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**On the blow-up of four-dimensional  
Ricci flow singularities**

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**On the blow-up of four-dimensional  
Ricci flow singularities**

by

**Davi Máximo Alexandrino Nogueira, B.S., M.S.**

**DISSERTATION**

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To my wife, Ilane, and my parents, Vicente & Heloisa,  
with love and admiration.

But his will is in the law of the Lord, and on his law he shall meditate day and night. And he shall be like a tree which is planted near the running waters, which shall bring forth its fruit, in due season. And his leaf shall not fall off: and all whatsoever he shall do shall prosper.

Psalms 1:2-3

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# On the blow-up of four-dimensional Ricci flow singularities

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In 2002, Feldman, Ilmanen, and Knopf [15] constructed the first example of a non-trivial (i.e. non-constant curvature) complete non-compact shrinking soliton, and conjectured that it models a Ricci flow singularity forming on a closed four-manifold. In this thesis, we confirm their conjecture and, as a consequence, show that limits of blow-ups of Ricci flow singularities on closed four-dimensional manifolds do not necessarily have non-negative Ricci curvature.

# Table of Contents

<b>Acknowledgments</b>	<b>vi</b>
<b>Abstract</b>	<b>viii</b>
<b>List of Figures</b>	<b>xi</b>
<b>Chapter 1. Introduction</b>	<b>1</b>
1.1 Ricci flow: from basics to singularities . . . . .	3
1.1.1 Ricci flow singularity formation . . . . .	4
1.1.2 Singular analysis . . . . .	6
1.2 Kähler geometry and Kähler-Ricci flow . . . . .	9
1.3 Summary of results . . . . .	10
<b>Chapter 2. Proof of Theorem A</b>	<b>15</b>
2.1 $U(2)$ -invariant Kähler metrics . . . . .	15
2.1.1 Curvature Terms . . . . .	17
2.1.2 The effect of Ricci flow . . . . .	18
2.1.3 Soliton metrics on $L$ and $M$ . . . . .	19
2.2 Dilation Variables, <i>Type-I</i> blow-up, and convergence modulo Diffeomorphisms . . . . .	20
2.2.1 Dilation Variables . . . . .	20
2.2.2 Equations in $\phi, \tau$ variables . . . . .	22
2.2.3 Comparison and convergence modulo diffeomorphisms for a large class of potentials . . . . .	24
2.3 End of proof of Theorem A . . . . .	34
2.4 The cone of metrics with non-negative Ricci curvature . . . . .	36
<b>Appendices</b>	<b>37</b>
<b>Appendix A. The Hamilton-Ivey pinching estimate</b>	<b>38</b>

<b>Appendix B.</b>	<b>Line bundles over <math>\mathbb{C}P^1</math></b>	<b>40</b>
<b>Appendix C.</b>	<b>Comparison principle</b>	<b>42</b>
<b>Bibliography</b>		<b>44</b>

## List of Figures

1.1	A forming singularity . . . . .	5
1.2	Blowing-up a singularity . . . . .	6

# Chapter 1

## Introduction

In recent decades, geometric flows have become an important subject of modern mathematics, especially for their connection with some central themes of differential geometry: (a) understanding spaces in terms of their curvature; (b) finding and classifying optimal geometries; and (c) studying rigidity properties of optimal geometries.

The basic idea behind this subject is to evolve a given geometry through a heat equation-like PDE such that, after a while, it becomes a *better geometry*. Along these lines, a quite successful evolution is *Ricci flow*: given a smooth closed<sup>1</sup> Riemannian manifold  $(M, g_0)$ , one evolves the metric  $g_0$  according to the (weakly) parabolic system

$$\frac{\partial}{\partial t}g = -2\text{Rc}(g), \quad g(0) = g_0, \quad (1.0.1)$$

where  $\text{Rc}(g)$  stands for the Ricci curvature of  $g$ . *Formally*, the Ricci curvature tensor takes the form

$$\text{Rc}(g) = O(g^{-1}\partial^2g + g^{-2}(\partial g)^2),$$

---

<sup>1</sup>Compact without boundary.

and thus Ricci flow resembles a quasilinear heat equation with a diffusion term and a quadratic reaction term. The diffusion term tends to spread the geometry evenly around, while the reaction tends to accumulate it. This competition indicates two possible cases: either the curvature will spread around as the flow converges to a constant curvature metric; or there will be a region of  $M$  where curvature is accumulated and a singularity is formed.

Consider, for example, the very first Ricci flow result:

**Theorem 1** (Hamilton [18]). *Let  $(M, g_0)$  be a closed three-manifold with positive Ricci curvature. Moreover, let  $g(t)$ ,  $t \in [0, T)$ , denote the unique maximal solution to the Ricci flow with initial metric  $g_0$ . Then the rescaled metrics  $\frac{1}{4(T-t)}g(t)$  converge to a metric of constant sectional curvature 1 as  $t \nearrow T$ .*

And from years later, the following higher-dimensional result, known in the literature as the *Sphere theorem* of Brendle and Schoen:

**Theorem 2** (Brendle, Schoen [6]). *Let  $(M, g_0)$  be a closed Riemannian manifold of dimension  $n \geq 4$  with strictly  $1/4$ -pinched sectional curvature in the pointwise sense. Let  $g(t)$ ,  $t \in [0, T)$ , denote the unique maximal solution to the Ricci flow with initial metric  $g_0$ . Then the rescaled metrics  $\frac{1}{2(n-1)(T-t)}g(t)$  converge to a metric of constant sectional curvature 1 as  $t \nearrow T$ .*

Since a manifold with constant sectional curvature 1 must be diffeomorphic to a quotient  $\mathbb{S}^n/\Gamma$  [35], where  $\mathbb{S}^n$  is the standard sphere of  $\mathbb{R}^{n+1}$  and  $\Gamma$  is a finite group of isometries acting freely, the above normalized convergence

results have the immediate non-trivial topological consequence: they classify, respectively, the diffeomorphism type of all closed three-manifold with positive Ricci curvature, and  $n$ -dimensional ( $n \geq 4$ ) closed Riemannian manifolds with strictly  $1/4$ -pinched sectional curvature in the pointwise sense.

*Remark 3.* From a topological standpoint, Theorem 2 is particularly interesting since it is *not* known whether the standard four-sphere admits more than one differentiable structure.

For general metrics, however, it is likely that a local singularity will form. In such cases, a careful singular analysis is usually required before making possible any topological application. We will discuss singularities soon after recalling some basics of Ricci flow analysis.

## 1.1 Ricci flow: from basics to singularities

We start by mentioning the basic results on the analysis of Ricci flow:

- (i) *Short-time existence and uniqueness:* the parabolic theory for Ricci flow states that for any given initial smooth metric  $g_0$  on a closed manifold  $M$ , one can always solve (1.0.1) uniquely (and smoothly) on a short interval of time.
- (ii) *Regularity:* If the flow becomes singular at some finite time  $T < \infty$  then

$$\lim_{t \nearrow T} \max_{x \in M} |\text{Rm}(x, t)| = \infty,$$

where  $\text{Rm}$  denotes the Riemann curvature tensor

In other words, it is always possible to smoothly extend Ricci flow  $g(t)$  past a finite time  $t_0$  as long as the norm of the Riemann curvature tensor remains bounded as  $t \nearrow t_0$ .

With the above regularity result in mind, we next illustrate why we expect Ricci flow to exist only for finite time. Starting from an arbitrary initial data  $(M^n, g)$ , consider the evolution of the *scalar curvature*<sup>2</sup>:

$$\frac{\partial}{\partial t} R = \Delta_{g(t)} R + 2|\text{Rc}(g(t))|^2 \geq \Delta_{g(t)} R + \frac{2}{n} R^2. \quad (1.1.1)$$

By the parabolic maximum principle we see that a finite time singularity is inevitable if the scalar curvature ever becomes everywhere positive.

*Remark 4.* For an initial metric with positive scalar curvature, in particular for any metric satisfying the assumptions of Theorem 1 or Theorem 2, one can see from the above that the Ricci flow can only exist for finite time. Therefore, letting  $g(t)$ ,  $t \in [0, T)$ , denote the unique maximal solution to the Ricci flow with initial metric  $g_0$  in the statements of Theorem 1 or Theorem 2 is actually not an extra assumption, but follows from the analytical theory of Ricci flow.

### 1.1.1 Ricci flow singularity formation

Normalized convergence results as Theorems 1 and 2 rely heavily on the curvature conditions of the initial metric  $g_0$ . In general, without curvature hypothesis on  $g_0$ , one expects *singularities* to form on certain regions of the

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<sup>2</sup>The scalar curvature is defined as the trace of the Ricci curvature

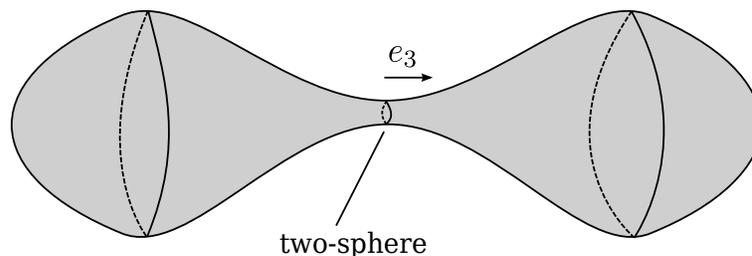


Figure 1.1: A forming singularity

manifold. To get a feel of this process, we start by considering a schematic process of singularity formation called *neckpinch* that was rigorously confirmed by Angenent and Knopf in [4]. We begin with the following “dumbbell” metric on the three-sphere  $\mathbb{S}^3$  and try to imagine its evolution under Ricci flow:

Let  $e_1, e_2, e_3$  be orthonormal vectors at the point indicated in the above picture, with  $e_3$  orthogonal to the indicated cross-sectional two-sphere  $\mathbb{S}^2$ . Note that the sectional curvatures  $K_{e_1 \wedge e_3}$  and  $K_{e_2 \wedge e_3}$  are slightly negative, while  $K_{e_1 \wedge e_2}$  is very positive. Therefore

$$\begin{aligned} \text{Rc}(e_1, e_1) &= K_{e_1 \wedge e_2} + K_{e_1 \wedge e_3} = \text{very positive} \\ \text{Rc}(e_2, e_2) &= K_{e_1 \wedge e_2} + K_{e_2 \wedge e_3} = \text{very positive} \\ \text{Rc}(e_3, e_3) &= K_{e_1 \wedge e_3} + K_{e_2 \wedge e_3} = \text{slightly negative} \end{aligned}$$

Hence, we expect distances to shrink rapidly in the  $e_1$  and  $e_2$  directions, but expand slowly in the  $e_3$  direction. Thus, the evolution wishes to quickly contract the cross-sectional  $\mathbb{S}^2$  indicated, while slowly stretching the neck. At later times, therefore, we expect the  $\mathbb{S}^2$ -neck to pinch and form a local singularity. At this point, we can see that whenever Ricci flow runs into a singularity, we need to have a good understanding about singularity formation before being able to make any topological conclusion.

