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Fuchsian Groups of Signature $(0:2,\ldots,2;1;0)$ with Rational Hyperbolic Fixed Points

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Fuchsian Groups of Signature $(0 : 2, \ldots, 2; 1; 0)$ with Rational Hyperbolic Fixed Points

by

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Dedicated to my family.

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Fuchsian Groups of Signature $(0 : 2, \ldots, 2; 1; 0)$ with Rational Hyperbolic Fixed Points

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We construct Fuchsian groups Γ of signature $(0 : 2, \ldots, 2; 1; 0)$ so that the set of hyperbolic fixed points of Γ will contain a given finite collection of elements in the boundary of the hyperbolic plane. We use this to establish that there are infinitely many non-commensurable non-cocompact Fuchsian groups Δ of finite covolume sitting in $\mathrm{PSL}_2(\mathbb{Q})$ so that the set of hyperbolic fixed points of Δ will contain a given finite collection of rational boundary points of the hyperbolic plane. We also give a parameterization of Fuchsian groups of signature $(0 : 2, 2, 2; 1; 0)$ and investigate when particular hyperbolic elements have rational fixed points. Moreover, we include a detailed list of the group elements and their *killer intervals* for the known pseudomodular groups that Long and Reid found; in addition, the list contains a new list of killer intervals for a pseudomodular group not found by Long and Reid.

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Chapter 1

Introduction

A prominent example of a Fuchsian group is the quotient $\mathrm{PSL}_2(\mathbb{Z}) = \mathrm{SL}_2(\mathbb{Z})/\{\pm I\}$ of modular group $\mathrm{SL}_2(\mathbb{Z})$. The group $\mathrm{PSL}_2(\mathbb{Z})$ is a discrete subgroup of $\mathrm{PSL}_2(\mathbb{R})$; as such, it acts discontinuously by fractional linear transformations of the upper half plane model of the hyperbolic plane. Much is known for this group; for example, the set $\mathbb{Q} \cup \{\infty\}$ is all fixed points of parabolics in $\mathrm{PSL}_2(\mathbb{Z})$ (see Example 5.1.1), and every real quadratic number is the fixed point of a hyperbolic element of $\mathrm{PSL}_2(\mathbb{Z})$ (see Lemma 3.1 of [2]).

Let Γ be a Fuchsian group, meaning a discrete subgroup of the group of orientation preserving isometries of H^2, the hyperbolic plane. A boundary point of H^2 fixed by a parabolic element of a Fuchsian group Γ is referred to as a *cusp* of Γ, and a line fixed by a hyperbolic element is referred to as an *axis* with endpoints called hyperbolic fixed points. For an arbitrary Fuchsian group, determining its set of cusps, hyperbolic fixed points, and/or axes is quite a challenge; for some of the literature addressing this type of problem, see [2] and its references.

Recall that Fuchsian groups Γ_1 and Γ_2 are *commensurable* if Γ_1 has a subgroup of finite index which is conjugate to a subgroup of finite index in Γ_2.

This work has been motivated by the following questions:

Questions. If Γ_1 and Γ_2 are finite covolume Fuchsian groups with the same set of cusps (or same set of axes), when are they commensurable?

In [2], Long and Reid exhibit four examples of mutually noncommensurable subgroups of $\mathrm{PSL}_2(\mathbb{Q})$, which are not commensurable with the modular group, but each of them have cusp set exactly $\mathbb{Q} \cup \{\infty\}$; they call such groups *pseudomodular*. It is still unknown whether or not there are infinitely many pseudomodular groups up to commensurability. Any other possible candidate for pseudomodular groups are (non-arithmetic) discrete subgroups $\Delta \leq \mathrm{PSL}_2(\mathbb{Q})$, since their cusp set is contained in $\mathbb{Q} \cup \{\infty\}$. For Fuchsian groups, a boundary point cannot both be a cusp and a hyperbolic fixed point. Hence arithmetic and pseudomodular groups cannot have rational hyperbolic fixed points. If one can exhibit a hyperbolic element of Δ that has rational fixed points, then Δ's cusp set is properly contained in $\mathbb{Q} \cup \{\infty\}$; thus showing Δ to be neither arithmetic nor pseudomodular. So Long and Reid asked how to predict when rational hyperbolic fixed points are present.

Another motivation arises from a question A. Rapinchuk asked: Are there infinitely many commensurability classes of finite covolume Fuchsian groups sitting in $\mathrm{PSL}_2(\mathbb{Q})$? Vinberg, answering this question in a preprint [5], has introduced a way to produce infinitely many noncommensurable finite covolume Fuchsian groups in $\mathrm{SL}_2(\mathbb{Q})$. His examples arise as the even subgroup of a group generated by reflections in the sides of quadrilaterals. To establish that he constructed infinitely many groups up to commensurability, Vinberg uses results (from [6]) about the least ring of definition for his examples.

We provide a new solution to Rapinchuk's question, which is different from Vinberg's, and the results also address the presence of rational hyperbolic fixed points. Namely, we will construct Fuchsian groups sitting in $\mathrm{PSL}_2(\mathbb{Q})$, all of which will possess a given finite set of rational hyperbolic fixed points. The result is

Theorem. *Let Y be a finite set of rational boundary points of the hyperbolic plane. Then there are infinitely many noncommensurable finite covolume Fuchsian groups sitting in* $\mathrm{PSL}_2(\mathbb{Q})$*, whose set of hyperbolic fixed points contains* Y.

This will be proved in Section 4.2. As a brief outline, let Y be a finite number of boundary points of the hyperbolic plane. We construct (with considerable freedom) examples of Fuchsian groups Γ of signature $(0 : 2, \ldots, 2; 1; 0)$ such that the set of hyperbolic fixed points of Γ contains Y (see Section 3.1); furthermore, when Y is a set of rational boundary points, and by restricting some of the freedom in the construction, one can guarantee $\Gamma \leq \mathrm{PGL}_2(\mathbb{Q})$. Then we address when the constructed groups are mutually noncommensurable, which relies on analyzing how they act on different trees (see Section 4.2). Specifically, we consider fixed points of the action of Γ on Serre's trees of $\mathrm{SL}_2(\mathbb{Q}_p)$ for primes $p \equiv 3 \mod 4$ (see Proposition 4.2.1). This perspective allows us to construct an infinite family of mutually noncommensurable groups in $\mathrm{PSL}_2(\mathbb{Q})$.

In Chapter 5, we discuss Fuchsian groups Γ of signature $(0 : 2, 2, 2; 1; 0)$. We present the known examples of pseudomodular groups, give a parameterization of the Γ's which define complete hyperbolic spheres with three cone points of order 2 and one cusp, and investigate when particular hyperbolic elements of Γ have rational fixed points.

Chapter 2

Preliminaries

In this chapter, we will review some preliminaries on hyperbolic geometry, Fuchsian groups, and Serre's trees of $\mathrm{SL}_2(\mathbb{Q}_p)$.

2.1 The Hyperbolic Plane

The hyperbolic plane H^2 is 2-dimensional hyperbolic space. We will define two standard models for the hyperbolic plane for notational purposes. Furthermore, we will be omitting the development of the explicit metrics for these models, but refer the reader to [3] for a treatise on the standard models for n-dimensional hyperbolic space. We will conclude this section by relating the two models.

Minkowski Space and the Hyperboloid Model

Let M^3 be a dimension 3 real vector space with a nondegenerate quadratic form $\langle \bullet, \bullet \rangle$ of signature $(2,1)$. We will notate $L = \{v : \langle v, v \rangle = 0\}$ (the set of *light-like vectors*) and $T = \{v : \langle v, v \rangle < 0\}$ (the set of *time-like vectors*).

Choose a basis (e_1, e_2, e_0) with $\langle e_i, e_j \rangle = 0$ if $i \neq j$, $\langle e_i, e_i \rangle = 1$ if $i \geq 1$, and $\langle e_0, e_0 \rangle = -1$. Such a basis is called a *Lorentz orthonormal basis*, and M^3

is denoted as $\mathbb{R}^{2,1}$ when such a basis is fixed; $\mathbb{R}^{2,1}$ is called *Minkowski space*. Let L^+ and T^+ be the sets of vectors with positive e_0-coordinate in L and T, respectively.

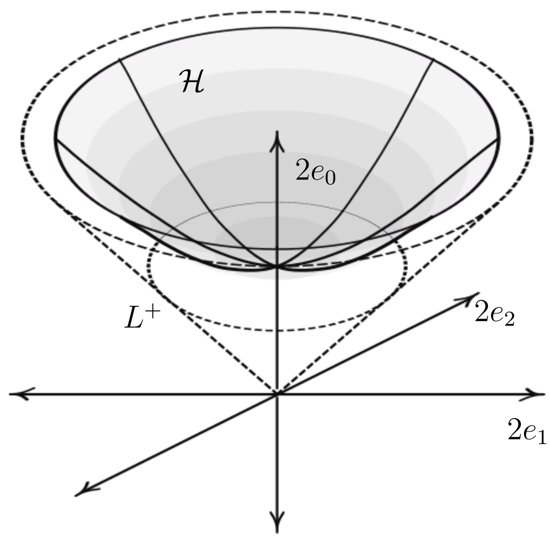

Figure 2.1: Dashed lines outline the light-cone L^+ and upper sheet of the hyperboloid $\mathcal{H}$ is shaded.

We let $\mathrm{O}(2,1)$ be the group of linear transformations of M^3 that preserves the quadratic form; furthermore, the subgroup $\mathrm{O}^+(2,1)$ of elements preserving the upper sheet of the hyperboloid $\mathcal{H} = \{v : \langle v, v\rangle = -1\} \cap T^+$ has index two in $\mathrm{O}(2,1)$. A well-known theorem (see [3] pg 64) is

Theorem 2.1.1. *The group* $\mathrm{O}^+(2,1)$ *is isomorphic to the groups of isometries of* H^2.

A ray from the origin in L^+ corresponds to a point on the boundary H^2. A vector $v \in L^+$ corresponds to a *horosphere*

$$\{w \in \mathcal{H} : \langle v, w \rangle = -1\},$$

and the corresponding *horoball* is

$$\{w \in \mathcal{H} : 0 > \langle v, w \rangle \geq -1\}.$$

Upper Half-Plane Model

An additional reference for the upper half plane model of the hyperbolic plane is [1]. The *upper half-plane model* of H^2 is $\mathbb{U}^2 = \{z \in \mathbb{C} : \Im(z) > 0\}$ with a specific metric.

The group $\mathrm{GL}_2^+(\mathbb{R})$, invertible 2 by 2 matrices having positive determinant, acts by the fractional linear transformations on $\mathbb{U}^2$; that is, for $z \in \mathbb{U}^2$,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az+b}{cz+d}, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2^+(\mathbb{R}),$$

which factors through to $\mathrm{PGL}_2^+(\mathbb{R}) = \mathrm{GL}_2^+(\mathbb{R})/Z(\mathrm{GL}_2^+(\mathbb{R})) \cong \mathrm{PSL}_2(\mathbb{R})$. A well-known theorem (see [3] Theorem 7.4.1) is

Theorem 2.1.2. *The group* $\mathrm{PSL}_2(\mathbb{R})$ *is isomorphic to the orientation preserving isometries of* H^2*, which is an index two subgroup of the group of isometries of* H^2*.*

Remark. At times, we implicitly identify $\mathrm{PGL}_2^+(\mathbb{R})$ with $\mathrm{PSL}_2(\mathbb{R})$ so that we can consider, e.g., the group $\mathrm{PGL}_2^+(\mathbb{Q}) \leq \mathrm{PGL}_2^+(\mathbb{R})$ as a subgroup of

$\mathrm{PSL}_2(\mathbb{R})$; furthermore, we will only explicitly distinguish between a matrix and its equivalence class when necessary. Moreover, conjugating subgroups of $\mathrm{PGL}_2^+(\mathbb{R})$ in $\mathrm{PGL}_2(\mathbb{R})$ gives another subgroup of $\mathrm{PGL}_2^+(\mathbb{R})$, since conjugation preserves the sign of the determinant.

