The Dissertation Committee for Mio Ilan Alter
certifies that this is the approved version of the following dissertation:

Differential $\mathbb{T}$-Equivariant K-Theory

Committee:

Daniel S. Freed, Supervisor

David Ben-Zvi

Andrew Blumberg

Karen Uhlenbeck

Andrew Neitzke

Jacques Distler

Differential $\mathbb{T}$-Equivariant K-Theory

by

Mio Ilan Alter, B.A.

DISSERTATION

Presented to the Faculty of the Graduate School of

The University of Texas at Austin

in Partial Fulfillment

of the Requirements

for the Degree of

DOCTOR OF PHILOSOPHY

THE UNIVERSITY OF TEXAS AT AUSTIN

May 2013

Dedicated to Julia.

Acknowledgments

I wish first to thank my advisor, Dan Freed, for being so generous with his time and knowledge and for his patience and support. I also wish to thank my parents for their boundless love, encouragement, and enthusiasm. Finally, I owe a debt of gratitude to my academic brother, Dan Carney, for being a purveyor of balance and humor and a great friend.

Differential $\mathbb{T}$-Equivariant K-Theory

Mio Ilan Alter, Ph.D.
The University of Texas at Austin, 2013

Supervisor: Daniel S. Freed

For $\mathbb{T}$ the circle group, we construct a differential refinement of $\mathbb{T}$-equivariant K-theory. We first construct a de Rham model for delocalized equivariant cohomology $H^\bullet_{\mathscr{D}}$ and a delocalized equivariant Chern character $\mathrm{Ch}_{\mathscr{D}} : K^\bullet_{\mathbb{T}} \to H^\bullet_{\mathscr{D}}$ based on [19] and [14]. We show that $\mathrm{Ch}_{\mathscr{D}}$ induces an isomorphism $\mathrm{Ch}_{\mathscr{D}} : K^\bullet_{\mathbb{T}} \otimes \mathbb{C} \to H^\bullet_{\mathscr{D}}$. We then construct a geometric model for differential $\mathbb{T}$-equivariant K-theory analogous to the model of differential K-theory in [27] and deduce its basic properties.

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Chapter 1

Introduction

Differential cohomology is a refinement of cohomology for smooth manifolds that includes local geometric as well as global topological information. For M a compact smooth manifold, the differential refinement $\check{H}^*(M)$ of ordinary integral cohomology $H^*(M;\mathbb{Z})$ consists roughly of integral cocycles and differential forms representing them. Thus, whereas $H^2(M;\mathbb{Z})$ is the discrete abelian group of isomorphism classes of complex line bundles on M, $\check{H}^2(M)$ is the abelian Lie group of isomorphism classes of line bundles with connection. Differential K-theory has been constructed by Hopkins-Singer [31], Klonoff [36], Freed-Lott [27], Bunke-Schick [20], and Simons-Sullivan [47]. A schematic description is as follows. Let β be a formal variable of $\deg\beta = -2$ and let $\mathcal{R}$ be the $\mathbb{Z}$-graded ring $\mathcal{R} = \mathbb{C}[\beta, \beta^{-1}]$. Let $K^\bullet(M)$ denote the $\mathbb{Z}$-graded ring of topological K-theory and let $\Omega(M;\mathcal{R})^\bullet$ denote the algebra of differential forms with values in $\mathcal{R}$. Differential K-theory fits into the following commutative diagram,

$$\begin{array}{ccc} \check{K}^\bullet(M) & \longrightarrow & \Omega(M;\mathcal{R})^\bullet_{\text{closed}} \\ \downarrow & & \downarrow [-]_{\text{dR}} \\ K^\bullet(M) & \xrightarrow[\text{Ch}]{} & H_{dR}(M;\mathcal{R})^\bullet \end{array}$$

where the bottom map is the Chern character $\mathrm{Ch} : K^\bullet(M) \to H_{dR}(M;\mathcal{R})^\bullet$, the right vertical map is given by taking the de Rham cohomology class and $\Omega(M;\mathcal{R})^\bullet_{\text{closed}}$ are the closed differential forms. One way to define differential K-theory is to require the above to be a homotopy pullback square. This makes clear how differential K-theory combines topological K-theory and differential forms. Freed and Lott construct a geometric model by generators and relations. The generators of $\breve{K}^0(M)$ are triples (E, ∇, η) where $E \to M$ is a vector bundle with connection ∇ and $\eta \in \Omega(M;\mathcal{R})^{-1}/\mathrm{im}\ (d)$; the relations come from short exact sequences of vector bundles. For $\breve{K}^{-1}(M)$, one does the same on $M \times S^1$, then uses Bott periodicity to define $\breve{K}^j(M)$ for any integer j. Ortiz ([44]) constructs differential G-equivariant K-theory for G a finite group. In this paper, we construct differential $\mathbb{T}$-equivariant K-theory for $\mathbb{T}$ the circle group.

Equivariant K-theory $K^\bullet_{\mathbb{T}}(M)$ is constructed as the Grothendieck group of isomorphism classes of $\mathbb{T}$-equivariant vector bundles. There is a standard equivariant de Rham cohomology $H_{\mathbb{T}}(M;\mathcal{R})^\bullet$ represented by equivariant differential forms. The equivariant K-theory $K^\bullet_{\mathbb{T}}(M)$ is a module over $K^\bullet_{\mathbb{T}}(pt) = R(\mathbb{T}) \cong \mathbb{Z}[t, t^{-1}]$, the representation ring of $\mathbb{T}$, and there is an augmentation homomorphism $\epsilon : R(\mathbb{T}) \to \mathbb{Z}$ given by evaluating at $t = 1$. The kernel of this homomorphism is called the augmentation ideal, denote it by I. There is an equivariant Chern character $\mathrm{Ch}_{\mathbb{T}} : K^\bullet_{\mathbb{T}}(M) \to H_{\mathbb{T}}(M;\mathcal{R})^\bullet$, but by the Atiyah-Segal Completion theorem [9], it factors through the completion of $K^\bullet_{\mathbb{T}}(M)$ at the ideal I and, in fact, $H^\bullet_{\mathbb{T}}(M)$ is isomorphic to this completion. Thus, if one attempted to construct differential $\mathbb{T}$-equivariant K-theory by forming a pullback diagram as above with $K^\bullet_{\mathbb{T}}(M)$ and $H_{\mathbb{T}}(M;\mathcal{R})^\bullet$, the additional geometric information in such a model would be a refinement of only a small piece of the topological theory. To repair this defect, we must use a "delocalized" equivariant de Rham cohomology that detects the whole group $\mathbb{T}$, not just a formal neighborhood of the identity. The idea of "globalizing" the Chern character has been studied for finite and discrete groups in [48], [11], and for compact Lie groups in[23], [28], [12], [19], [45] and [14]. We present a construction which combines the models of Brylinski [19] and Block-Getzler [14] and use it to construct differential $\mathbb{T}$-equivariant K-theory by analogy with Freed-Lott [27].

In addition to the mathematical appeal of marrying global topology and differential forms, there is some motivation for studying differential K-theory from theoretical physics. Charges of D-branes in type II string theory are elements of differential K-theory, [43], [25], and $\check{K}^*$ is also related to T-duality [34].

The format of the present paper is as follows. In section 2, we briefly review the constructions of $K^\bullet(M)$ and $K^\bullet_{\mathbb{T}}(M)$, recall their salient features, and also recall the construction of the classical Chern character. In section 3, we review equivariant cohomology and the equivariant Chern character. We conclude by reiterating the necessity of a delocalized theory. In section 4, we construct delocalized equivariant cohomology and the delocalized equivariant Chern character. We show that upon tensoring with $\mathbb{C}$, the latter is an isomorphism. In section 5, we construct differential $\mathbb{T}$-equivariant K-theory and deduce some of its basic properties.

Chapter 2

K-theory, Equivariant K-theory, and the Chern character

2.1 K-theory

In this section we rapidly recall the basic features of K-theory and set conventions. Let M be a compact smooth manifold, let $V(M)$ denote the set of isomorphism classes of finite rank complex vector bundles over M and let $\Delta \subset V(M) \times V(M)$ be the diagonal. The set $V(M)$ is a monoid under direct sum of bundles; we make a group by taking

$$K^0(M) := (V(M) \times V(M))/\Delta. \tag{2.1}$$

For vector bundles E, F, the isomorphism class of $E \otimes F$ depends only on the isomorphism classes of E and F. Thus, tensor product of bundles induces a commutative product on $V(M)$ so induces on $K^0(M)$ the structure of a commutative ring. This ring satisfies the standard universal property that if A a commutative ring, any map $\varphi : V(M) \to A$ that satisfies for all $[E], [F] \in V(M)$, $\varphi([E \oplus F]) = \varphi([E]) + \varphi([F])$ and $\varphi([E \otimes F]) = \varphi([E])\varphi([F])$ induces a unique ring homomorphism $K^0(M) \to A$.

Identifying S^n with $\mathbb{R}^n \cup \{pt\}$, for $i : M = M \times \{pt\} \hookrightarrow M \times S^n$, we define

$$K^{-n}(M) := \ker\{K^0(M \times S^n) \xrightarrow{i^*} K^0(M)\}. \tag{2.2}$$

For $H \to S^2 \cong \mathbb{C}\mathrm{P}^1$ the Hopf line bundle, the dual of the tautological line bundle, the class $\beta := [H] - [1] \in K^{-2}(pt)$ is called the *Bott class* and multiplication by it induces the Bott periodicity isomorphism

$$K^{-n}(X) \underset{\cong}{\xrightarrow{\beta}} K^{-n-2}(X) \tag{2.3}$$

for $n \geqslant 0$. One inductively defines $K^n(X) := K^{n-2}(X)$ for $n \geqslant 1$. We will write $K^\bullet(M)$ for the full $\mathbb{Z}$-graded ring

$$K^\bullet(M) := \bigoplus_{j \in \mathbb{Z}} K^j(M). \tag{2.4}$$

An equivalent description of $K^0(M)$ is as the free abelian group $\mathcal{F}$ generated by all (finite rank, complex) vector bundles on M modulo the subgroup $\mathcal{S}$ generated by short exact sequences. Recall that if

$$0 \longrightarrow E_1 \xrightarrow{i} E_2 \longrightarrow E_3 \longrightarrow 0$$

is a short exact sequence of vector bundles, there exists a splitting $s : E_3 \to E_2$. This determines an isomorphism

$$i \oplus s : E_1 \oplus E_3 \xrightarrow{\cong} E_2.$$

It follows that taking the quotient of $\mathcal{F}$ by the subgroup generated by short exact sequences identifies isomorphic bundles and identifies direct sum of bundles with addition in $\mathcal{F}$.

There is an equivalent description of $K^{-1}(M)$ in terms of pairs consisting of a bundle on M along with an automorphism. One then takes the free abelian group generated by such pairs (E, γ) modulo short exact sequences in which the maps of bundles commute with the automorphisms, and the relation $(E, \gamma) + (E, \gamma') = (E, \gamma\gamma')$. This description is related to bundles on $M \times S^1$ as follows.

Definition 2.1.1. If $E \to M$ is a vector bundle and $\gamma : E \to E$ is an automorphism, let $\pi : M \times I \to M$ be projection. Let E_γ denote the bundle $\pi^* E \to M \times I$ with the bundles $\pi^* E|_{M \times \{0\}}$ and $\pi^* E|_{M \times \{1\}}$ identified via γ.

Every bundle on $M \times S^1$ is isomorphic to one obtained from a bundle on M with an automorphism in this way. If $p : M \times S^1 \to M$ is projection, a pair (E, γ) determines an element

$$[E_\gamma] - [p^* E] \in K^{-1}(M) \tag{2.5}$$

in the first definition (2.2) of K-theory in degree -1.

2.2 Equivariant K-theory

For G a compact Lie group, G-equivariant K-theory of a smooth compact G-manifold M is defined as the Grothendieck group of *G-equivariant vector bundles* over M.

Definition 2.2.1. Let M be a smooth compact G-manifold. A *G-equivariant vector bundle* over M is a smooth G-manifold E with a map $\pi : E \to M$ satisfying the following three conditions

1. $\pi : E \to M$ is a vector bundle,

2. the projection map π is equivariant,

3. for each $g \in G$, the map $E_m \to E_{gm}$ is a linear map of vector spaces.

Let M be a compact smooth G-manifold, let $V_G(M)$ be the set of isomorphism classes of finite rank complex G-equivariant vector bundles on M and let $\Delta_G \subset V_G(M) \times V_G(M)$ be the diagonal. We define

$$K_G^0(M) := (V_G(M) \times V_G(M))/\Delta_G. \tag{2.6}$$

For $i : M = M \times \{pt\} \hookrightarrow M \times S^n$,

$$K_G^{-n}(M) := \ker\{K_G^0(M \times S^n) \xrightarrow{i^*} K_G^0(M)\} \tag{2.7}$$

where we consider $M \times S^n$ as a G-manifold with trivial G-action on the second factor. Bott periodicity also holds in equivariant K-theory, see [46]. We inductively define $K_G^n(M) = K_G^{n-2}(M)$ for $n \geqslant 1$ and write

$$K^\bullet(M) = \bigoplus_{j \in \mathbb{Z}} K_G^j(M) \tag{2.8}$$

for the full $\mathbb{Z}$-graded ring.

Again, one can equivalently describe $K_G^0(M)$ as the quotient of the free abelian group generated by all finite rank complex G-equivariant vector bundles by the subgroup generated by short exact sequences. Every G-equivariant vector bundle on $M \times S^1$ is isomorphic to one obtained from an equivariant vector bundle on M with an equivariant automorphism as in definition 2.1.1

and $K_G^{-1}(M)$ can similarly be described as generated by pairs consisting of an equivariant bundle with an equivariant automorphism.

Definition 2.2.2. Let $E_\gamma \to M \times S^1$ denote the bundle obtained from an equivariant bundle $E \to M$ and an equivariant automorphism $\gamma : E \to E$.

Two important cases of $K_G^\bullet$ are the extremes:

Proposition 2.9. *If G acts freely on M and $\pi : M \to M/G$ is the quotient map then*

$$\pi^* : K^\bullet(M/G) \longrightarrow K_G^\bullet(M) \tag{2.10}$$

is an isomorphism.

If $E \to M$ is an equivariant vector bundle, $E/G \to M/G$ is again a vector bundle. We have maps

$$Q : V_G(M) \to V(M/G) \quad \text{and} \quad \pi^* : V(M/G) \to V_G(M)$$

where $Q([E]) = [E/G]$ and $\pi^*[F] = [\pi^* F]$. Both Q and π^* are maps of monoids, $Q \circ \pi^* = Id$, and $\pi^* \circ Q$ is an isomorphism. It follows by the universal properties of $K^\bullet(M/G)$ and $K_G^\bullet(M)$ that they induce inverse isomorphisms. See [46] Proposition 2.1.

If G acts trivially on M, there is a map $K^\bullet(M) \to K_G^\bullet(M)$ given by considering a vector bundle as an equivariant vector bundle with trivial G-action. For $R(G)$ the representation ring of G, the map $M \to pt$ induces a homomorphism of rings $R(G) = K^\bullet_G(pt) \to K^\bullet_G(M)$. Combining these yields a homomorphism $\mu : K^\bullet(M) \otimes R(G) \to K^\bullet_G(M)$.

Proposition 2.11. *If G acts trivially on M, then the homomorphism*

$$\mu : K^\bullet(M) \otimes R(G) \longrightarrow K^\bullet_G(M) \tag{2.12}$$

is an isomorphism.

One constructs an inverse to μ by decomposing an equivariant vector bundle into its isotypical pieces. See [46] Proposition 2.2. For $G = \mathbb{T}$, this is particularly easy. A $\mathbb{T}$-vector bundle $E \to M$ decomposes as $E = \bigoplus_{i=1}^N E_i$ where $\mathbb{T}$ acts on E_i by the character $\tau \mapsto \tau^{k_i}$. Identifying the representation ring $R(\mathbb{T})$ with $\mathbb{Z}[t, t^{-1}]$ by the isomorphism which sends the defining representation of $\mathbb{T}$ to t, one sends

$$\begin{aligned} K^0_{\mathbb{T}}(M) &\longrightarrow K^0(M) \otimes \mathbb{Z}[t, t^{-1}] \\ [E] &\longmapsto \sum_{i=1}^N t^{k_i}[E_i]. \end{aligned} \tag{2.13}$$

On the right, $[E_i] \in K^0(M)$ means the class of E_i as a non-equivariant vector bundle. In degree $-j$ we do the same for bundles on $M \times S^j$.

Another important case is that of a homogeneous space. Let H be a closed subgroup of G. A representation $\mathbb{V}$ of H determines a G-vector bundle $G \times_H \mathbb{V} \to G/H$ and a the fiber $E|_{H \in G/H}$ of a G-vector bundle $E \to G/H$ is an H-representation. Thinking of a representation as a vector bundle over a point, we have

Proposition 2.14. *The above correspondence induces an isomorphism*

$$K_G^\bullet(G/H) \cong K_H^\bullet(pt) \cong R(H) \tag{2.15}$$

2.3 The classical Chern character

For M a compact smooth manifold we work with the differential graded algebra

$$(\Omega(M;\mathcal{R})^\bullet, d)$$

of differential forms on M with values in the graded ring $\mathcal{R} := \mathbb{C}[\beta, \beta^{-1}]$ where $\deg \beta = -2$. We grade by *total degree*: for $\eta \in \Omega^k(M;\mathbb{C})$,

$$\deg(\eta\beta^\ell) = k - 2\ell.$$

The cohomology $H(M;\mathcal{R})^\bullet$ of this complex is $\mathbb{Z}$-graded with

$$H(M;\mathcal{R})^0 \cong \prod_{k=0}^{\infty} H^{2k}(M;\mathbb{C})\beta^k \quad \text{and} \quad H(M;\mathcal{R})^{-1} \cong \prod_{k=0}^{\infty} H^{2k+1}(M;\mathbb{C})\beta^{k+1} \tag{2.16}$$

and multiplication by β and β^{-1} give periodicity isomorphisms. We write

$$H(M;\mathcal{R})^\bullet = \bigoplus_{j\in\mathbb{Z}} H(M;\mathcal{R})^j \tag{2.17}$$

for the full $\mathbb{Z}$-graded ring.

Let $E \to M$ be a vector bundle with connection $\nabla : \Omega^0(M;E) \to \Omega^1(M;E)$. Combining the de Rham d on forms with ∇ on sections of E, this extends uniquely to an operator

$$d_\nabla : \Omega^k(M;E) \to \Omega^{k+1}(M;E) \tag{2.18}$$

which satisfies the Leibnitz rule. A connection on E induces a connection on the dual bundle E^* and hence by the Leibnitz rule a connection on $E \otimes E^* \cong \mathrm{End}(E)$. We thus obtain an operator

$$d_\nabla : \Omega^k(M; \mathrm{End}(E)) \to \Omega^{k+1}(M; \mathrm{End}(E)). \tag{2.19}$$

The induced operator 2.19 can be expressed in terms of the operator 2.18 by, for $\alpha \in \Omega^k(M; \mathrm{End}(E))$,

$$d_\nabla \alpha = [d_\nabla, \alpha]. \tag{2.20}$$

Let $F = d_\nabla^2 \in \Omega^2(M; \mathrm{End}(E))$ be the curvature of ∇.

Definition 2.3.1. The *Chern character* of ∇ is the differential form

$$\mathrm{Ch}(\nabla) := \mathrm{tr}(e^{-\beta F}) = \sum_{j=0}^{\infty} \frac{1}{j!}(-\beta)^j \, \mathrm{tr}\, \underbrace{(F \wedge \cdots \wedge F)}_{j} \in \Omega(M; \mathcal{R})^0$$

Lemma 2.21. *For $\alpha \in \Omega^\bullet(M; \mathrm{End}(E))$,*

$$d\,\mathrm{tr}(\alpha) = \mathrm{tr}([d_\nabla, \alpha]).$$

Proof. On some open set $U \subset M$, we may write $\nabla = d + A$ with $A \in \Omega^1(U; \mathrm{End}(E))$. On U we have,

$$\mathrm{tr}([d_\nabla, \alpha]) = \mathrm{tr}([d + A, \alpha]) = \mathrm{tr}([d, \alpha]) + \mathrm{tr}([A, \alpha]) = \mathrm{tr}(d\alpha) + 0 = d\,\mathrm{tr}(\alpha)$$

since the trace vanishes on brackets. □

Lemma 2.22. *The form* $\mathrm{Ch}(\nabla)$ *is closed.*

Proof.

$$d\,\mathrm{Ch}(\nabla) = d\,\mathrm{tr}(e^{-\beta F}) = \mathrm{tr}([d_\nabla, e^{-\beta F}]) = \sum_{j=0}^{\infty} \frac{(-\beta)^j}{j!}\,\mathrm{tr}([d_\nabla, F]) = 0$$

since $[d_\nabla, F] = [d_\nabla, d_\nabla^2] = 0$. This last equation is the Bianchi identity. □

It follows that $\mathrm{Ch}(\nabla)$ determines a class $[\mathrm{Ch}(\nabla)] \in H(M;\mathcal{R})^0$. Next, we see that changing the connection changes the Chern character form by an explicit exact form. It will follow from this that the class $[\mathrm{Ch}(\nabla)]$ depends only on the bundle E and not on the particular connection chosen.

Let $\mathcal{C}_E$ be the affine space of all connections on E. It is modeled on the vector space $\Omega^1(M;\mathrm{End}(E))$. Let ∇' be another connection on E and let $\nabla_s : I \to \mathcal{C}_E$ be a smooth path with $\nabla_0 = \nabla$ and $\nabla_1 = \nabla'$. There is a canonical path, $\nabla + s(\nabla' - \nabla)$, from ∇ to ∇', but we will make a construction that holds for any smooth path and then deduce the dependence on the chosen path. Let $\pi : M \times I \to M$ be projection (where I is the interval $[0,1]$ with coordinate s). The path ∇_s determines a connection on $W = \pi^* E \to M \times I$,

$$\overline{\nabla} = \nabla_s + ds\partial_s. \tag{2.23}$$

See appendix B for the definition and properties of integration along the fiber.

Definition 2.3.2. The *Chern-Simons form* of the path ∇_s is

$$\mathrm{CS}(\nabla_s) = \int_{[0,1]} \mathrm{Ch}(\overline{\nabla}) \in \Omega(M;\mathcal{R})^{-1}.$$

More explicitly, if $\sigma \in \Omega^0(M \times I, W)$, $\overline{F}$ is the curvature of $\overline{\nabla}$, and F_s is the curvature of ∇_s, we have

$$\begin{aligned}
\overline{F}\sigma &= \overline{\nabla}^2\sigma \\
&= (\nabla_s + ds\partial_s)(\nabla_s + ds\partial_s)\sigma \\
&= \nabla_s^2\sigma + \nabla_s\{ds\partial_s\sigma\} + ds\partial_s\{\nabla_s\sigma\} + 0 \\
&= F_s\sigma - ds \wedge \{\nabla_s(\partial_s\sigma)\} + ds \wedge \left\{\left(\frac{d\nabla_s}{ds}\right)\sigma\right\} + ds \wedge \nabla_s(\partial_s\sigma) \\
&= F_s\sigma + ds \wedge \left\{\left(\frac{d\nabla_s}{ds}\right)\sigma\right\}
\end{aligned}$$

where we use the Leibnitz rule in the third line and $\frac{d\nabla_s}{ds} \in \Omega^1(M; \mathrm{End}(E))$. Thus,

$$\overline{F} = F_s - \frac{d\nabla_s}{ds} \wedge ds.$$

Computing, we find,

$$\begin{aligned}
\mathrm{Ch}(\overline{\nabla}) &= \mathrm{tr}(e^{-\beta\overline{F}}) \\
&= \mathrm{tr}(e^{-\beta F_s}e^{\beta(d\nabla_s/ds)\wedge ds}) \\
&= \mathrm{tr}\left(e^{-\beta F_s}\left(1 + \beta\frac{d\nabla_s}{ds} \wedge ds\right)\right) \\
&= \mathrm{tr}(e^{-\beta F_s}) + \beta\,\mathrm{tr}\left(e^{-\beta F_s}\frac{d\nabla_s}{ds}\right) \wedge ds
\end{aligned}$$

so

$$\mathrm{CS}(\nabla_s) = \int_{[0,1]} \beta\,\mathrm{tr}\left(\left(\frac{d\nabla_s}{ds}\right)e^{-\beta F_s}\right) ds \in \Omega(M; \mathcal{R})^{-1}. \tag{2.24}$$

It follows by Stokes' theorem that

$$d\,\mathrm{CS}(\nabla_s) = d\int_{[0,1]} \mathrm{Ch}(\overline{\nabla}) = \int_{[0,1]} d\,\mathrm{Ch}(\overline{\nabla}) + \int_{\partial[0,1]} \mathrm{Ch}(\overline{\nabla}) = \mathrm{Ch}(\nabla') - \mathrm{Ch}(\nabla).$$

Thus, the class $[\mathrm{Ch}(\nabla)] \in H(M; \mathcal{R})^0$ is independent of the chosen connection.

Definition 2.3.3. The Chern character of E is

$$\mathrm{Ch}(E) := [\mathrm{Ch}(\nabla)] \in H(M; \mathcal{R})^0.$$

Next, we show that changing the path changes the Chern-Simons form by an exact form.

Lemma 2.25. *If* $\alpha : S^1 \to \mathcal{C}_E$ *is a smooth loop of connections on* E*, then* $\mathrm{CS}(\alpha)$ *is exact.*

Proof. Let $p : M \times S^1 \to M$ and $\pi : M \times I \to M$ be the projections and let $V = p^*E \to M \times S^1$. The path α determines a connection $\overline{\nabla}$ on $\pi^*E \to M \times I$ as in equation 2.23. Since α is a loop, we may view $\overline{\nabla}$ as a connection on V. Observe that

$$\mathrm{CS}(\alpha) = \int_{S^1} \mathrm{Ch}(\overline{\nabla}) = p_* \mathrm{Ch}(\overline{\nabla}).$$

By Stokes' theorem,

$$d\,\mathrm{CS}(\alpha) = \mathrm{Ch}(\nabla) - \mathrm{Ch}(\nabla) = 0$$

so $\mathrm{CS}(\alpha)$ is closed. Moreover,

$$\begin{aligned} [\mathrm{CS}(\alpha)] &= [p_* \mathrm{Ch}(\overline{\nabla})] \\ &= p_* \mathrm{Ch}(V) \\ &= p_* \mathrm{Ch}(p^* E) \\ &= p_*(p^* \mathrm{Ch}(E)) \\ &= 0 \end{aligned}$$

by equation B.1. Therefore $\mathrm{CS}(\alpha)$ is exact. □

We may now make the following definition.

Definition 2.3.4. If ∇ and ∇' are two connections on E and ∇_s is any smooth path with $\nabla_0 = \nabla$ and $\nabla_1 = \nabla'$, then

$$\mathrm{CS}(\nabla', \nabla) := \mathrm{CS}(\nabla_s) \text{ mod im } (d) \in \Omega(M; \mathcal{R})^{-1}/\text{im } (d).$$

It follows that for connections $\nabla, \nabla', \nabla''$,

$$\mathrm{CS}(\nabla'', \nabla) = \mathrm{CS}(\nabla'', \nabla') + \mathrm{CS}(\nabla', \nabla). \tag{2.26}$$

The reason to remember the dependence of the Chern-Simons form on the path is the following.

Lemma 2.27. *If $\varphi : I \to \mathrm{Aut}(E)$, is a path of bundle automorphisms with $\varphi_0 := \varphi(0) = Id$, ∇ is a connection on E, and we set $\nabla' = \varphi_1^* \nabla$, then*

$$\mathrm{CS}(\nabla, \nabla') = 0 \in \Omega(M; \mathcal{R})^{-1}/im\ (d).$$

Proof. For $W = \pi^* E \to M \times I$, let

$$\overline{\nabla} = \varphi_s^* \nabla + ds\partial_s,$$

and let

$$\tilde{\nabla} = \nabla + ds\partial_s.$$

Then φ defines an automorphism of W which is φ_s on $W|_{M\times\{s\}}$ and

$$\overline{\nabla} = \varphi^* \tilde{\nabla}.$$

Writing $\overline{F}$ and $\widetilde{F}$ for the curvatures of the indicated connections, it follows that

$$\overline{F} = \varphi^{-1}\widetilde{F}\varphi.$$

Since the trace is conjugation invariant, this implies that

$$\mathrm{Ch}(\overline{\nabla}) = \mathrm{tr}(e^{-\beta\overline{F}}) = \mathrm{tr}(e^{-\beta\widetilde{F}}) = \mathrm{Ch}(\widetilde{\nabla}).$$

Observe that $\widetilde{\nabla}$ is determined by the constant path ∇ so by equation 2.24

$$\int_{[0,1]} \mathrm{Ch}(\widetilde{\nabla}) = \int_{[0,1]} \beta\,\mathrm{tr}\left(\left(\frac{d\nabla}{ds}\right) e^{-\beta F}\right) ds = 0$$

since the integrand is identically zero. It follows that

$$\mathrm{CS}(\varphi_s^*\nabla) = \int_{[0,1]} \mathrm{Ch}(\overline{\nabla}) = \int_{[0,1]} \mathrm{Ch}(\widetilde{\nabla}) = 0 \in \Omega(M;\mathcal{R})^{-1}.$$

Thus

$$\mathrm{CS}(\nabla', \nabla) = 0 \in \Omega(M;\mathcal{R})^{-1}/\mathrm{im}\ (d).$$

That is, the Chern-Simons form of the path $\varphi_s^*\nabla$ is identically zero so the Chern-Simons form of the two connections constructed using any path is zero modulo exact forms. □

2.4 The Chern character homomorphism

The Chern character determines a homomorphism $\mathrm{Ch} : K^\bullet(M) \to H(M;\mathcal{R})^\bullet$ of $\mathbb{Z}$-graded rings. Indeed, if $V, W \to M$ are vector bundles with connections ∇_V and ∇_W, and curvatures F_V and F_W, respectively, then $\nabla_V \oplus$

∇_W is a connection on $V \oplus W$ with curvature $F_V \oplus F_W$ and

$$\begin{aligned}
\mathrm{Ch}(\nabla_V \oplus \nabla_W) &= \mathrm{tr}(e^{\beta(F_V \oplus F_W)}) \\
&= \mathrm{tr}(e^{\beta F_V}) + \mathrm{tr}(e^{\beta F_W}) \\
&= \mathrm{Ch}(\nabla_V) + \mathrm{Ch}(\nabla_W).
\end{aligned} \tag{2.28}$$

Similarly,

$$\nabla_V \otimes \nabla_W := \nabla_V \otimes 1_W + 1_V \otimes \nabla_W \tag{2.29}$$

is a connection on $V \otimes W$ with curvature $F_V \otimes 1_W + 1_V \otimes F_W$ and

$$\begin{aligned}
\mathrm{Ch}(\nabla_V \otimes \nabla_W) &= \mathrm{tr}(e^{-\beta(F_V \otimes 1_W + 1_V \otimes F_W)}) \\
&= \mathrm{tr}(e^{-\beta(F_V \otimes 1_W)} e^{-\beta(1_V \otimes F_W)}) \\
&= \mathrm{tr}((e^{-\beta F_V} \otimes 1_W)(1_V \otimes e^{-\beta F_W})) \\
&= \mathrm{tr}(e^{-\beta F_V} \otimes e^{-\beta F_W}) \\
&= \mathrm{tr}\left(e^{-\beta F_V}\right) \wedge \mathrm{tr}\left(e^{-\beta F_W}\right) \\
&= \mathrm{Ch}(\nabla_V) \wedge \mathrm{Ch}(\nabla_W).
\end{aligned} \tag{2.30}$$

It follows from the universal property that Ch induces a ring homomorphism

$$\mathrm{Ch}^0 : K^0(M) \to H(M; \mathcal{R})^0. \tag{2.31}$$

We define

$$\mathrm{Ch}^{-n} : K^{-n}(M) \longrightarrow H(M; \mathcal{R})^{-n} \tag{2.32}$$

to be the composition

$$K^{-n}(M) \hookrightarrow K^0(M \times S^n) \xrightarrow{\mathrm{Ch}^0} H(M \times S^n; \mathcal{R})^0 \xrightarrow{p_*} H(M; \mathcal{R})^{-n} \tag{2.33}$$

where p_* is the integration along the fiber map defined in appendix B. We then inductively define $\mathrm{Ch}^n = \mathrm{Ch}^{n-2}$ for $n \geqslant 1$. The key fact is that

Key Fact (Atiyah-Hirzebruch [5]): The induced map

$$\mathrm{Ch} : K^\bullet(M) \otimes \mathbb{C} \longrightarrow H(M; \mathcal{R})^\bullet \tag{2.34}$$

is an isomorphism.