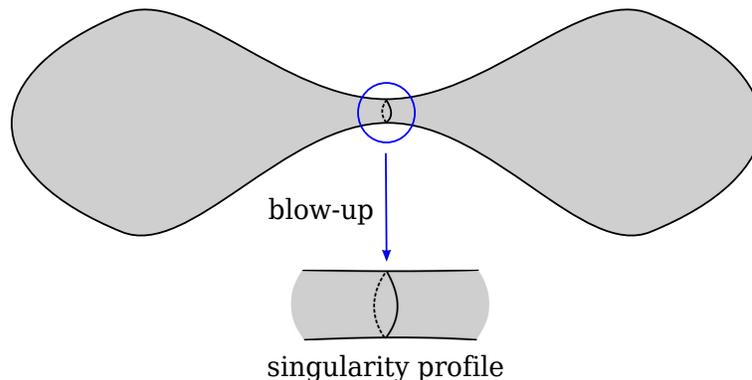


Figure 1.2: Blowing-up a singularity

### 1.1.2 Singular analysis

To analyze Ricci flow singularities, one follows the conventional wisdom of singular analysis from non-linear PDEs and does a *blow-up* at the singularity using the scaling symmetry of the equation. Depending on how much compactness is at hand, one can extract a *singularity model* from a sequence of such blow-ups, which will usually have a better geometry than the original Ricci flow.

Moreover, if one has a good knowledge about the possible singularity models, then one is able to understand the structure of the singularity formation and see how to perform surgery while controlling the geometry and the topology of the manifold, thus arriving at the so-called *Ricci flow with surgeries*.

The above blow-up analysis in three-dimensional Ricci flow has been proved quite successful by the work of Hamilton (*e.g.* see [19]) and Perelman ([27], [29], [28]) on the Poincaré and Geometrization conjectures (see also Cao

and Zhu [11], Morgan and Tian [26], Kleiner and Lott [21], or Tao [33]), and a lot of their theory carries on to higher-dimensional settings, which is an area of active research.

Indeed, for an  $n$ -dimensional Ricci flow  $g(t)$  on a maximal time interval  $[0, T)$  with  $T < \infty$ , it follows from Hamilton-Perelman's theory that one can choose a sequence of points  $p_i \in M$  and times  $t_i \nearrow T$  with

$$\lambda_i = |\text{Rm}|(p_i, t_i) = \sup_{x \in M, t \leq t_i} |\text{Rm}|(x, t) \longrightarrow \infty$$

such that the rescaled flows

$$g_i(t) = \lambda_i g \left( t_i + \frac{t}{\lambda_i} \right)$$

will converge (in a suitable sense, modulo sequential diffeomorphisms and up to subsequence) to a complete Ricci flow  $(N, g_\infty(t))$ , which one calls *singularity model*. Moreover, if the singularity is of *Type-I*, i.e., the curvature blows up like

$$\limsup_{t \nearrow T} \max_M |\text{Rm}|(T - t) < \infty,$$

by the work of Enders, Müller, and Topping in [14] the above limit  $(N, g_\infty(t))$  will be a non-flat *gradient shrinking soliton*, that is, a self-similar Ricci flow, where the metric  $g_\infty$  evolves only by scaling and diffeomorphism. This means there will exist a time dependent function  $f$  defined on  $N$  such that one can obtain the Ricci flow  $g_\infty(t)$  from pullbacks of an initial metric  $g_\infty(0)$  by diffeomorphisms  $\phi_t$  of  $N$  generated by  $\frac{1}{T-t} \nabla f$ , that is,

$$g_\infty(t) = (T - t) \phi_t^* g_\infty(0),$$

and one can check this is the case whenever the soliton equation

$$\mathrm{Rc}(g_\infty) + \nabla\nabla f = \frac{1}{2}g_\infty \tag{1.1.2}$$

is satisfied at some point in time (see Cao [10] for a survey on metrics satisfying the above equation).

In dimension three, the *Hamilton-Ivey pinching* estimate [19, 20] roughly states that if the flow has a region with very negative sectional curvature, then the most positive sectional curvature is much larger still (see Appendix A for a more detailed description). This implies that limits of blow-ups of three-dimensional Ricci flows will have non-negative sectional curvature and thus drastically constrains the singularities that can appear, making three-dimensional Ricci flow with surgeries plausible.

In higher dimensions, one has that such limits will have non-negative scalar curvature by the work of Chen [12], but this type of estimate is lacking for more useful curvature conditions.

One of the main result of this thesis is to prove that a similar estimate *cannot* exist for Ricci curvature in dimension four, *i.e.*, that limits of blow-ups of four-dimensional Ricci flows do not necessarily have non-negative Ricci curvature. We achieve this by solving a conjecture left by Feldman, Ilmanen, and Knopf in [15], which we state soon after some Kähler geometry preliminaries.

## 1.2 Kähler geometry and Kähler-Ricci flow

We next introduce Kähler manifolds and cite their properties that will be used in these thesis.

**Definition 5** (Kähler manifold). A Riemannian manifold  $(M^n, g)$  is called a *Kähler manifold* if it admits a smooth tensor  $J : TM \rightarrow TM$  such that  $J \circ J = -Id$  and  $J$  is compatible with metric  $g$ , *i.e.*,  $g(JX, JY) = g(X, Y)$  and  $\nabla J \equiv 0$ .

One can check from the definition that a Kähler manifold  $(M^n, g, J)$  must have dimension  $n = 2k$  and admit holomorphic coordinates modeled by  $\mathbb{C}^k$ . In addition, both  $g$  and  $Rc$  must be  $J$ -invariant and the 2-forms:

$$\omega(X, Y) = g(JX, Y), \quad \rho(X, Y) = \frac{1}{2}Rc(JX, Y)$$

are closed and real. We call  $\omega$  and  $\rho$  respectively by the *Kähler and Ricci forms*. As a consequence, in local coordinates  $z = (z^1, z^2, \dots, z^k)$ ,  $g$  and  $Rc$  are given by potentials:

$$g_{\alpha\bar{\beta}} = \partial_\alpha \partial_{\bar{\beta}} P \tag{1.2.1}$$

$$Rc_{\alpha\bar{\beta}} = \partial_\alpha \partial_{\bar{\beta}} (-\log \det g), \tag{1.2.2}$$

and the cohomology class  $[\rho] = c_1(M)$  is the first Chern class of  $M$ . In particular,  $[\rho]$  depends only on the underlying complex structure, in spite of the fact that we used the metric  $g$  to construct the Ricci form  $\rho$ .

Now consider a Kähler manifold  $(M^n, g_0, J_0)$  and suppose we are interested in evolving it by Ricci flow<sup>3</sup>

$$\frac{\partial}{\partial t}g = -\text{Rc}(g), \quad g(0) = g_0. \quad (1.2.3)$$

Writing down the above evolution (1.2.3) in local coordinates  $z = (z^1, z^2, \dots, z^k)$ ,  $2k = n$ , using potentials as in (1.2.1) for  $g$  and  $\text{Rc}(g)$ , one can see that (1.2.3) is equivalent to a complex parabolic Monge-Ampère equation, see *e.g.* Cao [8]. A spoil of that is that  $(M^n, g(t), J_0)$  remains Kähler as long as Ricci flow exists and, for that reason, we refer to (1.2.3) as *Kähler-Ricci flow*.

The main results of this thesis are concerned with Kähler-Ricci flow on certain two-dimensional complex manifolds, in other words, Ricci flow on four-dimensional real manifolds that admit a Kähler structure. They are related in spirit to a series of conjectures by Song and Tian [31] on singularity formation of Kähler-Ricci flow.

### 1.3 Summary of results

Let  $M$  be  $\mathbb{C}\mathbb{P}^2$  blown-up at one point and  $L$  be  $\mathbb{C}^2$  blown-up at zero. We invite the reader not familiar with these spaces to see Appendix B before reading what follows.

---

<sup>3</sup>In the context of Kähler geometry, Ricci flow appears in the literature as  $\partial_t g = -\text{Rc}(g)$  instead of the usual  $\partial_t g = -2\text{Rc}(g)$ . This only changes things by scaling.

We will think of  $M$  as  $\mathbb{C}^2 \setminus \{0\}$  with one  $\mathbb{C}\mathbb{P}^1$  glued at 0 (the section  $\Sigma_0$ ) and another at  $\infty$  (the section  $\Sigma_\infty$ ) and of  $L$  as  $\mathbb{C}^2 \setminus \{0\}$  with a  $\mathbb{C}\mathbb{P}^1$  glued only at 0. Both  $M$  and  $L$  are line bundles over  $\mathbb{C}\mathbb{P}^1$ ,  $M$  with line  $\mathbb{C}\mathbb{P}^1$  and  $L$  with line  $\mathbb{C}$ . We consider Kähler metrics on manifold  $M$  and evolve them by Ricci flow.

Let  $g(t)$  be a one-parameter family of Kähler metrics evolving by Kähler-Ricci flow (1.2.3) on  $M$ . The Kähler class  $[\omega(t)]$  of the metric  $g(t)$  will evolve by

$$\partial_t[\omega(t)] = -[\text{Rc}(\omega)] = -c_1(M), \quad (1.3.1)$$

where  $c_1(M)$  is the first Chern class of the complex surface  $M$ . In particular,

$$[\omega(t)] = [\omega(0)] - tc_1(M). \quad (1.3.2)$$

On  $M$ , the cohomology classes of the divisors  $[\Sigma_0]$  and  $[\Sigma_\infty]$  span  $H^{1,1}(M; \mathbb{R})$ , so any Kähler class  $[\omega]$  can be written uniquely as

$$[\omega] = b[\Sigma_\infty] - a[\Sigma_0]$$

for constants  $0 < a < b$ , and the first Chern class satisfies  $c_1(M) = -[\Sigma_0] + 3[\Sigma_\infty]$ . This and equation (1.3.2) give

$$[\omega(t)] = b(t)[\Sigma_\infty] - a(t)[\Sigma_0] \quad (1.3.3)$$

for  $a(t) = a(0) - t$  and  $b(t) = b(0) - 3t$ . Thus, if initially  $b(0) > 3a(0)$ , then  $a(t) \rightarrow 0$  as  $t \nearrow T = a(0)$  and the class  $[\omega(T)]$  will not be Kähler. This will mean that the  $\mathbb{C}\mathbb{P}^1$  of the section  $[\Sigma_0]$  has collapsed to a point and thus Ricci flow must have become singular at a time no later than  $t = T$ .

