The Dissertation Committee for Ti Zhou
certifies that this is the approved version of the following dissertation:

Essays on Pricing and Portfolio Choice in Incomplete Markets

Committee:

Thaleia Zariphopoulou, Supervisor

Todd Arbogast

Lorenzo Garlappi

Stathis Tompaidis

Gordan Zitkovic

Essays on Pricing and Portfolio Choice in Incomplete Markets

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Ti Zhou, B.S.

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To my family.

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Essays on Pricing and Portfolio Choice in Incomplete Markets

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This dissertation is a contribution to the pricing and portfolio choice theory in incomplete markets. It consists of three self-contained but interlinked essays.

In the first essay, we present a utility-based methodology for the valuation and the risk management of mortgage-backed securities subject to totally unpredictable prepayment risk. Incompleteness stems from its embedded prepayment option which affects the security's cash flow pattern. The prepayment time is constructed via deterministic or stochastic hazard rate. The relevant indifference price consists of a linear term, corresponding to the remaining outstanding balance, and a nonlinear one that incorporates the investor's risk aversion and the interest payments generated by the mortgage contract. The indifference valuation approach is also extended to the case of homogeneous mortgage pools.

In the second essay, using forward optimality criteria, we analyze a portfolio choice problem when the local risk tolerance is time-dependent and asymptotically linear in wealth. This class corresponds to a dynamic extension of the traditional (static) risk tolerances associated with the power, logarithmic and exponential utilities. We provide explicit solutions for the optimal investment strategies and wealth processes in an incomplete non-Markovian market with asset prices modelled as Ito processes. The methodology allows for measuring the investment performance in terms of a benchmark and alternative market views.

In the last essay, we extend the forward investment performance approach to study the optimal portfolio choice problem in an incomplete market driven by jump processes. The asset price is modelled by a one-dimensional Lévy-Itô process. We prove the existence of a forward performance process by restricting the local risk tolerance functions to be time-independent and linear in wealth. This yields only three types of performance measurement criteria, namely, exponential, power and logarithmic. The optimal portfolios are constructed via stochastic feedback controls under these criteria.

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Chapter 1

Introduction

In 1952, H. Markowitz's revolutionary paper [54] laid the groundwork for the modern mathematical theory of finance. Since then, pricing and portfolio choice have always been the main topics of its research. In 1969, R. Merton introduced stochastic calculus into the study of finance. He found a path-breaking method of solving the problem of optimal consumption and portfolio choice in a continuous-time setting (see [55] and [56]). His method involves reduction of the problem to a partial differential (Hamilton-Jacobi-Bellman) equation for the investor's indirect utility function for wealth. This formulation, a breakthrough in its own right, may well have influenced the way F. Black and M. Scholes approached the option pricing problem in 1973 (see [9]). Under the assumptions that the payoff of an a European call option could be perfectly replicated by dynamically rebalancing a portfolio consisting of underlying stock and risk-free bond, they chose to work in a continuous setting with the same stock price model assumed by Merton and reduced the valuation equation to a PDE. In the period 1979-1983, Harrison, Kreps, and Pliska further developed the risk-neutral pricing methodology and showed how to price numerous other derivative securities (see, for example, [34], and [35, 36]). The Black-Scholes-Merton framework provided, for the first time, a satisfying solution to an important practical problem of finding "fair" prices for financial derivatives with a solid theoretical foundation. It didn't take too long for the financial industry to adopt this theory and turn it into many immediate applications.

However, the assumption of perfect replication, or market completeness is neither financially realistic or theoretically robust. From a financial point of view, completeness implies that options are redundant assets and the very existence of the derivatives market becomes a mystery, if not a paradox in such models. From a theoretical point of view, the addition of jumps, stochastic volatilities, or credit-linked events to the model could easily introduce unhedgeable risks and ruin the completeness. Therefore, Market completeness is rather an exception than the rule. As the author is preparing this dissertation, there is an ongoing debate about whether the current global financial crisis is partially caused by the misuse and abuse of mathematics in finance. In the author's humble opinion, it is still too early to make any conclusion. But in late 1990s, the Nobel Laureate M. Miller commented on the failure of Long-Term Capital Management [1] by the following thought-provoking words:

