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Optimal Inventory and Pricing Decisions for Supply Chain Management

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Optimal Inventory and Pricing Decisions for Supply Chain Management

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Dedicated to my Mother Dunhe Yang.

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Optimal Inventory and Pricing Decisions for Supply Chain Management

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The dissertation contains two major research projects. In the first project, we first study a multi-period inventory planning problem. In each period, the firm under consideration can source from two possibly unreliable suppliers for a price-dependent demand. Our analysis suggests that the optimal procurement policy is neither a simple reorder-point policy nor a complex one without any structure, as previous studies suggest. Instead, we prove the existence of a reorder point for each supplier. No order is placed to that supplier for any inventory level above the reorder point and a positive order is issued to that supplier for *almost* every inventory level below the reorder point. We characterize conditions under which the optimal policy reveals monotone response to changes in the inventory level. Furthermore, two special cases of our model are examined in detail to demonstrate how our analysis generalizes a number of well-known results in the literature.

In the second project, we study a long-run inventory planning problem in which the retailer can replenish inventory and change price adjustment. We establish that it is optimal to change the price from low to high in each replenishment cycle, the optimal order-up-to level may decrease when the ordering cost increases, and fewer customers are served when the unit cost of procurement increases. Additionally, we provide efficient algorithms to compute the optimal stocking and pricing policies.

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Chapter 1

Introduction

We move to an age which is full of uncertainties and changes. Retailers have to adjust their sales policies based on inventory levels, demand uncertainties, competitors' strategies etc. On the other hand, customers' choices not only depend on prices of retailers but also rely on their trust on the retailers. This dissertation tries to answer a few issues in such complicated context.

In the first project, we study a multi-period inventory planning problem. Multi-souring can hedge against risk of supply unreliability. However, it is unclear what procurement policy the firm should follow when the demand of the product is also price-dependent. We formulate a multi-period procurement problem and developed a new procurement policy, named "near re-order point policy". It is neither a simple reorder-point policy nor a complex one without any structure, as previous studies suggest. Instead, we prove the existence of a reorder point for each supplier. No order is placed to that supplier for any inventory level above the reorder point and a positive order is issued to that supplier for *almost* every inventory level below the reorder point. We characterize conditions under which the optimal policy reveals monotone response to changes in the inventory level. Furthermore, two special cases of our model are examined in detail to demonstrate how our analysis generalizes a number of well-known results in the literature.

In the second project we study a long-run inventory planning problem in which the retailer can replenish inventory and change price dynamically. The study answers the following three questions: 1) How frequently should a retailers change prices? 2) When to change prices if the retail can only adjust price limited times? 3) How to change prices? Our analysis discovers that the frequency of price change is related to the adjustment cost. In the extreme case, the retailer will change price every time the inventory level changes. The retailer prefers to change price less frequently if the adjustment cost is higher. We also investigate the pattern of the optimal sales strategy. We show that it is optimal to change the price from low to high in each replenishment cycle, the optimal order-up-to level may decrease when the ordering cost increases, and fewer customers are served when the unit cost of procurement increases.

Chapter 2

Sourcing from Suppliers with Random Yield for Price-Dependent Demand

2.1 Introduction

[1] Supply as well as demand uncertainties are commonly present in practice. They impose different types of challenges in procurement planning. For example, firms have realized the importance of mitigating supply risk via multiple sourcing whereas chosen to shape demand via dynamic pricing to mitigate the risk of under- or over-stocking. In this study, we combine these two approaches into a problem of jointly pricing and multiple sourcing in a multi-period, single-product revenue maximization setting. We study the problem of an inventory manager who can replenish from two sources with random yields and price the product based on the stock level in each period over a finite planning horizon.

We show that the optimal procurement policy is neither a simple reorder-point policy nor a complex one without any structure. Instead, we prove the existence of a reorder point for each supplier. No order is placed to that supplier for any inventory level above the reorder point and a positive order is

[1]The project is a joint work with Professor Feng Qi and Professor Sridhar Seshadri.

issued to that supplier for *almost* every inventory level below the reorder point. We label such a policy as a "near reorder-point policy" because the optimal order quantity may equal zero at countable points below the reorder level. We also characterize conditions under which the optimal policy reveals monotone response to changes in the inventory level.

Two important special cases of this problem have been widely discussed in the literature. The first is the problem of supply diversification under exogenous price. This line of work was initiated by Anupindi and Akella [5] with further development by Dada et al. [27], Burke et al. [14], Federgruen and Yang [31], Federgruen and Yang [30], among others. For multi-period problems, it is known that under the assumption of a continuous demand distribution, the optimal policy is a reorder-point policy for each supplier. That is, a *strictly* positive order is placed when the inventory level is less than the reorder point for that supplier. In contrast, we establish that for discrete demand distribution the policy is in general a near reorder-point policy.

The second special case of our problem is when only a single unreliable supplier is involved and the demand is price dependent. For this model, Li and Zheng [48] prove that the optimal order quantity and price are nonincreasing in the inventory level when the demand has only an additive noise component. They also conjecture that the optimal policy is complex when a multiplicative demand noise is introduced. In this case, they suggest that there can be several strictly positive alternating ordering and no-ordering intervals of the inventory level. In contrast, we show that such positive no-ordering intervals cannot exist because the policy is a near reorder-point policy.

We consider two suppliers but our results extend to n suppliers. We allow for both additive and multiplicative demand uncertainty. We also discuss how the sequence of knowing the uncertainty and making the decisions impacts the policy behavior. Specifically, there are thirty-two potential cases. The manager can know the price uncertainty before or after the price decision and he also can know uncertainty regarding yields from suppliers before or after placing the order. There can be multiple suppliers or just a single supplier. The demand uncertainty can be additive or multiplicative. The distribution of the demand can be continuous or discrete. We provide results for all these cases.

The remainder of this chapter is organized as follows. In §2.2, we lay out the model and derive our main result—the optimality of a near reorder-point policy. In §2.3, we revisit two important special cases of our model to demonstrate that our policy characterization is exact. We also derive conditions under which the optimal price and order quantity decisions become monotone with respect to the inventory level, which allows us to generalize a number of previous results. In §2.4, we examine the case when either the supply yields are perfectly correlated or the supply and demand are perfectly correlated. Section 2.5 concludes the study.

2.2 Problem Formulation and the Near Reorder-point Policy

This section presents the general model. The problem is formulated in §2.2.1 and the main result is derived in §2.2.2.

2.2.1 Model Development

The manager faces a T-period planning problem. To simplify the notation, we consider a system with stationary parameters over time. At the beginning of period t, the (net) inventory level I_t is reviewed. The manager needs to replenish inventory for fulfilling an uncertain demand $\mathcal{D}(p_t)$, where p_t is the unit selling price that has to be determined. There are two potential suppliers, indexed by $i = 1, 2$, whose output yields may be uncertain. Specifically, if an order of $q_{t,i}$ is placed, the actual amount delivered from supplier i is $u_i q_{t,i}$, where $u_i \in [0, 1]$ is the random yield rate with $\mathbb{E}u_i = \bar{u}_i > 0$. Both the price p_t and orders $(q_{t,1}, q_{t,2})$ must be determined before demand and supply uncertainties are resolved. The unmet demand is backordered and the leftover inventory is carried over to the next period.

The manager pays c_i dollars for each unit delivered from supplier i. In other words, he pays an average of $\bar{c}_i = \bar{u}_i c_i$ for each unit ordered from supplier i. Without loss of generality, we assume $c_1 \leq c_2$. The manager also pays a surplus/shortage cost $H(\cdot)$ upon the demand realization. We assume that $H(\cdot) \geq 0$ is continuous and convex. Also, $H(0) = 0$, $|H(x_1) - H(x_2)| \leq c^H |x_1 - x_2|$ for some positive and finite c^H, and $\lim_{x \to \pm\infty} H(x) = \infty$.

The customer pays the current price p_t upon the realization of demand in period t. This price may not be the same as the price quoted at the time when he actually obtains the product if the demand is backordered. In other words, the delay in filling the order does not affect the price charged directly. Instead, the manager pays a penalty cost for the backorder via $H(\cdot)$, which includes the loss of goodwill or compensation to the customer. Such a pricing scheme is widely used in examining joint pricing and replenishment decisions. The demand in period t is given by

$$\mathcal{D}(p_t) = \varepsilon D(p_t) + \omega, \quad p_t \in [\underline{\underline{p}}, \overline{\overline{p}}], \tag{2.1}$$

where ε and ω are, respective, the multiplicative and additive demand noise terms. We assume that ω has mean zero and support $[\underline{\underline{\omega}}, \overline{\overline{\omega}}]$, and ε has mean one and support $[\underline{\underline{\varepsilon}}, \overline{\overline{\varepsilon}}]$. Moreover, ω and ε are independent. For any feasible choice of $p_t \in [\underline{\underline{p}}, \overline{\overline{p}}]$, the demand is nonnegative with probability one and the average demand $D(p_t)$ is finite. The average demand $D(p_t) \geq 0$, $p_t \in [\underline{\underline{p}}, \overline{\overline{p}}]$, has an inverse $p(d)$ that is decreasing over $[\underline{\underline{d}}, \overline{\overline{d}}]$, where $\underline{\underline{d}} = D(\overline{\overline{p}})$ and $\overline{\overline{d}} = D(\underline{\underline{p}})$. [2] Thus, choosing an average demand d is equivalent to charging a price $p(d)$. We shall assume that $\overline{\overline{p}} = p(\underline{\underline{d}}) > c_1$ so that it is profitable to procure and sell the product for some feasible selling price. We also require the revenue $R(d) = dp(d)$ to be finite and concave for $d \in [\underline{\underline{d}}, \overline{\overline{d}}]$. The pricing scheme and the demand model stated above have been commonly assumed in the literature (see, e.g., [48],[22], [18]). We do not restrict the demand $\mathcal{D}(p_t)$ to follow a

[2] Increasing and decreasing are in the weak sense unless otherwise specified.

continuous distribution. In reality, discreteness of demand distribution may arise due to several reasons, such as sales restrictions (e.g., bulky quantities required by wholesalers).

Now we can write the inventory dynamics for our problem as follows [15]:

$$I_{t+1} = I_t + u_1 q_1 + u_2 q_2 - \varepsilon d - \omega. \tag{2.2}$$

Let $V_t(I)$ be the optimal profit function in period t when the inventory level is I. We assume that $V_{T+1}(I) = 0$ for all I. Then, the optimality equation is given by

[9]
$$V_t(I) = \max_{\substack{q_1 \geq 0, q_2 \geq 0, \\ \underline{\underline{d}} \leq d \leq \bar{\bar{d}}}} J_t(I, q_1, q_2, d), \tag{2.3}$$

where

$$\begin{aligned} J_t(I, q_1, q_2, d) &= R(d) - \bar{c}_1 q_1 - \bar{c}_2 q_2 + \mathbb{E} L_t(I + u_1 q_1 + u_2 q_2 - \varepsilon d), && (2.4) \\ L_t(y) &= -\mathbb{E} H(y - \omega) + \alpha \mathbb{E} V_{t+1}(y - \omega), && (2.5) \end{aligned}$$

and $\alpha \in [0, 1]$ is the discount factor. We denote $q^*_{t,i}(I)$, $i = 1, 2$ and $d^*_t(I)$ to be, respectively, the optimal order quantities and average demand[3] in period t.

[3]If multiple optimal solution exists, we always choose the one with the smallest $q^*_{t,1}(I)$. If there are multiple optimal solutions containing the smallest $q^*_{t,1}(I)$, we choose the one with the smallest $q^*_{t,2}(I)$ and then the one with the smallest $d^*_t(I)$. Under such solution selection criterion, the optimal $q^*_{t,1}(I)$, $q^*_{t,2}(I)$ and $d^*_t(I)$ are continuous in I; see Lemma A.0.1 in the appendix.

For ease of exposition, we have assumed that the random yields and the demand noise terms each form an independent sequence of independent and identically distributed random variables. However, all the results derived carry through when each of the processes are Markov-modulated or when they are correlated. Moreover, all of our results hold when considering more than two suppliers.

2.2.2 A Near Reorder-point Policy

In this section, we prove that the optimal ordering policy for the model developed in the previous section is a near reorder-point policy. Under this policy, as stated in Theorem 2.2.1 below, there is a threshold $I^*_{t,i}$ for supplier i, $i = 1, 2$, in period t. No order is issued to supplier i if the inventory level is above $I^*_{t,i}$. If, however, the inventory level is below $I^*_{t,i}$, a positive order is placed to supplier i almost everywhere—there cannot be a non-degenerate interval below $I^*_{t,i}$ within which no order is issued. Later in §2.3, we will demonstrate that the policy characterized in Theorem 2.2.1 is *exact*. In other words, the optimal ordering decision is neither a simple threshold policy nor a complex one involving alternating ordering and no-ordering intervals as previous studies may suggest [48].

Theorem 2.2.1. (The Near Reorder-Point Policy) *There exists a finite optimal reorder point $I^*_{t,i} < +\infty$, $i = 1, 2$, such that*

$$q^*_{t,i}(I) \begin{cases} = 0 & I \in [I^*_{t,i}, +\infty) \bigcup X_i \\ > 0 & \textit{otherwise.} \end{cases}$$

where X_i is a countable set and thus has Lebesgue measure of zero.

We prove the theorem by establishing Lemmas 2.2.1 through 2.2.6 below, which also allow us to obtain insights into the problem. The idea is to define a benchmark problem, model $\mathcal{B}$, and compare its solution with that of our general model. In particular, the benchmark problem assumes the supplier 1's yield rate in period t is a deterministic value, $\bar{u}_1$, and the yields in any future periods stay at u_1. Therefore, the only difference between model $\mathcal{B}$ defined below and model $\mathcal{G}$ lies in the calculation immediate profit in period t. Specifically, the objective function for the benchmark problem can be stated as

$$[\mathcal{B}] \qquad V_t^B(I) = \max_{\substack{q_1 \geq 0, q_2 \geq 0, \\ \underline{\underline{d}} \leq d \leq \overline{\overline{d}}}} J_t^B(I, q_1, q_2, d),$$

$$J_t^B(I, q_1, q_2, d) = R(d) - \bar{c}_1 q_1 - \bar{c}_2 q_2 + \mathbb{E} L_t(I + \bar{u}_1 q_1 + u_2 q_2 - \varepsilon d).$$

Let $q_{t,1}^B(I)$, $q_{t,2}^B(I)$ and $d_t^B(I)$ denote the optimal decisions of the above problem. We term this benchmark problem as model $\mathcal{B}$, whereas the general problem defined in (2.3)-(2.5) as model $\mathcal{G}$.

The first observation, articulated in the next lemma, is that the cost functions involved in both problems are well-behaved.

Lemma 2.2.1. *The functions L_t, J_t, J_t^B, and V_t are concave. Both $J_t^B(I, q_1, q_2, d)$ and $J_t(I, q_1, q_2, d)$ are submodular in (I, q_i) and supermodular in (q_i, d) and (I, d), $i = 1, 2$.*

Lemma 2.2.1 suggests that the cost functions in both models are concave, which is a direct consequence of concavity of one-period costs. The concavity results allow for further exploration of the optimal policy analytically.

Moreover, the submodularity and supermodularity of the objective functions suggest that inventory and the order quantities are substitutes, whereas they are complements of the average demand. Intuitively, these relations may suggest smaller order quantities and a larger average demand when the inventory level is higher. However, as we show in §2.3, the optimal decisions may not be monotone with respect to the inventory level even in special cases of our model.

The next result indicates a simple strategy to order from supplier 1, who has a deterministic yield, in the benchmark problem $\mathcal{B}$.

Lemma 2.2.2. *There exists a* $\bar{y}^B_{t,1}$ *such that* $q^B_{t,1}(I) = \max\{(\bar{y}^B_{t,1} - I)/\bar{u}_1, 0\}$*. Moreover, there exist* $\bar{q}^B_{t,2}$ *and* $\bar{d}^B_t$*, such that* $q^B_{t,2}(I) = \bar{q}^B_{t,2}$ *and* $d^B_t(I) = \bar{d}^B_t$ *for* $I \leq \bar{y}^B_{t,1}$*.*

Because the yield rate of supplier 1 is deterministic in problem $\mathcal{B}$, determining the order quantity $q^B_{t,1}$ for a given I is equivalent to determining the post-order inventory position $y^B_{t,1} = I + \bar{u}_1 q^B_{t,1}$. Given $(I, q^B_{t,1})$, the optimal order from supplier 2 and the optimal average demand depend on $(I, q^B_{t,1})$ only via $y^B_{t,1}$. These observations allows us to compute the optimal $\bar{y}^B_{t,1}$ as the *base-stock level* for supplier 1, which is independent of the inventory level I. Whenever, the inventory level I is below the base-stock level $\bar{y}^B_{t,1}$, an order is issued to supplier 1 to bring the stock level up to $\bar{y}^B_{t,1}$. Correspondingly, a fixed order of $\bar{q}^B_{t,2}$ is placed to supplier 2 and a fixed price of $p(\bar{d}^B_t)$ is charged. When $I > \bar{y}^B_{t,1}$, however, no order is issued to supplier 1. In this case, the optimal order quantity from supplier 2 and the optimal average demand depend on the inventory level in a nonlinear manner in general, as implied from our later analysis in §2.3.2.

Compared to the benchmark model $\mathcal{B}$, our model $\mathcal{G}$ is less likely to order from supplier 1 as stated in the lemma below.

Lemma 2.2.3. *If* $q^B_{t,1}(I) = 0$*, then* $q^*_{t,1}(I) = 0$*.*

The intuition of Lemma 2.2.3 is as follows. When a positive order is placed to the supplier with random yield in model $\mathcal{G}$ there is a risk in the associated delivery quantity. This risk makes it even more undesirable to order in model $\mathcal{G}$ given that no order was placed with the supplier in model $\mathcal{B}$.

It is interesting to contrast this result to Theorem 4.4 in [48]. They prove that $q^B_{t,1}(I) = 0$ implies $q^*_{t,1}(I) = 0$ and $q^B_{t,1}(I) > 0$ implies $q^*_{t,1}(I) > 0$ for the special case of our models with a single supplier (i.e., $\bar{u}_2 = 0$) and without a multiplicative demand noise (i.e., $\varepsilon = 1$). In other words, the optimal policies call for either ordering from supplier 1 in both models, or not ordering from supplier 1 in both. The comparison in Lemma 2.2.3 allows for the possibility of ordering in model $\mathcal{B}$ (i.e., $q^B_{t,1}(I) > 0$), but not in model $\mathcal{G}$ (i.e., $q^*_{t,1}(I) = 0$). Such a possibility, as shown in our subsequent analysis, can indeed happen in our models.

Our next result further states that whether or not to place a positive order has a direct implication on the marginal value of the inventory.

Lemma 2.2.4. *Let $\tilde{q}_{t,i}(I)$ be the smallest unconstrained maximizer of $J_t(I, q_{t,1}, q_{t,2}, d_t)$ with $d_t = d_t^*(I)$ and $q_{t,j} = q_{t,j}^*(I)$, $j \neq i$.*

i) If $\tilde{q}_{t,i}(I) < 0$, then there exists a $\delta > 0$ such that $\frac{V_t(I) - V_t(I-\delta)}{\delta} \leq c_i$.

ii) If $\tilde{q}_{t,i}(I) > 0$, then there exists a $\delta > 0$ such that $\frac{V_t(I+\delta) - V_t(I)}{\delta} \geq c_i$.

By definition, we must have $q_{t,i}^*(I) = 0$ when $\tilde{q}_{t,i}(I) < 0$ and $q_{t,i}^*(I) = \tilde{q}_{t,i}(I) = 0$ when $\tilde{q}_{t,i}(I) > 0$. According to Lemma 2.2.4 ii), the marginal value of inventory must exceeds the unit ordering of the supplier whenever a positive order is issued to that supplier. Because the marginal value of inventory is decreasing (as $V_t(I)$ is concave), this observation indicates a reorder point for the supplier, which is bounded from the above.

Lemma 2.2.5. *There exists an $I_{t,i}^* < +\infty$, such that $q_{t,i}^*(I) = 0$ for $I \geq I_{t,i}^*$. $i = 1, 2$.*

With Lemmas 2.2.3–2.2.5 in hand, we can derive the following result which directly leads to Theorem 2.2.1.

Lemma 2.2.6. *Suppose that there exists an $\bar{I}$ such that $q_{t,1}^*(\bar{I}) = 0$ and $q_{t,1}^*(\bar{I} + \delta) > 0$ for any $\delta \in [0, \gamma_+)$ and some $\gamma_+ > 0$. Then, there exists a $\gamma_- > 0$ such that $q_{t,1}^*(\bar{I} - \delta) > 0$ for any $\delta \in (0, \gamma_-)$.*

According to Lemma 2.2.6, given that no order should be placed at the inventory level $\bar{I}$ and that a positive order is placed at $\bar{I} + \delta$, then a positive order must be placed around the neighborhood of $\bar{I}$. In other words, there cannot be a non-degenerate interval containing $\bar{I}$ such that no order is placed in this interval. The next lemma further ensures that there can be at most countable number of such $\bar{I}$.

Lemma 2.2.7. *Let X_1 be the set of $\bar{I}$ that satisfies the conditions in Lemma 2.2.6. Then X_1 is countable.*

To see how Lemmas 2.2.6–2.2.7 lead to Theorem 2.2.1, define $I^*_{t,1}$ to be the largest I such that $q^*_{t,1}(I-\delta) > 0$ for arbitrarily small positive δ. Note from Lemma 2.2.5 that $I^*_{t,1}$ is finite. Thus, $q^*_{t,1}(I) = 0$ for any $I \geq I^*_{t,1}$. Then $\bar{I}$ described in Lemma 2.2.6, if it exists, must be lower than $I^*_{t,1}$. Therefore, for $I < I^*_{t,1}$, we must have $q^*_{t,1}(I) > 0$ almost everywhere—leading to the result for supplier 1 in Theorem 2.2.1. The result for supplier 2 can be obtained similarly.

Remark 2.2.1. *Theorem 2.2.1 can be extended to n-supplier settings. All one needs to do is to repeat the argument in the proof of Lemma 2.2.6 with $q_{t,i} = q^*_{t,i}(I)$ for $i = \{2, \cdots, n\}$.*

In our model, we have assumed that the manager makes the pricing decision before observing the yield realization. This fits the situation when the manager does not have the flexibility of real-time price adjustment based on current sales. Such a situation may arise, for example, when the price change requires a certain approval procedure within the firm that takes time or when the sales process is not instantaneously visible to the manager. If, however, the manager is free to quickly adjust the price based on the demand, there can be two variations of our model, as stated in the remarks below.

Remark 2.2.2. *If the price decision is made before observing the yields and after observing the demand noise, then the optimal price decision is decreasing in the inventory level and the optimal order decisions follow a near reorder-point policy. See Lemma A.0.2 in the Appendix.*

Remark 2.2.3. *If both the price and ordering decisions are made after observing yields, then the optimal policy is a base stock list price policy [29].*

The policy described in Theorem 2.2.1 can be complex and reveal non-monotone behavior with respect to the inventory level. Before ending this section, we characterize sufficient conditions under which the optimal policy has simple structural properties.

Theorem 2.2.2. (Conditions for Simple Policy Structure)

i) If the demand $\varepsilon d + \omega$ has a continuous distribution for each $d \in [\underline{\underline{d}}, \overline{\overline{d}}]$, then the optimal order quantities follow a strict reorder-point policy, i.e., $q^*_{t,i}(I) > 0$ for $I < I^*_{t,i}$ and $q^*_{t,i}(I) = 0$ for $I \geq I^*_{t,i}$, $i = 1, 2$.

ii) If the demand has only an additive noise, i.e., $\varepsilon = 1$, the following results hold.

a. When the inventory level increases, the optimal average demand increases but the increase is smaller than the change in the inventory level. That is, for a small enough $\delta > 0$, $0 \leq d^*_t(I + \delta) - d^*_t(I) \leq \delta$.

b. If the yields are of the all-or-nothing type, i.e., $u_i \in \{0, \overline{\overline{u}}_i\}$, $i = 1, 2$, *then the optimal order quantity* $q^*_{t,i}(I)$ *is decreasing in* I, $i = 1, 2$.

Moreover, i) *and* ii-a) *can be extended to general multiple-supplier settings, while* ii-b) *holds only for the two-supplier case.*

Part i) suggests that when the demand has a continuous distribution, the policy characterized in Theorem 2.2.1 becomes a *strict* reorder-point policy. In this case, the reorder point $I^*_{t,i}$ clearly divides the space of the inventory level I into an ordering region $I < I^*_{t,i}$ and a nonordering region $I \geq I^*_{t,i}$. This represents a generalization of the results obtained by Anupindi and Akella [5] and Federgruen and Yang [30], who prove the optimality of strict reorder-point policies for the special case when price is exogenous.

With only an additive noise in demand, Li and Zheng [48] show that the optimal price is decreasing in the inventory level when there is only one supplier. Part ii-a) extends this observation to the case of multiple suppliers. Moreover, if the suppliers' yields are of the all-or-nothing type, the optimal order quantities are decreasing in the inventory level as indicated by part ii-b). This result generalizes a similar one in [5] for the model with exogenous price.

2.3 Two Special Cases and Exact Examples

In this section, we revisit two special cases of our model that have been analyzed in the literature and demonstrate how earlier results might not hold when the demand distribution is discrete or the demand noise is multiplicative. The first special case discussed in §2.3.1 involves two suppliers with random yields and exogenous price. The second special case treated in §2.3.2 assumes one supplier with random yield and endogenously determined price. In each case, we present an exact example for the near reorder-point policy characterized in the previous section.

2.3.1 A Model with Two Suppliers and Exogenous Price

In this case, the price is exogenously given and thus $d = \underline{\underline{d}} = \overline{\overline{d}}$. Anupindi and Akella [5] and Federgruen and Yang [30] analyze a version of this problem with continuous demand distribution and derive a strict reorder-point policy (recall Theorem 2.2.2(i)). Our analysis, however, suggests that such a policy is suboptimal in general when allowing for discrete demand distributions. Example 1 below shows that the optimal policy is exactly a near reorder-point policy:

Example 1: $c_1 = 5$, $c_2 = 6$, $H(x) = 0.5\max\{x, 0\} + 15\max\{-x, 0\}$, $d = 10$, $Pr\{u_1 = 0\} = Pr\{u_1 = 1\} = 0.5$, $Pr\{u_2 = 0.2\} = Pr\{u_2 = 1\} = 0.5$, $Pr\{\varepsilon = 0.5\} = Pr\{\varepsilon = 1.5\} = 0.5$ and $\omega = 0$. The solution of this problem with $T = 1$ is given by

I	$q_1^*(I)$	$q_2^*(I)$
$(-\infty, -\frac{15}{2})$	$4 - \frac{4}{5}I$	$5 - I$
$[-\frac{15}{2}, \frac{5}{2})$	$\frac{5}{2} - I$	$\frac{25}{2}$
$[\frac{5}{2}, 5)$	$-10 + 4I$	$25 - 5I$
$[5, 15)$	$15 - I$	0
$[15, \infty)$	0	0

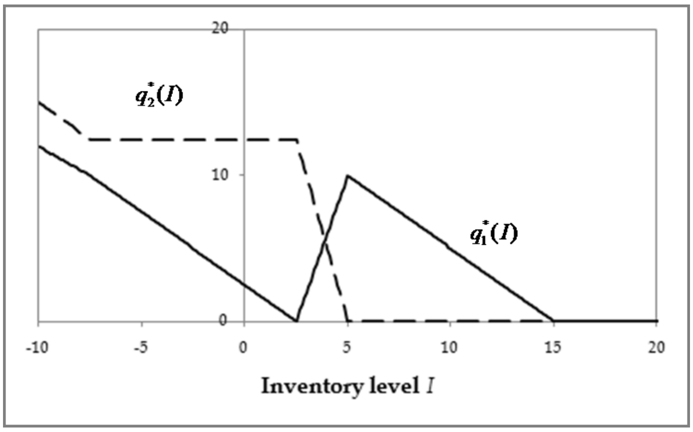


Figure 2.1: The optimal ordering decisions for Example 1.