In [15], Feldman, Ilmanen, and Knopf conjectured that indeed, at least for  $U(2)$ -invariant metrics, if  $b(0) > 3a(0)$  then a *Type-I* singularity will develop along  $\Sigma_0$  precisely at time  $t = T$  and the blow-up limit of such singularity is the gradient shrinking soliton they have constructed on  $L$ , the *FIK soliton*.

*Remark 6.* The FIK soliton was constructed in [15] and was the first example of a non-trivial ( *i.e.* non-constant curvature) complete non-compact shrinking soliton. It has Ricci curvature of mixed sign.

Since [15], a lot of investigation has been done on Kähler-Ricci flow of general Kähler manifolds. Of relevance to the work of this thesis are Tian and Zhang [34], Song and Weinkove [32], and the more recent Song [30].

When restricted to the Kähler-Ricci flow of  $U(2)$ -invariant metrics of  $M$  as above, [34] gives the singular time to be exactly  $T = a(0)$ , and in [32] the authors prove, among other things, that the singularity at  $t = T$  will develop only along  $\Sigma_0$ , with  $g(T)$  being a smooth metric on  $M \setminus \Sigma_0$ . Finally, Song [30] proved that such a singularity is *Type-I*, and using the compactness at hand, he argued that limits of blow ups of the flow will subconverge in the Cheeger-Gromov-Hamilton sense to a complete non-flat gradient shrinking Kähler-Ricci soliton on a manifold homeomorphic to  $\mathbb{C}^2$  blown up at one point. Moreover, the isometry group of this soliton contains the unitary group  $U(n)$ ; but since the complex structure might jump in the limit (see *e.g.* [13]), one is not able to argue using Cheeger-Gromov-Hamilton convergence that this soliton is in fact the FIK soliton.

The main result of this thesis is a proof of the Feldman-Ilmanen-Knopf conjecture for a large set of initial metrics:

**Theorem A.** *Let  $g(t)$  be metrics on  $M$  evolving by Kähler-Ricci flow (1.2.3). For a large open set<sup>4</sup> of  $U(2)$ -invariant initial metrics  $g(0)$  belonging to the Kähler class  $b(0)[\Sigma_\infty] - a(0)[\Sigma_0]$  with  $b(0) > 3a(0)$ , one has the following:*

- (i) *the flow is smooth until it becomes singular at time  $T = a(0)$ ,*
- (ii) *at  $t=T$  the flow develops a Type-I singularity in the region  $\Sigma_0$  and  $g(T)$  is a smooth Riemannian metric on  $M \setminus \Sigma_0$ ,*
- (iii) *parabolic dilations of  $g(t)$  converge uniformly on any parabolic neighborhood<sup>5</sup> of the singular set  $\Sigma_0$  to the evolution of the FIK soliton.*

As we have pointed out, (i) follows from the now standard general result of [34] and (ii) is proved by putting together the results in [32] and [30]. Our proof of (iii) is based on comparison principle techniques applied to the evolving metric potentials, and thus gives a Kähler limit with respect to the same original complex structure. We first prove convergence to the FIK metric in  $C^{0,1}$  topology without making any *Type-I* blow up assumptions. To prove higher regularity without making further restrictions on the class of initial data, we then use the *Type-I* blow up for the scalar curvature proved in [30].

---

<sup>4</sup>See Definition 18 in Section 3 for a precise statement on the set of initial metrics.

<sup>5</sup>This notion is made precise in Section 3 too. See also Remark 23.

*Remark 7.* We point out that comparison principle techniques have also been used to study another type of Ricci flow singularity, *neckpinches*, see Angenent and Knopf [4], [5], Angenent, Caputo, and Knopf [1], Angenent, Isenberg, and Knopf [2, 3], Wu [36], and Gu and Zhu [16].

Theorem A has the following two consequences. Since the FIK soliton has Ricci curvature of mixed sign near  $\Sigma_0$ :

**Theorem B.** *Limits of blow-ups of Ricci flow singularities on closed four-dimensional manifolds do not necessarily have non-negative Ricci curvature.*

Moreover, after constructing a metric on  $M$  with strictly positive Ricci curvature that satisfies the conditions of Theorem A, we will have:

**Corollary 8.** *Positive Ricci curvature is not preserved by Ricci flow in four dimensions or higher.*

The above results shows a contrast between Ricci flow in dimensions three and four. Moreover, Corollary 8 is related to a previous result of the author [25] and Zhang [37], which provides other examples that imply the same result stated in the corollary. Finally, the construction above provides explicit examples of solutions demonstrating the linear instability of the Cao-Koiso soliton [9, 22] that was proved by Hall and Murphy [17].

Theorems A and B of this thesis were proved in the reference [24], which was posted by the author on the arXiv in April of 2012, and has been accepted for publication in the *Journal für die reine und angewandte Mathematik*.

## Chapter 2

### Proof of Theorem A

#### 2.1 $U(2)$ -invariant Kähler metrics

In this section we consider rotationally symmetric Kähler metrics on  $\mathbb{C}^2 \setminus \{0\}$  to derive Kähler metrics on any given Kähler class of the complex surface  $M$ , following an ansatz introduced by Calabi [7, Section 3].

Let  $g$  be a  $U(2)$ -invariant Kähler metric on  $\mathbb{C}^2 \setminus \{0\}$ , the latter with complex coordinates  $z = (z^1, z^2)$ . Define  $u = |z^1|^2$ ,  $v = |z^2|^2$ , and  $w = u + v$ .

Since  $g$  is a Kähler metric and the second de Rham cohomology group  $H^2(\mathbb{C}^2 \setminus \{0\}) = 0$ , by the  $\partial\bar{\partial}$ -lemma one can find a global real smooth function  $P : \mathbb{C}^2 \setminus \{0\} \rightarrow \mathbb{R}$  such that

$$g_{\alpha\bar{\beta}} = \frac{\partial^2}{\partial z^\alpha \partial \bar{z}^\beta} P. \quad (2.1.1)$$

The further assumption of  $g$  being rotationally symmetric allows us to write  $P = P(r)$ , where  $r = \log w$ .<sup>1</sup> We then set  $\varphi(r) = P_r(r)$  (we use a subscript for the derivative since later  $P$  will be regarded as a function of time as well) and compute from (2.1.1)

$$g = [e^{-r} \varphi \delta_{\alpha\beta} + e^{-2r} (\varphi_r - \varphi) \bar{z}^\alpha z^\beta] dz^\alpha d\bar{z}^\beta, \quad (2.1.2)$$

---

<sup>1</sup>Depending on the purpose of the computation, one coordinate might be preferable than the other and we will use both  $r$  and  $w$  in the rest of the paper.

and:<sup>2</sup>

$$\begin{pmatrix} g_{1\bar{1}} & g_{1\bar{2}} \\ g_{2\bar{1}} & g_{2\bar{2}} \end{pmatrix} = \frac{1}{w^2} \begin{pmatrix} v\varphi + u\varphi_r & (\varphi_r - \varphi)\bar{z}^1 z^2 \\ (\varphi_r - \varphi)z^1 \bar{z}^2 & u\varphi + v\varphi_r \end{pmatrix}.$$

Because  $\det(g_{\alpha\bar{\beta}}) = e^{-2r}\varphi\varphi_r$ , one can quickly note that a potential  $P$  on  $\mathbb{C}^2 \setminus \{0\}$  gives rise to a Kähler metric as in (2.1.1) if, and only if,

$$\varphi > 0 \text{ and } \varphi_r > 0. \quad (2.1.3)$$

Given a metric as above, Calabi's Lemma [7, Section 3] tells us that  $g$  will extend to a smooth Kähler metric on the complex surface  $M$  if  $\varphi$  satisfies the following asymptotic properties – henceforth called *Calabi's conditions*:

- (i) There exists positive constants  $a_0$  and  $a_1$  such that  $\varphi$  has the expansion

$$\varphi(r) = a_0 + a_1 w + a_2 w^2 + O(|w|^3) \quad (2.1.4)$$

as  $r \rightarrow -\infty$ ;

- (ii) There exists a positive constant  $b_0$  and a negative constant  $b_1$  such that  $\varphi$  has the expansion

$$\varphi(r) = b_0 + b_1 w^{-1} + b_2 w^{-2} + O(|w|^{-3}) \quad (2.1.5)$$

as  $r \rightarrow \infty$ .

*Remark 9.* We note that  $\varphi_r > 0$  for  $r$  finite, but  $\varphi_r = 0$  for  $r = \pm\infty$ .

Summarizing the above:

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<sup>2</sup>The matrix  $(g)$  is actually a  $4 \times 4$  matrix:  $(g) = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}$ , where  $A = \begin{pmatrix} g_{1\bar{1}} & g_{1\bar{2}} \\ g_{2\bar{1}} & g_{2\bar{2}} \end{pmatrix}$ .