> The question is $[\cdots]$ whether the LTCM disaster was merely a unique and isolated event, a bad draw from nature's urn, or whether such disasters are the inevitable consequence of the Black-Scholes

[1] LTCM was a U.S. hedge fund founded in 1994. Board of directors members included M.Scholes and R. Merton, who shared the 1997 Nobel Prize in Economics. Initially enormously successful in its first years, in 1998 it lost $4.6 billion in less than four months leading to a massive bailout by other major banks and investment houses.

formula and the illusion it might give that all market participants can hedge away their risk at the same time.

In complete markets, there exits a unique equivalent martingale measure and pricing by replication comes down to calculation of an expectation under this measure. In incomplete markets, the main difficulty we are facing is *not* that there are infinitely many such measures; rather it is about how we could approximate a target payoff with a certain hedging strategy and minimize the risk associated with it. Different ways to measure risk lead to different approaches to hedging and pricing. Among others, *superhedging*, *mean-variance (quadratic) hedging* and *utility maximization* are the most popular methodologies employed in the literature. Each of these hedging strategies has a cost. Thus, the price of the option in incomplete markets will consist of two parts: the cost of hedging plus a risk premium to compensate for the investor's residual (unhedgeable) risk, which depends heavily on the investor's preferences. All the above suggests that pricing and portfolio choice are two inseparable problems in incomplete markets.

The problem of optimal portfolio choice itself is, also, of great interest. For example, the ultimate goal of modern investment institutions is not about hedging. Indeed, it is much more about return on capitals. The business of hedging only offers the lowest return to the investors. In a portfolio choice problem, investors seek the combination of securities that best suit their needs in an uncertain environment. In order to determine the optimum allocation, the investors need to model, estimate, assess and manage uncertainty.

This dissertation aims to focus on the *utility maximization* approach to study the pricing and portfolio choice problems in incomplete markets. To better explain the concepts introduced in this study, we first give a very brief review on the formulation of classical utility maximization problems. To this end, we follow the exposition in [60].

1.1 Classical utility maximization

Optimal asset allocation problems can be formulated as classical utility maximization (stochastic optimization) problems. They typically consist of a time horizon, a controlled process (investor's wealth) and an optimization criterion represented as the conditional expectation of a wealth functional, given a relevant filtration. Maximizing this expectation, over a given set of admissible policies, yields the so-called value function. Note that we occasionally refer to this type of problems as the Merton problem.

To facilitate the exposition, we denote the state controlled process by X^π, the set of admissible controls by $\mathcal{A}$ and the relevant filtration by $\mathcal{F}_t, 0 \leq t \leq T$. The criterion to be optimized is of the form $J = \mathbb{E}_{\mathbb{P}}^{(x,t)}[U(X_T^\pi)]$ with U being a concave and increasing function, often referred to as the investor's bequest or utility. The value function V is, in turn, defined as

$$V(x, t; T) = \sup_{\pi \in \mathcal{A}} \mathbb{E}_{\mathbb{P}}^{(x,t)}[U(X_T^\pi)]. \tag{1.1}$$

At $t = T$, it coincides with the utility datum and for previous times, it satisfies - under weak model assumptions - the Dynamic Programming Principle.

Namely,

$$V(X_t^{\pi^*}, t; T) = \begin{cases} \mathbb{E}_{\mathbb{P}}[V(X_s^{\pi^*}, s; T)|\mathcal{F}_t], & \text{for } t \le s < T \\ U(X_T^{\pi^*}), & \text{for } t = T, \end{cases}$$

where X^{π^*} stands for the optimized state process, with $X_t^{\pi^*} = x$. This problem has been mainly studied in the literature using Hamilton-Jacobi-Bellman equations (see [30]) or duality techniques and BSDEs (see, for example, [24, 43], [32] and [72]).

Note that the above value function is a *martingale* at the optimum, and a *supermartingale* otherwise. In essence, to specify the value function, one just needs to find a martingale that coincides with the utility at maturity.