Relating the Groups $\mathrm{O}^+(2,1)$ **and** $\mathrm{PGL}_2(\mathbb{R})$

We begin by considering vectors inside the light cone in Minkowski space. For every $v \in T^+$, there is a 2 by 2 real symmetric matrix whose determinant is $-\langle v, v\rangle$; namely,

$$v \mapsto \begin{pmatrix} \langle v, e_2 + e_0\rangle & \langle v, e_1\rangle \\ \langle v, e_1\rangle & \langle v, -e_2 + e_0\rangle \end{pmatrix}.$$

We have that $\mathrm{GL}_2(\mathbb{R})$ acts on 2 by 2 real symmetric matrices by similarity; that is, $\Sigma \mapsto A^{\mathrm{t}}\Sigma A$, where $A \in \mathrm{GL}_2(\mathbb{R})$ and Σ is a 2 by 2 real symmetric matrix. We use this to relate the hyperboloid and upper half plane models. The following example illustrates the way to relate the two models.

Example 2.1.3. Consider the isometry ρ_f, rotation by π with fixed point $f \in H^2$.

In the hyperboloid model, ρ_f corresponds to an element in $\mathrm{O}^+(2,1)$; let $f \in T^+$ be fixed by $\rho_f \in \mathrm{O}^+(2,1)$. Let Σ_f be the 2 by 2 real symmetric matrix associated to f,

$$\Sigma_f = \begin{pmatrix} \langle f, e_2 + e_0\rangle & \langle f, e_1\rangle \\ \langle f, e_1\rangle & \langle f, -e_2 + e_0\rangle \end{pmatrix}.$$

In the upper half plane model of H^2, say $a + bi$ is fixed by ρ_f as a matrix in

$\mathrm{SL}_2(\mathbb{R})$; that is,

$$(*) \qquad \rho_f = \begin{pmatrix} b & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} b & a \\ 0 & 1 \end{pmatrix}^{-1} = \frac{1}{b} \begin{pmatrix} a & -(b^2+a^2) \\ 1 & -a \end{pmatrix}.$$

We have that ρ_f, as the matrix from $(*)$, acts by similarity on 2 by 2 real symmetric matrices and fixes Σ_f when

$$b = -\frac{\sqrt{|\langle f, f \rangle|}}{\langle f, e_2 + e_0 \rangle} \quad \text{and} \quad a = -\frac{\langle f, e_1 \rangle}{\langle f, e_2 + e_0 \rangle}.$$

Note: in $\mathrm{PGL}_2(\mathbb{R})$, ρ_f can be represented by a matrix with entries in $\mathbb{Q}$ when $\langle f, f \rangle$, $\langle f, e_i \rangle$ are all in $\mathbb{Q}$.

Remark. Unless otherwise stated, the isometry denoted by ρ_f is rotation by π with fixed point $f \in H^2$. The lettering for a point in the closure of H^2 will denote a vector in Minkowski space $\mathbb{R}^{2,1}$, when considering the hyperboloid model of H^2; for example, we will write f for a vector in T^+ and its corresponding point in H^2 will also be written as f. We will only bring up this abuse of notation when necessary. Likewise, the lettering for isometries of H^2 will also denote its corresponding element in $\mathrm{O}^+(2,1)$ or $\mathrm{PGL}_2(\mathbb{R})$.

2.2 Fuchsian groups

In this section we give some results about Fuchsian groups (see [1]).

Let Γ be a Fuchsian group, meaning a discrete subgroup of the group of orientation preserving isometries of H^2. Orientation preserving isometries of H^2, denoted $\mathrm{Isom}^+(H^2)$, fall into three distinct categories, based on their fixed points.

Let $\gamma \in \mathrm{Isom}^+(H^2)$ and Γ be a Fuchsian group.

1. If γ fixes a point of H^2, then γ is called *elliptic*.

2. If γ fixes no points of H^2 and exactly one point of ∂H^2, then γ is called *parabolic*. The boundary point of H^2 fixed by a parabolic element $\gamma \in \Gamma$ is referred to as a *cusp* of Γ.

3. If γ fixes no points of H^2 and exactly two points of ∂H^2, then γ is called *hyperbolic*. The line fixed by a hyperbolic element γ is referred to as the *axis* of γ with endpoints called hyperbolic fixed points. The set of hyperbolic fixed points of hyperbolic elements in Γ is denoted by $\mathrm{HFix}(\Gamma)$.

The action of Γ on H^2 admits a connected fundamental domain; if this has finite area, then Γ has finite covolume. Furthermore, for a given finitely generated non-elementary Fuchsian group Γ, its *signature* is $(g : m_1, \ldots, m_r; s; t)$, where

g is the genus of H^2/Γ,

m_i's are the orders of the r conjugacy classes of maximal elliptic cyclic subgroups of Γ,

s is the number of conjugacy classes of maximal parabolic cyclic subgroups of Γ,

t is the numbers of conjugacy classes of maximal boundary hyperbolic cyclic subgroups in Γ.

Fuchsian groups Γ_1 and Γ_2 are *commensurable* if Γ_1 has a subgroup of finite index which is conjugate to a subgroup of finite index in Γ_2. The *commensurability class* of a Fuchsian group Γ is the collection of groups that are commensurable with Γ.

2.3 Serre's Tree for $\mathrm{SL}_2(K)$

As in Serre's book [4] Chapter II§1, let K denote a field with a discrete valuation v; recall that v is a homomorphism of $K^\times$ onto $\mathbb{Z}$, and $\mathcal{O}_v$ denotes the *valuation ring* of K, i.e., the set of $x \in K$ such that $v(x) \geq 0$ or $x = 0$. Fix an element $\pi \in K$ with $v(\pi) = 1$, the *uniformizer*. Furthermore, let k_v denote the *residue field* $\mathcal{O}_v/\pi\mathcal{O}_v$. If $K = \mathbb{Q}$, then most v subscripts are replaced with the letter p for the p-adic valuation v_p.

The Tree $\mathcal{T}_v$

Let V be a vector space of dimension 2 over K. A *lattice* in V is any finitely generated $\mathcal{O}_v$-submodule of V which generates the K-vector space V; such a module is free of rank 2. The group $K^\times$ acts on the set of lattices; we call the orbit of a lattice L under this action its class (at times notated $[L] = \Lambda$), and two lattices belonging to the same class are called equivalent. The set of lattice classes is denoted by $\mathcal{T}_v$, which is made into a combinatorial graph with edges between Λ_0 and Λ_1 when $[L_i] = \Lambda_i$ such that $L_0 \leq L_1$ and

$L_1/L_0 \cong \mathcal{O}_v/\pi\mathcal{O}_v$. Serre proved that $\mathcal{T}_v$ is a tree.

Example 2.3.1. Let L_0 and L_1 be two lattices of V. By the invariant factor theorem, there is an $\mathcal{O}_v$-basis $\{e_1, e_2\}$ of L_1 and integers n_1, n_2 such that $\{\pi^{n_1}e_1, \pi^{n_2}e_2\}$ is an $\mathcal{O}_v$-basis for L_0. It can be seen that $L_0 \subset L_1$ if and only if $n_i \geq 0$; if $L_0 \subset L_1$ then L_1/L_0 is isomorphic to $(\mathcal{O}_v/\pi^{n_1}\mathcal{O}_v) \oplus (\mathcal{O}_v/\pi^{n_2}\mathcal{O}_v)$.

Now consider xL_0 and yL_1, where $x, y \in K^\times$. In the above paragraph, we replace n_i with $n_i + c$, where $c = v(y/x)$. Thus, the integer $|n_1 - n_2|$ depends only on the lattice classes Λ_0 and Λ_1 of L_0 and L_1, respectively. The integer $|n_1 - n_2|$ is the distance between Λ_0 and Λ_1.

$\mathrm{GL}_2(K)$ action on $\mathcal{T}_v$

We continue with the notation above, but will often suppress the subscript v. For $B \in \mathrm{GL}_2(K)$ as a linear automorphism of $K^2 = V$, B maps a lattice onto a lattice, and B also maps equivalent lattices to equivalent lattices. So there is a natural action of $\mathrm{GL}_2(K)$ on the set of lattice classes. Under this action, the valuation of the determinant is an invariant in the following way:

Lemma 2.3.2. *Let L_0 and L_1 be lattices, and let $A, B \in \mathrm{GL}_2(K)$ such that both map L_0 onto L_1. Then $v(\det(A)) = v(\det(B))$.*

We omit the proof, since it follows directly from $B^{-1}A(L_0) = L_0$ and properties of v and det.

One can show $\mathrm{GL}_2(K)$ acts by isometries on $\mathcal{T}_v$.

Example 2.3.3. Let $\mathcal{O}^2$ be the standard lattice in $K^2 = V$. It can be shown that

$$\mathrm{Stab}_{\mathrm{GL}_2(K)}(\mathcal{O}^2) = \mathrm{GL}_2(\mathcal{O}).$$

In fact, if $A \in \mathrm{SL}_2(K)$ and $A[\mathcal{O}^2] = [\mathcal{O}^2]$, then $A\mathcal{O}^2 = x\mathcal{O}^2$, where $x \in K^\times$. Since $A \in \mathrm{SL}_2(K)$, $0 = v(\det A) = 2v(x)$, which implies x is a unit in $\mathcal{O}$; hence $A\mathcal{O}^2 = \mathcal{O}^2$, so $A \in \mathrm{SL}_2(\mathcal{O})$. Therefore,

$$\mathrm{Stab}_{\mathrm{SL}_2(K)}([\mathcal{O}^2]) = \mathrm{Stab}_{\mathrm{SL}_2(K)}(\mathcal{O}^2) = \mathrm{SL}_2(\mathcal{O}).$$

Let L be a lattice, and let $B \in \mathrm{GL}_2(K)$ such that $B\mathcal{O}^2 = L$. Then

$$\mathrm{Stab}_{\mathrm{GL}_2(K)}(L) = B\,\mathrm{Stab}_{\mathrm{GL}_2(K)}(\mathcal{O}^2)B^{-1} = B\mathrm{GL}_2(\mathcal{O})B^{-1}.$$

Let us consider the link of $[\mathcal{O}^2]$ in the tree $\mathcal{T}_v$; that is, $[L] = \Lambda \in \mathcal{T}_v$ such that $\pi\mathcal{O}^2 \subset L \subset \mathcal{O}^2$. Lattices in the link are in bijective correspondence with submodules of the quotient module $\mathcal{O}^2/\pi\mathcal{O}^2$ (which is a 2-dimensional vector space over the residue field $k_v = \mathcal{O}/\pi\mathcal{O}$). With a little more work, one can see that the vertices in the link are in bijection with 1-dimensional subspaces of $\mathcal{O}^2/\pi\mathcal{O}^2$.

If we identify 1-dimensional subspaces of $\mathcal{O}^2/\pi\mathcal{O}^2$ with elements of $k_v \cup \{\infty\}$, then $\mathrm{SL}_2(k_v)$ acts on $k_v \cup \{\infty\}$ by linear fractional transformations; that is,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az+b}{cz+d}, \qquad \text{for } z \in k_v \cup \{\infty\}.$$

We see $\mathrm{SL}_2(\mathcal{O})$ acts in this way also, since $\mathrm{SL}_2(\mathcal{O}) \longrightarrow \mathrm{SL}_2(k_v)$.

Example 2.3.4. Let $\rho = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. Then ρ acts on the link of $[\mathcal{O}^2]$; consider its action by linear fractional transformation

$$\rho z = \frac{-1}{z}.$$

So if -1 is not a square then ρ does not fix any vertices in the link of $[\mathcal{O}^2]$.

We conclude this chapter by stating (without proof) the results below about groups acting on trees; they will be needed in Chapter 4 (for justification see Serre's book [4] §6.5).