Chapter 3

Equivariant Cohomology and the Equivariant Chern Character

For M a compact smooth $\mathbb{T}$-manifold, we review Cartan's model of equivariant differential forms on M and the corresponding equivariant cohomology $H_{\mathbb{T}}(M;\mathcal{R})^{\bullet}$. We construct the equivariant Chern character $\mathrm{Ch}_{\mathbb{T}}$ and deduce the equivariant versions of the properties of the classical Chern character just seen. We discuss the Completion and Localization theorems in equivariant K-theory which prevent the induced homomorphism $\mathrm{Ch}_{\mathbb{T}} : K_{\mathbb{T}}^{\bullet}(M) \to H_{\mathbb{T}}(M;\mathcal{R})^{\bullet}$ from being a complex isomorphism and thus demands a "delocalized" theory.

3.1 Equivariant Differential Forms and Equivariant Cohomology

See [13] Chapter 7 for a thorough treatment of this material. Let M be a smooth manifold with a smooth action of the circle group $\mathbb{T}$ and let $\mathfrak{t} = i\mathbb{R}$ denote the Lie algebra of $\mathbb{T}$. Let $u \in \mathfrak{t}^* = (i\mathbb{R})^*$ be the standard generator of the dual Lie algebra: for $ir \in i\mathbb{R}$, $u(ir) = r$. Let $C(M)^{\bullet}$ denote the $\mathbb{Z}$-graded algebra

$$C(M)^{\bullet} := \Omega(M;\mathcal{R})[[u]]^{\bullet}. \tag{3.1}$$

We set $\deg u = 2$ so that for $\eta \in \Omega(M;\mathcal{R})^k$,

$$\deg(\eta u^\ell) = k + 2\ell. \tag{3.2}$$

The group $\mathbb{T}$ acts on differential forms by pullback. Extending linearly over u gives an action on $C(M)^\bullet$ and we write

$$C_\mathbb{T}(M)^\bullet := \Omega(M;\mathcal{R})^\mathbb{T}[[u]]^\bullet \tag{3.3}$$

for the subalgebra of $\mathbb{T}$-invariant power series. Let ξ denote the vector field on M corresponding to $i \in i\mathbb{R}$. Thus, for $m \in M$,

$$\xi_m = \left.\frac{d}{dt}\right|_{t=0} e^{it} \cdot m. \tag{3.4}$$

Define the operator $d_\mathbb{T}$ on $\omega \in C(M)^\bullet$ by

$$d_\mathbb{T}\omega = d\omega - u\iota_\xi\omega. \tag{3.5}$$

Explicitly,

$$d_\mathbb{T}\left(\sum_{k=0}^\infty \omega_k u^k\right) = \sum_{k=0}^\infty (d\omega_k)u^k - \sum_{k=0}^\infty (\iota_\xi\omega_k)u^{k+1}. \tag{3.6}$$

One readily checks that $\mathbb{T}$ abelian implies that for any $\tau \in \mathbb{T}$, $\tau_*\xi_m = \xi_{\tau\cdot m}$ so

$$\tau^*\iota_\xi = \iota_\xi\tau^*. \tag{3.7}$$

It follows that

$$\tau^* d_\mathbb{T} = d_\mathbb{T}\tau^* \tag{3.8}$$

and thus

$$d_\mathbb{T}(C_\mathbb{T}(M)^\bullet) \subseteq C_\mathbb{T}(M)^{\bullet+1}. \tag{3.9}$$

By Cartan's formula $\mathcal{L} = d\iota + \iota d$ we have $d_{\mathbb{T}}^2 = -u\mathcal{L}_\xi$ which vanishes on $\mathbb{T}$-invariant differential forms so

$$(C_{\mathbb{T}}(M)^\bullet, d_{\mathbb{T}}) \tag{3.10}$$

is a complex.

Definition 3.1.1. The complex $(C_{\mathbb{T}}(M)^\bullet, d_{\mathbb{T}})$ is called the *Cartan complex.* An element of this complex is called an *equivariant differential form.* A homogeneous element $\omega \in C_{\mathbb{T}}(M)^k$ consists of a power series

$$\omega = \sum_{j=0}^{\infty} \omega_j u^j \tag{3.11}$$

where the coefficients are homogeneous invariant differential forms

$$\omega_j \in (\Omega(M; \mathcal{R})^{k-2j})^{\mathbb{T}}. \tag{3.12}$$

Remark 3.13. The Cartan complex with complex coefficients is usually defined using polynomials in u rather than power series so that, for example, in degree $2k$ one has

$$\Omega(M; \mathbb{C})^{\mathbb{T}}[u]^{2k} = \left\{ \sum_{j=0}^{k} \omega_j u^j \,\middle|\, \omega_j \in \Omega^{k-2j}(M; \mathbb{C})^{\mathbb{T}} \right\}. \tag{3.14}$$

Our definition is related to the standard one by

$$C_{\mathbb{T}}(M)^0 = \prod_{k=0}^{\infty} \beta^k \Omega(M; \mathbb{C})^{\mathbb{T}}[u]^{2k} \tag{3.15}$$

and

$$C_{\mathbb{T}}(M)^{-1} = \prod_{k=0}^{\infty} \beta^{k+1} \Omega(M; \mathbb{C})^{\mathbb{T}}[u]^{2k+1} \tag{3.16}$$

We thus see that β is a bookkeeping device that allows us to shift all of the homogeneous pieces of the standard Cartan complex of even degree into degree 0 and of odd degree into degree -1. We allow power series in u because we are taking the direct product rather than the direct sum, but this is a convention that depends on what one means by the ring associated to a graded ring (see [41] Remark 1.2). The point is that β retains the ordinary cohomological degree and in each β degree we have the standard Cartan complex.

Definition 3.1.2. Equivariant cohomology is the (2-periodic) $\mathbb{Z}$-graded cohomology theory given by

$$H_{\mathbb{T}}(M;\mathcal{R})^{\bullet} := H(C_{\mathbb{T}}(M)^{\bullet}, d_{\mathbb{T}}). \tag{3.17}$$

Proposition 3.18. *Two important properites of* $\mathbb{T}$*-equivariant cohomology are*

1. *if* $f, g : M \to N$ *are smooth equivariant maps and there exists a smooth* $\mathbb{T}$*-equivariant homotopy from* f *to* g*, then the induced maps on equivariant cohomology are equal*

$$f^* = g^* : H_{\mathbb{T}}(N;\mathcal{R})^{\bullet} \longrightarrow H_{\mathbb{T}}(M;\mathcal{R})^{\bullet},$$

2. *if* U *and* V *are* $\mathbb{T}$*-invariant open sets such that* $M = U \cup V$*, then* $H_{\mathbb{T}}(M;\mathcal{R})^{\bullet}, H_{\mathbb{T}}(U;\mathcal{R})^{\bullet} \oplus H_{\mathbb{T}}(V;\mathcal{R})^{\bullet}$ *and* $H_{\mathbb{T}}(U \cap V;\mathcal{R})^{\bullet}$ *fit into a long exact Mayer-Vietoris sequence.*

Proof. For the first property, consider $M \times I$ as a $\mathbb{T}$-manifold with trivial action on the second factor. If $H : M \times I \to N$ is the homotopy and $\omega \in C_{\mathbb{T}}(N)^{\bullet}$, then $H^*\omega \in C_{\mathbb{T}}(M \times I)^\bullet$. Observe that, as in appendix B, if ξ is the vector field on N which generates the $\mathbb{T}$-action, then $\overline{\xi} = (\xi, 0)$ is the vector field which generates the $\mathbb{T}$-action on $N \times I$ and for any $\eta \in \Omega(N \times I; \mathcal{R})^\bullet$,

$$\iota_\xi \left(\int_{[0,1]} \eta \right) = \int_{[0,1]} (\iota_{\overline{\xi}} \eta). \tag{3.19}$$

It follows by Stokes' theorem and the previous observation that

$$g^*\omega - f^*\omega = d_{\mathbb{T}} \int_{[0,1]} H^*\omega. \tag{3.20}$$

For the second, one must choose an equivariant partition of unity subordinate to the cover $\{U, V\}$. These exist by, for example, [29] B.33. The result then follows by the same method as in ordinary de Rham cohomology (see [16], p.22). □

Example 3.21. Let $\mathbb{T}$ act on $M = S^1$ by double-speed rotation: for $\tau \in \mathbb{T}$ and $\lambda \in S^1$, $\tau \cdot \lambda = \tau^2 \lambda$. Since the $\mathbb{T}$-invariant 0-forms on S^1 are the constants $\mathcal{R}$, it follows that

$$\begin{aligned} C_{\mathbb{T}}(S^1)^0 &= \left\{ \sum_{k=0}^{\infty} \alpha_k u^k \,\middle|\, \alpha_k \in \mathcal{R}^{-2k} \right\} \\ &= \left\{ \sum_{k=0}^{\infty} a_k (\beta u)^k \,\middle|\, a_k \in \mathbb{C} \right\}. \end{aligned} \tag{3.22}$$

Since

$$d_{\mathbb{T}} \left(\sum_k a_k (\beta u)^k \right) = \sum_k (d a_k)(\beta u)^k = 0, \tag{3.23}$$

all equivariant 0-forms are closed. If η is the unique $\mathbb{T}$-invariant 1-form on S^1 satisfying $\iota_\xi \eta = \eta(\xi) = 1$, the equivariant -1-forms are the series whose coefficients are constant multiples of η,

$$C_{\mathbb{T}}(S^1)^{-1} = \left\{ \sum_{k=0}^{\infty} c_k \eta \beta^{k+1} u^k \,\middle|\, c_k \in \mathbb{C} \right\}. \tag{3.24}$$

Since

$$\begin{aligned} d_{\mathbb{T}} \sum_{k=0}^{\infty} c_k \eta \beta^{k+1} u^k &= \sum_{k=0}^{\infty} c_k (d\eta) \beta^{k+1} u^k - \sum_{k=0}^{\infty} c_k \eta(\xi) \beta^{k+1} u^{k+1} \\ &= \sum_{k=0}^{\infty} -c_k (\beta u)^{k+1} \end{aligned} \tag{3.25}$$

the exact equivariant 0-forms are those power series with zero constant term. The equivariantly closed equivariant -1-forms are clearly zero. Thus,

$$H_{\mathbb{T}}(S^1; \mathcal{R})^q = \left\{ \begin{array}{ll} \mathbb{C} & q = 0 \\ 0 & q = -1 \end{array} \right\} = H(pt; \mathcal{R})^q = H(S^1/\mathbb{T}; \mathcal{R})^q. \tag{3.26}$$

This illustrates a limitation of equivariant cohomology: it does not see the doubling of the action. This is a general phenomenon, equivariant cohomology does not distinguish between free actions (those with trivial stabilizer) and locally free actions (those with finite stabilizers). See appendix C.

For comparison, we compute the equivariant K-theory of the same setup.

Example 3.27. Let $\mathbb{T}$ again act on $M = S^1$ by double-speed rotation. Every complex line bundle on S^1 is topologically trivial, but there are equivariant line bundles which are necessarily isomorphic as line bundles, but not equivariantly isomorphic. If $T = S^1 \times \mathbb{C}$ is the equivariant line bundle with $\mathbb{T}$-action $\tau \cdot (x, \lambda) = (\tau^2 x, \tau\lambda)$, and $\underline{1} = S^1 \times \mathbb{C}$ is the equivariant line bundle with the trivial action $\tau \cdot (x, \lambda) = (\tau^2 x, \lambda)$, one can easily write down an isomorphism $T^{\otimes 2} \cong \underline{1}$. It follows that for $k \in \mathbb{Z}$, there are isomorphisms of equivariant line bundles

$$T^{\otimes k} \cong \begin{cases} \underline{1}, & k \text{ even} \\ T, & k \text{ odd} \end{cases} \tag{3.28}$$

and thus,

$$K^0_{\mathbb{T}}(S^1) \otimes \mathbb{C} = \mathbb{C}\{\underline{1}\} \oplus \mathbb{C}\{T\}. \tag{3.29}$$

Next, every equivariant bundle over $M \times S^1$ is isomorphic to one obtained from an equivariant bundle on M with an equivariant automorphism. If α is an automorphism of T or $\underline{1}$, equivariance implies that for $x \in M$ and $\tau \in \mathbb{T}$, $\alpha(\tau^2 x) = \alpha(x)$ so α must be constant. It follows that every equivariant vector bundle on $M \times S^1$ is isomorphic to a pullback and thus that the restriction map

$$i^* : K^0_{\mathbb{T}}(M \times S^1) \longrightarrow K^0_{\mathbb{T}}(M) \tag{3.30}$$

is injective. This implies that $K^{-1}_{\mathbb{T}}(M) := \ker i^* = 0$. We have found that

$$K^q_{\mathbb{T}}(S^1) \otimes \mathbb{C} \cong \begin{cases} \mathbb{C}\{\underline{1}\} \oplus \mathbb{C}\{T\} & q = 0 \\ 0 & q = -1. \end{cases} \tag{3.31}$$

Thus, $K^\bullet_{\mathbb{T}}$ does detect the doubling of the action whereas $H^\bullet_{\mathbb{T}}$ does not.

Let $E \to M$ be a $\mathbb{T}$-equivariant vector bundle and for $\tau \in \mathbb{T}$ let $L_\tau : E \to E$ denote the action of τ on E. Denote the sections of E by $\Omega^0(M; E)$; let

$$\Omega^\bullet(M; E) = \Omega^\bullet(M) \otimes_{\Omega^0(M)} \Omega^0(M; E) \tag{3.32}$$

be the differential forms with values in E and let

$$C(M;E)^\bullet := C(M)^\bullet \otimes_{\Omega^0(M)} \Omega^0(M;E) \tag{3.33}$$

be the equivariant differential forms with values in E. The group $\mathbb{T}$ acts on sections of E by, for $\sigma \in \Omega^0(M;E)$, $\tau \in \mathbb{T}$,

$$(\tau \cdot \sigma)(m) := L_\tau \sigma(\tau^{-1} \cdot m). \tag{3.34}$$

Hence, $\mathbb{T}$ acts on $\Omega^\bullet(M;E)$ by, for $\omega \in \Omega^\bullet(M;\mathbb{C})$,

$$\tau \cdot (\omega \otimes \sigma) = \tau^* \omega \otimes (\tau \cdot \sigma). \tag{3.35}$$

Extending the action to be linear over u and $\mathcal{R}$ induces an action on $C(M;E)^\bullet$. We then set

$$C^\bullet_{\mathbb{T}}(M;E) := C^\bullet(M;E)^{\mathbb{T}}. \tag{3.36}$$

Recall that ξ, defined in equation 3.4, is the vector field on M corresponding to the action of $i \in i\mathbb{R}$. We then also have the Lie derivative of the section σ in the direction of ξ,

$$\mathcal{L}^E_\xi \sigma = \left.\frac{d}{dt}\right|_{t=0} e^{it} \cdot \sigma. \tag{3.37}$$

If ∇ is a connection on E we say that ∇ is $\mathbb{T}$-*invariant* if it commutes with the $\mathbb{T}$-action on $C(M;E)^\bullet$, that is, for every $\tau \in \mathbb{T}$,

$$[\nabla, \tau \cdot] = 0. \tag{3.38}$$

Differentiating, it follows that if ∇ is invariant,

$$[\nabla, \mathcal{L}^E_\xi] = 0. \tag{3.39}$$

The space $\mathcal{C}_E^{\mathbb{T}}$ of invariant connections is a non-empty affine subspace of the affine space $\mathcal{C}_E$ of all connections. See appendix D.

Definition 3.1.3. For ∇ an invariant connection on E the correpsonding *equivariant connection* is the operator

$$\nabla^{\mathbb{T}} := \nabla - u\iota_{\xi} : C(M;E)^{\bullet} \longrightarrow C(M;E)^{\bullet+1}. \tag{3.40}$$

One checks that for $\alpha \in C(M)^j$ and $\theta \in C(M;E)$,

$$\nabla^{\mathbb{T}}(\alpha \wedge \theta) = d_{\mathbb{T}}\alpha \wedge \theta + (-1)^j \alpha \wedge \nabla^{\mathbb{T}}\theta. \tag{3.41}$$

As before, ∇ on E induces a connection ∇ on $\mathrm{End}(E)$. The corresponding equivariant connection is an operator

$$\nabla^{\mathbb{T}} : C(M;\mathrm{End}(E))^{\bullet} \longrightarrow C(M;\mathrm{End}(E))^{\bullet+1}. \tag{3.42}$$

As in the non-equivariant case (equations 2.18 and 2.19), the operators 3.40 and 3.42 are related by, for $\theta \in C_{\mathbb{T}}(M;\mathrm{End}(E))^{\bullet}$,

$$\nabla^{\mathbb{T}}\theta = [\nabla^{\mathbb{T}}, \theta]. \tag{3.43}$$

Definition 3.1.4. The *equivariant curvature* of the invariant connection ∇ is the operator

$$F^{\mathbb{T}} := (\nabla^{\mathbb{T}})^2 + u\mathcal{L}_{\xi}^{E} : C(M;E)^{\bullet} \longrightarrow C(M;E)^{\bullet+2}. \tag{3.44}$$

Lemma 3.45. *The equivariant curvature $F^{\mathbb{T}}$ is in $C_{\mathbb{T}}(M;\mathrm{End}(E))^2$ and satisfies the* equivariant Bianchi identity

$$\nabla^{\mathbb{T}}F^{\mathbb{T}} = 0. \tag{3.46}$$

Proof. To show that $F^{\mathbb{T}} \in C_{\mathbb{T}}(M; \mathrm{End}(E))^2$ we show that it commutes with multiplication by any $\alpha \in C(M)^\bullet$. Let $\epsilon(\alpha)$ denote exterior multiplication by α. We have

$$\begin{aligned}
[F^{\mathbb{T}}, \epsilon(\alpha)] &= [(\nabla^{\mathbb{T}})^2 + u\mathcal{L}_\xi^E, \epsilon(\alpha)] \\
&= [\nabla^{\mathbb{T}}, [\nabla^{\mathbb{T}}, \epsilon(\alpha)]] + u[\mathcal{L}_\xi^E, \epsilon(\alpha)] \\
&= [\nabla^{\mathbb{T}}, \epsilon(d_{\mathbb{T}}\alpha)] + u\epsilon(\mathcal{L}_\xi^E \alpha) \qquad (3.47) \\
&= \epsilon\{(d_{\mathbb{T}}^2 + u\mathcal{L}_\xi^E)\alpha\} \\
&= 0.
\end{aligned}$$

Next, the equivariant Bianchi identity follows from the decomposition

$$\begin{aligned}
\nabla^{\mathbb{T}} F^{\mathbb{T}} &= [\nabla^{\mathbb{T}}, (\nabla^{\mathbb{T}})^2 + u\mathcal{L}_\xi^E] \\
&= [\nabla^{\mathbb{T}}, (\nabla^{\mathbb{T}})^2] + u[\nabla, \mathcal{L}_\xi^E] - u^2[\iota_\xi, \mathcal{L}_\xi^E] \qquad (3.48)
\end{aligned}$$

where the second term vanishes by the invariance of ∇ and the third by the Cartan formula $\mathcal{L} = d\iota + \iota d$ on differential forms. □

Expanding the definition of the equivariant curvature, we have

$$F^{\mathbb{T}} = (\nabla)^2 - u\nabla\iota_\xi - u\iota_\xi\nabla + u\mathcal{L}_\xi^E. \qquad (3.49)$$

An element of $C(M; E)^\bullet$ is a sum of elements of the form $\omega \otimes \sigma$ for $\omega \in C(M)^\bullet$ and $\sigma \in \Omega^0(M; E)$ and it is easy to verify that

$$F^{\mathbb{T}}(\omega \otimes \sigma) = (F \wedge \omega) \otimes \sigma + u\omega \otimes (\mathcal{L}_\xi^E - \nabla_\xi)\sigma. \qquad (3.50)$$

Definition 3.1.5. The *moment* of the $\mathbb{T}$-action relative to the connection ∇ is

$$\mu := u\{\mathcal{L}_\xi^E - \nabla_\xi\} \in \mathfrak{t}^* \otimes \Omega^0(M; \mathrm{End}(E))^{\mathbb{T}}. \tag{3.51}$$

We may thus write

$$F^{\mathbb{T}} = F + \mu \in C_{\mathbb{T}}(M; \mathrm{End}(E))^2. \tag{3.52}$$

If ∇ is an invariant connection such that $\mathcal{L}_\xi^E = \nabla_\xi$, then we see that $F^{\mathbb{T}} = F$. Such a connection is called *basic* for the $\mathbb{T}$-action and always exists if the action has finite stabilizers. See appendix D for a discussion.

3.2 The Equivariant Chern character

In this subsection, we define the equivariant Chern character and derive properties analogous to those of the classical Chern character in subsection 2.3.

Lemma 3.53. *For* $\alpha \in C(M; \mathrm{End}(E))^\bullet$,

$$d_{\mathbb{T}} \operatorname{tr}(\alpha) = \operatorname{tr}([\nabla^{\mathbb{T}}, \alpha]). \tag{3.54}$$

Proof. The proof of Lemma 2.21 works here since on some open set $U \subset M$ we can write $\nabla^{\mathbb{T}} = d_{\mathbb{T}} + A$ for $A \in \Omega^1(U; \mathrm{End}(E))$. □

Definition 3.2.1. The *equivariant Chern character* of the invariant connection ∇ is the equivariant differential form

$$\mathrm{Ch}_{\mathbb{T}}(\nabla) = \operatorname{tr}\left(e^{-\beta F^{\mathbb{T}}}\right) \in C_{\mathbb{T}}(M)^0. \tag{3.55}$$

Lemma 3.56. *The form* $\mathrm{Ch}_{\mathbb{T}}(\nabla)$ *is equivariantly closed.*

Proof. The proof is the same as Lemma 2.22, this time using Lemma 3.53 and the equivariant Bianchi identity. □

Next, suppose that ∇ and ∇' are two invariant connections on $E \to M$ and let $W = \pi^* E \to M \times I$. If $\nabla_s : I \to \mathcal{C}_E^{\mathbb{T}}$ is a smooth path of invariant connections with $\nabla_0 = \nabla$ and $\nabla_1 = \nabla'$, ∇_s determines an invariant connection $\overline{\nabla}$ on W given by equation 2.23. The equivariant connection corresponding to $\overline{\nabla}$ is

$$\overline{\nabla}^{\mathbb{T}} = \nabla_s^{\mathbb{T}} + ds\partial_s = (\nabla_s - u\iota_{\overline{\xi}}) + ds\partial_s. \tag{3.57}$$

Using that $\iota_\xi ds = 0$, one derives as preceding equation 2.24 that

$$\overline{F}^{\mathbb{T}} = F_s^{\mathbb{T}} + ds \wedge \frac{d\nabla_s}{ds} \tag{3.58}$$

where F_s again denotes the curvature of ∇_s.

Definition 3.2.2. The equivariant Chern-Simons form of the path ∇_s is

$$\mathrm{CS}_{\mathbb{T}}(\nabla_s) := \int_{[0,1]} \mathrm{Ch}_{\mathbb{T}}(\overline{\nabla}) \in C_{\mathbb{T}}(M)^{-1}. \tag{3.59}$$

As for equation 2.24, one derives the explicit expression

$$\mathrm{CS}_{\mathbb{T}}(\nabla_s) = \int_{[0,1]} \beta \operatorname{tr}\left(\frac{d\nabla_s}{ds} e^{-\beta F_s^{\mathbb{T}}}\right) ds \tag{3.60}$$

from equation 3.58 and the definition of $\mathrm{Ch}_{\mathbb{T}}$.

Lemma 3.61.

$$d_{\mathbb{T}}\, \mathrm{CS}_{\mathbb{T}}(\nabla_s) = \mathrm{Ch}_{\mathbb{T}}(\nabla') - \mathrm{Ch}_{\mathbb{T}}(\nabla). \tag{3.62}$$

Proof. This follows by Stokes' theorem and equation 3.19. □

As before, we show that changing the path of invariant connections changes the equivariant Chern-Simons form by an equivariantly exact form.

Lemma 3.63. *If* $\alpha : S^1 \to \mathcal{C}_E^{\mathbb{T}}$ *is a loop of invariant connections, then* $\mathrm{CS}_{\mathbb{T}}(\alpha)$ *is exact.*

Proof. Again, let $p : M \times S^1 \to M$ be projection, let $V = p^*E$, and let $\overline{\nabla}$ be the connection determined by α on V. Then $\mathrm{CS}_{\mathbb{T}}(\alpha)$ is closed and the method of Lemma 2.25 carries over word for word, now using p_* in equivariant cohomology. Thus $[\mathrm{CS}_{\mathbb{T}}(\alpha)] = 0$ so $\mathrm{CS}_{\mathbb{T}}(\alpha)$ is exact. □

Definition 3.2.3. The equivariant Chern-Simons form of connections ∇', ∇ is

$$\mathrm{CS}_{\mathbb{T}}(\nabla', \nabla) := \mathrm{CS}_{\mathbb{T}}(\nabla_s) \text{ mod im } (d_{\mathbb{T}}) \in C_{\mathbb{T}}(M)^{-1}/\text{im } (d_{\mathbb{T}}) \tag{3.64}$$

for any smooth path ∇_s with $\nabla_0 = \nabla$ and $\nabla_1 = \nabla'$.

It again follows, in analogy with equation 2.26, that if ∇, ∇' and ∇'' are three connections that

$$\mathrm{CS}_{\mathbb{T}}(\nabla'', \nabla) = \mathrm{CS}_{\mathbb{T}}(\nabla'', \nabla') + \mathrm{CS}_{\mathbb{T}}(\nabla', \nabla) \tag{3.65}$$

Lemma 3.66. *If* $E \to M$ *is a* $\mathbb{T}$*-equivariant vector bundle with invariant connection* ∇, $\varphi : I \to \mathrm{Aut}(E)$ *is a family of automorphisms with* $\varphi_0 = Id$ *(where* $\varphi_s := \varphi(s)$*), then taking* $\nabla' = \varphi_1^*\nabla$,

$$\mathrm{CS}_{\mathbb{T}}(\nabla', \nabla) = 0 \in C_{\mathbb{T}}(M)^{-1}/im\ (d_{\mathbb{T}}). \tag{3.67}$$

Proof. The proof is the same as Lemma 2.27: the path of invariant connections $\varphi_s^*\nabla$ determines an invariant connection

$$\overline{\nabla} = \varphi_s^*\nabla + ds\partial_s \tag{3.68}$$

on $W = \pi^*E \to M \times I$. The corresponding equivariant connection is

$$\begin{aligned} \overline{\nabla}^{\mathbb{T}} &= \varphi_s^*\nabla + ds\partial_s - u\iota_{\overline{\xi}} \\ &= (\varphi_s^*\nabla - u\iota_{\overline{\xi}}) + ds\partial_s \\ &= (\varphi_s^*\nabla)^{\mathbb{T}} + ds\partial_s. \end{aligned} \tag{3.69}$$

Since $\varphi_s : E \to E$ covers the identity on M,

$$(\varphi_s^*\nabla)^{\mathbb{T}} = \varphi_s^*(\nabla^{\mathbb{T}}) \tag{3.70}$$

so

$$\overline{\nabla}^{\mathbb{T}} = \varphi_s^*(\nabla^{\mathbb{T}}) + ds\partial_s. \tag{3.71}$$

Let

$$\tilde{\nabla}^{\mathbb{T}} = \nabla^{\mathbb{T}} + ds\partial_s, \tag{3.72}$$

then considering φ as an automorphism of W, we may write

$$\overline{\nabla}^{\mathbb{T}} = \varphi_s^*(\nabla^{\mathbb{T}}) + ds\partial_s = \varphi^*(\tilde{\nabla}^{\mathbb{T}}). \tag{3.73}$$

As before, the path which determines $\tilde{\nabla}$ is constant so by equation 3.60,

$$\mathrm{CS}_{\mathbb{T}}(\varphi_s^*\nabla) = \int_{[0,1]} \mathrm{Ch}_{\mathbb{T}}(\overline{\nabla}) = \int_{[0,1]} \mathrm{Ch}_{\mathbb{T}}(\tilde{\nabla}) = 0 \tag{3.74}$$

since the last integrand vanishes. It follows that

$$\mathrm{CS}_{\mathbb{T}}(\nabla', \nabla) = 0 \in C_{\mathbb{T}}(M)^{-1}/\mathrm{im}\ (d_{\mathbb{T}}). \tag{3.75}$$

$\square$

3.3 The Equivariant Chern character homomorphism

If V and W are equivariant vector bundles over M with invariant connections ∇_V and ∇_W, respectively, then $\nabla_V \oplus \nabla_W$ and $\nabla_V \otimes \nabla_W$ (defined in equation 2.29) are invariant connections on $V \oplus W$ and $V \otimes W$, respectively, as in the non-equivariant case. They have curvatures $F_{V\oplus W} = F_V \oplus F_W$ and $F_{V\otimes W} = F_V \otimes 1_W + 1_V \otimes F_W$, respectively. One readily checks that the moments relative to these connections are

$$\mu_{V\oplus W} = \mu_V \oplus \mu_W \quad \text{and} \quad \mu_{V\otimes W} = \mu_V \otimes 1_W + 1_V \otimes \mu_W \tag{3.76}$$

from which it follows that the equivariant curvatures corresponding to these invariant connections are

$$F^{\mathbb{T}}_{V\oplus W} = F^{\mathbb{T}}_V \oplus F^{\mathbb{T}}_W \quad \text{and} \quad F^{\mathbb{T}}_{V\otimes W} = F^{\mathbb{T}}_V \otimes 1_W + 1_V \otimes F^{\mathbb{T}}_W. \tag{3.77}$$

The calculations 2.28 and 2.30 and the above formulae imply that

$$\mathrm{Ch}_{\mathbb{T}}(\nabla_V \oplus \nabla_W) = \mathrm{Ch}_{\mathbb{T}}(\nabla_V) + \mathrm{Ch}_{\mathbb{T}}(\nabla_W) \tag{3.78}$$

and

$$\mathrm{Ch}_{\mathbb{T}}(\nabla_V \otimes \nabla_W) = \mathrm{Ch}_{\mathbb{T}}(\nabla_V) \wedge \mathrm{Ch}_{\mathbb{T}}(\nabla_W). \tag{3.79}$$

It follows from the universal property that $\mathrm{Ch}_{\mathbb{T}}$ induces a ring homomorphism

$$\mathrm{Ch}^0_{\mathbb{T}} : K^0_{\mathbb{T}}(M) \to H_{\mathbb{T}}(M; \mathcal{R})^0 \tag{3.80}$$

and we define

$$\mathrm{Ch}^{-n}_{\mathbb{T}} : K^{-n}_{\mathbb{T}}(M) \longrightarrow H_{\mathbb{T}}(M; \mathcal{R})^{-n} \tag{3.81}$$

to be the composition

$$K_{\mathbb{T}}^{-n}(M) \hookrightarrow K_{\mathbb{T}}^{0}(M \times S^n) \underset{\mathrm{Ch}_{\mathbb{T}}^0}{\longrightarrow} H_{\mathbb{T}}(M \times S^n; \mathcal{R})^0 \underset{p_*}{\longrightarrow} H_{\mathbb{T}}(M; \mathcal{R})^{-n} \tag{3.82}$$

where p_* is the integration along the fiber map defined in appendix B.2. We set $\mathrm{Ch}_{\mathbb{T}}^n = \mathrm{Ch}_{\mathbb{T}}^{n-2}$ for $n \geqslant 1$.