The optimal policy for Example 1, depicted in Figure 2.1, is the near reorder-point policy characterized in Theorem 2.2.1. Example 1 also suggests that the monotone property derived in Theorem 2.2.2 ii-b) (with the all-or-nothing type yield) does not extend to the case when one suppliers' minimum yield is positive. In this example, the optimal order quantity from supplier 1 is zero when $I = 5/2$, while it is strictly positive around the neighborhood of 5/2. When the inventory level is relatively low, i.e., $I \in [-15/2, 5/2)$, supply uncertainties induce a high risk of stockout. In this case, the optimal policy tends to order more from supplier 2, who is more reliable but more expensive than supplier 1. As the inventory level increases within this range, the order quantity from supplier 2 is fixed at 25/2, while that from supplier 1 reduces linearly. In contrast, when the inventory level is relatively high, i.e., $I \in (5/2, 5)$, it is undesired to place a single large order from one supplier because now the concern is mainly on overstock. As the inventory level increases within this range, the order quantity from supplier 2 decreases dramatically. Instead, the optimal policy tends to allocate more quantity to supplier 1 to leverage its lower ordering cost.

In general, the optimal order quantities from both suppliers may increase or decrease with the inventory level. It is, however, impossible that both order quantities increase at the same inventory level, as stated in the next theorem.

Theorem 2.3.1. *When $d = \underline{\underline{d}} = \overline{\overline{d}}$, the optimal order quantities cannot be strictly increasing in the inventory level at the same time. That is, for any $I^a < I^b$, $q^*_{t,i}(I^a) < q^*_{t,i}(I^b)$ implies $q^*_{t,j}(I^a) > q^*_{t,j}(I^b)$ for $i, j \in \{1, 2\}$ and $i \neq j$.*

2.3.2 A Model with One Supplier and Endogenous Price

Li and Zheng [48] discuss the case with one supplier and endogenous price which corresponds to $u_2 = 0$. They derive a strict reorder-point policy under the assumption that the demand uncertainty is additive. They present a counter example to demonstrate that the optimal policy in the case with the multiplicative demand uncertainty is complex. In their example, it is optimal to order nothing over a strictly positive interval and order a strictly positive quantity for some inventory level above that interval. We know from Theorem 2.2.1 that a solution with a strictly positive no-order interval below the reorder point is suboptimal and the optimal policy should be a near reorder-point policy. In fact, a strictly threshold policy is optimal in their example as we show in Appendix. [4]

The optimal policy for the case with the multiplicative demand uncertainty can be illustrated using Example 2 below. This example suggests the optimal policy in general is neither a complex one without any structure nor a strict reorder-point policy:

Example 2: $T = 1$, $c_1 = 12.8$, $H(x) = 0.5\max\{x,0\} + 25\max\{-x,0\}$, $p(d) = 20 - 0.5d$, $Pr\{u_1 = 0.1\} = Pr\{u_1 = 0.4\} = 0.5$, $u_2 = 0$, $Pr\{\varepsilon = 0.5\} = Pr\{\varepsilon = 1.5\} = 0.5$ and $\omega = 0$. The solution of this problem with $T = 1$ is given by

I	$q_1^*(I)$	$d^*(I)$	$\mathbb{E}[I + uq_1^*(I) - \varepsilon d^*(I)]$
$(-\infty, 0)$	$-\frac{5}{2}I$	0	$\frac{3}{8}I$
$[0, \frac{1}{12})$	$20I$	$6I$	0
$[\frac{1}{12}, \frac{1}{4})$	$\frac{5}{2} - 10I$	$\frac{1}{2}$	$\frac{1}{8} - \frac{2}{3}I$
$[\frac{1}{4}, \frac{3}{4})$	0	$2I$	$-I$
$[\frac{3}{4}, \frac{9}{4})$	0	$\frac{3}{2}$	$-\frac{3}{2} + I$
$[\frac{9}{4}, 60)$	0	$\frac{2}{3}I$	$\frac{1}{3}I$
$[60, \infty)$	0	40	$-40 + I$

In general, both the optimal order quantity and price may not be monotone with respect to the inventory level. However, at least one of them is decreasing as the inventory level increases, as stated in the next theorem.

Theorem 2.3.2. *When $u_2 = 0$, the optimal order quantity $q_1^*(I)$ and the optimal price cannot be strictly increasing in the inventory level at the same*

[4] We replicate their example and find such a complex structure goes away when the computational accuracy is high enough.

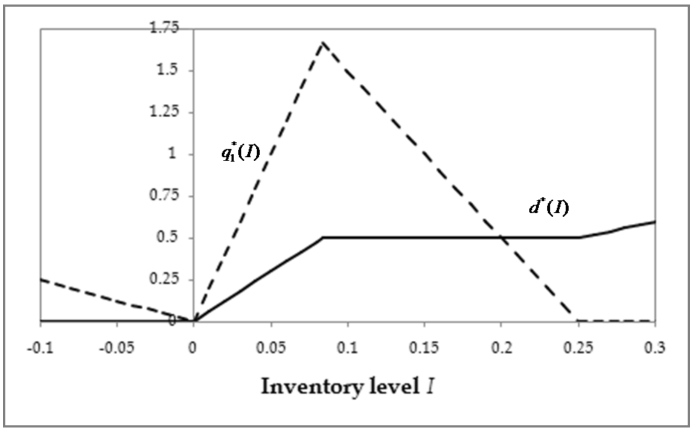


Figure 2.2: The optimal ordering and pricing decisions for Example 2.

time. That is, for any $I^a < I^b$, $q^*_{t,1}(I^a) < q^*_{t,1}(I^b)$ *implies* $d^*_t(I^a) < d^*_t(I^b)$ *and* $d^*_t(I^a) > d^*_t(I^b)$ *implies* $q^*_{t,1}(I^a) > q^*_{t,1}(I^b)$.

2.4 Effect of Correlation on the Optimal Policy

The complex policy behavior in our model is due to the *direct* interactions between the decisions and the various source of uncertainties via the terms u_1q_1, u_2q_2, and εd in the state dynamics (2.2). As pointed out in Theorem 2.2.2 and Remarks 2.2.2-2.2.3, the optimal decisions may become monotone with respect to the inventory level when some of such interactions are removed from the model. Our analysis so far has assumed that the random terms involved are independent. Intuitively, the source of uncertainties can be reduced when some of the random terms become perfectly correlated. The question is: will the optimal policy reveal monotone property when some of the random terms $\{u_1, u_2, \varepsilon\}$ become perfectly correlated? The answer, as we demonstrate below, depends on the sign of correlation— it is true under negative correlation but not under positive correlation. In §2.4.1, we study correlated yields with two suppliers and exogenously determined price. We also analyze the case when the yield of one supplier and the demand noise are correlated in §2.4.2.

2.4.1 Two Perfectly Correlated Suppliers and Exogenous Price

The yields from two different suppliers can correlated for various reasons. For example, the suppliers might procure components from a single source [54]. One such famous case was a small fire at a microchip plant owned by Philips in 2000. The plant supplied chips to both Ericsson and Nokia. Both Ericsson and Nokia had low yield due to the accident. When suppliers are in one region, their yields can be positively correlated. All Taiwanese LCD suppliers are exposed to the same natural disaster and the same political instability. If, however, the two sources of supplies are both internal manufacturing facilities, the firm may have limited engineering expertise or quality investment budget to allocate between these facilities [46]. In this case, the yields can be negatively correlated. To model these situations, we assume the yield of supplier 1 has a linear relation with the yield of supplier 2:

$$u_1 = au_2 + b, \tag{2.6}$$

where a and b are constants.

Theorem 2.4.1 characterizes the monotone properties of the optimal decisions which are also summarized in Table 2.1.

Theorem 2.4.1. (Perfect Correlated Supply Yields) *When* $d = \underline{\underline{d}} = \overline{\overline{d}}$*, the following results hold.*

i) When $b \geq 0$*,* $q^*_{t,1}(I)$ *is decreasing in* I*.*

ii) When $ab \leq 0$*,* $q^*_{t,2}(I)$ *is decreasing in* I*.*

iii) Suppose $c_1 \leq c_2$*. For* $b = 0$ *and* $a > 0$*, there exists an optimal solution in which* $q^*_{t,1}(I)$ *is decreasing in* I *and* $q^*_{t,2}(I) = 0$*. For* $b = -a\overline{\overline{u}}_2$ *and* $a < 0$*, there exists* $\bar{q}_{t,2}$ *such that* $q^*_{t,1}(I)$ *is deceasing in* I *and* $q^*_{t,2}(I) = \max\{\bar{q}_{t,2} - \frac{I}{\overline{\overline{u}}_2 + \underline{\underline{u}}_2}, 0\}$.

We interpret Theorem 2.4.1 with the help of Table 2.1, in which we classify the policy according to the stochastic relation between the yields and their correlation. The conditions inside the parentheses give rise to the corresponding sign of correlation and stochastic relation between the yields. We observe from the second row of Table 2.1 that both order quantities are decreasing in the inventory level when the yields are perfectly negatively correlated. (In this case, the orders to the suppliers might be complementary and thus reduce the supply risks.) If, however, the yields are positively correlated, the order to the supplier with the stochastically larger yield is decreasing in the inventory level, while the other order may increase or decrease with the inventory level. Thus, the more reliable supplier is used to hedge with inventory and the less reliable to miligate the increase in variance of ordering more from the other supplier.

Table 2.1: Interpretation of Theorem 2.4.1.

	$u_1 \leq^{st} u_2$	$u_1 \geq^{st} u_2$
Positive corr.	$(0 < a < 1,\ -\underline{u}_2 < b < 0)$	$(a > 1,\ b \geq 0)$
$a > 0$	$q^*_{t,1}(I) \uparrow\downarrow,\ q^*_{t,2}(I) \downarrow$	$q^*_{t,1}(I) \downarrow,\ q^*_{t,2}(I) \uparrow\downarrow$
Negative corr.		
$a < 0$	$q^*_{t,1}(I) \downarrow,\ q^*_{t,2}(I) \downarrow$	$q^*_{t,1}(I) \downarrow,\ q^*_{t,2}(I) \downarrow$

2.4.2 One Supplier and Endogenous Price with Perfect Correlation

In practice, the yield and demand can be correlated for various reasons. For example, the yield rate is decided by the technique and the equipment [6]. When a supplier realizes that the incoming demand is high, he might prefer to adopt a more reliable (high-tech) production line. In this case, the yield and the demand uncertainty are positively correlated. However, if the supplier is a small company, then a booming market might make the small company experience shortage of raw materials. In this case, the yield is negatively correlated with demand. To address the effect of the correlation, we focus on a single supplier ($u_2 = 0$) and assume that the price is endogenously determined. We assume that the yield rate u_1 and the multiplicative demand uncertainty ε are correlated as follows:

$$u_1 = a\varepsilon + b \tag{2.7}$$

for some constants a and b.

Theorem 2.4.2 characterizes the monotone properties of the optimal decisions which are also summarized in Table 2.2.

Theorem 2.4.2. (Perfectly Correlated Yield and Demand) *When $u_2 = 0$, the following results hold.*

i) $d_t^*(I)$ *is increasing in* I *when* $ab \leq 0$. *In particular, if* $b = 0$ *(then we must have* $a \geq 0$*),* $d_t^*(I) = \arg\max_{d \in [\underline{d}, \bar{\bar{d}}]} \{R(d) + (\bar{c}_1/a)d\}$ *is constant for* $I \leq I_{t,1}^*$ *and* $d_t^*(I)$ *is increasing in* I *for* $I > I_{t,1}^*$.

ii) $q_{t,1}^*(I)$ *is decreasing in* I *when* $b \geq 0$.

Table 2.2: Interpretation of Theorem 2.4.2.

	$\frac{u_1}{\bar{u}_1} \leq^{st} \varepsilon$	$\frac{u_1}{\bar{u}_1} \geq^{st} \varepsilon$
Positive corr.	$(0 < a < \bar{u}_1,\ -\underline{\varepsilon}\bar{u}_1 < b < 0)$	$(a > \bar{u}_1,\ b \geq 0)$
$a > 0$	$q_{t,1}^*(I) \uparrow\downarrow$, $p_t^*(I) \downarrow$	$q_{t,1}^*(I) \downarrow$, $p_t^*(I) \uparrow\downarrow$
Negative corr.		
$a < 0$	$q_{t,1}^*(I) \downarrow$, $p_t^*(I) \downarrow$	$q_{t,1}^*(I) \downarrow$, $p_t^*(I) \downarrow$

We interpret Theorem 2.4.2 with the help of Table 2.2, in which we classify the policy according to the stochastic relation between the yield and the demand and their correlation. The conditions inside the parentheses give rise to the corresponding sign of correlation and stochastic relation between the yield and the demand. We observe from the second row of Table 2.2 that both the order quantity and the optimal price are decreasing in the inventory level when u_1 and ε are perfectly negatively correlated. This is similar to the previous example in which the two risks affect one another. If, however, u_1 and ε are positively correlated, either price or the order quantity response might not be monotone. Specifically, when the price has a stochastic larger noise, the price is still decreasing in the inventory level, while the order quantity may increase or decrease with the inventory level.

2.5 Concluding Remarks

In the chapter, the price is decided before observing yield and demand. Our results can be extended to the case of pricing after observing demand and the case of placing the orders after observing the yield. Combining these, we summarize all cases in which there is a simple structure to the optimal policy in Table 2.3. For example, we can see that a strictly reorder-point policy is obtained when the demand has a continuous distribution. However, if the demand has a discrete distribution, the optimal order quantity in general is a near reorder-point policy. We can also see that both the order quantity and price are monotone in the case of a single supplier and additive demand uncertainty. However, the optimal order quantity follows a near reorder-point policy when the demand has multiplicative uncertainty or when there are multiple suppliers.

There are several dimensions along which our work could be extended. The chapter focuses on the finite horizon. The characterization of the optimal policy for the infinite horizon case would be interesting. In the chapter, we assume that suppliers do not carry inventory and are not strategic on their decisions. A possible extension is to consider suppliers' behavior under competition.

Table 2.3: Summary

Monotone Order Quantity	• Observing one supplier's yield before ordering, or • a single supplier and additive demand uncertainty, or • two all-or-nothing suppliers and additive demand uncertainty
Monotone Price	• Observing yields of two suppliers before pricing, or • deterministic yields, or • additive demand uncertainty, or • multiplicative demand uncertainty and deterministic yield
Reorder-point policy	• Continuous demand distribution, or • a single supplier and additive demand uncertainty

Chapter 3

Inventory-based dynamic pricing with costly price adjustment

3.1 Introduction

Rapid advances in technology have created many new opportunities for firms to change price and replenish inventory frequently. Firms realize that pricing and inventory decisions are interdependent in influencing the dynamics of demand and supply. A low price in the case of overstock typically allows for faster inventory turnover and reduced holding costs, and a high price in the case of stockout often alleviates the pressure of backlogging. Coordinating pricing and inventory decisions has become a major strategy for many firms [28][17]. In the meanwhile, price revision may be associated with significant cost (e.g. LevyBergen and DuttaVenable (1997) [47], Aguirregabiria(1999) [4], Chen and Hu(2012) [20]) that cannot be overlooked in evaluating the benefit of implementing an inventory-based pricing strategy. Moreover, firms must understand the effect of price revision cost on the design of combined inventory and pricing policy.

In this paper we study how a firm's ability to change prices frequently and efficiently may create additional revenue opportunities. We model a continuous-review stochastic control problem to make the pricing and inventory decisions. The customer arrives according to a Poisson process with a price-dependent arrival rate. A replenishment order is associated with a fixed ordering cost and a variable ordering cost. During each inventory replenishment cycle, the firm decides the amount of stock to order as well as how to adjust the price as customer demand arrives. Every time the price is revised, the firm pays a fixed adjustment cost. The goal is to find a combined pricing and ordering policy that yields a high profitability.

In our model, if the inverse demand arrive rate is linear or convex in the average interarrival time over a certain range, then the optimal price must be at one of the two ends of this range. This result suggests that we can replace any inverse demand rate function by its concave envelope. The resulting problem has the same optimal solution as the original problem. In other words, our formulation is general enough to allow for any price-demand relations, which represents a generalization of related studies in the literature. Moreover, our analysis holds for any monotone inventory holding cost functions, which need not be concave or convex.

We show that the optimal prices are always higher than the myopic price which maximizes the instantaneous gross (of any fixed costs) profit rate. Within an order cycle, the optimal price increases and the time-average gross profit rate increases as the inventory level drops. In general, the optimal order-up-to level increases when the variable ordering cost increases, the inventory holding cost increase and the demand increases. Interestingly, the optimal order-up-to level may not be monotone in the fixed ordering cost. This is due to the presence of a fixed price adjustment cost, which requires a careful coordination between price adjustment and inventory replenishment. We show that the scale economy in ordering is reflected in the replenishment cycle—A larger fixed ordering cost induces a longer average order cycle length.

When price adjustment is not costly, the optimal profit rate can be represented by the difference of the marginal price and inventory holding cost within each price segment, i.e., the range of stock level within which a single price is charged. In this case, the optimal pricing and ordering decisions can be computed easily. With costly price adjustment, however, the problem can be computationally challenging, because the profit function is not unimodal in the order quantity. We drive bounds on the optimal order quantity, which allows for narrowing down the search of the optimal solution. To allow for further computational tractability for problems with strong economies of scale in ordering, we consider a relaxation of the model in which we allow the inventory level to take continuous values. Under this relaxation, we uncover an interesting trade off between the fixed costs and the inventory holding cost that is analogous to the classical EOQ model. In particular, when the inventory holding cost is linear (convex, concave) in the stock level, the total of fixed ordering cost and price adjustment cost is equal to (greater than, less than) the total inventory holding cost within a replenishment cycle. We further show that by properly accounting the inventory holding cost, the optimal solution yields a same time-average profit for all price segments. This observation allows us to identify an efficient way to compute the optimal solution for the relaxed problem.

Through a numerical study, we demonstrate that the inventory-based price adjustment can yield significant profit improvement compared to the optimal static pricing policy. This is particularly the case when the fixed ordering cost is high, the variable ordering cost is high, the inventory holding cost is high, the demand is high, and the price adjustment cost is low. We also find that limited price adjustment can yield a benefit that is close to unlimited price adjustment.

The remainder of the paper is organized as follows. Section 3.2 reviews related literatures. We lay out the model in §3.3 and analyze the optimal policy in §3.4. In §3.5, we discussed a relaxed problem of our model. Section 3.6 concludes the study.

3.2 Literature Review

In practice, varying prices is often a natural mechanism for revenue management [38]. In most retail and industrial trade settings, firms use various forms of dynamic pricing such as promotion [7], auction [58], end-of-season sale [33], clearance price [55], personalized service [53], and price negotiations [43] to respond to market fluctuations and uncertainty in demand. Most of the studies in this area focus on understanding the firm's optimal selling strategy and do not consider inventory replenishment.

In many production and supply-chain management contexts, inventory can be replenished at a cost. In such cases, both pricing and inventory decisions need to be made. Pricing decisions are used to control demand, while replenishment decisions are used to control supply. The central problem is to optimally coordinate these demand and supply decisions. The study on combined pricing and inventory decisions dates back to Whitin(1955)[59], with further development along various dimensions including seasonal demand [39], fixed ordering cost [21][22], unreliable supply [51] [48] [32] and risk aversion [3] [24]. Excellent surveys are provided by Elmaghraby and Keskinocak(2003)[28], Yano and Gilbert(2003)[60], and ChenSimchi-Levi(2012) [23].

In a study similar to ours, Rajan, Rakesh and Steinberg(1992) [52] consider the relationship between pricing and ordering decisions for a monopolistic seller firm facing deterministic demand. Their paper provides guidance in determining when price changes during the cycle are worthwhile due to product aging, how often such changes should be made and how such changes affect ordering frequency and quantities. We consider uncertain demand arrival process and costly price adjustment. Consequently, the price revision in our model is based on the different inventory level, which is closer to practice. Chen, Wu and Yao(2010) [19] study a similar model in which the demand process is a price dependent Brownian motion. They consider three forms of demand-price relation, namely, linear, exponential and power functions. They also assume that the inventory holding cost is linear in the stock level. In contrast their study, our model allows for a general monotone price-demand function and a general monotone inventory holding cost function. Moreover, the consideration of costly price adjustment makes our problem much more challenging.

There are several papers model costly price adjustment in dynamic pricing. Celik, Muharremoglu and Savin(2009) [16] analyze the problem of selling a fixed stock of product over a finite horizon. They demonstrate the complexity of the optimal policy in the presence of price adjustment cost. Aguirregabiria(1999) [4] considers a periodic inventory replenishment system in which price adjustment is associated with a fixed cost. They demonstrate a cyclical price behavior in the optimal policy. Chen and Hu(2012) [20] extend this study by allowing different costs for markup and markdown. Under the assumption that the demand is deterministic, they design a polynomial time algorithms to maximize the total profit. Chen, Zhou and Chen(2011) [25] consider a more complex price adjustment cost which consists of a fixed component and a variable component. They derive solutions for two special cases of the problem, one without fixed price adjustment cost and one without inventory carryover. For the general model, they demonstrate the complexity of the optimal policy and propose a heuristic solution. Unlike the aforementioned studies, which assumes a linear price-demand relation, our model allows for any monotone demand function that need not be concave or convex.

3.3 The Model

The firm faces an infinite-horizon planning problem. The goal is to maximize the long-run average profit by appropriately adjusting the inventory level and the product price over time.

Customer demand arrives according to a Poisson process, whose rate $\lambda(p)$ depends on the price p currently chosen by the manager. We assume that the set of feasible prices $\mathcal{P}$ is bounded with $\underline{p} = \min \mathcal{P}$ and $\overline{p} = \max \mathcal{P}$. Also, $\lambda(p) > 0$ for $p \in \mathcal{P}$ and $\lambda(p)$ strictly decreasing. Then the expected interarrival time of the demand, $\tau(p) = 1/\lambda(p)$ is strictly increasing in p, and its inverse $p(\tau)$ is increasing over $\mathcal{T} = \{\tau : p(\tau) \in \mathcal{P}\}$. We also define $\underline{\tau} = \tau(\underline{p})$ and $\overline{\tau} = \tau(\overline{p})$. Therefore, charging a price $p(\tau)$ is equivalent to setting an expected interarrival time τ. Whenever he adjusts the price, the manager incurs a fixed cost $A \geq 0$.

The manager can pay a fixed cost K and a variable cost c to replenish inventory with a negligible delivery leadtime. The manager also incurs an inventory cost rate of $h(i)$ per unit time when the inventory level is i. We assume that $h(i)$ is strictly increasing in i with $h(0) = 0$ and $h(i) > 0$ for $i > 0$. Demand backlogging is not allowed.

In theory, the manager may adjust the price p at any time but at a cost A. However, because the demand process is Poisson, it is straightforward to see that we only need to restrict to policies that change price and inventory at a demand arrival epoch. Now consider the situation where a demand arrival just occurs and the current inventory level becomes i. Suppose that the average time until the next arrival is τ_i. Then the expected gross profit during τ_i can be computed as

$$\pi(i, \tau_i) = p(\tau_i) - h(i)\tau_i - c. \qquad (3.1)$$

We shall assume that $\pi(1, \overline{\tau}) > 0$ to rule out the trivial case where it is never optimal to order and sell any positive quantity. We further note that because holding inventory is costly (i.e., $h > 0$) and backlogging is not allowed, the manager should always replenish whenever the inventory level drops to zero. Then the manager's problem is to choose an order-up-to level S and a set of expected interarrival times $\vec{\tau} \equiv \{\tau_1, \cdots, \tau_S\}$ that maximizes his long-run average profit expressed as

$$V(S, \vec{\tau}) = \frac{\sum_{i=1}^{S} \pi(i, \tau_i) - A \sum_{i=1}^{S-1} \mathbb{1}_{\{\tau_i \neq \tau_{i+1}\}} - K}{\sum_{i=1}^{S} \tau_i}, \qquad (3.2)$$

where $\mathbb{1}_X$ is the indicator function for event X. Note that in computing the above profit function, we have assumed a stationary policy $(S, \vec{\tau})$ and applied the renewal reward theorem. For our subsequent analysis, we use $(S^*, \vec{\tau}^*)$ to denote the optimal decisions and V^* to denote the optimal average profit. We also define $\vec{\hat{\tau}}(S)$ to be the vector of optimal expected interarrival times and

$$\hat{V}(S) = V(S, \vec{\hat{\tau}}(S))$$

to be the corresponding average profit when the order-up-to level is fixed at S. In the case when multiple optimal solutions exist, we always choose the one with the smallest S^*.

3.4 Analysis of the Optimal Policy

In this section, we analyze the properties of the profit function and derive the optimal policy. In §§3.4.1, we characterize the optimal solution under several special cases, which allows us to rule out these cases in our analysis of the general model. In §§3.4.2, we specify the properties of the optimal policy and demonstrate the complexity of solving the problem. In §§3.4.3, we analyze how the optimal policy responds to the changes of the model parameters to obtain insights into the policy behaviors.

3.4.1 Preliminaries

In our model, we do not impose any condition on the price-demand relation. Our first result stated below suggests that it is without loss of generality to restrict to a concave $p(\tau)$.

Lemma 3.4.1. *Suppose*

$$p(\tau^b) \leq \frac{\tau^c - \tau^b}{\tau^c - \tau^a} p(\tau^a) + \frac{\tau^b - \tau^a}{\tau^c - \tau^a} p(\tau^c)$$

for some feasible $\tau^a, \tau^b, \tau^c \in \mathcal{T}$ *with* $\tau^a < \tau^b < \tau^c$*, then* $\hat{\tau}_i(S) \neq \tau^b$ *for any* $1 \leq i \leq S$ *and any* S*.*

To understand Lemma 3.4.1, we first consider the case where the feasible set of expected interarrival times is a compact interval, i.e., $\mathcal{T} = [\underline{\tau}, \overline{\tau}]$. If $p(\tau)$ is convex or linear over some interval $[\tau^a, \tau^c] \subset [\underline{\tau}, \overline{\tau}]$, then Lemma 3.4.1 states that the optimal $\tau_i(S)$ cannot be an interior point of this interval. With this result, we can replace $p(\tau)$ by its concave envelope $\bar{p}(\tau)$ defined by the following procedure. For each interval $[\tau^a, \tau^c]$ over which $p(\tau)$ is convex, define a pair $\{\tau^1, \tau^2\}$ with $\tau^1 \leq \tau^a < \tau^c \leq \tau^2$ such that

$$\begin{cases} \tau^1 = \min\{\tau^A : p(\tau) \leq \frac{\tau^2-\tau}{\tau^2-\tau^A} p(\tau^A) + \frac{\tau-\tau^1}{\tau^2-\tau^A} p(\tau^2) \text{ for any } \tau^A \leq \tau \leq \tau^2\}, \\ \tau^2 = \max\{\tau^C : p(\tau) \leq \frac{\tau^C-\tau}{\tau^C-\tau^1} p(\tau^1) + \frac{\tau-\tau^1}{\tau^C-\tau^1} p(\tau^C) \text{ for any } \tau^1 \leq \tau \leq \tau^C\}. \end{cases}$$

We replace the segment of $p(\tau)$ over $[\tau^1, \tau^2]$ by a linear segment defined as $\bar{p}(\tau) = \frac{\tau^2-\tau}{\tau^2-\tau^1} p(\tau^1) + \frac{\tau-\tau^1}{\tau^2-\tau^1} p(\tau^2)$ and keep $\bar{p}(\tau) = p(\tau)$ otherwise. Then it is easy to see that $\bar{p}(\tau)$ is concave. Moreover, based on Lemma 3.4.1, the optimal solution for the problem with the inverse demand $\bar{p}(\tau)$ coincides with that for our original problem.