**Lemma 10** (Calabi, [7]). *Any potential  $\varphi$  satisfying conditions (2.1.3), (2.1.4) and (2.1.5) will give rise to a  $U(2)$ -invariant Kähler metric on  $M$ . Moreover, this metric will belong in Kähler class  $b_0[\Sigma_\infty] - a_0[\Sigma_0]$ , and satisfy  $|\Sigma_0| = \pi a_0$  and  $|\Sigma_\infty| = \pi b_0$ .*

### 2.1.1 Curvature Terms

On Kähler manifolds, the Ricci tensor is given locally by

$$\text{Rc}_{\alpha\bar{\beta}} = -\frac{\partial^2}{\partial z^\alpha \partial \bar{z}^\beta} \log \det g.$$

In particular, for  $g$  as in (2.1.2), one has globally

$$\text{Rc}_{\alpha\bar{\beta}} = e^{-r} \psi \delta_{\alpha\beta} + e^{-2r} (\psi_r - \psi) \bar{z}^\alpha z^\beta \quad (2.1.6)$$

where  $\psi = -\partial_r(\log \det g) = 2 - \frac{\varphi_r}{\varphi} - \frac{\varphi_{rr}}{\varphi_r}$ . From equations (2.1.2) and (2.1.6), we compute the eigenvalues of the Ricci curvature endomorphism<sup>3</sup>

$$\begin{aligned} \lambda_1 &= \frac{\psi}{\varphi} \quad \text{with eigenvector} \quad U = \bar{z}^2 \frac{\partial}{\partial z^1} + \bar{z}^1 \frac{\partial}{\partial z^2} \\ \lambda_2 &= \frac{\psi_r}{\varphi_r} \quad \text{with eigenvector} \quad V = z^1 \frac{\partial}{\partial z^2} + z^2 \frac{\partial}{\partial z^1}. \end{aligned} \quad (2.1.7)$$

and, in particular, the scalar curvature

$$R(r, t) = \frac{2}{\varphi} \left( 2 - \frac{\varphi_r}{\varphi} - \frac{\varphi_{rr}}{\varphi_r} \right) + \frac{2}{\varphi_r} \left[ \left( -\frac{\varphi_r}{\varphi} \right)_r + \left( -\frac{\varphi_{rr}}{\varphi_r} \right)_r \right]. \quad (2.1.8)$$

Using Calabi's conditions we find for  $r$  near  $-\infty$

$$\begin{aligned} \lambda_1 &= \frac{1}{a_0} + O(e^r) \\ \lambda_2 &= -\frac{1}{a_0} - \frac{2a_2}{a_1^2} + O(e^r), \end{aligned} \quad (2.1.9)$$

---

<sup>3</sup>The map  $\text{Rc} : TM \rightarrow TM$  obtained by raising one index.

and for  $r$  near  $+\infty$

$$\begin{aligned}\lambda_1 &= \frac{3}{b_0} + O(e^{-r}) \\ \lambda_2 &= \frac{1}{b_0} + \frac{2b_2}{b_1^2} + O(e^{-r}).\end{aligned}\tag{2.1.10}$$

Moreover, for the Riemann curvature, a direct computation shows

$$\begin{aligned}R_{\alpha\bar{\beta}\gamma\bar{\delta}} &= e^{-4r} \left[ -\varphi_{rrr} + 4\varphi_{rr} - 2\varphi_r + 2\varphi - 4\frac{\varphi_r^2}{\varphi} + \frac{\varphi_{rr}^2}{\varphi_r} \right] \bar{z}^\alpha z^\beta \bar{z}^\gamma z^\delta \\ &\quad + e^{-3r} \left[ \varphi_r - \varphi_{rr} - \varphi + \frac{\varphi_r^2}{\varphi} \right] (\bar{z}^\alpha z^\beta \delta^{\gamma\delta} + \bar{z}^\alpha \delta^{\beta\gamma} z^\delta \\ &\quad + \delta^{\alpha\beta} \bar{z}^\gamma z^\delta + \delta_{\alpha\delta} z^\beta \bar{z}^\gamma) + e^{-2r} [-\varphi_r + \varphi] (\delta_{\alpha\beta} \delta_{\gamma\delta} + \delta_{\alpha\delta} \delta_{\beta\gamma})\end{aligned}\tag{2.1.11}$$

### 2.1.2 The effect of Ricci flow

From equations (2.1.2) and (2.1.6), one can see that  $g(t)$  evolves by Ricci flow  $\partial_t g = -\text{Rc}(g)$  if, and only if,  $\varphi$  evolves by  $\varphi_t = -\psi$ , that is,

$$\varphi_t = \frac{\varphi_{rr}}{\varphi_r} + \frac{\varphi_r}{\varphi} - 2.\tag{2.1.12}$$

*Remark 11.* Equation (2.1.12) looks alarming from the PDE point of view, as it might degenerate. On the other hand, we recall that for potentials  $\varphi$  that yield Kähler metrics on  $\mathbb{C}^2 \setminus \{0\}$  we have  $\varphi_r > 0$  and this condition is preserved, so (2.1.12) is parabolic.

The Kähler class will evolve as in (1.3.3) and

$$\varphi(-\infty, t) = a(t) = a_0 - t, \quad \varphi(+\infty, t) = b(t) = b_0 - 3t,$$

so the flow will become singular no later than  $t = a(0)$ . In fact, a general result in Kähler-Ricci flow, see [34], says that the Kähler-Ricci flow exists and is smooth up until the first time  $t = T$  where the Kähler class  $[w(T)]$  ceases

to be Kähler. In our case, because  $b(0) > 3a(0)$ , this happens precisely at  $T = a(0)$ . Furthermore, since  $|\Sigma_0| = \pi a(t)$ , the section  $\Sigma_0$  will vanish when  $t = T$  and, as it turns out,  $g(T)$  is still smooth in  $M \setminus \Sigma_0$ , see [32].

The following scale invariant estimate is an immediate consequence of the maximum principle and will be useful later.

**Lemma 12.** *Let  $F = \frac{\varphi_r}{\varphi}$ . Then,  $0 \leq F(r, t) \leq \max\{\max F(\cdot, 0), 1\}$ .*

*Proof.* Because of (2.1.3),  $F \geq 0$ . To prove the upper bound, let  $r_0$  be a local spatial maximum of  $F$ , at which we must have

$$F_{rr}(r_0) \leq 0 \text{ and } F_r(r_0) = 0.$$

We then compute the evolution of  $F$

$$F_t = \frac{F_{rr}}{\varphi_r} + \frac{F_r}{\varphi} \left(1 - \frac{F_r}{F^2}\right) + 2\frac{F}{\varphi}(1 - F),$$

and thus note that

$$F_t(r_0) = \frac{1}{\varphi} \left[ \frac{F_{rr}}{F} + 2F(1 - F) \right] \leq \frac{2}{\varphi} F(1 - F).$$

So, at a local spatial maximum where  $F(r_0) > 1$  (respectively  $\geq 1$ ),  $F(x, t)$  is decreasing (respectively non-increasing) in time, and the lemma follows since  $F \equiv 0$  in the boundary points  $r = -\infty, +\infty$ .  $\square$

### 2.1.3 Soliton metrics on $L$ and $M$

Following [15], consider a  $U(2)$ -invariant gradient shrinking soliton metric on  $L$  or  $M$ , normalized such that  $|\Sigma_0| = \pi$ . Then it must have a potential

$\varphi$  satisfying

$$\frac{\varphi_{rr}}{\varphi_r} + \frac{\varphi_r}{\varphi} - C\varphi_r + \varphi - 2 = 0, \quad (2.1.13)$$

for some constant  $C$  and with  $\lim \varphi = 1$  as  $r \rightarrow -\infty$  and another asymptotic condition as  $r \rightarrow +\infty$ , depending on whether one is looking for a soliton on  $M$  or  $L$ .

In the case one is looking on  $M$ , by the independent work of Koiso in [22] and Cao in [9],  $C$  must be a constant between  $\frac{1}{2}$  and 1 and there will exist only one such potential, modulo translations in  $r$ . This soliton has positive Ricci curvature, as Cao observed in [9], and satisfies  $a_0 = 1$ ,  $b_0 = 3$ .

For a soliton on  $L$ , Feldman-Ilmanen-Knopf proved that the  $C$  must be  $\sqrt{2}$  and that there exists only one such potential, again modulo translations in  $r$ . For our purposes, it is relevant to note that this soliton has Ricci curvature of mixed sign:  $\lambda_2 < 0$  for  $r$  near  $-\infty$ . In fact, using the soliton equation (2.1.13) and the expression (2.1.7) for the eigenvalue one finds that  $\lambda_2 = 1 - \sqrt{2} \frac{\varphi_{rr}}{\varphi_r}$  and since  $\lim_{r \rightarrow -\infty} \frac{\varphi_{rr}}{\varphi_r} = 1$ , we have that  $\lambda_2$  is negative near the section  $\Sigma_0$ .

## 2.2 Dilation Variables, *Type-I* blow-up, and convergence modulo Diffeomorphisms

### 2.2.1 Dilation Variables

What we know so far about the singularity formation is that it occurs along the section  $\Sigma_0$ , which shrinks to a point by [32], and that it is *Type-I* [30]. Hence it will be useful to use *parabolically dilated variables* that allow us to *zoom in on  $\Sigma_0$  in a Type-I fashion*.

Given an evolution  $\varphi(r, t)$  with singular time  $T$  as before, we define the *dilated time* variable  $\tau = -\log(T - t)$ , the *dilated spatial* variable  $\rho = r + \tau$  (which correspond to complex coordinates  $\zeta = e^{\tau/2}z$ ) and set

$$\phi(\rho, \tau) = e^\tau \varphi(r(\rho, \tau), t(\tau)). \quad (2.2.1)$$

The function  $\phi(\rho, \tau)$  evolves by

$$\phi_\tau = \frac{\phi_{\rho\rho}}{\phi_\rho} + \frac{\phi_\rho}{\phi} - \phi_\rho + \phi - 2 \quad (2.2.2)$$

and is a Kähler potential on  $M$  in complex coordinates  $\zeta$ . To see that, we note that it must satisfy  $\phi, \phi_\rho > 0$  for all  $\tau$ , and the Calabi conditions (2.1.4) and (2.1.5) — or, alternatively, that  $\phi$  represents the metric  $\bar{g}(\tau)$  on  $M$  equivalent to  $g(t)$  scaled by  $e^\tau = \frac{1}{T-t}$  and pulled back by the diffeomorphism  $z \rightarrow e^{-\tau/2}z$ .

Now that  $\rho, \tau$  are defined, we explain what we mean by zoom in  $\Sigma_0$  in a *Type-I* fashion, first on the level of the potential and then on the level of the metric.

For fixed  $\rho$ , we let  $t \nearrow T$  and thus have  $\tau \nearrow +\infty$  and  $(r(\rho, \tau), t(\tau)) \rightarrow (-\infty, T)$ . The dilation

$$\phi(\rho, \tau) = \frac{1}{T-t} \varphi(\rho - \tau, t)$$

is then a *Type-I* (note the factor  $\frac{1}{T-t}$ ) zoom in on how  $\varphi$  is going to zero along  $\Sigma_0$ , since  $\varphi(\rho - \tau, t) \rightarrow \varphi(-\infty, T) = 0$ .