3.4 Completion and Localization

Recall that in examples 3.21 and 3.27 of $\mathbb{T}$ acting on S^1 by double speed rotation, we found that $K_{\mathbb{T}}^{\bullet}$ sees the doubling of the action whereas $H_{\mathbb{T}}^{\bullet}$ does not. The equivariant Chern character cannot therefore be an isomorphism over $\mathbb{C}$. An explanation of this discrepancy is as follows. The ring $K_{\mathbb{T}}^{\bullet}(M) \otimes \mathbb{C}$ is a module over

$$K_{\mathbb{T}}^{\bullet}(pt) \otimes \mathbb{C} = R(\mathbb{T}) \otimes \mathbb{C} \cong \mathbb{C}[t, t^{-1}], \tag{3.83}$$

the complexified character ring of $\mathbb{T}$. For $\tau \in \mathbb{T}$, let $K_{\mathbb{T}}^{\bullet}(M) \otimes \mathbb{C}_{\tau}$ denote the localization at the ideal of characters which vanish at τ and let $K_{\mathbb{T}}^{\bullet}(M) \otimes \mathbb{C}_{\tau}^{\wedge}$ denote the formal completion at the same ideal. By the Atiyah-Segal completion theorem [9],

$$K_{\mathbb{T}}^{\bullet}(M) \otimes \mathbb{C}_{1}^{\wedge} \cong H_{\mathbb{T}}(M; \mathcal{R})^{\bullet}. \tag{3.84}$$

Thinking of $K_{\mathbb{T}}(M)^{\bullet} \otimes \mathbb{C}$ as a sheaf over $\operatorname{Spec} \mathbb{C}[t, t^{-1}] = \mathbb{C}^{\times}$, this indicates that the equivariant cohomology only detects the stalk of this sheaf over $1 \in \mathbb{T}$. Here $\mathbb{C}^{\times} = \mathbb{T}_{\mathbb{C}}$ is the complexification of the group $\mathbb{T}$; the module $K_{\mathbb{T}}^{\bullet}(M)$ corresponds to the unitary part, the restriction of the sheaf to $\mathbb{T} \subset \mathbb{C}^{\times}$. A special case of a theorem of Freed-Hopkins-Telemen ([26], Theorem 3.9) describes the stalks of this sheaf at other points of $\mathbb{T}$ in terms of equivariant cohomology, too. Writing M^τ for the submanifold with stabilizer subgroup $\langle \tau \rangle \subset \mathbb{T}$, it says that there is a natural isomorphism

$$K^\bullet_{\mathbb{T}}(M) \otimes \mathbb{C}^\wedge_\tau \cong K^\bullet_{\mathbb{T}}(M^\tau) \otimes \mathbb{C}^\wedge_\tau \cong H_{\mathbb{T}}(M^\tau; \mathcal{R})^\bullet \tag{3.85}$$

in which the first isomorphism is induced by the inclusion $M^\tau \hookrightarrow M$.

The idea of globalizing the Chern character $\mathrm{Ch} : K^\bullet_{\mathbb{T}}(M) \otimes \mathbb{C} \to H_{\mathbb{T}}(M; \mathcal{R})^\bullet$ to detect the whole sheaf and not just a single stalk has been studied for finite groups [48] and [11] and for compact Lie groups [19], [14], [26]. The idea of constructing a de Rham model of $K_{\mathbb{T}}(M) \otimes \mathbb{C}$ to receive such a map has been studied in [19] and [14]. We present a de Rham model and corresponding delocalized Chern character that are similar to those of Brylinski [19] and Block-Getzler [14].

Chapter 4

Delocalized Equivariant Cohomology

4.1 Delocalized equivariant differential forms and delocalized equivariant cohomology

Let M be a compact smooth manifold with a smooth action of $\mathbb{T}$. For $H \subset \mathbb{T}$ a subgroup, we write

$$M^H = \{m \in M \mid hm = m \text{ for all } h \in H\} \tag{4.1}$$

for the points fixed by H. For $H = \langle \tau \rangle$, we write $M^\tau := M^{\langle \tau \rangle}$. See appendix A for relevant facts about group actions. Two motivations for the complex we present are the following. First, there is an isomorphism

$$K^\bullet_{\mathbb{T}}(M^{\mathbb{T}}) \otimes \mathbb{C} \cong K^\bullet(M^{\mathbb{T}}) \otimes \mathbb{C}[t, t^{-1}] \xrightarrow[\cong]{\mathrm{Ch} \otimes Id} H(M^{\mathbb{T}}; \mathcal{R})^\bullet \otimes \mathbb{C}[t, t^{-1}] \tag{4.2}$$

by proposition 2.11 and the classical complex Chern character isomorphism. This suggests that we start with $\Omega(M^{\mathbb{T}}; \mathcal{R})^\bullet \otimes \mathbb{C}[t, t^{-1}]$. Second, equation 3.85 suggests that we add to this the Cartan complexes on all of the fixed point sets M^τ. We must then require that these agree in an appropriate sense on $M^{\mathbb{T}}$.

By Appendix A.6 only finitely many subgroups of $\mathbb{T}$ appear as stabilizer subgroups. Since every proper closed subgroup of $\mathbb{T}$ is finite cyclic, it follows that there are finitely many $\tau \in \mathbb{T}$ for which M^τ properly contains $M^{\mathbb{T}}$. Let

$$S_{\mathbb{T}}(M) = \{\tau \in \mathbb{T} \mid M^\tau - M^{\mathbb{T}} \neq \varnothing\} \tag{4.3}$$

be this finite set of group elements. We introduce the following definition.

Definition 4.1.1. A *delocalized equivariant differential form* is

- A finite Laurent series

$$\omega = \sum_{k=-M}^{N} \omega_k t^k \in \Omega(M^{\mathbb{T}}; \mathcal{R})^\bullet \otimes \mathbb{C}[t, t^{-1}] \tag{4.4}$$

  valued in differential forms on the fixed points $M^{\mathbb{T}}$. The grading is just that of the differential forms.

- A collection $\{\eta_\tau\}_{\tau \in S_{\mathbb{T}}(M)}$ of equivariant differential forms $\eta_\tau \in C_{\mathbb{T}}(M^\tau)^\bullet$ on the submanifolds M^τ.

These must be related by the following

$$\textit{Compatibility condition: } \eta_\tau|_{M^{\mathbb{T}}} = \sum_{k=-M}^{N} \tau^k \omega_k e^{-\beta i k u} \tag{4.5}$$

as elements of $C_{\mathbb{T}}(M^{\mathbb{T}})^\bullet$.

Definition 4.1.2. Let $\mathcal{A}_{\mathbb{T}}(M)^j$ denote the homogeneous delocalized equivariant differential forms of degree j, the abelian group of pairs $(\omega, \{\eta_\tau\}_{\tau \in S_{\mathbb{T}}(M)})$ as above in which ω and the η_τ are homogeneous of degree j. Let

$$(\mathcal{A}_{\mathbb{T}}(M)^\bullet, d_{\mathbb{T}}) \tag{4.6}$$

denote the full $\mathbb{Z}$-graded complex.

We extend the de Rham d and the wedge $\wedge$ to $\Omega(M^{\mathbb{T}};\mathcal{R})^{\bullet} \otimes \mathbb{C}[t,t^{-1}]$ to be linear over $\mathbb{C}[t,t^{-1}]$ and define the differential on $\mathcal{A}_{\mathbb{T}}^{\bullet}$ by

$$\delta(\omega, \{\eta_\tau\}) = (d\omega, \{d_{\mathbb{T}}\eta_\tau\}). \tag{4.7}$$

We may multiply delocalized forms by multiplying the series and wedging: the product

$$\left(\sum_{k=-M}^{N} \omega_k t^k, \{\eta_\tau\}\right) \cdot \left(\sum_{k=-M'}^{N'} \theta_j t^j, \{\xi_\tau\}\right) \tag{4.8}$$

is the delocalized equivariant differential form

$$\left(\sum_{k=-M}^{N} \sum_{j=-M'}^{N'} (\omega_k \wedge \theta_j) t^{k+j}, \{\eta_\tau \wedge \xi_\tau\}_{\tau\in S_{\mathbb{T}}(M)}\right). \tag{4.9}$$

Finally, $\mathbb{C}[t,t^{-1}]$ acts on $\mathcal{A}_{\mathbb{T}}(M)^{\bullet}$ by

$$t \cdot (\omega, \{\eta_\tau\}_{\tau\in S_{\mathbb{T}}}) = (t\omega, \{\tau\eta_\tau\}_{\tau\in S_{\mathbb{T}}(M)}) \tag{4.10}$$

where $\tau\eta_\tau$ means multiplication by the complex number $\tau \in \mathbb{T}$. Thus, $\mathcal{A}_{\mathbb{T}}(M)^{\bullet}$ becomes a differential $\mathbb{Z}$-graded algebra over $\mathbb{C}[t,t^{-1}]$.

Definition 4.1.3. The *delocalized equivariant cohomology* of M is the cohomology of this complex,

$$H_{\mathscr{D}}(M)^{\bullet} := H(\mathcal{A}_{\mathbb{T}}(M)^{\bullet}, \delta).$$

Remark 4.11. We denote the delocalized equivariant cohomology by $H_{\mathscr{D}}^{\bullet}$ and reserve $H_{\mathbb{T}}(-;\mathcal{R})^{\bullet}$ for the $\mathbb{T}$-equivariant cohomology defined in the previous section.

Proposition 4.12. *Delocalized equivariant cohomology satisfies the familiar properties*

1. *if $f, g : N \to M$ are smoothly homotopic smooth $\mathbb{T}$-maps then $f^* = g^* : H_{\mathscr{D}}(M)^\bullet \to H_{\mathscr{D}}(N)^\bullet$,*

2. *if $U, V \subset M$ are $\mathbb{T}$-invariant open sets with $M = U \cup V$, then $H_{\mathscr{D}}(M)^\bullet$, $H_{\mathscr{D}}(U)^\bullet \oplus H_{\mathscr{D}}(V)^\bullet$ and $H_{\mathscr{D}}(U \cap V)^\bullet$ fit into a long exact Mayer-Vietoris sequence,*

3. *for each n, the projection $p : M \times S^n \to M$ induces an integration along the fiber map,*

$$p_* : H_{\mathscr{D}}(M \times S^n)^\bullet \to H_{\mathscr{D}}(M)^{\bullet - n}$$

which satisfies the push-pull formula B.1.

Proof. We sketch the proof.

1. If N and M are smooth compact $\mathbb{T}$-manifolds, consider $N \times I$ as a $\mathbb{T}$-manifold with the trivial action on the second factor. Given maps $f, g : N \to M$, suppose that $H : N \times I \to M$ is a smooth homotopy with $H(n, 0) = f(n)$ and $H(n, 1) = g(n)$. We extend the operations of pullback and integration to $\Omega(-, \mathcal{R})^\bullet \otimes \mathbb{C}[t, t^{-1}]$ by requiring them to be linear over $\mathbb{C}[t, t^{-1}]$. Let $\alpha = [\omega, \{\eta_\tau\}] \in H_{\mathscr{D}}(M)^\bullet$ where

$$\omega = \sum_{k=-M}^{N} \omega_k t^k. \tag{4.13}$$

Then

$$H^*\omega = \sum_{k=-M}^{N} H^*\omega_k t^k \tag{4.14}$$

and we now check that

$$H^*(\omega, \{\eta_\tau\}) = (H^*\omega, \{H^*\eta_\tau\}) \tag{4.15}$$

satisfies the compatibility condition 4.5 and is thus an element of $\mathcal{A}_{\mathbb{T}}(N \times I)^\bullet$. Observe that

$$\begin{aligned} H^*\eta_\tau|_{N^{\mathbb{T}}\times I} &= H^*(\eta_\tau|_{M^{\mathbb{T}}}) \\ &= H^*\left(\sum_{k=-M}^{N} \tau^k \omega_k e^{-\beta iku}\right) \\ &= \sum_{k=-M}^{N} \tau^k (H^*\omega_k) e^{-\beta iku}. \end{aligned} \tag{4.16}$$

so $H^*(\omega, \{\eta_\tau\}) \in \mathcal{A}_{\mathbb{T}}(N \times I)^\bullet$. We now verify that

$$\int_{[0,1]} H^*(\omega, \{\eta_\tau\}) = \left(\int_{[0,1]} H^*\omega, \left\{\int_{[0,1]} H^*\eta_\tau\right\}\right) \tag{4.17}$$

is an element of $\mathcal{A}_{\mathbb{T}}(N)^\bullet$. The compatibility condition follows by integrating both sides of the above equation 4.16 and observing that

$$\int_{[0,1]} \left\{\sum_{k=-M}^{N} \tau^k (H^*\omega_k) e^{-\beta iku}\right\} = \sum_{k=-M}^{N} \tau^k \left\{\int_{[0,1]} (H^*\omega_k)\right\} e^{-\beta iku}. \tag{4.18}$$

Finally,

$$\begin{aligned} \delta\left(\int_{[0,1]} H^*\omega, \left\{\int_{[0,1]} \eta_\tau\right\}\right) &= \left(d\int_{[0,1]} H^*\omega, \left\{d_{\mathbb{T}}\int_{[0,1]} H^*\eta_\tau\right\}\right) \\ &= (g^*\omega - f^*\omega, \{g^*\eta_\tau - f^*\eta_\tau\}) \\ &= g^*(\omega, \{\eta_\tau\}) - f^*(\omega, \{\eta_\tau\}) \end{aligned} \tag{4.19}$$

Therefore $g^*\alpha = f^*\alpha$.

2. For the Mayer-Vietoris property, let M be a compact smooth $\mathbb{T}$-manifold, U, V an invariant open cover of M and let $\{\rho_U, \rho_V\}$ be an equivariant partition of unity subordinate to this cover. We must show that

$$0 \longrightarrow \mathcal{A}_{\mathbb{T}}(M)^{\bullet} \xrightarrow{j_U \oplus j_V} \mathcal{A}_{\mathbb{T}}(U)^{\bullet} \oplus \mathcal{A}_{\mathbb{T}}(V)^{\bullet} \xrightarrow{\mu} \mathcal{A}_{\mathbb{T}}(U \cap V)^{\bullet} \longrightarrow 0$$
$$(\alpha, \beta) \longmapsto \alpha - \beta$$

is a short exact sequence of cochain complexes. The key step is to show exactness at the last stage, that is, to show that the difference map μ is surjective. One does this by extending forms on $U \cap V$ to U by multiplying by ρ_V and to V by multiplying by ρ_U. Since multiplication by partitions of unity commutes with restriction of forms, one readily verifies that the same method shows that the above sequence of delocalized forms is short exact.

3. We indicated how integration along the fiber works in the proof of the first property. For $p : M \times S^n \to M$ and $(\theta, \{\xi_\tau\}) \in \mathcal{A}_{\mathbb{T}}(M \times S^n)^j$,

$$p_*(\theta, \{\xi_\tau\}) := (p_*\theta, \{p_*\xi_\tau\}) \in \mathcal{A}_{\mathbb{T}}(M)^{j-n}. \tag{4.20}$$

One verifies that $p_*\theta$ and $\{p_*\xi_\tau\}$ satisfy the compatibility condition as indicated in the proof of the first property. Since p_* is a chain map in ordinary cohomology and in equivariant cohomology, it is in delocalized equivariant cohomology so induces a map

$$\int_{S^n} = p_* : H_{\mathscr{D}}(M \times S^n)^{\bullet} \longrightarrow H_{\mathscr{D}}(M)^{\bullet - n} \tag{4.21}$$

□

The method of appendix B works in the present setting to show that for $pt \in S^n$ and $i : M \times \{pt\} \hookrightarrow M \times S^n$ the inclusion, integration over the fiber restricts to an isomorphism

$$\begin{array}{ccccccccc} 0 \longrightarrow & \ker i^* \cong H_{\mathscr{D}}(M)^{-n} \otimes H^n(S^n;\mathbb{C}) & \longrightarrow & H_{\mathscr{D}}(M \times S^n)^0 & \longrightarrow & H_{\mathscr{D}}(M)^0 & \longrightarrow 0 \\ & \cong \big\downarrow p_* & & & & & \\ & H_{\mathscr{D}}(M)^{-n} & & & & & \end{array} \tag{4.22}$$

Example 4.23. Let $M = pt$, then $S_{\mathbb{T}}(M) = \varnothing$ so

$$\mathcal{A}_{\mathbb{T}}(pt)^{\bullet} = \Omega(pt;\mathcal{R})^{\bullet} \otimes \mathbb{C}[t,t^{-1}] = \mathcal{R}^{\bullet} \otimes \mathbb{C}[t,t^{-1}]. \tag{4.24}$$

It follows that

$$H_{\mathscr{D}}(pt)^q = \begin{cases} \mathbb{C}[t,t^{-1}], & q = 0 \\ 0, & q = -1. \end{cases} \tag{4.25}$$

Example 4.26 (Free action)**.** Let $\mathbb{T}$ act freely on M, then $M^{\mathbb{T}} = \varnothing$, $S_{\mathbb{T}}(M) = \{1\}$, and $M^1 = M$ so

$$\mathcal{A}_{\mathbb{T}}(M)^{\bullet} = C_{\mathbb{T}}(M)^{\bullet}. \tag{4.27}$$

It follows from proposition C.7 that

$$H_{\mathscr{D}}(M)^{\bullet} = H_{\mathbb{T}}(M;\mathcal{R})^{\bullet} \cong H(M/\mathbb{T};\mathcal{R})^{\bullet}. \tag{4.28}$$

Example 4.29 (Locally free action)**.** Let $\mathbb{T}$ act on $M = S^1$ by double-speed rotation, that is, for $\tau \in \mathbb{T}$, $\lambda \in M$, $\tau \cdot \lambda = \tau^2\lambda$. In this case, $S_{\mathbb{T}}(M) = \{\pm 1\}$, $M^{\mathbb{T}} = \varnothing$, and $M^{\pm 1} = M$ so we get two copies of the Cartan complex,

$$\mathcal{A}_{\mathbb{T}}(S^1)^{\bullet} = C_{\mathbb{T}}(S^1)^{\bullet} \oplus C_{\mathbb{T}}(S^1)^{\bullet}, \tag{4.30}$$

one for each element of $S_{\mathbb{T}}(M)$. It follows that

$$H_{\mathscr{D}}(S^1)^\bullet = H_{\mathbb{T}}(S^1)^\bullet \oplus H_{\mathbb{T}}(S^1)^\bullet \tag{4.31}$$

and by Proposition C.7

$$H_{\mathscr{D}}(S^1)^\bullet \cong H(S^1/\mathbb{T};\mathcal{R})^\bullet \oplus H(S^1/\mathbb{T};\mathcal{R})^\bullet = H(pt;\mathcal{R})^\bullet \oplus H(pt;\mathcal{R})^\bullet. \tag{4.32}$$

Thus, $H_{\mathscr{D}}(S^1)^\bullet$ detects the doubling of the action.

4.2 The Delocalized Equivariant Chern character

Let $E \to M$ be a $\mathbb{T}$-equivariant vector bundle with invariant connection ∇, corresponding equivariant connection $\nabla^{\mathbb{T}}$, and equivariant curvature $F^{\mathbb{T}}$. For $\tau \in \mathbb{T}$, let $L_\tau : E \to E$ denote the action of τ on E. If $\tau \cdot m \neq m$ for some $m \in M$, although L_τ is invertible, it maps E_m to $E_{\tau \cdot m} \neq E_m$ so does not cover the identity on M. It is therefore not a vector bundle automorphism of E. However, since $\mathbb{T}$ acts trivially on $M^{\mathbb{T}}$, for all $\tau \in \mathbb{T}$,

$$L_\tau : E|_{M^{\mathbb{T}}} \to E|_{M^{\mathbb{T}}} \tag{4.33}$$

is a bundle automorphism. Letting τ vary, we obtain a homomorphism

$$L : \mathbb{T} \to \mathrm{Aut}(E|_{M^{\mathbb{T}}}) \tag{4.34}$$

and write $L_t := L(t)$. Similarly, for $\tau \in S_{\mathbb{T}}(M)$,

$$L_\tau : E|_{M^\tau} \to E|_{M^\tau} \tag{4.35}$$

is a bundle automorphism. On differential forms with values in E, $\mathbb{T}$ acts by a combination of pullback and its action on E. However, for $\tau \in S_{\mathbb{T}}(M)$, the pullback action of τ on $\Omega(M^\tau;\mathcal{R})^\bullet$ is trivial so τ acts on $\Omega(M;E|_{M^\tau})^\bullet$ simply by L_τ. We introduce the following definition. It is the equivariant Chern character of [14] (following proposition 4.3) adapted to our complex of delocalized equivariant differential forms.

Definition 4.2.1. The *delocalized equivariant Chern character* of ∇ is the delocalized equivariant form

$$\mathrm{Ch}_{\mathscr{D}}(\nabla) = \left(\mathrm{Ch}_{\mathscr{D}}(\nabla)_{\mathbb{T}}, \{\mathrm{Ch}_{\mathscr{D}}(\nabla)_\tau\}_{\tau \in S_{\mathbb{T}}(M)}\right) \in \mathcal{A}_{\mathbb{T}}(M)^0 \tag{4.36}$$

where

$$\mathrm{Ch}_{\mathscr{D}}(\nabla)_{\mathbb{T}} := \mathrm{tr}\left(L_t e^{-\beta F}\big|_{M^{\mathbb{T}}}\right) \in \Omega(M^{\mathbb{T}};\mathcal{R})^0 \otimes \mathbb{C}[t,t^{-1}] \tag{4.37}$$

and

$$\mathrm{Ch}_{\mathscr{D}}(\nabla)\tau := \mathrm{tr}\left(L_\tau e^{-\beta F^{\mathbb{T}}}\Big|_{M^\tau}\right) \in C_{\mathbb{T}}(M^\tau)^0. \tag{4.38}$$

As written, it is not obvious that $\mathrm{Ch}_{\mathscr{D}}(\nabla)_{\mathbb{T}}$ (equation 4.37) is an element of $\Omega(M^{\mathbb{T}};\mathcal{R})^0 \otimes \mathbb{C}[t,t^{-1}]$. We verify this first. We must then check that $\mathrm{Ch}_{\mathscr{D}}(\nabla)$ satisfies the compatibility condition 4.5.

For the first statement, recall from Proposition 2.11 that we may decompose $E|_{M^{\mathbb{T}}}$ into its isotypical components, that is, there exists an isomorphism of equivariant vector bundles

$$\varphi : V := \bigoplus_{i=1}^{N} V_i \to E|_{M^{\mathbb{T}}} \tag{4.39}$$

where each $V_i \to M^{\mathbb{T}}$ is an equivariant vector bundle on which L_λ acts by multiplication by λ^{k_i} for some $k_i \in \mathbb{Z}$. Endow $\bigoplus_i V_i$ with the pullback connection $\varphi^*\nabla$. Since ∇ is $\mathbb{T}$-invariant, $\varphi^*\nabla$ respects the direct sum decomposition of V so decomposes as

$$\varphi^*\nabla = \bigoplus_{i=1}^{N} \nabla_i. \tag{4.40}$$

Let F be the curvature of ∇. It follows that the curvature $F_{\varphi^*\nabla} = \varphi^{-1}F\varphi$ of $\varphi^*\nabla$ also respects the direct sum decomposition so we may write $F_{\varphi^*\nabla} = \bigoplus_i F_i$. We have

$$(F_{\varphi^*\nabla})^j = \bigoplus_{i=1}^{N} F_i^j = \bigoplus_{i=1}^{N} \underbrace{F_i \wedge \cdots \wedge F_i}_{j}. \tag{4.41}$$

Then, writing L_t for the $\mathbb{T}$-action on V as well as on $E|_{M^{\mathbb{T}}}$,

$$\begin{aligned}
\mathrm{Ch}_{\mathscr{D}}(\varphi^*\nabla)_{\mathbb{T}} &= \mathrm{tr}\left(L_t e^{-\beta F_{\varphi^*\nabla}}\right) \\
&= \mathrm{tr}\left(L_t e^{-\beta(\bigoplus_i F_i)}\right) \\
&= \sum_{i=1}^{N} \mathrm{tr}\left(L_t e^{-\beta F_i}\right) \\
&= \sum_{i=1}^{N} \mathrm{tr}\left(t^{k_i} e^{-\beta F_i}\right) \\
&= \sum_{i=1}^{N} t^{k_i}\,\mathrm{Ch}(\nabla_i) \in \Omega(M^{\mathbb{T}};\mathcal{R})^0 \otimes \mathbb{C}[t,t^{-1}].
\end{aligned} \tag{4.42}$$

Since φ is equivariant, we have

$$\begin{aligned}
\mathrm{Ch}_{\mathscr{D}}(\varphi^*\nabla)_{\mathbb{T}} &= \mathrm{tr}\left(L_t e^{-\beta F_{\varphi^*\nabla}}\right) \\
&= \mathrm{tr}\left(L_t e^{\varphi^{-1}(-\beta)F\varphi}\right) \\
&= \mathrm{tr}\left(L_t \varphi^{-1} e^{-\beta F}\varphi\right) \\
&= \mathrm{tr}\left(\varphi^{-1} L_t e^{-\beta F}\varphi\right) \\
&= \mathrm{tr}\left(L_t e^{-\beta F}\right) \\
&= \mathrm{Ch}_{\mathscr{D}}(\nabla)_{\mathbb{T}}.
\end{aligned} \tag{4.43}$$

Therefore, $\mathrm{Ch}_{\mathscr{D}}(\nabla)_{\mathbb{T}}$ is indeed an element of $\Omega(M^{\mathbb{T}};\mathcal{R})^0 \otimes \mathbb{C}[t,t^{-1}]$.

To verify the compatibility condition 4.5 we proceed as follows. Observe that on $E|_{M^{\mathbb{T}}} \to M^{\mathbb{T}}$, $\mathbb{T}$ acts trivially on the base, but nontrivially on the bundle. Since $\mathbb{T}$ acts trivially on $M^{\mathbb{T}}$, the operator ∇_ξ vanishes, but $\mathcal{L}^E_\xi$ need not. Thus, the equivariant curvature $F^{\mathbb{T}}$ of ∇ reduces to

$$F^{\mathbb{T}} = F + \mu = F + u\{\mathcal{L}^E_\xi - \nabla_\xi\} = F + u\mathcal{L}^E_\xi. \tag{4.44}$$

Since φ is equivariant, the equivariant curvature of $\varphi^*\nabla$ on V is

$$F^{\mathbb{T}}_{\varphi^*\nabla} = F_{\varphi^*\nabla} + u\mathcal{L}^V_\xi = \varphi F\varphi^{-1} + u\varphi\mathcal{L}^E_\xi\varphi^{-1} = \varphi F^{\mathbb{T}}\varphi^{-1}. \tag{4.45}$$

It follows from the calculation 4.43 that

$$\mathrm{Ch}_{\mathscr{D}}(\nabla)_\tau|_{M^{\mathbb{T}}} = \mathrm{tr}\left(L_\tau e^{-\beta F^{\mathbb{T}}}\Big|_{M^{\mathbb{T}}}\right) = \mathrm{tr}\left(L_\tau e^{-\beta F^{\mathbb{T}}_{\varphi^*\nabla}}\Big|_{M^{\mathbb{T}}}\right). \tag{4.46}$$

Now, $F^{\mathbb{T}}_{\varphi^*\nabla}$ respects the direct sum decomposition of V so may be written as

$$F^{\mathbb{T}}_{\varphi^*\nabla} = \bigoplus_{j=1}^{N}\{F_j + u\mathcal{L}^{V_j}_\xi\}. \tag{4.47}$$

Since $\mathbb{T}$ acts on V_j by multiplication, the Lie derivative on sections (equation 3.37) takes the following simplified form. For a section $\sigma : M^{\mathbb{T}} \to V_j$,

$$\mathcal{L}_\xi^{V_j} \sigma = \frac{d}{dt}\bigg|_{t=0} e^{k_j it}\sigma = k_j i\sigma, \tag{4.48}$$

so $\mathcal{L}_\xi^{V_j} = (k_j i) Id_{V_j} \in \Omega^0(M^{\mathbb{T}}; \mathrm{End}(V_j))$. It follows that

$$\begin{aligned} \mathrm{Ch}_{\mathscr{D}}(\nabla)_\tau|_{M^{\mathbb{T}}} &= \mathrm{tr}\left(L_\tau e^{-\beta(F^{\mathbb{T}}_{\varphi^*\nabla})}\right) \\ &= \sum_{j=1}^N \mathrm{tr}\left(\tau^{k_j} e^{-\beta(F_j + uk_j iId_{V_j})}\right) \\ &= \sum_{j=1}^N \tau^{k_j}\, \mathrm{tr}(e^{-\beta F_j} e^{-(\beta uk_j i)Id_{V_j}}) \\ &= \sum_{j=1}^N \tau^{k_j}\, \mathrm{tr}(e^{-\beta F_j}) e^{-\beta uk_j i} \\ &= \sum_{j=1}^N \tau^{k_j}\, \mathrm{Ch}(\nabla_j) e^{-\beta uk_j i}. \end{aligned} \tag{4.49}$$

By the calculations 4.42 and 4.43,

$$\mathrm{Ch}_{\mathscr{D}}(\nabla)_{\mathbb{T}} = \sum_{j=1}^N t^{k_j}\, \mathrm{Ch}(\nabla_j), \tag{4.50}$$

so the calculation 4.49 is exactly the compatibility condition 4.5. Therefore $\mathrm{Ch}_{\mathscr{D}}(\nabla)$ is indeed a delocalized equivariant differential form.

