{p}_l, r(0)] \cup [\underline{p}_h, \bar{p}_h]$ and $\tilde{F}(p)$ be the segment of $F(p)$ in the interval $[a, \underline{p}_h)$. There exist a search cost $\underline{s} > 0$ such that for all $s \leq \underline{s}$, $F(p)$ is a symmetric NE with $a > r(0)$.*

3.5 Conclusion

We have incorporated ambiguity over production costs from the perspective of the consumers to the existing models of price search in order to explore the effect that this particular type of uncertainty has on the prices observed in the market. We find that ambiguity allow the firms to attain higher profits under the low cost regime and prices are stochastically higher than otherwise expected, thus creating pricing asymmetries.

We showed that, for high search costs, there exist a unique symmetric NE with connected support where no endogenous search takes place both for the high and low production cost regimes. In particular, under the low production cost regime the support of the symmetric Nash equilibrium overlaps with that of the high production cost as firms can induce the belief of a high cost in consumers allowing them to extract extraordinary profits. Additionally, we characterize and show the existence of an equilibrium with disconnected support that induces costly search with positive probability as the search cost decreases for the low cost regime. In this type of equilibrium firms charge prices in the lower end of the support to capture a higher fraction of the market and in an interval to the left of the lower bound of the support for the high cost regime. However, the loss of profits due to the induced search in the latter interval must be compensated by a discrete increase in prices above the reservation price triggering costly search, thus creating an interval of prices that are not charged in equilibrium.

There are two main critiques to the model developed in this chapter. First, as in our model of consumption of durable goods under ambiguity, the use of the *Gilboa-*

Schmeilder representation implies an extreme attitude towards ambiguity that, can be argued, generates beliefs over the production cost too simplistic – either high or low with probability one. This "naive" belief structure can be relaxed, however, by using the α-*MEU* generalization to the maxmin expected utility representation proposed by Ghirardato et al. (2004). We infer that such generalization will not change our findings drastically. If we allow the agents to consider a common convex combination between the worst and best case scenarios instead of simply considering the worst case when they cannot distinguish between supports from the originally observed price the upper bound of the equilibrium pricing distribution will decrease, as well as the reservation price for the perceived high cost regime, but the qualitative results will remain the same. On the other hand, if we allow heterogeneity among the consumers with respect to the particular convex combination used to form the belief under ambiguity we would have to work with more than two possible reservation prices. However, as argued by Stahl (1996), the upper bound of the pricing distribution would lie precisely at one of those reservation prices, maintaining the essence of our results unchanged albeit with a greater degree of complexity. The second critique is the lack of dynamics in our model; which renders it unable to analyze price adjustments and inflation dynamics. Unfortunately, introducing a dynamic dimension to the model is more challenging, the main complication being to ensure the dynamic consistency of the updating process using *Gilboa-Schmeilder* type preferences. In fact, the correct way to model dynamically consistent transition matrices for a discrete state variable that follows a Markov process is still a matter of debate in the literature. Recently, Baliga, Hanany, and Klibanoff (Forthcoming) have proposed a model for updating beliefs under ambiguity using a smooth ambiguity representation (see Klibanoff et al. (2005)) that places special attention to dynamic consistency. The task presented to the agents in their model is a simple forecasting problem with a quadratic loss function, but the model could be extended to include it in the model discussed in this chapter as the mechanism used to update beliefs about the production cost. This possible extension would greatly simplify the dynamics of the model by bringing the updating problem to the more familiar Bayesian paradigm while addressing the concerns of extreme ambiguity attitudes at the same time.

Appendices

Appendix A

Proofs of Lemmata and Propositions in Chapter 1

A.1 Proof of Proposition 1.1

For the case where no firm has a captive market, i.e. $\theta_j = 0$ for all $j = 1, 2$ the problem reduces to a Bertrand game with a unique equilibrium strategies of $p_j = 0$. Clearly the equilibrium profits are also equal to zero. For the case where $\theta_m = 0$ the two markets are disconnected with both firms setting a monopoly price on its respective share of the market, i.e $p_j = r$ and the monopoly profits are simply given by: $\pi_j = \theta_j r$. The interesting case is when both firms share some fraction of the market, $\theta_m > 0$ and at least one of the firms has a captive market $\theta_j > 0$ for some j. The proof for this case requires a little more work.

Let $\underline{s}_j$ and $\bar{s}_j$ denote the lower and upper bounds of the support of firm j's equilibrium price distribution G_j and $\underline{p}_j$ be as defined in equation (1.5). For the rest of the proof we are going to set $\underline{p}_1 = \max\{\underline{p}_j, \underline{p}_k\}$, without loss of generality.

Lemma A.1. *The support of the equilibrium strategies satisfies* $0 < \underline{p}_1 \leq \underline{s}_k < \bar{s}_k \leq r$ *for all* $k = 1, 2$.

Proof. $0 < \underline{p}_1$ comes immediate from assuming $\theta_j > 0$ for some j and $\theta_m > 0$. It is easy to see that this game has no pure-strategy equilibrium, specifically that $\underline{s}_j < \bar{s}_j$.

For any $p_k \in (\underline{p}_j, r]$ firm j has the incentive undercut firm k's price; at $p_k = \underline{p}_j$ firm j's best response is to set $p_j = r$. Making $p_k = \underline{p}_j$ no longer be a best response. By setting $p_1 = r$ firm 1 can guarantee itself a profit of $\theta_1 r$ while setting any price higher would yield profits of zero, ruling out prices greater than r. By definition, any price below $\underline{p}_1$ is strictly dominated by $p_1 = r$ and hence the support of G_1 will never include prices below $\underline{p}_1$. Therefore $\underline{p}_1 \leq \underline{s}_1 < \bar{s}_1 \leq r$.

Since firm 1 will never charge a price below $\underline{p}_1$, that is $G_1(p) = 0$ for all $p < \underline{p}_1$, firm 2 can guarantee itself a positive profit by setting $p_2 = \underline{p}_1$, ruling out prices higher than r that yield zero profits. Similarly, for any $p_2 < \underline{p}_1$, $\pi_2(p_2, G_1) = (\theta_2 + \theta_m)p_2$ is a strictly increasing function of p_2, and

$$\lim_{p_2' \uparrow \underline{p}_1} \pi_2(p_2', G_1) = (\theta_2 + \theta_m)\underline{p}_1.$$

In other words, firm 2 can always undercut $\underline{p}_1$ and get the shared market so the equilibrium profits must satisfy $\pi_2^* \geq (\theta_2 + \theta_m)\underline{p}_1$ and $\pi_2^* > \pi_2(p_2, G_1)$ for all $p_2 < \underline{p}_1$ which in turns implies that $G_2(p) = 0$ for all $p < \underline{p}_1$. Therefore $\underline{p}_1 \leq \underline{s}_2 < \bar{s}_2 \leq r$. ■

Let $\alpha_j(p)$ denote the size of a mass point in firm j's equilibrium distribution at $p_j = p$.

Lemma A.2. *If $\bar{s}_2 \leq \bar{s}_1$ and $\alpha_2(\bar{s}_1) = 0$, then $\bar{s}_1 = r$.*

Proof. From Lemma A.1 we know $\bar{s}_j \leq r$. Assume $\bar{s}_1 < r$ and $\bar{s}_2 < \bar{s}_1$, then $\pi_1(\bar{s}_1, G_2) = \theta_1 \bar{s}_1 < \theta_1 r$ and firm 1 will profit from transferring all the mass in $\bar{s}_1$ to r. Now assume $\bar{s}_1 < r$ and $\bar{s}_2 = \bar{s}_1$, therefore $\alpha_2(\bar{s}_2) = 0$ and $\pi_1(p_1, G_2) = \theta_1 \bar{s}_1 < \theta_1 r$

for all $\bar{s}_2 \leq p_1 < r$ with probability one and firm 1 will profit from transferring all the mass in $\bar{s}_1$ to r. Hence a contradiction. $\blacksquare$

Lemma A.3. *If* $\alpha_2(\bar{s}_1) = 0$, *then* $\bar{s}_1 = \bar{s}_2$

Proof. Assume $\bar{s}_2 < \bar{s}_1$. By Claim A.2, we know that $\bar{s}_1 = r$. Furthermore, since firm 2 is charging a price less than $\bar{s}_2$ with probability 1, firm 1 has no incentive to put any mass on the interval $[\bar{s}_2, r)$, therefore $\lim_{p \uparrow r} G_1(p) = \lim_{p \uparrow \bar{s}_2} G_1(p)$, which implies that $\pi_2(\bar{s}_2, G_1) < \lim_{p \uparrow r} \pi_2(p, G_1)$, i.e. since firm 1 puts no mass between $\bar{s}_2$ and r, firm 2 will profit for increasing its price from $\bar{s}_2$ giving a contradiction. Similarly, $\bar{s}_1 < \bar{s}_2$ implies $\lim_{p \uparrow \bar{s}_2} G_2(p) = \lim_{p \uparrow \bar{s}_1} G_2(p)$ and therefore $\pi_1(\bar{s}_1, G_2) < \lim_{p \uparrow \bar{s}_2} \pi_1(p, G_2)$. $\blacksquare$

Lemma A.4. $\alpha_2(r) = 0$.

Proof. Assume $\alpha_2(r) > 0$. First we show that under this assumption G_1 will have no mass point at r. Assume not, $\alpha_1(r) > 0$, then then firm 1 has a profitable deviation by transferring the mass at r to a price $p' \in (\underline{p}_1, r)$. Now that we have made the case for $\alpha_1(r) = 0$, note that $\pi_2(r, G_1) = \theta_2 r$ with probability one, so any $\alpha_2(r) > 0$ is strictly dominated by putting that weight on $\underline{p}_1$. $\blacksquare$

Lemma A.5. *For all* j, G_j *has no mass points in the interval* $[\underline{p}_1, r)$.

Proof. From Lemma A.1 we know that $\text{Supp}(G_j) \subseteq [\underline{p}_1, r]$. Assume $\alpha_j(p') > 0$ at some $p' \in (\underline{p}_1, r)$ for some $j = 1, 2$; it is then profitable for firm k to transfer all the mass in an ε-neighborhood above p' to a δ-neighborhood below p' since the reduction in profits due to a lower price is more than compensated by the increase in the probability of having the lower price. Assume $\alpha_1(\underline{p}_1) > 0$; then it is profitable for firm 2 to transfer mass above $\underline{p}_1$ to $\underline{p}_1$ and get a tie with positive probability, thus reducing firm 1's profits at $\underline{p}_1$ making firm 1 transfer all the mass at this point to r a profitable deviation. Similarly when $\alpha_2(\underline{p}_1) > 0$; it is worthwhile to firm 1 to transfer the mass above $\underline{p}_1$ to r. Therefore if $\alpha_j(p') > 0$ at some $p' \in [\underline{p}_1, r)$ there will be a region above p' where the other firm does not put weight, so $\alpha_j(p') > 0$ could not be part of any equilibrium. ■

Lemmas A.2 through A.5 together imply then that firm 2's equilibrium strategy has no mass points throughout its support, and $G_1(r) = G_2(r) = 1$. Therefore firm 1's equilibrium profit is given by:

$$\pi_1^* = \int_{\underline{p}_1}^{r} \left[G_2(p_1)\theta_1 p_1 + (1 - G_2(p_1))(\theta_1 + \theta_m)p_1\right] g_1(p_1)dp_1$$

where the integration limits are given by Lemma A.1. Since all the prices charged in firm 1's equilibrium strategy must yield the same expected value for firm 1 to be willing to mix among them it must be the case that:

$$\pi_1^* = G_2(p_1)\theta_1 p_1 + (1 - G_2(p_1))(\theta_1 + \theta_m)p_1.$$

Using the fact that $G_2(r) = 1$ we can obtain the value of $\pi_1^* = \theta_1 r$. Substituting the value of π_1^* in the previous equation and solving for G_2 gives:

$$G_2(p) = \frac{\theta_1 + \theta_m}{\theta_m} - \frac{\theta_1 r}{\theta_m p}.$$

This expression equals zero at $\frac{\theta_1 r}{\theta_1 + \theta_m} = \underline{p}_1$, as defined in equation (1.5).