Geometrically, the metric  $\bar{g}(\tau)$  given by  $\phi(\rho, \tau)$  is just  $\frac{1}{T-t}g(t)$  modified by diffeomorphism, and if  $g(t)$  has a *Type-I* singularity at  $t = T$ ,  $\bar{g}(\tau)$  has

bounded curvature as  $\tau \nearrow +\infty$ . Moreover,

$$\begin{aligned} |\Sigma_0|_{\bar{g}(\tau)} &= \pi \\ |\Sigma_\infty|_{\bar{g}(\tau)} &= [(b_0 - 3a_0)e^\tau + 3]\pi, \end{aligned}$$

which indicates that as  $\tau \nearrow +\infty$ ,  $\Sigma_\infty$  is being blown away and suggests that the metrics  $\bar{g}(\tau)$  on  $M$  are becoming more and more like metrics on  $L$  where  $|\Sigma_0| = \pi$ , as one recalls from Appendix B.

Our goal is to prove that indeed the limit of these metrics will be modeled by the evolution of the FIK soliton metric on  $L$  constructed in [15]. But there are major difficulties that do not allow us to prove directly such convergence by working with  $\rho, \tau$ . One comes from the fact that the FIK potential is not stationary in these coordinates; in fact, the soliton is still moving by diffeomorphisms, or more precisely, it is translating in the  $\rho$  variable. The other is the fact that there actually exist a whole family of FIK potentials, generated by the translations in  $r$ . These difficulties lead us to the approach presented in the next section.

### 2.2.2 Equations in $\phi, \tau$ variables

Let  $\tau = -\log(T - t)$  and  $\rho = r + \tau$  be the dilation variables introduced above. Because  $\phi_\rho > 0$  along the flow, we can actually write  $\rho$  as a function of  $\phi$  at any fixed time  $\tau$  and thus consider the function

$$y(\phi, \tau) = \phi_\rho(\rho, \tau), \tag{2.2.3}$$

which, for any given  $\tau$ , is defined on the interval  $[1, (b_0 - 3a_0)e^\tau + 3]$ , satisfying  $y(1) = y((b_0 - 3a_0)e^\tau + 3) = 0$  and  $y(\phi)$  positive otherwise.

We next find the evolution equation of  $y(\phi, \tau)$ . First we note: since  $\phi_\rho = y$ , we have  $\phi_{\rho\rho} = y_\rho = y_\phi y$  and  $\phi_{\rho\rho\rho} = y_{\phi\phi} y^2 + y_\phi^2 y$ . And we then compute  $\partial_\tau|_\rho y$ , which in our notation means *the derivative of  $y$  with respect to  $\tau$  while fixing  $\rho$* .

$$\begin{aligned} \partial_\tau|_\rho y &= \phi_{\rho\tau} \\ &= \frac{\phi_{\rho\rho\rho}}{\phi_\rho} - \left(\frac{\phi_{\rho\rho}}{\phi_\rho}\right)^2 + \frac{\phi_{\rho\rho}}{\phi} - \left(\frac{\phi_\rho}{\phi}\right)^2 + \phi_\rho - \phi_{\rho\rho} \\ &= y_{\phi\phi} y + \frac{y_\phi y}{\phi} - \left(\frac{y}{\phi}\right)^2 + y - y_\phi y. \end{aligned}$$

Finally, since  $\partial_\tau|_\phi y = \partial_\tau|_\rho y - y_\phi \phi_\tau$ , we have

$$\begin{aligned} \partial_\tau|_\phi y &= y_{\phi\phi} y + \frac{y_\phi y}{\phi} - \left(\frac{y}{\phi}\right)^2 + y - y_\phi y - y_\phi \phi_\tau \\ &= y_{\phi\phi} y + (2 - \phi - y_\phi) y_\phi + y \left(1 - \frac{y}{\phi^2}\right). \end{aligned} \quad (2.2.4)$$

*Remark 13.* Because of Lemma 12,  $\frac{y}{\phi}$  is uniformly bounded in time.

The advantages of these variables are two-fold. First of all, they do not see translations in the  $\rho$  variable and thus the whole family of FIK potentials are represented by just one stationary potential  $\mathcal{Y} = \mathcal{Y}(\phi^{\text{FIK}})$ . Secondly, the non-linearities of (2.2.4) are mild when compared with (2.2.2), and this allow us to develop a barrier method based on the comparison principle in the next section.

*Remark 14.* In fact, one can check that all the FIK potentials satisfy the same equation when written using the coordinates as above

$$\mathcal{Y}(\phi) = \phi_\rho = \frac{\phi(\phi - 2) + \sqrt{2}(\phi - 1) + 1}{\sqrt{2}\phi},$$

and also that  $\partial_\tau|_\phi \mathcal{Y} = 0$ .

### 2.2.3 Comparison and convergence modulo diffeomorphisms for a large class of potentials

In this section we use techniques based on the comparison principle to prove an important step towards Theorem A, which is convergence modulo diffeomorphisms for a large class of potentials.

**Notation.** *In what follows we will write just  $\partial_\tau$  as short for  $\partial_\tau|_\phi$ . In particular,  $\partial_\tau$  and  $\partial_\phi$  are commuting derivatives.*

The first thing we prove is that for any initial data satisfying the conditions of Theorem A, one always has a  $C^1$  bound.

**Lemma 15.** *For any data  $y(\phi, \tau)$  coming from Ricci flow as in Theorem A, the derivative  $y_\phi$  is uniformly bounded in time.*

*Proof.* Since at the boundary  $y_\phi$  takes values 1 or  $-1$ , we only need to deal with interior spatial maxima and minima. We write the evolution equation for  $y_\phi$ :

$$y_{\phi\tau} = yy_{\phi\phi\phi} + [2 - \phi - y_\phi]y_{\phi\phi} - 2\frac{yy_\phi}{\phi^2} + \frac{2y^2}{\phi^3}. \quad (2.2.5)$$

If  $y_\phi$  has a negative local spatial minimum at  $\phi_0$ , then  $y_{\phi\phi}(\phi_0) = 0$ ,  $y_{\phi\phi\phi}(\phi_0) \geq 0$ , and thus

$$y_{\phi\tau}(\phi_0) = yy_{\phi\phi\phi}(\phi_0) - 2\frac{yy_\phi}{\phi^2}(\phi_0) + \frac{2y^2}{\phi^3}(\phi_0) > 0,$$

so  $y_\phi$  is uniformly bounded from below.

Finally, let  $\phi_0$  be a local spatial maximum of  $y_\phi$ , so that  $y_{\phi\phi}(\phi_0) = 0$  and  $y_{\phi\phi\phi}(\phi_0) \leq 0$ . By Remark 13,  $\frac{y}{\phi} < C$  for some constant  $C$  independent of time. Suppose  $y_\phi(\phi_0) > C$ , then

$$\begin{aligned} y_{\phi\tau}(\phi_0) &= yy_{\phi\phi\phi}(\phi_0) - 2\frac{yy_\phi}{\phi^2}(\phi_0) + \frac{2y^2}{\phi^3}(\phi_0) \\ &\leq 2\frac{y}{\phi^2} \left[ -y_\phi + \frac{y}{\phi} \right] < 0. \end{aligned}$$

We hence conclude  $y_\phi$  is uniformly bounded from above too and the lemma is proved.  $\square$

*Remark 16.* The above lemma, together with Remark 13, says that the evolution (2.2.4) has bounded coefficients on any compact interval  $[1, \phi_0]$ .

*Remark 17.* Because  $y_\phi = \frac{\phi_{\rho\rho}}{\phi_\rho}$ , the lemma will be useful to prove  $C^2$  bounds for  $\phi$ .

We next construct upper and lower barriers that for “most” initial data will trap the solution to evolution (2.2.4) and squeeze it to the FIK potential  $\mathcal{Y}(\phi) = \frac{\phi(\phi-2)+\sqrt{2}(\phi-1)+1}{\sqrt{2}\phi}$ .

**Definition 18** (*Metrics in the class  $\mathcal{C}$* ). Let  $\mathcal{C}$  be the class of all initial  $U(2)$ -invariant metrics on  $M$  belonging to the Kähler class  $b(0)[\Sigma_\infty] - a(0)[\Sigma_0]$  with

$b(0) > 3a(0)$ , and such that, moreover, the parabolic blow up  $\phi$  of the potential  $\varphi$  satisfies, when writing  $\phi_\rho$  in  $\phi$  coordinates as before,

$$y(\phi, 0) > \mathcal{Y}(\phi) - \frac{1}{5}\phi^2.$$

Let us remark that  $\mathcal{Y}(\phi) - \frac{1}{5}\phi^2$  will be exactly the initial barrier that we will use for the evolution (2.2.4), thus we are restricting ourselves to potentials that are initially strictly above it. This barrier is mostly negative, but is positive and small (strictly less than 0.06) on a small neighborhood of  $\phi = 2$ . This implies that a large family of initial data belongs to the class  $\mathcal{C}$ .

**Proposition 19.** *For any initial data  $y(\phi, 0)$  in the class  $\mathcal{C}$ , one has that  $y(\phi, \tau)$  remains in the class  $\mathcal{C}$  and converges uniformly on compact subsets to  $\mathcal{Y}(\phi)$  as  $\tau \nearrow \infty$ .*

We note that the “elliptic” operator in (2.2.4)

$$\mathcal{E}[y] = y_{\phi\phi}y + (2 - \phi - y_{\phi})y_{\phi} + y \left(1 - \frac{y}{\phi^2}\right)$$

can be written as a linear plus a quadratic part

$$\mathcal{E}[y] = \mathcal{L}[y] + \mathcal{Q}[y],$$

where  $\mathcal{L}[y] = (2 - \phi)y_{\phi} + y$  and  $\mathcal{Q}[y] = yy_{\phi\phi} - y_{\phi}^2 - \frac{y^2}{\phi^2}$ . The non-linearity of  $\mathcal{E}[\cdot]$  is such that for given functions  $y(\phi), s(\phi)$ :

$$\begin{aligned} \mathcal{E}[y + s] &= \mathcal{L}[y + s] + \mathcal{Q}[y + s] \\ &= \mathcal{L}[y] + \mathcal{L}[s] + \mathcal{Q}[y] + \mathcal{Q}[s] + \mathcal{M}[y, s] \\ &= \mathcal{E}[y] + \mathcal{E}[s] + \mathcal{M}[y, s], \end{aligned}$$

where  $\mathcal{M}[y, s] = sy_{\phi\phi} + ys_{\phi\phi} - 2y_{\phi}s_{\phi} - 2\frac{ys}{\phi^2}$  is bilinear in  $y$  and  $s$ . This mild non-linearity suggests the barrier approach.