Now consider the case where the feasible set of the expected interarrival times is discrete, i.e., $\mathcal{T} = \{\tau^1, \tau^2, \cdots\}$. Then one can perform the test with all $\tau^a, \tau^b, \tau^c \in \mathcal{T}$ and exclude the points τ^b satisfying the condition in Lemma 3.4.1. This leads to a new feasible set over which $p(\tau)$ is concave.

The next lemma further provides a lower bound on the optimal expected interarrival time that can be easily computed.

Lemma 3.4.2. *$\tau_i^* \geq \tau^M = \arg\max_{\tau \in \mathcal{T}} \{[p(\tau) - c]/\tau\}$[1] for all $1 \leq i \leq S^*$. Moreover, if $p(\tau)$ is concave, $[p(\tau) - c]/\tau$ is unimodal in τ.*

We note from (3.1) that τ^M also maximizes $\pi(i, \tau)/\tau = [p(\tau) - c]/\tau - ih$. In other words, τ^M is the *myopic* optimal expected interarrival time when we

[1]We always choose the smallest maximizer if multiple ones exist.

ignore the fixed cost of ordering K and the fixed cost of price adjustment A. Lemma 3.4.2 indicates that the presence of fixed costs induces a longer expected interarrival time, or alternatively a higher product price. We remark that τ^M is determined myopically because it does not take into account its impact on the future inventory level and inventory cost.

Lemma 3.4.3. *If $A \geq K$, then $\tau_i^* = \tau_1^*$ for all $1 \leq i \leq S^*$.*

Lemma 3.4.3 suggest that price adjustment is never optimal if the associated fixed cost is higher than that of ordering. Based on Lemmas 3.4.1, 3.4.2, and 3.4.3, it is without loss of generality that we assume a concave $p(\cdot)$, $\underline{\tau} = \tau^M$, and $A < K$ in our subsequent analysis.

3.4.2 Structure of the Optimal Solutions

Within each order cycle, the inventory level drops from the order-up-to level S to zero as the demand arrives. Intuitively, the manager should charge a higher price at a lower inventory level. This is formally established in the next lemma.

Proposition 3.4.1. *The following results hold:*

i) $\hat{\tau}_i(S)$ *is weakly decreasing in* i *for any* $S > 1$.

ii) $[p(\tau_i^*) - c]/\tau_i^*$ *is weakly decreasing in* i.

iii) $\pi(S^* + 1, \tau_{S^*}^*)/\tau_{S^*}^* \leq V^* < \pi(S^*, \tau_{S^*}^*)/\tau_{S^*}^*$.

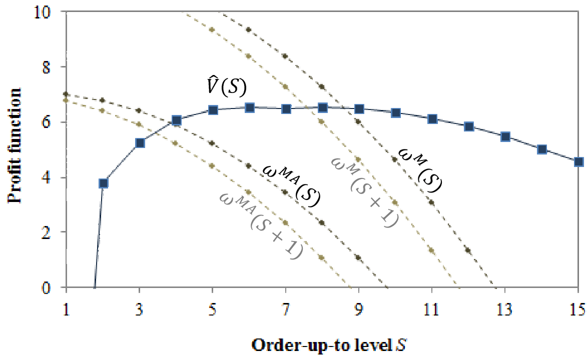
Proposition 3.4.1(i) states that the expected interarrival time and thus the price is weakly increasing as the inventory level decreases. This is because at the beginning of the order cycle, the inventory holding cost is high due to the high inventory level. Speeding up the demand arrival by charging a low price helps to reduce inventory. Toward the end of the order cycle, the inventory level and thus the inventory holding cost becomes low. Slowing down the demand arrival by posting a high price allows the manager to improve his profit. Such a policy leads to an increased time-average profit at each replenishment cycle, as Proposition 3.4.1(ii) further suggests. In particular, the time-average profit from selling the first unit, i.e., $\pi(S^*, \tau^*_{S^*})/\tau^*_{S^*}$, is lower than that of selling the subsequent units. This is mainly because a lower price is charged at a higher inventory level. Moreover, the time-average profit of selling any item should be no lower than the optimal profit rate V^* because of the fixed costs involved in the latter.

Proposition 3.4.2. *The function $V(S)$ satisfies the following:*

$$\begin{cases} \hat{V}(S) < \omega^M(S+1) & \text{if } S < S^*, \\ \omega^{MA}(S+1) \le \hat{V}(S) < \omega^M(S) & \text{if } S = S^*, \\ \hat{V}(S) \ge \omega^{MA}(S) & \text{if } S > S^*, \end{cases} \tag{3.3}$$

where $\omega^M(i) = \frac{\pi(i,\tau^M)}{\tau^M}$, $\omega^{MA}(i) = \frac{\pi(i,\tau^{MA})-A}{\tau^{MA}}$, *and* $\tau^{MA} = \arg\max_{\tau\in\mathcal{T}} \frac{p(\tau)-c-A}{\tau}$.

By definition, $\omega^M(i)$ is the gross profit rate (excluding all fixed costs) of selling the ith unit, and $\omega^{MA}(i)$ adjusts $\omega^M(i)$ by including a fixed price adjustment cost for each unit. Clearly, $\omega^M(i) \ge \omega^{MA}(i)$ and both can be computed easily. When it is not costly to adjust price, i.e, $A = 0$, $\omega^M(i) =$



Notes. $K = 10$, $c = 0.2$, $h(i) = 0.074h^2$, $A = 2.15$, $\tau \in \{0.4, 0.8, 1.2\}$, $p(0.4) = 5$, $p(0.8) = 8$ and $p(1.2) = 10$. The optimal solution is $S^* = 8$ and $(\tau_1^*, \cdots, \tau_7^*) = (0.8, 0.8, 0.8, 0.4, 0.4, 0.4, 0.4, 0.4)$.

Figure 3.1: The long-run profit function as characterized in Proposition 3.4.2

$\omega^{MA}(i)$. In this case, Proposition 3.4.2 suggests a simple way to search for the optimal order-up-to level.

When $A > 0$, the long-run profit function $\hat{V}(S)$ is not unimodal in S, as demonstrated in Figure 3.1. Proposition 3.4.2 suggests how to narrow down the range to search for an optimal S^* over the curve $\hat{V}(S)$. The first relation in (3.3) implies $S^* \geq 8$ and the last relation in (3.3) suggests $S^* \geq 4$. The optimal order-up-to level is $S^* = 8$ and the optimal policy calls for changing the price after selling 3 units during an replenishment cycle.

Lemma 3.4.4. *The function $\hat{V}(S)$ satisfies the following conditions.*

i) If $S \leq S^*$, then $\hat{V}(S) - \hat{V}(S_1) > -\delta\omega$ for all $S_1 < S$,

ii) If $S > S^*$, then $\hat{V}(S) - \hat{V}(S_2) > -\delta\omega$ for all $S < S_2$,

where $\delta\omega \equiv \omega^M(i) - \omega^{MA}(i)$ is independent of i.

In general, Lemma 3.4.4 further suggests the direction to search for the optimal order-up-to level. If we find $S_1 < S$ satisfying $\hat{V}(S) - \hat{V}(S_1) > -\delta\omega$, then the optimal order-up-to level is bigger than S. If we find $S_2 > S$ satisfying $\hat{V}(S) - \hat{V}(S_2) > -\delta\omega$, then the optimal order-up to level is less than S. We note that when there is not cost for price adjustment, i.e., $A = 0$, we have $\omega = 0$ and thus Lemma 3.4.4 implies that $\hat{V}(S)$ is unimodal.

Lemma 3.4.5. *$S^* \in [\underline{S}, \overline{S})$, where*

$$\begin{aligned} \overline{S} &\equiv \inf\left\{n : \tau^M \sum_{i=1}^{n} [h(n) - h(i)] \geq K\right\}, \\ \underline{S} &\equiv \sup\left\{n : \sum_{i=1}^{n} [A + \overline{\tau}h(n) - \underline{\tau}h(i)] < K\right\}. \end{aligned}$$

Lemma 3.4.5 specifies the upper and the lower bounds of the optimal order-up-to level S^*. Note that when the inventory holding cost is linear, i.e., $h(i) = hi$, then

$$\begin{aligned} \underline{S} &= \left\lfloor \sqrt{\frac{2K}{h(2\overline{\tau} - \underline{\tau})} + [\frac{2A - \underline{\tau}h}{2h(2\overline{\tau} - \underline{\tau})}]^2} + \frac{2A - \underline{\tau}h}{2h(2\overline{\tau} - \underline{\tau})} \right\rfloor, \\ \overline{S} &= \left\lceil \sqrt{\frac{2K}{\tau^M h} + \frac{1}{4}} + \frac{1}{2} \right\rceil. \end{aligned}$$

The bounds $\underline{S}$ and $\overline{S}$ increases when the set-up cost K increases and decreases when h increases. Also, $\underline{S}$ decreases when the adjustment cost A increases, and $\overline{S}$ decreases when the *myopic* optimal expected interarrival time τ^M increases.

3.4.3 Comparative Statics

In the section, we analyze how the optimal policy depend on the model parameters.

Proposition 3.4.3 (The Optimal Order-up-to Level)**.** *Consider two otherwise identical systems with*

i) variable ordering costs $c^a > c^b$*,*

ii) inventory holding costs $h^a(i) \geq h^b(i)$ *satisfying* $h^a(i) - h^b(i)$ *being weakly increasing in* i*,*

iii) fixed ordering costs $K^a < K^b$ *and* $A = 0$*, or*

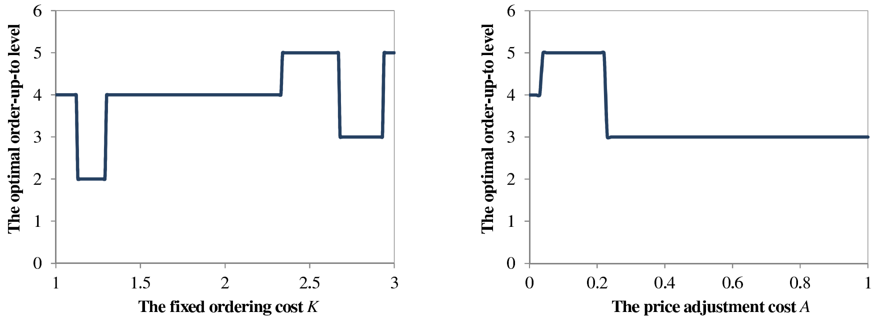
iv) inverse demand functions $p^i(\tau) = \tilde{p}(\gamma^i, \tau)$*,* $i = 1, 2$*, being submodular in* (γ^i, τ) *and increasing in* γ^i*, and* $\gamma^a < \gamma^b$*.*

Then $S^{a*} \leq S^{b*}$*.*

According to Proposition 3.4.3, a lower unit ordering cost, a higher inventory cost,[2] or a lower demand rate leads to a higher order-up-to level, which confirms with one's intuition. One may also expect that a higher fixed ordering cost would also induce a higher order-up-to level. Although this is true when there is no fixed cost of price adjustment, it is generally not the case otherwise. The left panel of Figure 3.2 provides a counterexample. We observe that with a positive price adjusting cost, it may become optimal to order less as the fixed cost becomes larger. With less units of items to

[2]Note that the condition that $h^a(i) - h^b(i)$ is weakly increasing is realistic and not restrictive. It simply says that the incremental cost of having one additional unit of inventory in system a should be no lower than that in system b. This is what one would expect if holding inventory is more costly in system a than in system b.

sell in a cycle, the optimal policy can potentially reduce the number of price changes and thus save on the price adjusting cost. This implies an interesting coordination problem between ordering and pricing strategies. As a result of such coordination, the optimal order-up-to level is not monotone with respect to the fixed ordering cost as well as the price adjustment cost, as demonstrated in Figure 3.2.

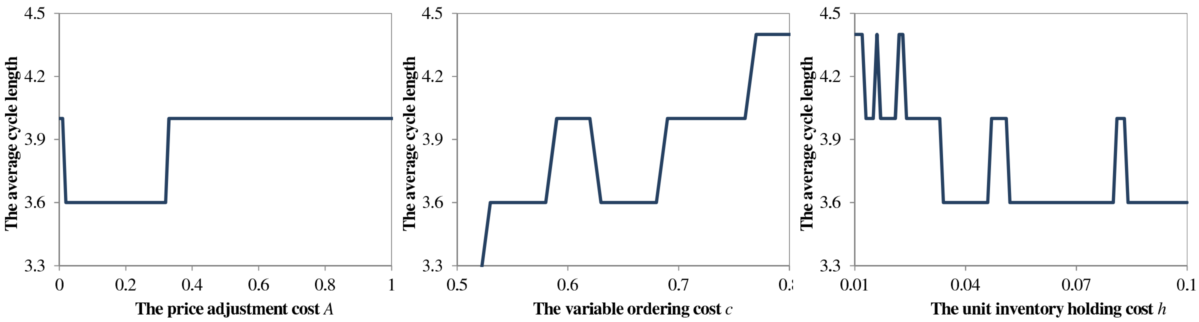


Notes. $c = 0.55$, $h(i) = 0.3i$, $K = 2.5$, $A = 0.2$, $\tau \in [0.4, 4]$, $p(\tau) = \ln(8.3\tau)$, $\tau \in \{0.4, 2.2., 4\}$.

Figure 3.2: The optimal order-up-to level as a function of model parameters

Proposition 3.4.4 (The Average Cycle Time)**.** *Consider two otherwise identical systems with fixed ordering costs $K^a > K^b$. Then $\sum_{i=1}^{S^{a*}} \tau^{a*} \geq \sum_{i=1}^{S^{b*}} \tau^{b*}$.*

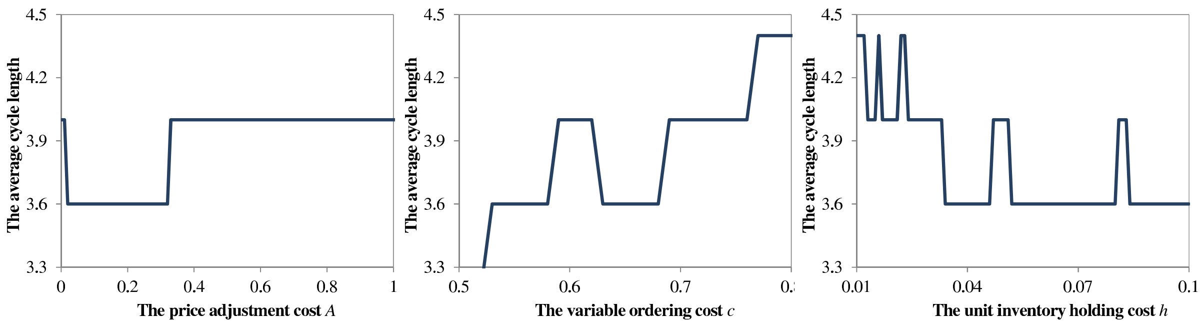
Proposition 3.4.4 suggests that as a result of the coordination between price and order, the average cycle length reduces as the fixed ordering cost decreases. Compared with Proposition 3.4.3, the economies of scale in ordering is reflected in time rather than in quantity. Figure 3.3 demonstrate that the average cycle length is generally not monotone with respect to the price adjustment cost or the variable ordering cost, even though the order-up-to level is.



Notes. $c = 0.6$, $h(i) = 0.05i^2$, $K = 3$, $A = 0$, $\tau \in \{0.4, 0.8, 1.2\}$, $p(0.4) = 1.8$, $p(0.8) = 2.5$ and $p(1.2) = 3.0$.

Figure 3.3: The optimal average cycle length as a function of model parameters

Proposition 3.4.5 (The Average Selling Speed)**.** *Consider two otherwise identical systems with variable ordering costs $c^a > c^b$. Then $\sum_{i=1}^{S^{a*}} \tau_i^* / S^{a*} \geq \sum_{i=1}^{S^{b*}} \tau_i^* / S^{b*}$.*



Notes. $c = 0.7$, $h(i) = 0.07i^2$, $K = 1.5$, $A = 0$, $\tau \in \{0.3, 0.8, 1.6\}$, $p(0.3) = 1.4$, $p(0.8) = 2.2$ and $p(1.6) = 3.1$.

Figure 3.4: The optimal average time to sell a unit as a function of model parameters

Proposition 3.4.6 (The Frequency of Price Adjustment)**.** *Consider two otherwise identical systems with fixed price adjustment costs $A^a > A^b$. Then,*

$$\frac{\sum_{i=1}^{S^{a*}-1} \mathbb{1}_{\{\tau_i^{a*} \neq \tau_{i+1}^{a*}\}}}{\sum_{i=1}^{S^{a*}} \tau_i^{a*}} \leq \frac{\sum_{i=1}^{S^{b*}-1} \mathbb{1}_{\{\tau_i^{b*} \neq \tau_{i+1}^{b*}\}}}{\sum_{i=1}^{S^{b*}} \tau_i^{b*}}.$$

Proposition 3.4.6 states that the frequency of changing price decreases as the adjustment cost increases.

3.5 Continuous Relaxation

As we have shown in §§3.4.2, the profit function $\hat{V}(S)$ may have a complex structure. In the presence of price adjustment cost, i.e., $A > 0$, $\hat{V}(S)$ is not unimodal. This imposes a challenge in solving the problem for systems exhibiting strong economies of scale in ordering (i.e., requiring a high order-up-to level). In this section, we consider a relaxation of the problem by allowing the inventory level to be continuous. When $p(\tau)$ is differentiable, one can evaluate the derivative of the profit function using this relaxation. In §§3.5.1, we first show that when price adjustment is not costly, one can easily obtain the exact optimal solution when $p(\tau)$ is differentiable. In §§3.5.2, we introduce the concept of price segment and reformulate the problem with continuous relaxation. In §§3.5.3, we analyze the structural properties of the relaxed problem.

3.5.1 The Special Case without Price Adjustment Cost

Proposition 3.5.1. *If $A = 0$ and price function $p(\tau)$ is differentiable, then*

$$\hat{V}(S) = p'(\hat{\tau}_i(S)) - h(i), \forall i. \tag{3.4}$$

From Proposition 3.5.1, we also can compute the optimal solution for different price functions as shown in Table 3.1. This proposition also reveals that the value function at a given inventory level can be regarded as a rate of profit. We shall see that the optimal solution has the property that the optimal profit rate is maintained constant if the inventory cost is accounted for properly!

The result in Proposition 3.5.1 allows for easy computation of the optimal solution. In Table 3.1 we provide solutions for three examples of commonly used demand functions when the inventory cost is linear, i.e., $h(i) = hi$. The detailed derivation of the solutions can be found in the appendix. We shall note that this table provides a simple guideline for changing prices when the cost of changing them is extremely small.

3.5.2 Definition of Price Segment

To treat the general model with $A \geq 0$, we introduce the notion of price segment. From Proposition 3.4.1, the price changes from high to low as the inventory level goes down. Denote $N = \sum_{i=1}^{S-1} \mathbb{1}_{\{\tau_i \neq \tau_{i+1}\}}$ as the number of price adjustment in one replenishment cycle. For each feasible solution $(S, \vec{\tau})$, we can define two vectors $\vec{x} = \{x_0, x_1, \cdots, x_{N+1}\}$ and $\vec{\iota} = \{\iota_0, \iota_1, \cdots, \iota_{N+1}\}$ as

Table 3.1: Examples of the optimal solution for the case without adjustment cost

Demand Function	$(\tilde{S}, \tilde{V})$ is the solution of	S^*	τ_i^*
Linear $p(\tau) = \alpha - \frac{\beta}{\tau}$	$\sqrt{Sh+V} = \frac{(\alpha-c)}{2\sqrt{\beta}}$ $S(\alpha - c) - K = \frac{4\sqrt{\beta}}{3h}[(\frac{(\alpha-c)}{2\sqrt{\beta}})^3 - (h+V)^{3/2}]$	$[\tilde{S}]$	$\sqrt{\frac{\beta}{ih+\tilde{V}}}$
Exponential $p(\tau) = \alpha \ln(\beta\tau)$ $(\beta\overline{\tau} > e)$	$1 + \ln(Sh+V) = \ln(\alpha\beta) - \frac{c}{\alpha}$ $S[\alpha\ln(\alpha\beta) - c] - K - \alpha$ $= \frac{\alpha}{h}[(Sh+V)\ln(Sh+V) - (h+V)\ln(h+V)]$	$[\tilde{S}]$	$\sqrt{\frac{\alpha}{ih+\tilde{V}}}$
Isoelastic $p(\tau) = \alpha\tau^\beta$ $(\beta < 1,\, 0 < \tau < 1)$	$(Sh+V)^{\frac{\beta}{1-\beta}} = \frac{\alpha^{\frac{1}{1-\beta}}(\beta^{\frac{\beta}{1-\beta}} - \beta^{\frac{1}{1-\beta}})}{c}$ $(Sc+K)\frac{h(1-\frac{\beta}{1-\beta})}{\alpha^{\frac{1}{1-\beta}}(\beta^{\frac{\beta}{1-\beta}} - \beta^{\frac{1}{1-\beta}})}$ $= (Sh+V)^{\frac{1-2\beta}{1-\beta}} - (h+V)^{\frac{1-2\beta}{1-\beta}}$	$[\tilde{S}]$	$\sqrt[1-\beta]{\frac{\alpha\beta}{ih+\tilde{V}}}$

follows.

$$
\begin{aligned}
x_0 &= 0, \quad x_{N+1} = S, \\
x_j &= \min\{l | \tau_{x_l+1} \neq \tau_{x_l}; \quad \tau_k = \tau_{x_{l-1}+1}, x_{l-1} \leq k \leq x_l\}, \quad 1 \leq j \leq N, \\
\iota_j &= \tau_{x_j}, \quad 1 \leq j \leq N+1.
\end{aligned}
$$

When the inventory level falls in $(x_{j-1}, x_j]$, items will be sold at the price $p(\iota_j)$. We call the interval $(x_{j-1}, x_j]$ the jth *price segment*. At each replenishment cycle, the inventory level changes from the $(N+1)$st price segment, to the Nth price segment, $\cdots$, until the first price segment as the inventory level decreases. Denote $H(x_{j-1}, x_j)$ as the average inventory holding cost per unit time when inventory level is in the interval $(x_{j-1}, x_j]$. That is,

$$
H(x_{j-1}, x_j) = \frac{\sum_{i=x_{j-1}+1}^{x_j} h(i)}{x_j - x_{j-1}}.
$$

Then, we can rewrite the long-run average profit $V(S, \hat{\tau})$ as

$$
V(S, \hat{\tau}) = V^\flat(N, \vec{\iota}, \vec{x}) = \frac{\sum_{j=1}^{N+1} \Big[(x_j - x_{j-1})\big[p(\iota_j) - \iota_j H(x_{j-1}, x_j)\big]\Big] - cx_{N+1} - K - NA}{\sum_{j=1}^{N+1}[(x_j - x_{j-1})\iota_j]}.
$$

For our subsequent analysis, we use $(N^*, \vec{\iota}^*, \vec{x}^*)$ to denote the optimal decisions that yields the optimal profit V^* to denote the optimal average profit. We also define $(\vec{\iota}(N), \vec{x}(N))$ to be the optimal decision when the number of price adjustment is equal to N.

Proposition 3.5.2. *If the price function $p(\iota)$ is differentiable, then*

$$V^* = p'(\iota_j^*) - H(x_{j-1}^*, x_j^*), \text{ for all } 0 \leq j \leq N^* + 1.$$

Proposition 3.5.2 shows that the long-run average profit equals the difference between the marginal price and average inventory holding cost over a price segment. In other words, the average profit rate for each price segment remains constant. This result is a generalization of Proposition 3.5.1. Note that when $A = 0$, it is optimal to adjust price whenever the inventory level changes. In this case, a price segment is defined as $(i-1, i]$ for each $1 \leq i \leq S$. The simple equation stated in Proposition 3.5.2 can be used to verify whether a pricing strategy is a candidate for being optimal.

3.5.3 Continuous Approximation

The problem as described above is very hard to solve when N is large. In this section, we explore the property of the approximate problem in which we assume that inventory level changes as a continuous variable and obtain a feasible solution by using a continuous approximation to the inventory level. Specifically, the average profit can be approximated as follows

$$W(N, \vec{\iota}, \vec{x}) = \frac{\sum_{j=1}^{N+1} \left[p(\iota_j)(x_j - x_{j-1}) - \iota_j \int_{x_{j-1}}^{x_j} h(x)dx\right] - cx_{N+1} - K - NA}{\sum_{j=1}^{N+1}[(x_j - x_{j-1})\iota_j]} \quad (3.5)$$

Note that in the above expression we have replaced the summation by the integration in computing the inventory holding cost. In other words, this approximation assumes that the demand arrives continuously at a price dependent rate.

For our subsequent analysis, we use $(N^*, \vec{\iota}^*, \vec{x}^*)$ to denote the optimal decisions for the approximated problem and W^* to denote the optimal average profit. We also define $(\vec{\hat{\iota}}(N), \vec{\hat{x}}(N))$ to be the optimal solution and $\hat{W}(N)$ to be the corresponding average profit when the number of price adjustment is N.

Proposition 3.5.3. *The equation* $h(z)[z-y] - \int_y^z h(x) = \frac{A}{\iota^M}, z \in (y, +\infty)$, *has a unique solution* $z(y)$, *where* $\iota^M = \arg\max_\iota \{[p(\iota) - c]/\iota\}$.

(i) If $\hat{W}(N) + h(z(\hat{x}_{N+1}(N))) < [p(\iota^M) - c]/\iota^M$ *then* $\hat{W}(N) < \hat{W}(N+1)$.

(ii) If $\hat{W}(N+1) + h(z(\hat{x}_{N+1}(N+1))) > [p(\iota^M) - c]/\iota^M$ *then* $\hat{W}(N) \geq \hat{W}(N+1)$.

Proposition 3.5.3 strengthens the results in Proposition 3.4.2 and Lemma 3.4.4 with the continuous relaxation of the problem. In the original model with a discrete inventory level, one searches for the optimal policy by changing the inventory one unit at a time. In the continuous relaxation, this is done by including or excluding a price segment.

Proposition 3.5.4. *Suppose that* $p(\cdot)$ *is differentiable. If* $h(\cdot)$ *is strictly concave (linear, strictly convex), then*

$$K + NA \quad < (=, >) \quad \sum_{j=1}^{N+1} \left[\hat{\iota}_j(N) \int_{\hat{x}_{j-1}(N)}^{\hat{x}_j(N)} h(x)dx \right]. \tag{3.6}$$

In the classical EOQ model, the fundamental trade off is to keep a balance between the fixed ordering cost and the inventory holding cost. As a result of this trade off, the optimal economical order quantity should lead to the set-up cost equal to the inventory holding cost within each order cycle. Proposition 3.5.4 suggest that a similar trade off in our model when the inventory holding cost is linear. The key difference in our model is the consideration of price adjustment. Therefore, in the left-hand side of (3.6), we shall also account for the price adjustment cost as a part of the fixed cost. The expression of inventory holding cost on the right-hand side is more complex than in the classical EOQ model. This is due to the fact that the rate of selling the product changes with the price. In general, the equality between the fixed cost and the inventory holding cost does not hold in our model, when the inventory holding cost is not linear. According to Proposition 3.5.4, the fixed cost is smaller (larger) than the inventory holding cost in an optimal solution when the inventory holding cost is concave (convex) in the stock level. Proposition 3.5.5 further discovers the impact of inventory holding cost on the optimal solution.