From Lemma A.5 we also know that firm 1's equilibrium strategy has no mass points in the interval $[\underline{p}_1, r)$. As above we can obtain the equilibrium strategy for firm 1 by

$$G_1(p_2)\theta_2 p_2 + (1 - G_1(p_2))(\theta_2 + \theta_m) = \pi_2^*.$$

Since firm 1 will never charge a price below $\underline{p}_1$, setting $G_1(\underline{p}_1) = 0$ we get:

$$\pi_2^* = (\theta_2 + \theta_m)\underline{p}_1 = \frac{(\theta_2 + \theta_m)\theta_1 r}{(\theta_1 + \theta_m)}.$$

Substituting this result in our indifference condition for π_2^* give us the result for G_1

$$G_1(p) = 1 - \left[\frac{(\theta_2 + \theta_m)\theta_1 r}{(\theta_1 + \theta_m)\theta_m p} - \frac{\theta_2}{\theta_m}\right].$$

Since the $\lim_{p\uparrow r} G_1(p) = \frac{\theta_2+\theta_m}{\theta_1+\theta_m}$, for G_1 to be well define it must be the case that $\alpha_1(r) = 1 - \frac{\theta_2+\theta_m}{\theta_1+\theta_m}$ obtaining the result in equations (1.6) and (1.7).

Uniqueness comes from the fact that our expression for G_1 and G_2 are the only distributions that satisfy Lemmas A.2 through A.5 and the indifference condition for an equilibrium. ■

A.2 Proof of Lemma 1.2

In order to establish the result on Lemma 1.2 we need to use the fact summarized in Claim A.6, i.e. for all levels of advertising set by firm j, the profit for firm k when forced to remain small are greater ore equal than the profits when large for $m_k = 0$.

Claim A.6. $\pi_k^S(m_j, 0) \geq \pi_k^L(m_j, 0)$ *for all* $m_j \in [0, 1]$

Proof. Evaluating the profit functions obtained in Lemma 1.1 using equations (1.1) - (1.3) we obtain:

$$\pi_k^S(m_j, 0) = \frac{s_k + s_m}{m_j + (s_j + s_m)(1 - m_j)} \left[(1 - s_0)m_j + s_j\right] r - \phi(0)$$

$$\pi_k^L(m_j, 0) = s_k(1 - m_j)r - \phi(0)$$

For the claim to be true it suffices to show

$$s_k(1 - m_j) \leq \frac{s_k + s_m}{m_j + (s_j + s_m)(1 - m_j)} \left[(1 - s_0)m_j + s_j\right]$$

Rewrite the previous expression as

$$\begin{aligned}
\left[s_k(1 - m_j)\right]\left[(1 - s_0)m_j + s_j + (s_k m_j + s_m)\right] &\leq s_k(1 - m_j))\left[(1 - s_0)m_j + s_j\right] \\
&\quad +(s_k m_j + s_m)\left[(1 - s_0)m_j + s_j\right] \\
\left[s_k(1 - m_j)\right](s_k m_j + s_m) &\leq (s_k m_j + s_m)\left[(1 - s_0)m_j + s_j\right]
\end{aligned}$$

If $(s_k m_j + s_m) = 0$ the condition is satisfied with strict equality for all m_j. If $(s_k m_j + s_m) \neq 0$ the conditions is satisfied since, by assumption $s_j \geq s_k$ and $(1 - s_0) \geq 0$. ■

Consider the case for $\hat{m}_k^{(m_j)} > 0$. Assume $\partial_{m_k}\pi_k^L|_{m_k=\hat{m}_k^{(m_j)}} \leq 0$ and $\partial_{m_k}\pi_k^S|_{m_k=\hat{m}_k^{(m_j)}} \geq 0$. The strict concavity of the profit functions implies that $\pi_k^L(m_j, m_k') > \pi_k^L(m_j, \hat{m}_k^{(m_j)}) = \pi_k^S(m_j, \hat{m}_k^{(m_j)}) > \pi_k^S(m_j, m_k')$ for some $m_k' < \hat{m}_k^{(m_j)}$. However, from Claim A.6 we know that $\pi_k^L(m_j, 0) \leq \pi_k^S(m_j, 0)$ and since the profit functions are continuous that would require for the profits to equal each other at some value of m_k less than $\hat{m}_k^{(m_j)}$, yielding a contradiction since $\hat{m}_k^{(m_j)}$ is uniquely defined by equation (1.14).

The case when $\hat{m}_k^{(m_j)} = 0$ arises only when $s_j = s_k$ and either $s_j + s_m = 1$ or $m_j = 0$. By substituting $s_j = s_k$ and $s_j + s_m = 1$ in equation (1.10) we can easily see that $\partial_{m_k} \pi_k^S |_{m_k = \hat{m}_k^{(m_j)}} < 0$. For the case when $s_j = s_k$ and $m_j = 0$ the condition specified in Lemma 1.2 reduces to

$$(1 - 2s_j - s_m) \leq \frac{\phi'(0)}{r} \Rightarrow \frac{s_j(1 - 2(s_j + s_m))}{(s_j + s_m)} < \frac{\phi'(0)}{r},$$

which is satisfied since $s_j/(s_j + s_m) < 1$ and $(1 - 2s_j - s_m) \leq \phi'(0)/r$ implies $(1 - 2(s_j + s_m)) \leq \phi'(0)/r$. The second part of Lemma 1.2 can be proven using similar arguments as above. ■

A.3 Proof of Proposition 1.15

First we verify that $m_k^* > \hat{m}_k^{(0)}$ is actually a best response to $m_j^* = 0$. As shown in Lemma 1.2 this is always the case when $\partial_{m_k} \pi_k^S |_{m_k = \hat{m}_k^{(0)}} \geq 0$ or:

$$\frac{r s_j \left(1 - (2s_j + 2s_m)\right)}{s_j + s_m} - \phi'(\hat{m}_k^{(0)}) \geq 0$$

for $m_j = 0$. Plugging $s_m = s_k = 0$ gives the first sufficient condition.

Now we proceed to verify that $m_j^* = 0$ is a best response for $m_k^* > \hat{m}_k^{(0)}$. It must be the case that, when facing the small firm's problem, firm j finds $m_j = 0$ optimal, that is: $\partial_{m_j} \pi_j^S |_{m_j = 0} \leq 0$ or:

$$-\frac{r \left(1 - 2s_j - 2s_m\right) \left(-s_k + m_k \left(-1 + s_j + s_k + s_m\right)\right)}{s_k + s_m - m_k \left(-1 + s_k + s_m\right)} - \phi'(0) \leq 0$$

for $m_j = 0$. Plugging $s_m = s_k = 0$ gives the second sufficient condition. Note that for $s_m = s_k = 0$ firm j's best response as the small firm is independent of m_k so it holds for all m_k and in particular for $m_k^* > \hat{m}_k^{(0)}$.

Finally, we need to verify that firm j prefers to remain small as a response to $m_k^* > \hat{m}_k^{(0)}$. Define $\hat{m}_j^{(m_k)}$ as the level of advertising required for firm j to match firm k's captive market size. Given $m_k^* > \hat{m}_k^{(0)}$, it is clear that $\hat{m}_j^{(m_k^*)} > 0$. Additionally, for $s_j > 0$ and $s_k = 0$, $\hat{m}_j^{(m_k^*)} < m_k^*$ since firm j's captive market is still greater than zero after firm k's advertises. Firm k's best response to $m_j^* = 0$ is obtained from equation (1.12):

$$r(1 - s_j) = \phi'(m_k^*).$$

Assume the previous equation has an interior solution with $m_k^* < 1$. Using Lemma 1.2, firm j will find optimal to remain small if

$$r(1 - s_j)(1 - m_k^*) \leq \phi'(\hat{m}_j^{(m_k^*)}) \tag{A.1}$$

or equivalently if

$$(1 - m_k^*) \leq \frac{\phi'(\hat{m}_j^{(m_k^*)})}{\phi'(m_k^*)}.$$

Since $\hat{m}_j^{(m_k^*)} < m_k^*$ the condition is always satisfied for all $m_k^* > 0$ which is in fact the case for $m_k^* > \hat{m}_k^{(0)} = s_j > 0$.

Assume firm k's maximization problem has a corner solution with $m_k^* = 1$. In this case equation (A.1) becomes

$$\phi'(\hat{m}_j^{(1)}) \geq 0;$$

which is always true by the convexity of ϕ. ■

A.4 Proof of Proposition 1.16

First, we solve the case for the small firm, denoted k without loss of generality. By setting $s_0 = 0$, which trivially implies $s_m = s_j = s_k = 0$ and using the definitions for the market shares after advertising – equations (1.1)-(1.3) – in equation (1.10) we have:

$$\partial_{m_k} \pi_k^s = 1 - 2m_k - \frac{\gamma}{1 - m_k}.$$

Clearly $\partial_{m_k} \pi_k^s > 0$ at $m_j = 0$. For $s_0 = 0$, $\hat{m}_k^{(m_j)} = m_j$, therefore, if $\partial_{m_k} \pi_k^S|_{m_k = m_j} \geq 0$ we fall in the case where the small firm has a corner solution at $m_k = m_j$ (see Figure 1.1(b)). Otherwise, the optimal advertising level for the small firm is given by $\partial_{m_k} \pi_k^S = 0$, or equivalently

$$m_k = \frac{2 - \sqrt{1 + 8\gamma}}{4}.$$

Denote this optimal level of advertising by Δ_1. Since π_k^S is strictly concave, $\partial_{m_k} \pi_k^S|_{m_k = m_j} \geq 0$ for all $m_j \leq \Delta_1$ and the best response function for the small firm is given by:

$$BR_k^S(m_j) = \begin{cases} m_j & \text{if } m_j \leq \Delta_1 \\ \Delta_1 & \text{otherwise} \end{cases} \tag{A.2}$$

Now consider the problem of the large firm, denoted by j. Similarly as above, $\hat{m}_j^{(m_k)} = m_k$, and we can find the first order condition for the large firm from equation (1.12):

$$\partial_{m_j} \pi_j^L = (1 - m_k) - \frac{\gamma}{1 - m_j}.$$

Note that, as m_j tends to 1, $\partial_{m_j} \pi_j^L$ approaches negative infinity which implies $m_j < 1$. If $\partial_{m_j} \pi_j^L|_{m_j = m_k} \leq 0$, the large firm will prefer to be small after advertising. Since we are restricting firm j to remain large, the only way for it to be small and large at the same time is if $m_j = m_k$ (see Figure 1.1(a)). Otherwise, the large firm will remain strictly larger than the small firm and the optimal level of advertising is:

$$m_j = 1 - \frac{\gamma}{1 - m_k}.$$

Denote this optimal level of advertising for the large firm by $\Phi(m_k)$. Therefore, the best response function for the large firm when restricted to remain large is given by:

$$BR_j^L(m_k) = \begin{cases} m_k & \text{if } m_k \geq \Phi(m_k) \\ \Phi(m_k) & \text{otherwise} \end{cases} \tag{A.3}$$

In order to obtain the unrestricted best response function for firm j, define Δ_2 as the value of m_k such that $\partial_{m_j} \pi_j^L|_{m_j = m_k} = 0$. From our particular case, $\Delta_2 = 1 - \sqrt{\gamma}$. Note that $0 < \Delta_1 < \Delta_2 < 1$. When $m_k < \Delta_1$, firm j will not voluntarily choose to be the small firm as shown by Lemma 1.2. Hence $m_k < \Delta_1 \Rightarrow BR_j(m_k) = \Phi(m_k)$. Similarly, when $m_k > \Delta_2$, firm j will not choose to be large, and it will be facing the small firm problem with $m_j = \Delta_1$, since firm k is setting an advertising level greater than Δ_1. For $\Delta_1 \leq m_k \leq \Delta_2$, firm j will choose to be either small or large depending on the value of π_j^L and π_j^S. The indifference point can be found by noting that π_j^S is independent of m_k and π_j^L is decreasing in m_k so it exist $\tilde{m}_k$ such that:

$$\Phi(\tilde{m}_k)(1 - \tilde{m}_k) + \gamma \log(1 - \Phi(\tilde{m}_k)) = \Delta_1(1 - \Delta_1) + \gamma \log(1 - \Delta_1)$$

with $\Delta_1 < \tilde{m}_k < \Delta_2$, and the unrestricted best response function for firm j is given by:

$$BR_j(m_k) = \begin{cases} \Delta_1 & \text{if } m_k \geq \tilde{m}_k \\ \Phi(m_k) & \text{otherwise} \end{cases} \tag{A.4}$$