Next we review the utility indifference valuation approach to pricing and hedging in incomplete markets.

1.2 Utility indifference price

Utility indifference valuation is based on the classical arguments of stochastic dominance. The essence is to incorporate an investor's attitude towards the risks that cannot be eliminated. By using the utility maximization criteria, it produces the so called *indifference price* which is the amount that makes an investor indifferent between the investment opportunities with and without holding the derivative at hand. This approach was first introduced in [39].

To illustrate the idea, we only consider a European type option with payoff H at maturity T. Herein, two utility maximization problems are introduced. The first one arises from the classical Merton problem of optimal investment, without taking into account the derivative, as we described in the previous section. The second problem corresponds to the situation in which the derivative is written at time t. Therefore, the writer's value function V^w is

$$V^w(x,t;T) = \sup_{\pi \in \mathcal{A}} \mathbb{E}_{\mathbb{P}}^{(x,t)}[U(X_T^\pi - H)]. \tag{1.2}$$

The writer's indifference price for the European claim H at time t with maturity T is then defined as the amount $h^w = h^w(x,t;T,H)$ which satisfies

$$V^w(x + h^w, t; T) = V(x,t;T)$$

with V and V^w defined in (1.1) and (1.2), respectively. In the same manner, we could define the buyer's indifference price $h^b = -h^w(x,t;T,-H)$.

Several remarks are noteworthy here. First, the indifference price is not linear due to the nonlinear pricing mechanism, namely,

$$h^w(x,t;T,H1) + h^w(x,t;T,H2) \neq h^w(x,t;T,H1+H2).$$

Second, the indifference price depends generally on the initial wealth of the investor except for the case with exponential utility. Third, buying and selling are not symmetric since utility function weighs gains and losses in an asymmetric way. We refer the readers to [15] for a detailed exposition of this approach.

1.3 Forward performance

In the classical formulation of utility maximization (cf. (1.1)), the utility refers to the investor's risk aversion at the time horizon T. Advocates of the use of such utility function in pricing and portfolio choice often refers to the classic work of Von Neumann and Morgenstern [83] to justify this approach. But since the utility function is defined in isolation to the investment opportunities given to the investor, it is not clear to the investor how to specify her own utility. Technically speaking, explicit solutions to the classical utility maximization problems can only be derived under very restrictive model and utility assumptions. The general non-Markovian models concentrate only on the mathematical existence of optimal allocations and on the dual representation of utility.

Moreover, from the utility indifference pricing theory, we have observed that fixing the trading horizon made the valuation of options with arbitrary maturities impossible due to the concern of intertemporal inconsistency.

Motivated by the above, we need to define an investment performance criterion for any intermediate rebalancing period which also maintains the intertemporal consistency. At the optimum, the investor must be indifferent to the investment horizon. In line with the classical formulation, only at the optimum, the investor achieves on average her performance objectives; suboptimally, she experiences decreasing future performance.

An alternative approach to stochastic optimization was thus proposed by Musiela and Zariphopoulou (see [57] and [60, 62]). In contrast to the traditional framework, a datum is assigned at initial time and the investment performance measurement is constructed forward in time. It is then called a *forward performance.* To summarize, mathematically, an $\mathcal{F}_t$-adapted process $U_t(x)$ is a forward performance process if for each $t \geq 0$, $U_t(x)$ is an increasing and concave function of x, and for each self-financing strategy π,

$$\mathbb{E}_{\mathbb{P}}[U_s(X_s^{\pi})|\mathcal{F}_t] \leq U_t(X_t^{\pi}), \quad s \geq t,$$

there exists a self financing strategy π^*, for which,

$$\mathbb{E}_{\mathbb{P}}[U_s(X_s^{\pi^*})|\mathcal{F}_t] = U_t(X_t^{\pi^*}), \quad s \geq t,$$

and $U_0(x) = u_0(x)$, where u_0 is a concave and increasing (deterministic) function.

1.4 Outline of this dissertation

The rest of this dissertation consists of three self-contained but interlinked essays which are adapted or modified from the author's original research papers [86], [85] (joint with his supervisor T. Zariphopoulou) and [87].