Proposition 3.5.5. *If the price function $p(\iota)$ is differentiable, then for all $j \leq N+1$,*

$$\hat{W}(N) = \pi_j^E(N, \hat{x}_{j-1}(N), \hat{x}_j(N), \hat{\iota}_j(N)) - \frac{H_j^E(N, \vec{\iota}(N), \vec{x}(N))}{[\hat{x}_j(N) - \hat{x}_{j-1}(N)]\hat{\iota}_j(N)}, \tag{3.7}$$

where

$$\begin{aligned}
\pi_j^E(N, x_{j-1}, x_j, \iota_j) &= \frac{p(\iota_j) - c}{\iota_j} - \frac{\int_{x_{j-1}}^{x_j} h(x)dx}{x_j - x_{j-1}} \\
H_j^E(N, \vec{\iota}, \vec{x}) &= \iota_j \left[(x_j - x_{j-1})h(x_j) - \int_{x_{j-1}}^{x_j} h(x)dx \right], \\
&\quad + (x_j - x_{j-1}) \sum_{m=j+1}^{N+1} \left[\iota_m \Big(h(x_m) - h(x_{m-1}) \Big) \right].
\end{aligned}$$

The *segment holding cost* H_j^E accounts the total inventory holding cost paid for the items sold in the jth price segment, which is computed in a way analogous to the concept of echelon holding cost. The first term of H_j^E corresponds to the holding cost during the $j$the price segment as these items are sold and the second term calculates the cost of holding these items in previous price segments. The *segment gross profit rate* π_j^E is the time-average gross profit rate over the jth price segment. Therefore, Proposition 3.5.5 suggests a relation among the optimal profit, the segment gross profit rate and the time-average segment holding cost. This relation says that the difference of the last remains unchanged across different price segments and it equals the optimal profit. This observation provides the following insight to the seller firm: given the current inventory position, the firm should set the price based on the segment inventory cost instead of the inventory cost at paid during the current price segment. Using this idea, we allocate the fixed cost to price segment j proportionately according to the segment holding cost. The allocation provides a method for solving the continuous approximation in a nice and understandable way (see the details in the appendix).

Also notice that the proof of Proposition 3.5.5 only uses the first-order condition for optimality with respect to x_j. Hence, more can be asserted about the profit rates as stated below.

Corollary 3.5.1. *Given $\vec{\iota}$, if $\vec{x}$ are determined optimally and the fixed costs are allocated proportional to the segment holding cost, then the profit rate across the segments are equal.*

3.5.4 Numerical Study: The Value of Price Adjustment

In this section, we examine the effect of each parameter in the model on the value gained from changing prices within ordering cycles. In most retail practice, the number of different prices is usually very limited in view of various implementation considerations. In the subsequent analysis, we focus on how to obtain the optimal price segments and optimal selling prices for a given upper bound $\overline{N}$ on the number of price adjustment. We use the method developed in §3.5 to compute the optimal policy for the problem with continuous relaxation and compare the results with the static pricing model $\hat{W}(0)$, in which a fixed optimal price $\hat{p}(0)$ is maintained throughout the replenishment cycle.

In Table 3.2, we analyze the effect of different model parameters. We observe that when it becomes more costly to change price (i.e., when A increases), the number of price points in a cycle reduces. However, the cycle length CT^*, the average selling speed $\bar{\tau}^*$ and the average price $\bar{p}^*$ may increase or decrease in A. These observations are consistent with our discussion in §§3.4.3. Compared with the case of static pricing, allowing for changing price can lead to more profit improvement when the cost of price adjustment is lower, as shown in the last column of the table.

Table 3.2: Effect of model parameters on the value of price adjustment

Parameter	$(\iota^*_{N^*+1}, \cdots, \iota^*_1)$	CT^*	$\bar{\iota}^*$	$\bar{p}^*$	W^*	$\hat{p}(0)$	$\hat{W}(0)$	$\lvert \frac{W^*-\hat{W}(0)}{\hat{W}(0)} \rvert \times 100\%$
$A =$ 0	(1.5, 1.65, 1.81, 2.00, 2.21, 2.45, 2.71, 3.05, 3.44, 3.88, 4.42)	38.47	2.22	8.47	0.8413	8.51	0.7109	18.34%
0.15	(1.50, 1.88, 2.39, 3.11, 4.13)	38.80	2.22	8.47	0.8423	8.51	0.7109	15.76%
0.30	(1.50, 2.62, 3.64)	38.31	2.19	8.41	0.8155	8.51	0.7109	14.70%
0.60	(1.50, 2.28, 3.69)	38.78	2.20	8.43	0.7999	8.51	0.7109	12.51%
1.5	(1.53, 3.12)	37.91	2.14	8.34	0.7726	8.51	0.7109	8.67%
$K =$ 20	(1.55, 3.21)	27.91	1.93	7.97	1.2534	7.81	1.2348	1.51%
30	(1.50, 2.07, 2.97)	31.14	1.98	8.13	1.0874	8.05	1.0409	4.47%
40	(1.50, 2.28, 3.69)	38.78	2.20	8.43	0.7999	8.51	0.7109	12.51%
45	(1.50, 2.39, 4.13)	42.82	2.32	8.58	0.6772	8.74	0.5679	19.26%
$c =$ 0.5	(1.50)	28.28	1.50	7.17	1.62	7.17	1.6162	0.00%
1.0	(1.50, 2.32)	32.25	1.74	7.67	1.32	7.17	1.2829	3.23%
1.5	(1.50, 2.64)	34.41	1.89	7.94	1.0477	7.17	0.9496	10.33%
2.0	(1.50, 2.28, 3.69)	38.78	2.20	8.43	0.7999	8.51	0.7109	12.51%
$h =$ 0.01	(1.50)	109.54	1.50	7.17	2.7144	7.17	2.7144	0.00%
0.05	(1.50, 1.92)	53.04	1.64	7.49	1.8234	7.24	1.8120	0.62%
0.15	(1.50, 2.28, 3.69)	38.78	2.20	8.43	0.7999	8.51	0.7109	12.51%
0.20	(1.50, 2.58, 5.00)	37.68	2.55	8.82	0.4892	9.13	0.3545	37.99%
α = 3.5	(1.53, 2.61, 5.00)	44.83	2.71	7.93	0.3521	8.36	0.2357	49.38%
4.0	(1.50, 2.28, 3.69)	38.78	2.20	8.43	0.7999	8.51	0.7109	12.51%
4.5	(1.50, 2.49)	33.89	1.86	8.89	1.3093	8.80	1.2424	5.38%
5.0	(1.50, 2.17)	31.88	1.72	9.56	1.8541	9.17	1.8130	2.27%
$\beta =$ 3.0	(1.84, 2.91, 5.00)	46.85	3.10	8.58	0.3608	9.13	0.2658	36.69%
3.5	(1.60, 2.62, 4.71)	43.33	2.62	8.49	0.5755	8.77	0.4799	19.92%
4.0	(1.50, 2.28, 3.69)	38.78	2.20	8.43	0.7999	8.51	0.7109	12.51%
4.5	(1.50, 2.65)	34.55	1.90	8.43	1.0325	8.31	0.9556	8.04%

Basic Setting: $p(\iota) = \alpha \ln(\beta\iota)$ for $\iota \in [1.5, 5]$, $\alpha = \beta = 4$, $K = 40$, $c = 2$, $A = 0.6$, $h(i) = 0.15i$, and $\bar{N} = 10$. $CT^* = \sum_{i=0}^{N^*}[(x^*_{j+1} - x^*_j)\iota^*_j]$, $\bar{\iota}^* = \frac{\sum_{i=0}^{N^*}[(x^*_{j+1}-x^*_j)\iota^*_j]}{x^*_{N^*+1}}$ and $\bar{p}^* = \frac{\sum_{i=0}^{N^*}[(x^*_{j+1}-x^*_j)p(\iota^*_j)]}{x^*_{N^*+1}}$.

When the fixed ordering cost K, the variable ordering cost c, or the unit inventory holding cost h becomes higher, inventory based price adjustment yields a larger value. Moreover, the difference among price points also becomes larger, as implied in the second column of Table 3.2. Even though the response of the optimal policy to these three cost parameters is very similar, the explanation is very different. An increased K implies a strong scale economy in ordering and an extended cycle length. In this case, it is desirable to further reduced the price at the beginning of the cycle to speed up the selling and reduce inventory. When it is close to the end of the cycle, a much higher price is charged to ensure profitability when inventory level is lower. Consequently, price adjustment becomes more useful when K is large. An increased c is immediately translated to a reduced gross margin, which induces the firm to increase the price. However, in view of the inventory holding cost, it is not economical to uniformly reduce the product price over the replenishment cycle. Instead, the firm should keep the low price at the beginning of the cycle and only increase the price toward to end to compensate for the increased unit cost. Therefore, the firm's ability to change price becomes important when c is high. Intuitively, a larger h may induce the firm to charge a lower price at the beginning of the cycle to speed up the selling process. However, this is not always true. From Table 3.2, we observe that the firm would rather make up the increased holding cost by further increasing the price toward the end of the cycle.

The profit improvement from changing price decreases as α or β increases. From the second column of Table 3.2, we observe that the difference of price points becomes smaller as α or β increases. This is because the speed of selling the product (i.e., $1/\iota$) becomes less sensitive to the price. Interesting α and β has a different effect on the average selling price—The firm can sells at a higher average price when α increases or β decreases.

In Table 3.3, we compute the percentage gap between the optimal profit rate and that under limited number of price adjustments. Recall our discussion for Proposition 3.4.6. The optimal policy calls for more frequent price change when the adjustment cost is low. Therefore, it makes sense for us to focus on the case when $A = 0$ such that it is optimal to revise the price frequently.

Table 3.3: Effect of Upper bound $\overline{N}$ for Different Adjustment Cost

Parameter		$\frac{\dot{W}(0)}{\dot{W}(100)} \times 100\%$	$\frac{\dot{W}(1)}{\dot{W}(100)} \times 100\%$	$\frac{\dot{W}(2)}{\dot{W}(100)} \times 100\%$	$\frac{\dot{W}(3)}{\dot{W}(100)} \times 100\%$
$A =$	0	84.46%	96.56%	98.75%	99.32%
	0.15	86.20%	98.06%	99.83%	100.00%
	0.30	87.18%	98.67%	100.00%	100.00%
	0.60	88.88%	99.58%	100.00%	100.00%
	1.5	92.02%	100.00%	100.00%	100.00%
$K =$	20	96.49%	99.48%	99.81%	99.90%
	30	91.89%	98.47%	99.44%	99.72%
	40	84.45%	96.56%	98.75%	99.32%
	45	79.04%	95.12%	98.20%	99.09%
$c =$	1.0	94.90%	99.35%	99.76%	99.88%
	1.5	89.80%	98.41%	99.42%	99.70%
	2.0	84.45%	96.56%	98.75%	99.32%
	2.5	78.08%	94.33%	97.63%	98.81%
$h =$	0.01	100.00%	100.00%	100.00%	100.00%
	0.05	98.60%	99.83%	99.92%	99.93%
	0.15	84.45%	96.56%	98.75%	99.32%
	0.20	66.14%	92.04%	97.28%	98.78%
$\alpha =$	3.5	60.20%	90.74%	96.83%	98.58%
	4.0	84.45%	96.56%	98.75%	99.32%
	4.5	92.47%	98.78%	99.56%	99.78%
	5.0	96.34%	99.53%	99.83%	99.92%
$\beta =$	3.0	67.05%	93.21%	97.50%	98.73%
	3.5	77.60%	94.21%	97.59%	98.80%
	4.0	84.45%	96.56%	98.75%	99.32%
	4.5	89.50%	98.34%	99.40%	99.70%

Basic Setting: $p(\iota) = \alpha \ln(\beta\iota)$ for $\iota \in [1.5, 5]$, $\alpha = \beta = 4$, $K = 40$, $c = 2$, $A = 0$, and $h(i) = 0.15i$.

As a general observation from Table 3.3, price adjustment is subject to rapid diminishing returns—The additional value obtained from allowing for one more price revision within an ordering cycle decreases rapidly with the maximum number of price adjustments $\bar{N}$. In all instances reported in the table, almost 99% of the optimal profit can be achieved by allowing when $N = 4$. Note that in computing the results in Table 3.3, we have set $A = 0$ in our basic setting. From the first part of the table that the number of price adjustment needed to obtain almost 100% optimal profit rate reduces as the price adjustment cost increases. In other words, when price adjustment is costly, even less number of price revisions are needed to achieve a high profit. This is consistent with most retail practice that a very limited number of price points are observed for a certain product.

The result in Table 3.3 further suggests that one can find a close-to-optimal solution for our problem by testing limited number of price points using the continuous relaxation of the model. This allows for dramatically reduction of the problem size.

3.6 Concluding Remarks

Many firms now have the flexibility to change price, order products or do both at any time. However, the flexibility on pricing poses challenges for making the optimal pricing and replenishment decisions. Our paper explores this question and provides general instructions for making these decisions. Our study shows that the optimal pricing strategy is to charge a low price at the beginning and change to high price later. We also see the effect of the cost of changing prices. The total replenishment cycle increases as the setup cost increases. The average inter-arrival time increases as the unit cost increases.

To investigate more, we discuss two special cases. One is without adjustment cost. We find the firm prefers changing price at each inventory level. The optimal solution has the property that the rate of profit is maintained constant if the inventory cost is accounted for properly. We provide the expression of the optimal order-up-to level and the optimal long-run average profit for different price functions.

For the general case, we provide the continuous approximation solution. We show that in the optimal solution the fixed cost may be larger or small than the holding cost within a replenishment cycle, depending on the inventory holding cost function. We investigate the profit rate at each price segment and discover that in the optimal solution the difference between the profit rate of each price segment and the optimal long-run average profit equals the echelon inventory cost of each price segment. We proved a formulation which can be easily solved using standard package.

It would be interesting to see whether changing prices within a cycle is as effective when there is positive leadtime. Leadtime equals zero in our paper. As we know, involvement of leadtime would make the problem much more difficult. It is due to the fact that the long-run average problem can not directly be reduced to one replenishment cycle any longer. The general method is to design a heuristic pricing replenishment policy and compare it with the optimal pricing and replenishment policy. We hope to do so on future work.

The restriction on the arrival process creates the benefit of simplifying the problem from changing price anytime to changing price based on the inventory level. If demand does not follow a Poisson Process, we can not make such simplification. However, if firms change the price based on their current inventory level, then our results can be extended to any arrival process.

Our results also lend themselves to empirical testing regarding the optimality of the pricing and stocking policies. Firms can use data about the inventory position to calculate echelon inventory cost. Further, they can test whether their marginal profit of each price segment minus the long-run average profit equals echelon inventory cost. If not, then firms can get more profit by adjusting the price at the corresponding price segments.

Chapter 4

Conclusion

Two parts of dissertation discuss how to adjust price in different environments.

The first project discusses a periodic review inventory model. The retailer can replenish inventory and change price at each period. We demonstrate that the optimal order policy is a near-reorder point policy. On the other hand, the optimal price may increase as the inventory decreases. We also summarize optimal policies under different supplier uncertainties and demand uncertainties.

The second project discusses the optimal sale strategy during the replenishment cycle. The retailer can adjust price at any time. When demand follows Poisson distribution, it is equivalent to adjust price at each inventory level. We find that under the optimal policy, the retailer prefers to adjust price at each inventory level when the adjustment cost is zero. However, the retailer only needs to adjust price at limited times when the adjustment cost is strictly positive. It is interesting to find that the optimal profit rate will improve largely even when the price is only adjusted once.

When a customer was cheated, he may not visit the retailer any longer.

Appendices

Appendix A

Appendix for Chapter 2

Proof of Lemma 2.2.1. When $t = T$, we have $V_{T+1}(I) = 0$ and

$$J_T(I, q_1, q_2, d) = R(d) - c_1 q_1 - c_2 q_2 - \mathbb{E}H(I + u_1 q_1 + u_2 q_2 - \varepsilon d - \omega)$$

is concave. Because concavity is preserved under maximization, $V_T(I)$ is concave. Now suppose that $V_{t+1}(I)$ is concave. It follows immediately that $J_t(I, q_1, q_2, d)$ is concave and thus $V_t(I)$ is concave. It then follows that $L_t(y)$ is concave and thus $J_t^B(I, q_1, q_2, d)$ is jointly concave.

Since L_t in (2.4) is concave, $L_t(I + u_1 q_1 + u_2 q_2 - \varepsilon d)$ is submodular in (I, q_i) and supermodular in (q_i, d) and (I, d), $i = 1, 2$. Hence, J_t and J_t^B are submodular in (I, q_i) and supermodular in (q_i, d) and (I, d).

Lemma A.0.1. *Suppose that $F(I, q_1, q_2, d)$ is jointly concave in (I, q_1, q_2, d). Let*

$$\begin{aligned}
\Theta &= \left\{(I, \tilde{q}_1, \tilde{q}_2, \tilde{d}) | F(I, \tilde{q}_1, \tilde{q}_2, \tilde{d}) = \max_{q_1 \geq 0, q_2 \geq 0, \underline{\underline{d}} \leq d \leq \bar{\bar{d}}} F(I, q_1, q_2, d)\right\}, \\
q_1^*(I) &= \inf_{\tilde{q}_1} \left\{\tilde{q}_1 | (I, \tilde{q}_1, \tilde{q}_2, \tilde{d}) \in \Theta\right\}, \\
q_2^*(I) &= \inf_{\tilde{q}_2} \left\{\tilde{q}_2 | (I, \tilde{q}_1, \tilde{q}_2, \tilde{d}) \in \Theta, \tilde{q}_1 = q_1^*(I)\right\}, \\
d^*(I) &= \inf_{\tilde{d}} \left\{\tilde{d} | (I, \tilde{q}_1, \tilde{q}_2, \tilde{d}) \in \Theta, \tilde{q}_1 = q_1^*(I), \tilde{q}_2 = q_2^*(I)\right\}.
\end{aligned}$$

Then $q_1^*(I)$, $q_2^*(I)$, and $d^*(I)$ are continuous in I.

Proof By definition, $(q_1^*(I), q_2^*(I), d^*(I))$ is a maximizer of $F(I, q_1, q_2, d)$. Because $F(I, q_1, q_2, d)$ is jointly concave in (I, q_1, q_2, d), Θ is a convex set [11]. Therefore, $q_1^*(I) = \min_{\tilde{q}_1} \left\{ \tilde{q}_1 | (I, \tilde{q}_1, \tilde{q}_2, \tilde{d}) \in \Theta \right\}$ is continuous in I.

Now we note that $F(I, q_1^*(I), q_2, d)$ is jointly concave in (I, q_2, d). Let

$$\Theta_1 = \left\{ (I, \tilde{q}_2, \tilde{d}) | F(I, q_1^*(I), \tilde{q}_2, \tilde{d}) = \max_{q_2 \geq 0, \underline{\underline{d}} \leq d \leq \overline{\overline{d}}} F(I, q_1, q_2, d) \right\}.$$

Then Θ_1 is a convex set and $q_2^*(I) = \min\{\tilde{q}_2 | (I, \tilde{q}_2, \tilde{d}) \in \Theta_1\}$ is continuous in I.

Finally, we have $F(I, q_1^*(I), q_2^*(I), d)$ being jointly concave in (I, d). Let

$$\Theta_2 = \left\{ \tilde{d} | F(I, q_1^*(I), q_2^*(I), \tilde{d}) = \max_{\underline{\underline{d}} \leq d \leq \overline{\overline{d}}} F(I, q_1, q_2, d) \right\}.$$

Then Θ_2 is a convex set and $d^*(I) = \min\{\tilde{d} | (I, \tilde{d}) \in \Theta_2\}$ is continuous in I.

Proof of Lemma 2.2.2. Define $y_1 = I + \bar{u}_1 q_1$ and

$$\begin{aligned} \Phi_t(y_1, q_2, d) &= J_t^B(I, (y_1 - I)/\bar{u}_1, q_2, d) - \bar{c}_1 I \\ &= R(d) - \bar{c}_1 y_1 - \bar{c}_2 q_2 + \mathbb{E} L_t(y_1 + u_2 q_2 - \varepsilon d). \end{aligned} \tag{A.1}$$

We observe that the right-hand side does not depend on I and is concave in (y_1, q_2, d). Let $\tilde{d}(y_1)$ and $\tilde{q}_2(y_1)$ denote the maximizer of Φ_t for a given y_1. It follows that $\Phi_t(y_1, \tilde{q}_2(y_1), \tilde{d}(y_1))$ is concave in y_1 and has a maximizer, which we denote by $\bar{y}_{t,1}^B$. We also denote $\bar{q}_{t,2}^B = \tilde{q}_2^B(\bar{y}_{t,1}^B)$ and $\bar{d}_t^B = \tilde{d}(\bar{y}_{t,1}^B)$. Thus, the optimal ordering decision for supplier 1 follows a base-stock policy, i.e.,

$q^B_{t,1}(I) = (\bar{y}^B_{t,1} - I)^+/\bar{u}_1$. Moreover, when $I \leq \bar{y}^B_{t,1}$, we must have $I + \bar{u}_1 q^B_1(I) = \bar{y}^B_{t,1}$, which implies $q^B_{t,2}(I) = \tilde{q}^B_2(\bar{y}^B_{t,1}) = \bar{q}^B_{t,2}$ and $d^B_t(I) = \tilde{d}(\bar{y}^B_{t,1}) = \bar{d}^B_t$.

Proof of Lemma 2.2.3. We suppose $q^*_{t,1}(I) > 0$ and $q^B_{t,1}(I) = 0$. Then, we must have

$$\begin{aligned}
& J^B_t(0, q^B_{t,2}(I), d^B_t(I)) \\
= \; & R(d^B_t(I)) - \bar{c}_2 q^B_{t,2}(I) + \mathbb{E}L_t(I + u_2 q^B_{t,2}(I) - \varepsilon d^B_t(I)) \\
= \; & J_t(0, q^B_{t,2}(I), d^B_t(I)) \\
\leq \; & J_t(q^*_{t,1}(I), q^*_{t,2}(I), d^*_t(I)) \\
= \; & R(d^*_t(I)) - \bar{c}_1 q^*_{t,1}(I) - \bar{c}_2 q^*_{t,2}(I) + \mathbb{E}L_t(I + u_1 q^*_{t,1}(I) + u_2 q^*_{t,2}(I) - \varepsilon d^*_t(I)) \\
\leq \; & R(d^*_t(I)) - \bar{c}_1 q^*_{t,1}(I) - \bar{c}_2 q^*_{t,2}(I) + \mathbb{E}L_t(I + \bar{u}_1 q^*_{t,1}(I) + u_2 q^*_{t,2}(I) - \varepsilon d^*_t(I)).
\end{aligned}$$

The first inequality follows from the facts that for model $\mathcal{G}$, $(0, q^B_{t,2}(I), d^B_t(I))$ is a feasible solution and $(q^*_{t,1}(I), q^*_{t,2}(I), d^*_t(I))$ is the optimal solution. The second inequality follows from the concavity of L_t and Jensen's inequality. If the inequality is strict, then the above relation suggests that in model $\mathcal{B}$, $(q^*_{t,1}(I), q^*_{t,2}(I), d^*_t(I))$ yields a higher profit than $(0, q^B_{t,2}(I), d^B_t(I))$. This contradicts the optimality of $(0, q^B_{t,2}(I), d^B_t(I))$. If equality holds in the above relation, then $q^*_{t,1}(I) = 0$ is also an optimal solution. Therefore, we conclude the proof.

Proof of Lemma 2.2.4. We prove the result for supplier 1 and that for supplier 2 follows in a similar way. To see i), we note that $\tilde{q}_{t,1}(I) < 0$

implies, for a small enough δ,

$$
\begin{aligned}
0 &\geq J_t(I, 0, q^*_{t,2}(I), d^*_t(I)) - J_t(I, -\delta/\bar{u}_1, q^*_{t,2}(I), d^*_t(I)) \\
&= V_t(I) - [R(d^*_t(I)) - \bar{c}_1(-\delta/\bar{u}_1) - \bar{c}_2 q^*_{t,2}(I) \\
&\quad +\mathbb{E}L_t(I + u_1(-\delta/\bar{u}_1) + u_2 q^*_{t,2}(I) - \varepsilon d^*_t(I))] \\
&= -c_1\delta + V_t(I) - [R(d^*_t(I)) - \bar{c}_2 q^*_{t,2}(I) + \mathbb{E}L_t(I - u_1\delta/\bar{u}_1 + u_2 q^*_{t,2}(I) - \varepsilon d^*_t(I))] \\
&\geq -c_1\delta + V_t(I) - [R(d^*_t(I)) - \bar{c}_2 q^*_{t,2}(I) + \mathbb{E}L_t(I - \bar{u}_1\delta/\bar{u}_1 + u_2 q^*_{t,2}(I) - \varepsilon d^*_t(I)) \\
&= -c_1\delta + V_t(I) - [R(d^*_t(I)) - \bar{c}_2 q^*_{t,2}(I) + \mathbb{E}L_t(I - \delta + u_2 q^*_{t,2}(I) - \varepsilon d^*_t(I))] \\
&\geq -c_1\delta + V_t(I) - V_t(I - \delta).
\end{aligned}
$$

The second inequality follows from Jensen' inequality. The last inequality follows from the maximality of $V_t(I - \delta)$. Hence, we obtain part i).

We show part ii) by contradiction. There does not exists a $\delta > 0$ satisfying $\frac{V_t(I+\delta)-V_t(I)}{\delta} \geq c_1$. We must have $\frac{V_t(I+\delta)-V_t(I)}{\delta} < c_1$ for any $\delta \geq 0$ and thus for any $\delta \in [0, q^*_{t,1}(I)\overline{\overline{u}}_1)]$. It follows, for each realization of $u_1 = \check{u}_1 \neq 0$,

$$
\begin{aligned}
c_1\check{u}_1 q^*_{t,1}(I) &> V_t(I + \check{u}_1 q^*_{t,1}(I)) - V_t(I) \\
&\geq J_t(I + \check{u}_1 q^*_{t,1}(I), 0, q^*_{t,2}(I), d^*_t(I)) - J_t(I, q^*_{t,1}(I), q^*_{t,2}(I), d^*_t(I))) \\
&= \bar{c}_1 q^*_{t,1}(I) + \mathbb{E}L_t(I + \check{u}_1 q^*_{t,1}(I) + u_2 q^*_{t,2}(I) - \varepsilon d^*_t(I)) \\
&\quad -\mathbb{E}L_t(I + u_1 q^*_{t,1}(I) + u_2 q^*_{t,2}(I) - \varepsilon d^*_t(I)). \qquad \text{(A.2)}
\end{aligned}
$$

The second inequality follows from the maximality of V_t. Taking expectation over $\check{u}_1$, we obtain

$$\begin{aligned} c_1\bar{u}_1 q^*_{t,1}(I) &\geq \bar{c}_1 q^*_{t,1}(I) + \mathbb{E}L_t(I + u_1 q^*_{t,1}(I) + u_2 q^*_{t,2}(I) - \varepsilon d^*_t(I)) \\ &\quad -\mathbb{E}L_t(I + u_1 q^*_{t,1}(I) + u_2 q^*_{t,2}(I) - \varepsilon d^*_t(I)) \\ &= \bar{c}_1 q^*_{t,1}(I) = c_1\bar{u}_1 q^*_{t,1}(I). \end{aligned}$$

Since equality holds in the above relation, we must have equality in (A.2) for each $\check{u}_1$. It follows that $V_t(I+\delta) - V_t(I) = c_1\delta$ for $\delta \in [0, q^*_{t,1}(I)\bar{\bar{u}}_1)]$. This leads to a contradiction.