*Proof of Proposition 19.* Let

$$\mathcal{Y}(\phi) = \frac{\phi(\phi - 2) + \sqrt{2}(\phi - 1) + 1}{\sqrt{2}\phi}$$

be the FIK potential and consider  $s(\phi, \tau) = -\lambda(\tau)\phi^2$ , where  $\dot{\lambda} = -\delta\lambda$  for some  $\delta \in \mathbb{R}$ . We compute  $(\partial_{\tau} - \mathcal{E})(\mathcal{Y} + s)$ :

$$\begin{aligned} (\partial_{\tau} - \mathcal{E})[\mathcal{Y} + s] &= \partial_{\tau}s - \mathcal{L}[s] - \mathcal{Q}[s] - \mathcal{M}[\mathcal{Y}, s] \\ &= \lambda (\delta\phi^2 + \mathcal{L}[\phi^2] - \lambda\mathcal{Q}[\phi^2] + \mathcal{M}[\mathcal{Y}, \phi^2]), \end{aligned}$$

and once substituting  $\mathcal{L}[\phi^2]$ ,  $\mathcal{Q}[\phi^2]$ , and  $\mathcal{M}[\mathcal{Y}, \phi^2]$ , we have

$$(\partial_{\tau} - \mathcal{E})[\mathcal{Y} + s] = \lambda \left( (\delta + 3\lambda - 1)\phi^2 + 2(2 - \sqrt{2})\phi - 3\phi^{-1}(2 - \sqrt{2}) \right). \quad (2.2.6)$$

Let  $y(\phi, \tau)$  be a solution coming from Ricci flow. Suppose that  $y(\phi, 0)$  belongs to the class  $\mathcal{C}$  of initial data.

Choose  $\lambda(0) = 1/5$  and let  $\delta$  be a positive number smaller than  $10^{-6}$  to be fixed. Then, by (2.2.6) the function

$$y_1(\phi, \tau) = \mathcal{Y}(\phi) - \frac{1}{5}e^{-\delta\tau}\phi^2$$

satisfies  $(\partial_{\tau} - \mathcal{E})[y_1] < 0$  for all times  $\tau > 0$ , i.e.,  $y_1(\phi, \tau)$  is a subsolution to our evolution problem.

By the comparison principle (Appendix C), if a solution  $y$  of  $(\partial_{\tau} - \mathcal{E})y = 0$  initially starts above  $y_1$ , then  $y$  will stay above  $y_1$  for all later times, as long as the boundary data behave as such.

A general solution  $y$  that initially belongs to class  $\mathcal{C}$  satisfies the assumption  $y(\phi, 0) > y_1(\phi, 0)$  and has the boundary conditions:

$$y(1) = y((b_0 - 3a_0)e^\tau + 3) = 0.$$

It is clear that  $y(1) = 0 > y_1(1)$  for all times. Also:

$$y_1(\phi, \tau) = \mathcal{Y} - \frac{1}{5} \exp(-\delta\tau)\phi^2 < \phi - \frac{1}{5} \exp(-\delta\tau)\phi^2 < 0$$

if  $\phi > 5 \exp(\delta\tau)$  (here we are using that  $\mathcal{Y}(\phi)$  is always below  $\phi$ ). Because  $(b_0 - 3a_0)e^\tau + 3 > 5 \exp(\delta\tau)$ , if  $\delta$  is chosen to be small enough, then we have  $y((b_0 - 3a_0)e^\tau + 3) = 0 > y_1((b_0 - 3a_0)e^\tau + 3)$ . Thus, the subsolution  $y_1$  stays below  $y$  at the boundary, and therefore everywhere, for all later times  $\tau > 0$ . In particular, because  $\lambda$  is decreasing in magnitude, this implies  $y(\phi, \tau)$  belongs to  $\mathcal{C}$  for all later times.

Furthermore, for the same initial data  $y(\phi, 0) \in \mathcal{C}$ , we choose

$$y_2(\phi, \tau) = \mathcal{Y} + \lambda_0 e^{-\tau/2} \phi^2$$

where  $\lambda_0$  is big enough so that  $y_2(\phi, 0) = \mathcal{Y} + \lambda_0 \phi^2 > y(\phi, 0)$ . Moreover, by equation (2.2.6) we will have that  $(\partial_\tau - \mathcal{E})[y_2] > 0$  for all  $\tau > 0$ . Moreover, one can check that the boundary data of  $y_2$  stay above those of  $y$  for all times. Thus, again using comparison,  $y_2$  stays above  $y$  for all later times.

Hence we have proved that

$$y_1(\phi, \tau) \leq y(\phi, \tau) \leq y_2(\phi, \tau), \tag{2.2.7}$$

and since on compact intervals for  $\phi$  we have  $y_1(\phi, \tau) \nearrow \mathcal{Y}(\phi)$  and  $y_2(\phi, \tau) \searrow \mathcal{Y}(\phi)$  uniformly, we must have  $y(\phi, \tau)$  converging to  $\mathcal{Y}(\phi)$  uniformly on compact subsets and the proposition thus is proved.  $\square$

Proposition 19 gives us uniform  $C^0$ -convergence for  $y(\phi) = \phi_\rho$  on compact subsets. We next prove that one actually has uniform  $C^{1,1}$ -convergence. For that we will use the *Type-I* blow up of the scalar curvature proved by Song in [30], i.e., there exists a constant  $C > 0$  such that

$$-C \leq R_{g(t)} \leq \frac{C}{T-t}.$$

In particular, for the dilated flow  $\bar{g}(\tau)$  this implies

$$-Ce^{-\tau} \leq R_{\bar{g}(\tau)} \leq C,$$

and since  $R_{\bar{g}(\tau)} = \frac{4}{\phi}(1 - y_\phi) - 2y_{\phi\phi}$ , we must have  $y_{\phi\phi}$  is uniformly bounded by Lemma 15.

**Proposition 20.** *For any initial data  $y(\phi, 0)$  in the class  $\mathcal{C}$ , one has that  $y(\phi, \tau)$  converges uniformly in the  $C^{1,1}$  topology on compact subsets to  $\mathcal{Y}(\phi)$  as  $\tau \nearrow \infty$ .*

*Proof.* Since we have uniform  $C^2$  bounds for  $y$  on compact intervals, the  $C^{1,1}$ -convergence follows from the following standard argument and Proposition 19. Since the spatial derivative  $y_{\phi\phi}$  is uniformly bounded in time, for any sequence of times,  $y_\phi$  will converge uniformly up to subsequence. Moreover, since the convergence is uniform, the limit of  $y_\phi$  along any such subsequence must be

the spatial derivative  $\mathcal{Y}_\phi$  of the stationary state  $\mathcal{Y}$ . Because the latter does not depend on the subsequence, we have that  $y_\phi$  converges uniformly in time to the derivative  $\mathcal{Y}_\phi$ . Since  $y$  converges uniformly in  $C^1$  while  $y_{\phi\phi}$  is uniformly bounded, the proposition is then proved.  $\square$

*Remark 21.* For the reader's convenience, we point out the relation between the Riemann curvature and derivatives of  $y$ . By the rotational symmetry of  $\bar{g}(\tau)$ , we reduce our analysis to a point of the form  $(z^1, z^2) = (\xi, 0)$ , and use (2.1.11) to compute

$$\begin{aligned} |\text{Rm}(\bar{g}(\tau))| &\leq 2|R_{1\bar{1}1\bar{1}}| + 2|R_{2\bar{2}2\bar{2}}| + 2|R_{1\bar{1}2\bar{2}}| \\ &\leq 2 \left| \frac{1}{\phi_\rho^2} \left( -\phi_{\rho\rho\rho} + \frac{\phi_{\rho\rho}^2}{\phi_\rho} \right) \right| + 4 \left| \frac{-\phi_\rho + \phi}{\phi^2} \right| + 2 \left| -\frac{\phi_{\rho\rho}}{\phi\phi_\rho} + \frac{\phi_\rho}{\phi^2} \right| \\ &= 2 \left| \frac{1}{\phi_\rho} \left( \frac{\phi_{\rho\rho}}{\phi_\rho} \right)_\rho \right| + \frac{4}{\phi} \left| -\frac{\phi_\rho}{\phi} + 1 \right| + \frac{2}{\phi} \left| -\frac{\phi_{\rho\rho}}{\phi_\rho} + \frac{\phi_\rho}{\phi} \right|. \end{aligned} \quad (2.2.8)$$

Moreover, since  $\bar{g}(\tau) = \frac{1}{T-t}g(t)$ , we note that by Song's [30] *Type-I* result there exists a uniform constant  $C > 0$  such that

$$|\text{Rm}(\bar{g}(\tau))| \leq C.$$

Recalling that  $\frac{\phi_{\rho\rho}}{\phi_\rho} = y_\phi$ ,  $\frac{1}{\phi_\rho} \left( \frac{\phi_{\rho\rho}}{\phi_\rho} \right)_\rho = y_{\phi\phi}$ , and also that one has  $\frac{\phi_\rho}{\phi}$  uniformly bounded by Lemma 12, we see from the above that *Type-I* blow up is what one really needs to establish a second derivative bound for  $y$ .