Chapter 2 studies the utility indifference valuation problem of mortgage-backed securities in the presence of prepayment risk. The prepayment time is modelled through the hazard rate in its reduced form. We solve the problem explicitly by using HJB equation. This case study provides new insights into pricing and hedging for a rich class of securities which are subject to credit risk. Chapter 3 further develops the forward investment performance approach and studies the portfolio choice problem when the local risk tolerance is time-dependent and asymptotically linear in wealth. We provide explicit solutions for the optimal investment strategies and wealth processes in a non-Markovian market with asset prices modelled as Ito processes. Chapter 4 extends the concepts of forward performance to a market model driven by jump processes. We prove the existence of a forward performance process by restricting the local risk tolerance functions to be time-independent and linear in wealth. In the appendix, we show the existence and uniqueness of a classical solution to a semilinear parabolic PDE which turns out to be useful in Chapter 2.

Chapter 2

Indifference valuation of mortgage-backed securities in the presence of prepayment risk

2.1 Introduction

The mortgage market has experienced explosive growth over the past 25 years and is today a major component of the US bond market. The most significant structural change of this market is its innovation in terms of the design of new mortgage instruments and the development of products that use pools of mortgage as collateral for the issuance of a security. The Federal Government has played an important role in it. Three agencies, Ginnie Mae (The Government National Mortgage Association), Fannie Mae (the Federal National Mortgage Association), and Freddie Mac (the Federal Home Loan Mortgage Corporation) were created to issue and insure those new securities. The introduction and the use of such securities, called *mortgage-backed securities* (MBS), removes to some extent the default risk, facilitates the flow of mortgage loans, and standarizes the mortgages sold in the second market. On the demand side, MBS have come to represent a significant portion of fixed-income holdings for many institutional investors over the past decade. [1]

[1]As the author is preparing this chapter, there is an ongoing global financial crisis which began with the subprime mortgage market meltdown in the US. The reasons for this crisis

The most common type of mortgage-backed security is the *mortgage pass-through*, a security that passes the borrower's mortgage payments through the trustee before being disbursed to the investors. The possibility that mortgages will prepay and force investors to seek alternative investments, usually with lower returns, is the so called *prepayment risk.*

This prepayment risk is assumed to be unpredictable since the borrower's prepayment decisions are often due to factors not directly inferred from market. The purpose herein is to provide a utility-based methodology for the valuation and the risk management of MBS subject to totally unpredictable prepayment risk. The study is focused on the optimal decisions of the investor holding the security. The market is incomplete due to the fact that the cash flow of a mortgage contract is affected by the mortgagor's decision to prepay. Through this prepayment option, the mortgagor could fully, or partially, prepay his outstanding principle instead of paying the scheduled installment. Hence, the investor holding the MBS will lose some of the expected interest payments. The prepayment time for a mortgage contract can be fully observed across time but is not predictable.

In the existing literature, mortgage contracts have been primarily valued via arbitrage free arguments. The associated models can be classified into two categories (see, [67, 68]). The first category, is primarily, based on an *empirical approach.* The main idea is to first specify a stochastic model for in-

are varied and complex. Some believe that the securitization practices in the mortgage market should be of blame.

terest rates and possibly other econometric factors, and then add a statistical model describing how the mortgagor's prepayment behavior depends on these factors. Because the models are in general complicated and the individual prepayment behavior is path-dependent, Monte Carlo simulation is used to estimate the expected value of the discounted cash flow, (see, among others, [73, 74], [48] and [49]). The second category uses an *option based approach.* The underlying idea is to incorporate the optimal behavior with respect to the mortgagor's decision about when to refinance. Intuition based on the theory of early exercise for American options is frequently used. We refer the reader to [27, 28] and [78].

Dating back at least to [73, 74], it was recognized that the random prepayment time can be described through the so called hazard rate, that is, the conditional rate of prepayment given the current state of factors observable in the market. Recent developments in the credit risk area yield a plethora of results for the modeling of totally inaccessible random times of defaults through the hazard processes. It is called the intensity based approach or reduced form approach[2].