Proof of Lemma 2.2.5. The result would follow if there exists an I_t^{UB} such that $q^*_{i,t}(I) = 0$, $i = 1, 2$, for any $I \geq I_t^{UB}$. Note from Lemma 2.2.4(ii) and the concavity of $V_t(I)$, $q^*_{i,t}(I) = 0$ if $\frac{V_t(I+\delta)-V_t(I)}{\delta} \leq \min\{c_1, c_2\} = c_1$ for any $\delta > 0$. We show that there exists an I_t^{UB} such that $\frac{V_t(I+\delta)-V_t(I)}{\delta} \leq \min\{c_1, c_2\} = c_1$ for $I \geq I_t^{UB}$ and any $\delta > 0$. This is clearly true for period $T+1$. Suppose it is true for period $t+1$. Note that $\frac{H(I+\nu)-H(I)}{\nu} \geq 0$ for $I \geq 0$ and $\nu > 0$. Hence, there exists an I_t^{UB} such that

$$\frac{\mathbb{E}[L_t(I_t^{UB} + \delta - \varepsilon\bar{\bar{d}}) - L_t(I_t^{UB} - \varepsilon\bar{\bar{d}})]}{\delta} \leq 0 + \alpha c_1 \leq c_1.$$

This implies

$$
\begin{aligned}
& V_t(I_t^{UB}+\delta) - V_t(I_t^{UB}) \\
= & \max_{q_1,q_2\geq 0,\ \underline{\underline{d}}\leq d\leq \overline{\overline{d}}} J_t(I_t^{UB}+\delta, q_1, q_2, d) - \max_{q_1,q_2\geq 0,\ \underline{\underline{d}}\leq d\leq \overline{\overline{d}}} J_t(I_t^{UB}, q_1, q_2, d) \\
\leq & \max_{q_1,q_2\geq 0,\ \underline{\underline{d}}\leq d\leq \overline{\overline{d}}} \Big[J_t(I_t^{UB}+\delta, q_1, q_2, d) - J_t(I_t^{UB}, q_1, q_2, d) \Big] \\
= & \max_{q_1,q_2\geq 0,\ \underline{\underline{d}}\leq d\leq \overline{\overline{d}}} \mathbb{E}\Big[L_t(I_t^{UB}+\delta+u_1q_1+u_2q_2-\varepsilon d) - L_t(I_t^{UB}+u_1q_1+u_2q_2-\varepsilon d) \Big] \\
\leq & \ \mathbb{E}\Big[L_t(I_t^{UB}+\delta-\varepsilon d) - L_t(I_t^{UB}-\varepsilon d) \Big] \\
\leq & \ c_1\delta.
\end{aligned}
$$

Hence, we conclude the proof.

Proof of Lemma 2.2.6. We prove the result by contradiction. Suppose that the result is not true. Then by the continuity of $q_{1,t}^*(I)$ established in Lemma A.0.1, there exists a γ^- such that $q_{t,1}^*(\bar{I}-\delta_1)=0$ for any $\delta_1 \in (0,\gamma^-)$. Now choose a $\delta_1 \in [\underline{\delta}, \overline{\delta}] \in (0,\gamma^-)$ satisfying $\underline{\delta} < \overline{\delta}$. Because $q_{t,1}^*(\bar{I}-\delta_1)=0$, we must have

$$
\begin{aligned}
V_t(\bar{I}-\delta_1) &= R(d_t^*(\bar{I}-\delta_1)) - \bar{c}_2 q_{t,2}^*(\bar{I}-\delta_1) \\
&\quad + \mathbb{E}L_t(\bar{I}-\delta_1+u_2q_{t,2}^*(\bar{I}-\delta_1)-\varepsilon d_t^*(\bar{I}-\delta_1)) \\
&= J_t^B(\bar{I}-\delta_1, 0, q_{t,2}^*(\bar{I}-\delta_1), d_t^*(\bar{I}-\delta_1)) \\
&\leq \max_{q_2\geq 0, \underline{\underline{d}}\leq d\leq \overline{\overline{d}}} J_t^B(\bar{I}-\delta_1, 0, q_2, d).
\end{aligned}
$$

Because $q_{t,1}^*(\bar{I}+\delta)>0$, we have from Lemma 2.2.3, $q_{t,1}^B(\bar{I}+\delta)>0$ and thus $q_{t,1}^B(\bar{I}-\delta_1)>0$. Moreover, from Lemma 2.2.2, we must have the base-stock level $\bar{y}^B_{t,1} > \bar{I}$, and $q^B_{t,1}(I) = (\bar{y}^B_{t,1} - I)/\bar{u}_1$ for $I \le \bar{I}$. Now define $\tilde{J}^B_t(I, q_1) = \max_{q_2 \ge 0, \underline{\underline{d}} \le d \le \bar{\bar{d}}} J^B_t(I, q_1, q_2, d)$. Then $\tilde{J}^B_t(I, q_1)$ is jointly concave in (I, q_1) and the smallest maximizer of $\tilde{J}^B_t(\bar{I} - \delta_1, q_1)$ is $q^B_{t,1}(\bar{I} - \delta_1) = (\bar{y}^B_{t,1} - \bar{I} + \delta_1)/\bar{u}_1 > \delta_1/\bar{u}_1$. Therefore,

$$\begin{aligned} V_t(\bar{I} - \delta_1) &\le \tilde{J}^B_t(\bar{I} - \delta_1, 0) \le \tilde{J}^B_t(\bar{I} - \delta_1, \delta_1/\bar{u}_1) - \kappa \\ &= \max_{q_2 \ge 0, \underline{\underline{d}} \le d \le \bar{\bar{d}}} \{R(d) - \bar{c}_1\delta_1/\bar{u}_1 - \bar{c}_2 q_2 + \mathbb{E}L_t(\bar{I} + u_2 q_2 - \varepsilon d)\} - \kappa \\ &\le V_t(\bar{I}) - c_1\delta_1 - \kappa, \end{aligned} \tag{A.3}$$

where $\kappa = \tilde{J}^B_t(\bar{I}, 0) - \tilde{J}^B_t(\bar{I}, -\underline{\delta}/\bar{u}_1) > 0$ as the base-stock level $\bar{y}^B_{t,1} > \bar{I}$. To see the second equality, we note that $\kappa = \tilde{J}^B_t(\bar{I} - \delta_1, \delta_1/\bar{u}_1) - \tilde{J}^B_t(\bar{I} - \delta_1, (\delta_1 - \underline{\delta})/\bar{u}_1) \le \tilde{J}^B_t(\bar{I} - \delta_1, \delta_1/\bar{u}_1) - \tilde{J}^B_t(\bar{I} - \delta_1, 0)$. Therefore, (A.3) implies

$$V_t(\bar{I}) - V_t(\bar{I} - \delta_1) > c_1\delta_1. \tag{A.4}$$

Let $\tilde{q}_{t,1}(I)$ be the smallest unconstrained maximizer of $J_t(I, q_1, q^*_{t,2}(I), d^*_t(I))$. Because $q^*_{t,1}(\bar{I} - \delta_1) = 0$, we must have $\tilde{q}_{t,1}(\bar{I} - \delta_1) \le 0$. Now if $\tilde{q}_{t,1}(\bar{I} - \delta_1) < 0$, then from Lemma 2.2.4 and the concavity of $V_t(I)$, we have $V_t(\bar{I}) - V_t(\bar{I} - \delta_1) \le c_1\delta_1$, which contradicts (A.4). Hence, we obtain $\tilde{q}_{t,1}(\bar{I} - \delta_1) = 0$ for $\delta_1 \in [\underline{\delta}, \bar{\delta}]$.

Now we note that V_t is concave so that there can be at most countable points at which V_t is not differentiable [10]. For a differentiable point $I \in [\bar{I} - \bar{\delta}, \bar{I} - \underline{\delta}]$, we have

$$\begin{aligned} V'_t(I) &= \frac{\partial}{\partial I}\mathbb{E}L_t(I + u_1 q^*_{t,1}(I) + u_2 q^*_{t,2}(I) - \varepsilon d^*_t(I)) \\ &= \frac{\partial}{\partial I}\mathbb{E}L_t(I + u_2 q^*_{t,2}(I) - \varepsilon d^*_t(I)) \end{aligned}$$

is well defined. Because $\tilde{q}_{t,1}(I) = 0$, we derive by the envelope theorem

$$\begin{aligned}\left.\frac{\partial J_t(I, q, q^*_{t,2}(I), d^*_t(I))}{\partial q}\right|_{q=0} &= -\bar{c}_1 + \frac{\partial}{\partial I}\mathbb{E}u_1 L_t(I + u_2 q^*_{t,2}(I) - \varepsilon d^*_t(I)) \\ &= -\bar{c}_1 + \bar{u}_1 \frac{\partial}{\partial I}\mathbb{E}L_t(I + u_2 q^*_{t,2}(I) - \varepsilon d^*_t(I)) = 0.\end{aligned}$$

This implies that $V'_t(I) = \bar{c}_1/\bar{u}_1 = c_1$. This contradicts (A.4).

Proof of Lemma 2.2.7. Because $V_t(I)$ is concave, it can have at most countable number of non-differentiable points. Let N denote the set of non-differentiable points. We show that $X_1 \subset N$. Suppose this is not true. Then there exist an $\bar{I}$ with $V'_t(I)$ well-defined over the neighborhood of $\bar{I}$ such that $\bar{I}$ satisfies the conditions in Lemma 2.2.6. In other words, $q^*_{t,1}(\bar{I}) = 0$ and $q^*_{t,1}(\bar{I} + \delta) > 0$ for any arbitrarily small δ. From the proof of Lemma A.0.1, $\tilde{q}_{1,t}(I)$ is continuous. Hence, we must have $\tilde{q}_{t,1}(\bar{I}) = q^*_{t,1}(\bar{I}) = 0$. By the envelope theorem, we have

$$V'_t(\bar{I}) = \frac{\partial \mathbb{E}L_t(\bar{I} + u_2 q^*_{t,2}(\bar{I}) - \varepsilon d^*_t(\bar{I}))}{\partial \bar{I}}.$$

Hence, the right-hand side of the above is well defined. The first-order condition of $\tilde{q}_{t,1}(\bar{I}) = 0$ leads to

$$-\bar{c}_1 + \bar{u}_1 \frac{\partial \mathbb{E}L_t(\bar{I} + u_2 q^*_{t,2}(\bar{I}) - \varepsilon d^*_t(\bar{I}))}{\partial \bar{I}} = 0.$$

Now consider the benchmark problem $\mathcal{B}$. The above relation implies that

$$\left.\frac{\partial J^B_t(\bar{I}, q_1, q^*_{t,2}(\bar{I}), d^*_t(\bar{I}))}{\partial q_1}\right|_{q_1=0} = -\bar{c}_1 + \bar{u}_1 \frac{\partial \mathbb{E}L_t(\bar{I} + u_2 q^*_{t,2}(\bar{I}) - \varepsilon d^*_t(\bar{I}))}{\partial \bar{I}} = 0.$$

It is also clear that

$$J^B_t(\bar{I}, 0, q^*_{t,2}(\bar{I}), d^*_t(\bar{I})) = \max_{q_2 \geq 0, \underline{\underline{d}} \leq d \leq \bar{\bar{d}}} J^B_t(\bar{I}, 0, q_2, d).$$

Therefore, $q^B_{t,1}(\bar{I}) = 0$, $q^B_{t,2}(\bar{I}) = q^*_{t,2}(\bar{I})$ and $d^B_t(\bar{I}) = d^*_t(\bar{I})$. By Lemma 2.2.2, we must have $q^B_{t,1}(I) = 0$ for any $I \geq \bar{I}$. Moreover, by Lemma 2.2.3, we deduce that $q^*_{t,1}(I) = 0$ for any $I \geq \bar{I}$. This leads to a contradiction.

Lemma A.0.2. *If the pricing decision is made before observing (u_1, u_2) and after observing (ε, ω), then the optimal price is decreasing in the inventory level and the optimal orders follow a near reorder-point policy.*

Proof of Lemma A.0.2. Let $\tilde{V}_t(I)$ be the optimal profit function in period t when the inventory level is I. The optimality equation is given by

$$\begin{aligned}
\tilde{V}_t(I) &= \max_{q_1 \geq 0, q_2 \geq 0} \tilde{W}_t(I, q_1, q_2) \\
\tilde{W}_t(I, q_1, q_2) &= \mathbb{E}_{\varepsilon,\omega}[\max_{\underline{d} \leq d \leq \overline{\overline{d}}} \tilde{J}_t(I, q_1, q_2, d)] \\
\tilde{J}_t(I, q_1, q_2, d) &= R(d) - \bar{c}_1 q_1 - \bar{c}_2 q_2 - \mathbb{E}_{u_1,u_2} H(I + u_1 q_1 + u_2 q_2 - \varepsilon d - \omega) \\
&\quad + \alpha \mathbb{E}_{u_1,u_2} \tilde{V}_{t+1}(I + u_1 q_1 + u_2 q_2 - \varepsilon d - \omega)
\end{aligned}$$

Note that $\tilde{J}_t(I, q_1, q_2, d)$ is supermodular in (I, d). Hence, the optimal price is increasing in I. Finally, the optimal ordering decisions can be derived in a way similar to that of Theorem 2.2.1.

Proof of Theorem 2.2.2. To see part i), we note that the optimal profit function $V_t(I)$ is differentiable in I when $\varepsilon d + \omega$ has a continuous distribution. The result follows immediately from the proof of Lemma 2.2.7.

For part ii-a), we first show that $d^*_t(I)$ is increasing in I. We have

$$V_t(I) = \max_{d \in [\underline{d}, \overline{\overline{d}}]} \{R(d) + G_t(I - d)\}, \tag{A.5}$$

where

$$G_t(z) = \max_{q_1 \geq 0, q_2 \geq 0} \{-\bar{c}_1 q_1 - \bar{c}_2 q_2 + \mathbb{E} L_t(z + u_1 q_1 + u_2 q_2)\}.$$

Since L_t is concave, it is clear that $G_t(z)$ is concave in z and thus $G_t(I-d)$ is supermodular. We deduce that $R(d) + G_t(I,d)$ is supermodular, which leads to the desired result.

Next we prove that the change in $d_t^*(I)$ is less than the corresponding change in I. We can write (A.5) as

$$V_t(I) = \max_{z \in [I-\overline{\overline{d}}, I-\underline{\underline{d}}]} R(I-z) + G_t(z).$$

Since R and G_t are concave, $R(I-z) + G_t(z)$ is supermodular in (I,z). It follows that the maximizer $z_t^*(I) = I - d_t^*(I)$ is increasing in I. Hence, we obtain the desired result.

To prove part ii-b), let $\theta_i = \Pr\{u_i = \overline{\overline{u}}_i\}$, $i = 1,2$, and $z = I - d$. We have

$$J_t(I, q_1, q_2, I-z) = R(I-z) + (1-\theta_1)(1-\theta_2)L_t(z) + \overline{G}_t(z, q_1, q_2),$$

where

$$\begin{aligned} \overline{G}_t(z, q_1, q_2) &= -c_1\theta_1\overline{\overline{u}}_1 q_1 - c_2\theta_2\overline{\overline{u}}_2 q_2 + \theta_1(1-\theta_2)L_t(z + \overline{\overline{u}}_1 q_1) \\ &\quad + (1-\theta_1)\theta_2 L_t(z + \overline{\overline{u}}_2 q_2) + \theta_1\theta_2 L_t(z + \overline{\overline{u}}_1 q_1 + \overline{\overline{u}}_2 q_2). \end{aligned}$$

Note that for fixed (I,d), the optimal (q_1, q_2) maximizes $\overline{G}_t(z, q_1, q_2)$ and depends on (I, d) only via $z = I - d$. Now define $y_1 = z + \overline{\overline{u}}_1 q_1$. Then,

$$\begin{aligned}
& \max_{q_1 \geq 0, q_2 \geq 0} \overline{G}_t(z, q_1, q_2) \\
= & \max_{y_1 \geq z, q_2 \geq 0} \overline{G}_t(z, (y_1 - z)/\overline{\overline{u}}_1, q_2) \\
= & \max_{q_2 \geq 0} \{c_1\theta_1 z - c_2\theta_2\overline{\overline{u}}_2 q_2 + (1-\theta_1)\theta_2 L_t(z + \overline{\overline{u}}_2 q_2) + Q(z, q_2)\},
\end{aligned} \tag{A.6}$$

where

$$Q(z, q_2) = \max_{y_1 \geq z} -c_1\theta_1 y_1 + \theta_1(1-\theta_2)L_t(y_1) + \theta_1\theta_2 L_t(y_1 + \overline{\overline{u}}_2 q_2).$$

Let $\bar{y}_1(q_2)$ denote the unconstrained maximizer of $Q(y_1, q_2)$. Then, for given (z, q_2) the optimal y_1 is $\max\{\bar{y}_1(q_2), z\}$. It is easily seen that $Q(y_1, q_2)$ is jointly concave and thus $Q(z, q_2)$ is concave. If $\bar{y}_1(q_2) \geq z)$, then $Q(z, q_2)$ submodular in (z, q_2) because L_t is concave. If, however, $\bar{y}_1(q_2) < z$, then then $Q(z, q_2)$ is concave in q_2 and constant in z. In either case, the function inside the maximum of (A.6) is submodular in (q_2, z) and thus the optimal $q_2(z)$ is decreasing in z. We further recall part i) that the optimal $z_t^*(I) = I - d_t^*(I)$ is increasing in I. We deduce that $q_{2,t}^*(I) = q_2(z_t^*(I)$ is decreasing in I. Now note that $q_{1,t}^*(I) = \max\{\bar{y}_1(q_2(z_t^*(I))) - z_1^*(I), 0\}/\overline{\overline{u}}_1$. Since $q_{1,t}^* = 0$ when $I \geq I_{t,1}^*$, we focus on the case $I < I_{t,1}^*$. Let $y_2 = z + \overline{\overline{u}}_2 q_2$. For $I < I_{t,1}^*$,

$$\begin{aligned}
& \max_{q_1 \geq 0, q_2 \geq 0} \overline{G}_t(z, q_1, q_2) \qquad \text{(A.7)} \\
= & \max_{y_2 \geq z} \{c_1\theta_1 z - c_2\theta_2(y_2 - z) + (1-\theta_1)\theta_2 L_t(y_2) + \tilde{Q}((y_2 - z)/\overline{\overline{u}}_2)\}
\end{aligned}$$

where

$$\tilde{Q}(q_2) = \max_{y_1} \{-c_1\theta_1 q_1 + \theta_1(1-\theta_2)L_t(y_1) + \theta_1\theta_2 L_t(y_1 + \overline{\overline{u}}_2 q_2)\}.$$

Because L_t is concave, $\tilde{Q}$ is concave. It follows that the function inside the maximum of (A.7) is supermodular in (y_2, z) and thus the optimal $y_2(z)$ is increasing in z. We further recall part i) that the optimal $z_t^*(I) = I - d_t^*(I)$ is increasing in I. We deduce that $z_t^*(I) + \overline{\overline{u}}_2 q_{2,t}^*(I)$ is increasing in I. Also, we can rewrite $\tilde{Q}$ as follows,

$$\tilde{Q}(q_2) = \max_{q_1}\{-c_1\theta_1 q_1 + \theta_1(1-\theta_2)L_t(z + \overline{\overline{u}}_1 q_1) + \theta_1\theta_2 L_t(z + \overline{\overline{u}}_2 q_2 + \overline{\overline{u}}_1 q_1)\}. \tag{A.8}$$

The function inside the maximum of (A.8) is submodular in (z, q_1) and $(z + \overline{\overline{u}}_2 q_2, q_1)$. Because both $z_t^*(I) + \overline{\overline{u}}_2 q_{2,t}^*(I)$ and $z_t^*(I)$ increase in I, $q_{1,t}^*(I)$ decrease in I. We conclude the proof.

Proof of Theorem 2.3.1. We prove the result using contradiction by assuming that $q_{t,2}^*(I^a) \le q_{t,2}^*(I^b)$. We must have

$$\begin{aligned}
0 &< J_t(I^b, q_{t,1}^*(I^b), q_{t,2}^*(I^b), d_t) - J_t(I^b, q_{t,1}^*(I^a), q_{t,2}^*(I^b), d_t) \\
&= -c_1(q_{t,1}^*(I^b) - q_{t,1}^*(I^a)) + \mathbb{E}L_t(I^b + u_1 q_{t,1}^*(I^b) + u_2 q_{t,2}^*(I^b) - \varepsilon d_t) \\
&\quad -\mathbb{E}L_t(I^b + u_1 q_{t,1}^*(I^a) + u_2 q_{t,2}^*(I^b) - \varepsilon d_t) \\
&\le -c_1(q_{t,1}^*(I^b) - q_{t,1}^*(I^a)) \\
&\quad +\mathbb{E}L_t\Big[I^b + u_1 q_{t,1}^*(I^b) + u_2 q_{t,2}^*(I^b) - \varepsilon d_t - \Big(I^b - I^a + u_2\big(q_{t,2}^*(I^b) - q_{t,2}^*(I^a)\big)\Big)\Big] \\
&\quad -\mathbb{E}L_t\Big[I^b + u_1 q_{t,1}^*(I^a) + u_2 q_{t,2}^*(I^b) - \varepsilon d_t - \Big(I^b - I^a + u_2\big(q_{t,2}^*(I^b) - q_{t,2}^*(I^a)\big)\Big)\Big] \\
&= -c_1(q_{t,1}^*(I^b) - q_{t,1}^*(I^a)) + \mathbb{E}L_t\big(I^a + u_1 q_{t,1}^*(I^b) + u_2 q_{t,2}^*(I^a) - \varepsilon d_t\big) \\
&\quad -\mathbb{E}L_t\big(I^a + u_1 q_{t,1}^*(I^a) + u_2 q_{t,2}^*(I^a) - \varepsilon d_t\big) \\
&= J_t(I^a, q_{t,1}^*(I^b), q_{t,2}^*(I^a), d_t) - J_t(I^a, q_{t,1}^*(I^a), q_{t,2}^*(I^a), d_t) \le 0.
\end{aligned}$$

The first inequality follow from the optimality of $(q^*_{t,1}(I^b), q^*_{t,2}(I^b))$ for $J_t(I^b, q_{t,1}, q_{t,2}, d_t)$. The second inequality follows from the concavity of L_t. The third equality comes from the optimality of $(q^*_{t,1}(I^a), q^*_{t,2}(I^a))$ for $J_t(I^a, q_{t,1}, q_{t,2}, d_t)$.

Moreover, in the view of above proof, we can obtain that the results of i and ii-a hold for multiple suppliers and the result of ii-b only holds for two suppliers. We conclude the proof.

Proof of Theorem 2.3.2. Suppose that $d^*_t(I^a) \geq d^*_t(I^b)$ and $q^*_{t,1}(I^a) < q^*_{t,1}(I^b)$ for $I^a < I^b$. Then we must have

$$\begin{aligned}
0 &< J_t(I^b, q^*_{t,1}(I^b), 0, d^*_t(I^b)) - J_t(I^b, q^*_{t,1}(I^a), 0, d^*_t(I^b)) \\
&= -c_1(q^*_{t,1}(I^b) - q^*_{t,1}(I^a)) + \mathbb{E}L_t(I^b + u_1 q^*_{t,1}(I^b) - \varepsilon d^*_t(I^b)) \\
&\quad -\mathbb{E}L_t(I^b + u_1 q^*_{t,1}(I^a) - \varepsilon d^*_t(I^b)) \\
&\leq -c_1(q^*_{t,1}(I^b) - q^*_{t,1}(I^a)) \\
&\quad +\mathbb{E}L_t\Big[I^b + u_1 q^*_{t,1}(I^b) - \varepsilon d^*_t(I^b) - \Big(I^b - I^a + \varepsilon\big(d^*_t(I^a) - d^*_t(I^b)\big)\Big)\Big] \\
&\quad -\mathbb{E}L_t\Big[I^b + u_1 q^*_{t,1}(I^a) - \varepsilon d^*_t(I^b) - \Big(I^b - I^a + \varepsilon\big(d^*_t(I^a) - d^*_t(I^b)\big)\Big)\Big] \\
&= -c_1(q^*_{t,1}(I^b) - q^*_{t,1}(I^a)) + \mathbb{E}L_t\big(I^a + u_1 q^*_{t,1}(I^b) - \varepsilon d^*_t(I^a)\big) \\
&\quad -\mathbb{E}L_t\big(I^a + u_1 q^*_{t,1}(I^a) - \varepsilon d^*_t(I^a)\big) \\
&= J_t(I^a, q^*_{t,1}(I^b), 0, d^*_t(I^a)) - J_t(I^a, q^*_{t,1}(I^a), 0, d^*_t(I^a)) \leq 0.
\end{aligned}$$

The first inequality follow from the optimality of $(q^*_{t,1}(I^b), d^*_t(I^b))$ for $J_t(I^b, q_{t,1}, 0, d_t)$. The second inequality follows from the concavity of L_t. The third equality comes from the optimality of $(q^*_{t,1}(I^a), d^*_t(I^a))$ for $J_t(I^a, q_{t,1}, 0, d_t)$. We reach a contradiction. We conclude the proof.

Proof of Theorem 2.4.1. If $a = 0$, then $u_1 = b$ is deterministic. From Lemma 2.2.2, we know $q^*_{t,1}(I)$ follows a base-stock policy and is thus decreasing in I. Now note that for $I \leq \bar{y}^B_{t,1}$, $q^*_{t,2}(I) = \bar{q}^B_{t,2}$ is constant in I. For $I > \bar{y}^B_{t,1}$, we have $q^*_{t,1}(I) = 0$ and $J_t(I, 0, q_2, d)$ is submodular in (I, q_2) by Lemma 2.2.1. Hence, $q^*_{t,2}(I)$ is decreasing in I. In particular, if $a = b = 0$, then $u_1 = 0$. In this case, the problem reduces to one with only supplier 2 and thus $q^*_{t,2}(I)$ is decreasing in I as pointed out by Henig and Gerchak [41].

Now we examine the case when $a \neq 0$. To see part i), we first note that $q^*_{t,2}(I) = 0$ for $I \geq I^*_{t,2}$. In this case, $q^*_{t,1}(I)$ is decreasing in I since $J_t(I, q_1, 0, d)$ is submodular in (I, q_1) by Lemma 2.2.1. If $I < I^*_{t,2}$, then $q^*_{t,2}(I) = \tilde{q}_{t,2}(I)$ where $\tilde{q}_{t,2}(I)$ is defined in Lemma 2.2.4. Now define $x \equiv aq_1 + q_2$ (or $q_2 = x - aq_1$) and $\mathcal{J}_t(I, q_1, x) = J_t(I, q_1, x - aq_1, d)$. Also let $\mathcal{W}_t(I, q_1) = \max_x \mathcal{J}_t(I, q_1, x)$ with $\tilde{x}(I, q_1)$ denoting the corresponding maximizer. It is easily seen that $\mathcal{J}_t$ is concave and thus $\mathcal{W}_t$ is concave. The result would follow provided that $\mathcal{W}_t(I, q_1)$ is submodular in (I, q_1), i.e. for any $I^a \leq I^b$ and $q^a_1 \leq q^b_1$,

$$\mathcal{W}_t(I^a, q^a_1) + \mathcal{W}_t(I^b, q^b_1) \leq \mathcal{W}_t(I^a, q^b_1) + \mathcal{W}_t(I^b, q^a_1). \tag{A.9}$$

It is clear that (A.9) holds when $I^a = I^b$. Therefore, we focus on the case when $I^a < I^b$. Define $x^a = \tilde{x}(I^a, q_1^a)$ and $x^b = \tilde{x}(I^b, q_1^b)$. We deduce

$$\begin{aligned}
&\mathcal{W}_t(I^a, q_1^a) + \mathcal{W}_t(I^b, q_1^b) - 2R(d) \\
&= -(\bar{c}_1 + a\bar{c}_2)q_1^a - \bar{c}_2 x^a + \mathbb{E}L_t(I^a + bq_1^a + u_2 x^a - \varepsilon d) \\
&\quad -(\bar{c}_1 + a\bar{c}_2)q_1^b - \bar{c}_2 x^b + \mathbb{E}L_t(I^b + bq_1^b + u_2 x^b - \varepsilon d) \\
&= -(\bar{c}_1 + a\bar{c}_2)q_1^b - \bar{c}_2 x^a + \mathbb{E}L_t(I^a - b(q_1^b - q_1^a) + bq_1^b + u_2 x^a - \varepsilon d) \\
&\quad -(\bar{c}_1 + a\bar{c}_2)q_1^a - \bar{c}_2 x^b + \mathbb{E}L_t(I^b + b(q_1^b - q_1^a) + bq_1^a + u_2 x^b - \varepsilon d) \\
&\le \mathcal{W}_t(I^a - b(q_1^b - q_1^a), q_1^b) + \mathcal{W}_t(I^b + b(q_1^b - q_1^a), q_1^a) - 2R(d). \qquad \text{(A.10)}
\end{aligned}$$

The inequality follows from the maximality of $\mathcal{W}_t$.