Next, let $\psi : (\widetilde{E}, \widetilde{\nabla}) \to (E, \nabla)$ be an isomorphism of equivariant bundles with connection over M, meaning that ψ is an equivariant isomorphism and $\widetilde{\nabla} = \psi^*\nabla$. Write $\widetilde{F}^{\mathbb{T}}$ and $F^{\mathbb{T}}$ for the equivariant curvatures of $\widetilde{\nabla}$ and ∇, respectively. By equation 4.45,

$$\widetilde{F}^{\mathbb{T}} = \psi^{-1} F^{\mathbb{T}} \psi \tag{4.51}$$

so by the conjugation invariance of the trace,

$$\mathrm{Ch}_{\mathscr{D}}(\tilde{\nabla}) = \mathrm{Ch}_{\mathscr{D}}(\nabla). \tag{4.52}$$

Lemma 4.53. *The delocalized form* $\mathrm{Ch}_{\mathscr{D}}(\nabla)$ *is closed.*

Proof. By the invariance of ∇ (and the fact that τ acts trivially on M^τ), $\nabla^{\mathbb{T}} L_\tau = L_\tau \nabla^{\mathbb{T}}$ on $C_{\mathbb{T}}(M^\tau; \mathrm{End}(E|_{M^\tau}))^\bullet$. Similarly, $\nabla L_t = L_t \nabla$ on $\Omega(M^{\mathbb{T}}; \mathrm{End}(E|_{M^{\mathbb{T}}}))^\bullet$. It follows by Lemma 2.21 that on $M^{\mathbb{T}}$,

$$d\,\mathrm{tr}\left(L_t e^{-\beta F}\right) = \mathrm{tr}\left([\nabla, L_t e^{-\beta F}]\right) = \mathrm{tr}\left(L_t[\nabla, e^{-\beta F}]\right) = 0 \tag{4.54}$$

since $[\nabla, e^{-\beta F}] = 0$ by the Bianchi identity as in Lemma 2.22. Similarly, on M^τ by Lemma 3.53,

$$d_{\mathbb{T}}\,\mathrm{tr}\left(L_\tau e^{-\beta F^{\mathbb{T}}}\right) = \mathrm{tr}\left(L_\tau [\nabla^{\mathbb{T}}, e^{-\beta F^{\mathbb{T}}}]\right) = 0 \tag{4.55}$$

by the equivariant Bianchi identity as in Lemma 3.56. Therefore,

$$\delta\,\mathrm{Ch}_{\mathscr{D}}(\nabla) = (d\,\mathrm{Ch}_{\mathscr{D}}(\nabla)_{\mathbb{T}}, \{d_{\mathbb{T}}\,\mathrm{Ch}_{\mathscr{D}}(\nabla)_\tau\}) = 0. \tag{4.56}$$

□

Definition 4.2.2. Let ∇ and ∇' be two invariant connections on E, $\pi : M \times I \to M$ projection and $W = \pi^* E \to M \times I$. Let ∇_s be a smooth path of invariant connections with $\nabla_0 = \nabla$ and $\nabla_1 = \nabla'$ and let $\overline{\nabla}$ be the invariant connection on W corresponding to ∇_s. Define

$$\mathrm{CS}_{\mathscr{D}}(\nabla_s) := \int_{[0,1]} \mathrm{Ch}_{\mathscr{D}}(\overline{\nabla}) \in \mathcal{A}_{\mathbb{T}}(M)^{-1}. \tag{4.57}$$

One derives an explicit expression for $\mathrm{CS}_{\mathscr{D}}(\nabla_s)$ as done in equation 2.24. By equation 3.58, for each $\tau \in S_{\mathbb{T}}(M)$,

$$\begin{aligned} \mathrm{CS}_{\mathscr{D}}(\nabla_s)_\tau &= \int_{[0,1]} \mathrm{tr}\left(L_\tau \exp\left\{ -\beta \left(F_s^{\mathbb{T}} + ds \wedge \frac{d\nabla_s}{ds} \right) \right\} \bigg|_{M^\tau} \right) \\ &= \int_{[0,1]} \beta \, \mathrm{tr}\left(\frac{d\nabla_s}{ds} L_\tau \exp\{-\beta F_s^{\mathbb{T}}\} \bigg|_{M^\tau} \right) ds. \end{aligned} \tag{4.58}$$

Simiarly,

$$\mathrm{CS}_{\mathscr{D}}(\nabla_s)_{\mathbb{T}} = \int_{[0,1]} \beta \, \mathrm{tr}\left(\frac{d\nabla_s}{ds} L_t \exp\{-\beta F_s\} \bigg|_{M^{\mathbb{T}}} \right) ds. \tag{4.59}$$

Lemma 4.60.

$$\delta \, \mathrm{CS}_{\mathscr{D}}(\nabla_s) = \mathrm{Ch}_{\mathscr{D}}(\nabla') - \mathrm{Ch}_{\mathscr{D}}(\nabla). \tag{4.61}$$

Proof. We decompose W into its $\mathbb{T}$-eigenbundles on $(M \times I)^{\mathbb{T}} = M^{\mathbb{T}} \times I$ and its τ-eigenbundles on $(M \times I)^\tau = M^\tau \times I$ and use the previous results. As in equation 4.39 and the discussion there, on $M^{\mathbb{T}} \times I$ there is an equivariant isomorphism

$$\varphi : \bigoplus_{i=1}^N W_i \to W|_{M^{\mathbb{T}} \times I} \tag{4.62}$$

and

$$\varphi^* \overline{\nabla} = \bigoplus_{i=1}^N \overline{\nabla}_i. \tag{4.63}$$

Write $(\varphi^* \overline{\nabla})|_{M^{\mathbb{T}} \times \{0\}} = \bigoplus_{i=1}^N \nabla_i$ and $(\varphi^* \overline{\nabla})|_{M^{\mathbb{T}} \times \{1\}} = \bigoplus_{i=1}^N \nabla'_i$. Then by equation 4.42

$$\mathrm{CS}_{\mathscr{D}}(\nabla_s)_{\mathbb{T}} = \int_{[0,1]} \mathrm{Ch}_{\mathscr{D}}(\overline{\nabla})_{\mathbb{T}} = \sum_{i=1}^N t^{k_i} \int_{[0,1]} \mathrm{Ch}(\overline{\nabla}_i) \tag{4.64}$$

so

$$d \, \mathrm{CS}_{\mathscr{D}}(\gamma)_{\mathbb{T}} = \sum_{i=1}^N t^{k_i} \{ \mathrm{Ch}(\nabla'_i) - \mathrm{Ch}(\nabla_i) \} = \mathrm{Ch}_{\mathscr{D}}(\nabla')_{\mathbb{T}} - \mathrm{Ch}_{\mathscr{D}}(\nabla)_{\mathbb{T}}. \tag{4.65}$$

By a similar argument, we may decompose $W|_{M^\tau \times I}$ into its τ-eigenbundles and write

$$\mathrm{CS}_{\mathscr{D}}(\nabla_s)_\tau = \int_{[0,1]} \mathrm{Ch}_{\mathscr{D}}(\overline{\nabla})_\tau = \sum_{j=1}^{M} \tau^{m_j} \int_{[0,1]} \mathrm{Ch}_{\mathbb{T}}(\overline{\nabla}_j). \tag{4.66}$$

It follows that

$$d_{\mathbb{T}} \, \mathrm{CS}_{\mathscr{D}}(\nabla_s)_\tau = \mathrm{Ch}_{\mathscr{D}}(\nabla')_\tau - \mathrm{Ch}_{\mathscr{D}}(\nabla)_\tau \tag{4.67}$$

and thus

$$\delta \, \mathrm{CS}_{\mathscr{D}}(\nabla_s) = \mathrm{Ch}_{\mathscr{D}}(\nabla') - \mathrm{Ch}_{\mathscr{D}}(\nabla). \tag{4.68}$$

□

Lemma 4.69. *If $\alpha : S^1 \to \mathcal{C}_E^{\mathbb{T}}$ is a loop if invariant connections, then $\mathrm{CS}_{\mathscr{D}}(\alpha)$ is exact.*

Proof. The formal proof of Lemma 2.25 again applies word for word now using integration along the fiber in delocalized equivariant cohomology. □

Definition 4.2.3. The delocalized equivariant Chern-Simons form of a pair of invariant connections ∇' and ∇ is

$$\mathrm{CS}_{\mathscr{D}}(\nabla', \nabla) := \mathrm{CS}_{\mathscr{D}}(\nabla_s) \text{ mod im } (\delta) \in \mathcal{A}_{\mathbb{T}}(M)^{-1}/\text{im } (\delta) \tag{4.70}$$

for any smooth path ∇_s of invariant connections with $\nabla_0 = \nabla$ and $\nabla_1 = \nabla'$.

It again follows by analogy with equations 2.26 and 3.65 that if ∇'', ∇' and ∇ are three connections

$$\mathrm{CS}_{\mathscr{D}}(\nabla'', \nabla) = \mathrm{CS}_{\mathscr{D}}(\nabla'', \nabla') + \mathrm{CS}_{\mathscr{D}}(\nabla', \nabla). \tag{4.71}$$

Lemma 4.72. *If $\varphi : I \to \mathrm{Aut}(E)$ is a smooth path of automorphisms with $\varphi_0 = Id$, then taking $\nabla' = \varphi_1^* \nabla$,*

$$\mathrm{CS}_{\mathscr{D}}(\nabla', \nabla) = 0 \in \mathcal{A}_{\mathbb{T}}(M)^{-1}/im\ (\delta). \tag{4.73}$$

Proof. The proof is the same as that of Lemma 3.66, the corresponding statement in equivariant cohomology. Using the notation of 3.66,

$$\mathrm{CS}_{\mathscr{D}}(\varphi_s^* \nabla) = \int_{[0,1]} \mathrm{Ch}_{\mathscr{D}}(\overline{\nabla}) = \int_{[0,1]} \mathrm{Ch}_{\mathscr{D}}(\tilde{\nabla}) = 0 \tag{4.74}$$

since the path which determines $\tilde{\nabla}$ is constant so by the formulae 4.58 and 4.59, the last integrand vanishes. □

4.3 The delocalized equivariant Chern character is a complex isomorphism

It follows from the calculations of sections 2.4 and 3.3 that $\mathrm{Ch}_{\mathscr{D}}$ takes direct sum to addition and tensor product to wedge. Thus, by the universal property $\mathrm{Ch}_{\mathscr{D}}$ induces a ring homomorphism

$$\mathrm{Ch}^0_{\mathscr{D}} : K^0_{\mathbb{T}}(M) \to H_{\mathscr{D}}(M)^0. \tag{4.75}$$

We again define

$$\mathrm{Ch}^{-n}_{\mathscr{D}} : K^{-n}_{\mathbb{T}}(M) \to H_{\mathscr{D}}(M)^{-n} \tag{4.76}$$

by the composition

$$K^{-n}_{\mathbb{T}}(M) \hookrightarrow K^0_{\mathbb{T}}(M \times S^n) \xrightarrow[\mathrm{Ch}^0_{\mathscr{D}}]{} H_{\mathscr{D}}(M \times S^n)^0 \xrightarrow[p_*]{} H_{\mathscr{D}}(M)^{-n} \tag{4.77}$$

where p_* is the integration along the fiber map defined in equation 4.21. We then set $\mathrm{Ch}^n_{\mathscr{D}} = \mathrm{Ch}^{n-2}_{\mathscr{D}}$ for $n \geqslant 1$. We show in this section that this homomorphism induces an isomorphism upon tensoring with $\mathbb{C}$.

Theorem 4.78. *For any compact smooth $\mathbb{T}$-manifold M, $\mathrm{Ch}_{\mathscr{D}}$ induces a ring homomorphism*

$$\mathrm{Ch}_{\mathscr{D}} : K^\bullet_{\mathbb{T}}(M) \otimes \mathbb{C} \longrightarrow H_{\mathscr{D}}(M)^\bullet \tag{4.79}$$

which is an isomorphism.

Brylinski constructs an equivariant Chern character in his model and proves that it is an isomorphism [19] so this is not a new theorem. There is a quasi-isomorphism from our complex to his so the two versions of delocalized equivariant cohomology agree and the delocalized Chern characters agree under the induced isomorphism. However, rather than appeal to this isomorphism, we give a direct proof which uses the theorem of Freed-Hopkins-Teleman (equation 3.85, [26] theorem 3.9) mentioned at the beginning of the section.

Proof. We make a Mayer-Vietoris argument. Let U be an invariant neighborhood of $M^{\mathbb{T}}$ which equivariantly deformation retracts onto it and let $V = M - M^{\mathbb{T}}$. Then $\{U, V\}$ is an invariant open cover of M and it follows from Proposition 2.11 that

$$K^\bullet_{\mathbb{T}}(U) \otimes \mathbb{C} \cong K^\bullet(U) \otimes \mathbb{C}[t, t^{-1}]. \tag{4.80}$$

Since

$$\mathcal{A}_{\mathbb{T}}(M^{\mathbb{T}})^{\bullet} = \Omega(M^{\mathbb{T}}; \mathcal{R})^{\bullet} \otimes \mathbb{C}[t, t^{-1}] \tag{4.81}$$

the homotopy invariance of $H^{\bullet}_{\mathscr{D}}$ implies that

$$H_{\mathscr{D}}(U)^{\bullet} \cong H(U; \mathcal{R})^{\bullet} \otimes \mathbb{C}[t, t^{-1}]. \tag{4.82}$$

It follows that

$$\begin{array}{ccc} K^{\bullet}_{\mathbb{T}}(U) \otimes \mathbb{C} & \xrightarrow{\mathrm{Ch}_{\mathscr{D}}} & H_{\mathscr{D}}(U)^{\bullet} \\ \downarrow \cong & & \downarrow \cong \\ K^{\bullet}(U) \otimes \mathbb{C}[t, t^{-1}] & \xrightarrow{\mathrm{Ch} \otimes Id} & H(U; \mathcal{R})^{\bullet} \otimes \mathbb{C}[t, t^{-1}] \end{array} \tag{4.83}$$

commutes. Since the classical Chern character is a complex isomorphism, it follows that

$$\mathrm{Ch}_{\mathscr{D}} : K^{\bullet}_{\mathbb{T}}(U) \otimes \mathbb{C} \longrightarrow H_{\mathscr{D}}(U)^{\bullet} \tag{4.84}$$

is an isomorphism.

Remark 4.85. Note that V equivariantly deformation retracts onto the complement of an open tubular neighborhood of $M^{\mathbb{T}}$, a closed subset of M which is thus compact. We replace V with this homotopy equivalent compact set in the rest of the proof.

Observe that since $V^{\mathbb{T}} = \varnothing$ so the complex of delocalized equivariant differential forms is just the direct sum of the Cartan complexes indexed by $S_{\mathbb{T}}(V)$,

$$(\mathcal{A}^{\bullet}_{\mathbb{T}}(V), \delta) = \left(\bigoplus_{\tau \in S_{\mathbb{T}}(V)} C_{\mathbb{T}}(V^{\tau})^{\bullet}, d_{\mathbb{T}} \right) \tag{4.86}$$

hence

$$H_{\mathscr{D}}(V)^\bullet = \bigoplus_{\tau \in S_{\mathbb{T}}(V)} H_{\mathbb{T}}(V^\tau; \mathcal{R})^\bullet. \tag{4.87}$$

If $E \to V$ is an equivariant vector bundle and ∇ is any invariant connection on E, we saw in the previous section that for each $\tau \in S_{\mathbb{T}}(V)$, we can decompose E over V^τ into its isotypical subbundles. Thus, writing ∇^τ for the restriction of ∇ to $E|_{V^\tau}$ there exists an equivariant isomorphism of bundles with connection

$$(E|_{V^\tau}, \nabla^\tau) \xrightarrow{\cong} \bigoplus_{i=1}^{N_\tau} (E_i^\tau, \nabla_i^\tau) \tag{4.88}$$

and using this isomorphism, we may write the delocalized Chern character as the map

$$\begin{aligned} K^0_{\mathbb{T}}(V) \otimes \mathbb{C} &\xrightarrow{\mathrm{Ch}_{\mathscr{D}}} \bigoplus_\tau H_{\mathbb{T}}(V^\tau; \mathcal{R})^0 \\ [E] &\longmapsto \bigoplus_\tau \sum_{i=1}^{N_\tau} \tau^{k_i} [\mathrm{Ch}_{\mathbb{T}}(\nabla_i^\tau)] \end{aligned} \tag{4.89}$$

The Freed-Hopkins-Teleman theorem ([26] theorem 3.9) describes the completion of twisted G-equivariant K-theory in terms of G-equivariant cohomology. The untwisted version of this theorem for $G = \mathbb{T}$ states that for X any finite $\mathbb{T}$-CW complex there are natural isomorphisms

$$K^\bullet_{\mathbb{T}}(X) \otimes \mathbb{C}_q^\wedge \xrightarrow{\cong} K^\bullet_{\mathbb{T}}(X^\tau) \otimes \mathbb{C}_q^\wedge \xrightarrow{\cong} H_{\mathbb{T}}(X^\tau; \mathcal{R})^\bullet. \tag{4.90}$$

Here $q \in \mathbb{C}^\times$ and $\tau \in \mathbb{T} \subset \mathbb{C}^\times$ is a generator of the unitary part of the algebraic subgroup generated by q, that is, the intersection of that algebraic subgroup with $\mathbb{T}$. The first map is induced by inclusion. By [33], every compact smooth $\mathbb{T}$-manifold can be given the structure of a $\mathbb{T}$-CW complex so we may apply the theorem to V. Since the $\mathbb{T}$-action on V has finitely many (finite) stabilizers,

$K^\bullet_{\mathbb{T}}(V) \otimes \mathbb{C}^\wedge_q = 0$ unless $q = \tau \in S_{\mathbb{T}}(V)$. Thus $K^\bullet_{\mathbb{T}}(V) \otimes \mathbb{C}$ is the global sections of a skyscraper sheaf supported on $S_{\mathbb{T}}(V) \subset \mathbb{T}$. It follows that this sheaf is the direct sum of its stalks

$$K^\bullet_{\mathbb{T}}(V) \otimes \mathbb{C} \cong \bigoplus_{\tau \in S_{\mathbb{T}}(V)} K^\bullet_{\mathbb{T}}(V) \otimes \mathbb{C}^\wedge_\tau. \tag{4.91}$$

By 4.90, there is thus a natural isomorphism

$$\psi_{\mathrm{FHT}} : K^\bullet_{\mathbb{T}}(V) \otimes \mathbb{C} \xrightarrow{\cong} \bigoplus_\tau H_{\mathbb{T}}(V^\tau; \mathcal{R})^\bullet. \tag{4.92}$$

We must now show that our homomorphism (4.89) is the isomorphism (4.92). The τ-component of the isomorphism (4.92) can be expressed as follows. Let $E\mathbb{T} \to B\mathbb{T}$ be the universal principal $\mathbb{T}$-bundle. First, ψ_{FHT} decomposes into τ-eigenbundles. Writing E^τ for $E|_{V^\tau}$ it is

$$\begin{array}{ccccc} K^\bullet_{\mathbb{T}}(V) \otimes \mathbb{C} & \longrightarrow & K^\bullet_{\mathbb{T}}(V^\tau) \otimes \mathbb{C} & \longrightarrow & K^\bullet_{\mathbb{T}}(V^\tau) \otimes \mathbb{C} \\ {[E]} & \longmapsto & [E^\tau] & \longmapsto & \sum_{i=1}^{N_\tau} \tau^{k_i} [E^\tau_i] \end{array} \tag{4.93}$$

It then completes

$$K^\bullet_{\mathbb{T}}(V^\tau) \otimes \mathbb{C} \longrightarrow K^\bullet_{\mathbb{T}}(V^\tau) \otimes \mathbb{C}^\wedge_\tau = K^\bullet(V^\tau \times_{\mathbb{T}} E\mathbb{T}) \otimes \mathbb{C} \tag{4.94}$$

and identifies the completion with the ordinary K-theory of the Borel quotient $V^\tau \times_{\mathbb{T}} E\mathbb{T}$ by the Atiyah-Segal Competion theorem [9]. Finally, it maps this to equivariant cohomology by the ordinary Chern character

$$K^\bullet(V^\tau \times_{\mathbb{T}} E\mathbb{T}) \otimes \mathbb{C} \xrightarrow{\mathrm{Ch} \otimes Id} H(V^\tau \times_{\mathbb{T}} E\mathbb{T}; \mathcal{R})^\bullet =: H^{\mathrm{top}}_{\mathbb{T}}(V^\tau; \mathcal{R})^\bullet. \tag{4.95}$$

This last group is the topological definition of $\mathbb{T}$-equivariant cohomology. Bott and Tu [17] prove the compatibility of equivariant characteristic classes constructed using the topological definition of equivariant cohomology and those constructed as we have done in the Cartan model. To elaborate briefly, there are two ways to obtain equivariant cohomology classes on V^τ from an equivariant vector bundle. One is to use equivariant geometric objects associated to the equivariant bundle on V^τ to obtain classes in the cohomology of the Cartan model as we have done. The other is to construct a (non-equivariant) vector bundle on $V^\tau \times_{\mathbb{T}} E\mathbb{T}$, then construct its characteristic classes in singular cohomology to obtain cohomology classes on $V^\tau \times_{\mathbb{T}} E\mathbb{T}$. There is an isomorphism between the cohomology of the Cartan model and the topological definition of $\mathbb{T}$-equivariant cohomology called the equivariant de Rham isomorphism. See [17] and [30]. Bott and Tu show that the two constructions described commute with this isomorphism. It follows that the following diagram in which the vertical map is the equivariant de Rham isomorphism

$$\begin{array}{ccccc} & & & & H_{\mathbb{T}}(V^\tau;\mathcal{R})^\bullet \\ & \nearrow^{\mathrm{Ch}_{\mathbb{T}}\otimes Id} & & & \downarrow \cong \\ K^\bullet_{\mathbb{T}}(V^\tau)\otimes\mathbb{C} & \to & K^\bullet_{\mathbb{T}}(V^\tau)\otimes\mathbb{C}^\wedge_\tau & \xrightarrow[\mathrm{Ch}\otimes Id]{} & H^{\mathrm{top}}_{\mathbb{T}}(V^\tau;\mathcal{R})^\bullet \end{array} \tag{4.96}$$

commutes. Precomposing with the map which restricts from V to V^τ then decomposes into τ-eigenbundles, we obtain

$$\begin{array}{ccccccc} & & & & & & H_{\mathbb{T}}(V^\tau;\mathcal{R})^\bullet \\ & & & \nearrow^{\mathrm{Ch}_{\mathbb{T}}\otimes Id} & & & \downarrow \cong \\ K^\bullet(V)\otimes\mathbb{C} & \to & K^\bullet_{\mathbb{T}}(V^\tau)\otimes\mathbb{C} & \to & K^\bullet_{\mathbb{T}}(V^\tau)\otimes\mathbb{C}^\wedge_\tau & \xrightarrow[\mathrm{Ch}\otimes Id]{} & H^{\mathrm{top}}_{\mathbb{T}}(V^\tau;\mathcal{R})^\bullet \\ [E] & \longmapsto & \sum_{i=1}^{N_\tau} \tau^{k_i}[E_i^\tau] & & & & \end{array} \tag{4.97}$$

The bottom row is the isomorphism ψ_{FHT} and from equation (4.89) it is clear that the diagonal is the delocalized Chern character $\mathrm{Ch}_{\mathscr{D}}$. Therefore

$$\mathrm{Ch}_{\mathscr{D}} : K_{\mathbb{T}}(V)^{\bullet} \otimes \mathbb{C} \longrightarrow H_{\mathscr{D}}(V; \mathcal{R})^{\bullet}$$

is an isomorphism. Since the action is locally free on $U \cap V$, $\mathrm{Ch}_{\mathscr{D}}$ is an isomorphism on $U \cap V$ as well. It follows by the Mayer-Vietoris sequence and the Five Lemma that

$$\mathrm{Ch}_{\mathscr{D}} : K^{\bullet}_{\mathbb{T}}(M) \otimes \mathbb{C} \longrightarrow H_{\mathscr{D}}(M; \mathcal{R})^{\bullet}$$

is an isomorphism. $\square$

Chapter 5

Differential $\mathbb{T}$-equivariant K-theory

We construct differential equivariant K-theory by generators and relations as before. This time, we take a free abelian group modulo a subgroup generated by certain short exact sequences rather than pairs modulo the diagonal. This is inspired by and completely analogous to the construction of Freed-Lott [27].

5.1 Even differential $\mathbb{T}$-equivariant K-theory

Suppose that

$$0 \longrightarrow E_1 \xrightarrow{\iota} E_2 \longrightarrow E_3 \longrightarrow 0 \tag{5.1}$$

is a short exact sequence of equivariant vector bundles with invariant connections $\{\nabla_i\}_{i=1}^3$, respectively, and let

$$\sigma : E_3 \to E_2 \tag{5.2}$$

be a splitting. Then

$$\iota \oplus \sigma : E_1 \oplus E_3 \to E_2 \tag{5.3}$$

is an isomorphism.

Definition 5.1.1. The triple Chern-Simons form of the three connections is

$$\mathrm{CS}_{\mathscr{D}}(\nabla_1, \nabla_2, \nabla_3) := \mathrm{CS}_{\mathscr{D}}((\iota \oplus \sigma)^* \nabla_2, \nabla_1 \oplus \nabla_3) \in \mathcal{A}_{\mathbb{T}}(M)^{-1}/\mathrm{im}\ (\delta). \quad (5.4)$$

We see that this is independent of the chosen splitting as follows. Since the space of splittings is affine, if $\sigma' : E_3 \to E_2$ is another splitting, there exists a path γ with $\gamma(0) = \sigma$ and $\gamma(1) = \sigma'$. Then

$$\varphi(s) = (\iota \oplus \gamma(s))^{-1}(\iota \oplus \sigma) : E_1 \oplus E_3 \to E_1 \oplus E_3 \quad (5.5)$$

is a path of automorphisms with $\varphi(0) = Id$. It follows from the additivity of $\mathrm{CS}_{\mathscr{D}}$ (equation 4.71) and lemma 4.72 that the sum

$$\mathrm{CS}_{\mathscr{D}}((\iota \oplus \sigma)^* \nabla_2, \nabla_1 \oplus \nabla_3) + \mathrm{CS}_{\mathscr{D}}(\nabla_1 \oplus \nabla_3, (\iota \oplus \sigma')^* \nabla_2) \quad (5.6)$$

is equal to

$$\mathrm{CS}_{\mathscr{D}}((\iota \oplus \sigma)^* \nabla_2, (\iota \oplus \sigma')^* \nabla_2) \quad (5.7)$$

which is zero. Thus,

$$\mathrm{CS}_{\mathscr{D}}((\iota \oplus \sigma)^* \nabla_2, \nabla_1 \oplus \nabla_3) = \mathrm{CS}_{\mathscr{D}}((\iota \oplus \sigma')^* \nabla_2, \nabla_1 \oplus \nabla_3). \quad (5.8)$$

Definition 5.1.2. The group $\check{K}^0_{\mathbb{T}}(M)$ is the abelian group given by the following generators and relations. A generator is a triple

$$\mathscr{E} = (E, \nabla, \eta) \quad (5.9)$$

where

- $E \to M$ is an equivariant vector bundle,

- ∇ is an invariant connection on E,

- $\eta \in \mathcal{A}_{\mathbb{T}}(M)^{-1}/\mathrm{im}\ (\delta)$.

The relations are $\mathscr{E}_2 = \mathscr{E}_1 + \mathscr{E}_3$ whenever there is a short exact sequence 5.1 of equivariant vector bundles and

$$\eta_2 = \eta_1 + \eta_3 + \mathrm{CS}_{\mathscr{D}}(\nabla_1, \nabla_2, \nabla_3) \in \mathcal{A}_{\mathbb{T}}(M)^{-1}/\mathrm{im}\ (\delta). \tag{5.10}$$

Definition 5.1.3. The group $\check{K}^j_{\mathbb{T}}(M)$ for j even is defined as above with $\eta \in \mathcal{A}_{\mathbb{T}}(M)^{j-1}/\mathrm{im}\ (\delta)$. In this case, equation 5.10 becomes

$$\eta_2 = \eta_1 + \eta_3 + \beta^{-j/2}\,\mathrm{CS}_{\mathscr{D}}(\nabla_1, \nabla_2, \nabla_3) \in \mathcal{A}_{\mathbb{T}}(M)^{j-1}/\mathrm{im}\ (\delta). \tag{5.11}$$

Let $\mathcal{A}_{\mathbb{T}}(M)^{-j}_K \subset \mathcal{A}_{\mathbb{T}}(M)^{-j}_{\mathrm{closed}}$ denote the union of affine spaces of (delocalized equivariantly) closed differential forms whose cohomology class lies in the image of $\mathrm{Ch}^{-j}_{\mathscr{D}} : K^{-j}_{\mathbb{T}}(M) \to H_{\mathscr{D}}(M)^{-j}$. Observe that two generators (E, ∇, η) and (E, ∇, η') in degree 0 are equivalent if and only if there exists an automorphism $\varphi : E \to E$ and

$$\eta' = \eta + \mathrm{CS}_{\mathscr{D}}(\varphi^*\nabla, \nabla). \tag{5.12}$$

In this case, there exists an equivariant bundle $E_\varphi \to M \times S^1$ with invariant connection $\overline{\nabla}$ such that

$$\mathrm{CS}_{\mathscr{D}}(\varphi^*\nabla, \nabla) = \int_{S^1} \mathrm{Ch}_{\mathscr{D}}(\overline{\nabla}) \bmod \mathrm{im}\ (\delta) \tag{5.13}$$

so

$$\eta' - \eta \in \mathcal{A}_{\mathbb{T}}(M)^{-1}_K. \tag{5.14}$$

Define the *characteristic class* (or forgetful) map on generators by

$$\begin{aligned} \check{K}^0_{\mathbb{T}}(M) &\xrightarrow{c} K^0_{\mathbb{T}}(M) \\ [E,\nabla,\eta] &\longmapsto [E] \end{aligned} \tag{5.15}$$

There is also a *curvature* map,

$$\begin{aligned} \check{K}^0_{\mathbb{T}}(M) &\xrightarrow{\omega} \mathcal{A}_{\mathbb{T}}(M)^0_K \\ [E,\nabla,\eta] &\mapsto \mathrm{Ch}_{\mathscr{D}}(\nabla) - \delta\eta \end{aligned} \tag{5.16}$$

It follows from equation 5.12 that ω takes the same value on equivalent generators so is well-defined. By analogy with [40] we define flat $\check{K}_{\mathbb{T}}$-theory in degree -1 to be the kernel of this map

$$\check{K}^{-1}_{\mathbb{T},\mathrm{flat}}(M) := \ker\omega. \tag{5.17}$$

5.2 Odd differential $\mathbb{T}$-equivariant K-theory

Let $E \to M$ be an equivariant vector bundle with invariant connection ∇ and let $\gamma : E \to E$ be an equivariant automorphism of E. Let ∇_s be a smooth path of connections with $\nabla_0 = \nabla$ and $\nabla_1 = \gamma^*\nabla = \gamma^{-1}\nabla\gamma$. Let

$$\overline{\nabla}^{\gamma} = \nabla_s + ds\partial_s. \tag{5.18}$$

Now let γ_1 and γ_2 be two automorphisms of E. Le ∇_{s_1} be a smooth path from ∇ to $\gamma_1^*\nabla$ and let ∇_{s_2} be a smooth path from ∇ to $(\gamma_1\gamma_2)^*\nabla$ and let

$$\overline{\nabla}^{\gamma_1,\gamma_2} = s_1\nabla_{s_1} + s_2\nabla_{s_2} + ds_1\partial_{s_1} + ds_2\partial_{s_2} \tag{5.19}$$

For

$$\Delta = \{(s_1,s_2) \in \mathcal{A}^2 \mid s_1 \geqslant 0, s_2 \geqslant 0, \text{ and } s_1 + s_2 \leqslant 1\} \tag{5.20}$$

the standard 2-simplex and $\pi : M \times \Delta \to M$ projection, $\overline{\nabla}^{\gamma_1,\gamma_2}$ is a connection on $\pi^* E \to M \times \Delta$. Let

$$\mathrm{CS}_{\mathscr{D}}(\nabla, \gamma_1, \gamma_2) := \int_{\Delta} \mathrm{Ch}_{\mathscr{D}}(\overline{\nabla}^{\gamma_1,\gamma_2}) \in \mathcal{A}_{\mathbb{T}}(M)^{-2}/\mathrm{im}\ (\delta). \tag{5.21}$$

Let $\{E_i\}_{i=1}^3$ be equivariant vector bundles with invariant connections $\{\nabla_i\}_{i=1}^3$ and equivariant automorphisms $\{\gamma_i\}_{i=1}^3$ which fit into a diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E_1 & \xrightarrow{\iota} & E_2 & \longrightarrow & E_3 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \gamma_1} & & \downarrow{\scriptstyle \gamma_2} & & \downarrow{\scriptstyle \gamma_3} & & \\ 0 & \longrightarrow & E_1 & \xrightarrow{\iota} & E_2 & \longrightarrow & E_3 & \longrightarrow & 0 \end{array} \tag{5.22}$$

in which the rows are exact and the squares commute. The triple Chern-Simons form is defined in degree -1 as follows. Let $\pi : M \times I \to M$ be projection and let $\sigma : E_3 \to E_2$ be a splitting which makes 5.22 commute. Then $\nabla_1 \oplus \nabla_3$ and $(\iota \oplus \sigma)^* \nabla_2$ are connections on $E_1 \oplus E_3$. Let $W = \pi^*(E_1 \oplus E_3)$ and let $\Gamma = \pi^*(\gamma_1 \oplus \gamma_3)$ be the induced automorphism of W. Choose a smooth path of connections from $\pi^*(\nabla_1 \oplus \nabla_3)$ to $\pi^*(\iota \oplus \sigma)^* \nabla_2$. This determines a connection ∇_W on W.

Definition 5.2.1. The triple Chern-Simons form in degree -1 is the form

$$\mathrm{CS}_{\mathscr{D}}(\{\nabla_i\}_{i=1}^3, \{\gamma_i\}_{i=1}^3) := \int_{[0,1]} \mathrm{CS}_{\mathscr{D}}(\Gamma^* \nabla_W, \nabla_W) \in \mathcal{A}_{\mathbb{T}}(M)^{-2}/\mathrm{im}\ (\delta). \tag{5.23}$$

Definition 5.2.2. The group $\check{K}_{\mathbb{T}}^{-1}(M)$ is the abelian group given by the following generators and relations. A generator is a quadruple

$$\mathscr{E} = (E, \nabla, \gamma, \eta) \tag{5.24}$$

where

- $E \to M$ is an equivariant vector bundle,
- ∇ is an invariant connection on E,
- $\gamma : E \to E$ is an equivariant automorphism
- $\eta \in \mathcal{A}_{\mathbb{T}}(M)^{-2}/\text{im }(\delta)$.

The relations are

1. $\mathscr{E}_2 = \mathscr{E}_1 + \mathscr{E}_3$ whenever there is a commutative diagram of equivariant vector bundles 5.22 and
$$\eta_2 = \eta_1 + \eta_3 + \text{CS}_{\mathscr{D}}(\{\nabla_i\}_{i=1}^3, \{\gamma_i\}_{i=1}^3),$$
2. $(E, \nabla, \gamma_1, 0) + (E, \nabla, \gamma_2, 0) = (E, \nabla, \gamma_1\gamma_2, \text{CS}_{\mathscr{D}}(\nabla, \gamma_1, \gamma_2))$.

Definition 5.2.3. The group $\check{K}_{\mathbb{T}}^j(M)$ for j odd is defined as above with $\eta \in \mathcal{A}_{\mathbb{T}}(M)^{j-1}/\text{im }(\delta)$ and the relations suitably shifted by a power of β.

Let $E \to M$ is a bundle with two commuting (equivariant) automorphisms γ and φ. Let $\overline{\nabla}^{\gamma}$ be the connection constructed as in equation (5.18) on $E_\gamma \to M \times S^1$. Let $p : M \times S^1 \to M$ be projection, then since $\gamma\varphi = \varphi\gamma$, $\Phi := p^*\varphi$ determines an automorphism of E_γ. Choose a path from $\overline{\nabla}^{\gamma}$ to $\Phi^*\overline{\nabla}^{\gamma}$ and let $\tilde{\nabla}$ be the corresponding connection on $(E_\gamma)_\Phi \to M \times S^1 \times S^1$.