The author in [33] first applied the intensity based methodology in mortgage valuation and showed how to unify the empirical approach and the option based approach. He, also, derived some explicit formulae for the mortgage's value and a nonlinear equation for the endogenous mortgage rate. In

[2]The interested reader may find a detailed exposition in the book [8] and the survey paper [7].

essence, his work resembles to a hybrid model. Another work, based on the intensity of prepayment times, is given in [44]. Therein a valuation formula for the mortgage pool with a large number of homogeneous mortgage contracts is derived.

An alternative approach to pricing and risk management is based on partial equilibrium. This approach produces the so called indifference price which is the amount that makes an investor indifferent between the investment opportunities with and without holding the security at hand. It is based on classical arguments of stochastic dominance and was first used in [39] for the valuation of European calls in the presence of transaction costs. Since then, there has been significant advance in understanding the theoretical implications of the utility based pricing rule in incomplete markets, using different solution methods. For a concise exposition, we refer the reader to the volume [15].

The utility maximization problem that incorporates prepayments is formulated in a similar way as the optimal investment problem with uncertain time horizon. The latter is an extension of the classical Merton problem and has been studied with PDE methods in [10] or with duality in [13] (see, also, [71], [47] and [11]).

Some more relevant work to our problem is given in [7]. Therein, the authors considered the indifference valuation of a general defaultable contingent claim and applied the intensity based approach. The indifference prices associated with two different filtrations were derived using arguments from duality and the theory of BSDE. In [76], the author also gave a concrete example of the investment problem for defaultable bonds with unpredictable recovery. In a direction related to our approach, the author in [84] studied the writer's indifference price of a catastrophe bond in an incomplete market with an affine term structure.

In our model, we consider a fixed rate mortgage contract in which the mortgage rate and the monthly payment do not vary over the life of the mortgage. For tractability, we assume that only two securities are available for trading, a risky one and a riskless money account. The price of the former is modeled as a lognormal process with deterministic coefficients while the money market account's interest rate is taken to be zero. We, also, allow for another source of randomness represented via a stochastic factor affecting non-traded state variables. The state variables can be observed by the investors and affect the cash flow of the mortgage-backed securities.

We are interested in the indifference price for an investor who wishes to purchase the MBS (the buyer). The totally inaccessible prepayment time for the mortgage contract is modeled by its hazard rate that can be either deterministic or stochastic. Moreover, we assume the financial institution that issues the MBS (the writer) has *more* information about the prepayment time than the investors. Hence *asymmetry* emerges between buyer's and writer's indifference prices.

A sufficient condition for the hazard rate to be deterministic is to have prepayment time independent of the information about the traded assets and the non-traded factor. This can be used to model the prepayment of those passive mortgagors who prepay for extrinsic reasons. Under this assumption, we derive the buyer's indifference price for the fixed rate mortgage contract. A probabilistic representation of the indifference price, before the prepayment time, is given in terms of the sum of a linear and a nonlinear term. The linear part is the remaining outstanding balance of the mortgage contract. The nonlinear part has static insurance-type certainty equivalent characteristics and prices the interest payments generated by the mortgage contract.

The stochastic hazard rate of prepayment corresponds to those active mortgagors who prepay for intrinsic reasons and take advantage of the opportunities available in the market.

The stochastic hazard rate is modeled as a non-negative function of a diffusion process correlated with the risky asset. The buyer's indifference price before prepayment turns out to solve a quasilinear equation. As before, the price is, in this case represented as the sum of a linear and a nonlinear term. The linear term is the remaining outstanding balance of the mortgage contract. The nonlinear term is given as a distortion of the solution to a second-order parabolic equation of *reaction-diffusion* type. This reaction-diffusion equation is, in turn, analyzed in detail. We, further, look at the payoff decomposition of the fixed rate mortgage contract subject to totally unpredictable prepayment risk. We find that besides the usual unhedgeable risks emerging in the incomplete market, there is an additional instantaneous loss at the prepayment time.