Moreover, by the concavity of $\mathcal{W}_t(I, q)$, we obtain

$$\frac{\mathcal{W}_t(I^b, q_1^b) - \mathcal{W}_t(I^a - b(q_1^b - q_1^a), q_1^b)}{I^b - I^a + b(q_1^b - q_1^a)} \ge \frac{\mathcal{W}_t(I^b, q_1^b) - \mathcal{W}_t(I^a, q_1^b)}{I^b - I^a}, \qquad \text{(A.11)}$$

$$\frac{\mathcal{W}_t(I^b + b(q_1^b - q_1^a), q_1^a) - \mathcal{W}_t(I^a, q_1^a)}{I^b - I^a + b(q_1^b - q_1^a)} \le \frac{\mathcal{W}_t(I^b, q_1^a) - \mathcal{W}_t(I^a, q_1^a)}{I^b - I^a}. \qquad \text{(A.12)}$$

Relations in (A.10), (A.11) and (A.12) imply

$$\frac{\mathcal{W}_t(I^b, q_1^b) - \mathcal{W}_t(I^a, q_1^b)}{I^b - I^a} \le \frac{\mathcal{W}_t(I^b, q_1^a) - \mathcal{W}_t(I^a, q_1^a)}{I^b - I^a},$$

which leads to (A.9).

To see part ii), we note that $u_2 = -\frac{1}{a}u_1 - \frac{b}{a}$. Hence, a similar argument as that in part i) yields the result.

To see parts iii), we note that $u_1 = au_2$ when $b = 0$. We have

$$\begin{aligned}
\mathcal{J}_t(I, q_1, x) &= R(d) - c_1 a\bar{u}_2 q_1 - c_2\bar{u}_2(x - aq_1) \\
&\quad + \mathbb{E}L_t(I + au_2 q_1 + u_2(x - aq_1) - \varepsilon d) \\
&= R(d) + a\bar{u}_2(c_2 - c_1)q_1 - \bar{c}_2 x + \mathbb{E}L_t(I + u_2 x - \varepsilon d).
\end{aligned}$$

The right-hand side is separable in q_1 and x. Hence, the optimal $x_t^*(I)$ maximizes the last two terms for $x \geq 0$. Since L_t is concave, it is clear that $x_t^*(I)$ is decreasing in I. Also $q_{t,1}^*(I)$ maximize $a\bar{u}_2(c_2 - c_1)q_1$ over $q_1 \in [0, x_t^*(I)/a]$. If $c_2 \geq c_1$, we must have $q_{t,1}^*(I) = x_t^*(I)/a$ is decreasing in I, which, in turn, implies $q_{t,2}^*(I) = x_t^*(I) - aq_{t,1}^*(I) = 0$. If, however, $c_2 < c_1$, then $q_{t,I}^* = 0$ and $q_{t,2}^*(I) = x_t^*(I)$ is decreasing in I.

For $b = -a(\overline{\overline{u}}_2 + \underline{\underline{u}}_2)$ and $a < 0$, we have $q_{t,1}^*(I)$ is decreasing in I from the part i. From the Theorem 2.2.1, Lemma 2.2.4 and the fact $c_1 \leq c_2$, we know that $\tilde{q}_{t,1}(I) \geq 0$ and $\tilde{q}_{t,2}(I) \geq 0$ when $I \leq I_{t,2}^*$. From the continuity of $q_{t,2}^*(I)$, we only need to prove that there exists $\bar{q}_{t,2}$ and an optimal solution satisfying that $q_{t,2}^*(I) = \bar{q}_{t,2} - \frac{I}{\overline{\overline{u}}_2 + \underline{\underline{u}}_2}$ when $I \leq I_{t,2}^*$. Define $y_1 = I - a(\overline{\overline{u}}_2 + \underline{\underline{u}}_2)q_1$ and

$$
\begin{aligned}
\Phi_t(y_1, x) &= \mathcal{J}_t(I, q_1, x) - \big[(c_2 - c_1)\frac{\bar{u}_2}{\overline{\overline{u}}_2 + \underline{\underline{u}}_2} + c_1\big]I \\
&= R(d) - c_1(a\bar{u}_2 - a(\overline{\overline{u}}_2 + \underline{\underline{u}}_2))q_1 \\
&\quad -c_2\bar{u}_2(x - aq_1) - \big[(c_2 - c_1)\frac{\bar{u}_2}{\overline{\overline{u}}_2 + \underline{\underline{u}}_2} + c_1\big]I \\
&\quad +\mathbb{E}L_t[I + (au_2 - a(\overline{\overline{u}}_2 + \underline{\underline{u}}_2))q_1 + u_2(x - aq_1) - \varepsilon d] \\
&= R(d) - \big[(c_2 - c_1)\frac{\bar{u}_2}{\overline{\overline{u}}_2 + \underline{\underline{u}}_2} + c_1\big](I - a(\overline{\overline{u}}_2 + \underline{\underline{u}}_2)q_1) - c_2\bar{u}_2 x \\
&\quad +\mathbb{E}L_t[I - a(\overline{\overline{u}}_2 + \underline{\underline{u}}_2)q_1 + u_2 x - \varepsilon d] \\
&= R(d) - \big[(c_2 - c_1)\frac{\bar{u}_2}{\overline{\overline{u}}_2 + \underline{\underline{u}}_2} + c_1\big]y_1 - c_2\bar{u}_2 x + \mathbb{E}L_t[y_1 + u_2 x - \varepsilon d].
\end{aligned}
$$

We observe that the right-hand side does not depend on I and is concave in (y_1, x). Let $(\bar{y}_1, \bar{x})$ to denote the maximizer of Φ_t. We must exist an optimal solution satisfying $I - a(\overline{\overline{u}}_2 + \underline{\underline{u}}_2)q_1^*(I) = \bar{y}_1$ and $aq_1^*(I) + q_2^*(I) = \bar{x}$ when $I \leq I_{t,2}^*$. Define $\bar{q}_{t,2} = \bar{x} + \frac{\bar{y}_1}{\overline{\overline{u}}_2 + \underline{\underline{u}}_2}$. Thus, $q_2^*(I) = \bar{x} - aq_1^*(I) = \bar{x} - \frac{I - \bar{y}_1}{\overline{\overline{u}}_2 + \underline{\underline{u}}_2} = \bar{q}_{t,2} - \frac{I}{\overline{\overline{u}}_2 + \underline{\underline{u}}_2}$. We conclude the proof.

Proof of Theorem 2.4.2. The proof is similar to the one for Theorem 2.4.1. The added complexity here is that we have to consider the decision $d_t(I)$ which is bounded from above.

Example 3 Li and Zheng [48]: $T = 1$, $c_1 = 1$, $H(x) = h(\max\{x, 0\})^2 + s(\max\{-x, 0\})^2$ with $h = 2$, $s = 9$, $p(d) = 20 - d$, $Pr\{u_2 = 0\} = 1$, $Pr\{\varepsilon = 0\} = Pr\{\varepsilon = 2\} = 0.5$ and $\omega = 0$. The yield rate u_1 is uniform(0,0.5).

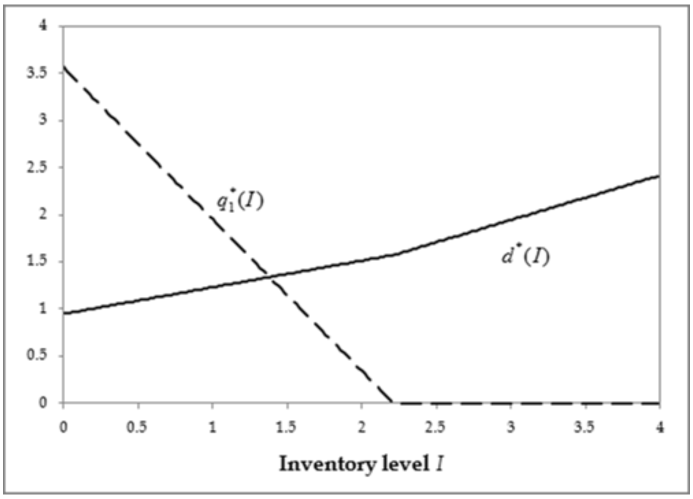


Figure A.1: The example provided in Li and Zheng [48].

Appendix B

Appendix for Chapter 3

Lemma B.0.1. *If $A_1/B_1 < A_2/B_2$ with $B_1 \geq 0$ and $B_2 \geq 0$, then the following results hold:*

i) $A_1/B_1 < (A_1 + A_2)/(B_1 + B_2) < A_2/B_2$.

ii) If $B_2 > B_1$, *then* $(A_2 - A_1)/(B_2 - B_1) > A_2/B_2 > A_1/B_1$. *If* $B_1 > B_2$, *then* $(A_1 - A_2)/(B_1 - B_2) < A_2/B_2$.

iii) $g(n) = \frac{A_1+nA_2}{B_1+nB_2}$ *is increasing in* n.

Proof. The proof for part (i) is straightforward. Part (ii) follows because

$$g'(n) = \frac{A_2(B_1 + nB_2) - B_2(A_1 + nA_2)}{(B_1 + nB_2)^2} = \frac{A_2B_1 - A_1B_2}{(B_1 + nB_2)^2} > 0.$$

□

Derivations for the solutions presented in Table 3.1.

Linear demand function $p(\tau) = \alpha - \frac{\beta}{\tau}$. We have $p'(\tau) = \frac{\beta}{\tau^2}$. From (3.4), we have $\tau_i = \sqrt{\frac{\beta}{ih+V}}$. Hence, given S, we have

$$V = \frac{S(\alpha - c) - K - \sqrt{\beta}\sum_{i=1}^{S}(\sqrt{ih+V} + \frac{ih}{\sqrt{ih+V}})}{\sqrt{\beta}\sum_{i=1}^{S}\frac{1}{\sqrt{h+V}}}$$

$$S(\alpha - c) - K = \sqrt{\beta}\sum_{i=1}^{S}(\sqrt{ih+V} + \frac{ih+V}{\sqrt{ih+V}})$$

$$S(\alpha - c) - K = 2\sqrt{\beta}\sum_{i=1}^{S}\sqrt{ih+V}$$

If S is a large number, we can use an integral to approximate the right-hand side. That is to say

$$S(\alpha - c) - K \doteq 2\sqrt{\beta}\int_1^S \sqrt{xh+V}dx$$

$$S(\alpha - c) - K \doteq \frac{4\sqrt{\beta}}{3h}[(Sh+V)^{3/2} - (h+V)^{3/2}]$$

Take the derivative of both sides with respect to S, we have

$$\alpha - c = \frac{4\sqrt{\beta}}{3h}[(Sh+V)^{1/2}(h + \frac{dV}{dS}) - (h+V)^{1/2}\frac{dV}{dS}]$$

$$\frac{dV}{dS} = \frac{1}{\sqrt{SH+V} - \sqrt{S+V}}[\frac{(\alpha-c)h}{2\sqrt{\beta}} - h\sqrt{Sh+V}].$$

We obtain that $\frac{dV}{dS} > 0$, when $\sqrt{Sh+V} < \frac{(\alpha-c)h}{2\sqrt{\beta}}$; $\frac{dV}{dS} = 0$, when $\sqrt{Sh+V} = \frac{(\alpha-c)h}{2\sqrt{\beta}}$; $\frac{dV}{dS} < 0$, when $\sqrt{Sh+V} > \frac{(\alpha-c)h}{2\sqrt{\beta}}$.

Therefore (S^*, V^*) are a solution of the following equation system

$$\sqrt{Sh+V} = \frac{(\alpha-c)}{2\sqrt{\beta}} \tag{B.1}$$

$$S(\alpha - c) - K = \frac{4\sqrt{\beta}}{3h}[(\frac{(\alpha-c)}{2\sqrt{\beta}})^3 - (h+V)^{3/2}] \tag{B.2}$$

The optimal expected interarrival time can be obtained by $\tau_i^* = \sqrt{\frac{\beta}{ih+V^*}}$.

Exponential demand function $p(\tau) = \alpha \ln(\beta\tau)$, $(\beta\overline{\tau} > e)$. We have $p'(\tau) = \frac{\alpha}{\tau}$. From (3.4), we have $\tau_i = \sqrt{\frac{\alpha}{ih+V}}$. Hence, given S, we have

$$V = \frac{-Sc - K + \alpha \sum_{i=1}^{S}[\ln(\frac{\alpha\beta}{ih+V}) - \frac{ih}{ih+V}]}{\alpha \sum_{i=1}^{S} \frac{1}{ih+V}}$$

$$Sc + K = \alpha[\sum_{i=1}^{S} \ln(\frac{\alpha\beta}{ih+V})] - \alpha[\sum_{i=1}^{S} \frac{ih+V}{ih+V})]$$

$$S(c+\alpha) + K = \alpha[\sum_{i=1}^{S} \ln(\frac{\alpha\beta}{ih+V})]$$

$$S[\alpha(\ln(\alpha\beta) - 1) - c] - K = \alpha[\sum_{i=1}^{S} \ln(ih+V)]$$

If S is a large number, we can use an integral to approximate the right-hand side. That is to say

$$S[\alpha(\ln(\alpha\beta) - 1) - c] - K = \alpha[\int_{i=1}^{S} \ln(ih+V)]dx$$

$$S[\alpha(\ln(\alpha\beta) - 1) - c] - K = \frac{\alpha}{h}\{(Sh+V)[\ln(Sh+V) - 1] - (h+V)[\ln(h+V) - 1]\}$$

$$S[\alpha(\ln(\alpha\beta) - 1) - c] - K + \alpha(S-1) = \frac{\alpha}{h}[(Sh+V)\ln(Sh+V) - (h+V)\ln(h+V)]$$

$$S[\alpha \ln(\alpha\beta) - c] - K - \alpha = \frac{\alpha}{h}[(Sh+V)\ln(Sh+V) - (h+V)\ln(h+V)]$$

Take the derivative of both sides with respect to S, we have

$$\alpha \ln(\alpha\beta) - c = \frac{\alpha}{h}[(\ln(Sh+V)+1)(h + \frac{dV}{dS}) - (\ln(h+V)+1)\frac{dV}{dS}]$$

$$\frac{dV}{ds}\frac{1}{h}\ln(\frac{Sh+V}{h+V}) = \ln(\alpha\beta) - \frac{c}{\alpha} - [\ln(Sh+V)+1]$$

We obtain that $\frac{dV}{dS} > 0$, when $1 + \ln(Sh+V) < \ln(\alpha\beta) - \frac{c}{\alpha}$; $\frac{dV}{dS} = 0$, when $1 + \ln(Sh+V) < \ln(\alpha\beta) - \frac{c}{\alpha}$; $\frac{dV}{dS} < 0$, when $1 + \ln(Sh+V) > \ln(\alpha\beta) - \frac{c}{\alpha}$.

Therefore (S^*, V^*) are a solution of the following equation system

$$1 + \ln(Sh + V) = \ln(\alpha\beta) - \tfrac{c}{\alpha} \quad \text{(B.3)}$$

$$S[\alpha \ln(\alpha\beta) - c] - K - \alpha = \tfrac{\alpha}{h}[(Sh + V)\ln(Sh + V) - (h + V)\ln(h + V)] \quad \text{(B.4)}$$

The optimal expected interarrival time can be obtained by $\tau_i^* = \sqrt{\frac{\alpha}{ih+V^*}}$.

Isoelastic demand function $p(\tau) = \alpha\tau^\beta$, *$(\beta < 1,\ 0 < \tau < 1)$.* We have $p'(\tau) = \frac{\alpha\beta}{(\tau)^{1-\beta}}$. From (3.4), we have $\tau_i = \sqrt[1-\beta]{\frac{\alpha\beta}{ih+V}}$. Hence, given S, we have

$$V = \frac{-Sc - K + \sum_{i=1}^{S}[\alpha(\frac{\alpha\beta}{ih+V})^{\beta/(1-\beta)} - ih\sqrt[1-\beta]{\frac{\alpha\beta}{ih+V}}]}{\sum_{i=1}^{S}\sqrt[1-\beta]{\frac{\alpha\beta}{ih+V}}}$$

$$Sc + K = \sum_{i=1}^{S}[\alpha(\frac{\alpha\beta}{ih+V})^{\beta/(1-\beta)}] - (\alpha\beta)^{1/(1-\beta)}\sum_{i=1}^{S}[\frac{ih+V}{\sqrt[1-\beta]{ih+V}}]$$

$$Sc + K = \sum_{i=1}^{S}[\alpha(\frac{\alpha\beta}{ih+V})^{\beta/(1-\beta)}] - (\alpha\beta)^{1/(1-\beta)}\sum_{i=1}^{S}[\frac{1}{ih+V}]^{\beta/(1-\beta)}$$

$$Sc + K = \alpha^{\frac{1}{1-\beta}}(\beta^{\frac{\beta}{1-\beta}} - \beta^{\frac{1}{1-\beta}})\sum_{i=1}^{S}(ih+V)^{-\frac{1}{1-\beta}}$$

If S is a large number, we can use an integral to approximate the right-hand side. That is,

$$Sc + K = \alpha^{\frac{1}{1-\beta}}(\beta^{\frac{\beta}{1-\beta}} - \beta^{\frac{1}{1-\beta}})\int_{i=1}^{S}(xh+V)^{-\frac{\beta}{1-\beta}}dx$$

$$Sc + K = \frac{\alpha^{\frac{1}{1-\beta}}(\beta^{\frac{\beta}{1-\beta}} - \beta^{\frac{1}{1-\beta}})}{h(1-\frac{\beta}{1-\beta})}[(Sh+V)^{\frac{1-2\beta}{1-\beta}} - (h+V)^{\frac{1-2\beta}{1-\beta}}]$$

Take the derivative of both sides with respect to S, we have

$$c = \frac{\alpha^{\frac{1}{1-\beta}}(\beta^{\frac{\beta}{1-\beta}} - \beta^{\frac{1}{1-\beta}})}{h}[(Sh+V)^{-\frac{\beta}{1-\beta}}(h + \frac{dV}{dS}) - (h+V)^{-\frac{\beta}{1-\beta}}\frac{dV}{dS}]$$

$$\frac{dV}{dS}\frac{1}{h}[(h+V)^{-\frac{\beta}{1-\beta}} - (Sh+V)^{-\frac{\beta}{1-\beta}}] = \frac{1}{(Sh+V)^{\frac{\beta}{1-\beta}}} - \frac{c}{\alpha^{\frac{1}{1-\beta}}(\beta^{\frac{\beta}{1-\beta}} - \beta^{\frac{1}{1-\beta}})}$$

We obtain that $\frac{dV}{dS} > 0$, when $(Sh+V)^{\frac{\beta}{1-\beta}} < \frac{\alpha^{\frac{1}{1-\beta}}(\beta^{\frac{\beta}{1-\beta}}-\beta^{\frac{1}{1-\beta}})}{c}$; $\frac{dV}{dS} = 0$, when $(Sh+V)^{\frac{\beta}{1-\beta}} = \frac{\alpha^{\frac{1}{1-\beta}}(\beta^{\frac{\beta}{1-\beta}}-\beta^{\frac{1}{1-\beta}})}{c}$; $\frac{dV}{dS} < 0$, when $(Sh+V)^{\frac{\beta}{1-\beta}} > \frac{\alpha^{\frac{1}{1-\beta}}(\beta^{\frac{\beta}{1-\beta}}-\beta^{\frac{1}{1-\beta}})}{c}$.

Therefore (S^*, V^*) are a solution of the following equation system

$$(Sh+V)^{\frac{\beta}{1-\beta}} = \frac{\alpha^{\frac{1}{1-\beta}}(\beta^{\frac{\beta}{1-\beta}}-\beta^{\frac{1}{1-\beta}})}{c} \tag{B.5}$$

$$Sc+K = \frac{\alpha^{\frac{1}{1-\beta}}(\beta^{\frac{\beta}{1-\beta}}-\beta^{\frac{1}{1-\beta}})}{h(1-\frac{\beta}{1-\beta})}[(Sh+V)^{\frac{1-2\beta}{1-\beta}} - (h+V)^{\frac{1-2\beta}{1-\beta}}] \tag{B.6}$$

The optimal expected interarrival time can be obtained by $\tau_i^* = \sqrt[1-\beta]{\frac{\alpha\beta}{ih+V^*}}$.

□

Computation of the Optimal Solution for the Continuous Relaxation

Let ω_j be the ratio of the length of the jth price segment to the length of the replenishment cycle. To find the optimal selling price, we can write the problem as follows.

$$\textbf{(MP)} \quad \max_{x_j;\omega_j;\iota_j;\zeta_j} \quad W = \sum_{j=1}^{N+1}\left\{\omega_j\left[\frac{p(\iota_j)-c}{\iota_j} - \frac{\int_{x_{j-1}}^{x_j} h(x)dx}{x_j - x_{j-1}} - \frac{\zeta_j}{(x_j - x_{j-1})\iota_j}\right]\right\}$$

$$s.t. \quad \sum_{j=1}^{N+1}\zeta_j = K + NA,$$

$$\omega_j = \frac{(x_j - x_{j-1})\iota_j}{\sum_{m=1}^{N+1}[(x_m - x_{m-1})\iota_m]}, \forall j$$

$$x_j > x_{j-1}, \forall j$$

Here, ζ_j can be represent as a method to allocate the fixed cost $(K + NA)$ to $N+1$ price segments.

We can use BARON to obtain optimal solutions for **MP**. In all the examples reported in §3.5.4, the optimal solution of **MP** can be obtained in no more than 10 seconds.

Proof of Lemma 3.4.1. Suppose that the result is not true. Then we must have $\hat{\tau}_j(S) = \tau^b$ for some j. Let $i_1 = \min\{i : \hat{\tau}_i = \tau^b\}$ and $i_2 = \max\{i : \hat{\tau}_i = \tau^b\}$. We can find $\hat{\tau}_i = \tau^a$ for $1 \le i \le i_1 - 1$ and $\hat{\tau}_i = \tau^c$ for $i_2 + 1 \le i \le S$. Now define two vectors as follows:

$$\tau_i^A = \begin{cases} \tau^a & \text{for } i_1 \le i \le i_2 \\ \hat{\tau}_i(S) & \text{otherwise,} \end{cases} \quad \text{and} \quad \tau_i^C = \begin{cases} \tau^c & \text{for } i_1 \le i \le i_2 \\ \hat{\tau}_i(S) & \text{otherwise.} \end{cases}$$

Denote $\lambda = \frac{\tau^c - \tau^b}{\tau^c - \tau^a}$. Then $\hat{\tau}_i(S) = \lambda \tau_i^A + (1-\lambda)\tau_i^C$ and $p(\hat{\tau}_i(S)) \le \lambda p(\tau_i^A) + (1-\lambda)p^(\tau_i^C)$ for $i \le i \le S$. We have

$$\pi(i, \hat{\tau}_i(S)) \le \lambda \pi(i, \tau_i^A) + (1-\lambda)\pi(i, \tau_i^C).$$

By the optimality of $\hat{\tau}_i(S)$, we have for $I = A, C$,

$$\hat{V}(S) \ge V(S, \vec{\tau}_S^I) \ge \frac{\sum_{i=1}^S \pi(i, \tau_i^I) - A \sum_{i=2}^S \mathbb{1}_{\hat{\tau}_i(S) \ne \hat{\tau}_{i-1}(S)} - K - cS}{\sum_{i=1}^S \tau_i^I}.$$

Note that the second inequality follows because $\sum_{i=2}^S \mathbb{1}_{\hat{\tau}_i(S) \ne \hat{\tau}_{i-1}(S)} \le \sum_{i=2}^S \mathbb{1}_{\tau_i^I \ne \tau_{i-1}^I}$.

Moreover, strictly inequality holds for $I = A$. We derive

$$\begin{aligned}
\hat{V}(S) &\geq \max\left\{ \frac{\lambda\big[\sum_{i=1}^{S}\pi(i,\tau_i^A) - A\sum_{i=2}^{S}\mathbb{1}_{\hat{\tau}_i(S)\neq\hat{\tau}_{i-1}(S)} - K - cS\big]}{\lambda\sum_{i=1}^{S}\tau_i^A}, \right. \\
&\qquad \left. \frac{(1-\lambda)\big[\sum_{i=1}^{S}\pi(i,\tau_i^C) - A\sum_{i=2}^{S}\mathbb{1}_{\hat{\tau}_i(S)\neq\hat{\tau}_{i-1}(S)} - K - cS\big]}{(1-\lambda)\sum_{i=1}^{S}\tau_i^C} \right\} \\
&> \frac{\lambda\big[\sum_{i=1}^{S}\pi(i,\tau_i^A) - A\sum_{i=2}^{S}\mathbb{1}_{\hat{\tau}_i(S)\neq\hat{\tau}_{i-1}(S)} - K - cS\big]}{\lambda\sum_{i=1}^{S}\tau_i^A + (1-\lambda)\sum_{i=1}^{S}\tau_i^C} \\
&\quad + \frac{(1-\lambda)\big[\sum_{i=1}^{S}\pi(i,\tau_i^C) - A\sum_{i=2}^{S}\mathbb{1}_{\hat{\tau}_i(S)\neq\hat{\tau}_{i-1}(S)} - K - cS\big]}{\lambda\sum_{i=1}^{S}\tau_i^A + (1-\lambda)\sum_{i=1}^{S}\tau_i^C} \\
&\geq \frac{\big[\sum_{i=1}^{S}\pi(i,\hat{\tau}_i(S)) - A\sum_{i=2}^{S}\mathbb{1}_{\hat{\tau}_i(S)\neq\hat{\tau}_{i-1}(S)} - K - cS\big]}{\sum_{i=1}^{S}\hat{\tau}_i(S)} = \hat{V}(S).
\end{aligned}$$

This is impossible and hence we conclude the proof. □

Proof of Lemma 3.4.2. We prove the result using contradiction. Suppose that the result is not true. Then $j = \min\{i|\tau_i^* < \tau^M, 0 \leq i \leq S^*\}$ is well defined. Let $l = \max\{i|\tau_{j+i-1}^* = \tau_j^*, 1 \leq i \leq S^* - j + 1\}$. We consider two cases.

Case 1: $l = S$. In this case, we must have $j = 1$ and thus $\tau_i^* = \tau_1^* < \tau^M$. We derive

$$\begin{aligned}
V^* &= \frac{p(\tau_1^*) - c}{\tau_1^*} - \frac{\sum_{i=1}^{S^*} h(i)}{S^*} - \frac{K}{\tau_1^* S^*} \\
&< \frac{p(\tau^M) - c}{\tau^M} - \frac{\sum_{i=1}^{S^*} h(i)}{S^*} - \frac{K}{\tau^M S^*} = V(S^*, \tau^M, \cdots, \tau^M).
\end{aligned}$$

The inequality follows from the maximality of τ^M and $\tau_1^* < \tau^M$. The above relation contradicts the optimality of V^*. Hence, this case is not possible.