By direct inspection of the response functions given by equations (A.2) and (A.4) we can see that the only equilibrium in pure strategies is when the small firm sets $m_k = \Delta_1$ and the large firm sets

$$m_j = \Phi(\Delta_1) = 2\Delta_1.$$

Finally, from equation (1.1) we can obtain the size of the market after advertising, $s_1 = 1 - \gamma$. Since neither firm will ever charge a price above the reservation price in equilibrium this fraction also corresponds to the proportion of consumers buying one of the products after advertising. $\blacksquare$

Appendix B

Proofs of Lemmata and Propositions in Chapter 2

B.1 Proof of Corollary 2.2

From the definition of $\pi(x)$ in equation 2.1 we have:

$$\frac{\partial \pi(x)}{\partial x} = -(q - \psi(1))\varphi e^x.$$

With $\psi(1) = \frac{\sigma^2}{2} + \mu$. Thus the derivative of $\pi(x)$ with respect to x is negative, provided $q > \psi(1)$, which is the same restriction imposed by equation (2.3). $\pi(x)$ changes sign as:

$$\lim_{x \to \infty} \pi(x) = -\infty, \text{ and}$$

$$\lim_{x \to -\infty} \pi(x) = qU > 0.$$

Equation (2.5) follows from direct evaluation of the integral defining $W(x)$ in Theorem 2.1. By doing a few simple manipulations and rearrangement we write

$$V_2^* = U - Ue^{\beta^+(x-x^*)} - \varphi\left(e^x - e^{\beta^+(x-x^*)+x^*}\right)\frac{(q-\psi)}{q}\left(\frac{\beta^-}{\beta^- - 1}\right)\left(\frac{\beta^+}{\beta^+ - 1}\right).$$

Using the fact that $\frac{q}{(q-\psi)} = \left(\frac{\beta^-}{\beta^- - 1}\right)\left(\frac{\beta^+}{\beta^+ - 1}\right)$ we can further simplify V_2^* to

$$V_2^* = U - Ue^{\beta^+(x-x^*)} - \varphi e^x + e^{\beta^+(x-x^*)}\varphi e^{x^*}$$

and equation (2.6) follows by substituting equation (2.5) in the previous expression.

■

B.2 Proof of Lemma 2.3

From equation (2.7) we have:

$$\partial_x \Pi(x) = -(1+\varphi)e^x + \frac{U\beta^+}{\beta^+ - 1} e^{\beta^+(x-x^*)} \tag{B.1}$$

Using equation (2.5), rewrite equation B.1 as:

$$\partial_x \Pi(x) = -(1+\varphi)e^x \left[\frac{1}{e^{x^*}} \frac{qU(\beta^- - 1)}{\varphi\beta^-(q-\psi(1))}\right] + \frac{U\beta^+}{\beta^+ - 1} e^{\beta^+(x-x^*)}.$$

From the fact that $\frac{q}{q-\psi(1)} = \frac{\beta^+}{\beta^+-1}\frac{\beta^-}{\beta^- -1}$, the previous expression can rewritten as:

$$\partial_x \Pi(x) = \frac{U\beta^+}{\beta^+ - 1}\left(Z^{\beta^+} - \frac{1+\varphi}{\varphi} Z\right). \tag{B.2}$$

with $Z = e^{x-x^*}$. Since $\beta^+ > 1$, Z^{β^+} is a convex function of Z while $\frac{1+\varphi}{\varphi}Z$ is linear in Z. At $x = x^*$, $Z = 1$ and

$$\left(Z^{\beta^+} - \frac{1+\varphi}{\varphi} Z\right) = 1 - \frac{1+\varphi}{\varphi} < 0$$

since $\varphi > 0$ by definition. Furthermore,

$$\lim_{x \to -\infty} \left(Z^{\beta^+} - \frac{1+\varphi}{\varphi} Z\right) = 0$$

Since a convex function and a linear function cross in at most two points, it must be the case that $\left(Z^{\beta^+} - \frac{1+\varphi}{\varphi} Z\right) < 0$ for all $x \leq x^*$, which establishes the result from equation (B.2) $\blacksquare$

B.3 Proof of Lemma 2.5

Using equation (2.7), and after some manipulations we have

$$G(x) = U - \frac{e^x(1+\varphi)(\beta^- - 1)}{\beta^-} + \frac{U(\beta^- - \beta^+)}{(\beta^+ - 1)\beta^-} e^{\beta^+(x-x^*)} \tag{B.3}$$

Let x_* be a candidate solution for $G(x_*) = 0$. Using equation (2.5) we can then write:

$$U - \frac{e^{x_*}(1+\varphi)\left(\beta^- - 1\right)}{\beta^-}\left(\frac{1}{e^{x^*}}\right)\left(\frac{qU}{(q-\psi)\varphi}\frac{\left(\beta^- - 1\right)}{\beta^-}\right) + \frac{U\left(\beta^- - \beta^+\right)}{\left(\beta^+ - 1\right)\beta^-}e^{\beta^+(x_* - x^*)} = 0,$$

which can be simplified to

$$U - \frac{\left(\beta^- - 1\right)\beta^+}{\beta^-\left(\beta^+ - 1\right)}\left(\frac{(1+\varphi)U}{\varphi}\right)e^{x_* - x^*} + \frac{U\left(\beta^- - \beta^+\right)}{\left(\beta^+ - 1\right)\beta^-}e^{\beta^+(x_* - x^*)} = 0.$$

The previous expression is equivalent to:

$$1 - \frac{\left(\beta^- - 1\right)\beta^+}{\beta^-\left(\beta^+ - 1\right)}\left(\frac{(1+\varphi)}{\varphi}\right)e^{x_* - x^*} + \frac{\left(\beta^- - \beta^+\right)}{\left(\beta^+ - 1\right)\beta^-}e^{\beta^+(x_* - x^*)} = 0. \tag{B.4}$$

Set $\delta = x_* - x^*$ and rewrite equation (B.4) in the form of $e^{\delta}F(\delta)$ with

$$F(\delta) = e^{-\delta} - \frac{\left(\beta^- - 1\right)\beta^+}{\beta^-\left(\beta^+ - 1\right)}\left(\frac{(1+\varphi)}{\varphi}\right) + \frac{\left(\beta^- - \beta^+\right)}{\left(\beta^+ - 1\right)\beta^-}e^{\left(\beta^+ - 1\right)\delta}.$$

Note that $F(\delta) = 0$ has a negative solution at δ^* and changes sign as δ crosses δ^* if and only if $G(x_*) = 0$ at some $x_* < x^*$ and changes sign at x_*.

Claim B.1. *At $\delta = 0$, $F(0) < 0$.*

Proof. Evaluating $F(\delta)$ at zero,

$$\begin{aligned} 1 - \frac{\left(\beta^- - 1\right)\beta^+}{\beta^-\left(\beta^+ - 1\right)}\left(\frac{1+\varphi}{\varphi}\right) + \frac{\left(\beta^- - \beta^+\right)}{\left(\beta^+ - 1\right)\beta^-} &< 0 \\ \frac{\left(\beta^+ - 1\right)\beta^- + \left(\beta^- - \beta^+\right)}{\left(\beta^- - 1\right)\beta^+} &< \frac{1+\varphi}{\varphi} \\ 1 &< \frac{1+\varphi}{\varphi} \end{aligned}$$

Which is clearly satisfied for $\varphi > 0$. ■

Hence $F(\delta)$ is negative in a left neighborhood of 0 since F is a continuous function. The restriction in the parameters stated in equation (2.3) imply that $\beta^+ > 1$. Therefore

$$\lim_{\delta \to -\infty} F(\delta) = +\infty.$$

Again, by continuity of the function, $F(\delta)$ must have a zero $\delta^* < 0$. Furthermore,

$$\frac{\partial^2 F(\delta)}{\partial \delta^2} = e^{-\delta} + \frac{e^{\delta\left(-1+\beta^+\right)}\left(\beta^- - \beta^+\right)\left(\beta^+ - 1\right)}{\beta^-} > 0,$$

thus $F(\delta)$ is convex, which guarantees that F changes sign as it passes δ^*, and the uniqueness of δ^*. ■

B.4 Proof of Corollary 2.6

Equation (2.10) is obtained using Lemma 2.5 and equation (B.3) therein evaluated at x_*. Equation (2.11) follows from directly calculating the integral in Theorem 2.4

$$V_1^*(x) = \left[U - e^{x_*}(1+\varphi) + \frac{U}{(\beta^+ - 1)} e^{\beta^+(x_* - x^*)}\right] e^{\beta^-(x - x_*)},$$

and recognizing the term in brackets as $\Pi(x_*)$ in equation (2.7). ■

B.5 Proof of Lemma 2.7

First, let us establish the following result:

Claim B.2. $\partial_\mu x^*_{(\mu)} = -\frac{\partial_\mu \beta^+_{(\mu)}}{\left(\beta^+_{(\mu)} - 1\right)\beta^+_{(\mu)}}$

Proof. Using the fact that $\frac{q}{(q-\psi_\mu(1))} = \frac{\beta^+_{(\mu)}}{\left(\beta^+_{(\mu)}-1\right)}\frac{\beta^-_{(\mu)}}{\left(\beta^-_{(\mu)}-1\right)}$ rewrite equation (2.15) as:

$$e^{x^*_{(\mu)}} = \frac{U\beta^+_{(\mu)}}{\varphi\left(\beta^+_{(\mu)}-1\right)} \tag{B.5}$$

Taking the derivative with respect to μ on both sides of the previous expression we have:

$$\partial_\mu x^*(\mu) = -\left(\frac{1}{\left(\beta^+_{(\mu)}-1\right)}\right)\frac{U\partial_\mu\beta^+_{(\mu)}}{\varphi\left(\beta^+_{(\mu)}-1\right)}e^{-x^*_{(\mu)}}$$

The result follows from substituting $e^{-x^*_{(\mu)}}$ using equation (B.5) in the previous expression. ■

Taking the derivative of equation (2.17) with respect to μ we obtain:

$$\partial_\mu\Pi(x|\mu) = \left(\left(x-x^*_{(\mu)}\right)\partial_\mu\beta^+_{(\mu)} - \beta^+_{(\mu)}\left[\partial_\mu x^*_{(\mu)} + \frac{\partial_\mu\beta^+_{(\mu)}}{\left(\beta^+_{(\mu)}-1\right)\beta^+_{(\mu)}}\right]\right)\frac{Ue^{\beta^+_{(\mu)}\left(x-x^*_{(\mu)}\right)}}{\left(\beta^+_{(\mu)}-1\right)}$$

From Claim B.2, the term in brackets is equal to zero and

$$\partial_\mu\Pi(x|\mu) = \left(\left(x-x^*_{(\mu)}\right)\partial_\mu\beta^+_{(\mu)}\right)\frac{Ue^{\beta^+_{(\mu)}\left(x-x^*_{(\mu)}\right)}}{\left(\beta^+_{(\mu)}-1\right)} > 0$$

since the option to sell the durable good is evaluated in its continuation region, $x < x^*_{(\mu)}$, $\beta^+_{(\mu)} > 1$ by assumption, and it is easy to verify that $\partial_\mu\beta^+_{(\mu)} < 0$ from its definition. ■

B.6 Proof of Lemma 2.9

Before proceeding with the proof, let us establish a useful result.