Observe now that the two generators $(E, \nabla, \gamma, \eta)$ and $(E, \nabla, \gamma, \eta')$ are equivalent if and only if there is an automorhpism $\varphi : E \to E$ which makes the diagram

$$\begin{array}{ccc} E & \xrightarrow{\varphi} & E \\ \downarrow \gamma & & \downarrow \gamma \\ E & \xrightarrow[\varphi]{} & E \end{array} \tag{5.25}$$

commute and

$$\eta' - \eta = \mathrm{CS}_{\mathscr{D}}(\Phi^* \overline{\nabla}^{\gamma}, \overline{\nabla}^{\gamma}) = \int_{S^1} \int_{S^1} \mathrm{Ch}_{\mathscr{D}}(\tilde{\nabla}) \tag{5.26}$$

so

$$\eta' - \eta \in \mathcal{A}_{\mathbb{T}}(M)^{-2}_{K}. \tag{5.27}$$

The characteristic class and curvature maps are defined in degree -1 on generators by

$$\begin{aligned} \check{K}^{-1}_{\mathbb{T}}(M) &\xrightarrow{c} K^{-1}_{\mathbb{T}}(M) \\ [E, \nabla, \gamma, \eta] &\longmapsto [E, \gamma] \end{aligned} \tag{5.28}$$

and

$$\begin{aligned} \check{K}^{-1}_{\mathbb{T}}(M) &\xrightarrow{\omega} \mathcal{A}_{\mathbb{T}}(M)^{-1} \\ [E, \nabla, \gamma, \eta] &\rightarrow \mathrm{Ch}^{-1}_{\mathscr{D}}(\overline{\nabla}^{\gamma}) - \delta\eta \end{aligned} \tag{5.29}$$

For non-equivariant differential K-theory $\check{K}^{\bullet}$, we adopt the model of [27] excluding the hermitian metrics. Thus, a generator of $\check{K}^0(M)$ is a triple (E, ∇, η) where $E \to M$ is a vector bundle with connection ∇ and $\eta \in \Omega(M; \mathcal{R})^{-1}$. A generator of $\check{K}^{-1}(M)$ is a quadruple $(E, \nabla, \gamma, \eta)$ where $\gamma : E \to E$ is an automorphism. The relations are the basis for our relations so are completely analogous. There are analogous non-equivariant characteristic class and curvature maps and $\check{K}^{\bullet}_{\mathrm{flat}}$ is the kernel of the non-equivariant curvature map.

Lott indicates in [40] that $\check{K}^\bullet_{\mathrm{flat}}(M)$ is isomorphic to $K^\bullet(M;\mathbb{C}/\mathbb{Z})$. The proof is apparent from the description in [7] section 5 and [35] section 7.21. We present the details in the next section and use the same method to construct an isomorphism

$$\check{K}^\bullet_{\mathbb{T},\mathrm{flat}}(M) \longrightarrow \check{K}^\bullet(M;\mathbb{C}/\mathbb{Z}). \tag{5.30}$$

See appendix E for a description of the models we adopt for K-theory and equivariant K-theory with $\mathbb{C}/\mathbb{Z}$ coefficients.

5.3 A map $\check{K}^\bullet_{\mathrm{flat}}(M) \to K^\bullet(M;\mathbb{C}/\mathbb{Z})$

Let M be a smooth manifold. There is a curvature map given in degree 0 by

$$\begin{aligned} \check{K}^0(M) &\xrightarrow{\omega} \Omega(M;\mathcal{R})^0_K \\ [E,\nabla,\eta] &\longmapsto \mathrm{Ch}(\nabla) - d\eta \end{aligned} \tag{5.31}$$

and in degree -1 by

$$\begin{aligned} \check{K}^{-1}(M) &\xrightarrow{\omega} \Omega(M;\mathcal{R})^{-1}_K \\ [E,\nabla,\gamma,\eta] &\longmapsto \mathrm{Ch}^{-1}(\nabla) - d\eta \end{aligned} \tag{5.32}$$

of which $\check{K}^{j-1}_{\mathrm{flat}}(M)$ is the kernel in degree j. We will construct a map

$$F : \check{K}^\bullet_{\mathrm{flat}}(M) \longrightarrow K^\bullet(M;\mathbb{C}/\mathbb{Z}) \tag{5.33}$$

as follows. We describe the map in degree -1 first. There is a map

$$K^{-1}(M;\mathbb{Q}) \xrightarrow{(\rho,-\iota)} K^{-1}(M;\mathbb{Q}/\mathbb{Z}) \oplus H(M;\mathcal{R})^{-1} \tag{5.34}$$

described in appendix E and

$$K^{-1}(M;\mathbb{C}/\mathbb{Z}) = \mathrm{coker}(\rho,-\iota). \tag{5.35}$$

We will construct a map

$$\begin{array}{ccc} \check{K}^{-1}_{\text{flat}}(M) & \xrightarrow{(\Gamma,\Upsilon)} & K^{-1}(M;\mathbb{Q}/\mathbb{Z}) \oplus H(M;\mathcal{R})^{-1} \\ [\mathcal{E}] - [\mathcal{E}'] & \longmapsto & (a,b) \end{array} \tag{5.36}$$

that depends on choices, then show that making different choices changes (a,b) to $(a+\rho(c), b-\iota(c))$ for some $c \in K^{-1}(M;\mathbb{Q})$ so that (a,b) is unique up to an element of the image of $(\rho, -\iota)$.

We first construct Γ. Let $\mathcal{E} = (E, \nabla, \eta)$ and $\mathcal{E}' = (E', \nabla', \eta')$ represent an element $[\mathcal{E}] - [\mathcal{E}'] \in \check{K}^{-1}_{\text{flat}}(M)$. Then

$$\operatorname{Ch}(\nabla) - d\eta = \operatorname{Ch}(\nabla') - d\eta'. \tag{5.37}$$

It follows that $\operatorname{rank} E = \operatorname{rank} E'$. The Chern character $\operatorname{Ch} : K^0(M;\mathbb{Z}) \to H(M;\mathcal{R})^0$ has kernel the torsion subgroup of $K^0(M;\mathbb{Z})$ and $\operatorname{Ch}([E]-[E'])$ is represented by $\operatorname{Ch}(\nabla) - \operatorname{Ch}(\nabla') = d(\eta - \eta')$ so $[E]-[E']$ is a torsion element of $K^0(M;\mathbb{Z})$. Thus, for some n,

$$[nE] - [nE'] = 0 \tag{5.38}$$

so for some $k = nm$, there exists an isomorphism

$$n(E \oplus \underline{\mathbb{C}}^m) \xrightarrow{\cong} n(E' \oplus \underline{\mathbb{C}}^m) \tag{5.39}$$

Adjusting our representatives $\mathcal{E}$ and $\mathcal{E}'$ by adding $(\underline{\mathbb{C}}^m, d, 0)$ to both, we may assume that E and E' are bundles such that there exist an isomorphism $nE \to nE'$. Let φ be such an isomorphism. This defines an element

$$[E, E', \varphi] \in K^{-1}(M;\mathbb{Z}/n\mathbb{Z}) \tag{5.40}$$

We set

$$a = \Gamma([\mathcal{E}] - [\mathcal{E}']) \tag{5.41}$$

to be the image of $[E, E', \varphi]$ in the colimit $K^{-1}(M; \mathbb{Q}/\mathbb{Z})$.

We now construct the other component Υ of our map. There is one obvious connection $\nabla^{\oplus n}$ on nE. The isomorphism $\varphi : nE \to nE'$ gives another: $\varphi^*(\nabla'^{\oplus n})$. Let

$$\zeta = \eta - \eta' + \frac{\mathrm{CS}(\nabla^{\oplus n}, \varphi^* \nabla'^{\oplus n})}{n} \in \Omega(M; \mathcal{R})^{-1} \tag{5.42}$$

Then

$$\begin{aligned} d\zeta &= d\eta - d\eta' + \frac{\mathrm{Ch}(\nabla^{\oplus n}) - \mathrm{Ch}(\varphi^* \nabla'^{\oplus n})}{n} \\ &= d\eta - d\eta' + \frac{n\,\mathrm{Ch}(\nabla) - n\,\mathrm{Ch}(\nabla')}{n} \\ &= 0 \end{aligned} \tag{5.43}$$

so ζ defines a cohomology class in $H(M; \mathcal{R})^{-1}$; we set

$$b := \Upsilon([\mathcal{E}] - [\mathcal{E}']) = [\zeta]. \tag{5.44}$$

Suppose now that we repeat the above construction, but this time using a different isomorphism $\psi : nE \to nE'$. Let us add subscripts to our constructions to distinguish those constructed with φ from those constructed with ψ. Thus, we now denote a, ζ and b by $a_\varphi, \zeta_\varphi, b_\varphi$. Then $\psi = \varphi\gamma$ where $\gamma = \varphi^{-1}\psi$ is an automorphism of nE. Let us look at the Υ component first. Observe that $\psi^*(\nabla'^{\oplus n}) = \gamma^*\varphi^*(\nabla'^{\oplus n})$ so up to exact forms

$$\mathrm{CS}(\nabla^{\oplus n}, \psi^*(\nabla'^{\oplus n})) = \mathrm{CS}(\nabla^{\oplus n}, \varphi^*(\nabla'^{\oplus n})) + \mathrm{CS}(\varphi^*(\nabla'^{\oplus n}), \gamma^*\varphi^*(\nabla'^{\oplus n})). \tag{5.45}$$

We thus see that

$$\zeta_\psi - \zeta_\varphi = \frac{\mathrm{CS}(\varphi^*\nabla'^{\oplus n}, \gamma^*(\varphi^*\nabla'^{\oplus n}))}{n} \in \Omega(M;\mathcal{R})^{-1}. \tag{5.46}$$

We must show that this term represents an element in the image of $j = \mathrm{Ch}^{-1}$: $K^{-1}(M;\mathbb{Q}) \to H(M;\mathcal{R})^{-1}$. We will construct an element of $K^{-1}(M;\mathbb{Q})$ of which it is the image. Let $V_\gamma = (nE)_\gamma \to M \times S^1$ (using the notation of definition 2.1.1) and let ∇_s be a path of connections constant in a neighborhood of 0 and 1 with $\nabla_0 = \varphi^*\nabla'^{\oplus n}$ and $\nabla_1 = \gamma^*(\varphi^*\nabla'^{\oplus n})$. Let $\overline{\nabla} = \nabla_s + ds\partial_s$. Then $\overline{\nabla}$ defines a connection on V_γ. Let $p : M \times S^1 \to M$ be projection and set

$$c = \frac{[V_\gamma] - [p^*nE]}{n} \in K^{-1}(M;\mathbb{Q}). \tag{5.47}$$

Then $\iota(c)$ is represented by

$$\frac{1}{n}\int_{S^1} \mathrm{Ch}(\overline{\nabla}) = \frac{\mathrm{CS}(\gamma^*(\varphi^*\nabla'^{\oplus n}), \varphi^*\nabla'^{\oplus n})}{n} \in \Omega(M;\mathcal{R})^{-1}. \tag{5.48}$$

Thus

$$\begin{aligned}\zeta_\psi &= \zeta_\varphi + \frac{\mathrm{CS}(\varphi^*\nabla'^{\oplus n}, \gamma^*(\varphi^*\nabla'^{\oplus n}))}{n} \\ &= \zeta_\varphi - \frac{\mathrm{CS}(\gamma^*(\varphi^*\nabla'^{\oplus n}), \varphi^*\nabla'^{\oplus n})}{n}\end{aligned} \tag{5.49}$$

so

$$b_\psi = b_\varphi - j(c). \tag{5.50}$$

Next, if we construct the first component Γ using ψ instead of φ, a_ψ is the image of $[E, E', \psi] \in K^{-1}(M;\mathbb{Z}/n\mathbb{Z})$ in the colimit $K^{-1}(M;\mathbb{Q}/\mathbb{Z})$. We must show that $a_\psi = a_\varphi + \rho(c)$. For $p : M \times S^1 \to M$, the element $c \in K^{-1}(M;\mathbb{Q})$ is the image of

$$\widetilde{c} = [V_\gamma] - [p^* nE] \in A_n = K^{-1}(M;\mathbb{Z}) \tag{5.51}$$

in the colimit which defines K-theory with $\mathbb{Q}$-coefficients so $\rho(c)$ is the image of the reduction mod n of $\widetilde{c}$ in the colimit $K^{-1}(M;\mathbb{Q}/\mathbb{Z})$. We have

$$\begin{array}{ccccc} \widetilde{c} \in A_n = K^{-1}(M;\mathbb{Z}) & \longrightarrow & K^{-1}(M;\mathbb{Z}/n\mathbb{Z}) \ni & \widetilde{c} \bmod n \\ \downarrow & & \downarrow & \downarrow \\ c \in K^{-1}(M;\mathbb{Q}) & \overset{\rho}{\dashrightarrow} & K^{-1}(M;\mathbb{Q}/\mathbb{Z}) \quad \ni & \rho(c) \end{array} \tag{5.52}$$

and we show in appendix E (equation E.11) that

$$[V, V', \widetilde{\psi}] - [V, V', \widetilde{\varphi}] = \widetilde{c} \bmod n. \tag{5.53}$$

It follows that

$$a_\psi - a_\varphi = \rho(c) \tag{5.54}$$

as desired.

In degree 0, the construction is very similar. If $[\mathcal{E}] - [\mathcal{E}']$ is in the kernel of the curvature map in degree 0, let $\mathcal{E} = (E, \nabla, \gamma, \eta)$ and $\mathcal{E}' = (E', \nabla', \gamma', \eta')$. Then (E, ∇, γ) defines a bundle $E_\gamma \to M \times S^1$ with connection $\overline{\nabla}^\gamma$ and similarly for (E', ∇', γ') and the element $[E_\gamma] - [E'_{\gamma'}] \in K^0(M \times S^1)$ is torsion. Choosing an isomorphism $\varphi : nE_\gamma \to nE'_{\gamma'}$, we obtain an element of $K^0(M;\mathbb{Z}/n\mathbb{Z})$. This determines the first component of F in degree 0. For the second, set

$$\zeta_\varphi = \eta - \eta' + \frac{\beta^{-1}}{n} \int_{S^1} \mathrm{CS}(\varphi^* \overline{\nabla}^{\gamma \oplus n}, \overline{\nabla}^{\gamma \oplus n}) \in \Omega(M;\mathcal{R})^0 \tag{5.55}$$

Showing that the image of F is unique up to the image of $(\rho, -\iota)$ is completely analogous to the degree -1 case.

Finally, suppose that we carry out the above construction this time for $m \neq n$. Let us choose an isomorphism $\tilde{\varphi} : mE \to mE'$ and let $(a_{\tilde{\varphi}}, b_{\tilde{\varphi}})$ be the resulting element of $K^{-1}(M; \mathbb{Q}/\mathbb{Z}) \oplus H(M; \mathcal{R})^{-1}$. Then we get an isomorphism $n\tilde{\varphi} : nmE \to nmE'$ and a corresponding element $(a_{n\tilde{\varphi}}, b_{n\tilde{\varphi}})$. Then $[E, E', n\tilde{\varphi}] \in K^{-1}(M; \mathbb{Z}/nm\mathbb{Z})$ and $[E, E', n\tilde{\varphi}] = n[E, E', \tilde{\varphi}]$ so the images of $[E, E', n\tilde{\varphi}]$ and $n[E, E', \tilde{\varphi}]$ in the colimit are equal,

$$a_{n\tilde{\varphi}} = a_{\tilde{\varphi}} \in K^{-1}(M; \mathbb{Q}/\mathbb{Z}). \tag{5.56}$$

Similarly,

$$\begin{aligned} \zeta_{n\tilde{\varphi}} &= \eta - \eta' + \frac{\mathrm{CS}(n\tilde{\varphi}^*(\nabla'^{\oplus nm}), \nabla^{\oplus nm})}{nm} \\ &= \eta - \eta' + \frac{n\,\mathrm{CS}(\tilde{\varphi}^*(\nabla'^{\oplus m}), \nabla^{\oplus m})}{nm} \bmod \mathrm{im}\ (d) \\ &= \zeta_{\tilde{\varphi}} \bmod \mathrm{im}\ (d) \end{aligned} \tag{5.57}$$

so

$$b_{n\tilde{\varphi}} = b_{\tilde{\varphi}} \in H(M; \mathcal{R})^{-1} \tag{5.58}$$

It follows that for $\varphi : nE \to nE'$ the original isomorphism of our construction,

$$a_{m\varphi} = a_{\varphi} \quad \text{and} \quad b_{m\varphi} = b_{\varphi}. \tag{5.59}$$

Thus $(a_{\varphi}, b_{\varphi})$ corresponds to the isomorphism $m\varphi : nmE \to nmE'$ and $(a_{\tilde{\varphi}}, b_{\tilde{\varphi}})$ corresponds to the isomorphism $n\tilde{\varphi} : nmE \to nmE'$. It follows that the difference

$$(a_{m\varphi}, b_{m\varphi}) - (a_{n\tilde{\varphi}}, b_{n\tilde{\varphi}}) \tag{5.60}$$

is in the image of $(\rho, -\iota)$ so the constructions using n and m determine the same element of the cokernel $K^{-1}(M;\mathbb{C}/\mathbb{Z})$.

5.4 The map $\check{K}^{\bullet}_{\text{flat}}(M) \to K^{\bullet}(M;\mathbb{C}/\mathbb{Z})$ is an isomorphism

We will show that the map constructed in the previous section is an isomorphism by showing that both $\check{K}^{\bullet}_{\text{flat}}(M)$ and $K^{\bullet}(M;\mathbb{C}/\mathbb{Z})$ fit into long exact sequences, map one long exact sequence to the other, and use the Five Lemma. $K^{\bullet}(M;\mathbb{C}/\mathbb{Z})$ fits into the long exact coefficient sequence

$$\cdots \longrightarrow K^{-1}(M;\mathbb{C}) \to K^{-1}(M;\mathbb{C}/\mathbb{Z}) \to K^{0}(M;\mathbb{Z}) \longrightarrow \cdots \tag{5.61}$$

The sequence (E.6) is the long exact sequence corresponding to the coefficient exact sequence $0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0$. One constructs the long exact sequence corresponding to $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$ from this by taking colimits as one constructs $K^{\bullet}(X;\mathbb{Q}/\mathbb{Z})$ from $K^{\bullet}(X;\mathbb{Z}/n\mathbb{Z})$. From this, it is easy to construct the sequence (5.61). The following lemma is due to Karoubi; it appears in [35] section 7.21.

Lemma 5.62. *The flat differential K-theory group $\check{K}^{\bullet}_{\text{flat}}(M)$ fits into the long exact sequence*

$$\cdots \longrightarrow H(M;\mathcal{R})^{-1} \xrightarrow{j} \check{K}^{-1}_{\text{flat}}(M) \xrightarrow{k} K^{0}(M;\mathbb{Z}) \xrightarrow{i} H(M;\mathcal{R})^{0} \xrightarrow{j} \cdots \tag{5.63}$$

where

1. $j : H(M;\mathcal{R})^{-1} \to \check{K}^{-1}_{\text{flat}}(M)$ *is given by, for α a closed -1-form with*

$\alpha = [\eta]$,

$$j(\alpha) = [\underline{\mathbb{C}}^n, d, \eta] - [\underline{\mathbb{C}}^n, d, 0]$$

2. k is the forgetful map,

$$k([E, \nabla, \eta] - [E', \nabla', \eta']) = [E] - [E'],$$

3. i is the map induced by the inclusion $\mathbb{Z} \hookrightarrow \mathbb{C}$ followed by the Chern character isomorphism $\mathrm{Ch} : K^\bullet(M; \mathbb{C}) \to H(M; \mathcal{R})^\bullet$, *and*

4. $j : H(M; \mathcal{R})^0 \to \check{K}^0_{\mathrm{flat}}(M)$ is given by, for α a closed 0*-form with $\alpha = [\eta]$,*

$$j(\alpha) = [\underline{\mathbb{C}}^n, d, Id, \eta] - [\underline{\mathbb{C}}^n, d, Id, 0]$$

Proof. We first verify exactness at $H(M; \mathcal{R})^{-1}$, $\check{K}^{-1}_{\mathrm{flat}}(M)$, and $K^0(M; \mathbb{Z})$.

1. Exactness at $K^0(M; \mathbb{Z})$. The kernel of i is the torsion subgroup of $K^0(M; \mathbb{Z})$ so by definition of $\check{K}^{-1}_{\mathrm{flat}}(M)$, $ik = 0$. If $[E] - [\underline{\mathbb{C}}^r] \in K^0(M; \mathbb{Z})$ is in the kernel of i, then if ∇ is any connection on E and d is the trivial connection on $\underline{\mathbb{C}}^r$, $\mathrm{Ch}(\nabla) - \mathrm{Ch}(d)$ represents the image under i so is an exact form. Thus, there is a -1-form η such that $\mathrm{Ch}(\nabla) - \mathrm{Ch}(d) = \mathrm{Ch}(\nabla) - \mathrm{rank}\, E = d\eta$. Then

$$[E] - [\underline{\mathbb{C}}^r] = k\{[E, \nabla, \eta] - [\underline{\mathbb{C}}^r, d, 0]\}. \tag{5.64}$$

2. Exactness at $\check{K}^{-1}_{\mathrm{flat}}(M)$. It is clear that $kj = 0$. If

$$\mathcal{E} - \mathcal{E}' = [E, \nabla, \eta] - [E', \nabla', \eta'] \in \check{K}^{-1}_{\mathrm{flat}}(M) \tag{5.65}$$

is in the kernel of k, then there exists an isomorphism $\alpha : E \oplus \underline{\mathbb{C}}^k \to E' \oplus \underline{\mathbb{C}}^k$ and

$$\mathcal{E} - \mathcal{E}' = [E \oplus \underline{\mathbb{C}}^k, \nabla \oplus d, \eta] - [E' \oplus \underline{\mathbb{C}}^k, \nabla' \oplus d, \eta']. \tag{5.66}$$

Then

$$[E' \oplus \underline{\mathbb{C}}^k, \nabla' \oplus d, \eta'] = [E \oplus \underline{\mathbb{C}}^k, \nabla \oplus d, \eta''] \tag{5.67}$$

where $\eta'' = \eta' + \mathrm{CS}(\alpha^*(\nabla' \oplus d), \nabla \oplus d)$ so

$$\begin{aligned}\mathcal{E} - \mathcal{E}' &= [E \oplus \underline{\mathbb{C}}^k, \nabla \oplus d, \eta] - [E \oplus \mathbb{C}^k, \nabla \oplus d, \eta''] \\ &= [E, \nabla, \eta] - [E, \nabla, \eta''] \end{aligned} \tag{5.68}$$

Now, let F be a complement to E so there exists an isomorphism $E \oplus F \cong \underline{\mathbb{C}}^r$ and let ∇_F be a connection on F. Then adjusting our representatives $\mathcal{E}, \mathcal{E}'$ by adding $(F, \nabla_F, 0)$ to both we have

$$\mathcal{E} - \mathcal{E}' = [\underline{\mathbb{C}}^r, \nabla \oplus \nabla_F, \eta] - [\underline{\mathbb{C}}^r, \nabla \oplus \nabla_F, \eta'']. \tag{5.69}$$

Changing representatives one last time we have

$$\begin{aligned}\mathcal{E} - \mathcal{E}' &= [\underline{\mathbb{C}}^r, d, \widetilde{\eta}] - [\underline{\mathbb{C}}^r, d, \widetilde{\eta}''] \\ &= [\underline{\mathbb{C}}^r, d, \widetilde{\eta} - \widetilde{\eta}''] - [\underline{\mathbb{C}}^r, d, 0] \\ &= j([\widetilde{\eta} - \widetilde{\eta}'']) \end{aligned} \tag{5.70}$$

where $\widetilde{\eta} = \eta + \mathrm{CS}(\nabla \oplus \nabla_F, d)$ and similarly for $\widetilde{\eta}''$. Therefore the sequence is exact at $\check{K}^{-1}_{\mathrm{flat}}(M)$.

3. Exactness at $H(M;\mathcal{R})^{-1}$. Let γ be an automorphism of the trivial bundle of rank r over $M \times I$ and let $W_\gamma \to M \times S^1$ be the bundle obtained by gluing the ends via γ as in the previous section. Every element of $K^{-1}(M;\mathbb{Z})$ of of the form $[W_\gamma] - [\underline{\mathbb{C}}^r]$. Let ∇_s be a path of connections from d to $\gamma^* d$ and let $\overline{\nabla}$ be the corresponding connection on W_γ. Then

$$i\{[W_\gamma] - [\underline{\mathbb{C}}^r]\} = \int_{S^1} \mathrm{Ch}(\overline{\nabla}) = \mathrm{CS}(\gamma^* d, d) \tag{5.71}$$

so

$$ji\{[W_\gamma] - [\underline{\mathbb{C}}^r]\} = [\underline{\mathbb{C}}^r, d, \mathrm{CS}(\gamma^* d, d)] - [\underline{\mathbb{C}}^r, d, 0]. \tag{5.72}$$

One readily deduces from the relations which define $\check{K}^0(M)$ by considering the exact sequence

$$0 \longrightarrow \underline{\mathbb{C}}^r \xrightarrow{\gamma} \underline{\mathbb{C}}^r \longrightarrow 0 \longrightarrow 0 \tag{5.73}$$

that these two triples are equivalent so their difference is zero. Therefore $ji = 0$.

Next, suppose that for $\alpha = [\eta] \in H(M;\mathcal{R})^{-1}$, $j(\alpha) = 0$. Then

$$(\underline{\mathbb{C}}^r, d, \eta) \sim (\underline{\mathbb{C}}^r, d, 0). \tag{5.74}$$

It follows that there exists a short exact sequence of vector bundles over M

$$0 \longrightarrow \underline{\mathbb{C}}^r \xrightarrow{\gamma} \underline{\mathbb{C}}^r \longrightarrow 0 \longrightarrow 0 \tag{5.75}$$

and

$$\eta = 0 + 0 + \mathrm{CS}(\gamma^* d, d) \tag{5.76}$$

so indeed $\alpha = \mathrm{Ch}^{-1}([W_\gamma] - [\underline{\mathbb{C}}^r]) = i([W_\gamma] - [\underline{\mathbb{C}}^r])$.

To show exactness of the next portion

$$\cdots \longrightarrow H(M;\mathcal{R})^0 \xrightarrow{j} \check{K}^0_{\text{flat}}(M) \xrightarrow{k} K^1(M;\mathbb{Z}) \xrightarrow{i} H(M;\mathcal{R})^1 \xrightarrow{j} \cdots \quad (5.77)$$

of the sequence is completely analogous. Since all of the groups are periodic with period 2, this completes the proof. □

Lemma 5.78. *The following diagram commutes*

$$\begin{array}{ccccccccc}
K^{-1}(M;\mathbb{Z}) & \xrightarrow{i} & H(M;\mathcal{R})^{-1} & \xrightarrow{j} & \check{K}^{-1}_{\text{flat}}(M) & \xrightarrow{k} & K^0(M;\mathbb{Z}) & \xrightarrow{i} & H(M;\mathcal{R})^0 \\
\downarrow{=} & & \downarrow{\cong} & & \downarrow{F} & & \downarrow{=} & & \downarrow{\cong} \\
K^{-1}(M;\mathbb{Z}) & \xrightarrow[\mathbb{Z}\hookrightarrow\mathbb{C}]{} & K^{-1}(M;\mathbb{C}) & \xrightarrow[\pi]{} & K^{-1}(M;\mathbb{C}/\mathbb{Z}) & \xrightarrow[b]{} & K^0(M;\mathbb{Z}) & \xrightarrow[\mathbb{Z}\hookrightarrow\mathbb{C}]{} & K^0(M;\mathbb{C})
\end{array}$$

Proof. The vertical maps marked $\cong$ are the inverses of the Chern character isomorphisms in the given degree. It is clear that the squares on the ends commute. The map π is given by the composition

$$\begin{array}{ccccc}
K^{-1}(M;\mathbb{C}) & \longrightarrow & K^{-1}(M;\mathbb{Q}/\mathbb{Z}) \oplus H(M;\mathcal{R})^{-1} & \longrightarrow & K^{-1}(M;\mathbb{C}/\mathbb{Z}) \\
\mathcal{E} & \longmapsto & (0, \mathrm{Ch}^{-1}(\mathcal{E})) & \longmapsto & [0, \mathrm{Ch}^{-1}(\mathcal{E})]
\end{array} \quad (5.79)$$

where Ch^{-1} is the Chern character in degree -1 (not the inverse) and brackets indicate the image in the quotient group. It follows that for $\alpha = [\eta] \in H(M;\mathcal{R})^{-1}$, going around the square

$$\begin{array}{ccc}
H(M;\mathcal{R})^{-1} & \xrightarrow{j} & \check{K}^{-1}_{\text{flat}}(M) \\
\downarrow & & \downarrow{F} \\
K^{-1}(M;\mathbb{C}) & \xrightarrow[\pi]{} & K^{-1}(M;\mathbb{C}/\mathbb{Z})
\end{array} \quad (5.80)$$

counter-clockwise is given by

$$\alpha \longmapsto [0, \alpha]. \quad (5.81)$$

Going around clockwise is given by

$$\alpha \longmapsto Fj(\alpha) = F([\underline{\mathbb{C}}^r, d, \eta] - [\underline{\mathbb{C}}^r, d, 0]) = [a, b] \tag{5.82}$$

To construct the first component of $F([\underline{\mathbb{C}}^r, d, \eta] - [\underline{\mathbb{C}}^r, d, 0])$, we can choose $n = 1$ and the isomorphism $\underline{\mathbb{C}}^r \to \underline{\mathbb{C}}^r$ to be the identity. Then $a = \Gamma j(\alpha) = 0$. The second component

$$b = \Upsilon j(\alpha) \tag{5.83}$$

is the cohomology class of $\eta + \mathrm{CS}(d, d)$, but $\mathrm{CS}(d, d)$ is an exact form so

$$b = [\eta] = \alpha \tag{5.84}$$

and we have

$$Fj(\alpha) = [0, \alpha] \tag{5.85}$$

so indeed, the square commutes.

Finally, the map marked b is the Bockstein homomorphism. In our model of $K^{-1}(M; \mathbb{C}/\mathbb{Z})$ it is given as follows. $K^{-1}(M; \mathbb{Q}/\mathbb{Z})$ is defined as the colimit of the diagram of groups $A_n = K^{-1}(M; \mathbb{Z}/n\mathbb{Z})$ where, if $m = kn$ there is a unique map $A_n \to A_m$ which is multiplication by k. Returning briefly to the notation $A = B = M \times S^2$ and $g = 1_M \times \Sigma f_n$, $K^{-1}(M; \mathbb{Z}/n\mathbb{Z}) = K^0(B, A, g)$. For each n there is a map, the connecting homomorphism in the long exact sequence,

$$\begin{array}{c} K^0(B, A, g) = K^{-1}(M; \mathbb{Z}/n\mathbb{Z}) \longrightarrow K^0(M; \mathbb{Z}) \cong K^0(X \times S^2, X \times \{pt\}) \\ [V, V', \tilde{\varphi}] \longmapsto [V] - [V']. \end{array} \tag{5.86}$$

These induce a map on the colimit $K^{-1}(M;\mathbb{Q}/\mathbb{Z}) \to K^0(M;\mathbb{Z})$ by the universal property. Precomposing with projection gives a map

$$\widetilde{b} : K^{-1}(M;\mathbb{Q}/\mathbb{Z}) \oplus H(M;\mathcal{R})^{-1} \longrightarrow K^{-1}(M;\mathbb{Q}/\mathbb{Z}) \longrightarrow K^0(M;\mathbb{Z}) \tag{5.87}$$

When we constructed F, we showed that the difference of two elements

$$[V, V', \widetilde{\varphi}], [V, V', \widetilde{\psi}] \in K^{-1}(M;\mathbb{Z}/n\mathbb{Z}) \tag{5.88}$$

is the reduction mod n of an element $\widetilde{c} \in K^{-1}(M;\mathbb{Z})$ which was a lift of $c \in K^{-1}(M;\mathbb{Q})$ and deduced that the difference of the images $\overline{[V, V'\widetilde{\varphi}]}, \overline{[V, V', \widetilde{\psi}]} \in K^{-1}(M;\mathbb{Q}/\mathbb{Z})$ is $\rho(c)$ where $\rho : K^{-1}(M;\mathbb{Q}) \to K^{-1}(M;\mathbb{Q}/\mathbb{Z})$ is the reduction mod $\mathbb{Z}$. It follows that $\widetilde{b}$ is constant on the fibers of the quotient map so descends to a map

$$\begin{array}{ccc} K^{-1}(M;\mathbb{Q}/\mathbb{Z}) \oplus H(M;\mathcal{R})^{-1} & \xrightarrow{\widetilde{b}} & K^0(M;\mathbb{Z}) \\ \downarrow & \nearrow_{b} & \\ K^{-1}(M;\mathbb{C}/\mathbb{Z}) & & \end{array} \tag{5.89}$$

Let $[\mathcal{E}] - [\mathcal{E}'] \in \check{K}^{-1}_{\text{flat}}(M)$ with $\mathcal{E} = (E, \nabla, \eta)$ and $\mathcal{E}' = (E', \nabla', \eta')$ and let $\overline{[E, E', \varphi]} = \Gamma([\mathcal{E}] - [\mathcal{E}'])$ denote the image of $[E, E', \varphi] \in K^{-1}(M;\mathbb{Z}/n\mathbb{Z})$ in the colimit $K^{-1}(M;\mathbb{Q}/\mathbb{Z})$. Unraveling the definitions of F and b we see that

$$bF(\mathcal{E} - \mathcal{E}') = b[\overline{[E, E', \varphi]}, [\zeta_\varphi]] = [E] - [E'] \in K^0(M;\mathbb{Z}) \tag{5.90}$$

is indeed the forgetful map k so the remaining square does commute.