Case 2: $1 \le l < S$. Consider any $X \subset \{1, \cdots, S^*\}$. We can partition X into at most $s \le S$ number of sets $\{x_1, \cdots, x_s\}$ such that $\tau_i^* = \tau_{k_1}^*$, where $k_1 = \min\{x_k\}$ for $k = 1, 2, \cdots, s$. Then,

$$\frac{\sum_{i\in X} \pi(i,\tau_i^*)}{\sum_{i\in X}\tau_i^*} = \frac{\sum_{k=1}^{s}\left[\tau_{k_1}^* \sum_{i\in x_k} \frac{\pi(i,\tau_{k_1}^*)}{\tau_{k_1}^*}\right]}{\sum_{i\in X}\tau_i^*} \le \frac{\sum_{k=1}^{s}\left[\tau_{k_1}^* \sum_{i\in x_k} \frac{\pi(i,\tau^M)}{\tau^M}\right]}{\sum_{i\in X}\tau_i^*} = \frac{\pi(i,\tau^M)}{\tau^M}.$$

Now define $I = \{i : 1 \le i \le j-1 \text{ or } j+l \le i \le S\}$ and $N^A = \sum_{i=1}^{S-1} \mathbb{1}_{\{\tau_i^* \neq \tau_{i+1}^*\}}$. We derive

$$\begin{aligned}
& V(S^*, \tau_1^*, \cdots, \tau_{j-1}^*, \tau^M, \cdots, \tau^M, \tau_{j+l+1}^*, \cdots, \tau_{S^*}^*) \\
= \; & \frac{\sum_{i\in I}\pi(i,\tau_i^*) + \sum_{i\notin I}\pi(i,\tau^M)}{\sum_{i\in I}\tau_i^* + l\tau^M} - \frac{AN^A + K}{\sum_{i\in I}\tau_i^* + l\tau^M} \\
> \; & \frac{\sum_{i\in I}\pi(i,\tau_i^*)/\tau^M + \sum_{i\notin I}\pi(i,\tau_j^*)/\tau_j^*}{\sum_{i\in I}\tau_i^*/\tau^M + l} - \frac{AN^A + K}{\sum_{i\in I}\tau_i^* + l\tau_j^*} \\
= \; & \frac{\sum_{i\in I}\pi(i,\tau_i^*) + (\tau^M/\tau_j^*)\sum_{i\notin I}\pi(i,\tau_j^*)}{\sum_{i\in I}\tau_i^* + (\tau^M/\tau_j^*)l\tau_j^*} - \frac{AN^A + K}{\sum_{i\in I}\tau_i^* + l\tau_j^*} \\
\ge \; & \frac{\sum_{i=1}^{S^*}\pi(i,\tau_i^*)}{\sum_{i=1}^{S^*}\tau_i^*} - \frac{AN^A + K}{\sum_{i=1}^{S^*}\tau_i^*} = V^*
\end{aligned}$$

The first inequality follows from $\tau_j^* < \tau^M$ and the second equality is obtained by applying Lemma B.0.1(iii). The above relation contradicts the optimality of V^*.

Finally, the unimodality of $[p(\tau) - c]/\tau$ in τ can be easily verified for a concave $p(\tau)$. Hence, we conclude the proof. □

Proof of Lemma 3.4.3. We prove the result by contradiction. Suppose $N^A = \sum_{i=1}^{S^*-1} \mathbb{1}_{\tau_i^* \neq \tau_{i+1}^*} > 1$. Then we can define a sequence of indices $\{l_0, \cdots, l_{N^A+1}\}$

with $1 = l_0 < \cdots < l_{N^A+1} = S^* + 1$ such that $\tau_i^* = \tau_{l_k-1}^*$ for $l_{k-1} \le \tau \le l_k - 1$ and $1 \le k \le N^A + 1$. Define

$$\hat{j} = \arg \max_{0 \le j \le N^A} \frac{\sum_{i=l_j}^{l_{j+1}-1} \pi(i, \tau_{l_j}^*) - A}{(l_{j+1} - l_j)\tau_{l_j}^*}.$$

Then,

$$\begin{aligned} V(l_{\hat{j}+1} - l_{\hat{j}}, \tau_{\hat{j}}^*, \cdots, \tau_{\hat{j}}^*) &= \frac{\sum_{i=1}^{l_{\hat{j}+1}-l_{\hat{j}}} \pi(i, \tau_{\hat{j}}^*) - K}{(l_{\hat{j}+1} - l_{\hat{j}})\tau_{\hat{j}}^*} \ge \frac{\sum_{i=l_{\hat{j}}}^{l_{\hat{j}+1}-1} \pi(i, \tau_{\hat{j}}^*) - A + (A - K)}{(l_{\hat{j}+1} - l_{\hat{j}})\tau_{\hat{j}}^*} \\ &\ge \frac{\sum_{k=0}^{N^A} \left[\sum_{i=l_k}^{l_{k+1}-1} \pi(i, \tau_{l_k}^*) - A\right]}{\sum_{k=0}^{N^A} (l_{k+1} - l_k)\tau_{l_k}^*} + \frac{A - K}{(l_{\hat{j}+1} - l_{\hat{j}})\tau_{\hat{j}}^*} \\ &\ge \frac{\sum_{k=0}^{N^A} \left[\sum_{i=l_k}^{l_{k+1}-1} \pi(i, \tau_{l_k}^*) - A\right] + (A - K)}{\sum_{k=0}^{N^A} (l_{k+1} - l_k)\tau_{l_k}^*} = V^*. \end{aligned}$$

The first inequality follows because $\pi(i, \tau)$ is weakly decreasing in i, the second inequality from Lemma B.0.1, and the third inequality form $A > K$. The above relation contradicts the optimality of V^*. Hence, we conclude the proof. □

Proof of Proposition 3.4.1. Suppose the result in part (i) is not true. Then there exists some inventory level j, $1 < j \le S$, such that $\hat{\tau}_j(S) > \hat{\tau}_{j-1}(S)$. Note that we can always choose j to be the maximum of such inventory levels, i.e., $j = \max\{i : \hat{\tau}_i(S) > \hat{\tau}_{i-1}(S)\}$. Define $l_1 = \min\{i : \hat{\tau}_i(S) = \hat{\tau}_{j-1}(S), 1 \le i \le j - 1\}$ and $l_2 = \max\{i : \hat{\tau}_i(S) = \hat{\tau}_j(S), j \le i \le S\}$. Now we construct another vector $(\check{\tau}_1, \cdots, \check{\tau}_S)$ as follows:

$$\check{\tau}_i = \begin{cases} \hat{\tau}_{j-1}(S) & \text{if } l_1 + l_2 - j + 1 \le i \le l_2, \\ \hat{\tau}_j(S) & \text{if } l_1 \le i \le l_1 + l_2 - j, \\ \hat{\tau}_i(S) & \text{if } 1 \le i < l_1 \text{ or } l_2 < i \le S. \end{cases}$$

By construction we have $\sum_{i=1}^{S} \hat{\tau}_i(S) = \sum_{i=1}^{S} \check{\tau}_i$ and thus $\sum_{i=1}^{S} p(\hat{\tau}_i(S)) = \sum_{i=1}^{S} p(\check{\tau}_i)$. Because $h(i)$ is strictly increase, we have $\sum_{i=1}^{S} h(i)\hat{\tau}_i(S) > \sum_{i=1}^{S} h(i)\check{\tau}_i$.

Also, by maximality of j, we deduce $\sum_{i=1}^{S-1} \mathbb{1}_{\{\hat{\tau}_i(S) \neq \hat{\tau}_{i+1}(S)\}} \geq \sum_{i=1}^{S-1} \mathbb{1}_{\{\check{\tau}_i \neq \check{\tau}_{i+1}\}}$. Therefore,

$$\begin{aligned} V(S, \vec{\hat{\tau}}(S)) &= \frac{\sum_{i=1}^{S}(p(\hat{\tau}_i(S)) - h(i)\hat{\tau}_i(S)) - A\sum_{i=1}^{S-1} \mathbb{1}_{\{\hat{\tau}_i(S) \neq \hat{\tau}_{i+1}(S)\}} - K - cS}{\sum_{i=1}^{S} \hat{\tau}_i(S)} \\ &< \frac{\sum_{i=1}^{S}(p(\check{\tau}_i) - h(i)\check{\tau}_i) - A\sum_{i=1}^{S-1} \mathbb{1}_{\{\check{\tau}_i \neq \check{\tau}_{i+1}\}} - K - cS}{\sum_{i=1}^{S} \check{\tau}_i} \\ &= V(S, \vec{\check{\tau}}). \end{aligned}$$

It contradicts with the optimality of $\vec{\hat{\tau}}(S)$. Thus, we obtain part (i).

Part (ii) follows immediately because $\tau^M < \tau_1^* \leq \cdots \leq \tau_{S^*}^*$ and $[p(\tau) - c]/\tau$ is unimodal in τ with the maximizer τ^M.

Part (iii) can be obtained by contradiction. If $V^* \geq \frac{S^*, \pi(S^*, \tau_{S^*}^*)}{\tau_{S^*}^*}$, then from Lemma B.0.1, we have

$$\hat{V}(S^* - 1) \geq V(S^* - 1, \tau_1^*, \cdots, \tau_{S^*-1}^*) - \frac{A\mathbb{1}_{\{\hat{\tau}_{S^*-1}^* \neq \hat{\tau}_(S^*)^*\}}}{\sum_{i=1}^{S^*-1} \tau_i^*} \geq V^*$$

It contradicts with the optimality of V^*.

Next we show $\pi(S^*+1, \tau_{S^*}^*)/\tau_{S^*}^* \leq V^*$. We assume $\pi(S^*+1, \tau_{S^*}^*)/\tau_{S^*}^* > V^*$. From Lemma B.0.1(ii),

$$\hat{V}(S+1) \geq V(S+1, \tau_1^*, \cdots, \tau_{S^*}^*, \tau_{S^*}^*) > V^*.$$

It contradicts with the optimality of V^*. We conclude the proof. □

Proof of Proposition 3.4.2. We first show $V^* < \omega^M(S^*)$. Suppose $V^* \geq$

$\omega^M(S^*)$. We have

$$\begin{aligned} V(S^*-1,\tau_1^*,\cdots,\tau_{S^*-1}^*) &= \frac{\sum_{i=1}^{S^*-1}\pi(i,\tau_i^*) - K - A\sum_{i=1}^{S^*-2} 1_{\{\tau_i^*\neq\tau_{i+1}^*\}}}{\sum_{i=1}^{S^*-1}\tau_i^*} \\ \geq & \ \frac{\sum_{i=1}^{S^*-1}\pi(i,\tau_i^*) - K - A\sum_{i=1}^{S^*-2} 1_{\{\tau_i^*\neq\tau_{i+1}^*\}} + \pi(i,\tau_{S^*}^*) - A\mathbb{1}_{\{\tau_{S^*-1}^*\neq\tau_{S^*}^*\}}}{\sum_{i=1}^{S^*-1}\tau_i^* + \tau_{S^*}^*} \\ = & \ V^*. \end{aligned}$$

The inequality follows from Lemma B.0.1(ii) and the fact that $V^* \geq \omega^M(S^*) > \frac{\pi(S^*,\tau_{S^*}^*)-A\mathbb{1}_{\{\tau_{S^*-1}^*\neq\tau_{S^*}^*\}}}{\tau_{S^*}^*}$. The above relation contradicts the fact that S^* is the smallest maximizer. Hence, we must have $\omega^M(S^*) > V^*$.

Next we prove $\omega^{MA}(S^*+1) \leq V^*$. Suppose $\omega^{MA}(S^*+1) > V^*$. Then,

$$\begin{aligned} & V(S^*+1,\tau_1^*,\cdots,\tau_{S^*}^*,\tau^{MA}) \\ \geq & \ \frac{\sum_{i=1}^{S^*}\pi(i,\tau_i^*) - K - A\sum_{i=1}^{S^*-1}\mathbb{1}_{\{\tau_i^*\neq\tau_{i+1}^*\}} + \pi(S^*+1,\tau^{MA}) - A}{\sum_{i=1}^{S^*}\tau_i^* + \tau^{MA}} \\ > & \ \frac{\sum_{i=1}^{S^*}\pi(i,\tau_i^*) - K - A\sum_{i=1}^{S^*-1}\mathbb{1}_{\{\tau_i^*\neq\tau_{i+1}^*\}}}{\sum_{i=1}^{S^*}\tau_i^*} = V^*. \end{aligned}$$

The first equality follows from $\mathbb{1}_{\{\tau_{S^*}^*\neq\tau^{MA}\}} \leq 1$. The second inequality follows from Lemma B.0.1(i) and the observation that $V^* < \omega^{MA}(S^*+1) = \frac{\pi(S^*+1,\tau^{MA})-c-A}{\tau^{MA}}$. The above relation contradicts the optimality of V^* and thus $\omega^{MA}(S^*+1) \leq V^*$.

Now we show that $\hat{V}(S) \geq \omega^{MA}(S)$ for $S > S^*$.

$$\begin{aligned} \hat{V}(S^*+1) &\geq V(S^*+1,\tau_1^*,\cdots,\tau_{S^*}^*,\tau^{MA}) \\ &\geq \frac{\sum_{i=1}^{S^*}\pi(i,\tau_i^*) - K - A\sum_{i=1}^{S^*-1}\mathbb{1}_{\{\tau_i^*\neq\tau_{i+1}^*\}} + \pi(S^*+1,\tau^{MA}) - A}{\sum_{i=1}^{S^*}\tau_i^* + \tau^{MA}} \\ &\geq \frac{\pi(S^*+1,\tau^{MA}) - A}{\tau^{MA}} = \omega^{MA}(S^*+1). \end{aligned}$$

The first inequality follows from the optimality of $\hat{V}(S^*+1)$. The second inequality follows from $1_{\{\tau^*_{S^*} \neq \tau^{MA}\}} \leq 1$. The last inequality follows from Lemma B.0.1(ii) $\hat{V}(S^*) \geq \omega^{MA}(S^*+1) = \frac{\pi(S^*+1,\check{\tau}^{MA})-A}{\check{\tau}^{MA}}$. Note that $\omega^{MA}(i) > \omega^{MA}(i+1)$. Thus, we can repeat the same argument to show $\hat{V}(i) \geq \omega^{MA}(i)$ for all $i \geq S^*+1$.

Finally, we note for $S < S^*$, $\hat{V}(S) < \hat{V}(S^*) < \omega^M(S^*) \leq \omega^M(S+1)$. Hence, we conclude the proof. □

Proof of Lemma 3.4.4. To see part (i), let $S^\natural = \max\{i|\hat{V}(i) < \omega^{MA}(i)\}$. In view of the proof of Proposition 3.4.2, we have $V(S^\natural - 1) < V(S^\natural) < \omega^{MA}(S^\natural) < \omega^{MA}(S^\natural - 1)$. By induction, we have for all $S_1 < S \leq S^\natural$,

$$V(S_1) < V(S) \leq V(S^\natural) < \omega^{MA}(S^\natural) \leq \omega^{MA}(S) < \omega^{MA}(S_1)$$

Next we consider the case $S_1 < S$ and $S^\natural < S \leq S^*$. Using Proposition 3.4.2, we have $\omega^{MA}(S) \leq \hat{V}(S) < \omega^M(S)$ for all $S^\natural < S \leq S^*$ and $\hat{V}(S_1) < \hat{V}(S^*) < \omega^M(S^*)$. Hence,

$$\begin{aligned}
&V(S) - V(S_1) > \omega^{MA}(S) - \omega^M(S^*) \\
&= \omega^{MA}(S) - \omega^{MA}(S^*) - (\omega^M(S^*) - \omega^{MA}(S^*)) \\
&\geq 0 - \delta\omega = -\delta\omega
\end{aligned}$$

The first inequality comes from $\omega^{MA}(S) \geq \hat{V}(S)$ and $\hat{V}(S_1) < \omega^M(S^*)$. The second inequality comes from that $S \leq S^*$ and $\omega^{MA}(.)$ is a strictly decreasing function.

To see the part (ii), from Proposition 3.4.2, we have $\hat{V}(S) \geq \omega^{MA}(S)$ when $S > S^*$. Hence, $\hat{V}(S) + \delta\omega \geq \omega^M(S) > \omega^M(i)$ for all $i > S$. Hence,

$$
\begin{aligned}
&\hat{V}(S) + \delta\omega \\
&> \max_{S+1 \leq j \leq S_2} \{V(S, \hat{\tau}_1(S_2), \cdots, \hat{\tau}_S(S_2)), \frac{\pi(j, \hat{\tau}_j(S_2)) - A\mathbb{1}_{\hat{\tau}_{j-1}(S_2) \neq \hat{\tau}_j(S_2)}}{\hat{\tau}_j(S_2)}\} \\
&= \max_{S+1 \leq j \leq S_2} \left\{ \frac{\sum_{i=1}^{S} \pi(i, \hat{\tau}_i(S_2)) - K - A \sum_{i=1}^{S-1} \mathbb{1}_{\hat{\tau}_i(S_2) \neq \hat{\tau}_{i+1}(S_2)}}{\sum_{i=1}^{S} \hat{\tau}_i(S_2)}, \right. \\
&\qquad \left. \frac{\pi(j, \hat{\tau}_j(S_2)) - A\mathbb{1}_{\hat{\tau}_{j-1}(S_2) \neq \hat{\tau}_j(S_2)}}{\hat{\tau}_j(S_2)} \right\} \\
&\geq \frac{\sum_{i=1}^{S} \pi(i, \hat{\tau}_i(S_2))}{\sum_{i=1}^{S} \hat{\tau}_i(S_2) + \sum_{j=S+1}^{S_2} \hat{\tau}_j(S_2)} - K - A \sum_{i=1}^{S-1} \mathbb{1}_{\hat{\tau}_i(S_2) \neq \hat{\tau}_{i+1}(S_2)} \\
&\qquad + \sum_{j=S+1}^{S_2} [\pi(j, \hat{\tau}_j(S_2)) - A\mathbb{1}_{\hat{\tau}_{j-1}(S_2) \neq \hat{\tau}_j(S_2)}]\Big] \\
&= \hat{V}(S_2).
\end{aligned}
$$

We conclude the proof of the part (ii). □

Proof of Lemma 3.4.5. Suppose there is a finite $\bar{S}$ such that $\tau^M \sum_{i=1}^{\bar{S}} [h(\bar{S}) - h(i)] \geq K$,

$$
V(\bar{S}, \tau^M, \cdots, \tau^M) = \frac{\bar{S}(p(\tau^M) - c) - K - \sum_{i=1}^{\bar{S}} h(i)}{\bar{S}\tau^M} \geq \frac{\bar{S}(p(\tau^M) - c) - \bar{S}h(\bar{S})\tau^M}{\bar{S}\tau^M} = \omega^M(\bar{S})
$$

Therefore, $\hat{V}(\bar{S}) \geq \omega^M(\bar{S})$. From Proposition 2, we have $S^* < \bar{S}$.

Suppose there is a finite $\underline{S}$ such that $\sum_{i=1}^{\underline{S}} [A + \overline{\tau} h(\underline{S}) - \underline{\tau} h(i)] < K$.

Combining with $A \geq 0$ and the optimality of $\omega^{MA}(S)$, we have

$$\begin{aligned}
\hat{V}(\underline{S}) &\leq \frac{\sum_{i=1}^{\underline{S}}[p(\hat{\tau}_i(\underline{S})) - c - A]}{\sum_{i=1}^{\underline{S}} \hat{\tau}_i(\underline{S})} - \frac{K - \underline{S}A + \sum_{i=1}^{\underline{S}}[h(i)\hat{\tau}_i(\underline{S})]}{\sum_{i=1}^{\underline{S}} \hat{\tau}_i(\underline{S})} \\
&\leq \omega^{MA}(\underline{S}) + h(\underline{S}) - \frac{K - \underline{S}A + [\sum_{i=1}^{\underline{S}} h(i)]\underline{\tau}}{\underline{S}\overline{\tau}} \\
&= \omega^{MA}(\underline{S}) - \frac{K - \sum_{i=1}^{\underline{S}}[h(S)\overline{\tau} - h(i)\underline{\tau} + A]}{\underline{S}\overline{\tau}} < \omega^{MA}(\underline{S})
\end{aligned}$$

Therefore, $\hat{V}(\underline{S}) < \omega^{MA}(\underline{S})$. From Proposition 2, we have $S^* \geq \underline{S}$. $\square$

Proof Proposition 3.4.3. By Lemma 3.4.1, we only need to treat the case when τ is over a compact interval. To see part (i), we note that τ^M continues and weakly increases in c. Therefore, if $c^a - c^b$ is sufficiently small, we have $\tau^{Mb} \leq \tau^{Ma} \leq \sum_{i=1}^{S^{b*}} \tau_i^{b*}/S^{b*} + \epsilon \mathbb{1}_{\{\tau_1^{b*} = \tau^{Mb}\}}$ for any $\epsilon > 0$. We consider two cases:

Case 1: If $\tau_1^{b*} = \tau^{Mb}$, then $\tau_i^{b*} = \tau^{Mb}$ for all $1 \leq i \leq S^{b*}$ by Lemma 3.4.2. Also

$$\begin{aligned}
\hat{V}^a(S^{b*}) - \omega^{Ma}(S^{b*}+1) &\geq V^a(S^{b*}, \tau^{Ma}, \cdots, \tau^{Ma}) - \omega^{Ma}(S^{b*}+1) \\
&= -\frac{K}{S\tau^{Ma}} - \frac{\sum_{i=1}^{S^{b*}} h(i)}{S^{b*}} + h(S^{b*}+1) \\
&\geq -\frac{K}{S\tau^{Mb}} - \frac{\sum_{i=1}^{S^{b*}} h(i)}{S^{b*}} + h(S^{b*}+1) \\
&= V^b(S^{b*}, \tau^{Mb}, \cdots, \tau^{Mb}) - \omega^{Mb}(S^{b*}+1) \\
&= V^{b*} - \frac{\pi^b(S^{b*}+1, \tau^{Mb})}{\tau^{Mb}} \geq 0
\end{aligned}$$

The first inequality comes from the optimality of $\hat{V}^a(S^{b*})$, the second from $\tau^{Ma} \geq \tau^{Mb}$, and the third from Proposition 3.4.1. Together with Proposition 3.4.2 (i), we obtain $S^{a*} \leq S^{b*}$.

Case 2: If $\tau_1^{b*} > \tau^{Mb}$, then $\sum_{i=1}^{S^{b*}} \tau_i^{b*}/S^{b*} \geq \max\{\tau^{Ma}, \tau_{S^{b*}}^{b*}\}$. Suppose $S^{a*} > S^{b*}$. We claim $\tau_{S^{a*}}^{a*} \geq \tau_{S^{b*}}^{b*}$. To see that, suppose $\tau_{S^{a*}}^{a*} < \tau_{S^{b*}}^{b*}$. Let $l^a = \max\{i : \tau_i^{a*} > \tau_{S^{a*}}^{a*}, 1 \leq i \leq S^{a*}\}$. Because $V^{a*} \geq V^a(S^{a*}, \tau_1^{a*}, \cdots, \tau_{l^a}^{a^*}, \tau_{S^{b*}}^{b^*}, \cdots, \tau_{S^{b*}}^{b^*})$, we derive from Lemma B.0.1(ii)

$$V^{a*} \geq \frac{\left[\sum_{i=l^a+1}^{S^{b*}} \pi^b(i, \tau_{S^{b*}}^{b*}) + A\mathbb{1}_{\{\tau_{l^a}^{a^*} = \tau_{S^{b*}}^{b^*}\}}\right] - \sum_{i=l^a+1}^{S^{b*}} \pi^a(i, \tau_{S^{a*}}^{a*})}{(S^{a*} - l^a)\tau_{S^{b*}}^{b*} - (S^{a*} - l^a)\tau_{S^{a*}}^{a*}} \geq \frac{p(\tau_{S^{b*}}^{b*}) - p(\tau_{S^{a*}}^{a*})}{\tau_{S^{b*}}^{b*} - \tau_{S^{a*}}^{a*}}.$$

Likewise, we can derive $V^{b*} < \frac{p(\tau_{S^{b*}}^{b*}) - p(\tau_{S^{a*}}^{a*})}{\tau_{S^{b*}}^{b*} - \tau_{S^{a*}}^{a*}}$. These relations imply $V^{b*} < V^{a*}$, which contradicts the fact that $c^a > c^b$. Thus, we must have $\tau_{S^{a*}}^{a*} \geq \tau_{S^{b*}}^{b*}$.

Now note from Proposition 3.4.1, $V^{b^*} \geq \pi^b(S^{b^*}+1, \tau_{S^{b*}}^{b*})/\tau_{S^{b*}}^{b*}$ and $V^{a*} < \pi^{\tilde{b}}(S^{\tilde{b}*}, \tau_{S^{a*}}^{a*})/\tau_{S^{a*}}^{a*}$. Then,

$$\begin{aligned} V^{b*} - V^{a*} &> \frac{\pi^b(S^{b^*}+1, \tau_{S^{b*}}^{b*})}{\tau_{S^{b*}}^{b*}} - \frac{\pi^a(S^{a*}, \tau_{S^{a*}}^{a*})}{\tau_{S^{a*}}^{a*}} \\ &= \frac{p(\tau_{S^{b*}}^{b*}) - c^b}{\tau_{S^{b*}}^{b*}} - \frac{p(\tau_{S^{a*}}^{a*}) - c^a}{\tau_{S^{a*}}^{a*}} - h(S^{b^*}+1) + h(S^{a*}) \\ &\geq \frac{p(\tau_{S^{b*}}^{b*}) - c^b}{\tau_{S^{b*}}^{b*}} - \frac{p(\tau_{S^{a*}}^{a*}) - c^a}{\tau_{S^{a*}}^{a*}} \geq \frac{c^a - c^b}{\max\{\tau^{Ma}, \tau_{S^{b*}}^{b*}\}}. \end{aligned}$$

The first inequality follows from the assumption $S^{a*} > S^{b*}$. To see the second inequality, we consider two cases: (a) If $\tau_{S^{b*}}^{b*} \geq \tau^{Ma}$, then $\tau^{Ma} \leq \tau_{S^{b*}}^{b*} \leq \tau_{S^{a*}}^{a*}$. It follows that $[p(\tau_{S^{b*}}^{b*}) - c^b]/\tau_{s^{b*}}^{b*} - (c^a - c^b)/\tau_{s^{b*}}^{b*} \geq [p(\tau_{S^{a*}}^{a*}) - c^a]/\tau_{s^{a*}}^{a*}$. (b) If $\tau_{S^{b*}}^{b*} < \tau^{Ma}$, then $\tau^{Mb} \leq \tau_{S_b^*}^{b*} < \tau^{Ma}$. It follows that $[p(\tau_{S^{b*}}^{b*}) - c^b]/\tau_{s^{b*}}^{b*} \geq [p(\tau^{Ma}) - c^a]/\tau^{Ma} + (c^a - c^b)/\tau^{Ma} \geq [p(\tau_{S^{a*}}^{a*}) - c^a]/\tau_{s^{a*}}^{a*} + (c^a - c^b)/\tau^{Ma}$.

However,

$$V^{b*} - V^{\tilde{b}*} < V^{b*} - V^{\tilde{b}}(S^{b*}, \tau_1^{b*}, \cdots, \tau_{S^{b*}}^{b^*}) = \frac{(c^a - c^b)S^{b*}}{\sum_{i=1}^{S^{b*}} \tau_i^{b*}} \leq \frac{c^a - c^b}{\max\{\tau^{Ma}, \tau_{S^{b*}}^{b*}\}}, \tag{B.7}$$

which leads to a contradiction. Therefore, $S^{a*} \le S^{b*}$ and obtain part (i).