Claim B.3.

$$\partial_\eta \Pi(y_*) - \lambda^-_{(\eta)} \Pi(y_*) \partial_\eta y_* = 0. \tag{B.6}$$

Proof. Combining equations (2.16) and (2.17)

$$\Pi(y_*) = U - (1+\varphi)e^{y_*} + \frac{U}{\left(\beta^+_{(\bar{\mu})} - 1\right)} e^{\beta^+_{(\bar{\mu})}(y_* - y^*)}. \tag{B.7}$$

Taking the derivative with respect to μ we have:

$$\partial_\eta \Pi(y_*) = \left(\frac{e^{\beta^+_{(\bar{\mu})}(y_* - y^*)} U \beta^+_{(\bar{\mu})}}{\beta^+_{(\bar{\mu})} - 1} - (1+\varphi)e^{y_*} \right) \partial_\eta y_*$$

Plugging it in equation (B.6)

$$\left(\frac{e^{\beta^+_{(\bar{\mu})}(y_* - y^*)} U \beta^+_{(\bar{\mu})}}{(\beta^+_{(\bar{\mu})} - 1)\lambda^-_{(\eta)}} - \frac{(1+\varphi)e^{y_*}}{\lambda^-_{(\eta)}} - \Pi(y_*) \right) \lambda^-_{(\eta)} \partial_\eta y_* = 0.$$

Substituting $\Pi(y_*)$ using equation (B.7), and after some basic algebra, we can write the previous expression as:

$$\left(U - \frac{e^{y_*}(1+\varphi)\left(\lambda^-_{(\eta)} - 1\right)}{\lambda^-_{(\eta)}} + \frac{U\left(\lambda^-_{(\eta)} - \beta^+_{(\bar{\mu})}\right)}{\left(\beta^+_{(\bar{\mu})} - 1\right)\lambda^-_{(\eta)}} e^{\beta^+_{(\bar{\mu})}(y_* - y^*)} \right) \lambda^-_{(\eta)} \partial_\eta y_* = 0.$$

By equation (2.21) we can immediately see that the term in parenthesis is equal to zero. ■

Rewrite $\mathscr{V}_1^*$ in equation (2.20) as a function of an arbitrary drift $\eta \in [\nu - \sigma\kappa, \nu + \sigma\kappa]$

$$\mathscr{V}_1^*(x|\bar{\mu}) = \Pi(y_*) e^{\lambda^-_{(\eta)}(x - y_*)}. \tag{B.8}$$

Applying the chain rule to equation (B.8)

$$\begin{aligned}\partial_\eta \mathscr{V}_1^*(x|\bar{\mu}) &= e^{\lambda^-_{(\eta)}(x-y_*)}\left(\partial_\eta \Pi(y_*) + \Pi(y_*)\left((x-y_*)\partial_\eta \lambda^-_{(\eta)} - \lambda^-_{(\eta)}\partial_\eta y_*\right)\right)\\ &= e^{\lambda^-_{(\eta)}(x-y_*)}\left(\partial_\eta \Pi(y_*) - \lambda^-_{(\eta)}\Pi(y_*)\partial_\eta y_{*)} + (x-y_*)\Pi(y_*)\partial_\eta \lambda^-_{(\eta)}\right),\end{aligned}$$

and using Claim B.3

$$\partial_\eta \mathscr{V}_1^*(x|\bar{\mu}) = e^{\lambda^-_{(\eta)}(x-y_*)}\left((x-y_*)\Pi(y_*)\partial_\eta \lambda^-_{(\eta)}\right) < 0$$

for all $x > y_*$. The latter follows from the fact that $\Pi(y_*) > 0$ since, by definition, it is the value of the termination payoff evaluated at the exercise threshold and it will never be optimal to exercise an option with negative instant payoff, and we can easily verify that $\partial_\eta \lambda^-_{(\eta)} < 0$. ■

B.7 Proof of Proposition 2.11

Equation (2.15) give us the close form solution for y^*. By taking the derivative with respect to $\bar{\mu}$ we obtain:

$$\partial_{\bar{\mu}} y^* = \frac{\partial_{\bar{\mu}}\beta^-_{(\bar{\mu})}}{\beta^-_{(\bar{\mu})}\left(\beta^-_{(\bar{\mu})} - 1\right)} + \frac{1}{(q-\chi(1))}.$$

Using the definition $\beta^- = -\frac{\mu}{\sigma^2} - \frac{\sqrt{\mu^2+2q\sigma^2}}{\sigma^2}$ we can directly calculate the value of the derivative of y^*, which is given by:

$$\partial_{\bar{\mu}} y^* = \frac{\sigma^2}{2q\sigma^2 + \bar{\mu}^2 - \bar{\mu}\sqrt{\bar{\mu}^2+2q\sigma^2} - \sigma^2\sqrt{\bar{\mu}^2+2q\sigma^2}},$$

We want to show that $\partial_{\bar{\mu}} y^* > 0$, or equivalently that

$$\begin{aligned} 2q\sigma^2 + \bar{\mu}^2 - \bar{\mu}\sqrt{\bar{\mu}^2 + 2q\sigma^2} - \sigma^2\sqrt{\bar{\mu}^2 + 2q\sigma^2} &> 0 \\ 2q\sigma^2 + \bar{\mu}^2 &> \sqrt{\bar{\mu}^2 + 2q\sigma^2}\left(\bar{\mu} + \sigma^2\right) \\ \sqrt{\bar{\mu}^2 + 2q\sigma^2} &> \left(\bar{\mu} + \sigma^2\right). \end{aligned}$$

If $(\bar{\mu} + \sigma^2) < 0$, the proof is complete as the right hand side of the previous equation must be positive. On the other hand, if $(\bar{\mu} + \sigma^2)$ is positive we can square both sides of the previous equation and see that the condition is verified as long as $2q > 2\bar{\mu} + \sigma^2$, which corresponds to the restriction on parameters set by equation (2.3). ■

B.8 Proof of Proposition 2.12

Before proving the statement in the proposition we establish a necessary and sufficient condition on $(\tilde{y}_* - \tilde{x}^*)$ for $\partial_{\bar{\eta}}\tilde{y}_* > 0$ and $\partial_{\underline{\mu}}\tilde{y}_* > 0$.

Claim B.4. $\frac{\tilde{\lambda}^- - \tilde{\beta}^+}{\tilde{\lambda}^-} e^{\tilde{\beta}^+(\tilde{y}_* - \tilde{x}^*)} < 1$.

Proof. From equation (2.22) we have:

$$\frac{\left(\tilde{\lambda}^- - \tilde{\beta}^+\right)}{\tilde{\lambda}^-} e^{\tilde{\beta}^+(\tilde{y}_* - \tilde{x}^*)} = \frac{(1+\varphi)\left(\tilde{\lambda}^- - 1\right)\left(\tilde{\beta}^+ - 1\right)}{U\tilde{\lambda}^-} e^{\tilde{y}_*} - \left(\tilde{\beta}^+ - 1\right)$$

Using equation (B.5) we obtain:

$$\frac{\left(\tilde{\lambda}^- - \tilde{\beta}^+\right)}{\tilde{\lambda}^-} e^{\tilde{\beta}^+(\tilde{y}_* - \tilde{x}^*)} = \left(\frac{1+\varphi}{\varphi}\right)\frac{\tilde{\beta}^+\left(\tilde{\lambda}^- - 1\right)}{\tilde{\lambda}^-} e^{\tilde{y}_* - \tilde{x}^*} - \left(\tilde{\beta}^+ - 1\right).$$

Let $Z(y) = e^{y - \tilde{x}^*}$, and rewrite the previous expression as

$$\frac{\left(\tilde{\lambda}^- - \tilde{\beta}^+\right)}{\tilde{\lambda}^-} \left(Z\left(\tilde{y}_*\right)\right)^{\tilde{\beta}^+} = \left(\frac{1+\varphi}{\varphi}\right)\frac{\tilde{\beta}^+\left(\tilde{\lambda}^- - 1\right)}{\tilde{\lambda}^-} Z\left(\tilde{y}_*\right) - \left(\tilde{\beta}^+ - 1\right) \tag{B.9}$$

Note that as y tends to negative infinity, $Z(y)$ approaches zero, and $Z(y) < 1$ since the solution to equation (2.22) must be less than $\tilde{x}^*$. Additionally, the left hand side of the equation (B.9) is a convex function of Z, as $\tilde{\beta}^+ > 1$, while the right hand side is a linear function of Z, and these functions cross only once in the interval $(0, 1)$ (see the proof of of Lemma 2.5). At $Z = 0$,

$$\frac{\left(\tilde{\lambda}^- - \tilde{\beta}^+\right)}{\tilde{\lambda}^-} (Z)^{\tilde{\beta}^+} > \left(\frac{1+\varphi}{\varphi}\right) \frac{\tilde{\beta}^+ \left(\tilde{\lambda}^- - 1\right)}{\tilde{\lambda}^-} Z - \left(\tilde{\beta}^+ - 1\right).$$

Let Z' be the value of Z such that $\frac{(\tilde{\lambda}^- - \tilde{\beta}^+)}{\tilde{\lambda}^-}(Z')^{\tilde{\beta}^+} = 1$. Then, if

$$\frac{\left(\tilde{\lambda}^- - \tilde{\beta}^+\right)}{\tilde{\lambda}^-} (Z')^{\tilde{\beta}^+} < \left(\frac{1+\varphi}{\varphi}\right) \frac{\tilde{\beta}^+ \left(\tilde{\lambda}^- - 1\right)}{\tilde{\lambda}^-} Z' - \left(\tilde{\beta}^+ - 1\right),$$

it must be the case that $\left(Z\left(\tilde{y}_*\right)\right)^{\tilde{\beta}^+} < 1$. Therefore, the proof will be complete if we can show that

$$\left(\frac{1+\varphi}{\varphi}\right) \frac{\tilde{\beta}^+ \left(\tilde{\lambda}^- - 1\right)}{\tilde{\lambda}^-} \left(\frac{\tilde{\lambda}^-}{\tilde{\lambda}^- - \tilde{\beta}^+}\right)^{1/\tilde{\beta}^+} - \left(\tilde{\beta}^+ - 1\right) > 1,$$

or equivalently:

$$\left(\frac{1+\varphi}{\varphi}\right) \left(\frac{\tilde{\lambda}^- - 1}{\tilde{\lambda}^-}\right) > \left(\frac{\tilde{\lambda}^- - \tilde{\beta}^+}{\tilde{\lambda}^-}\right)^{1/\tilde{\beta}^+} \tag{B.10}$$

As $\tilde{\beta}^+$ approaches 1, equation (B.10) reduces to

$$\left(\frac{1+\varphi}{\varphi}\right) \left(\frac{\tilde{\lambda}^- - 1}{\tilde{\lambda}^-}\right) > \frac{\tilde{\lambda}^- - 1}{\tilde{\lambda}^-}$$

which is satisfied for $\varphi > 0$. The left hand side of equation (B.10) is independent of $\tilde{\beta}^+$, so all that remains to verify is that the right hand side is decreasing in $\tilde{\beta}^+$.

$$\frac{\partial}{\partial \tilde{\beta}^+} \left(\left(\frac{\tilde{\lambda}^- - \tilde{\beta}^+}{\tilde{\lambda}^-}\right)^{1/\tilde{\beta}^+} \right) = - \frac{\left(\frac{\tilde{\lambda}^- - \tilde{\beta}^+}{\tilde{\lambda}^-}\right)^{1/\tilde{\beta}^+} \left[\tilde{\beta}^+ + \left(\tilde{\lambda}^- - \tilde{\beta}^+\right) \log\left(\frac{\tilde{\lambda}^- - \tilde{\beta}^+}{\tilde{\lambda}^-}\right)\right]}{\left(\tilde{\beta}^+\right)^2 \left(\tilde{\lambda}^- - \tilde{\beta}^+\right)},$$

which is negative if and only if the term in brackets is less than zero, *i.e.* if

$$\log\left(\frac{\tilde{\lambda}^-}{\tilde{\lambda}^- - \tilde{\beta}^+}\right) < \frac{\tilde{\beta}^+}{\tilde{\lambda}^- - \tilde{\beta}^+}. \tag{B.11}$$

Assume $\tilde{\beta}^+ \geq \tilde{\lambda}^-(1-e)$. Then $\left(\frac{\tilde{\lambda}^-}{\tilde{\lambda}^- - \tilde{\beta}^+}\right) \leq \frac{1}{e}$ and $\log\left(\frac{\tilde{\lambda}^-}{\tilde{\lambda}^- - \tilde{\beta}^+}\right) \leq -1$. Since $\tilde{\lambda}^- < 0$ and $\tilde{\beta}^+ > 1$, $\frac{\tilde{\beta}^+}{\tilde{\lambda}^- - \tilde{\beta}^+} \in (-1, 0)$ and equation (B.11) is satisfied. Assume $\tilde{\beta}^+ < \tilde{\lambda}^-(1-e)$. Since $\frac{\tilde{\beta}^+}{\tilde{\lambda}^- - \tilde{\beta}^+}$ is decreasing in $\tilde{\beta}^+$,

$$\lim_{\tilde{\beta}^+ \to \left(\tilde{\lambda}^-(1-e)\right)^-} \left(\frac{\tilde{\beta}^+}{\tilde{\lambda}^- - \tilde{\beta}^+}\right) = -1 + \frac{1}{e}, \text{ and}$$

$$\lim_{\tilde{\beta}^+ \to \left(\tilde{\lambda}^-(1-e)\right)^-} \left(\log\left(\frac{\tilde{\lambda}^-}{\tilde{\lambda}^- - \tilde{\beta}^+}\right)\right) = -1.$$