Proposition 2.3.5. *Let G be a group generated by the elements a_i, b_i and let A, B be the subgroups of G generated by the a_i and b_i, respectively. Assume that G acts on a tree X so that $\mathrm{Fix}_X(A) \neq \emptyset$, $\mathrm{Fix}_X(B) \neq \emptyset$ and so that, for each pair (i, j) the automorphism $a_i b_j$ has a fixed point. Then G has a fixed point; that is, $\mathrm{Fix}_X(G) \neq \emptyset$.*

Corollary 2.3.6. *Let G act on a tree X, and suppose that G is generated by a finite number of elements $s_1, \ldots, s_n$ such that the s_j and $s_i s_j$ have fixed points. Then G has a fixed point.*

Chapter 3

The Construction of Γ

In this chapter, we will construct a fundamental domain for a Fuchsian group Γ of signature $(0 : 2, \ldots, 2; 1; 0)$ so that the set of hyperbolic fixed points of Γ, notated by $\mathrm{HFix}(\Gamma)$, will contain a given finite set Y of elements in boundary of the hyperbolic plane. We will also find some (sufficient) conditions for Γ to be a subgroup of $\mathrm{PGL}_2(\mathbb{Q})$; enabling us to produce an infinite family of Fuchsian groups sitting in $\mathrm{PSL}_2(\mathbb{Q})$.

3.1 The Construction of a Fundamental Domain

Before the construction begins, we introduce some notation and a lemma. Let H^2 be the hyperbolic plane. We will write $\overline{xy}$ to be the closure of the geodesic line with end points x, y in the closure of H^2. The isometry denoted by ρ_f is rotation by π with fixed point $f \in H^2$. Let v_0 and v_∞ be distinct elements in ∂H^2; then we have an order $\leq$ on $\partial H^2 \setminus \{v_\infty\}$ (i.e. the order on the real line in the upper half plane model).

Lemma 3.1.1. *Let $x < v < y < u$ in $\partial H^2 \setminus \{v_\infty\}$. For each f in the interior of $\overline{cy}$, where $c = \overline{vu} \cap \overline{xy}$, one constructs $t = \rho_f(u)$ and $w = \rho_f(v)$. Then $x < v < t < y < w < u$ in ∂H^2, $f = \overline{xy} \cap \overline{tu}$, and $f \in \overline{vw}$.*

We omit the proof, but include a figure to illustrate the lemma.

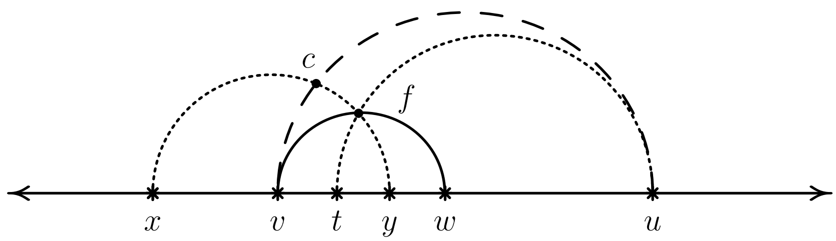

Figure 3.1: Lemma 3.1.1 drawn in upper half plane model.

As an overview of the construction (Figure 3.2 illustrates an example), we start with a finite set of points in the boundary of the hyperbolic plane (the set $Y = \{y_i\}$). We use Lemma 3.1.1 to sequentially construct the vertices and edges (the solid lines in Figure 3.2) of an ideal convex polygon in H^2. We then show that the ideal convex polygon constructed is a fundamental domain for a discrete group Γ generated by isometries that are rotations by π. Furthermore, Γ is guaranteed to possess a set of hyperbolic elements (the dash lines are their axes in Figure 3.2) whose fixed points contain the initially given set of boundary points.

Below we will have the notational convention: $f_i \in H^2$, $\rho_i = \rho_{f_i}$ is rotation by π with fixed point f_i, and $y_i, x_i, v_i \in \partial H^2$. As a slight variation when considering the hyperboloid model of the hyperbolic plane (as in Chapter 4), f_i, x_i, y_i, v_i will be vectors in Minkowski space $\mathbb{R}^{2,1}$.

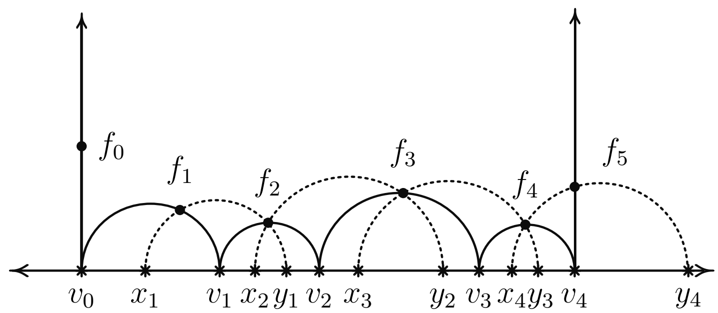

Figure 3.2: The solid lines bound a fundamental domain for Γ in the upper half plane model with $Y = \{y_1, \ldots, y_4\}$.

The Construction of a Fundamental Domain for Γ

We begin the construction; let Y be a finite set of $n-1$ points in the boundary of the hyperbolic plane, and let $Y = \{y_i\}$ so that $v_0 < y_1 < \cdots < y_{n-1} \neq v_\infty$ in the boundary of the hyperbolic plane.

$1^{\mathbf{st}}$ **Step:** Choose x_1 such that $v_0 < x_1 < y_1$ in ∂H^2, and then choose $f_1 \in \overline{x_1 y_1}$. Define $v_1 = \rho_1(v_0)$; note $v_0 < x_1 < v_1 < y_1$ in ∂H^2, and $f_1 \in \overline{v_0 v_1}$.

When $n > 2$, let $i \in \{2, \ldots, n-1\}$.

$i^{\mathbf{th}}$ **Step:** Let $x_{i-1} < v_{i-1} < y_{i-1} < y_i$ in ∂H^2. By Lemma 3.1.1, one can choose a $f_i \in \overline{x_{i-1} y_{i-1}}$ and construct $x_i = \rho_i(y_i)$ and $v_i = \rho_i(v_{i-1})$, so that $v_{i-1} < x_i < y_{i-1} < v_i < y_i$ and $f_i \in \overline{v_{i-1} v_i}$.

$n^{\mathbf{th}}$ **Step:** Let $x_{n-1} < v_{n-1} < y_{n-1} \neq v_\infty$ in ∂H^2. Now construct $f_n = \overline{x_{n-1}y_{n-1}} \cap \overline{v_{n-1}v_\infty}$, and define $v_n = \rho_n(v_{n-1})$.

Last Step: Given $\rho_n \cdots \rho_1(v_0) = v_n$, construct $f_0 \in \overline{v_0 v_n}$ and ρ_0 so that $\rho_n \cdots \rho_1 \rho_0$ is parabolic fixing v_n.

Remark. When this construction is done with vectors in Minkowski space $\mathbb{R}^{2,1}$, v_n and v_∞ are linearly dependent light-like vectors, and in the last step one can see for $f_0 \in \text{span}\,\{v_n + v_0\}$, the element ρ_0 as a Lorentz transformation maps v_n to v_0 (as vectors in $\mathbb{R}^{2,1}$), which shows $\rho_n \cdots \rho_1 \rho_0$ is parabolic fixing v_n.

We have an ideal $n+1$ sided convex polygon P with vertices $\{v_i\}$ (see Figure 3.2); furthermore, f_i is on the edge $\overline{v_{i-1}v_i}$, and f_0 is on the edge $\overline{v_n v_0}$. Since ρ_i is rotation by π with fixed point $f_i \in H^2$, ρ_i maps $\overline{v_{i-1}v_i}$ to itself (likewise, ρ_0 maps $\overline{v_n v_0}$ to itself); that is, ρ_i maps the directed edge $\overline{f_i v_{i-1}}$ to the directed edge $\overline{f_i v_i}$, and ρ_0 maps the directed edge $\overline{f_0 v_0}$ to the directed edge $\overline{f_0 v_n}$. By Poincaré s Polyhedron Theorem (see section §9.8 in [1]), the group Γ generated by $\{\rho_1, \ldots, \rho_n, \rho_0\}$ is discrete, and P is a fundamental domain for Γ. In the **Last Step**, we made $\rho_n \ldots \rho_1 \rho_0$ parabolic fixing v_n; thus H^2/Γ is a complete finite area once punctured 2-sphere with $n+1$ cone points of order 2.

For $1 \leq i < n$, the element $\rho_i \rho_{i+1}$ is hyperbolic with axis $\overline{x_i y_i}$, since f_i and f_{i+1} both lie on the geodesic line $\overline{x_i y_i}$ (by construction); therefore, y_i is a hyperbolic fixed point for $\rho_i \rho_{i+1} \in \Gamma$.

By Lemma 3.1.1, one sees that there are infinitely many choices for each f_i (for $1 \leq i < n$), producing infinitely many such Fuchsian groups Γ; establishing the following:

Proposition 3.1.2. *Let Y be a finite set of $n-1$ points in ∂H^2. Then there are infinitely many Fuchsian groups Γ of finite covolume that have the signature $(0 : \underbrace{2, \ldots, 2}_{n+1}; 1; 0)$ such that $Y \subset \mathrm{HFix}(\Gamma)$.*

3.2 Slight Modifications of the Construction

In this section, we mention two modifications of the construction. The first modification allows us to choose the $x_i \in \partial H^2$ instead of the fixed points of the isometries use to generate Γ. This could be preferred when addressing questions having to do with axes. The second modification can (slightly) simplify producing a parabolic element stabilizing v_∞, since we do not need to construct ρ_0; the parabolic element is built from the isometries $\rho_1, \ldots \rho_n$ and $\tau\rho_i\tau^{-1}$, where τ is the reflection through the geodesic $\overline{v_0 v_\infty}$.

First Modification of the Construction

This modification allows us to choose the $x_i \in \partial H^2$ instead of choosing the f_i. We first need a lemma; we omit the proof, but again refer the reader to Figure 3.1.

Lemma 3.2.1. *Let $x < v < y < u$ in $\partial H^2 \setminus \{v_\infty\}$. For each t in ∂H^2 such that $v < t < y$, one constructs $f = \overline{xy} \cap \overline{tu}$ and $w = \rho_f(v)$. Then*

$x < v < t < y < w < u$ in ∂H^2 and $f \in \overline{vw}$.

Let Y be a finite set of $n-1$ points in the boundary of the hyperbolic plane, and let $Y = \{y_i\}$ so that $v_0 < y_1 < \cdots < y_{n-1} \neq v_\infty$ in the boundary of the hyperbolic plane.

We use $1^{\mathbf{st}}$ **Step** and the **Last Step** as given in previous section, and we modify $i^{\mathbf{th}}$ **Step** to

> When $n > 2$, let $i \in \{2, \ldots, n-1\}$.
>
> $i^{\mathbf{th}}$ **Modified Step:** Let $x_{i-1} < v_{i-1} < y_{i-1} < y_i$ in ∂H^2. By Lemma 3.2.1, one can choose a $x_i \in \partial H^2$ such that $v_{i-1} < x_i < y_{i-1}$ and construct $f_i = \overline{x_{i-1}y_{i-1}} \cap \overline{x_i y_i}$ and $v_i = \rho_i(v_{i-1})$, so that $v_{i-1} < x_i < y_{i-1} < v_i < y_i$ and $f_i \in \overline{v_{i-1}v_i}$.

This modification constructs an ideal $n+1$ sided convex polygon P with vertices $\{v_0, v_1, \ldots, v_{n-1}, v_n = v_\infty\}$ just as in the previous section.

Second Modification of the Construction

Let Y be a finite set of $n-1$ points in the boundary of the hyperbolic plane, and let $Y = \{y_i\}$ so that $v_0 < y_1 < \cdots < y_{n-1} \neq v_\infty$ in the boundary of the hyperbolic plane.

We use $1^{\mathbf{st}}$ **Step** through $n^{\mathbf{th}}$ **Step**, and we replace the **Last Step**.