The proof that the portion of the diagram centered on $F : \check{K}^0_{\text{flat}}(M) \to K^0(M;\mathbb{C}/\mathbb{Z})$ commutes is again entirely analogous. This completes the proof.

$\square$

Corollary 5.91. *The map*

$$F : \check{K}^{\bullet}_{\mathrm{flat}}(M) \longrightarrow K^{\bullet}(M;\mathbb{C}/\mathbb{Z}) \tag{5.92}$$

is an isomorphism.

Proof. Both sequences are exact and the two maps on either side of $F : \check{K}^{\bullet}_{\mathrm{flat}}(M) \to K^{\bullet}(M;\mathbb{C}/\mathbb{Z})$ are isomorphisms. The Five Lemma then implies that F is an isomorphism. ◻

5.5 An isomorphism $\check{K}^{\bullet}_{\mathbb{T},\mathrm{flat}}(M) \to K^{\bullet}_{\mathbb{T}}(M;\mathbb{C}/\mathbb{Z})$

For M now a smooth $\mathbb{T}$-manifold, we have the analogous equivariant curvature map ω defined in equations 5.16 and 5.29. We set $\check{K}^{j-1}_{\mathbb{T},\mathrm{flat}}(M)$ to be the kernel of the equivariant curvature map in degree j.

We construct a map

$$F : \check{K}^{\bullet}_{\mathbb{T},\mathrm{flat}}(M) \longrightarrow K^{\bullet}_{\mathbb{T}}(M;\mathbb{C}/\mathbb{Z}) \tag{5.93}$$

as before. The previous construction holds in the equivariant setting because the kernel of the delocalized Chern character $\mathrm{Ch}_{\mathscr{D}} : K^{\bullet}_{\mathbb{T}}(M) \to H_{\mathscr{D}}(M)^{\bullet}$ is again the $\mathbb{Z}$-torsion subgroup. Thus if $[\mathcal{E}]-[\mathcal{E}'] \in \check{K}^{-1}_{\mathbb{T},\mathrm{flat}}(M)$ with $\mathcal{E} = (E,\nabla,\eta)$ and $\mathcal{E}' = (E',\nabla',\eta')$, we deduce as before that for some n, $[nE]-[nE'] = 0$ in $K^{0}_{\mathbb{T}}(M)$ so may once again assume that there exists an isomorphism $\varphi : nE \to nE'$. This is how we construct the first component of F. We construct the second component as before now using delocalized equivariant Chern-Simons forms. This produces an element $(a,b) = (a_\varphi, b_\varphi) \in K^{-1}_{\mathbb{T}}(M;\mathbb{Q}/\mathbb{Z}) \oplus H_{\mathscr{D}}(M)^{-1}$. Repeating the construction with $\psi = \varphi\gamma$, we have already indicated that $a_\psi = a_\varphi + \rho(c)$ where

$$c = \frac{[V_\gamma] - [p^* nE]}{n} \in K^{-1}_{\mathbb{T}}(M;\mathbb{Q}) \tag{5.94}$$

and showing that $b_\psi = b_\varphi - \iota(c)$ is completely analogous to the non-equivariant argument. The construction in degree 0 is also analogous to the non-equivariant one. We thus get a well-defined map

$$F : \check{K}^{\bullet}_{\mathbb{T},\mathrm{flat}}(M) \longrightarrow K^{\bullet}_{\mathbb{T}}(M;\mathbb{C}/\mathbb{Z}) \tag{5.95}$$

The groups $K^{\bullet}_{\mathbb{T}}(M;\mathbb{C}/\mathbb{Z})$ fit into a long exact sequence corresponding to the coefficient short exact sequence $0 \to \mathbb{Z} \to \mathbb{C} \to \mathbb{C}/\mathbb{Z} \to 0$. The groups $\check{K}^{\bullet}_{\mathbb{T},\mathrm{flat}}(M)$ fit into an exact sequence analogous to the non-equivariant flat differential K-groups.

Lemma 5.96. *In the diagram,*

$$\begin{array}{ccccccccc}
K^{-1}_{\mathbb{T}}(M;\mathbb{Z}) & \xrightarrow{i} & H_{\mathscr{D}}(M;\mathcal{R})^{-1} & \xrightarrow{j} & \check{K}^{-1}_{\mathbb{T},\mathrm{flat}}(M) & \xrightarrow{k} & K^{0}_{\mathbb{T}}(M;\mathbb{Z}) & \xrightarrow{i} & H_{\mathscr{D}}(M;\mathcal{R})^{0} \\
{\scriptstyle =}\downarrow & & {\scriptstyle \cong}\downarrow & & {\scriptstyle F}\downarrow & & {\scriptstyle =}\downarrow & & {\scriptstyle \cong}\downarrow \\
K^{-1}_{\mathbb{T}}(M;\mathbb{Z}) & \xrightarrow[\mathbb{Z}\hookrightarrow\mathbb{C}]{} & K^{-1}_{\mathbb{T}}(M;\mathbb{C}) & \xrightarrow[\pi]{} & K^{-1}_{\mathbb{T}}(M;\mathbb{C}/\mathbb{Z}) & \xrightarrow[b]{} & K^{0}_{\mathbb{T}}(M;\mathbb{Z}) & \xrightarrow[\mathbb{Z}\hookrightarrow\mathbb{C}]{} & K^{0}_{\mathbb{T}}(M;\mathbb{C})
\end{array}$$

the rows are exact and the diagram commutes.

Sketch of proof. Proving that the non-equivariant sequence was exact used that every bundle has a complement, that $\mathrm{Ch} : K^{\bullet}(M;\mathbb{C}) \to H(M;\mathcal{R})^{\bullet}$ is an isomorphism, that every element of $K^{-1}(M)$ can be constructed from an automorphism of a bundle over M, and the relations which define $\check{K}^\bullet(M)$. The above facts hold in the equivariant setting and the relations which define $\check{K}^\bullet_{\mathbb{T}}(M)$ are completely analogous so we obtain an analogous exact sequence for $\check{K}^\bullet_{\mathbb{T},\text{flat}}(M)$. The proof that the diagram commutes is identical to the non-equivariant case. ☐

Corollary 5.97. *The map*

$$F : \check{K}^\bullet_{\mathbb{T},\text{flat}}(M) \longrightarrow K^\bullet_{\mathbb{T}}(M; \mathbb{C}/\mathbb{Z}) \tag{5.98}$$

is an isomorphism.

5.6 Exact Sequences for $\check{K}^\bullet_{\mathbb{T}}$

Differential K-theory fits into exact sequences

$$0 \longrightarrow \frac{\Omega(M; \mathcal{R})^{\bullet-1}}{\Omega(M; \mathcal{R})^{\bullet-1}_K} \longrightarrow \check{K}^\bullet(M) \longrightarrow K^\bullet(M) \longrightarrow 0 \tag{5.99}$$

and

$$0 \longrightarrow K^{\bullet-1}(M; \mathbb{C}/\mathbb{Z}) \longrightarrow \check{K}^\bullet(M) \longrightarrow \Omega(M; \mathcal{R})^\bullet_K \longrightarrow 0 \tag{5.100}$$

We show that the equivariant theory fits into analogous exact sequences.

Let

$$P_{\mathbb{T}}(M)^0 = \{(c, \alpha) \in K^0_{\mathbb{T}}(M) \times \mathcal{A}_{\mathbb{T}}(M)^0_K \mid \mathrm{Ch}_{\mathscr{D}}(c) = [\alpha]\}. \tag{5.101}$$

Proposition 5.102. *There are exact sequences*

$$0 \longrightarrow \frac{\mathcal{A}_{\mathbb{T}}(M)^{\bullet-1}}{\mathcal{A}_{\mathbb{T}}(M)^{\bullet-1}_K} \longrightarrow \check{K}^\bullet_{\mathbb{T}}(M) \xrightarrow{c} K^\bullet_{\mathbb{T}}(M) \longrightarrow 0 \tag{5.103}$$

$$0 \longrightarrow K_{\mathbb{T}}^{\bullet-1}(M;\mathbb{C}/\mathbb{Z}) \to \check{K}_{\mathbb{T}}^{\bullet}(M) \xrightarrow{\omega} \mathcal{A}_{\mathbb{T}}(M)_K^{\bullet} \longrightarrow 0 \tag{5.104}$$

and

$$0 \longrightarrow \frac{H_{\mathscr{D}}(M)^{\bullet-1}}{\mathrm{Ch}_{\mathscr{D}}^{\bullet-1} K_{\mathbb{T}}^{\bullet-1}(M)} \xrightarrow{\tilde{h}} \check{K}_{\mathbb{T}}^{\bullet}(M) \xrightarrow{\chi} P_{\mathbb{T}}(M)^{\bullet} \longrightarrow 0. \tag{5.105}$$

Proof. In degree 0: in the sequence 5.103, the homomorphism c is clearly surjective since every equivariant vector bundle has an invariant connection. Define

$$h : \mathcal{A}_{\mathbb{T}}(M)^{-1} \to \check{K}_{\mathbb{T}}^{0}(M) \tag{5.106}$$

by

$$h(\eta) = [\underline{\mathbb{C}}, d, \eta] - [\underline{\mathbb{C}}, d, 0]. \tag{5.107}$$

Here $\underline{\mathbb{C}}$ is the equivariantly trivial bundle of rank and d is the trivial connection. By equation 5.12, the kernel of h is $\mathcal{A}_{\mathbb{T}}(M)_K^{-1}$ so h induces a injective map

$$\overline{h} : \frac{\mathcal{A}_{\mathbb{T}}(M)^{-1}}{\mathcal{A}_{\mathbb{T}}(M)_K^{-1}} \to \check{K}_{\mathbb{T}}^{0}(M) \tag{5.108}$$

It is clear that $c \circ \overline{h} = 0$. We must show that $\ker c = \operatorname{im} \overline{h}$. Suppose that $\mathcal{E} - \mathcal{E}'$ is in $\ker c$ and let (E, ∇, η) represent $\mathcal{E}$ and (E', ∇', η') represent $\mathcal{E}'$. Then $[E] = [E']$ in $K_{\mathbb{T}}^0(M)$ so there exists an isomorphism $\varphi : E \oplus \underline{\mathbb{C}}^n \to E' \oplus \underline{\mathbb{C}}^n$ for some n. Note that here $\underline{\mathbb{C}}^n$ is a topologically trivial, but not necessarily equivariantly trivial, bundle. We have

$$\begin{aligned}\mathcal{E} - \mathcal{E}' &= [E, \nabla, \eta] - [E', \nabla', \eta'] \\ &= [E \oplus \underline{\mathbb{C}}^n, \nabla \oplus d, \eta] - [E' \oplus \underline{\mathbb{C}}^n, \nabla' \oplus d, \eta'] \\ &= [E \oplus \underline{\mathbb{C}}^n, \nabla \oplus d, \eta] - [E \oplus \underline{\mathbb{C}}^n, \tilde{\nabla}, \eta']\end{aligned} \tag{5.109}$$

Let F be an equivariant bundle such that there is an equivariant isomorphism $E \oplus \underline{\mathbb{C}}^n \oplus F \cong \underline{\mathbb{C}}^r$. If ∇_F is any invariant connection on F, then adding $(F, \nabla_F, 0)$ to both representatives we have

$$\begin{aligned} \mathcal{E} - \mathcal{E}' &= [\underline{\mathbb{C}}^r, \nabla_1, \eta] - [\underline{\mathbb{C}}^r, \nabla_2, \eta'] \\ &= [\underline{\mathbb{C}}^r, \nabla_1, \eta] - [\underline{\mathbb{C}}^r, \nabla_1, \tilde{\eta}'] \qquad (5.110) \\ &= [\underline{\mathbb{C}}^r, \nabla_1, \eta - \tilde{\eta}'] - [\underline{\mathbb{C}}^r, \nabla_1, 0] \end{aligned}$$

(5.111)

where $\tilde{\eta}' = \eta' + \mathrm{CS}_{\mathscr{D}}(\nabla_1, \nabla_2)$. Finally, it is clear from the relations which define $\check{K}^0_T$ that

$$[\underline{\mathbb{C}}^r, \nabla_1, \eta - \tilde{\eta}'] - [\underline{\mathbb{C}}^r, \nabla_1, 0] = [\underline{\mathbb{C}}, d, \eta - \tilde{\eta}'] - [\underline{\mathbb{C}}, d, 0] \qquad (5.112)$$

so

$$\mathcal{E} - \mathcal{E}' = \overline{h}(\tilde{\eta} - \tilde{\eta}'). \qquad (5.113)$$

Therefore the sequence is exact.

The sequence 5.104 is exact by corollary 5.97 that $\ker \omega$ is isomorphic to $\mathbb{C}/\mathbb{Z}$-K-theory in degree -1.

The proof that the sequence 5.105 is exact is very similar to the discussion the section 5.3. The map $\widetilde{h}$ is is just the map h on cohomology classes: for $\alpha = [\eta] \in H_{\mathscr{D}}(M)^{-1}$,

$$\widetilde{h}(\alpha) = [\underline{\mathbb{C}}^n, d, \eta] - [\underline{\mathbb{C}}^n, d, 0].$$

By equation 5.12 and the discussion there, if $\widetilde{h}([\eta]) - \widetilde{h}([\eta']) = 0$, $\eta - \eta' \in \mathcal{A}_{\mathbb{T}}(M)^{-1}_{K}$ so $[\eta - \eta'] \in \operatorname{Ch}^{-1}_{\mathscr{D}} K^{-1}_{\mathbb{T}}(M) \subset H_{\mathscr{D}}(M)^{-1}$. Thus,

$$\widetilde{h} : \frac{H_{\mathscr{D}}(M)^{-1}}{\operatorname{Ch}^{-1}_{\mathscr{D}} K^{-1}_{\mathbb{T}}(M)} \longrightarrow \check{K}^{0}_{\mathbb{T}}(M)$$

is injective. The map χ is defined by

$$\chi(\mathcal{E} - \mathcal{E}') = ([E] - [E'], \operatorname{Ch}_{\mathscr{D}}(\nabla) - \delta\eta - \operatorname{Ch}_{\mathscr{D}}(\nabla') + \delta\eta')$$

where $\mathcal{E}$ is represented by (E, ∇, η) and $\mathcal{E}'$ is represented by (E', ∇', η'). It is clear that $\chi \circ \widetilde{h} = 0$. We must show that $\ker \chi = \operatorname{im} \widetilde{h}$. The kernel of χ consists of differences $\mathcal{E} - \mathcal{E}'$ for which

1. $[E] = [E']$ and
2. $\operatorname{Ch}_{\mathscr{D}}(\nabla) - \operatorname{Ch}_{\mathscr{D}}(\nabla') = \delta\eta - \delta\eta'$

The second condition implies that $\operatorname{rank} E = \operatorname{rank} E'$. That $[E] = [E']$ means that there exists an isomorphism $E \oplus \underline{\mathbb{C}}^n \to E' \oplus \underline{\mathbb{C}}^n$. By adjusting the original representatives, we may assume that there exists an isomorphism $\varphi : E \to E'$. Observe that, as in the construction in section 5.3, the second condition implies that

$$\zeta := \eta - \eta' + \operatorname{CS}_{\mathscr{D}}(\varphi^* \nabla', \nabla) \in \mathcal{A}_{\mathbb{T}}(M)^{-1}$$

is closed. Then

$$\begin{aligned}
\mathcal{E} - \mathcal{E}' &= [E, \nabla, \eta] - [E', \nabla', \eta'] \\
&= [E, \nabla, \eta] - [E, \nabla, \eta' - \mathrm{CS}_{\mathscr{D}}(\varphi^* \nabla', \nabla)] \\
&= [\underline{\mathbb{C}}^n, \widetilde{\nabla}, \eta] - [\underline{\mathbb{C}}^n, \widetilde{\nabla}, \eta' - \mathrm{CS}_{\mathscr{D}}(\varphi^* \nabla', \nabla)] \\
&= [\underline{\mathbb{C}}^n, d, \eta + \mu] - [\underline{\mathbb{C}}^n, d, \eta' - \mathrm{CS}_{\mathscr{D}}(\varphi^* \nabla', \nabla) + \mu] \\
&= [\underline{\mathbb{C}}^n, d, \zeta] - [\underline{\mathbb{C}}^n, d, 0] \\
&= \widetilde{h}([\zeta])
\end{aligned}$$

where in the third line we added a complementary bundle to E and $\mu = \mathrm{CS}_{\mathscr{D}}(\widetilde{\nabla}, d)$. Therefore, the sequence 5.105 is exact.

Showing that the sequences are exact in degree -1 is almost completely analogous. One must use the second relation defining $\check{K}_{\mathbb{T}}^{-1}$ to show, for example, that

$$[E, d, \gamma, \eta] - [E, d, \gamma, \eta'] = [\underline{\mathbb{C}}^n, d, Id, \eta] - [\underline{\mathbb{C}}^n, d, Id, \eta']$$

to deduce the exactness of 5.103. Since all groups are periodic with period 2, this completes the proof. □

5.7 Some Calculations

We again identify the representation ring $R(\mathbb{T})$ of the circle with $\mathbb{Z}[t, t^{-1}]$ as in proposition 2.11.

Proposition 5.114. *For $\mathbb{T}$ acting on the point and $R(\mathbb{T})$ the representation* *ring of the circle, we have*

$$\check{K}^j_{\mathbb{T}}(pt) = \begin{cases} R(\mathbb{T}) & j = 0 \\ (R(\mathbb{T}) \otimes \mathbb{C})/R(\mathbb{T}) & j = -1. \end{cases}$$

Proof. When $j = 0$, $\mathcal{A}_{\mathbb{T}}(pt)^{j-1} = 0$ so by sequence 5.103 and proposition 2.11,

$$\check{K}^0_{\mathbb{T}}(pt) \cong K^0_{\mathbb{T}}(pt) \cong K^0(pt) \otimes \mathbb{Z}[t, t^{-1}] \cong \mathbb{Z}[t, t^{-1}].$$

When $j = -1$, $K^{-1}_{\mathbb{T}}(pt) = 0$ so by sequence 5.103,

$$\check{K}^{-1}_{\mathbb{T}}(pt) \cong \mathcal{A}_{\mathbb{T}}(M)^{-2}/\mathcal{A}_{\mathbb{T}}(M)^{-2}_{K} \cong \mathbb{C}[t, t^{-1}]/\mathbb{Z}[t, t^{-1}].$$

□

It is reasonable to ask if for free actions, by analogy with Proposition 2.9, $\check{K}^*_{\mathbb{T}}(M)$ is isomorphic to $\check{K}^*(M/\mathbb{T})$. The answer is *no*: although every equivariant vector bundle on M is isomorphic to a pullback, not every connection is pulled back. There are many more (delocalized equivariant) differential forms on M than there are (ordinary) differential forms on $M/\mathbb{T}$. It is also reasonable to ask if for trivial actions, by analogy with Proposition 2.11, $\check{K}^*_{\mathbb{T}}(M)$ is isomorphic to $\check{K}^*(M) \otimes R(\mathbb{T})$. The answer is *yes*.

Proposition 5.115. *If $\mathbb{T}$ acts trivially on M,*

$$\check{K}^*_{\mathbb{T}}(M) \cong \check{K}^*(M) \otimes R(\mathbb{T}).$$

Proof. For $[E, \nabla, \eta] \in \check{K}^0_{\mathbb{T}}(M)$, we may decompose (E, ∇) into its eigenbundles, $(E, \nabla) \cong \left(\bigoplus_{i=1}^N E_i, \bigoplus_{i=1}^N \nabla_i\right)$, where $\mathbb{T}$ acts on E_i by $\tau \mapsto \tau^{k_i}$. Since

$\mathcal{A}_{\mathbb{T}}(M)^{-1} = \Omega(M;\mathcal{R})^{-1} \otimes \mathbb{C}[t,t^{-1}]$, we may write

$$\eta = \sum_{i=1}^{N} \eta_i t^{k_i} + \sum_{i=N+1}^{N+M} \eta_i t^{\ell_i}$$

for $\eta_i \in \Omega(M;\mathcal{R})^{-1}$. In this decomposition of η we have merely separated the characters which appear in the decomposition of E from those which do not. Then

$$[E,\nabla,\eta] = \sum_{i=1}^{N}[E_i,\nabla_i,\eta_i t^{k_i}] + \sum_{i=N+1}^{N+M}[0,0,\eta_i t^{\ell_i}].$$

Define

$$\psi : \check{K}^0_{\mathbb{T}}(M) \longrightarrow \check{K}^0(M) \otimes_{\mathbb{Z}} \mathbb{Z}[t,t^{-1}]$$

by

$$[E,\nabla,\eta] \longmapsto \sum_{i=1}^{N}[E_i,\nabla_i,\eta_i]t^{k_i} + \sum_{i=N+1}^{N+M}[0,0,\eta_i]t^{\ell_i}$$

where $[E_i,\nabla_i,\eta_i] \in \check{K}^0(M)$ is the element obtained by forgetting the $\mathbb{T}$-action on E. Tensoring the sequence 5.99 with $R(\mathbb{T}) \cong \mathbb{Z}[t,t^{-1}]$ over $\mathbb{Z}$ preserves exactness since a Laurent polynomial is zero if and only if all of its coefficients are zero. This gives the bottom row in the following diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \dfrac{\Omega(M;\mathcal{R})^{-1} \otimes R(\mathbb{T})}{\Omega(M;\mathcal{R})^{-1}_K \otimes R(\mathbb{T})} & \longrightarrow & \check{K}^0_{\mathbb{T}}(M) & \xrightarrow{c_{\mathbb{T}}} & K^0_{\mathbb{T}}(M) & \longrightarrow & 0 \\
 & & \downarrow = & & \downarrow \psi & & \downarrow \cong & & \\
0 & \longrightarrow & \dfrac{\Omega(M;\mathcal{R})^{-1} \otimes R(\mathbb{T})}{\Omega(M;\mathcal{R})^{-1}_K \otimes R(\mathbb{T})} & \longrightarrow & \check{K}^0(M) \otimes R(\mathbb{T}) & \xrightarrow{c\otimes Id} & K^0(M) \otimes R(\mathbb{T}) & \longrightarrow & 0
\end{array}$$

in which the top row is the sequence 5.103. We denote the forgetful map in the top row by $c_{\mathbb{T}}$ to distinguish it from the corresponding map in the non-equivariant theory. It is clear that this diagram commutes so by the Five Lemma, ψ is an isomorphism in degree 0.

For a generator $[E, \nabla, \gamma, \eta] \in \check{K}_{\mathbb{T}}^{-1}(M)$, since γ is an equivariant automorphism, it respects the decomposition of (E, ∇) into eigenbundles. We may thus define ψ analogously in degree -1 and the same argument shows that it is again an isomorphism. $\square$

Appendices

Appendix A

Preliminaries on group actions

Let M be a compact smooth manifold with a smooth action of a compact Lie group G. The following rapid review is based on [49] and appendix B of [29]. By "subgroup of G" we will mean "*closed* subgroup of G". For $m \in M$, we will write $G_m \subset G$ for the stabilizer subgroup $\{g \in G \mid gm = m\}$ and for $H \subset G$

$$M^H := \{m \in M \mid hm = m \text{ for all } h \in H\}$$

for the points fixed by H.

Proposition A.1 ([49] Prop 5.4)**.** *For $m \in M$, the map $\psi_m : G/G_m \to M$, $gG_m \mapsto g \cdot m$, is an embedding. Consequently, the orbit $G \cdot m$ is an embedded submanifold G-diffeomorphic to G/G_m.*

Proof. The action map $G \to M$, $g \mapsto g \cdot m$ is smooth, has constant rank, and factors as $G \xrightarrow{\pi} G/G_m \xrightarrow{\psi_m} M$. It follows that $\psi_m : G/G_m \to M$ is smooth. It is injective and has constant rank so it is an immersion. Since G is compact, ψ_m is an embedding. □

Proposition A.2 (GGK B.26)**.** *For $m \in M^G$ a fixed point, there exists a G-diffeomorphism from an open neighborhood of the origin in T_mM to an open neighborhood of m in M.*

Proof. Let $U \subset M$ be an invariant open set containing m and let $f : U \to T_mM$ be any smooth function with $df_m = Id : T_mM \to T_mM$. Then $F : U \to T_mM$, $F(u) = \int_G g_* f(g^{-1}u)dg$ is smooth, G-equivariant, and has $dF_m = Id$. By the Implicit Function Theorem, we may invert F on a neighborhood of m to obtain the desired G-diffeomorphism. □

Observe that for $m \in M$, T_mM is a representation of G_m which decomposes as

$$T_mM = (T_mM)^{G_m} \oplus W = T_m(G \cdot m) \oplus W$$

Proposition A.3 (Equivariant Tubular Neighbhorhood Theorem, GGK B.24)**.** *For $m \in M$, choose a G_m-invariant metric. There exists a disc $D \subset W$ and a G-diffeomorphism $\varphi : G \times_{G_m} D \to U \subset M$ onto an open neighborhood U of the orbit $G \cdot m$ such that $\varphi[g, 0] = g \cdot m$.*

Proof. By the previous proposition, there exists a G_m-diffeomorphism

$$\begin{array}{c} \{\text{open neighborhood of } 0 \in T_mM = T_m(G \cdot m) \oplus W\} \\ \Big\downarrow \psi \\ \{\text{open neighborhood of } m \in M\}. \end{array} \tag{A.4}$$

Choose a G_m-invariant metric on M and let $D' \subset W$ be a disc about the origin contained in the domain of ψ. Define $\varphi : G \times_{G_m} D' \to M$ by

$$\varphi[g, v] = g \cdot \psi(v).$$

Then φ is well defined and a local diffeomorphism at $[e, 0]$. Since φ is G-equivariant, it is a local diffeomorphism at $[g, 0]$ for all $g \in G$. To see that there exists a disc $D \subset D'$ and neighborhood $U \subset U'$ of the orbit $G \cdot m$ such that $\varphi : G \times_{G_m} D \to U$ is a diffeomorphism, we argue by contradiction. Suppose that φ is not injective for any $D \subset D'$. Then there exist $v_n, w_n \in W$ with $v_n, w_n \to 0$ and $g_n, h_n \in G$ such that $[g_n, v_n] \neq [h_n, w_n]$ but for which

$$g_n \cdot \psi(v_n) = \varphi[g_n, v_n] = \varphi[h_n, w_n] = h_n \cdot \psi(w_n).$$

We may assume without loss of generality that $h_n = e$ for all n. (If not, we take $\widetilde{g}_n = h_n^{-1} g_n$ and $\widetilde{h}_n = e$.) The G-action determines a map

$$\begin{aligned} \alpha : G \times M &\longrightarrow M \times M \\ (g, x) &\longmapsto (g \cdot x, x). \end{aligned}$$

Since the action is proper, α is a proper map. We have

$$\alpha(g_n, \psi(v_n)) = (g_n \cdot \psi(v_n), \psi(v_n)) = (\psi(w_n), \psi(v_n)).$$

This sequence in $M \times M$ converges to (m, m). If $K \subset M \times M$ is a compact set containing (m, m), then $\alpha^{-1}(K)$ is compact so contains a convergent subsequence $(g_{n_j}, \psi(v_{n_j})) \to (g_\infty, m)$. On the one hand,

$$\varphi[g_{n_j}, v_{n_j}] \xrightarrow{j \to \infty} \varphi[g_\infty, 0] = g_\infty \cdot \psi(0) = g_\infty \cdot m.$$

On the other hand,

$$\varphi[e, w_{n_j}] \xrightarrow{j \to \infty} \varphi[e, 0] = m.$$

Since $\varphi[g_n, v_n] = \varphi[e, w_n]$, it follows that $g_\infty \cdot m = m$. Therefore $g_\infty \in G_m$ so

$$[g_\infty, 0] = [e, 0] \in G \times_{G_m} D'.$$

We have shown that $[g_{n_j}, v_{n_j}]$ and $[e, w_{n_j}]$ are two sequences in $G \times_{G_m} D'$ which converge to $[e, 0]$ and

$$\varphi[g_{n_j}, v_{n_j}] = \varphi[e, w_{n_j}].$$

Thus, φ is not injective on any neighborhood of $[e, 0]$ which contradicts that it is a local diffeomorphism at that point. It follows that there exists a disc $D \subset D'$ and a neighborhood $U \subset U'$ of the orbit $G \cdot m$ such that

$$\varphi : G \times_{G_m} D \to U$$

is a G-diffeomorphism. □

Proposition A.5. *For any non-trivial subgroup H, M^H is a disjoint union of closed submanifolds.*

Proof. That the action is continuous implies that M^H is closed. Let F be a connected component of M^H and let $m \in F$. Consider M as an H-manifold and apply Proposition A.2 to obtain an H-diffeomorphism $\psi : V \to U$ for open sets $0 \in V \subset T_mM$ and $m \in U \subset M$. Then $\psi : V \cap (T_mM)^H \to U \cap M^H$ is a diffeomorphism. □

The *orbit type* of $m \in M$ is the conjugacy class of its stabilizer G_m in G. The orbit type is constant on an orbit because the stabilizer of $g \cdot m$ is gG_mg^{-1}.