Hence, equation (B.11) holds at the upper bound of $\tilde{\beta}^+$. As $\tilde{\beta}^+$ tends to 1, equation (B.11) approaches

$$\log\left(\frac{\tilde{\lambda}^-}{\tilde{\lambda}^- - 1}\right) < \frac{1}{\tilde{\lambda}^- - 1}.$$

In order to see that the previous condition is satisfied for all $\tilde{\lambda}^- < 0$, define $f(\tilde{\lambda}^-) = \log\left(\frac{\tilde{\lambda}^-}{\tilde{\lambda}^- - 1}\right) - \frac{1}{\tilde{\lambda}^- - 1}$. Clearly,

$$\lim_{\tilde{\lambda}^- \to -\infty} f(\tilde{\lambda}^-) = 0, \text{ and}$$

$$\frac{df(\tilde{\lambda}^-)}{d\tilde{\lambda}^-} = \frac{1}{\tilde{\lambda}^-(\tilde{\lambda}^- - 1)^2} < 0.$$

Therefore, equation (B.11) also holds at the lower bound of $\tilde{\beta}^+$. Finally, it suffices to show that $\log\left(\frac{\tilde{\lambda}^-}{\tilde{\lambda}^- - 1}\right)$ is decreasing in $\tilde{\beta}^+$ to guarantee that equation (B.11) holds throughout the interval $(1, \tilde{\lambda}^-(1-e))$.

$$\frac{\partial}{\partial \tilde{\beta}^+}\left(\log\left(\frac{\tilde{\lambda}^-}{\tilde{\lambda}^- - \tilde{\beta}^+}\right)\right) = \frac{1}{\tilde{\lambda}^- - \tilde{\beta}^+} < 0.$$

$\blacksquare$

Now we proof the statements in Proposition 2.12. First, we show that $\partial_{\bar{\eta}}\tilde{y}_* > 0$. Equation (2.22) (implicitly) defines the exercise threshold $\tilde{y}_*$. Taking the implicit derivative with respect to $\bar{\eta}$ and after some rearrangement of terms we obtain:

$$\left(\frac{e^{\tilde{\beta}^+(\tilde{y}_*-\tilde{x}^*)}U\tilde{\beta}^+\left(\tilde{\lambda}^- - \tilde{\beta}^+\right)}{\left(\tilde{\beta}^+ - 1\right)\tilde{\lambda}^-} - \frac{e^{\tilde{y}_*}(1+\varphi)\left(\tilde{\lambda}^- - 1\right)}{\tilde{\lambda}^-}\right)\partial_{\bar{\eta}}\tilde{y}_* =$$
$$\left(\frac{e^{\tilde{y}_*}(1+\varphi)\left(\tilde{\lambda}^- - 1\right)}{\tilde{\lambda}^-} - \frac{\tilde{\beta}^+ e^{\tilde{\beta}^+(\tilde{y}_*-\tilde{x}^*)}U\left(\tilde{\lambda}^- - 1\right)}{\left(\tilde{\beta}^+ - 1\right)\tilde{\lambda}^-}\right)\frac{\partial_{\bar{\eta}}\tilde{\lambda}^-}{\tilde{\lambda}^-\left(\tilde{\lambda}^- - 1\right)}$$

Using equation (2.21) we can replace $\frac{(1+\varphi)\left(\tilde{\lambda}^- - 1\right)e^{\tilde{y}_*}}{\tilde{\lambda}^-}$ in both sides of the previous expression which allow us to rewrite the derivative of $\tilde{x}^*$ with respect to $\bar{\eta}$ as:

$$\partial_{\bar{\eta}}\tilde{y}_* = \left(\frac{\left(\tilde{\lambda}^- - \tilde{\beta}^+\right)}{\tilde{\lambda}^-}e^{\tilde{\beta}^+(\tilde{y}_*-\tilde{x}^*)} - 1\right)^{-1}\left(1 - e^{\tilde{\beta}^+(\tilde{y}_*-\tilde{x}^*)}\right)\frac{\partial_{\bar{\eta}}\tilde{\lambda}^-}{\tilde{\lambda}^-\left(\tilde{\lambda}^- - 1\right)} \qquad \text{(B.12)}$$

The first term in the right hand side of the previous expression is negative from Claim B.4. Since $(\tilde{y}_* - \tilde{x}^*) < 0$ and $\tilde{\beta}^+ > 1$ the second term is positive. It can be easily verified that $\partial_{\bar{\eta}}\tilde{\lambda}^- < 0$ and $\tilde{\lambda}^- < 0$ by definition. Therefore $\partial_{\bar{\eta}}\tilde{y}_* > 0$.

Finally, we show that $\partial_{\underline{\mu}}\tilde{y}_* > 0$. Taking the implicit derivative with respect to $\bar{\mu}$ and using the same equation (2.22) to replace $\frac{e^{\tilde{y}_*}\left(\lambda^-_{(\eta)} - 1\right)(1+\varphi)}{\lambda^-_{(\eta)}}$ we can write the derivative of $\tilde{x}^*$ with respect to $\bar{\mu}$ as:

$$\left(\frac{\left(\tilde{\lambda}^- - \tilde{\beta}^+\right) e^{\tilde{\beta}^+(\tilde{y}_* - \tilde{x}^*)}}{\tilde{\lambda}^-} - 1\right) \partial_{\underline{\mu}} \tilde{y}_* =$$

$$\left[\left(\tilde{\beta}^+ \partial_{\underline{\mu}} \tilde{x}^* - \partial_{\underline{\mu}} \tilde{\beta}^+ \left(\tilde{y}_* - \tilde{x}^*\right)\right)\left(\tilde{\lambda}^- - \tilde{\beta}^+\right) + \frac{\left(\tilde{\lambda}^- - 1\right) \partial_{\underline{\mu}} \tilde{\beta}^+}{\left(\tilde{\beta}^+ - 1\right)}\right] \frac{e^{\tilde{\beta}^+(\tilde{y}_* - \tilde{x}^*)}}{\left(\tilde{\beta}^+ - 1\right) \tilde{\lambda}^-} \tag{B.13}$$

The term in brackets can be further reduce to $\partial_{\underline{\mu}} \tilde{\beta}^+ (1 - (\tilde{\lambda}^- - \tilde{\beta}^+)(\tilde{y}_* - \tilde{x}^*))$ by plugging in $\partial_{\underline{\mu}} \tilde{x}^*$ as obtained in Claim B.2. $\partial_{\underline{\mu}} \tilde{y}_* > 0$ if $(1 - (\tilde{\lambda}^- - \tilde{\beta}^+)(\tilde{y}_* - \tilde{x}^*)) < 0$, since $\partial_{\underline{\mu}\tilde{\beta}^+} < 0$, or equivalently if:

$$(\tilde{y}_* - \tilde{x}^*) < \frac{1}{\tilde{\lambda}^- - \tilde{\beta}^+}.$$

From Claim B.4 we know that $\frac{\tilde{\lambda}^- - \tilde{\beta}^+}{\tilde{\lambda}^-} e^{\tilde{\beta}^+(\tilde{y}_* - \tilde{x}^*)} < 1$ which implies that

$$(\tilde{y}_* - \tilde{x}^*) < \frac{1}{\tilde{\beta}^+} \log\left(\frac{\tilde{\lambda}^-}{\tilde{\lambda}^- - \tilde{\beta}^+}\right) < \frac{1}{\tilde{\lambda}^- - \tilde{\beta}^+}$$

as shown in the proof of Claim B.4 ■

B.9 Proof of Propositon 2.13

Taking the derivative of equation (2.23) with respect to $\underline{\mu}$ we obtain:

$$\begin{aligned} \partial_{\underline{\mu}} \mathscr{V}_1^*(x) = & -e^{\tilde{\lambda}^-(x - \tilde{y}_*)} \tilde{\lambda}^- \left(U - e^{\tilde{y}_*}(1+\varphi) + \frac{e^{(\tilde{y}_* - \tilde{x}^*)\tilde{\beta}^+} U}{\tilde{\beta}^+ - 1}\right) \partial_{\underline{\mu}} \tilde{y}_* \\ & + e^{\tilde{\lambda}^-(x - \tilde{y}_*)} \left(-e^{\tilde{y}_*}(1+\varphi) \partial_{\underline{\mu}} \tilde{y}_* - \frac{U \partial_{\underline{\mu}} \tilde{\beta}^+ e^{(\tilde{y}_* - \tilde{x}^*)\tilde{\beta}^+}}{\left(\tilde{\beta}^+ - 1\right)^2}\right) \\ & + e^{\tilde{\lambda}^-(x - \tilde{y}_*)} \left(\frac{U e^{(\tilde{y}_* - \tilde{x}^*)\tilde{\beta}^+} \left(\tilde{\beta}^+ \left(\partial_{\underline{\mu}} \tilde{y}_* - \partial_{\underline{\mu}} \tilde{x}^*\right) + \left(\tilde{y}_* - \tilde{x}^*\right) \partial_{\underline{\mu}} \tilde{\beta}^+\right)}{\tilde{\beta}^+ - 1}\right) \end{aligned} \tag{B.14}$$

Taking the implicit derivative of equation (2.22) with respect to $\underline{\mu}$ we can get

$$\frac{Ue^{(\tilde{y}_*-\tilde{x}^*)\tilde{\beta}^+}\left(\tilde{\beta}^+\left(\partial_{\underline{\mu}}\tilde{y}_*-\partial_{\underline{\mu}}\tilde{x}^*\right)+(\tilde{y}_*-\tilde{x}^*)\,\partial_{\underline{\mu}}\tilde{\beta}^+\right)}{\tilde{\beta}^+-1}=$$

$$\frac{e^{\tilde{y}_*}\left(\tilde{\lambda}^--1\right)(1+\varphi)\partial_{\underline{\mu}}\tilde{y}_*}{\left(\tilde{\lambda}^--\tilde{\beta}^+\right)}+\frac{U\ \partial_{\underline{\mu}}\tilde{\beta}^+e^{\tilde{\beta}^+(\tilde{y}_*-\tilde{x}^*)}}{\left(\tilde{\beta}^+-1\right)}\left(\frac{1}{\left(\tilde{\beta}^+-1\right)}+\frac{1}{\left(\tilde{\lambda}^--\tilde{\beta}^+\right)}\right)$$

Substituting the previous expression in equation (B.14) and using equation (2.22) to replace

$$\frac{U\left(\tilde{\lambda}^--\tilde{\beta}^+\right)}{\left(\tilde{\beta}^+-1\right)\tilde{\lambda}^-}e^{\tilde{\beta}^+(\tilde{y}_*-\tilde{x}^*)}=-U+\frac{(1+\varphi)\left(\tilde{\lambda}^--1\right)}{\tilde{\lambda}^-}e^{\tilde{y}_*}.$$

we can write $\partial_{\underline{\mu}}\mathscr{V}_1^*(x)$ as

$$\partial_{\underline{\mu}}\mathscr{V}_1^*(x)=\left(\partial_{\underline{\mu}}\tilde{y}_*\left(\tilde{\lambda}^--\left(\tilde{\lambda}^--\tilde{\beta}^+\right)e^{\tilde{\beta}^+(\tilde{y}_*-\tilde{x}^*)}\right)+\frac{\partial_{\underline{\mu}}\tilde{\beta}^+}{\left(\tilde{\beta}^+-1\right)}e^{\tilde{\beta}^+(\tilde{y}_*-\tilde{x}^*)}\right)\frac{Ue^{\tilde{\lambda}^-(x-\tilde{y}_*)}}{\left(\tilde{\lambda}^--\tilde{\beta}^+\right)} \tag{B.15}$$

From Proposition 2.12, and Claim B.4 therein, $\left(\tilde{\lambda}^--\left(\tilde{\lambda}^--\tilde{\beta}^+\right)e^{\tilde{\beta}^+(\tilde{y}_*-\tilde{x}^*)}\right)<0$ and $\partial_{\underline{\mu}}\tilde{y}_*>0$. It is easy to verify that $\partial_{\underline{\mu}}\tilde{\beta}^+<0$, thus $\partial_{\underline{\mu}}\mathscr{V}_1^*(x)>0$. ■