Combining Proposition 20 with Song's *Type-I* result [30] we have the following:

**Theorem 22.** *Let  $g(0)$  be a metric on  $M$  belonging to the class  $\mathcal{C}$ . Then the Ricci flow (1.2.3) starting at  $g(0)$  will develop a Type-I singularity along  $\Sigma_0$ . Moreover, parabolic dilations  $\bar{g}(\tau)$  of  $g(t)$  as in (2.2.1) will converge, modulo diffeomorphisms and uniformly on a corresponding time-dependent neighborhood of the singular region  $\Sigma_0$ , to one of the FIK solitons.*

*Proof.* We let  $\varphi$  be in the class of initial data  $\mathcal{C}$  and consider the dilation variables  $\rho, \tau$  as well as the dilated potential  $\phi(\rho, \tau)$ . We apply the following gauge-fixing construction: by the implicit function theorem, we can find a smooth function  $C(\tau)$  such that if we define  $\mu = \rho + C(\tau)$  and then define

$$\Phi(\mu, \tau) = \phi(\mu - C(\tau), \tau),$$

we will have  $\Phi(0, \tau)$  constant in time, say equal to 2. Moreover,  $\Phi_\mu$  is related to  $\Phi$  just as  $\phi_\rho$  is related to  $\phi$ , and thus we have by Proposition 20,  $y(\Phi)$  converges  $C^{1,1}$  uniformly to  $\mathcal{Y}(\Phi)$ , so that  $\Phi$  converges  $C^{2,1}$  uniformly (in compact  $\Phi$  intervals, so in particular for  $-\infty \leq \mu \leq 0$ ) to the unique  $\Phi^{\text{FIK}}$  that satisfies  $\Phi^{\text{FIK}}(0) = 2$ .

Finally, since we know that the singularity along  $\Sigma_0$  is *Type-I* [30], the Riemann curvature  $\text{Rm}(\bar{g}(\tau))$  will be bounded uniformly in the region  $-\infty \leq \mu \leq 0$ . Shi's local estimates for higher derivatives of the Riemann curvature under Ricci flow then dictate that one actually has bounds of any higher order and that thus convergence is smooth.  $\square$

A few remarks are now in order.

*Remark 23.* For any fixed  $\rho \in \mathbb{R}$ , consider the parabolic neighborhood

$$N(\rho) = \{z \in \mathbb{C}^2 \mid \rho(z) \leq \rho\} = \{z \in \mathbb{C}^2 \mid |z|^2 \leq e^\rho(T - t)\}.$$

Theorem 22 says that for metrics of the class  $\mathcal{C}$ , there exist diffeomorphisms  $\Psi_\tau$  (corresponding precisely to the  $C(\tau)$  change of gauge) such that as  $\tau \nearrow \infty$

$$\Psi_\tau^* \bar{g}(\tau) \longrightarrow g_{\text{FIK}}$$

uniformly on the neighborhood  $\Psi_\tau^{-1}(N(\rho))$ . In the next section, we will prove that the diffeomorphisms  $\Psi_\tau^{-1}$  correspond to the 1-parameter family of diffeomorphisms by which the FIK solitons move under Ricci flow, i.e., we will prove that for large  $\tau$  one has asymptotically

$$C(\tau) = (\sqrt{2} - 1)\tau \pm \text{constant}.$$

This will conclude the proof of Theorem A that parabolic dilations  $\bar{g}(\tau)$  of  $g(t)$  converge to the flow of an FIK soliton uniformly on any parabolic neighborhood  $N(\rho)$ .

*Remark 24.* If we write  $\varphi$  in non-logarithmic coordinates as  $\varphi(r, t) = f(w, t)$ , where  $w = e^r$ , and expand  $\Phi(\mu, \tau) = \phi(\mu - C(\tau), \tau) = e^\tau \varphi(\mu - C(\tau) - \tau, t)$  around  $w = 0$  we get

$$\begin{aligned} \Phi(\mu, \tau) &= e^\tau f(e^{\mu - C(\tau) - \tau}, t) \\ &= 1 + e^\mu e^{-C(\tau)} f_w(0, t) + \cdots \end{aligned}$$

Because  $\phi$  must converge smoothly to FIK for fixed  $\mu$ , we must have that <sup>4</sup>

$$f_w(0, t) \sim e^{C(\tau)}.$$

*Remark 25.* Finally, we note that by Theorem 22 the scalar curvature at the singular region  $\Sigma_0$  must blow up like:

$$R_{g(t)} \Big|_{\Sigma_0} = \frac{4 - 2\sqrt{2}}{T - t} + O((T - t)^{\alpha-1}), \quad (2.2.9)$$

for some  $\alpha > 0$ . In fact, by the barrier argument in Proposition 19, there must exist positive constants  $C_0$  and  $\delta_0$  such that  $|y - \mathcal{Y}| \leq C_0 e^{-\delta_0 \tau}$  holds uniformly in time on a fixed interval of the form  $[1, \phi_0]$ , and by Theorem 22, the higher derivatives of  $y$  are uniformly bounded in time on  $[1, \phi_0]$ . Moreover, we can use an interpolation inequality of the form: (*e.g.*, Corollary 7.21 of [23])

$$\|\partial_\phi(y - \mathcal{Y})\|_p \leq \epsilon \|\partial_{\phi\phi}(y - \mathcal{Y})\|_p + \frac{C}{\epsilon} \|y - \mathcal{Y}\|_p, \quad (2.2.10)$$

on  $[1, \phi_0]$ , where  $p > 1$ ,  $\|\cdot\|_p$  is the usual (spatial)  $L^p$ -norm,  $\epsilon$  is any positive number, and  $C$  is a universal constant that does not depend on  $y, \mathcal{Y}, \epsilon$ , or  $p$ , and we can thus argue that whenever  $\|\partial_{\phi\phi}(y - \mathcal{Y})\|_p^{1/2} \neq 0$ , by setting  $\epsilon = \frac{\|y - \mathcal{Y}\|_p^{1/2}}{\|\partial_{\phi\phi}(y - \mathcal{Y})\|_p^{1/2}}$  in (2.2.10), one has

$$\|\partial_\phi(y - \mathcal{Y})\|_p \leq (1 + C) \|y - \mathcal{Y}\|_p^{1/2} \|\partial_{\phi\phi}(y - \mathcal{Y})\|_p^{1/2},$$

and if  $\|\partial_{\phi\phi}(y - \mathcal{Y})\|_p^{1/2} = 0$ , what we want follows directly from (2.2.10). Furthermore, because  $[1, \phi_0]$  has finite measure, we can send  $p$  to infinity and

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<sup>4</sup>This notation means that there exist constants  $0 < C_1 < C_2$  such that  $C_1 e^{C(\tau) - \tau} \leq f_w(0, t) \leq C_2 e^{C(\tau) - \tau}$  as  $\tau \nearrow \infty$ , *et cetera*.

pass the above inequality to the limit:

$$\|\partial_\phi(y - \mathcal{Y})\|_\infty \leq (1 + C)\|y - \mathcal{Y}\|_\infty^{1/2}\|\partial_{\phi\phi}(y - \mathcal{Y})\|_\infty^{1/2}, \quad (2.2.11)$$

and since  $\|\partial_{\phi\phi}(y - \mathcal{Y})\|_\infty^{1/2}$  is uniformly bounded in time and  $\|y - \mathcal{Y}\|_\infty \leq C_0 e^{-\delta_0 \tau}$  in  $[1, \phi_0]$ , we find positive constants  $C_1, \delta_1$  such that:

$$\|y_\phi - \mathcal{Y}_\phi\|_\infty \leq C_1 e^{-\delta_1 \tau}.$$

Moreover, by bootstrapping the argument above, we can also find positive constants  $\delta_2, C_2$  such that on  $[1, \phi_0]$ :

$$\|y_{\phi\phi} - \mathcal{Y}_{\phi\phi}\|_\infty \leq C_2 e^{-\delta_2 \tau}.$$

Finally, since

$$(T - t)R_{g(t)}\Big|_{\Sigma_0} = R_{\bar{g}(\tau)}\Big|_{\Sigma_0} = -2y_{\phi\phi}(1, \tau),$$

and  $\mathcal{Y}_{\phi\phi}(1) = \sqrt{2} - 2$ ,  $e^{-\tau} = T - t$ , (2.2.9) follows for  $\alpha = \delta_2 > 0$ .

### 2.3 End of proof of Theorem A

In this section, we finish the proof of Theorem A. This is done by using Theorem 22 and the remarks following it.

Let  $\varphi(r, t)$  be a potential belonging to the class  $\mathcal{C}$ ,  $\phi(\rho, \tau)$  its corresponding dilated potential as in (2.2.1), and  $\Phi(\mu, \tau)$  and  $C(\tau)$  as in Theorem 22. By Remark 23, and since (unnormalized) FIK potentials move under Ricci flow by the diffeomorphisms

$$z \longrightarrow e^{-\sqrt{2}\tau/2}z,$$

it is enough to prove that asymptotically for large  $\tau$  one has

$$C(\tau) = (\sqrt{2} - 1)\tau \pm \text{constant}. \quad (2.3.1)$$

Note that Remark 24 tells us that if we write  $\varphi$  in non-logarithmic coordinates as  $\varphi(r, t) = f(w, t)$ , we must have then

$$f_w(0, t) \sim e^{C(\tau)}.$$

This allows us to use estimate (2.2.9) on the blow-up of the scalar curvature along  $\Sigma_0$  to study  $C(\tau)$  for large  $\tau$ . In fact, recalling the Ricci eigenvalues,

$$R_{g(t)} \Big|_{\Sigma_0} = 2\lambda_1 \Big|_{\Sigma_0}(t) + 2\lambda_2 \Big|_{\Sigma_0}(t),$$

where  $\lambda_1 \Big|_{\Sigma_0}(t) = \frac{1}{T-t}$ , estimate (2.2.9) tells us that the eigenvalue  $\lambda_2 \Big|_{\Sigma_0}(t)$  must blow up like

$$\lambda_2 \Big|_{\Sigma_0}(t) = \frac{1 - \sqrt{2}}{T - t} + O((T - t)^{\alpha-1}), \quad (2.3.2)$$

for some  $\alpha > 0$ . Moreover, we can compute  $\lambda_2 \Big|_{\Sigma_0}(t)$  directly from (2.1.7) in the coordinate  $w$  and find

$$\lambda_2 \Big|_{\Sigma_0}(t) = -\frac{f_{wt}(0, t)}{f_w(0, t)}. \quad (2.3.3)$$

Integrating (2.3.3) and using (2.3.2) we find that

$$f_w(0, t) \sim (T - t)^{1-\sqrt{2}} = e^{(\sqrt{2}-1)\tau}, \quad (2.3.4)$$

and this gives (2.3.1) by Remark 24. Theorem A is then proved.