Proposition A.6 (GGK B.39)**.** *M has finitely many orbit types.*

Proof. Cover M by equivariant tubular neighborhoods of orbits and choose a finite subcover. If we can show that for an H-vector space W, $G \times_H W$ has only finitely many orbit types, it follows that M does. Thus, consider $M = G \times_H W$. We proceed by induction on the dimension of M. If $\dim M = 1$, then either $H = G$ and $\dim W = 1$ or $\dim G/H = 1$ and $W = 0$. In the first case, W is a 1-dimensional vector space on which $H = G$ acts. If there extists a $g \in G$ and a nonzero $w \in W$ such that $g \cdot w = w$, then by linearity, g acts trivially on W. If K is the kernel of the representation $G \to \mathrm{Aut}(W)$, then for any nonzero $w \in W$, $K \subset G_w$ and $G_w \subset K$ so $G_w = K$. It follows that the only orbit types are (K) and (G), corresponding to the nonzero vectors and the zero vector, respectively. In the second case, $W = 0$. Since

$$[e, 0] = [aa^{-1}, 0] = a \cdot [a, 0],$$

$[e, 0]$ and $[a, 0]$ have the same orbit type. The stabilizer $G_{[e,0]} = H$ so the orbit type of any $[a, 0] \in M$ is (H). Thus, the only orbit type is (H). For the inductive step, assume that every G-manifold of dimension $< n$ has finitely many orbit types. Let $M = G \times_H W$ have dimension n and fix an H-invariant inner product on W (so H acts unitarily on W). For $[a, w] \in M$, if $w = 0$, $[a, w]$ has orbit type (H). If $w \neq 0$, $[a, w]$ has the same orbit type as $[a, w/\|w\|]$: that the action is linear implies $H_w = H_{w/\|w\|}$. Let $S^W \subset W$ be the unit sphere, and let $N = G \times_H S^W$. Then N has dimension $n - 1$ and $[a, w/\|w\|] \in N$. By hypothesis, N has only finitely many orbit types. Since the orbit type of every point in M is the orbit type of a point in N, M has only finitely many orbit types. □

Appendix B

Integration along the fiber

B.1 Non-equivariant case

See [16] page 37 for a thorough discussion and proofs. Let M be a compact smooth manifold, let S^1 be the circle with its standard orientation and let $p : M \times S^1 \to M$ be projection. Integration over the circle defines a map

$$\int_{S^1} = p_* : \Omega(M \times S^1; \mathcal{R})^j \longrightarrow \Omega(M; \mathcal{R})^{j-1}$$

By Stokes' theorem,

$$p_* d = dp_*$$

so p_* is a chain map

$$\int_{S^1} = p_* : (\Omega(M \times S^1; \mathcal{R})^\bullet, d) \longrightarrow (\Omega(M; \mathcal{R})^{\bullet-1}, d)$$

and thus induces a map in cohomology

$$\int_{S^1} = p_* : H(M \times S^1; \mathcal{R})^\bullet \longrightarrow H(M; \mathcal{R})^{\bullet-1}$$

called integration over the fiber (or push forward). Let p and π be the projections

$$\begin{array}{ccc} & M \times S^1 & \\ {}^{p}\swarrow & & \searrow^{\pi} \\ M & & S^1 \end{array}$$

then by the Kunneth theorem ([16] p.47), there is an isomorphism

$$H(M;\mathcal{R})^\bullet \otimes H(S^1;\mathbb{C})^\bullet \xrightarrow{\cong} H(M\times S^1;\mathcal{R})^\bullet$$
$$\omega\otimes\eta \longmapsto p^*\omega\wedge\pi^*\eta$$

Fix a point $pt \in S^1$ and let $i : M = M\times\{pt\} \to M\times S^1$ be the inclusion, then we obtain an exact sequence

$$0 \longrightarrow \ker i^* \to H(M\times S^1;\mathcal{R})^\bullet \xrightarrow{i^*} H(M;\mathcal{R})^\bullet \longrightarrow 0$$

where i^* is surjective because $i^*p^* = Id$. By the Kunneth theorem, we have

$$H(M\times S^1;\mathcal{R})^0 \cong (H(M;\mathcal{R}^0)\otimes H^0(S^1;\mathbb{C})) \oplus (H(M;\mathcal{R})^{-1}\otimes H^1(S^1;\mathbb{C}))$$

If ds is a volume form on S^1 with volume 1, then $i^*ds = 0$. We may thus identify $\ker i^*$ with the second summand above. Finally, since integration over the circle is an isomorphism $H^1(S^1;\mathbb{C}) \to \mathbb{C}$, the integration map

$$p_* : H(M;\mathcal{R})^{-1}\otimes H^1(S^1;\mathbb{C}) \to H(M;\mathcal{R})^{-1}$$

is an isomorphism. We thus see that p_* is an isomorphism

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \ker i^* & \longrightarrow & H(M\times S^1;\mathcal{R})^0 & \xrightarrow{i^*} & H(M;\mathcal{R})^0 & \longrightarrow & 0 \\ & & \cong\big\downarrow p_* & & & & & & \\ & & H(M;\mathcal{R})^{-1} & & & & & & \end{array}$$

on the kernel of i^*. Observe that under the Kunneth isomorphism, p_* is the map

$$p_*(p^*\omega\wedge\pi^*\eta) = \omega\wedge p_*\eta \tag{B.1}$$

We will refer to this formula as the *push-pull* formula. It says that $H(M \times S^1; \mathcal{R})^\bullet$ and $H(M; \mathcal{R})^\bullet$ are both $H(M; \mathcal{R})^\bullet$-modules and that p_* is a map of modules. A special case is the formula

$$p_*(p^*\omega) = 0.$$

If we replace S^1 by S^n above, the same argument shows that p_* defines an isomorphism

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \ker i^* & \longrightarrow & H(M \times S^n; \mathcal{R})^0 & \xrightarrow{i^*} & H(M; \mathcal{R})^0 & \longrightarrow & 0 \\ & & \cong \big\downarrow p_* & & & & & & \\ & & H(M; \mathcal{R})^{-n} & & & & & & \end{array}$$

where now i is the inclusion $M \times \{pt\} \to M \times S^n$.

B.2 Equivariant case

Let M be a compact smooth $\mathbb{T}$-manifold and consider $M \times S^1$ as a $\mathbb{T}$-manifold with trivial action on the second factor. Let ξ be the vector field on M which generates the $\mathbb{T}$-action. There is a canonical isomorphism $T(M \times S^1) \cong TM \oplus TS^1$ under which the vector field which generates to the $\mathbb{T}$-action on $M \times S^1$ is $(\xi, 0)$. We will denote both the vector fields on M and $M \times S^1$ by ξ and rely on context. Extending integration to be linear over u gives a map

$$\int_{S^1} = p_* : C_{\mathbb{T}}(M \times S^1)^j \longrightarrow C_{\mathbb{T}}(M)^{j-1}$$

One readily checks that for $\eta \in C_{\mathbb{T}}(M \times S^1)^\bullet$

$$\iota_\xi \int_{S^1} \eta = \int_{S^1} \iota_\xi \eta.$$

It follows that

$$d_{\mathbb{T}} p_* = p_* d_{\mathbb{T}}$$

so p_* is a chain map

$$\int_{S^1} = p_* : (C_{\mathbb{T}}(M \times S^1)^\bullet, d_{\mathbb{T}}) \longrightarrow (C_{\mathbb{T}}(M)^{\bullet-1}, d_{\mathbb{T}})$$

and thus induces a map in cohomology

$$p_* : H_{\mathbb{T}}(M \times S^1; \mathcal{R})^\bullet \longrightarrow H_{\mathbb{T}}(M; \mathcal{R})^{\bullet-1}$$

The Kunneth theorem holds in equivariant cohomology as well so the discussion of the previous section extends to the present context. More generally, if $p : M \times S^n \to M$ is projection and $i : M \times \{pt\} \to M \times S^n$ is inclusion, p_* defines an isomorphism

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \ker i^* & \longrightarrow & H_{\mathbb{T}}(M \times S^n; \mathcal{R})^0 & \xrightarrow{i^*} & H_{\mathbb{T}}(M; \mathcal{R})^0 & \longrightarrow & 0 \\ & & \cong \big\downarrow p_* & & & & & & \\ & & H_{\mathbb{T}}(M; \mathcal{R})^{-n} & & & & & & \end{array}$$

which also satisfies a push-pull formula analogous to equation B.1.

Appendix C

The Complex of Basic Forms and Equivariant Cohomology

Let G be a compact Lie group with Lie algebra $\mathfrak{g}$ and M a compact smooth G-manifold. For $X \in \mathfrak{g}$ we denote by the same letter the corresponding vector field on M.

Definition C.0.1. A differential form $\omega \in \Omega^\bullet(M)$ is *basic* if it is

- invariant, $g^*\omega = \omega$ for all $g \in G$, and
- horizontal, $\iota_X \omega = 0$ for all $X \in \mathfrak{g}$.

Let $\Omega^\bullet(M)_{\text{bas}} \subset \Omega^\bullet(M)$ denote the subalgebra of basic differential forms. If the action is free, then M/G is a smooth manifold, $\pi : M \to M/G$ is a principal G-bundle, and the basic forms on M are exactly those forms pulled back from M/G.

C.1 Koszul's theorem

Koszul's theorem states that the complex of basic forms on M computes the real cohomology of the quotient M/G. Let $\pi : M \to M/G$ be the quotient map. The argument goes as follows. For $k \in \mathbb{Z}^{\geqslant 0}$, let $\mathcal{F}^k \to M/G$ be the sheaf that to an open set $U \subset M/G$ assigns

$$\mathcal{F}^k(U) = \Omega^k(\pi^{-1}U)_{\mathrm{bas}}.$$

One proves a Poincaré lemma which shows that the complex of sheaves

$$0 \longrightarrow \underline{\mathbb{R}} \longrightarrow \mathcal{F}^0 \xrightarrow{d} \mathcal{F}^1 \xrightarrow{d} \mathcal{F}^2 \xrightarrow{d} \cdots$$

is an acyclic resolution of the constant sheaf $\underline{\mathbb{R}} \to M/G$. It follows by the uniqueness of sheaf cohomology that the complex of basic forms on M computes the real cohomology of M/G.

Theorem C.1 (Koszul's Poincaré lemma)**.** *Let ω be a basic form on M. If ω is closed on an equivariant tubular neighborhood of an orbit in M, then it is exact on that neighborhood.*

Koszul describes the proof in [37]. We include it for completeness.

Proof. Let $p \in M$ be a point with stabilizer $H \subset G$ and let $G \cdot p$ be the orbit of p. By Proposition A.3 the orbit $G \cdot p$ has an equivariant tubular neighborhood of the form $N = G \times_H V$ where V is representation of H. N is the vector bundle associated to the principal H-bundle $G \to G/H$. It follows that pullback by the projection map $\pi_H : G \times V \to G \times_H V$ gives an isomorphism

$$\pi_H^* : \Omega^\bullet(G \times_H V) \xrightarrow{\cong} \Omega^\bullet(G \times V)_{H\,\mathrm{bas}}$$

where $H\,\mathrm{bas}$ denotes the forms which are basic for the H-action. In fact, G also acts on $G \times V$ by its action on (the left) on G and this commutes with the

H action so there is a $G \times H$ action on $G \times V$. This descends to a G-action on $N = G \times_H V$. Pullback by π_H restricts to an isomorphism

$$\pi_H^* : \Omega^\bullet(G \times_H V)_{G\,\mathrm{bas}} \xrightarrow{\cong} \Omega^\bullet(G \times V)_{G\times H\,\mathrm{bas}}$$

Now, $G \times V \to (G \times V)/G \cong V$ is an (H-equivariant) principal G-bundle. Our diagram looks like

$$\begin{array}{ccc} & G \times V & \\ {}^{\pi_H}\swarrow & & \searrow^{\pi_G} \\ G \times_H V & & G \times_G V \cong V \end{array}$$

It follows that pullback by π_G defines an isomorphism

$$\pi_G^* : \Omega^\bullet(V)_{H\,\mathrm{bas}} \xrightarrow{\cong} \Omega^\bullet(G \times V)_{G\times H\,\mathrm{bas}}$$

We have thus identified the G-basic forms on the neighborhood N of the orbit of p with the H-basic forms on V. Now that we are in a vector space, we can use the standard proof of the Poincaré lemma (see [22] chapter 4, section 3). Let ω be a closed H-basic form on V and let $F : V \times [0,1] \to V$ be the map $F(v,t) = tv$. Then F is an H-equivariant homotopy from the identity map on V to the constant map. Let $\iota_i : V \to V \times \{i\}$ be the inclusions for $i = 0, 1$, and let

$$K : \Omega^\bullet(V \times [0,1]) \longrightarrow \Omega^{\bullet-1}(V)$$

be the map

$$\eta \longmapsto \int_{[0,1]} \eta.$$

Let $\overline{\omega} = F^*\omega \in \Omega^\bullet(V \times [0,1])$, then we have

$$\iota_1^*\overline{\omega} - \iota_0^*\overline{\omega} = dK\overline{\omega} + Kd\overline{\omega}$$

which reduces to

$$\omega = d(K\overline{\omega}).$$

Now, the H-action on $V \times [0,1]$ is trivial on the second factor so for any $X \in \mathrm{Lie}(H)$, $\iota_X K = K\iota_X$. Then, since pushforward by F just scales the vector field X, one readily checks that $K\overline{\omega}$ is basic. This completes the proof. □

It follows by the uniqueness of sheaf cohomology that

Theorem C.2 (Koszul)**.**

$$H(\Omega^\bullet(M;\mathbb{R})_{G\,\mathrm{bas}}, d) \cong H^\bullet(M/G;\mathbb{R}).$$

C.2 The Weil Model, the Cartan Model, and Locally Free Actions

Let M be a compact smooth $\mathbb{T}$-manifold and let $\mathfrak{t} = \mathrm{Lie}(\mathbb{T}) = i\mathbb{R}$. As usual, we let ξ denote both $i \in \mathfrak{t}$ and the vector field on M which it generates. The Weil complex of M is

$$W^\bullet(M) = \Omega^\bullet(M) \otimes \Lambda^\bullet(\mathfrak{t}^*) \otimes \mathrm{Sym}^\bullet(\mathfrak{t}^*)$$

Let $\theta \in \Lambda^1(\mathfrak{t}^*)$ and $u \in \mathrm{Sym}^1(\mathfrak{t}^*)$ denote the generators, thus $\theta(\xi) = u(\xi) = 1$. The grading is by total degree, $\deg\theta = 1$ and $\deg u = 2$. Then

$$W^\bullet(M) = \Omega^\bullet(M) \otimes (\mathbb{R} \oplus \mathbb{R}\theta) \otimes \mathbb{R}[u]$$

The differential is the graded derivation which extends the de Rham d on forms and satisfies

$$d\theta = u \quad \text{and} \quad du = 0.$$

We extend the contraction operator and Lie derivative by

$$\iota_\xi \theta = \theta(\xi) = 1 \quad \text{and} \quad \iota_\xi u = 0$$

and define $\mathcal{L}_\xi = d\iota_\xi + \iota_\xi d$. An element of the Weil complex is *basic* if it is invariant and horizontal: that is, annihilated by $\mathcal{L}_\xi$ and ι_ξ. Let $W(M)^\bullet_{\text{bas}}$ denote the subcomplex of basic forms.

Theorem C.3 (Cartan, Mathai-Quillen)**.** *There is a quasi-isomorphism*

$$\begin{aligned} (W^\bullet(M)_{\text{bas}}, d) &\xrightarrow{\varphi_{\text{MQ}}} (\Omega(M)^{\mathbb{T}}[u]^\bullet, d_{\mathbb{T}}) \\ \alpha - \theta\iota_\xi\alpha &\longmapsto \alpha \\ a - \theta\iota_\xi a &\longleftarrow a \end{aligned}$$

Now suppose that the $\mathbb{T}$-action on M is locally free meaning that it has finite stabilizers.

Theorem C.4 (Cartan)**.** *If $\mathbb{T}$ acts on M locally freely, the inclusion*

$$j : (\Omega^\bullet(M)_{\text{bas}}, d) \hookrightarrow (W^\bullet(M)_{\text{bas}}, d)$$

induces an isomorphism

$$j : H(\Omega^\bullet(M)_{\text{bas}}, d) \xrightarrow{\cong} H(W^\bullet(M)_{\text{bas}}, d).$$

In this case, the vector field ξ which generates the action is nowhere zero. Let $\gamma \in \Omega^1(M)$ be an invariant one-form satisfying $\gamma(\xi) \equiv 1$. The idea is that γ and $d\gamma$ will play the roles of θ and u, respectively. A proof of the corresponding theorem for G an arbitrary compact connected Lie group appears in [30] chapter 5. We deduced this direct proof from the discussion there.

Proof. Let $\widetilde{\omega}$ be a homogeneous element of $W^\bullet(M)_{\mathrm{bas}}$ of degree d. Then we can express $\widetilde{\omega}$ as

$$\widetilde{\omega} = \sum_{r=0}^{n} \widetilde{\omega}_r u^r.$$

Since $\widetilde{\omega}$ is basic, the forms $\widetilde{\omega}_r$ are basic so we can write them as

$$\widetilde{\omega}_r = \omega_r - \theta \iota_\xi \omega_r$$

for homogeneous elements $\omega_r \in \Omega^{d-2r}(M)^{\mathbb{T}}$. Define

$$\pi : W^d(M)_{\mathrm{bas}} \longrightarrow \Omega^d(M)_{\mathrm{bas}}$$

by

$$\pi(\widetilde{\omega}) = \sum_{r=0}^{n} (\omega_r - \gamma \wedge \iota_\xi \omega_r) \wedge (d\gamma)^r.$$

That is, we replace θ with γ and u with $d\gamma$, respectively. One readily checks that π is a chain map. It may be that terms in this sum vanish for $r \leqslant n$. We will define a chain homotopy

$$Q : W^d(M)_{\mathrm{bas}} \to W^{d-1}(M)_{\mathrm{bas}}$$

such that

$$dQ + Qd = Id - j\pi. \tag{C.5}$$

We define it by

$$Q(\widetilde{\omega}) = \sum_{r=1}^{n} \sum_{j=0}^{r-1} (\theta - \gamma) \wedge \widetilde{\omega}_r \wedge (d\gamma)^j u^{r-1-j}$$

Observe that $Q(\widetilde{\omega})$ is basic since $\iota_\xi(\theta - \gamma) = 0 = \iota_\xi u$ and the forms $\widetilde{\omega}_r$ and $d\gamma$

are basic. We verify equation C.5.

$$
\begin{aligned}
dQ(\widetilde{\omega}) =& d\left(\sum_{r=1}^{n}\sum_{j=0}^{r-1}(\theta-\gamma)\wedge\widetilde{\omega}_r\wedge(d\gamma)^j u^{r-1-j}\right)\\
=& \sum_{r=1}^{n}\sum_{j=0}^{r-1}(u-d\gamma)\wedge\widetilde{\omega}_r\wedge(d\gamma)^j u^{r-1-j}\\
& -\sum_{r=1}^{n}\sum_{j=0}^{r-1}(\theta-\gamma)\wedge\{d\omega_r-u\iota_\xi\omega_r+\theta d\iota_\xi\omega_r\}\wedge(d\gamma)^j u^{r-1-j}\\
=& \sum_{r=1}^{n}\sum_{j=0}^{r-1}\widetilde{\omega}_r\wedge(d\gamma)^j u^{r-j}\\
& -\sum_{r=1}^{n}\sum_{j=0}^{r-1}\widetilde{\omega}_r\wedge(d\gamma)^{j+1} u^{r-1-j}\\
& \underbrace{-\sum_{r=1}^{n}\sum_{j=0}^{r-1}\theta d\omega_r\wedge(d\gamma)^j u^{r-1-j}}_{A}\\
& \underbrace{+\sum_{r=1}^{n}\sum_{j=0}^{r-1}\theta\iota_\xi\omega_r\wedge(d\gamma)^j u^{r-j}}_{B}\\
& \underbrace{+\sum_{r=1}^{n}\sum_{j=0}^{r-1}\gamma\wedge d\omega_r\wedge(d\gamma)^j u^{r-1-j}}_{C}\\
& \underbrace{-\sum_{r=1}^{n}\sum_{j=0}^{r-1}\gamma\wedge\iota_\xi\omega_r\wedge(d\gamma)^j u^{r-j}}_{D}\\
& \underbrace{-\sum_{r=1}^{n}\sum_{j=0}^{r-1}\theta\gamma\wedge\iota_\xi d\omega_r\wedge(d\gamma)^j u^{r-1-j}}_{E}
\end{aligned}
$$

where we used in the last line that $d\iota_\xi = -\iota_\xi d$ since the forms ω_r are invariant so their Lie derivatives vanish. To calculate $Qd(\widetilde{\omega})$, we have to write $d\widetilde{\omega}$ as a sum of polynomials in u with basic coefficients. We have

$$\begin{aligned} d\widetilde{\omega} &= \sum_{r=0}^{n}(d\omega_r - u\iota_\xi\omega_r + \theta d\omega_r)u^r \\ &= \sum_{r=0}^{n}(d\omega_r - \theta\iota_\xi d\omega_r)u^r - \sum_{r=0}^{n}\iota_\xi\omega_r u^{r+1} \end{aligned}$$

Then

$$\begin{aligned} Qd(\widetilde{\omega}) &= \sum_{r=1}^{n}\sum_{j=0}^{r-1}(\theta-\gamma)\wedge(d\omega_r - \theta\iota_\xi d\omega_r)\wedge(d\gamma)^j u^{r-1-j} \\ &\quad - \sum_{r=0}^{n}\sum_{j=0}^{r}(\theta-\gamma)\wedge(\iota_\xi\omega_r)\wedge(d\gamma)^j u^{r-j} \\ &= \underbrace{\sum_{r=1}^{n}\sum_{j=0}^{r-1}\theta d\omega_r\wedge(d\gamma)^j u^{r-1-j}}_{A} \\ &\quad \underbrace{-\sum_{r=1}^{n}\sum_{j=0}^{r-1}\gamma\wedge d\omega_r\wedge(d\gamma)^j u^{r-1-j}}_{C} \\ &\quad \underbrace{+\sum_{r=1}^{n}\sum_{j=0}^{r-1}\theta\gamma\wedge\iota_\xi d\omega_r\wedge(d\gamma)^j u^{r-1-j}}_{E} \\ &\quad \underbrace{-\sum_{r=0}^{n}\sum_{j=0}^{r}\theta\iota_\xi\omega_r\wedge(d\gamma)^j u^{r-j}}_{B'} \\ &\quad \underbrace{+\sum_{r=0}^{n}\sum_{j=0}^{r}\gamma\wedge\omega_r\iota_\xi\wedge(d\gamma)^j u^{r-j}}_{D'} \end{aligned}$$

Thus, in $dQ + Qd$, the terms marked A, C, and E cancel. For the remaining pairs of terms we have

$$B + B' = \sum_{r=1}^{n} \sum_{j=0}^{r-1} \theta \iota_\xi \omega_r \wedge (d\gamma)^j u^{r-j}$$
$$- \sum_{r=0}^{n} \sum_{j=0}^{r} \theta \iota_\xi \omega_r \wedge (d\gamma)^j u^{r-j}$$
$$= - \sum_{r=0}^{n} \theta \iota_\xi \omega_r \wedge (d\gamma)^r$$

and

$$D + D' = - \sum_{r=1}^{n} \sum_{j=0}^{r-1} \gamma \wedge \iota_\xi \omega_r \wedge (d\gamma)^j u^{r-j}$$
$$+ \sum_{r=0}^{n} \sum_{j=0}^{r} \gamma \wedge \iota_\xi \omega_r \wedge (d\gamma)^j u^{r-j}$$
$$= \sum_{r=0}^{n} \gamma \wedge \iota_\xi \omega_r \wedge (d\gamma)^r$$

We are left with

$$(dQ + Qd)\widetilde{\omega} = \sum_{r=1}^{n} \sum_{j=0}^{r-1} \widetilde{\omega}_r \wedge (d\gamma)^j u^{r-j}$$
$$- \sum_{r=1}^{n} \sum_{j=0}^{r-1} \widetilde{\omega}_r \wedge (d\gamma)^{j+1} u^{r-1-j}$$
$$- \sum_{r=0}^{n} \theta \iota_\xi \omega_r \wedge (d\gamma)^r$$
$$+ \sum_{r=0}^{n} \gamma \wedge \iota_\xi \omega_r \wedge (d\gamma)^r$$
$$= \sum_{r=1}^{n} \widetilde{\omega}_r u^r - \sum_{r=1}^{n} \widetilde{\omega}_r \wedge (d\gamma)^r$$
$$- \sum_{r=0}^{n} \theta \iota_\xi \omega_r \wedge (d\gamma)^r + \sum_{r=0}^{n} \gamma \wedge \iota_\xi \omega_r \wedge (d\gamma)^r$$

Note that $\sum_{r=0}^{n} \widetilde{\omega}_r \wedge (d\gamma)^r$ is not quite $j\pi(\widetilde{\omega})$ because the coefficients are

$$\widetilde{\omega}_r = \omega_r - \theta\iota_\xi\omega_r \quad \text{rather than} \quad \omega_r - \gamma \wedge \iota_\xi\omega_r.$$

The last two sums exactly make the necessary substitution. Thus

$$\begin{aligned}(dQ + Qd)\widetilde{\omega} &= \sum_{r=1}^{n} \widetilde{\omega}_r u^r - \left(\sum_{r=1}^{n} \widetilde{\omega}_r \wedge (d\gamma)^r \right. \\ &\qquad \left. + \sum_{r=0}^{n} \theta\iota_\xi\omega_r \wedge (d\gamma)^r - \sum_{r=0}^{n} \gamma \wedge \iota_\xi\omega_r \wedge (d\gamma)^r \right) \\ &= \sum_{r=0}^{n} \widetilde{\omega}_r u^r - \sum_{r=0}^{n} (\omega_r - \gamma \wedge \iota_\xi\omega_r) u^r \\ &= \widetilde{\omega} - j\pi(\widetilde{\omega})\end{aligned}$$

Since π is the identity on $\Omega^\bullet(M)_{\text{bas}}$, $\pi j = Id$. Going the other way, $j\pi$ is chain homotopic to the identity so j and π are homotopy inverses to each other. Therefore j induces an isomorphism in cohomology. □

Observe that the inclusions commute with the Mathai-Quillen map

$$\begin{array}{ccc} (W^\bullet(M)_{\text{bas}}, d) & \xrightarrow{\varphi_{\text{MQ}}} & (\Omega(M)^{\mathbb{T}}[u]^\bullet, d_{\mathbb{T}}) \\ \uparrow & \nearrow & \\ (\Omega^\bullet(M)_{\text{bas}}, d) & & \end{array}$$

We will thus also denote by j the inclusion of the basic forms into the Cartan complex. This is a chain map because on basic forms $d_{\mathbb{T}} = d$. By Koszul's theorem we may identify the cohomology of the quotient with the cohomology of the complex of basic forms

$$H^\bullet(M/\mathbb{T}; \mathbb{R}) = H(\Omega^\bullet(M)_{\text{bas}}, d).$$

We thus obtain

Corollary C.6. *The inclusion*

$$j : (\Omega^\bullet(M)_{\mathrm{bas}}, d) \hookrightarrow (\Omega(M)^{\mathbb{T}}[u]^\bullet, d_{\mathbb{T}})$$

induces an isomorphism

$$j : H^\bullet(M/\mathbb{T};\mathbb{R}) = H(\Omega^\bullet(M)_{\mathrm{bas}}, d) \longrightarrow H(\Omega(M)^{\mathbb{T}}[u]^\bullet, d_{\mathbb{T}}) = H^\bullet_{\mathbb{T}}(M;\mathbb{R}).$$

It follows that the inclusion induces an isomorphism with periodic coefficients, too.

Corollary C.7. *The inclusion*

$$\begin{array}{ccc} j : \Omega(M;\mathcal{R})^\bullet_{\mathrm{bas}} & \hookrightarrow & C_{\mathbb{T}}(M)^\bullet \\ \downarrow{=} & & \downarrow{=} \\ \prod_{k=0}^{\infty} \beta^{k-\bullet}\Omega^{2k-\bullet}(M;\mathbb{C})_{\mathrm{bas}} & \hookrightarrow & \prod_{k=0}^{\infty} \beta^{k-\bullet}\Omega(M;\mathbb{C})^{\mathbb{T}}[u]^{2k-\bullet} \end{array}$$

induces an isomorphism

$$j : H(M/\mathbb{T};\mathcal{R})^\bullet \longrightarrow H_{\mathbb{T}}(M;\mathcal{R})^\bullet.$$

Appendix D

Invariant and basic connections

Let $\pi : P \to M$ be a principal G-bundle. Recall that for $p \in P$ there is a well-defined vertical tangent space $V_p = \ker \pi_* \subset T_pP$ and writing $P_m := \pi^{-1}(m)$, there are diffeomorphisms $\varphi_p : G \to P_{\pi(p)}$ given by $g \mapsto p \cdot g$. Identifying $\mathfrak{g}$ with the left-invariant vector fields on G, recall that the Maurer-Cartan form of G is the $\mathfrak{g}$-valued one-form ω_{MC} that takes $v \in T_gG$ to the unique left-invariant vector field that it generates. A connection on P is a one-form $\omega \in \Omega^1(P; \mathfrak{g})$ satisfying

1. $\omega_{p \cdot g} = \mathrm{Ad}_{g^{-1}} \omega_p$, and

2. $\varphi_p^* \omega = \omega_{MC}$.

Equivalently, a connection is a smooth G-invariant distribution H such that $H_p \oplus V_p = T_pP$; the correspondence is $H_p \leftrightarrow \ker \omega_p$. By the second condition, for each $p \in P$, a connection gives a splitting $\omega_p : T_pP \to \mathfrak{g}$ of the short exact sequence

$$0 \longrightarrow \mathfrak{g} \xrightarrow{\varphi_p} T_pP \xrightarrow{\pi_*} T_{\pi(p)}M \longrightarrow 0.$$

A local section $\sigma : U \to P$ gives a local trivialization $\varphi : U \times G \to P|_U$, $(m, g) \mapsto \sigma(m) \cdot g$, $U \times G$ has canonical horizontal subspaces, and pushing these forward by φ gives a connection on $P|_U$. Since the space of splittings of a short exact sequence of vector spaces is an affine space, we may average local connections with partitions of unity to obtain a connection on P.

Suppose now that M is a $\mathbb{T}$-manifold and that there is a $\mathbb{T}$-action on P which covers the action on M. A connection $\omega \in \Omega^1(P;\mathfrak{g})$ on P is $\mathbb{T}$-invariant if $\tau^*\omega = \omega$ for all $\tau \in \mathbb{T}$. If $\theta \in \Omega^1(P;\mathfrak{g})$ is any connection, $\int_{\mathbb{T}} \tau^*\theta d\mu$ is an invariant connection. It follows that the space of invariant connections $\mathcal{C}_P^{\mathbb{T}}$ on P is a non-empty affine subspace of the space $\mathcal{C}_P$ of all connections on P.

Recall that if $E \to M$ is a vector bundle and $P \to M$ is its principal G-bundle of frames, then we may write E as an associated vector bundle $E \cong P \times_G \mathbb{V}$ for a representation $\rho : G \to \mathrm{Aut}(\mathbb{V})$. Sections $\sigma(m) = [p, s(p)]$ of E are thus in bijection with functions $s : P \to \mathbb{V}$ satisfying $s(p{\cdot}g) = \rho(g)^{-1}s(p)$. If ω is a connection on P, it determines a connection on E by

$$\sigma \mapsto \nabla\sigma \Leftrightarrow s \mapsto ds + \dot{\rho}(\omega)s.$$

From this expression, one readily checks that an invariant connection on E is one induced from an invariant connection on P.

Consider again a smooth compact $\mathbb{T}$-manifold M and a principal G-bundle $P \to M$ with an action of $\mathbb{T}$ covering the action on M. Suppose now that the action on M is locally free, has only finite stabilizers, and let $\overline{\xi}$ and ξ be the vector fields on P and M, respectively, which generate the $\mathbb{T}$-action. Since $\overline{\xi}$ covers ξ and ξ is nowhere zero it follows that $\overline{\xi}$ is nowhere vertical. A *basic* connection on P is a connection $\omega \in \Omega^1(P;\mathfrak{g})$ which is a basic differential form for the $\mathbb{T}$-action on P. Thus, ω is invariant, $\tau^*\omega = \omega$, for all $\tau \in \mathbb{T}$, and horizontal with respect to the $\mathbb{T}$-action, $\iota_{\overline{\xi}}\omega = 0$.

We show the existence of basic connections as follows. The real span of the vector field $\overline{\xi}$ determines a real sub-line bundle $L \subset TP$ and for all $p \in P$, $L_p \cap V_p = 0$. To make our connection $\mathbb{T}$-horizontal, we must choose splittings $s_p : T_pP \to \mathfrak{g}$ such that $L_p \subset \ker s_p$. For $\sigma : U \to P$ a local section and $\varphi : U \times G \to P|_U$ the corresponding local trivialization, L determines a smooth, nowhere vertical, real line bundle $\widetilde{L} \subset T(U \times G)$. Choosing horizontal subspaces which contain $\widetilde{L}$ gives a connection ω_U on $P|_U$ which satisfies $\iota_{\overline{\xi}}\omega_U = 0$. Since being horizontal is a linear condition, averaging horizontal local connection forms by partitions of unity again gives a connection ω on P which satisfies $\iota_{\overline{\xi}}\omega = 0$. Finally, setting

$$\widetilde{\omega} = \int_{\mathbb{T}} \tau^*\omega d\mu$$

yields a basic connection.

Finally, if the $\mathbb{T}$-action on M has finite stabilizers and E is a $\mathbb{T}$-equivariant vector bundle, P its principal frame bundle and ω is a basic connection on P, then the induced connection ∇ on E satisfies

$$\begin{aligned}\nabla_{\xi}\sigma - \mathfrak{L}^E_{\xi}\sigma &\Leftrightarrow \iota_{\overline{\xi}}(ds + \dot{\rho}(\omega)s) - \mathfrak{L}^P_{\overline{\xi}}s \\ &\Leftrightarrow \iota_{\overline{\xi}}ds + \dot{\rho}(\iota_{\xi}\omega)s - (\iota_{\overline{\xi}}ds + d\iota_{\overline{\xi}}s) \\ &\Leftrightarrow \iota_{\overline{\xi}}ds + 0 - \iota_{\overline{\xi}}ds - 0 = 0\end{aligned}$$

Thus, a basic connection on E, one which satisfies $\nabla_{\xi} - \mathfrak{L}^E_{\xi} = 0$, is one induced from a basic connection on P.