Suppose we have completed $1^{\mathbf{st}}$ **Step** through $n^{\mathbf{th}}$ **Step** of the previous section.

Modified Last Step: Let τ be the isometry which reflects through the geodesic $\overline{v_0 v_\infty}$. Now define $\tilde{v}_i = \tau v_i$ and $\tilde{f}_i = \tau f_i$.

This constructs an ideal $2n$ sided convex polygon $\tilde{P}$ with vertices

$$\{v_0, v_1, \ldots, v_{n-1}, v_n = v_\infty, \tilde{v}_{n-1}, \ldots, \tilde{v}_1\}.$$

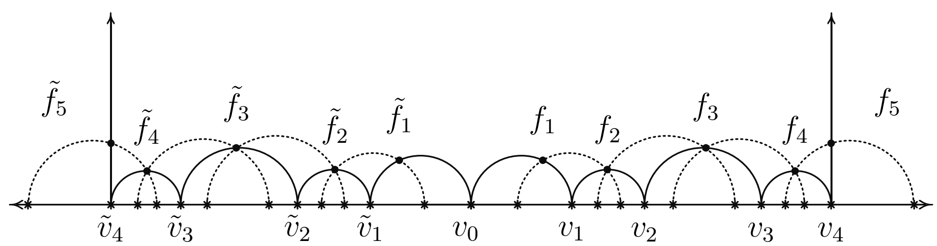

Figure 3.3: The solid lines bound a fundamental domain $\tilde{P}$ in the upper half plane model

Let $\tilde{\Gamma}$ be generated by the rotations by π about the f_i and $\tilde{f}_i$. One can show that the following element is parabolic fixing v_∞,

$$\tilde{\rho}_n \ldots \tilde{\rho}_1 \rho_1 \ldots \rho_n,$$

where $\tilde{\rho}_i = \rho_{\tilde{f}_i}$. With a similar argument to that in Section 3.1, $\tilde{\Gamma}$ is discrete with a fundamental domain $\tilde{P}$. This gives a slight variant of Proposition 3.1.2, where the change is in the relationship of the number of points in Y to the signature of the Fuchsian groups of finite covolume.

Proposition 3.2.2. *Let Y be a finite set of $n-1$ points in ∂H^2. Then there are infinitely many Fuchsian groups $\tilde{\Gamma}$ of finite covolume that have the signature $(0 : \underbrace{2, \ldots, 2}_{2n}; 1; 0)$ such that $Y \subset \mathrm{HFix}(\tilde{\Gamma})$.*

Chapter 4

Acting on the Tree of $\mathrm{SL}_2(\mathbb{Q}_p)$

In this chapter, we will be interested in the construction of Γ (in Section 3.1) carried out in Minkowski space. We will relate the properties of the self inner products of the fixed vectors for the $\rho_i \in \Gamma$ to: first, when Γ lies in $\mathrm{PGL}_2(\mathbb{Q})$ (see the proof of Proposition 4.1.1), and second, the behavior of Γ's action on the trees of $\mathrm{SL}_2(\mathbb{Q}_p)$ for different prime $p \equiv 3 \mod 4$ (see Proposition 4.2.1). The chapter will conclude by establishing a commensurability result (see Theorem 4.2.3) about the family of Γ produced by the construction for a given set of rational boundary points Y.

4.1 When Γ lies in $\mathrm{PGL}_2(\mathbb{Q})$

In addition to the notation of Section 2.1, let $\{e_1, e_2, e_0\}$ be a Lorentz orthonormal basis, and let the $\mathbb{Q}$-linear combination of $\{e_1, e_2, e_0\}$ be denoted by $\mathbb{Q}^{2,1}$; furthermore, let $L^+_{\mathbb{Q}} = L^+ \cap \mathbb{Q}^{2,1}$ and $T^+_{\mathbb{Q}} = T^+ \cap \mathbb{Q}^{2,1}$.

Proposition 4.1.1. *Let $v_0 < y_1 < \cdots < y_{n-1} \neq v_\infty$ in $L^+_{\mathbb{Q}}$. Then there are infinitely many non-cocompact Fuchsian groups Δ of finite covolume sitting in* $\mathrm{PSL}_2(\mathbb{Q})$ *such that* $\{y_1, \ldots, y_{n-1}\} \subset \mathrm{HFix}(\Delta)$.

Proof. We follow the construction of Γ in Minkowski space. In the $i^{\mathbf{th}}$ **Step** (for $1 \le i < n$) of Section 3.1, we additionally require the choice of x_1 and the f_i, as vectors in Minkowski space, to lie in $\mathbb{Q}^{2,1}$. Furthermore, f_n and f_0 will be in $\mathbb{Q}^{2,1}$, since all x_i, v_i, y_i will be in $\mathbb{Q}^{2,1}$. Thus Γ will sit in $\mathrm{PGL}_2(\mathbb{Q})$. Even with these additional requirements in the construction, there are still infinitely many choices for each f_i (for $1 \le i < n$), producing infinitely many such Fuchsian groups Γ.

For each Γ, $\{y_i\} \subset \mathrm{HFix}(\Gamma)$, and let

$$\Delta = \ker \{\Gamma \longrightarrow \mathrm{PGL}_2(\mathbb{Q})/\mathrm{PSL}_2(\mathbb{Q})\},$$

which is of finite index in Γ; therefore, it follows $\{y_i\} \subset \mathrm{HFix}(\Delta)$. $\square$

Acting with Global Fixed Points on $\mathcal{T}_v$

Recall the notation of Section 2.3 about the tree of SL_2 over a local field.

Lemma 4.1.2. *Let Γ be generated by a finite number of ρ_j, where $\rho_j = C_j \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) C_j^{-1}$ with $C_j \in \mathrm{GL}_2(K)$.*

Then the following are equivalent statements about the action of Γ on the tree $\mathcal{T}_v$:

1. $\Gamma \le \mathrm{Stab}(\Lambda)$ *for some* $\Lambda \in \mathcal{T}_v$*;*
2. $\bigcap C_j \,\mathrm{Fix}_{\mathcal{T}_v}\left(\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)\right) \neq \emptyset$*;*
3. *For each pair* (m,k)*,* $\mathrm{Fix}_{\mathcal{T}_v}(\rho_m \rho_k) \neq \emptyset$*.*

Proof. (1) $\Rightarrow$ (2): Assume $\Gamma \leq \mathrm{Stab}(\Lambda)$ for some $\Lambda \in \mathcal{T}_v$; that is, each ρ_j fixes Λ. So we have $C_j^{-1}\Lambda \in \mathrm{Fix}_{\mathcal{T}_v}\left(\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)\right)$. Therefore,

$$\Lambda \in \bigcap C_j \, \mathrm{Fix}_{\mathcal{T}_v}\left(\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)\right).$$

(2) $\Rightarrow$ (1): Now suppose there exists $\Lambda \in \mathcal{T}_v$ and $\Lambda_j \in \mathrm{Fix}_{\mathcal{T}_v}\left(\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)\right)$ such that $\Lambda = C_j\Lambda_j$ for each j. From this, one sees $C_j \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) C_j^{-1}$ fixes Λ; that is, each generator of Γ fixes Λ.

(3) $\Leftrightarrow$ (1): see Corollary 2.3.6 (or [4], I§6.5). $\square$

Lemma 4.1.3. *Let Γ be generated by a finite number of ρ_j, where $\rho_j = C_j \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) C_j^{-1}$ with $C_j = \left(\begin{smallmatrix} b_j & a_j \\ 0 & 1 \end{smallmatrix}\right) \in \mathrm{GL}_2(K)$, as in (∗) of Example 2.1.3.*

When -1 is not a square in K, the following are equivalent:

1. $\Gamma \leq \mathrm{Stab}(\Lambda)$ *for some* $\Lambda \in \mathcal{T}_v$*;*

2. *For each pair (m,k),*

$$C_k^{-1}C_m \in \mathrm{GL}_2(\mathcal{O}_v);$$

3. *For each pair (m,k),*

$$v(a_m - a_k) \geq v(b_m) = v(b_k).$$

Proof. When -1 is not a square in K, we have that $\mathrm{Fix}_{\mathcal{T}_v}\left(\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)\right) = \{[\mathcal{O}_v^2]\}$, where $\mathcal{O}_v^2$ is the standard lattice.

Let ρ_m and ρ_k be two generators of Γ. Then $\mathrm{Fix}_{\mathcal{T}_v}(\rho_m\rho_k) \neq \emptyset$ if and only if

$$C_m \, \mathrm{Fix}_{\mathcal{T}_v}\left(\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)\right) \bigcap C_k \, \mathrm{Fix}_{\mathcal{T}_v}\left(\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)\right) \neq \emptyset,$$

and that holds only when $C_k^{-1}C_m \in \mathrm{GL}_2(\mathcal{O}_v)$. By Lemma 4.1.2, (1) and (2) are equivalent.

To complete the proof note that $C_k^{-1}C_m = \begin{pmatrix} \frac{b_m}{b_k} & \frac{a_m-a_k}{b_k} \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_2(\mathcal{O}_v)$ if and only if

$$v\left(\frac{b_m}{b_k}\right) = 0 \qquad \text{and} \qquad v\left(\frac{a_m - a_k}{b_k}\right) \geq 0.$$

□

4.2 When Global Fixed Points Do Not Exists

Proposition 4.2.1. *Let $v_0 < y_1 < y_2 < \cdots < y_{n-1} \neq v_\infty$ in $L^+_{\mathbb{Q}}$ ($n > 2$), and let a prime $p \equiv 3 \mod 4$. Then there are non-cocompact Fuchsian groups Δ of finite covolume sitting in $\mathrm{PSL}_2(\mathbb{Q})$ with $\{y_1, \ldots, y_{n-1}\} \subset \mathrm{HFix}(\Delta)$, each of which stabilize no vertex in the tree $\mathcal{T}_p$.*

Proof. Consider the construction of Γ in Section 3.1 in Minkowski space; below we will describe additional requirements for choosing x_1 and the f_i.

In the $1^{\mathbf{st}}$ **Step**, choose $x_1 \in L^+_{\mathbb{Q}}$ so $v_0 < x_1 < y_1$. When choosing f_1, additionally require that $f_1 \in \mathrm{span}_{\mathbb{Q}}\{x_1, y_1\}$ and $|\langle f_1, f_1\rangle|$ is in the rational square class $p(\mathbb{Q}^\times)^2 = \{p\alpha^2 : \alpha \in \mathbb{Q}\}$, which is possible because $\mathrm{span}_{\mathbb{Q}}\{x_1, y_1\}$ is isotropic.

In the $i^{\mathbf{th}}$ **Step** (for $1 < i < n$), one specifies a rational square class, say $n_i(\mathbb{Q}^\times)^2 \in \mathbb{Q}^\times/(\mathbb{Q}^\times)^2$ (n_i a square free integer) such that $p \nmid n_i$. When choosing f_i, additionally require that $f_i \in \mathrm{span}_{\mathbb{Q}}\{x_{i-1}, y_{i-1}\}$ and $|\langle f_i, f_i\rangle| \in n_i(\mathbb{Q}^\times)^2$, which is also possible because $\mathrm{span}_{\mathbb{Q}}\{x_{i-1}, y_{i-1}\}$ is isotropic.