To see part (ii), suppose $S^{a*} > S^{b*}$. We claim $\tau^{a*}_{S^{a*}} \ge \tau^{b*}_{S^{b*}}$. To see that, suppose $\tau^{a*}_{S^{a*}} < \tau^{b*}_{S^{b*}}$. Let $l^a = \min\{i : \tau^{a*}_i = \tau^{a*}_{S^{a*}}, 1 \le i \le S^{a*}\}$. Because $V^{a*} \ge V^a(S^{a*}, \tau^{a*}_1, \cdots, \tau^{a*}_{l_a}, \tau^{b*}_{S^{b*}}, \cdots, \tau^{b*}_{S^{b*}})$, we derive from Lemma B.0.1(ii)

$$V^{a*} \ge \frac{[\sum_{i=l^a}^{S^{b*}} \pi^b(i, \tau^{b*}_{S^{b*}}) + A\mathbb{1}_{\tau^{a*}_{l^a-1}=\tau^{b*}_{S^{b*}}}] - \sum_{i=l^a}^{S^{a*}} \pi^a(i, \tau^{a*}_{S^{a*}})}{(S^{a*} - l^a + 1)\tau^{b*}_{S^{b*}} - (S^{a*} - l^a + 1)\tau^{a*}_{S^{a*}}} \ge \frac{p(\tau^{b*}_{S^{b*}}) - p(\tau^{a*}_{S^{a*}})}{\tau^{b*}_{S^{b*}} - \tau^{a*}_{S^{a*}}}.$$

Likewise, we can derive $V^{b*} < \frac{p(\tau^{b*}_{S^{b*}}) - p(\tau^{a*}_{S^{a*}})}{\tau^{b*}_{S^{b*}} - \tau^{a*}_{S^{a*}}}$. These relations imply $V^{a*} > V^{b*}$, which contradicts the fact that $h^a(i) \ge h^b(i)$ for all $i > 0$. Thus, we must have $\tau^{a*}_{S^{a*}} \ge \tau^{b*}_{S^{b*}}$.

We further note

$$\begin{aligned} V^{b*} - V^{a*} &> \frac{\pi^b(S^{b*} + 1, \tau^{b*}_{S^{b*}})}{\tau^{b*}_{S^{b*}}} - \frac{\pi^a(S^{a*}, \tau^{a*}_{S^{a*}})}{\tau^{a*}_{S^{a*}}} \\ &= \frac{p(\tau^{b*}_{S^{b*}}) - c}{\tau^{b*}_{S^{b*}}} - \frac{p(\tau^{a*}_{S^{a*}}) - c}{\tau^{a*}_{S^{a*}}} + h^a(S^{a*}) - h^b(S^{b*} + 1) \\ &\ge h^a(S^{a*}) - h^b(S^{b*} + 1) \ge h^a(S^{b*} + 1) - h^b(S^{b*} + 1). \end{aligned}$$

The first inequality comes from Proposition 3.4.1(iii) and the second from Lemma 3.4.2 and $\tau^{a*}_{S^{a*}} \ge \tau^{b*}_{S^{b*}} \ge \tau^M$. However,

$$\begin{aligned} & V^{b*} - V^{a*} < V^{b*} - V^a(S^{b*}, \tau^{b*}_1, \cdots, \tau^{b*}_{S^{b*}}) \\ &= \frac{\sum_1^{S^{b*}} [h^a(i) - h^b(i)]\tau^{b*}_i}{\sum_1^{S^{b*}} \tau^{b*}_i} \le h^a(S^{b*} + 1) - h^b(S^{b*} + 1). \end{aligned}$$

The last two relations contradicts each other. Hence, we conclude part (ii).

To see part (iii), we note that $\hat{V}^b(S) < \hat{V}^a(S)$ for any $S \ge 1$ and $V^{b*} < V^{a*}$. Suppose $S^{b*} < S^{a*}$, then from the proof of part (i), we have

$$\hat{V}^a(S^{b*}) > \hat{V}^b(S^{b*}) \ge \omega^M(S^{b*} + 1) \ge \omega^M(S^{a*}) > \hat{V}^a(S^{a*}) \ge \hat{V}^a(S^{b*}).$$

This leads to a contradiction, and thus we must have $S^{b*} \geq S^{a*}$.

Finally, we show part (iv). Without loss of generality, suppose $\tilde{p}(\tau)$ is continuous in τ. First, we will show that τ^M decreases in γ. For every $\gamma^c > \gamma^d$ and $\tau^c > \tau^d$, we have

$$\begin{aligned} & \frac{\tilde{p}(\gamma^c,\tau^c)}{\tau^c} - \frac{\tilde{p}(\gamma^d,\tau^c)}{\tau^c} = \frac{\tilde{p}(\gamma^c,\tau^c) - \tilde{p}(\gamma^d,\tau^c)}{\tau^c} \\ < & \frac{\tilde{p}(\gamma^c,\tau^d) - \tilde{p}(\gamma^d,\tau^d)}{\tau^c} \leq \frac{\tilde{p}(\gamma^c,\tau^d) - \tilde{p}(\gamma^d,\tau^d)}{\tau^d} = \frac{\tilde{p}(\gamma^c,\tau^d)}{\tau^d} - \frac{\tilde{p}(\gamma^d,\tau^d)}{\tau^d}. \end{aligned}$$

The first inequality comes from the submodularity of $\tilde{p}(\gamma,\tau)$, the second inequality from $\tau^c \geq \tau^d$. Hence, $\frac{\tilde{p}(\gamma,\tau)}{\tau}/\tau$ is submodular in (γ,τ). Thus, $(\tilde{p}(\gamma,\tau) - c)/\tau$ is also submodular on (γ,τ). Therefore, τ^M decreases in γ.

Without loss of generality, we assume $\gamma^b - \gamma^a > 0$ is small enough satisfying that

$$\tau^{Mb} \leq \tau^{Ma} < \min\{\tau^\theta\}\mathbb{1}_{\tau_1^{b*} > \tau^{Mb}} + \epsilon \mathbb{1}_{\tau_1^{b*} = \tau^{Mb}},$$

where $\tau^\theta = \min\{\tau : \frac{\tilde{p}(\tau)-c}{\tau} = \frac{\sum_{i=1}^{S^{b*}}[\tilde{p}(\tau_i^{b*})-c]}{\sum_{i=1}^{S^{b*}}\tau_i^{b*}}, \tau \geq \tau^{Mb}\}$ and $\epsilon > 0$. To show that $S^{a*} \leq S^{b*}$, we analyze two cases: *Case 1*, $\tau_1^{b*} = \tau^{Mb}$ and *Case 2*, $\tau_1^{b*} > \tau^{Mb}$.

Case 1: $\tau_1^{b*} = \tau^{Mb}$. From Proposition 3.4.1, we have $\tau_i^{b*} = \tau^{Mb}$ for all $1 \leq i \leq S^{b*}$. We also have

$$\begin{aligned} & \hat{V}^a(S^{b*}) - \omega^{Ma}(S^{b*}+1) \\ \geq & V^a(S^{b*}, \tau^{Ma}, \cdots, \tau^{Ma}) - \omega^{Ma}(S^{b*}+1) = -\frac{K}{S\tau^{Ma}} - \frac{\sum_{i=1}^{S^{b*}} h(i)}{S^{b*}} + h(S^{b*}+1) \\ \geq & -\frac{K}{S\tau^{Mb}} - \frac{\sum_{i=1}^{S^{b*}} h(i)}{S^{b*}} + h(S^{b*}+1) = V^b(S^{b*}, \tau^{Mb}, \cdots, \tau^{Mb}) - \omega^{Mb}(S^{b*}+1) \\ = & V^{b*} - \frac{\pi^b(S^{b*}+1, \tau^{Mb})}{\tau^{Mb}} \geq 0. \end{aligned}$$

The first inequality comes from the optimality of $\hat{V}^a(S^{b*})$. The second inequality comes from $\tau^{Ma} \geq \tau^{Mb}$. The third inequality comes from Proposition 3.4.1. Combing $\hat{V}^a(S^{b*}) > \omega^{Ma}(S^{b*}+1)$ with Proposition 3.4.2 (i), we have that $S^{a*} \leq S^{b*}$.

Case 2: $\tau_1^{b*} > \tau^{Mb}$. Suppose $S^{a*} > S^{b*}$. Let $l^a = \min\{i : \tau_i^{a*} = \tau_{S^{a*}}^{a*}, 1 \leq i \leq S^{a*}\}$. First, we show $\tau_{S^{a*}}^{a*} \geq \tau_{S^{b*}}^{b*}$. Suppose $\tau_{S^{a*}}^{a*} < \tau_{S^{b*}}^{b*}$. Because $V^{a*} \geq V^a(S^{a*}, \tau_1^{a*}, \cdots, \tau_{l^a}^{a*}, \tau_{S^{b*}}^{b*}, \cdots, \tau_{S^{b*}}^{b*})$, we derive from Lemma B.0.1(ii)

$$V^{a*} \geq \frac{\left[\sum_{i=l^a}^{S^{b*}} \pi^a(i, \tau_{S^{b*}}^{b*}) + A\mathbb{1}_{\{\tau_{l^a}^{a*}=\tau_{S^{b*}}^{b*}\}}\right] - \sum_{i=l^a}^{S^{b*}} \pi^a(i, \tau_{S^{a*}}^{a*})}{(S^{a*}-l^a+1)\tau_{S^{b*}}^{b*} - (S^{a*}-l^a+1)\tau_{S^{a*}}^{a*}} \geq \frac{\tilde{p}(\gamma^a, \tau_{S^{b*}}^{b*}) - \tilde{p}(\gamma^a, \tau_{S^{a*}}^{a*})}{\tau_{S^{b*}}^{b*} - \tau_{S^{a*}}^{a*}}.$$

Likewise, we can derive $V^{b*} < \frac{\tilde{p}(\gamma^b, \tau_{S^{b*}}^{b*}) - \tilde{p}(\gamma^b, \tau_{S^{a*}}^{a*})}{\tau_{S^{b*}}^{b*} - \tau_{S^{a*}}^{a*}}$. Because $\tilde{p}(\gamma, \tau)$ is submodular, we have $\tilde{p}(\gamma^a, \tau_{S^{b*}}^{b*}) - \tilde{p}(\gamma^a, \tau_{S^{a*}}^{a*}) \geq \tilde{p}(\gamma^b, \tau_{S^{b*}}^{b*}) - \tilde{p}(\gamma^b, \tau_{S^{a*}}^{a*})$ and thus $V^{a*} \geq V^{b*}$. This is impossible as $p^b(\tau) > p^a(\tau)$. Therefore, we must have $\tau_{S^{a*}}^{a*} \geq \tau_{S^{b*}}^{b*}$.

Also from Proposition 3.4.1, $V^{b*} \geq \pi^b(S^{b*}+1, \tau_{S^{b*}}^{b*})/\tau_{S^{b*}}^{b*}$ and $V^{a*} < \pi^a(S^{a*}, \tau_{S^{a*}}^{a*})/\tau_{S^{a*}}^{a*} \leq \omega^{Ma}$. We have

$$\begin{aligned}
V^{b*} - V^{a*} &> \frac{\pi^b(S^{b*}+1, \tau_{S^{b*}}^{b*})}{\tau_{S^{b*}}^{b*}} - \frac{\pi^a(S^{a*}, \tau_{S^{a*}}^{a*})}{\tau_{S^{a*}}^{a*}} \\
&= \frac{\tilde{p}(\gamma^b, \tau_{S^{b*}}^{b*}) - c}{\tau_{S^{b*}}^{b*}} - \frac{\tilde{p}(\gamma^a, \tau_{S^{a*}}^{a*}) - c}{\tau_{S^{a*}}^{a*}} - h(S^{b*}+1) + h(S^{a*}) \\
&\geq \frac{\tilde{p}(\gamma^b, \tau_{S^{b*}}^{b*}) - c}{\tau_{S^{b*}}^{b*}} - \frac{\tilde{p}(\gamma^a, \tau_{S^{a*}}^{a*}) - c}{\tau_{S^{a*}}^{a*}} \\
&\geq \frac{\tilde{p}(\gamma^b, \tau_{S^{b*}}^{b*}) - \tilde{p}(\gamma^a, \tau_{S^{b*}}^{b*})}{\tau_{S^{b*}}^{b*}}.
\end{aligned}$$

The second inequality follows from the relation $h(S^{a*}) \geq h(S^{b*}+1)$ as $S^{a*} > S^{b*}$. To see the third inequality, we consider two cases. (a) If $\tau_{S^{b*}}^{b*} \geq \tau^{Ma}$, then

$\tau^{a*}_{S^{a*}} \geq \tau^{b*}_{S^{b*}} \geq \tau^{Ma}$. It follows $\frac{\tilde{p}(\gamma^a, \tau^{b*}_{S^{b*}})-c}{\tau^{b*}_{S^{b*}}} \geq \frac{\tilde{p}(\gamma^a, \tau^{a*}_{S^{a*}})-c}{\tau^{a*}_{S^{a*}}}$. (b) If $\tau^{b*}_{S^{b*}} < \tau^{Ma}$, then $\tau^{Ma} > \tau^{b*}_{S^{b*}} \geq \tau^{Mb}$. By Lemma 3.4.2, we obtain $\frac{\tilde{p}(\gamma^b, \tau^{b*}_{S^{b*}})-c}{\tau^{b*}_{S^{b*}}} \geq \frac{\tilde{p}(\gamma^b, \tau^{Ma})-c}{\tau^{Ma}}$.

We further note

$$
\begin{aligned}
V^{b*} - V^{a*} &< V^{b*} - V^{a}(S^{b*}, \tau^{b*}_1, \cdots, \tau^{b*}_{S^{b*}}) = \frac{\sum_{i=1}^{S^{b*}} [\tilde{p}(\gamma^b, \tau^{b*}_i) - \tilde{p}(\gamma^a, \tau^{b*}_i)]}{\sum_{i=1}^{S^{b*}} \tau^{b*}_i} \\
&\leq \frac{\tilde{p}(\gamma^b, \tau^{b*}_{S^{b*}}) - \tilde{p}(\gamma^a, \tau^{b*}_{S^{b*}})}{\tau^{b*}_{S^{b*}}} \qquad \text{(B.8)}
\end{aligned}
$$

We reach a contradiction and thus we obtain part (iv). □

Proof of Proposition 3.4.4. From Proposition 3.4.3, we have $S^{a*} \geq S^{b*}$. Suppose $\sum_{i=1}^{S^{b*}} \tau^{b*} > \sum_{i=1}^{S^{a*}} \tau^{a*}$. Because $V^{b*} \geq V^{b}(S^{a*}, \tau^{a*}_1, \cdot, \tau^{a^*}_{S^{a*}})$, we derive from Lemma B.0.1(ii).

$$
V^{b*} < \frac{\sum_{i=1}^{S^{b*}} \pi(i, \tau^{b*}_i) - A \sum_{i=1}^{S^{b*}-1} \mathbb{1}_{\{\tau^{b*}_i \neq \tau^{b*}_{i+1}\}} - \left[\sum_{i=1}^{S^{a*}} \pi(i, \tau^{a*}_i) - A \sum_{i=1}^{S^{a*}-1} \mathbb{1}_{\{\tau^{a*}_i \neq \tau^{a*}_{i+1}\}} \right]}{\sum_{i=1}^{S^{b*}} \tau^{b*}_i - \sum_{i=1}^{S^{a*}} \tau^{a*}_i}.
$$

Also, because $V^{a*} \geq V(S^{b*}, \tau^{b*}_1, \cdots, \tau^{b*}_{S^{b*}})$, we derive from Lemma B.0.1(ii).

$$
V^{a*} > \frac{\sum_{i=1}^{S^{b*}} \pi(i, \tau^{b*}_i) - A \sum_{i=1}^{S^{b*}-1} \mathbb{1}_{\{\tau^{b*}_i \neq \tau^{b*}_{i+1}\}} - \left[\sum_{i=1}^{S^{a*}} \pi(i, \tau^{a*}_i) - A \sum_{i=1}^{S^{a*}-1} \mathbb{1}_{\{\tau^{a*}_i \neq \tau^{a*}_{i+1}\}} \right]}{\sum_{i=1}^{S^{b*}} \tau^{b*}_i - \sum_{i=1}^{S^{a*}} \tau^{a*}_i}.
$$

The two relations above imply $V^{a*} > V^{b*}$, which leads to a contradiction. Hence, we must have $\sum_{i=1}^{S^{b*}} \tau^{b*} \leq \sum_{i=1}^{S^{a*}} \tau^{a*}$. □

Proof of Proposition 3.4.5. We have

$$\begin{aligned} 0 &\leq V^{a*} - V^{a}(S^{b*}, \tau_1^{b*}, \cdots, \tau_{S^{b*}}^{b*}) + V^{b*} - V^{b}(S^{a*}, \tau_1^{a*}, \cdots, \tau_{S^{a*}}^{a*}) \\ &= V^{a*} - V^{b}(S^{a*}, \tau_1^{a*}, \cdots, \tau_{S^{a*}}^{a*}) + V^{b*} - V^{a}(S^{b*}, \tau_1^{b*}, \cdots, \tau_{S^{b*}}^{b*}) \\ &= c^{a}\left[\frac{S^{b*}}{\sum_{i=1}^{S^{b*}} \tau_i^{b*}} - \frac{S^{a*}}{\sum_{i=1}^{S^{a*}} \tau_i^{a*}}\right] + c^{b}\left[\frac{S^{a*}}{\sum_{i=1}^{S^{a*}} \tau_i^{a*}} - \frac{S^{b*}}{\sum_{i=1}^{S^{b*}} \tau_i^{b*}}\right] \\ &= (c^{a} - c^{b})\left[\frac{S^{b*}}{\sum_{i=1}^{S^{b*}} \tau_i^{b*}} - \frac{S^{a*}}{\sum_{i=1}^{S^{a*}} \tau_i^{a*}}\right] \end{aligned}$$

Hence, $\frac{\sum_{i=1}^{S^{a*}} \tau_i^{a*}}{S^{a*}} \geq \frac{\sum_{i=1}^{S^{b*}} \tau_i^{b*}}{S^{b*}}$. □

Proof of Proposition 3.4.6. The result follows directly from the relation

$$V^{a*} + V^{b*} \geq V^{a}(S^{b*}, \tau_1^{b*}, \cdots, \tau_{S^{b*}}^{b*}) + V^{b}(S^{a*}, \tau_1^{a*}, \cdots, \tau_{S^{a*}}^{a*}).$$

Since the argument is similar to that of Proposition 3.4.4, we omit the details.

□

Proof of Proposition 3.5.1. Differentiating the objective function (3.2) with respect to τ_i, $(1 \leq i \leq S)$, we have

$$\frac{p'(\tau_i) - h(i)}{\sum_{l=1}^{S} \tau_l} - V\frac{1}{\sum_{l=1}^{S} \tau_l} = 0,$$

which leads to $V = p'(\tau_i) - h(i)$, $\forall i$. □

Proof of Proposition 3.5.2. Differentiating the objective function $V^{\flat}$ with respect to ι, we obtain

$$\frac{[p'(\iota_j) - H(x_{j-1}, x_j)dx](x_j - x_{j-1})}{\sum_{l=1}^{N+1}[(x_l - x_{l-1})\iota_l]} - \frac{(x_j - x_{j-1})V}{\sum_{l=1}^{N+1}[(x_l - x_{l-1})\iota_l]} = 0,$$

which lead to $V = p'(\iota_j) - H(x_{j-1}, x_j)$. $\square$

Proof of Proposition 3.5.3. Let $G(z, y) = h(z)(z-y) - \int_y^z h(x)dx$. Then, $G_1'(z, y) = (z-y)h'(z) + h(z) - h(z) = (z-y)h'(z) > 0$ and $\lim_{z \to y} G(z, y) = 0$. Hence, for a fixed y, there exists an unique solution $z(y)$ of $G(z, y) = A/\iota^M$, $z > y$.

Next, we claim

$$\frac{\frac{A}{\iota^M} + \int_y^z h(x)dx}{z - y} \geq \frac{\frac{A}{\iota^M} + \int_y^{z(y)} h(x)dx}{z(y) - y}, \quad \forall z > y. \tag{B.9}$$

To see that, we note

$$\frac{\partial}{\partial z}\left(\frac{\frac{A}{\iota^M} + \int_y^z h(x)dx}{z - y}\right) = \frac{(z-y)h(z) - \frac{A}{\iota^M} - \int_y^z h(x)dx}{(z-y)^2}(z-y)^2 = \frac{G(z, y) - \frac{A}{\iota^M}}{(z-y)^2}.$$

Also, $G(z, y) - A/\iota^M < 0$ when $z \in (y, z(y))$ and $G(z, y) - A/\iota^M > 0$ when $z \in (z(y), +\infty)$.

To prove part (i), we denote $y_1 = \hat{x}_{N+1}(N)$. Then

$$\begin{aligned}
\hat{W}(N) &< \frac{p(\iota^M) - c}{\iota^M} - h(z(y_1)) \\
&= \frac{p(\iota^M) - c}{\iota^M} - \frac{\frac{A}{\iota^M} + \int_{z(y_1)}^{y_1} h(x)dx}{z(y_1) - y_1} \\
&= \frac{[p(\iota^M) - c](z(y_1) - y_1) - \iota^M \int_{y_1}^{z(y_1)} h(x)dx - A}{[z(y_1) - y_1]\iota^M}
\end{aligned}$$

Applying Lemma B.0.1, we obtain

$$\hat{W}(N) < W(N+1, \hat{\iota}_0(N), \cdots, \hat{\iota}_{N+1}(N), \iota^M, \hat{x}_1(N), \cdots, \hat{x}_{N+1}(N), z(\hat{x}_{N+1}(N))) \leq \hat{W}(N+1),$$

leading to part (i).

To prove part (ii), we assume that $W(N) < \hat{W}(N+1)$ holds. We have

$$\hat{W}(N+1) > W(N, \hat{\iota}_1(N+1), \cdots, \hat{\iota}_{N+1}(N+1), \hat{x}_1(N+1), \cdots, \hat{x}_{N+1}(N+1)).$$

Let $y_2 = \hat{x}_{N+1}(N+1)$. Applying Lemma B.0.1, we deduce

$$\begin{aligned}
\hat{W}(N+1) &< \frac{(\hat{x}_{N+2}(N+1) - y_2)\big[p(\hat{\iota}_{N+2}(N+1)) - c\big] - \hat{\iota}_{N+2}(N+1)\int_{y_2}^{\hat{x}_{N+2}(N+1)} h(x)dx}{[\hat{x}_{N+2}(N+1) - y_2]\hat{\iota}_{N+2}(N+1)} \\
&= \frac{p(\hat{\iota}_{N+2}(N+1)) - c}{\hat{\iota}_{N+2}(N+1)} - \frac{\int_{y_2}^{\hat{x}_{N+2}(N+1)} h(x)dx + \frac{A}{\hat{\iota}_{N+2}(N+1)}}{\hat{x}_{N+2}(N+1) - y_2} \\
&\leq \frac{p(\iota^M) - c}{\iota^M} - \frac{\int_{y_2}^{\hat{x}_{N+2}(N+1)} h(x)dx + \frac{A}{\hat{\iota}_{N+2}(N+1)}}{\hat{x}_{N+2}(N+1) - y_2} \\
&\leq \frac{p(\iota^M) - c}{\iota^M} - \frac{\int_{y_2}^{z(y_2)} h(x)dx + \frac{A}{\hat{\iota}_{N+2}(N+1)}}{z(y_2) - y_2} \\
&= \frac{p(\iota^M) - c}{\iota^M} - h(z(y_2)).
\end{aligned}$$

This leads to a contradiction and thus we obtain part (ii). □

Proof of Proposition 3.5.4. Let $G(y) = \int_0^y h(x)dx - \frac{yh(y)}{2}$. We have

$$G'(y) = h(y) - \frac{h(y) + yh'(y)}{2} = \frac{h(y) - h(0) - (y-0)h'(y)}{2}.$$

It is easy to see that $G'(y) > (=, <) 0$ when $h(x)$ is strictly concave (linear, strictly concave).

The first order condition of x_j yields

$$\begin{aligned}
&\frac{p(\iota_{N+1}) - \iota_{N+1}h(x_{N+1}) - c}{\sum_{i=1}^{N+1}[(x_j - x_{j-1})\iota_j]} - \frac{\iota_{N+1}W}{\sum_{i=1}^{N+1}[(x_j - x_{j-1})\iota_j]} = 0, \\
&\frac{p(\iota_j) - p(\iota_{j+1}) - h(x_j)(\iota_j - \iota_{j+1})}{\sum_{i=1}^{N+1}[(x_j - x_{j-1})\iota_j]} - \frac{(\iota_j - \iota_{j+1})W}{\sum_{i=1}^{N+1}[(x_j - x_{j-1})\iota_j]} = 0, \ \forall j \leq N.
\end{aligned}$$

We deduce

$$\frac{p(\iota_{N+1})-c}{\iota_{N+1}} = h(x_{N+1}) + W, \tag{B.10}$$

$$\frac{p(\iota_j)-p(\iota_{j+1})}{\iota_j-\iota_{j+1}} = h(x_j) + W, \ \forall j \leq N. \tag{B.11}$$

In view of (3.5), we have

$$\begin{aligned}
&W[x_{N+1}\iota_{N+1} + \sum_{i=1}^{N} x_j(\iota_j - \iota_{j+1})] = W\sum_{i=1}^{N+1}[(x_j - x_{j-1})\iota_j] \\
&= [p(\iota_{N+1}) - c]x_{N+1} + \sum_{i=1}^{N} x_j[p(\iota_j - p(\iota_{j+1})] - \sum_{j=1}^{N+1}\big[\iota_j\int_{x_{j-1}}^{x_j} h(x)dx\big] - [K+NA] \\
&= [W + h(x_{N+1})]x_{N+1}\iota_{N+1} + \sum_{i=1}^{N}\big[[W + h(x_j)]x_j(\iota_j - \iota_{j+1})\big] \\
&\qquad -\sum_{j=1}^{N+1}\big[\iota_j\int_{x_{j-1}}^{x_j} h(x)dx\big] - [K+NA].
\end{aligned}$$

Simplifying the equation above, we obtain

$$K + NA - \sum_{j=1}^{N+1}\big[\iota_j\int_{x_{j-1}}^{x_j} h(x)dx\big] = 2\sum_{j=1}^{N+1}\Big[\iota_j\big[G(x_{j-1}) - G(x_j)\big]\Big]. \tag{B.12}$$

Hence the result follows. □

Proof of Proposition 3.5.5. The first-order condition of x_j, $1 \leq i \leq N+1$, yields

$$\frac{p(\iota_{N+1})-c}{\iota_{N+1}} - W = h(x_{N+1}),$$

$$\frac{p(\iota_j)-p(\iota_{j+1})}{\iota_j-\iota_{j+1}} - W = h(x_j), \ \ j \leq N.$$

Hence,

$$p(\iota_{N+1}) - c - W\iota_{N+1} = \iota_{N+1}h(x_{N+1}),$$
$$p(\iota_j) - p(\iota_{j+1}) - W(\iota_j - \iota_{j+1}) = (\iota_j - \iota_{j+1})h(x_j), \;\; j \leq N.$$

For all $1 \leq j \leq N+1$, summing up the first $(N+2-j)$ equations leads to

$$p(\iota_j) - c - W\iota_j = \iota_{N+1}h(x_{N+1}) + \sum_{m=j}^{N}\Big[(\iota_m - \iota_{m+1})h(x_m)\Big].$$

Hence

$$\begin{aligned}
&(p(\iota_j) - c - W\iota_j)(x_j - x_{j-1}) - \iota_j\int_{x_{j-1}}^{x_j} h(x)dx \\
&= \iota_j\Big[(x_j - x_{j-1})h(x_j) - \int_{x_{j-1}}^{x_j} h(x)dx\Big] + (x_j - x_{j-1})\sum_{m=j+1}^{N}\big[\iota_{m+1}(h(x_{m+1}) - h(x_m))\big]. \\
&\Big[\frac{p(\iota_j) - c}{\iota_j} - \frac{\int_{x_{j-1}}^{x_j} h(x)dx}{x_j - x_{j-1}} - W\Big](x_j - x_{j-1})\iota_j \\
&= \iota_j\Big[(x_j - x_{j-1})h(x_j) - \int_{x_{j-1}}^{x_j} h(x)dx\Big] + (x_j - x_{j-1})\sum_{m=j}^{N+1}\big[\iota_m(h(x_m) - h(x_{m-1}))\big].
\end{aligned}$$

we conclude the proof. □

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