Appendix E

Equivariant K-theory with $\mathbb{C}/\mathbb{Z}$-coefficients

E.1 Relative K-theory

Given a continuous map $g : A \to B$ of compact Hausdorff topological spaces, we can think of the K-theory of B relative to A (with respect to g) as the reduced K-theory of the mapping cone $C(g) = B \cup_g CA$. The cofiber sequence of g gives the usual long exact sequence

$$\widetilde{K}^0(A) \longleftarrow \widetilde{K}^0(B) \longleftarrow \widetilde{K}^0(C(g)) \longleftarrow \widetilde{K}^{-1}(A) \longleftarrow \widetilde{K}^{-1}(B) \dots \tag{E.1}$$

relating $\widetilde{K}^\bullet(A), \widetilde{K}^\bullet(B)$ and $\widetilde{K}^\bullet(C(g))$. Another helpful description of the relative K-theory is as equivalence classes of triples (E, E', φ) where $E, E' \to B$ are vector bundles and $\varphi : g^*E \to g^*E'$ is an isomorphism. This is an obvious generalization of the situation in which g is an inclusion.

Definition E.1.1. Given $g : A \to B$, let $C(B, A, g)$ be the set of triples (E, E', φ) where $E, E' \to B$ are vector bundles and $\varphi : g^*E \to g^*E'$ is an isomorphism. An isomorphism of triples $\mathcal{E} \cong \mathcal{F} = (F, F', \psi)$ is a pair of isomorphisms $E \to E'$ and $F \to F'$ which make the diagram

$$\begin{array}{ccc} g^*E & \xrightarrow{\varphi} & g^*E' \\ \downarrow & & \downarrow \\ g^*F & \xrightarrow[\psi]{} & g^*F' \end{array}$$

commute. An elementary triple is one of the form $\mathcal{P} = (P, P, Id)$. Let $\sim$ be the equivalence relation $\mathcal{E} \sim \mathcal{F}$ if and only if there exist elementary triples $\mathcal{P}, \mathcal{Q}$ such that

$$\mathcal{E} \oplus \mathcal{P} \cong \mathcal{F} \oplus \mathcal{Q}.$$

Let $K^0(B, A, g)$ be the semi-group of equivalence classes under direct sum. Since every vector bundle has a complement, we can represent any element of $K^0(B, A, g)$ by a triple in which one of the bundles is trivial.

An element $(E, \underline{\mathbb{C}}^k, \varphi) \in C(B, A, g)$ defines a vector bundle $E \cup_\varphi \underline{\mathbb{C}}^k$ on $C(g) = B \cup_g CA$: it is $E \to B$ and the trivial bundle $\underline{\mathbb{C}}^k \to CA$ glued over the base of the cone $A \subset CA$ via the trivialization $\varphi : g^* E \to \underline{\mathbb{C}}^k$. Define

$$p : K^\bullet(B, A, g) \longrightarrow \widetilde{K}^\bullet(C(g))$$

by

$$[E, \underline{\mathbb{C}}^k, \varphi] \longmapsto [E \cup_\varphi \underline{\mathbb{C}}^k] - [\underline{\mathbb{C}}^k]$$

Observe that when g is an inclusion, $K^\bullet(B, A, g) \cong \widetilde{K}^\bullet(B/A)$ and p is pullback by the quotient map $B \cup CA \to B \cup CA/CA \cong B/A$ in which case it is an isomorphism.

Proposition E.2. *The map p is an isomorphism of semi-groups.*

Proof. We construct an inverse

$$q : \widetilde{K}^\bullet(C(g)) \longrightarrow K^\bullet(B, A, g)$$

to p as follows. Write the cone on A as

$$CA = A \times [0,1]/\{(a,0) \sim *\}$$

and let $pt \in CA$ be the cone point. Let

$$V = \{(a,t) \in CA \mid t \leqslant 1/2\}$$

be a closed neighborhood of pt. By Excision

$$\widetilde{K}^\bullet(C(g)) = K^\bullet(B \cup_g CA, pt) \cong K^\bullet(B \cup_g CA, V) \cong K^\bullet(B \cup_g CA \backslash \{pt\}, V \backslash \{pt\})$$

An element $\mathcal{E} \in K^\bullet(B \cup_g CA \backslash \{pt\}, V \backslash \{pt\})$ is represented by a triple $(E, \underline{\mathbb{C}}^k, \varphi)$ where

$$\varphi : E|_{V\backslash\{pt\}} \longrightarrow \underline{\mathbb{C}}^k|_{V\backslash\{pt\}}$$

is an isomorphism. Since $CA\backslash\{pt\}$ deformation retracts to A, we may assume that E is constant along the cone, that is, that $E|_{CA\backslash\{pt\}}$ is pulled back from the base $A \subset CA$; if not, we can choose an isomorphism to a bundle that is pulled back and change φ accordingly to obtain an equivalent triple. Since E is a bundle on the punctured mapping cone, the pullback of E from B to $A \subset CA$ is the restriction of E to $A \subset CA$, that is, $g^*E|_B = E|_{A \subset CA}$. Now, identifying A with $A \times \{1/2\} \subset CA$ we have

$$E|_{A\times\{1/2\}} = E_{A\times\{0\}} = g^*E|_B$$

so restricting φ to $A \times \{1/2\} \subset V$ gives an isomorphism

$$\varphi' := \varphi|_{A\times\{1/2\}} : g^*E|_B \longrightarrow \underline{\mathbb{C}}^k$$

We thus define q by

$$\begin{array}{rcl} q : \widetilde{K}^\bullet(C(g)) & \xrightarrow{\cong} K^\bullet(C(g)\backslash\{pt\}, V\backslash\{p\}) \longrightarrow & K^\bullet(B, A, g) \\ [E] - [\underline{\mathbb{C}}^k] \longmapsto & [E, \underline{\mathbb{C}}^k, \varphi] \longmapsto & [E|_B, \underline{\mathbb{C}}^k, \varphi'] \end{array}$$

Now, $qp = Id$. Going the other way, since we assumed that E was constant along the cone, φ extends to the whole cone to give an isomorphism

$$Id \cup \varphi^{-1} : E|_B \cup_{\varphi'} \underline{\mathbb{C}}^k \longrightarrow E$$

from which it follows that $pq = Id$. □

Remark E.3. Giving $K^0(B, A, g)$ the group structure inherited from p, it is a group and p is an isomorphism of groups.

A homotopy between two elements of $C(B, A, g)$ is an element of $C(B\times I, A \times I, g \times Id)$ which restricts to the two given elements at the ends.

Proposition E.4. *Homotopic elements of* $C(B, A, g)$ *define the same element of* $K^0(B, A, g)$.

This follows from the homotopy invariance of $\widetilde{K}^0(C(g))$.

We will frequently appeal to this description of relative K-theory, even when the map is an inclusion. We will see that this gives a useful perspective on K-theory with $\mathbb{Z}/n\mathbb{Z}$-coefficients.

E.2 K-theory with $\mathbb{Z}/n\mathbb{Z}$-coefficients

Let $f_n : S^1 \to S^1$ be the map $z \mapsto z^n$ and let Cf_n be the mapping cone of f_n. That is,

$$CS^1 := (S^1 \times I)/(S^1 \times \{0\})$$

and

$$Cf_n := (CS^1 \sqcup S^1)/\{(t,1) \sim f_n(t)\} = CS^1 \cup_{f_n} S^1.$$

For X a topological space without basepoint we define

$$K^0(X;\mathbb{Z}/n\mathbb{Z}) := K^0(X \times Cf_n, X \times \{pt\}),$$

$$K^{-j}(X;\mathbb{Z}/n\mathbb{Z}) := K^0(X \times C\Sigma^j f_n, X \times \{pt\}).$$

and

$$K^j(X;\mathbb{Z}/n\mathbb{Z}) := K^{j-2}(X;\mathbb{Z}/n\mathbb{Z})$$

for $j \geqslant 1$. To understand this definition, observe that the cofiber sequence for f_n yields a sequence

$$X \times S^1 \xrightarrow{1_X \times f_n} X \times S^1 \rightarrow X \times Cf_n \rightarrow X \times S^2 \xrightarrow{1_X \times \Sigma f_n} X \times S^2 \cdots \tag{E.5}$$

For X a space without basepoint, $X^+ = X \sqcup *$ and S^n the n-sphere with a fixed base point pt,

$$K^{-n}(X) := \widetilde{K}^0(S^n(X^+)) \cong K^0(X \times S^n, X \times \{pt\})$$

It follows that applying $K^0(-, X \times \{pt\})$ to the sequence of spaces (E.5) (to the right of $X \times Cf_n$) yields a long exact sequence

$$K^0(X;\mathbb{Z}/n\mathbb{Z}) \xleftarrow{\delta} K^0(X;\mathbb{Z}) \xleftarrow{n} K^0(X;\mathbb{Z}) \leftarrow K^{-1}(X;\mathbb{Z}/n\mathbb{Z}) \xleftarrow{\delta} \cdots \tag{E.6}$$

The "reduction mod n" map δ is the connecting homomorphism of the long exact sequence. We will make explicit the construction shortly.

Observe also that there are homeomorphisms

$$(X \times Cf_n)/X \times \{pt\} \cong C(X \times S^1) \sqcup X \times S^1/\{(x,t,1) \sim (x, f_n(t))\} = C(1_X \times f_n)$$

and more generally

$$(X \times C\Sigma^j f_n)/X \times \{pt\} \cong C(1_X \times \Sigma^j f_n)$$

so that

$$K^{-j}(X;\mathbb{Z}/n\mathbb{Z}) \cong K^{-j}(C(1_X \times \Sigma^j f_n), pt) \cong \widetilde{K}^{-j}(C(1_X \times \Sigma^j f_n))$$

Thus,

$$K^0(X;\mathbb{Z}/n\mathbb{Z}) \cong \widetilde{K}^0(C(1_X \times f_n)) \cong K^0(X \times S^1, X \times S^1, 1_X \times f_n)$$

so elements of $K^0(X;\mathbb{Z}/n\mathbb{Z})$ are equivalence classes of triples (E, E', φ) where E, E' are vector bundles over $X \times S^1$ and

$$\varphi : (1_X \times f_n)^* E \longrightarrow (1_X \times f_n)^* E'$$

is an isomorphism. Similarly, elements of $K^{-1}(X;\mathbb{Z}/n\mathbb{Z})$ are equivalences classes of triples (F, F', ψ) where F, F' are vector bundles over $X \times S^2$ and

$$\psi : (1_X \times \Sigma f_n)^* F \longrightarrow (1_X \times \Sigma f_n)^* F'$$

is an isomorphism.

Now let H be the Hopf bundle over S^2 and let $\xi = [H] \in K^0(S^2)$. Then

$$\xi^n - 1 = n(\xi - 1)$$

so

$$\xi^n + (n-1) = n\xi.$$

It follows that the bundles $H^{\otimes n} \oplus (n-1)\underline{\mathbb{C}}$ and nH are stably equivalent. Since they have the same degree and rank there exists an isomorphism

$$H^{\otimes n} \oplus (n-1)\underline{\mathbb{C}} \to nH. \tag{E.7}$$

Fix such an isomorphism α_n for each $n \geqslant 2$ and identify $(\Sigma f_n)^* H$ with $H^{\otimes n}$. Let E, E' be bundles over X and α an isomorphism $nE \to nE'$. Let

$$V := E \otimes (H \oplus (n-1)\underline{\mathbb{C}}) \quad \text{and} \quad V' := E' \otimes (H \oplus (n-1)\underline{\mathbb{C}})$$

Then V, V' are bundles over $X \times S^2$ and writing $g = 1_X \times \Sigma f_n$,

$$g^* V = E \otimes (H^{\otimes n} \oplus (n-1)\underline{\mathbb{C}}) \xrightarrow{Id \otimes \alpha_n} E \otimes nH \cong nE \otimes H,$$

similarly for $g^* V'$, and $\alpha : nE \to nE'$ yields a definite isomorphism $g^* V \to g^* V'$. We have just shown that

Proposition E.8 (Atiyah-Patodi-Singer). *A pair of bundles $E, E' \to X$ with an isomorphism $\alpha : nE \to nE'$ defines an element of $K^{-1}(X; \mathbb{Z}/n\mathbb{Z})$.*

We now describe the "reduction mod n" map. Let $A = B = X \times S^2$ and let $g = 1_X \times \Sigma f_n$ and consider the diagram

$$\begin{array}{ccccc}
 & & \widetilde{K}^0(B \cup_g CA) & & \\
 & \overset{m^*}{\nearrow} & \cong \uparrow p & \overset{k^*}{\searrow} & \\
K^0(B \cup_g CA, B) & \xrightarrow[h]{} & K^0(B, A, g) & \xrightarrow[j]{} & K^0(B) \\
\cong \downarrow \theta & & & & \\
K^{-1}(A) = \widetilde{K}^0(S(A^+)) & & & &
\end{array}$$

1. θ is the isomorphism induced by the quotient map $B \cup_g CA/B \cong SA$,

2. m^*, k^* are induced by the obvious inclusions,

3. p is the isomorphism described in the previous section

4. $h = qm^*$ is defined to make the triangle commute,

5. j is the forgetful map $[E, E', \varphi] \mapsto [E] - [E']$.

Since k^* is restriction to $B \subset B \cup_g CA$, it is clear that the right-hand triangle commutes. The construction of h and the fact that the row is exact now follows the standard construction of the connecting homomorphism in the long exact sequence of a pair (when the map is inclusion, see [3] Proposition 2.4.4)

Let $i : B \hookrightarrow C(g) = B \cup_g CA$ be inclusion and identify

$$K^0(B \cup_g CA, B) = K^0(B \cup_g CA, B, i).$$

In this description, m^* is the map "forget the isomorphism over B"; it is then clear that $jh = 0$ since $k^*m^* = 0$: it takes bundles on $C(g)$ with an isomorphism over B to the difference of their restrictions to B. To see that $\ker j = \operatorname{im} h$, suppose that $\mathcal{E} \in K^0(B, A, g)$ and $j(\mathcal{E}) = 0$. If $(E, \underline{\mathbb{C}}^k, \varphi)$ represents $\mathcal{E}$ then $[E] - [\underline{\mathbb{C}}^k] = 0$ in $K^0(B)$. Thus, there exists an isomorphism

$$\psi : E \oplus \underline{\mathbb{C}}^r \longrightarrow \underline{\mathbb{C}}^{k+r}$$

of bundles over B. There is a canonical isomorphism $g^*(E \oplus \underline{\mathbb{C}}^r) \to g^*E \oplus \underline{\mathbb{C}}^r$; let

$$\tilde{\varphi} = \varphi \oplus Id : g^*E \oplus \underline{\mathbb{C}}^r \xrightarrow{\cong} \underline{\mathbb{C}}^{k+r}.$$

Then

$$[E \oplus \underline{\mathbb{C}}^r, \underline{\mathbb{C}}^{k+r}, \tilde{\varphi}] = \mathcal{E} \in K^0(B, A, g).$$

Set

$$\tilde{\mathcal{E}} = [E \oplus \underline{\mathbb{C}}^r \cup_{\tilde{\varphi}} \underline{\mathbb{C}}^{k+r}, \underline{\mathbb{C}}^{k+r}, \psi] \in K^0(B \cup_g CA, B, i).$$

Then identifying $\tilde{K}^0(B \cup_g CA)$ with $K^0(B \cup_g CA \backslash \{pt\}, V \backslash \{pt\})$ by the Excision isomorphism

$$m^* \tilde{\mathcal{E}} = [E \oplus \underline{\mathbb{C}}^r \cup_{\tilde{\varphi}} \underline{\mathbb{C}}^{k+r}, \underline{\mathbb{C}}^{k+r}, Id] \in K^0(B \cup_g CA \backslash \{pt\}, V \backslash \{pt\})$$

so

$$h(\tilde{\mathcal{E}}) = qm^*(\tilde{\mathcal{E}}) = [E \oplus \underline{\mathbb{C}}^r, \underline{\mathbb{C}}^{k+r}, \tilde{\varphi}] = \mathcal{E} \in K^0(B, A, g).$$

In short, h is given by

$$h[E \oplus \underline{\mathbb{C}}^r \cup_{\tilde{\varphi}} \underline{\mathbb{C}}^{k+r}, \underline{\mathbb{C}}^{k+r}, \psi] = [E \oplus \underline{\mathbb{C}}^r, \underline{\mathbb{C}}^{k+r}, \tilde{\varphi}].$$

Recalling that $A = B = X \times S^2$, the "reduction mod n" map is thus the composition

$$\delta : K^{-1}(X; \mathbb{Z}) \longrightarrow K^{-1}(A) \xrightarrow{\theta^{-1}} K^0(B \cup_g CA, B) \xrightarrow{h} K^0(B, A, g) = K^{-1}(X; \mathbb{Z}/n\mathbb{Z})$$

where the first map is the Bott periodicity isomorphism.

We have seen that if E and E' are vector bundles over X such that there exists an isomorphism $\varphi : nE \to nE'$, then (E, E', φ) defines an element of $K^{-1}(X; \mathbb{Z}/n\mathbb{Z})$. If we choose a different isomorphism $\psi : nE \to nE'$, then (E, E', ψ) defines another element of $K^{-1}(X; \mathbb{Z}/n\mathbb{Z})$. These two elements are not in general the same. We identify their difference as the reduction mod n of a specific integral class.

Let $\gamma : nE \to nE$ be an automorphism such that $\psi = \varphi\gamma$. Observe that $\varphi\gamma \oplus Id$ and $\varphi \oplus \gamma$ are both isomorphisms $nE \oplus nE \to nE' \oplus nE$. An explicit homotopy $\varphi\gamma \oplus Id \simeq \varphi \oplus \gamma$ is given by

$$\begin{pmatrix} \varphi\cos^2(\pi t/2) + \varphi\gamma\sin^2(\pi t/2) & \sin(\pi t/2)\cos(\pi t/2)(\varphi\gamma - \varphi) \\ \sin(\pi t/2)\cos(\pi t/2)(\gamma - Id) & \sin^2(\pi t/2) + \gamma\cos^2(\pi t/2) \end{pmatrix}$$

For this to be an isomorphism of bundles for all $t \in [0,1]$ it must be an isomorphism on every fiber. Let $U \subset X$ be an open set on which both bundles are trivializable and choose trivializations $nE|_U \to U \times \mathbb{C}^r$ and $nE'|_U \to U \times \mathbb{C}^r$. Let $\varphi_U, \gamma_U : U \times \mathbb{C}^r \to U \times \mathbb{C}^r$ be φ and γ in these trivializations, then $\varphi_U(x), \gamma_U(x)$ are isomorphisms for each $x \in U$ and over U, the given homotopy factors as the composition of

$$\begin{pmatrix} \varphi_U & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos(\pi t/2) & \sin(\pi t/2) \\ -\sin(\pi t/2) & \cos(\pi t/2) \end{pmatrix} \tag{E.9}$$

and

$$\begin{pmatrix} 1 & 0 \\ 0 & \gamma_U \end{pmatrix} \begin{pmatrix} \cos(\pi t/2) & -\sin(\pi t/2) \\ \sin(\pi t/2) & \cos(\pi t/2) \end{pmatrix} \tag{E.10}$$

which makes clear that it is an isomorphism for all $t \in [0,1]$.

Now, let

$$V = E \otimes (H \oplus (n-1)\underline{\mathbb{C}}) \quad \text{and} \quad V' = E' \otimes (H \oplus (n-1)\underline{\mathbb{C}})$$

be the bundles over $X \times S^2$ corresponding to E and E'. Identify g^*V with $nE \otimes H$ by the chosen isomorphism $1_E \otimes \alpha_n$ and similarly identify g^*V' with

$nE' \otimes H$. Then

$$\widetilde{\varphi} = \varphi \otimes 1_H : g^*V \longrightarrow g^*V'$$

and

$$[V, V', \widetilde{\varphi}] \in K^{-1}(X; \mathbb{Z}/n\mathbb{Z})$$

is the element determined by (E, E, φ). Similarly, $[V, V', \widetilde{\psi}]$ is the element determined by (E, E', ψ) where $\widetilde{\psi} = \psi \otimes 1_H$. The automorphism γ also induces an automorphism $\widetilde{\gamma} = \gamma \otimes 1_H : g^*V \to g^*V$ and thus an element $[V, V, \widetilde{\gamma}] \in K^{-1}(X; \mathbb{Z}/n\mathbb{Z})$. The homotopy $\varphi \oplus \gamma \simeq \varphi\gamma \oplus Id$ gives a homotopy $\widetilde{\varphi} \oplus \widetilde{\gamma} \simeq \widetilde{\varphi}\widetilde{\gamma} \oplus Id$. By the homotopy invariance of $K^{-1}(X; \mathbb{Z}/n\mathbb{Z})$ we see that

$$\begin{aligned}
[V, V', \widetilde{\varphi}] + [V, V, \widetilde{\gamma}] &= [V \oplus V, V' \oplus V, \widetilde{\varphi} \oplus \widetilde{\gamma}] \\
&= [V \oplus V, V' \oplus V, \widetilde{\varphi}\widetilde{\gamma} \oplus Id] \\
&= [V, V', \widetilde{\varphi}\widetilde{\gamma}] + [V, V, Id] \\
&= [V, V', \widetilde{\varphi}\widetilde{\gamma}] \\
&= [V, V', \widetilde{\psi}]
\end{aligned}$$

We now construct an element of $K^{-1}(X; \mathbb{Z})$ of which $[V, V, \widetilde{\gamma}]$ is the reduction mod n. Let $p : M \times S^1 \to M$ be projection. The automorphism $\gamma : nE \to nE$ determines a bundle $V_\gamma = (nE)_\gamma \to M \times S^1$ thus an element

$$[V_\gamma] - [p^*nE] \in K^{-1}(X; \mathbb{Z}).$$

Under the Bott periodicity isomorphism,

$$\begin{aligned}
K^{-1}(X; \mathbb{Z}) &\longrightarrow K^{-1}(A) \\
[V_\gamma] - [p^*nE] &\longmapsto [V_\gamma \otimes H] - [p^*nE \otimes H] - ([V_\gamma] - [p^*nE])
\end{aligned}$$

Analogously to the previous construction, this is the difference of two elements: the one obtained from the bundle $nE \otimes H \to X \times S^2$ and the automorphism $\gamma \otimes 1_H$ and the one obtained as before from (the pullback of) $nE \to X \times S^2$ and the automorphism γ. From this description, it is not hard to see that

$$\begin{array}{ccc} K^{-1}(A) & \xrightarrow{\qquad h\theta^{-1} \qquad} & K^0(B, A, g) \\ [V_\gamma \otimes H] - [p^* nE \otimes H] - ([V_\gamma] - [p^* nE]) & \longmapsto & [V, V, \tilde{\gamma}] - [p^* nE, p^* nE, \gamma] \end{array}$$

where V is as above. For the given bundles $nE \otimes H$ and nE over $A = X \times S^2$, we just have to find bundles over $B = X \times S^2$ of which these are the pullbacks via g. The intermediate step that we have skipped is to find complements to the given bundles on A so that we can extend them over the cone CA. In $K^0(B, A, g)$, the bundles on A need not extend over the cone so we can just subtract off these extra terms again. Finally, the last term $[nE, nE, \gamma]$ is zero in $K^{-1}(X; \mathbb{Z}/n\mathbb{Z})$: since $g^* nE = nE$, the triple (nE, nE, γ) is equivalent to the triple (nE, nE, Id) as illustrated by the diagram

$$\begin{array}{ccc} nE & \xrightarrow{\gamma} & nE \\ \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle Id} \\ nE & \xrightarrow[Id]{} & nE \end{array}$$

Therefore,

$$[V, V', \widetilde{\psi}] - [V, V', \tilde{\varphi}] = [V, V, \tilde{\gamma}] \in K^{-1}(X; \mathbb{Z}/n\mathbb{Z})$$

is the reduction mod n of

$$[V_\gamma] - [p^* nE] \in K^{-1}(X; \mathbb{Z}). \tag{E.11}$$

E.3 K-theory with $\mathbb{C}/\mathbb{Z}$-coefficients

The rational numbers $\mathbb{Q}$ can be constructed as the colimit of the diagram of abelian groups $A_n = \mathbb{Z}$ for $n \in \mathbb{Z}^{>0}$ where, if $m = nk$, there is a unique map $f_{nm} : A_n \to A_m$ which is multiplication by k. $K^\bullet(X;\mathbb{Q})$ is defined as the colimit of the analogous diagram with $A_n = K^\bullet(X;\mathbb{Z})$ in which the unique map $A_n \to A_m$ is again multiplication by k. The maps

$$\begin{aligned} Q_n : A_n &\longrightarrow K^\bullet(X) \otimes \mathbb{Q} \\ a &\longmapsto a \otimes \frac{1}{n} \end{aligned} \tag{E.12}$$

induce an isomorphism

$$Q : K^\bullet(X;\mathbb{Q}) \longrightarrow K^\bullet(X) \otimes \mathbb{Q} \tag{E.13}$$

We identify $K^\bullet(X;\mathbb{Q})$ with $K^\bullet(X) \otimes \mathbb{Q}$ under this isomorphism. Similarly, $\mathbb{Q}/\mathbb{Z}$ can be constructed as the colimit of the diagram with $A_n = \mathbb{Z}/n\mathbb{Z}$ and if $m = nk$ there is a unique map $A_n \to A_m$ which is multiplication by k. $K^\bullet(X;\mathbb{Q}/\mathbb{Z})$ is then defined as the colimit of the analogous diagram with $A_n = K^\bullet(X;\mathbb{Z}/n\mathbb{Z})$. The reduction mod $\mathbb{Z}$ map

$$\rho : K^\bullet(X;\mathbb{Q}) \longrightarrow K^\bullet(X;\mathbb{Q}/\mathbb{Z})$$

is the map induced on colimits from the morphism of diagrams which on the n^{th} group is reduction mod n. Setting $K^\bullet(M;\mathbb{C}) := K^\bullet(M) \otimes \mathbb{C}$ there is the obvious injection

$$i : K^\bullet(X;\mathbb{Q}) \hookrightarrow K^\bullet(X;\mathbb{C})$$

Together, these yield a map

$$(\rho, -i) : K^\bullet(X;\mathbb{Q}) \longrightarrow K^\bullet(X;\mathbb{Q}/\mathbb{Z}) \oplus K^\bullet(X) \otimes \mathbb{C}.$$

We define

$$K^\bullet(X;\mathbb{C}/\mathbb{Z}) = \operatorname{coker}(\rho, -i)$$

When $M = pt$, it is not hard to see that in even degrees $(\rho, -\iota)$ is an injective map $\mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \oplus \mathbb{C}$ with cokernel $\mathbb{C}/\mathbb{Z}$ and the groups are all zero in odd degrees.

If $X = M$ is a smooth manifold, since $\mathrm{Ch} : K^\bullet(M) \otimes \mathbb{C} \to H(M;\mathcal{R})^\bullet$ is an isomorphism, we can make the same construction with

$$\iota = \mathrm{Ch} \circ i : K^\bullet(M;\mathbb{Q}) \hookrightarrow H_{\mathscr{D}}(M)^\bullet.$$

Then

$$\operatorname{coker}(\rho, -\iota) \cong K^\bullet(M;\mathbb{C}/\mathbb{Z}).$$

This is the model we use.

E.4 $\mathbb{T}$-equivariant K-theory with $\mathbb{C}/\mathbb{Z}$ coefficients

We proceed by direct analogy with the non-equivariant case. There are no new constructions, we merely indicate what we used at each step of the previous construction and why the same works in the equivariant setting. If X is a $\mathbb{T}$-space and Y is any space, we make $X \times Y$ a $\mathbb{T}$-space with trivial action on the second factor. The formulation of relative cohomology for any equivariant map $g : A \to B$ of $\mathbb{T}$-spaces goes through as before as does the isomorphism $K^0_{\mathbb{T}}(B, A, g) \cong \widetilde{K}^0_{\mathbb{T}}(C(g))$. For $f_n : S^1 \to S^1$ and Cf_n as above, we define $\mathbb{T}$-equivariant K-theory with $\mathbb{Z}/n\mathbb{Z}$ coefficients by

$$K^0_{\mathbb{T}}(X; \mathbb{Z}/n\mathbb{Z}) := K^0(X \times Cf_n, X \times \{pt\})$$

and

$$K^{-j}_{\mathbb{T}}(X; \mathbb{Z}/n\mathbb{Z}) := K^0(X \times C\Sigma^j f_n, X \times \{pt\})$$

for $j > 0$ and set $K^j_{\mathbb{T}}(X; \mathbb{Z}/n\mathbb{Z}) = K^{j-2}_{\mathbb{T}}(X; \mathbb{Z}/n\mathbb{Z})$ for $j > 1$. We see that $K^0_{\mathbb{T}}(X; \mathbb{Z}/n\mathbb{Z})$ is represented by triples (E, E', φ) where $E, E' \to X \times S^1$ are equivariant vector bundles and

$$\varphi : (1_X \times f_n)^* E \longrightarrow (1_X \times f_n)^* E'$$

is an isomorphism. Elements of $K^{-1}(X; \mathbb{Z}/n\mathbb{Z})$ are similarly represented by pairs of bundles over $X \times S^2$ with an isomorphism of pullbacks. Since Bott periodicity holds in equivariant K-theory, the same argument shows that a pair of bundles $E, E' \to X$ with an isomorphism $nE \to nE'$ defines an element of $K^{-1}_{\mathbb{T}}(X; \mathbb{Z}/n\mathbb{Z})$. If $E, E' \to X$ are two bundles and $\varphi, \psi : nE \to nE'$ are two isomorphisms, write $\psi = \varphi\gamma$ where $\gamma : nE \to nE$ is an automorphism. As before, let $V = E \otimes (H \oplus (n-1)\mathbb{C})$ and $V' = E' \otimes (H \oplus (n-1)\mathbb{C})$ be the corresponding bundles over $X \times S^2$ and let $\widetilde{\varphi} = \varphi \otimes 1_H$ and similarly for $\widetilde{\psi}$ and $\widetilde{\gamma}$. Let $\pi : X \times I \to X$ and $p : X \times S^1 \to X$ be projections and let $V_\gamma = \pi^* nE/\gamma \to X \times S^1$ as before. Then the same argument shows that

$$[V, V', \widetilde{\psi}] - [V, V', \widetilde{\varphi}] = [V, V, \widetilde{\gamma}] \in K^{-1}_{\mathbb{T}}(X; \mathbb{Z}/n\mathbb{Z})$$

is the reduction mod n of

$$[V_\gamma] - [p^* nE] \in K_{\mathbb{T}}^{-1}(X; \mathbb{Z}).$$

Equivariant K-theory with $\mathbb{Q}$ and $\mathbb{Q}/\mathbb{Z}$ coefficients are defined as the analogous colimits and equivariant K-theory with $\mathbb{C}/\mathbb{Z}$ coefficients as the analogous cokernel

$$K_{\mathbb{T}}^{\bullet}(X; \mathbb{C}/\mathbb{Z}) := \{K_{\mathbb{T}}^{\bullet}(X; \mathbb{Q}/\mathbb{Z}) \oplus K_{\mathbb{T}}^{\bullet}(X) \otimes \mathbb{C}\}/K_{\mathbb{T}}^{\bullet}(X) \otimes \mathbb{Q}$$

When $X = M$ is a compact smooth $\mathbb{T}$-manifold, since $\mathrm{Ch}_{\mathscr{D}} : K^{\bullet}(M; \mathbb{C}) \to H_{\mathscr{D}}(M; \mathcal{R})^{\bullet}$ is an isomorphism, we again take

$$K_{\mathbb{T}}^{\bullet}(M; \mathbb{C}/\mathbb{Z}) \cong \{K_{\mathbb{T}}^{\bullet}(X; \mathbb{Q}/\mathbb{Z}) \oplus H_{\mathscr{D}}(M)^{\bullet}\}/K_{\mathbb{T}}^{\bullet}(X) \otimes \mathbb{Q}.$$

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Vita

Mio Ilan Alter was born in New York City on November 22nd, 1979, the son of Herbert H. Alter and Dr. Ann Ilan Alter. He graduated from the Bronx High School of Science, then attended the University of Chicago, where he received a Bachelor of Arts in Cinema Studies in 2002. Returning to New York, he worked at Anthology Film Archives and the notorious Kim's Video before enrolling at Columbia University. At Columbia, he studied Physics and Mathematics, completing the equivalent of a Bachelor of Science in Mathematics in 2007. He began his graduate studies in Mathematics at the University of Texas at Austin in August, 2007.

Permanent address: 1170 Bedford street
Austin, TX 78702

This dissertation was typeset with LaTeX[†] by the author.

[†]LaTeX is a document preparation system developed by Leslie Lamport as a special version of Donald Knuth's TeX Program.