## 2.4 The cone of metrics with non-negative Ricci curvature

In this section we prove Corollary 8 by constructing a metric on  $M$  with strictly positive Ricci curvature and belonging to the class  $\mathcal{C}$ . We recall that by (2.1.10) one has for the eigenvalues of Ricci and  $r$  near  $+\infty$  that:

$$\begin{aligned}\lambda_1 &= \frac{3}{b_0} + O(e^{-r}) \\ \lambda_2 &= \frac{1}{b_0} + \frac{2b_2}{b_1^2} + O(e^{-r})\end{aligned}$$

Let  $\varphi_{KC}$  be the potential for the Cao-Koiso soliton, which has positive Ricci curvature everywhere. The metric  $\varphi^{KC}$  is not the metric we are looking for only because  $b_0 = 3a_0$ . In fact, one can check explicitly that the Cao-Koiso metric is above the barrier as required in Definition 18. Thus we can perturb  $\varphi^{KC}$  by a small amount near  $r = +\infty$ , to obtain a metric potential  $\varphi$  with  $b_0 > 3a_0$ . Since the perturbation is only made near  $r + \infty$ , where the above expansion for the eigenvalues holds,  $\varphi$  will still have strictly positive Ricci curvature everywhere, and also belong to the class  $\mathcal{C}$ .

*Remark 26.* The construction above provides explicit examples of solutions demonstrating the linear instability of the Cao-Koiso soliton that was proved by Hall and Murphy [17].

## Appendices

## Appendix A

### The Hamilton-Ivey pinching estimate

In what follows, we make a precise statement of the Hamilton-Ivey curvature pinching estimate.

As mentioned in the introduction, the estimate roughly says that if a solution to the Ricci flow on a three-manifold becomes singular (*i.e.*, the curvature goes to blows up) as time  $t$  approaches the maximal time  $T$ , then the most negative sectional curvature will be small compared to the most positive sectional curvature.

For a precise statement, we work with the eigenvalues of the Riemann curvature tensor instead of the sectional curvatures. The first difficulty in addressing this estimate in higher dimensions is precisely the fact that the curvature tensor is a more complex object in higher dimensions, and the relationship between its eigenvalues and the sectional curvatures is more complicated.

**Theorem 27** (Hamilton [19], Ivey [20]). *Suppose we have a solution to Ricci flow on a three-manifold which is complete with bounded curvature for each  $t \geq 0$ . Assume at  $t = 0$  that the eigenvalues  $\lambda \geq \mu \geq \nu$  of the curvature operator at each point are bounded below by  $\nu \geq -1$ . The scalar curvature  $R = \lambda + \mu + \nu$  is their sum. Then at all points and all times  $t \geq 0$  we have*

*the pinching estimate  $R \geq (-\nu)[\log(-\nu) - 3]$ , whenever  $\nu < 0$ .*

For the proof of this theorem, we refer to the reader the original papers [19] and [20], as well as the more recent [11], [26], [21], or [33].

## Appendix B

### Line bundles over $\mathbb{CP}^1$

On the complex projective space  $\mathbb{CP}^1$  with projective coordinates  $[z_1 : z_2]$ , let  $\varphi_1 : \mathcal{U}_1 = \{[z_1 : z_2] \in \mathbb{CP}^1 : z_1 \neq 0\} \rightarrow \mathbb{C}$  and  $\varphi_2 : \mathcal{U}_2 = \{[z_1 : z_2] \in \mathbb{CP}^1 : z_2 \neq 0\} \rightarrow \mathbb{C}$  denote its usual charts given by  $\varphi_1([z_1 : z_2]) = z_2/z_1$  and  $\varphi_2([z_1 : z_2]) = z_1/z_2$ .

We consider two topologically distinct line bundles over  $\mathbb{CP}^1$  denoted by  $L$  and  $M$  described as follows.  $L$  has the complex line  $\mathbb{C}$  as fibers:

$$L = [(\mathcal{U}_1 \times \mathbb{C}) \sqcup (\mathcal{U}_2 \times \mathbb{C})] \text{ quotient by } \sim$$

where  $\mathcal{U}_1 \times \mathbb{C} \ni ([z_1 : z_2]; \xi) \sim ([y_1 : y_2], \eta) \in \mathcal{U}_2 \times \mathbb{C}$  if, and only if,  $[z_1 : z_2] = [y_1 : y_2]$  and  $\eta = \left(\frac{y_2}{z_1}\right) \xi$ . The manifold  $M$  has fibers  $\mathbb{C} \cup \{\infty\}$ :

$$M = [(\mathcal{U}_1 \times \mathbb{CP}^1) \sqcup (\mathcal{U}_2 \times \mathbb{CP}^1)] / \sim$$

where  $\mathcal{U}_1 \times \mathbb{CP}^1 \ni ([z_1 : z_2]; \xi) \sim ([y_1 : y_2], \eta) \in \mathcal{U}_2 \times \mathbb{CP}^1$  if, and only if,  $[z_1 : z_2] = [y_1 : y_2]$  and  $\eta = \left(\frac{y_2}{z_1}\right) \xi$ .

For our geometric purposes, we think of  $L$  and  $M$  in the following alternative manner. On  $M$ , consider the global sections  $\Sigma_0 = \{[z_1 : z_2]; 0\}$  and  $\Sigma_\infty = \{[z_1 : z_2]; \infty\}$  and define a map  $\Psi : \mathbb{C}^2 \setminus \{0\} \rightarrow \widehat{M}$ , where  $\widehat{M} =$

$M \setminus (\Sigma_0 \cup \Sigma_\infty)$ , given as

$$\Psi : (z_1, z_2) \mapsto ([z_1 : z_2]; z_\alpha)$$

if  $z_\alpha \neq 0$ . Because  $([z_1 : z_2]; z_\alpha) \sim ([z_1 : z_2]; z_\beta)$  whenever  $z_\alpha \neq 0$  and  $z_\beta \neq 0$ ,  $\Psi$  is well defined. Moreover, one can check that  $\Psi$  is biholomorphism. We then think of  $M$  as  $\mathbb{C}^2 \setminus \{0\}$  with one  $\mathbb{C}\mathbb{P}^1$  glued at 0 (the section  $\Sigma_0$ ) and another at  $\infty$  (the section  $\Sigma_\infty$ ) and of  $L = \widehat{M} \cup \Sigma_0$  as  $\mathbb{C}^2 \setminus \{0\}$  with a  $\mathbb{C}\mathbb{P}^1$  glued at 0.

# Appendix C

## Comparison principle

The comparison principle used in Section 3 for equation (2.2.4) is similar in spirit to Lemma 3 in [1]. For the reader's convenience, we outline the proof in what follows.

Since the evolving function  $y$  in Section 3 is non-negative for all times, any point of contact between  $y$  and one of the barriers must be a point where the barrier is non-negative. Moreover, one can check that the barriers used have spatial second derivative bounded in time. We can then reduce our analysis to the following:

**Proposition 28.** *Let  $y^-(\phi, \tau)$  and  $y^+(\phi, \tau)$  be non-negative sub- and super-solutions, respectively, of  $(\partial_\tau - \mathcal{E})[\cdot]$  on the interval  $[1, (b_0 - 3a_0)e^\tau + 3]$ . Suppose that either  $y^+$  or  $y^-$  satisfy  $y_{\phi\phi} < C$  for some constant  $C < \infty$  on a compact space-time set  $[1, (b_0 - 3a_0)e^{\bar{\tau}} + 3] \times [0, \bar{\tau}]$ . Moreover, assume that*

$$(i) \quad y^+(\phi, 0) > y^-(\phi, 0) \text{ in } (1, b_0 - 3a_0 + 3);$$

$$(ii) \quad y^+(1, \tau) \geq y^-(1, \tau) \text{ and } y^+((b_0 - 3a_0)e^\tau + 3, \tau) \geq y^-((b_0 - 3a_0)e^\tau + 3, \tau), \\ \text{for any } \tau \in [0, \bar{\tau}].$$

*Then, one must have  $y^+(\phi, \tau) \geq y^-(\phi, \tau)$  in  $(1, (b_0 - 3a_0)e^{\bar{\tau}} + 3) \times [0, \bar{\tau}]$ .*

*Proof.* Suppose first that  $y_{\phi\phi}^+ < C$ . For some  $\lambda > 0$  to be chosen later and any  $\alpha > 0$ , define

$$w = e^{-\lambda\tau}(y^+ - y^-) + \alpha \quad (\text{C.0.1})$$

Then  $w > 0$  on the parabolic boundary of our evolution. We will prove  $w$  is also positive in the interior of our domain and the lemma will follow by letting  $\alpha \searrow 0$ .

Assuming the contrary, there must be an interior point  $\phi_0$  and a first time  $\tau_0$  such that  $w(\phi_0, \tau_0) = 0$ . Then  $w_\tau(\phi_0, \tau_0) \leq 0$ , and at  $(\phi_0, \tau_0)$ :

$$y^+ = y^- - \alpha e^{\lambda\tau_0}, \quad y_\phi^+ = y_\phi^-, \quad y_{\phi\phi}^+ \geq y_{\phi\phi}^-.$$

Thus, at that point, we have  $0 \geq e^{\lambda\tau} w_\tau$  and

$$\begin{aligned} e^{\lambda\tau} w_\tau &= y_\tau^+ - y_\tau^- - \lambda(y^+ - y^-) \\ &\geq y^-(y_{\phi\phi}^+ - y_{\phi\phi}^-) + (y^- - y^+) \left[ \lambda - y_{\phi\phi}^+ - 1 + \frac{y^+ + y^-}{\phi^2} \right]. \end{aligned}$$

But if we use the uniform bound  $y_{\phi\phi}^+ < C$ , we have

$$0 \geq w_\tau > \alpha e^{\lambda\tau} \left[ \lambda - C - 1 + \frac{y^+ + y^-}{\phi^2} \right],$$

and, since  $\frac{y^+ + y^-}{\phi^2} \geq 0$ , this is a contradiction for any  $\lambda > C + 1$ . The result then follows in the case  $y_{\phi\phi}^+ < C$ .

To prove the lemma in the case that the subsolution  $y^-$  satisfy the uniform bound  $y_{\phi\phi}^- < C$ , one uses the fact that at a first interior zero  $(\phi_0, \tau_0)$  of  $w$  one has:

$$e^{\lambda\tau} w_\tau \geq y^+(y_{\phi\phi}^+ - y_{\phi\phi}^-) + (y^- - y^+) \left[ \lambda - y_{\phi\phi}^- - 1 + \frac{y^+ + y^-}{\phi^2} \right].$$

□

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