Now note (as in Example 2.1.3) that ρ_{f_i} as an element of $\mathrm{PGL}_2(\mathbb{Q})$ is given by the matrix

$$\rho_{f_i} = \begin{pmatrix} b_i & a_i \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} b_i & a_i \\ 0 & 1 \end{pmatrix}^{-1},$$

where

$$b_i = -\frac{\sqrt{|\langle f_i, f_i \rangle|}}{\langle f_i, e_2 + e_0 \rangle} \quad \text{and} \quad a_i = -\frac{\langle f_i, e_1 \rangle}{\langle f_i, e_2 + e_0 \rangle}.$$

Since b_1 has a factor $\sqrt{p}$, and each b_i $(1 < i < n)$ does not,

$$v_p(b_i) \neq v_p(b_1).$$

Moreover, -1 is not a square in $\mathbb{Q}_p$ (since $p \equiv 3 \mod 4$). By Lemma 4.1.3, Γ stabilizes no vertex in $\mathcal{T}_p$, and by construction $\{y_i\} \subset \mathrm{HFix}(\Gamma)$. Now let Δ be the kernel of $\Gamma \longrightarrow \mathrm{PGL}_2(\mathbb{Q})/\mathrm{PSL}_2(\mathbb{Q})$, which is of finite index in Γ; then Δ also stabilizes no vertex in $\mathcal{T}_p$ and $\{y_i\} \subset \mathrm{HFix}(\Delta)$. $\square$

Remark 4.2.2. For each Δ constructed in the proof of Proposition 4.2.1, there is an integer m such that Δ stabilizes a vertex of $\mathcal{T}_q$ for all primes $q > m$. To see this, choose m large enough so that m is greater than all the denominators of the entries of a matrix representing ρ_{f_i}, for each i, as an element of $\mathrm{PGL}_2(\mathbb{Q})$

Commensurability Result

Theorem 4.2.3. *Let Y be a finite set of rational points in the boundary of the hyperbolic plane. Then there are infinitely many noncommensurable non-cocompact Fuchsian groups Δ of finite covolume sitting in $\mathrm{PSL}_2(\mathbb{Q})$ so that $Y \subset \mathrm{HFix}(\Delta)$.*

Proof. We can let Y be a finite set of two or more rational points in the boundary of the hyperbolic plane (just add points if fewer than 2 are given). Let $Y = \{y_i\}$, so that $v_0 < y_1 < y_2 < \cdots < y_{n-1} \neq v_\infty$ in $L^+_{\mathbb{Q}}$ $(n > 2)$. Now let the family $\{\Delta\}$ be the set of non-cocompact Fuchsian groups of finite covolume sitting in $\mathrm{PSL}_2(\mathbb{Q})$ such that $\{y_1, \ldots, y_{n-1}\} \subset \mathrm{HFix}(\Delta)$, which are constructed in the proof of Proposition 4.1.1.

Assume for the purpose of contradiction that there is a finite number k of commensurability classes in family $\{\Delta\}$. Let $\{\Delta_1, \ldots, \Delta_k\}$ be distinct representatives from the k commensurability classes.

From Remark 4.2.2, there is an integer m such that each Δ_j $(1 \leq j \leq k)$ stabilizes a vertex in $\mathcal{T}_q$ for all $q > m$. By Dirichlet's theorem on arithmetic progressions, we can choose a prime $p > m$ and $p \equiv 3 \mod 4$. By Proposition 4.2.1, there is $\Delta_{k+1} \in \{\Delta\}$ with $\{y_i\} \subset \mathrm{HFix}(\Delta_{k+1})$ and so that Δ_{k+1} does not stabilize any vertex of $\mathcal{T}_p$. Therefore, each Δ_j $(1 \leq j \leq k)$ stabilizes a vertex in $\mathcal{T}_p$ but Δ_{k+1} does not. For subgroups of $\mathrm{PSL}_2(\mathbb{Q})$, the presence or absence of fixed points in $\mathcal{T}_p$ descends to finite index subgroups, and is invariant under conjugation. Thus Δ_{k+1} is not commensurable with any of the Δ_j $(1 \leq j \leq k)$, which contradicts the assumption there are a finite number of commensurability classes in the family $\{\Delta\}$. $\square$

Remark. By using Proposition 4.2.1 and Remark 4.2.2, one can inductively constructs an infinite family where the members lie in different commensurability classes.

As mentioned in the introduction, a boundary point cannot both be a cusp and a hyperbolic fixed point, for Fuchsian groups; thus a direct corollary of the theorem, in relationship to the set of cusps.

Corollary 4.2.4. *Let Y be finite set of rationals. Then there are infinitely many noncommensurable non-cocompact Fuchsian groups of finite covolume sitting in* $\mathrm{PSL}_2(\mathbb{Q})$ *whose cusp set is properly contained in* $(\mathbb{Q} \setminus Y) \cup \{\infty\}$.

Chapter 5

Fuchsian Groups of Signature $(0:2,2,2;1;0)$

Let Γ be a Fuchsian group of signature $(0:2,2,2;1;0)$. First, we discuss the few known non-arithmetic Γ's with cusp set precisely $\mathbb{Q}\cup\{\infty\}$ (i.e. the known pseudomodular groups). Second, we establish a parameterization of the Γ defining complete hyperbolic orbifold $\mathbf{S}^2\setminus\{\infty\}$ with three cone points of order 2 and one cusp, which identifies when Γ sits in $\mathrm{PGL}_2(\mathbb{Q})$. We conclude by using this parameterization to investigate when particular hyperbolic elements of Γ have rational fixed points.

5.1 The Set of Cusps

We begin this section with a classic example,

Example 5.1.1. In the case of the modular group $\mathrm{PSL}_2(\mathbb{Z})$, the two matrices

$$S=\begin{pmatrix}0&-1\\1&0\end{pmatrix}\qquad\text{and}\qquad P=\begin{pmatrix}1&1\\0&1\end{pmatrix}$$

generate $\mathrm{PSL}_2(\mathbb{Z})$. Now consider the action of SP^m on the $n/d\in\partial\mathbb{U}^2$;

$$SP^m\frac{n}{d}=\frac{-d}{n+md},$$

where n,d,m are integers. If n and d are coprime integers, then so too are d and $n+md$.

Assuming d is nonzero, we can choose m so that $0 \leq n + md \leq d - 1$. We replace n/d with $-d/(n+md)$ and iterate this process. At each step in the iteration, we decrease the denominator of a rational number by applying to it an element of the modular group. Therefore, starting with any $n/d \in \partial\mathbb{U}^2$ (n and d coprime integers), we use this iterative process to yield an element in $\mathrm{PSL}_2(\mathbb{Z})$ which maps $n/d \mapsto \infty$. This establishes the cusp set for $\mathrm{PSL}_2(\mathbb{Z})$ is $\mathbb{Q} \cup \{\infty\}$.

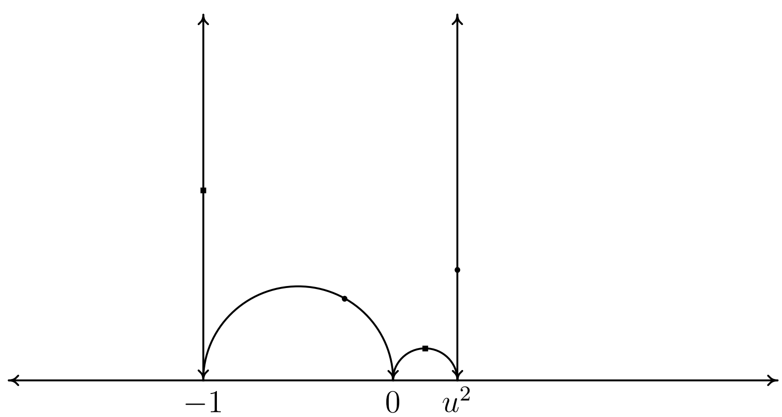

Figure 5.1: A fundamental domain for $\Delta(u^2, 2\tau)$ in $\mathbb{U}^2$

In [2], Long and Reid consider a family of Fuchsian groups $\Delta(u^2, 2\tau)$ that define hyperbolic tori with one cusp. Namely, for u^2 and τ real numbers with $0 < u^2 < \tau - 1$, the group $\Delta(u^2, 2\tau)$ is the subgroup of $\mathrm{PSL}_2(\mathbb{R})$ generated by the hyperbolic elements

$$g_1 = \frac{1}{\sqrt{\tau - 1 - u^2}} \begin{pmatrix} \tau - 1 & u^2 \\ 1 & 1 \end{pmatrix}, \quad g_2 = \frac{1}{\sqrt{u^2(\tau - 1 - u^2)}} \begin{pmatrix} u^2 & u^2 \\ 1 & \tau - u^2 \end{pmatrix}.$$

This was made so that

$$g_1 g_2^{-1} g_1^{-1} g_2 = \begin{pmatrix} -1 & -2\tau \\ 0 & -1 \end{pmatrix};$$

hence $\Delta(u^2, 2\tau)$ defines a complete hyperbolic torus with one cusp.

To investigate the cusp set of $\Delta(u^2, 2\tau)$ when $u^2, \tau \in \mathbb{Q}$, Long and Reid emulated the ideas of Example 5.1.1. For a few $\Delta(u^2, 2\tau)$, they produced an iterative process of $\Delta(u^2, 2\tau)$'s action on $\partial\mathbb{U}^2$, which reduces the denominator of $r \in \partial\mathbb{U}^2 \cap \mathbb{Q}$, when the interval $[0, 2\tau]$ is covered by what they call killer intervals (defined in the next section). Through computer experimentation, they were able to find $\Delta(5/7, 2\cdot 3)$, $\Delta(2/5, 2\cdot 2)$, $\Delta(3/7, 2\cdot 2)$, and $\Delta(3/11, 2\cdot 2)$ each have killer intervals covering $[0, 2\tau]$; and then established the following:

Theorem 5.1.2 (Long & Reid 2002)**.** *The groups*

$$\Delta(5/7, 2\cdot 3),\ \Delta(2/5, 2\cdot 2),\ \Delta(3/7, 2\cdot 2),\ \Delta(3/11, 2\cdot 2)$$

are four mutually noncommensurable subgroups, which are not commensurable with the modular group, and each of them have cusp set exactly $\mathbb{Q} \cup \{\infty\}$.

This led them to define

Definition 5.1.3. *A discrete group* $\Delta \leq \mathrm{PGL}_2(\mathbb{Q})$ *is pseudomodular if* Δ *is not commensurable with the modular group and has cusp set precisely* $\mathbb{Q} \cup \{\infty\}$.

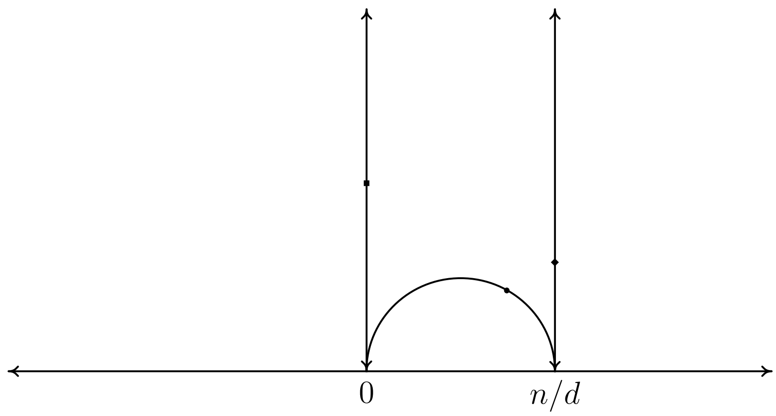

Figure 5.2: A fundamental domain for $\Gamma(n/d, t/s)$ in $\mathbb{U}^2$

For each $\Delta(u^2, 2\tau)$, there is a $\mathbb{Z}/2\mathbb{Z}$-supergroup, $\Gamma(u^2, \tau)$ of signature $(0:2,2,2;1;0)$. In fact, $\Gamma(n/d, t/s)$ (where $u^2 = n/d$ and $\tau = t/s$) of signature $(0:2,2,2;1;0)$ is the subgroup of $\mathrm{PSL}_2(\mathbb{R})$ generated

$$\rho_0 = \frac{1}{\sqrt{k_0}}\begin{pmatrix} 0 & -n \\ d & 0 \end{pmatrix}$$
$$\rho_1 = \frac{1}{\sqrt{k_1}}\begin{pmatrix} -dns & n^2 s \\ d(ns-dt) & dns \end{pmatrix}$$
$$\rho_2 = \frac{1}{\sqrt{k_2}}\begin{pmatrix} -ns & n(t-s) \\ -ds & ns \end{pmatrix}$$

where k_0, k_1, k_2 are the absolute values of the determinants of the above matrices, respectively. One can show

$$\rho_2\rho_1\rho_0 = \begin{pmatrix} 1 & t/s \\ 0 & 1 \end{pmatrix}.$$

Since the cusp set is preserved under passage to a subgroup of finite index, we will focus our attention to the family of $\Gamma(n/d, t/s)$ of signature $(0 : 2, 2, 2; 1; 0)$; such discrete groups defines a complete hyperbolic orbifold $\mathbf{S}^2 \setminus \{\infty\}$ with three cone points of order 2 and one cusp.