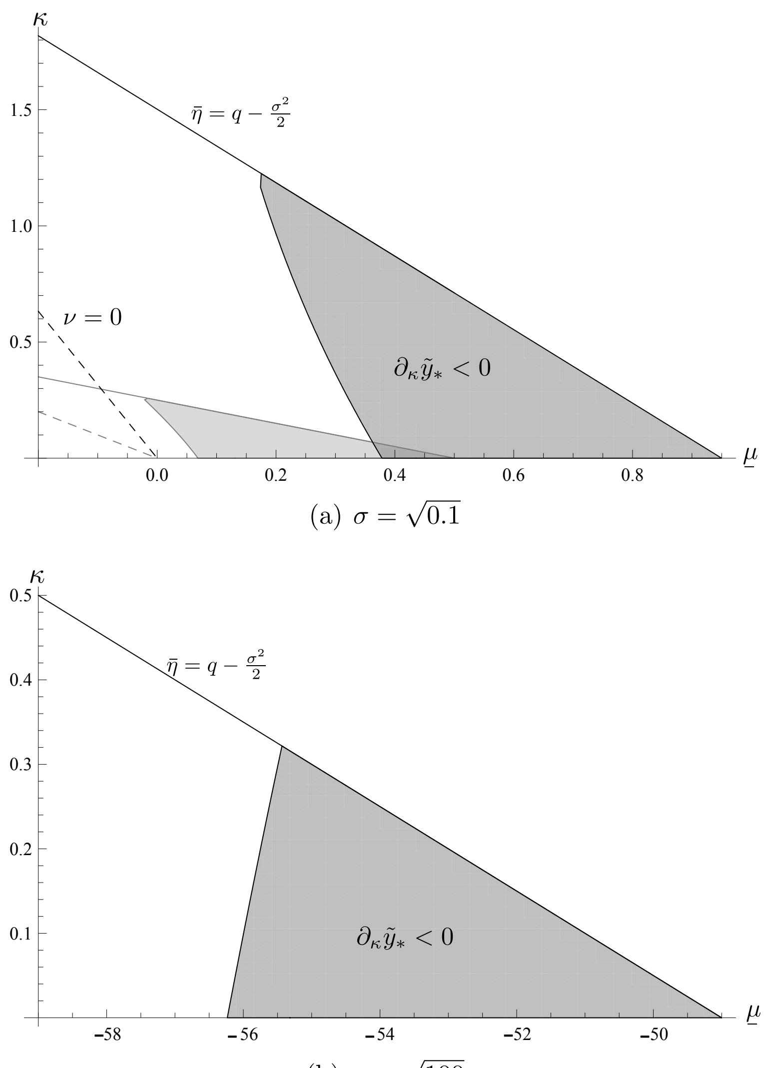
Appendix C

Change in the buyer's threshold: Numerical results

In this appendix we use numerical methods to evaluate the effect that a change in κ has on the optimal buying threshold, $\tilde{y}_*$. From our Result 2.10 we have that the *ex ante* drift used to evaluate the option to resale the good is given by $\underline{\mu} = \nu - \sigma\kappa$ while the drift used to compute the value of the option to buy the good is $\bar{\eta} = \nu + \sigma\kappa$. An increase in κ will reduce $\underline{\mu}$ and increase $\bar{\eta}$. Formally, $\partial_\kappa \bar{\mu} = -\sigma$ and $\partial_\kappa \eta = \sigma$. The total change of $\tilde{y}_*$ with respect to κ is

$$\frac{dy_*}{d\kappa} = \sigma\partial_{\bar{\eta}}\tilde{y}_* - \sigma\partial_{\underline{\mu}}\tilde{y}_*. \tag{C.1}$$

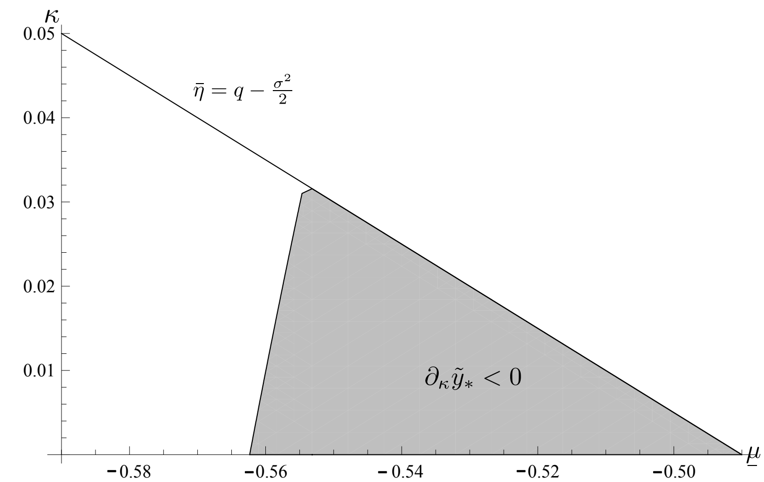
Using equations (B.12) and (B.13) we can evaluate the previous expression for a given set of parameters. To facilitate the exposition of the results we generate region plots highlighting the set of parameters for which the $\frac{dy_*}{d\kappa} < 0$. We vary the values of σ, q and φ to examine the result at "extreme" values of these parameters. In all the computations U was normalized to 1. Whenever possible the region depicted in Figure 2.3 is included in the figures in this appendix for comparison.



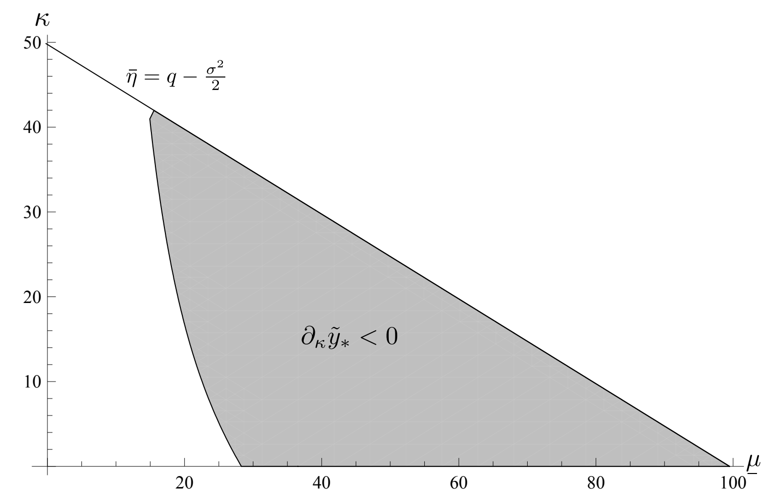
(a) $\sigma = \sqrt{0.1}$

(b) $\sigma = \sqrt{100}$

Figure C.1: *Region Plot for $\partial_\kappa \tilde{y}^* < 0$ for extreme values of σ ($q = 1$, $\varphi = 0.5$)*

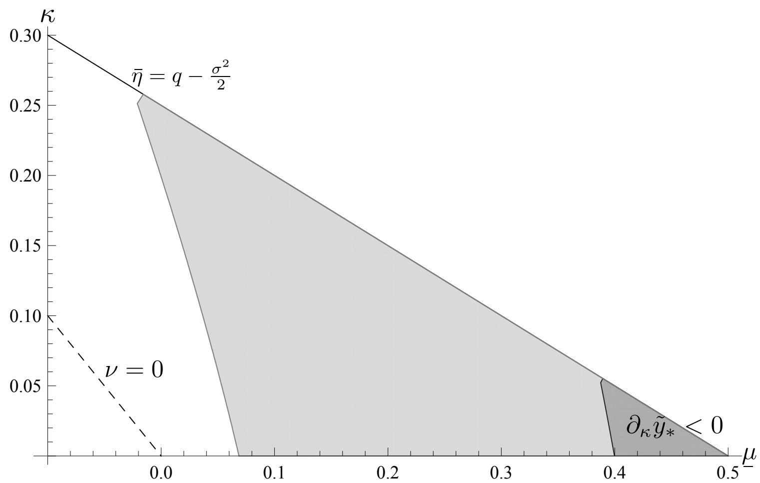


(a) $q = 0.1$

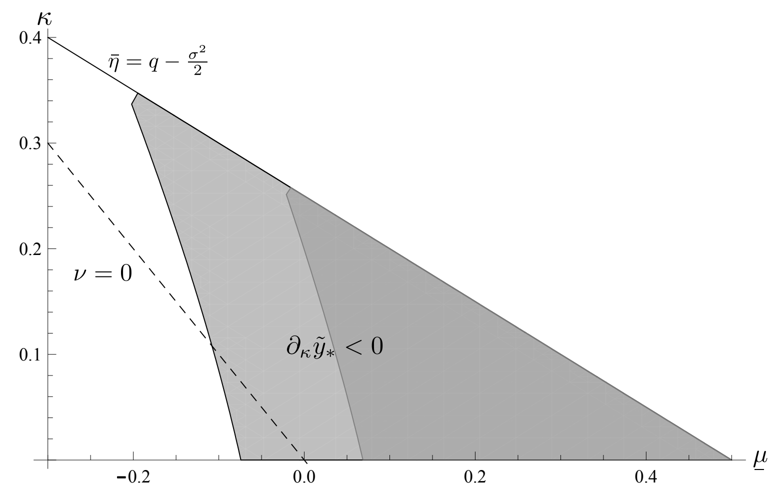


(b) $q = 100$

Figure C.2: *Region Plot for $\partial_\kappa \tilde{y}^* < 0$ for extreme values of q ($\sigma = 1$, $\varphi = 0.5$)*



(a) $\varphi = 0.1$



(b) $\varphi = 0.9$

Figure C.3: *Region Plot for $\partial_\kappa \tilde{y}^* < 0$ for extreme values of φ ($\sigma = 1$, $q = 1$)*

Appendix D

Proofs of Lemmata and Propositions in Chapter 3

D.1 Proof of Lemma 3.2

Before proceeding to the proof of Lemma 3.2 let us first establish the following result:

Claim D.1. *Let $\underline{p}$ be the lower bound of a symmetric NE strategy $F(p;c)$, then $c < \underline{p} \leq v$.*

Proof. Clearly, setting a price below marginal cost will yield negative profits so no equilibrium distribution can have $\underline{p} < c$. Assume $\underline{p} = c$, then $\mathbb{E}\pi(\underline{p}, F) = 0$. Since there is a fraction $(1 - \lambda)$ of consumers with $s > 0$, there exist an ε such that consumers observing an initial price $p' \in (c, \varepsilon)$ will stop and patronize the firm and $\mathbb{E}\pi(p', F) > 0$, a contradiction to the assumption $\underline{p} = c$.

Assume $\underline{p} > v$, then $\mathbb{E}\pi(p, F) = 0$ for all $p \in \operatorname{supp} F(p;c)$, and in particular for $p = \underline{p}$. We have shown, however, that there exist $p' \in (c, \varepsilon)$ such that $\mathbb{E}\pi(p', F) > 0$ and such deviation is always possible as $v > c_h$. Hence a contradiction. ■

Assume $\alpha(\underline{p}) > 0$. We now show that firm j can transfer all the mass at $\underline{p}$ to a price p' arbitrarily close to $\underline{p}$ resulting in a profitable deviation for each of the cost regimes. Such a price always exist since $\underline{p} > c$. Additionally, consumers will always buy one of the goods in the market for all $p \leq \underline{p}$ as $\underline{p} \leq v$ by Claim D.1.

Case D.1.1 ($c = c_h$)**.** Consider the case when $c = c_h$ and firms set prices according to $F(p; c_h)$. By definition $\underline{p} \in \operatorname{supp} F(p; c_h)$ and all consumers have a unique reservation price $r(1)$. Undercutting $\underline{p}$ by p' will result on consumers observing $p' \notin \operatorname{supp} F(p; c_h)$ to form a belief $\phi(p') = 0$ and its corresponding reservation price $r(0)$.

(i) Assume $\underline{p} \leq r(1)$, then, from equation (3.5) the expected profits at $\underline{p}$ satisfy:

$$\mathbb{E}\pi(\underline{p}, F) < \left((\lambda + \mu_{-j}) \cdot 1 + \frac{1-\lambda}{N}\right)(\underline{p} - c_h), \tag{D.1}$$

since at $\underline{p}$ any firm will get the fraction of consumers undertaking search (costly and non-costly) with probability less than one.

If $\underline{p} \leq r(0)$, setting p' will have no effect on the search decision of consumers observing p' and

$$\lim_{p' \uparrow \underline{p}} \mathbb{E}\pi(p', F) = \left((\lambda + \mu_{-j}) \cdot 1 + \frac{1-\lambda}{N}\right)(\underline{p} - c_h), \tag{D.2}$$

which is greater than $\mathbb{E}\pi(\underline{p}, F)$, from equation (D.1). Hence a contradiction.