5.2 Killer Intervals and Pseudomodular Groups

Let an element $\gamma \in \mathrm{PGL}_2(\mathbb{Q})$, and let

$$\gamma = k_\gamma \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

where $a, b, c, d \in \mathbb{Z}$, $c \neq 0$, and $\gcd\{a, b, c, d\} = 1$; we can view γ in $\mathrm{PSL}_2(\mathbb{R})$ by

$$\frac{1}{\sqrt{|ad - cd|}} \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Now consider its action on $r/s \in \partial\mathbb{U}^2 \cap \mathbb{Q}$;

$$\gamma\frac{r}{s} = \frac{ar + bs}{cr + ds};$$

thus $-d/c = \alpha/\beta$ (where $\gcd(\alpha, \beta) = 1$) is mapped to ∞ by γ. We define the *killer interval* of γ, say I_γ, to be the interval

$$I_\gamma = \{r/s : |cr + ds| < |s|\} = \left(\frac{\alpha}{\beta} - \frac{1}{|\beta| \gcd(|c|, |d|)}, \frac{\alpha}{\beta} + \frac{1}{|\beta| \gcd(|c|, |d|)}\right);$$

we also define $\gcd(|c|, |d|)$ to be the *contraction constant* of γ.

Theorem 5.2.1 (Long & Reid 2002)**.** *Let Γ be a Fuchsian group which sits in $\mathrm{PGL}_2(\mathbb{Q})$ such that H^2/Γ has one cusp, and let p generate $\mathrm{Stab}(\infty)$ with τ the translation length of p.*

If the interval $[0, \tau]$ is covered by killer intervals, then the cusp set of Γ is all of $\mathbb{Q} \cup \{\infty\}$.

Proof. Given a rational number $\tilde{r}/\tilde{s}$, there is a power of p which maps $\tilde{r}/\tilde{s}$ to $r/s \in [0, \tau]$ (an operation which does not increase the denominator). By hypothesis, there is a killer interval I_γ for $\gamma \in \Gamma$ such that $r/s \in I_\gamma$; that is, $\gamma(r/s)$ has denominator strictly less than s.

By iterating the above paragraph, we can reduce the denominator to zero in a finite number of steps. Let γ_i be the elements of the killer intervals used in this procedure. Therefore, one can produce a $w \in \Gamma$ that is spelled with letters $\{p, \gamma_1, \ldots, \gamma_k\}$ such that w maps $\tilde{r}/\tilde{s}$ to ∞. $\square$

In the appendix, there is a detailed list of the group elements and their killer intervals that cover $[0, \tau]$ for

$$\Gamma(u^2, \tau) \quad \text{for each } (u^2, \tau) \in \{(2/5, 2), (3/7, 2), (3/11, 2), (5/7, 3)\},$$

which are the (u^2, τ) for the pseudomodular groups found in Theorem 5.1.2. A new result is also included in the appendix; namely, a list of the killer intervals for $\Gamma(4/7, 2)$ that cover $[0, 2]$. After finding enough killer intervals for $\Gamma(4/7, 2)$ to show it is pseudomodular, it was noticed that $\Gamma(4/7, 2)$ is conjugate to $\Gamma(3/7, 2)$.

In fact, there are many different values for u^2 and t that produce groups $\Gamma(u^2, \tau)$ in the same conjugacy class (for example, see Lemma 2.1 of [2]); this can be established by an argument using traces. Though, $\Gamma(u_1^2, \tau_1)$ and $\Gamma(u_2^2, \tau_2)$ might be shown to be conjugate from such an argument, the element which conjugates $\Gamma(u_1^2, \tau_1)$ to $\Gamma(u_2^2, \tau_2)$ is usually not calculated. Without such an element, we do not have a way to transfer killer intervals from one group to the conjugate group.

Furthermore, it is difficult to find a collection of killer intervals that cover $[0, \tau]$ even for $\Gamma(u^2, \tau)$ which are known to possess such a collection. For example, the pseudomodular group $\Gamma(3/11, 2)$ is conjugate to $\Gamma(8/11, 2)$ (by a trace argument), but without the element which conjugates one to the other, a list of killer intervals which cover $[0, 2]$ for $\Gamma(8/11, 2)$ is still unknown. Moreover, trying to produce enough killer intervals which cover $[0, 2]$ for $\Gamma(8/11, 2)$ is not a trivial matter, one has to find the "right" killer intervals. Note: I have more than two thousand elements of $\Gamma(8/11, 2)$ with killer intervals in $[0, 2]$, but these are not enough killer intervals to cover $[0, 2]$; though, we can cover $[0, 2]$ with less than 45 killer intervals for $\Gamma(3/11, 2)$.

5.3 Hyperbolic Orbifolds $\mathbf{S}^2 \backslash \{\infty\}$ with three cone points of order 2

In this section, we establish a parameterization of the Γ (up to conjugation) that defining complete hyperbolic orbifold $\mathbf{S}^2 \setminus \{\infty\}$ with three cone points of order 2 and one cusp, which identifies when Γ sits in $\mathrm{PGL}_2(\mathbb{Q})$. We conclude by using this parameterization to investigate when particular hyperbolic elements of Γ have rational fixed points.

Let M be a complete hyperbolic orbifold $\mathbf{S}^2\backslash\{\infty\}$ with three cone points of order 2 and one cusp, and let $\mathbb{U}^2$ be the upper-half space model for hyperbolic plane. Let $\eta : \pi_1^o(M) \longrightarrow \text{Isom}(\mathbb{U}^2)$ and $\delta : \tilde{M} \longrightarrow \mathbb{U}^2$ be the corresponding developing map, and the universal orbifold covering projection $\kappa : \tilde{M} \longrightarrow M$. Then η maps $\pi_1^o(M)$ (the fundamental orbifold group) isomorphically onto a discrete group $\Gamma_\eta \leq \text{Isom}(\mathbb{U}^2)$ such that δ induces isometry from M to $\mathbb{U}^2/\Gamma_\eta$ (see Theorem 13.3.10 in [3] pg 722).

As an overview, we will consider the elliptic fixed points of Γ_η in $\mathbb{U}^2$. After an isometry of $\mathbb{U}^2$, we select three fixed points $\{s, g, r\}$ and their elliptic isometries $\{\rho_s, \rho_g, \rho_r\}$ where M is isometric to $\mathbb{U}^2/\Gamma$ and $\Gamma = \langle \rho_s, \rho_g, \rho_r \rangle$ with $\rho_r\rho_g\rho_s$ generating the stabilizer ∞. Roughly speaking, s will represent the cone point that is the closest to the cusp of M and g will represent the cone point that is the furthest. Then we use an isometry of $\mathbb{U}^2$, to put s at i, and so that s, g, r have a specific relationship to each other. Next, we consider Γ as a subgroup of $\text{O}^+(2,1)$ and show there is a way to parameterize Γ by a vector v_1 on the light cone. We conclude by using the coordinates of v_1 to see when particular hyperbolic elements of Γ have rational fixed points.

The Group $\Gamma = \langle \rho_s, \rho_g, \rho_r \rangle$

Let M, η, δ, κ, and Γ_η be define as above, and let $u_1, u_2, u_3 \in M$ be the 3 cone points of order 2. There is an isometry φ of $\mathbb{U}^2$ such that

a. $\varphi^{-1}\infty$ is a cusp of Γ_η,

b. $1 \geq \Im(f)$ for all $f \in \varphi(\delta(\kappa^{-1}\{u_1, u_2, u_3\}))$,

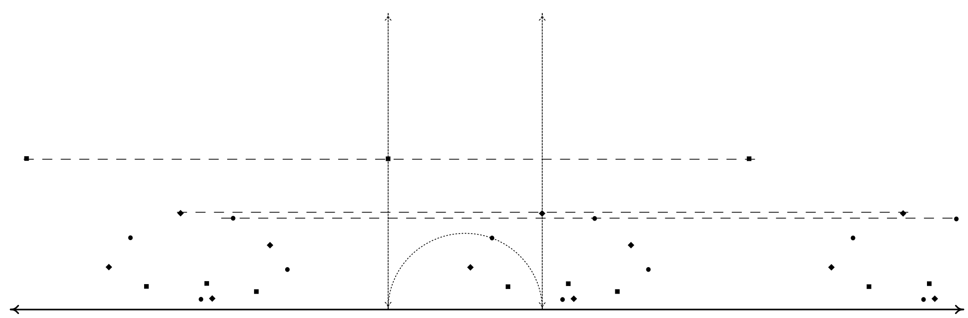

Figure 5.3: Viewing the cone points in $\mathbb{U}^2$.

In addition, there are $s, r, g, g_-, g_+ \in \mathbb{U}^2$ and a labeling of $\{u_1, u_2, u_3\} = \{u_s, u_r, u_g\}$ where

1. $s \in \varphi(\delta(\kappa^{-1}\{u_s\}))$ such that

$$\begin{aligned} s &= i \\ 1 \geq m_r &= \max\left\{\Im(f) : f \in \varphi(\delta(\kappa^{-1}\{u_r\}))\right\} \\ m_r &\geq \max\left\{\Im(f) : f \in \varphi(\delta(\kappa^{-1}\{u_g\}))\right\} \end{aligned}$$

2. $r \in \varphi(\delta(\kappa^{-1}\{u_r\}))$ such that

$$\Im(r) = \max\left\{\Im(f) : f \in \varphi(\delta(\kappa^{-1}\{u_r\}))\right\}$$
$$\Re(r) = \min\left\{|\Re(f)| : \Im(f) = \Im(r) \ \& \ f \in \varphi(\delta(\kappa^{-1}\{u_r\}))\right\}$$
$$\Re(r) > 0$$

3. $g_-, g_+ \in \varphi(\delta(\kappa^{-1}\{u_g\}))$ such that

$$\Im(g_-) = \Im(g_+) = \max\left\{\Im(f) : f \in \varphi(\delta(\kappa^{-1}\{u_g\}))\right\},$$
$$|\Re(g_-)| = \min\left\{|\Re(f)| : \Im(f) = \Im(g_-) \ \& \ f \in \varphi(\delta(\kappa^{-1}\{u_g\}))\right\},$$
$$\Re(g_-) < 0 < \Re(r) < \Re(g_+);$$

moreover, for each pair $f_1 \neq f_2 \in \varphi(\delta(\kappa^{-1}\{u_g\}))$ that $\Im(f_1) = \Im(f_2) = \Im(g_-) = \Im(g_+)$,

$$|\Re(g_+) - \Re(g_-)| \leq |\Re(f_1) - \Re(f_2)|,$$

4. $g \in \varphi(\delta(\kappa^{-1}\{u_g\}))$ and

$$\Im(g) = \max\left\{\Im(f) : 0 < \Re(f) < \Re(r) \ \& \ f \in \varphi(\delta(\kappa^{-1}\{u_g\}))\right\};$$

Let $\rho_s, \rho_g, \rho_r \in \mathrm{Isom}(\mathbb{U}^2)$ be elliptic elements of order two with fix points s, g, r respectively; furthermore, $\rho_s\infty = 0$, $\rho_r\infty = v_1$, and $\rho_g v_1 = 0$. Moreover, $p = \rho_r\rho_g\rho_s$ is parabolic and generates the stabilizer ∞ for $pg_- = g_+$. Let Γ be generated by $\{\rho_s, \rho_g, \rho_r\}$, and let $\pi : \mathbb{U}^2 \longrightarrow \mathbb{U}^2/\Gamma$, which induces an isometry from M to $\mathbb{U}^2/\Gamma$. Establishing the following:

Proposition 5.3.1. *Let M be a complete finite area H^2-orbifold that is a 2-sphere with 3 cone points of order 2 and one cusp. Then M is isometric to $\mathbb{U}^2/\Gamma$, where $\Gamma = \langle \rho_s, \rho_g, \rho_r \rangle$ (defined above).*

The Group $\Gamma = \langle \rho_s, \rho_g, \rho_r \rangle$ in $\mathrm{O}^+(2,1)$

We will consider $\Gamma = \langle \rho_s, \rho_g, \rho_r \rangle$ (as defined above) in Minkowski space $\mathbb{R}^{2,1}$. Here s, g, r are vectors in $\mathbb{R}^{2,1}$, and $\rho_s, \rho_g, \rho_r \in \mathrm{O}^+(2,1)$ are elliptic elements of order two with fix vectors s, g, r, respectively. Since $s = i$ in $\mathbb{U}^2$, let $s = (0,0,1)$, which is in the ray $(0,0,1)$; furthermore, we can set $v_2 = (0,1,1)$ and $v_0 = (0,-1,1)$. Thus $\rho_s v_2 = v_0$. So here v_2 and v_0 in L^+ represents ∞ and 0 in $\mathbb{U}^2$, respectively.

Moreover, $p = \rho_r \rho_g \rho_s$ is parabolic and generates the stabilizer of v_2. So we let $\rho_r v_2 = v_1$ (and $\rho_g v_1 = v_0$), where $v_i \in L^+$. Since $\rho_r v_2 = v_1$, $\rho_g v_1 = v_0$, and $\rho_s v_2 = v_0$, r is in the ray of $f_2 = v_1 + v_2$, g is in the ray of $f_1 = v_0 + v_1$, and s is in the ray of $f_0 = v_2 + v_0$; thus $\rho_s = \rho_{f_0}$, $\rho_g = \rho_{f_1}$, and $\rho_r = \rho_{f_2}$. Note: with v_0 and v_2 fixed, f_0 is fixed; furthermore, f_1 and f_2 are determined by v_1. We define $\Gamma(v_1)$ in $\mathrm{O}^+(2,1)$ to be the group

$$\Gamma(v_1) = \langle \rho_{f_0}, \rho_{f_1}, \rho_{f_2} \rangle,$$

where $f_i = v_{i-1} + v_i$ for $i \equiv 0, 1, 2 \mod 3$.

Since $s = i$ and $1 \geq \Im(r) \geq \Im(g)$ in $\mathbb{U}^2$, r and g are not in the interior of the horoball $\{z \in \mathbb{U}^2 : \Im(z) > 1\}$ (in Minkowski space, it is the horoball for

$(0,1,1) = v_2 \in L^+$). This corresponds (in Minkowski space) to

$$\frac{r}{\sqrt{|\langle r,r\rangle|}}, \frac{g}{\sqrt{|\langle g,g\rangle|}} \notin \{f \in \mathcal{H} : 0 > \langle v_2, f\rangle > -1\},$$

or in terms of v_i,

$$\frac{v_2+v_1}{\sqrt{2\,|\langle v_2,v_1\rangle|}}, \frac{v_0+v_1}{\sqrt{2\,|\langle v_0,v_1\rangle|}} \notin \{f \in \mathcal{H} : 0 > \langle v_2, f\rangle > -1\}.$$

So we get

$$(\dagger) \qquad |v_2, v_1| \geq 2 \geq \sqrt{2}\sqrt{|\langle v_0, v_1\rangle|} - |\langle v_2, v_1\rangle|.$$

If we write $v_1 = (\xi_1, \xi_2, \xi_0)$ in L^+, where $\xi_1 > 0$, then $(\dagger)$ becomes

$$|\xi_2 - \xi_0| \geq 2 \geq \sqrt{2}\sqrt{|\xi_2 + \xi_0|} - |\xi_2 - \xi_0|.$$

This establishes the following:

Proposition 5.3.2. *Let M be a complete finite area H^2-orbifold that is a 2-sphere with 3 cone points of order 2 and one cusp. Then M is isometric to $\mathcal{H}/\Gamma(v_1)$, where $\Gamma(v_1)$ is defined above with $v_1 = (\xi_1, \xi_2, \xi_0) \in L^+$ satisfying $\xi_1 > 0$ and*

$$|\xi_2 - \xi_0| \geq 2 \geq \sqrt{2}\sqrt{|\xi_2 + \xi_0|} - |\xi_2 - \xi_0|.$$

Note: if $v_1 = (\xi_1, \xi_2, \xi_0) \in L^+$ with $\xi_1 > 0$, then $\xi_1 = \sqrt{\xi_0^2 - \xi_2^2}$. Now from the note at the end of Example 2.1.3, and that $\langle f_i, f_i\rangle$ and $\langle f_i, e_j\rangle$ being in $\mathbb{Q}$ relates to v_1 being in $L^+_{\mathbb{Q}}$ (since $f_i = v_{i-1} + v_i$, where $i \equiv 0, 1, 2 \mod 3$), we have the following:

Proposition 5.3.3. *Let* $\Gamma(v_1)$ *(defined above) with* $v_1 = (\xi_1, \xi_2, \xi_0)$ *in* L^+ *satisfying* $\xi_1 > 0$ *and*

$$|\xi_2 - \xi_0| \geq 2 \geq \sqrt{2}\sqrt{|\xi_2 + \xi_0|} - |\xi_2 - \xi_0|\,.$$

If $v_1 \in L^+_{\mathbb{Q}}$, *then* $\Gamma(v_1)$ *(when considered in* $\mathrm{Isom}(\mathbb{U}^2)$*) sits in* $\mathrm{PGL}_2(\mathbb{Q})$.

Here is the hypothesis of Proposition 5.3.3

$$(\ddagger) \qquad \begin{cases} v_0 = (0,-1,1), \quad v_2 = (0,1,1), \quad v_1 = (\xi_1, \xi_2, \xi_0) \in L^+_{\mathbb{Q}}, \quad \xi_1 > 0, \\ \qquad\qquad |\xi_2 - \xi_0| \geq 2 \geq \sqrt{2}\sqrt{|\xi_2 + \xi_0|} - |\xi_2 - \xi_0|\,, \end{cases}$$

Proposition 5.3.4. *If* $v_1 = (\xi_1, \xi_2, \xi_0)$ *satisfies (‡) and any of*

$$\left|1 - 2\xi_2 + \xi_0^2\right|, \qquad \left|1 + 2\xi_2 + \xi_0^2\right|, \quad \textit{or} \quad \left|1 - 2\xi_0 + \xi_2^2\right|$$

are squares of a rational number, then $\Gamma(v_1)$ *contains hyperbolic elements with rational fixed points and (thus)* $\Gamma(v_1)$ *is neither arithmetic nor pseudomodular.*

Proof. Let $f_i = v_{i-1} + v_i$, where $i \equiv 0, 1, 2 \mod 3$. Now consider the time-like subspaces $A_{ij} = \mathrm{span}\{f_i, f_j\}$ for $i \neq j$. Thus A_{ij} is the axis of the element $\rho_{f_i}\rho_{f_j}$ (for $i \neq j$). We have that A_{ij} is isomorphic to $\mathbb{Q}^{1,1}$ if $\langle f_i, f_i\rangle\langle f_j, f_j\rangle - (\langle f_i, f_j\rangle)^2 = \alpha_{ij}$, where $\sqrt{|\alpha_{ij}|} \in \mathbb{Q}$. After a small calculation, one find that α_{01}, α_{12}, and α_{20} corresponds to $-4(1 - 2\xi_2 + \xi_0^2)$, $-4(1 - 2\xi_0 + \xi_2^2)$, and $-4(1 + 2\xi_2 + \xi_0^2)$, respectively. Thus one of the $|\alpha_{ij}|$ a square of a rational number when one of the three $|1 - 2\xi_2 + \xi_0^2|$, $|1 + 2\xi_2 + \xi_0^2|$, or $|1 - 2\xi_0 + \xi_2^2|$ is square of a rational number. $\square$

Proposition 5.3.5. *Let v_1 satisfies (‡), $f_i = v_{i-1} + v_i$, where $i \equiv 0, 1, 2 \mod 3$, and $f \in \bigcup \mathrm{Orb}_{\Gamma(v_1)}(f_i) \setminus \{f_0\}$.*

If $\langle f, f\rangle + (\langle(e_0, f\rangle)^2$ is a square of a rational number, then $\Gamma(v_1)$ contains hyperbolic elements with rational fixed points and (thus) $\Gamma(v_1)$ is neither arithmetic nor pseudomodular.

Proof. Let $f = (\alpha, \beta, \eta)$ then $\tilde{f} = \rho_{f_0}(f) = (-\alpha, -\beta, \eta)$. Now consider $A = \mathrm{span}\left\{f, \tilde{f}\right\}$, which is the axis for $\rho_f \rho_{\tilde{f}} \in \Gamma(v_1)$. We have that A is isomorphic to $\mathbb{Q}^{1,1}$ if $\left|\langle f, f\rangle\langle \tilde{f}, \tilde{f}\rangle - (\langle f, \tilde{f}\rangle)^2\right| = 4\eta^2(\alpha^2 + \beta^2 - \eta^2 + \eta^2)$ is a square of a rational number, which happens when $\langle f, f\rangle + (\langle e_0, f\rangle)^2$ is a square of a rational number. $\square$

Appendix

Recall $\Gamma(n/d, t/s)$ ($u^2 = n/d$ and $\tau = t/s$ in relation to Long and Reid's notation) of signature $(0 : 2, 2, 2; 1; 0)$ is the subgroup of $\mathrm{PSL}_2(\mathbb{R})$ generated ρ_0, ρ_1, and ρ_2, with $p = \rho_2\rho_1\rho_0$ parabolic fixing ∞. For convenience, we will include the group element which is associated to the killer interval listed. In the first column will be the group element γ, which is spelled with the convention: $P = p^{-1}$ and $s = \rho_0, g = \rho_1$ and $r = \rho_2$, in the second column (listed in ascending order) will be the fraction which is mapped to infinity by γ, and in the last two columns will be the contraction constant c_γ and the killer interval I_γ. Note: these were computer-generated, and not all words are reduced.