If $\underline{p} > r(0)$, setting p' will make the fraction of consumers with reservation price $r(0)$ to engage in costly search. As only firm j is deviating from $F(p; c_h)$, $\mu_j = (1-\lambda)/N$, and the search intensity is given by $\lambda + \mu_{-j} + (1-\lambda)/N$. Since $p' < \underline{p}$ firm j will get all the consumers searching for the lowest price with probability one and

$$\lim_{p' \uparrow \underline{p}} \mathbb{E}\pi(p', F) = \left(\left(\lambda + \mu_{-j} + \frac{1-\lambda}{N}\right) \cdot 1\right)(\underline{p} - c_h). \tag{D.3}$$

which is equal to the expected profits obtained in (D.2).

(ii) Assume $\underline{p} > r(1)$ then the expected profits at $\underline{p}$ satisfy:

$$\mathbb{E}\pi(\underline{p}, F) < \left(\left(\lambda + \mu_{-j} + \frac{1-\lambda}{N}\right) \cdot 1\right)(\underline{p} - c_h), \tag{D.4}$$

Similarly as above: the expected profits of setting p' for $\underline{p} \leq r(0)$ and $\underline{p} > r(0)$ are given by equations (D.2) and (D.3), respectively; and p' dominates $\underline{p}$. Hence a contradiction.

Case D.1.2 ($c = c_l$)**.** Consider the case when $c = c_l$ and firms set prices according to $F(p; c_l)$.

First assume that undercutting $\underline{p}$ will not affect the beliefs of consumers, *i.e.* there exist an $\varepsilon > 0$ such that $B_\varepsilon(\underline{p}) \cap \operatorname{supp} F(p; c_h) = \emptyset$, where $B_\varepsilon(\underline{p})$ is an open ball centered at $\underline{p}$. Since undercutting $\underline{p}$ has no effect on the search intensity of the market, setting p' dominates $\underline{p}$ as whatever fraction was previously shared with positive probability now will patronize firm j with probability one while the fraction of consumers buying from firm j, if any, will continue to do so since their reservation price remains unchanged.

Now assume that for all $\varepsilon > 0$, $B_\varepsilon(\underline{p}) \cap \operatorname{supp} F(p; c_h) \neq \emptyset$ so marginally undercutting $\underline{p}$ by p' will result on consumers observing p' to form a belief $\phi(p') \neq \phi(\underline{p})$ and its corresponding reservation price $r(\phi(p'))$.

If $r(\phi(p')) \geq r(\phi(\underline{p}))$, no additional search will be triggered by setting $p' < \underline{p}$. When $r(\phi(\underline{p})) \leq \underline{p}$, the expected profits at $\underline{p}$ satisfy equation (D.1) while the expected profits at p' are given by (D.2). Similarly, for $r(\phi(\underline{p})) > \underline{p}$, the expected profits at $\underline{p}$

satisfy equation (D.4), and the expected profits at p' are given by (D.3). Yielding a contradiction.

Finally, if $r(\phi(p')) < r(\phi(\underline{p}))$, setting p' will make the fraction of consumers with reservation price $r(\phi(p'))$ to engage in costly search. As only firm j is deviating from $F(p; c_h)$, $\mu_j = (1-\lambda)/N$, and the search intensity is given by $\lambda + \mu_{-j} + (1-\lambda)/N$. Since $p' < \underline{p}$ firm j will get all the consumers searching for the lowest price with probability one and the expected profits will satisfy equation (D.3) which are grater than the expected profits at $\underline{p}$, since it satisfies the equivalent conditions given by equations (D.2) and (D.4). Contradicting the assumption of $\underline{p}$ being the lower bound of a NE strategy. ■

D.2 Proof of Proposition 3.3.

In order to prove Proposition 3.3 it is easier to first establish the following result:

Lemma D.2. *If $F(p; c)$ is a symmetric NE pricing strategy with path connected support, then $r(\phi(p)) \geq p$ for all $p \in \operatorname{supp} F$.*

Proof. Assume not, then there exist a $p' \in \operatorname{supp} F$ such that $r(\phi(p')) < p'$. If $p' = \underline{p}$, then $r(\phi(\underline{p})) < \underline{p}$ implies that a consumer observing $\underline{p}$ will find optimal to engage in costly search, a contradiction as the expected gains of search at $\underline{p}$ are always zero. From Lemma 3.2, F has zero mass at $\underline{p}$, and since a cdf is right continuous, it must be the case that F is continuous in the interval $[\underline{p}, \underline{p} + \varepsilon)$ for some $\varepsilon > 0$. If $p' \in (\underline{p}, \underline{p} + \varepsilon)$

then

$$\lim_{p\uparrow p'}\left((\lambda+\mu_{-j})[1-F(p)]^{N-1}+\frac{1-\lambda}{N}\right)(p-c)=$$
$$\left(\lambda+\mu_{-j}+\frac{1-\lambda}{N}\right)[1-F(p')]^{N-1}(p'-c),$$

which is true if and only if $F(p')=0$, a contradiction as $p'>\underline{p}$ and F is strictly increasing since the support is path connected. Let q be the smallest $p'\in[(\underline{p}+\varepsilon),\bar{p}]$ satisfying $r(\phi(p'))<p'$. If there is no mass point at q,

$$\mathbb{E}\pi(q)=\left(\lambda+\mu_{-j}+\frac{1-\lambda}{N}\right)[1-F(q)]^{N-1}(q-c),$$

and undercutting q gives

$$\lim_{p\uparrow q}\mathbb{E}\pi(p)=\left((\lambda+\mu_{-j})[1-F(q)]^{N-1}+\frac{1-\lambda}{N}\right)(q-c)$$

which implies that $\lim_{p\uparrow q}\mathbb{E}\pi(p)>\mathbb{E}\pi(q)$ since $F(q)>0$. Hence a contradiction to F being an equilibrium.

If q is charged with positive probability, transferring all the mass at q to $(q-\nu)$, for an arbitrarily small ν, will generate a discrete increase in the expected fraction of consumers engaging in search by eliminating the possibility of a tie (see Varian (1980) Proposition 3). Additionally, by undercutting q the $(1-\lambda)/N$ fraction of consumers that were previously searching now will patronize the firm charging $(q-\nu)$ with probability one as $r(\phi(q-\nu))\geq q-\nu$, further increasing the market share of the firm for a marginally small decrease in per-costumer profits. Thus contradicting the assumption of equilibrium. ■

For the remainder of the proof we can use the standard argument to show that F is atomless throughout its support. Assume $F(p)$ has a a mass point at some $p' \in (\underline{p}, \bar{p}]$ (the case of $p' = \underline{p}$ is covered in Lemma 3.2). Then a firm can always undercut p' by transferring all the mass at p' to a price arbitrarily close. From the result in Lemma D.2, we do not need to worry about potential increases in the search intensity of the market as a result of undercutting p' since $r(\phi(p' - \varepsilon)) \geq p' - \varepsilon$ and the fraction of consumers patronizing a firm charging p' will continue to do so at $p' - \varepsilon$. Since the number of mass points in any distribution is countable and $\underline{p} > c$ from Claim D.1 such a price always exist. Therefore undercutting p' is a profitable deviation since it increases the market share discretely for a continuous decrease in profits per-costumer. A contradiction to $F(p)$ being an equilibirum. ■

D.3 Proof of Lemma 3.5.

$\mathbb{E}\pi(p;c) = 0$ for all $p > v$ and $\mathbb{E}(\underline{p};c) > 0$ from Claim D.1, therefore $\underline{p} \leq v$. In Section 3.2.1 we established that $\boldsymbol{r}$ has at most two elements. First, consider the case when $\boldsymbol{r} = \{r(1)\}$. Since $\boldsymbol{r}$ has only one element, $r(\phi(\bar{p})) = r(1)$. From Lemma D.2, $\bar{p} \leq r(\phi(\bar{p}))$, assume $\bar{p} < r(\phi(\bar{p}))$, then firm j setting $p_j = \bar{p}$ will get its captive share of the market $(1-\lambda)/N$ and lose all the *shoppers* with probability one since F is atomless at $\bar{p}$ form Proposition 3.3. It is profitable for firm j to increase its price to $\bar{p} < p' \leq \min\{r(\phi(\bar{p})), v\}$ as its market share will remain unchanged. A contradiction to $\bar{p}$ being the upper bound of $F(p;c)$.

Now consider the case when $\boldsymbol{r} = \{r(0), r(1)\}$. Assume $\bar{p}$ lies between the two reservation prices, if $r(\phi(\bar{p})) = \max\{r(0), r(1)\}$, we can show that charging a price above $\bar{p}$ gives a profitable deviation using the same argument as above. Similarly, $\bar{p} < \min\{r(0), r(1)\}$ implies $\bar{p} < r(\phi(\bar{p}))$, yielding a contradiction. ■

D.4 Proof of Lemma 3.6.

Assume otherwise, *i.e.* $\underline{p}_h = \underline{p}_l = \underline{p}$ and $\bar{p}_h = \bar{p}_l = \bar{p}$. From Lemma 3.5 we know that when $c = c_h$ the upper bound of $F(p; c_h)$ is equal to the unique reservation price and no endogenous search will occur in equilibrium, that is $\mu = 0$. Additionally, we know that any equilibrium distribution is atomless throughout the support from Proposition 3.3. The necessary indifference condition (IC) for $\bar{p}$ and $\underline{p}$ in the equilibrium support of $F(p; c_h)$ then implies:

$$\bar{p} - \underline{p} = \frac{N\lambda}{1-\lambda}(\underline{p} - c_h). \tag{D.5}$$

If the stores have the same equilibrium support when the costs are high or low, consumers are unable to distinguish among the two cost regimes, and they all behave as in the $c = c_h$ case and the IC condition in this case implies:

$$\underline{p} - c_l = \frac{1-\lambda}{N\lambda}(\bar{p} - \underline{p}).$$

Pluggin equation (D.5) we obtain $\underline{p} - c_l = \underline{p} - c_h$. A contradiction since $c_h > c_l$.

■

D.5 Proof of Proposition 3.7.

From Proposition 3.3, F is atomless. The expected equilibrium profits are given by:

$$\mathbb{E}\pi(p; c_h) = \int_{\underline{p}_h}^{\bar{p}_h} \left(\left(\lambda[1 - F(p; c_h)]^{N-1} + \frac{1-\lambda}{N} \right) (p - c_h) \right) dF(p; c_h), \tag{D.6}$$

where $\bar{p}_h = \min\{r_h, v\}$ from Lemma 3.5, which implies that no endogenous search will occur in equilibrium so $\mu = 0$. By definition of an equilibrium all prices in the support of F should yield the same expected value so

$$\mathbb{E}\pi(p; c_h) = \left(\lambda[1 - F(p; c_h)]^{N-1} + \frac{1-\lambda}{N} \right) (p - c_h)$$

for all p and in particular

$$\mathbb{E}\pi(\bar{p}_h; c_h) = \left(\frac{1-\lambda}{N} \right) (\bar{p}_h - c_h);$$

since at $\bar{p}$ the probability of being the store charging the lowest price is zero. Equating the two conditions for the equilibrium profits, and solving for F we get:

$$F(p; c_h) = 1 - \left[\frac{1-\lambda}{\lambda N} \left(\frac{\bar{p}_h - p}{p - c_h} \right) \right]^{\frac{1}{N-1}}.$$

Setting $F(\underline{p}; c_h) = 0$ we obtain the lower bound as a function of $\bar{p}$:

$$b(\bar{p}) = \left(\frac{1-\lambda}{1 + \lambda(N-1)} \right) (\bar{p}_h - c_h) + c_h.$$

Uniqueness comes from the fact that our expression for F is the only atomless distribution that satisfy the indifferent conditions for an equilibrium. ■

D.6 Proof of Lemma 3.8

(a) Assume $\bar{p}_l \in [\underline{p}_h, \bar{p}_h)$. A firm charging $p_j = \bar{p}_l$ will lose all *shoppers* with probability one from Proposition 3.3. Additionally, consumers observing p_j cannot distinguish between c_h and c_l since $p_j \in \operatorname{supp} F(p; c_h)$, so there exist a profitable deviation by increasing p_j to $\bar{p}_h$ as consumers will continue to buy from firm j without searching. Assume $\bar{p}_l < \underline{p}_h$. Similarly as above a firm setting a price at $\bar{p}_l$ will serve no *shoppers*. By charging a price outside the support of $F(p; c_h)$, firm j is revealing to the consumers observing p_j that the price is in fact low. However, by increasing its price to $p' \in \operatorname{supp} F(p; c_h)$, firm j can increase its profits since the fraction of consumers observing p' will believe that the cost is high and buy from firm j without searching, which take us to the previous case. Assume $\bar{p}_l > \bar{p}_h$, then firms can increase profits when the cost is high by charging $\bar{p}_l$ instead of $\bar{p}_h$, a contradiction to $\bar{p}_h$ being the upper bound of an equilibrium strategy when $c - c_h$.