Killer Intervals for $\Gamma(2/5, 2)$

γ	$\gamma^{-1}(\infty)$	c_γ	I_γ
s	0	5	$(-1/5, 1/5)$
srg	1/5	1	$(0, 2/5)$
r	2/5	1	$(1/5, 3/5)$
$sprsr$	3/5	9	$(26/45, 28/45)$
$srsr$	7/10	1	$(3/5, 4/5)$
$grsr$	4/5	3	$(11/15, 13/15)$
sr	1	5	$(4/5, 6/5)$
$sppsr$	6/5	3	$(17/15, 19/15)$
$rsrsP$	13/10	1	$(6/5, 7/5)$
$sPsrspsr$	7/5	9	$(62/45, 64/45)$
$srsP$	8/5	1	$(7/5, 9/5)$
$grsP$	9/5	1	$(8/5, 2)$
$srpsrsP$	2	5	$(9/5, 11/5)$

Killer Intervals for $\Gamma(3/7, 2)$

γ	$\gamma^{-1}(\infty)$	c_γ	I_γ
s	0	7	$(-1/7, 1/7)$
$rsrg$	1/7	3	$(2/21, 4/21)$
srg	3/14	1	$(1/7, 2/7)$
$srpppsrpsrg$	2/7	9	$(17/63, 19/63)$
r	3/7	1	$(2/7, 4/7)$
$srsrpr$	4/7	3	$(11/21, 13/21)$
$srsr$	5/7	1	$(4/7, 6/7)$
$srPsr$	6/7	1	$(5/7, 1)$
sr	1	7	$(6/7, 8/7)$
$srpppsr$	8/7	1	$(1, 9/7)$
$rsrsP$	9/7	1	$(8/7, 10/7)$
$\vdots$			

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cont. for $\Gamma(3/7,2)$

γ	$\gamma^{-1}(\infty)$	c_γ	I_γ
$rsrgrsrsP$	10/7	3	(29/21, 31/21)
$srsP$	11/7	1	(10/7, 12/7)
$srPPrgsgrppsrsP$	12/7	9	(107/63, 109/63)
$grsP$	25/14	1	(12/7, 13/7)
$srPPrgsgr$	13/7	3	(38/21, 40/21)
$srpsrsP$	2	7	(13/7, 15/7)

Killer Intervals for $\Gamma(4/7,2)$

γ	$\gamma^{-1}(\infty)$	c_γ	I_γ
s	0	7	(−1/7, 1/7)
$srgsrg$	1/7	1	(0, 2/7)
srg	2/7	1	(1/7, 3/7)
$srgsrgrpsrg$	3/7	3	(8/21, 10/21)
r	4/7	1	(3/7, 5/7)
$rpsrsrpr$	5/7	9	(44/63, 46/63)
$srsr$	11/14	1	(5/7, 6/7)
$rpsrsr$	6/7	3	(17/21, 19/21)
sr	1	7	(6/7, 8/7)
$srgsrgr$	8/7	3	(23/21, 25/21)
$rsrsP$	17/14	1	(8/7, 9/7)
$srgsrgrsrsP$	9/7	9	(80/63, 82/63)
$srsP$	10/7	1	(9/7, 11/7)
$grspsrsP$	11/7	3	(32/21, 34/21)
$grsP$	12/7	1	(11/7, 13/7)
$rpsrsrpsrsP$	13/7	1	(12/7, 2)
$srpsrsP$	2	7	(13/7, 15/7)

Killer Intervals for $\Gamma(3/11, 2)$

γ	$\gamma^{-1}(\infty)$	c_γ	I_γ
s	0	11	$(-1/11, 1/11)$
$rsrg$	1/11	3	$(2/33, 4/33)$
srg	3/22	1	$(1/11, 2/11)$
$srpppsrprsrsPpsrg$	2/11	36	$(71/396, 73/396)$
r	3/11	1	$(2/11, 4/11)$
$rsrgpr$	4/11	9	$(35/99, 37/99)$
gsr	45/121	1	$(4/11, 46/121)$
$rsprsrpr$	21/55	6	$(25/66, 127/330)$
$srsrpr$	13/33	3	$(38/99, 40/99)$
$srPsrpr$	9/22	2	$(17/44, 19/44)$
rsr	3/7	11	$(32/77, 34/77)$
$rsprsr$	5/11	3	$(14/33, 16/33)$
$rsrsr$	27/55	1	$(26/55, 28/55)$
$srPsrprsrsr$	129/253	1	$(128/253, 130/253)$
$srpppsrprsrsPpr$	17/33	6	$(101/198, 103/198)$
$srsPprsrsr$	41/77	1	$(40/77, 6/11)$
$srsPpsPsrsPppsrppsrsPpsrg$	6/11	4	$(23/44, 25/44)$
$srsr$	7/11	1	$(6/11, 8/11)$
$srPPrgsgrppsrsr$	8/11	6	$(47/66, 49/66)$
$srPsr$	9/11	1	$(8/11, 10/11)$
$srPsrpsrPsr$	10/11	1	$(9/11, 1)$
sr	1	11	$(10/11, 12/11)$
$srpppsrpsrpppsr$	12/11	1	$(1, 13/11)$
$srpppsr$	13/11	1	$(12/11, 14/11)$
$rsprsrpsrpppsr$	14/11	6	$(83/66, 85/66)$
$rsrsP$	15/11	1	$(14/11, 16/11)$
$srpppsrprsrsP$	16/11	4	$(63/44, 65/44)$
$rsrsPprsrsP$	113/77	1	$(16/11, 114/77)$
$srsPpsPsrsPppsrprsrsP$	49/33	6	$(293/198, 295/198)$
$srsrsPprsrsP$	61/41	11	$(670/451, 672/451)$
$srsrpsrppppsrsrPrgsgrprsrsP$	377/253	1	$(376/253, 378/253)$
$ssPsrsPprsrsP$	83/55	1	$(82/55, 84/55)$
$\vdots$			

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cont. for $\Gamma(3/11,2)$

γ	$\gamma^{-1}(\infty)$	c_γ	I_γ
$srsPpsPsrsPppsr$	17/11	3	$(50/33, 52/33)$
$srsrsP$	11/7	11	$(120/77, 122/77)$
$srsrpsrppppsrsrPrgsgr$	35/22	2	$(69/44, 71/44)$
$srppsrsrsP$	53/33	3	$(158/99, 160/99)$
$srsPpsPsrsPppsrsrPrgsgr$	89/55	6	$(533/330, 107/66)$
$spsrsPppsr$	197/121	1	$(196/121, 18/11)$
$srsrpspsrsPppsr$	18/11	9	$(161/99, 163/99)$
$srsP$	19/11	1	$(18/11, 20/11)$
$srsPpsPsrsPppsrppsrsP$	20/11	36	$(719/396, 721/396)$
$grsP$	41/22	1	$(20/11, 21/11)$
$srPPrgsgr$	21/11	3	$(62/33, 64/33)$
$srpsrsP$	2	11	$(21/11, 23/11)$

Killer Intervals for $\Gamma(5/7,3)$

γ	$\gamma^{-1}(\infty)$	c_γ	I_γ
s	0	7	$(-1/7, 1/7)$
$srsrprrsrgPrsrg$	1/7	125	$(124/875, 18/125)$
$rsrg$	5/28	1	$(1/7, 3/14)$
srg	5/21	1	$(4/21, 2/7)$
$srsrprrsrg$	2/7	25	$(7/25, 51/175)$
$sPPrps$	185/637	1	$(184/637, 186/637)$
$srsrprpsrsrpsrg$	45/154	5	$(16/55, 113/385)$
$srPPsrpsrg$	40/133	1	$(39/133, 41/133)$
g	5/16	7	$(17/56, 9/28)$
$srppppsrpsrg$	30/91	1	$(29/91, 31/91)$
$srPPrgsgrpspppsPr$	55/161	1	$(54/161, 8/23)$
$sprps$	130/371	1	$(129/371, 131/371)$
$srppsprps$	55/156	7	$(32/91, 193/546)$
$rsrsPgrgsg$	3190/9023	1	$(3189/9023, 3191/9023)$

$\vdots$

cont. next pg. $\longrightarrow$

cont. for $\Gamma(5/7,3)$

γ	$\gamma^{-1}(\infty)$	c_γ	I_γ
$srsrprrsrgpsprps$	$255/721$	25	$(6374/18025, 6376/18025)$
$rsrgPgrgsg$	$5735/16212$	1	$(2867/8106, 478/1351)$
$grgsg$	$635/1792$	1	$(317/896, 159/448)$
$srsrprpsrsrpsrppppsrpsrg$	$5/14$	25	$(62/175, 9/25)$
gsg	$35/97$	7	$(244/679, 246/679)$
$rsrsPrsgsg$	$120/329$	1	$(17/47, 121/329)$
$rPrsgsg$	$235/637$	1	$(18/49, 236/637)$
$rsgsg$	$115/308$	1	$(57/154, 29/77)$
$srsrprsgsg$	$45/119$	5	$(32/85, 226/595)$
$srsrPrPr$	$155/406$	1	$(11/29, 78/203)$
$rsrsPPrPr$	$65/168$	1	$(8/21, 11/28)$
rPr	$20/49$	1	$(19/49, 3/7)$
$rsrsPpprPr$	$3/7$	25	$(74/175, 76/175)$
$rsPsrsrps$	$10/21$	1	$(3/7, 11/21)$
$srsPPr$	$85/161$	1	$(12/23, 86/161)$
$srsrprrsrgrsrsPpsrg$	$15/28$	5	$(37/70, 19/35)$
$sPPr$	$20/37$	7	$(139/259, 141/259)$
$srPPrgsgrpsprsrsPppsPPr$	$65/119$	1	$(64/119, 66/119)$
$rPPr$	$55/98$	1	$(27/49, 4/7)$
$srsrprPPr$	$4/7$	25	$(99/175, 101/175)$
r	$5/7$	1	$(4/7, 6/7)$
$srsrprrsrgppr$	$6/7$	125	$(107/125, 751/875)$
$srsrppr$	$25/28$	1	$(6/7, 13/14)$
$rsrsPppr$	$13/14$	5	$(32/35, 33/35)$
$srsrgsr$	$125/133$	1	$(124/133, 18/19)$
$srPPrgsgrppr$	$20/21$	5	$(33/35, 101/105)$
spr	$25/26$	7	$(87/91, 88/91)$
$rsrsPgsr$	$75/77$	1	$(74/77, 76/77)$
$rPgsr$	$125/126$	1	$(62/63, 1)$
$srPgsr$	1	175	$(174/175, 176/175)$
gsr	$50/49$	1	$(1, 51/49)$
$\vdots$			

cont. next pg. $\longrightarrow$

cont. for $\Gamma(5/7, 3)$

γ	$\gamma^{-1}(\infty)$	c_γ	I_γ
$rpsrppppsrppgsr$	270/259	1	$(269/259, 271/259)$
$srsrgsgsr$	300/287	1	$(299/287, 43/41)$
$rsrsPgsgsr$	1550/1477	1	$(1549/1477, 1551/1477)$
$srsrprrsrgpgsr$	125/119	25	$(3124/2975, 3126/2975)$
$rsrgPgsgsr$	2825/2688	1	$(353/336, 471/448)$
$rPgsgsr$	2525/2401	1	$(2524/2401, 2526/2401)$
$gsgsr$	325/308	1	$(81/77, 163/154)$
$srsrprpr$	15/14	5	$(37/35, 38/35)$
$rsgsr$	25/23	7	$(174/161, 176/161)$
$rsprPrsrsPps$	100/91	1	$(99/91, 101/91)$
$rsrsrpr$	125/112	1	$(31/28, 9/8)$
$srsrpr$	8/7	5	$(39/35, 41/35)$
$srPPrgsgrpgsr$	25/21	1	$(8/7, 26/21)$
rsr	5/4	7	$(17/14, 9/7)$
$srppppsrrsrsr$	9/7	5	$(44/35, 46/35)$
$rsrsr$	10/7	1	$(9/7, 11/7)$
$srsr$	11/7	1	$(10/7, 12/7)$
$srsrprpsrsr$	12/7	5	$(59/35, 61/35)$
$srPPsr$	13/7	1	$(12/7, 2)$
sr	2	7	$(13/7, 15/7)$
$srppppsr$	15/7	1	$(2, 16/7)$
$srsrprpsrsrpsrppppsr$	16/7	5	$(79/35, 81/35)$
$rsrsP$	17/7	1	$(16/7, 18/7)$
$srsrprrsrgrsrsP$	18/7	25	$(449/175, 451/175)$
$srsP$	37/14	1	$(18/7, 19/7)$
$rsrsPppsrsP$	19/7	5	$(94/35, 96/35)$
$grsP$	58/21	1	$(19/7, 59/21)$
$rsgrsP$	73/26	7	$(255/91, 256/91)$
$srppppsrpgrsP$	453/161	1	$(452/161, 454/161)$
$sprPPrgsgr$	2033/721	1	$(2032/721, 2034/721)$
$srsrprpsrsrpsrppppsrpgrsP$	79/28	25	$(141/50, 494/175)$
$srpppsrpsprPPrgsgr$	1917/679	1	$(1916/679, 274/97)$
$rsrsrpsrsP$	337/119	1	$(48/17, 338/119)$
$srPPrgsgr$	20/7	5	$(99/35, 101/35)$
$srpsrsP$	3	7	$(20/7, 22/7)$

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