(b) From Lemma 3.6 it must be the case that $\underline{p}_h \neq \underline{p}_l$. Assume $\underline{p}_l > \underline{p}_h$. From the indifference condition (IC) required in equilibrium we have:

$$\bar{p} - \underline{p}_h = \frac{N\lambda}{1-\lambda}(\underline{p}_h - c_h). \tag{D.7}$$

As $\underline{p}_l > \underline{p}_h$ together with *(a)* implies $\operatorname{supp} F(p; c_h) \subset \operatorname{supp} F(p; c_l)$ so consumers will form beliefs as in the high cost case and $\mu = 0$ for all $p \in \operatorname{supp} F(p; c_l)$, and the IC condition implies

$$\bar{p} - \underline{p}_l = \frac{N\lambda}{1-\lambda}(\underline{p}_l - c_l). \tag{D.8}$$

Dividing equation (D.7) by (D.8) we obtain:

$$\frac{\underline{p}_h - c_h}{\underline{p}_l - c_l} = \frac{\bar{p} - \underline{p}_h}{\bar{p} - \underline{p}_l} > 1.$$

Which in turn requires $\underline{p}_h - \underline{p}_l > c_h - c_l > 0$, or equivalently $\underline{p}_h > \underline{p}_l$. A contradiction to our assumption $\underline{p}_l > \underline{p}_h$. ■

D.7 Proof of Lemma 3.10

Before proceeding to the proof, let us establish the following result:

Claim D.3. $\gamma(\lambda) < 1$ *for all* $\lambda > 0$.

Proof. First we show that $\gamma(\lambda)$ is strictly decreasing in λ.

$$\partial_\lambda \gamma(\lambda) = \frac{N\lambda - (1 + \lambda(N-1)) \log\left[\frac{1+(N-1)\lambda}{1-\lambda}\right]}{N\lambda^2(1+(N-1)\lambda)}$$

the previous expression is negative if and only if

$$\begin{aligned} \frac{N\lambda}{(1+(N-1)\lambda)} &< \log\left[\frac{1+(N-1)\lambda}{1-\lambda}\right] \\ &< \frac{1+(N-1)\lambda}{1-\lambda}, \end{aligned}$$

or equivalently if

$$\begin{aligned} (1+(N-1)\lambda)^2 &> (N\lambda)(1-\lambda) \\ 1 + \lambda(N-2) + \lambda^2((N-1)^2 + N) &> 0. \end{aligned}$$

which is always true for $N \geq 2$. Now that we establish that $\gamma(\lambda)$ is strictly decreasing it suffices to show that

$$\lim_{\lambda \to 0} \gamma(\lambda) = 1.$$

which can be obtained using L'Hopital's rule, and

$$\partial_\lambda \left[(1-\lambda) \log\left(\frac{1+(N-1)\lambda}{1-\lambda}\right)\right]\Bigg|_{\lambda=0} = N.$$

which completes the proof. ■

(i) An explicit expression for $H(p)$ is given by

$$H(p) = p - \frac{(1-\lambda)}{N\lambda}\left[(\bar{p}_h - p) + (\bar{p}_h - c_l)\log\left[\left(\frac{p - c_l}{\bar{p}_h - c_l}\right)\left(\frac{1+\lambda(N-1)}{1-\lambda}\right)\right]\right] - s. \tag{D.9}$$

Evaluating equation(D.9) at $\bar{p}_h$ we obtain:

$$\begin{aligned} H(\bar{p}_h) &= \bar{p}_h - \frac{(1-\lambda)}{N\lambda}\left[(\bar{p}_h - c_l)\log\left[\left(\frac{1+\lambda(N-1)}{1-\lambda}\right)\right]\right] - s \\ &= (1-\gamma(\lambda))\bar{p}_h + \gamma(\lambda)c_l - s. \end{aligned}$$

From our derivation of r_h, $s = (1-\gamma(\lambda))(r_h - c_h)$, and $\bar{p}_h \leq r_h$ by equation (3.6), which implies $s \leq (1-\gamma(\lambda))(\bar{p}_h - c_h)$, since $\gamma(\lambda) < 1$. Therefore

$$H(\bar{p}_h) \geq \gamma(\lambda)c_l + (1-\gamma(\lambda))c_h > 0.$$

(ii) Using equation (D.9)

$$H(\underline{p}_h) = \underline{p}_h - \frac{(1-\lambda)}{N\lambda}\left[(\bar{p}_h - \underline{p}_h) + (\bar{p}_h - c_l)\log\left[\left(\frac{\underline{p}_h - c_l}{\bar{p}_h - c_l}\right)\left(\frac{1+\lambda(N-1)}{1-\lambda}\right)\right]\right] - s.$$

Using the expression for $\underline{p}_h$ as a function of $\bar{p}_h$ presented in Proposition 3.7, and after some algebra we obtain

$$H(\underline{p}_h) = (c_h - c_l) + (\bar{p}_h - c_l)\left(\frac{(1-\lambda)}{N\lambda}\left(\log\left[\left(\frac{\underline{p}_h - c_l}{\bar{p}_h - c_l}\right)\right]\right) - \gamma(\lambda)\right) - s,$$

and the condition established by equation (3.9) follows. ■

D.8 Proof of Lemma 3.11

First, we show that $H(p)$ is a strictly convex function. This follows directly from the second derivative

$$\frac{d^2H(p)}{dq^2} = \frac{(1-\lambda)\,(\bar{p}_h - c_l)}{N\lambda\,(p-c_l)^{\,2}} > 0$$

for $\lambda \in (0,1)$. The second part of the proof consist of showing that the first derivative at $\underline{p}_h$ is equal to zero.

$$\left.\frac{dH(p)}{dp}\right|_{\underline{p}_l} = 1 - \frac{(1-\lambda)}{N\lambda}\left(\frac{\bar{p}_h - \underline{p}_l}{\underline{p}_l - c_l}\right).$$

Using the expression for $\underline{p}_l$ as a function of $\bar{p}_h$ presented in Proposition 3.9, we have

$$\begin{aligned}
\underline{p}_h - \underline{p}_l &= \underline{p}_h - \left[\frac{1-\lambda}{1+\lambda(N-1)}\,(\bar{p}_h - c_l) + c_l\right] \\
&= \left(\frac{\lambda N}{1+\lambda(N-1)}\right)(\bar{p}_h - c_l),
\end{aligned}$$

and

$$\underline{p}_l - c_l = \frac{1-\lambda}{1+\lambda(N-1)}\,(\bar{p}_h - c_l)\,;$$

therefore

$$\left(\frac{\bar{p}_h - \underline{p}_l}{\underline{p}_l - c_l}\right) = \frac{\lambda N}{1-\lambda},$$

and $dH(\underline{p}_l) = 0$ follows. ■

D.9 Proof of Lemma 3.12

Propositions 3.7 and 3.9 give the unique $F(p;c_h)$ and $F(p;c_l)$ with connected support, respectively. By definition, $F(p;c_h)$ first order stochastic dominates $F(p;c_l)$

if and only if

$$F(p;c_h) \leq F(p;c_l)$$

for all p. By direct verification of the previous condition:

$$\begin{aligned}
1 - \left[\frac{1-\lambda}{\lambda N}\left(\frac{\bar{p}_h - p}{p - c_h}\right)\right]^{\frac{1}{N-1}} &\leq 1 - \left[\frac{1-\lambda}{\lambda N}\left(\frac{\bar{p}_h - p}{p - c_l}\right)\right]^{\frac{1}{N-1}} \\
\left(\frac{\bar{p}_h - p}{p - c_h}\right) &\geq \left(\frac{\bar{p}_h - p}{p - c_l}\right) \\
\left(\frac{p - c_l}{p - c_h}\right) &\geq 1.
\end{aligned}$$

which is satisfied as $c_h > c_l$. ■

D.10 Proof of Proposition 3.13

We show the existence of this type of equilibrium by construction. First, let firm j set a price $p_j \in [a, \underline{p}_h)$ while the rest of the firms set their prices according to $F(p)$. Recall that μ_{-j} is the expected fraction of consumers engaging in costly search as a result of the $-j$ firms setting prices according to $F(p)$. For $\tilde{F}(p)$ to part of an equilibrium as described in Proposition 3.13 firm j should be indifferent between setting any price in the interval $[a, \underline{p}_h)$ and $p_j = \bar{p}$, that is:

$$\left[1 - \tilde{F}(p)\right]\left[\lambda + \mu_{-j} + \frac{1-\lambda}{N}\right](p_j - c_l) = \left(\frac{1-\lambda}{N}\right)(\bar{p}_h - c_l). \tag{D.10}$$

Where μ_{-j} is given by:

$$\mu_{-j} = \sum_{n=2}^{N}(1-\mathbb{Q})^{N-n}\mathbb{Q}^{n-1}\left[\frac{(n-1)(1-\lambda)}{N}\right], \tag{D.11}$$

and $\mathbb{Q} = \tilde{F}(\underline{p}_h) - \tilde{F}(a)$. Clearly, μ_{-j} is a function of the size of the interval $[a, \underline{p}_h)$, through $\mathbb{Q}$, and $\mu_{-j} < \frac{(N-1)(1-\lambda)}{N}$. Otherwise, if $\mu_{-j} = \frac{(N-1)(1-\lambda)}{N}$ the total level of search (costly and non costly) would be equal to one which imply that all firms set prices in the interval $[a, \underline{p}_h)$ with probability one. Fix an arbitrary value for μ_{-j} for now so $\tilde{F}(p)$ can be calculated from equation D.10.

In order to calculate $\hat{F}(p)$ firm j should be indifferent between setting a price in the support of $\hat{F}$ and and $p_j = \bar{p}$ and

$$\left[\left(1 - \hat{F}(p)\right)(\lambda + \mu_{-j}) + \frac{1-\lambda}{N}\right](p_j - c_l) = \left(\frac{1-\lambda}{N}\right)(\bar{p}_h - c_l). \tag{D.12}$$

For F to be an equilibrium, $r(0)$ must be consistent with μ_{-j}, that is, the price level determining the range of prices at which consumers will not search must induce a level of endogenous search equal to μ_{-j}, and μ_{-j} must be consistent with $r(0)$, in other words, given the level of endogenous search, $r(0)$ is an optimal reservation price. For a given starting point μ_{-j} we can obtain the implied consistent reservation price, $r(0)$, from equation 3.4, which in turn will determine the value of $\tilde{F}(a)$ since, by construction $\hat{F}(r(0)) = \tilde{F}(a)$, and $\tilde{F}(a)$ will update the initial value of μ_{-j} through $\mathbb{Q}$. To show that a fix point to this problem exists note that when μ_{-j} tends to $\frac{(N-1)(1-\lambda)}{N}$, each individual consumer who observe a price consistent with the low cost regime has less incentives to search as higher levels of search will push the highest price charged by the firms towards c_l, thus increasing $r(0)$, which in turns reduces the size of the interval inducing costly search pressing μ_{-j} downwards. Similarly, for values of μ_{-j} close to zero, firms have a higher incentive to raise prices (using the mix strategy F) and the gains of search increases, thus reducing $r(0)$ and increasing the size of the interval with costly search, pushing μ_{-j} upwards. Since both equations 3.4 and D.11 are continuous functions, its composition is also a continuous function and we can use the mid point theorem to verify the existence of the fix point.

In order for our equilibrium to be well defined we need to verify that $a > r(0)$, but note that as $s \to 0$, $r(0)$ approaches $\underline{p}_l$. However, a cannot get arbitrarily close to p_l since reducing the value of a increases μ_{-j} which is determined by the fix point explained above. Therefore, for sufficiently low values of s, $a > r(0)$ will be satisfied.

■

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