In realistic situations, a mortgage-backed security is backed by a *pool* of underlying mortgage contracts. The problem of such a pool with homogeneous mortgages is studied under the assumptions that the hazard rate for a single prepayment time is deterministic and different prepayment times are conditionally independent. Herein, a general formula for the buyer's indifference price for a mortgage pool with n mortgage contracts before the first prepayment time is derived. Consistently, the price is, again, the sum of a linear and a nonlinear term. The linear term is now the remaining outstanding balance of all n mortgage contracts while the nonlinear one is represented as a nested expectation of distorted interest payments.

The chapter is organized as follows. In Section 2.2, we introduce the model, define the basic fixed rate mortgage contract and construct the totally inaccessible prepayment time from its hazard rate. In Section 2.3, we consider two expected utility problems and define the indifference price for a single mortgage contract. In Section 2.4 and 2.5, we provide representation results for the indifference prices with deterministic and stochastic hazard rate of prepayment. In Section 2.6, we extend the analysis to the case of homogeneous mortgage pools. We provide conclusions and directions for further research in Section 2.7.

2.2 The model

In the probability space $(\Omega, \mathcal{G}, \mathbb{P})$, we introduce two independent standard Brownian motions $(W_t, W_t^{\perp}, t \geq 0)$. The reference filtration $\mathbb{F} = (\mathcal{F}_t, t \geq$

0) generated by both W and $W^{\perp}$ is $\mathbb{P}$-augmented and right-continuous. We assume that $\mathcal{F}_t \subset \mathcal{G}$, for any $t > 0$. Note, however, that it is not necessarily true that $\mathcal{F}_\infty = \mathcal{G}$. We consider a market in which a risky asset and a riskless money account are traded. The short rate is taken to be zero. The case of non-zero being handled by standard rescaling argument. The price of the risky asset, S_t, solves

$$dS_t = \mu(t)S_t dt + \sigma(t)S_t dW_t \tag{2.1}$$

where all the coefficients are deterministic, continuous and bounded.[3] Also, the inverse σ^{-1} is bounded. Notice that the financial market without considering the mortgage contract is complete.

In the sequel (see (2.24)), the Brownian motion $W^{\perp}$ will serve as an additional to W source of randomness for some non-traded state variable in the market, e.g. the labor income, housing price or mortgage rate.

2.2.1 The basic mortgage contract

Before we explore the general case of a mortgage pool, we consider the basic mortgage contract of duration $[0, T]$. A borrower takes a loan, say of P_0 dollars at initial time $t = 0$ and accepts the obligation to pay scheduled coupons at rate c, continuously till T. The typical maturity time for a mortgage contract is 30 years. The *contracted mortgage rate* is m. Usually, the

[3] While allowing for time dependent coefficients μ, σ might look like an unnecessary technical feature, it is worth mentioning that such cases would be appropriate to model securities other than a stock, e.g. a risky bond.

mortgage rate consists of two parts: the *servicing fee* and the *real interest rate*. Herein, without loss of generality, we ignore the former. We also consider fixed rate mortgage contracts, namely, contracts in which the interest rate and the monthly payment do not vary over the life of the mortgage.

If $P(t)$ denotes the scheduled remaining outstanding principal for a certain fixed-rate mortgage, then

$$\begin{cases} dP(t) = (mP(t) - c)dt \\ P(0) = P_0, \quad P(T) = 0. \end{cases} \tag{2.2}$$

Note that, because we have both initial and terminal conditions for this ODE, the mortgage rate m and the installment rate c are not independently chosen.

2.2.2 Prepayment risk and prepayment time

The investor holding the mortgage-backed securities will encounter prepayment risk. There are different reasons to prepay the mortgage before maturity, the most important of which are *home turnovers* and *refinancings.* Other reasons include *defaults*, which typically average less than 0.5%, and *partial prepayments*, referring to mortgagor's paying more than the scheduled payment each month to obtain a faster equity buildup. Like defaults, partial prepayment are also typically low (less than 0.5%).[4] In view of this, we will not

[4]Defaults and foreclosure activity increased dramatically in year 2006-2008. For example, during 2007, nearly 1.3 million U.S. housing properties were subject to foreclosure activity, up 79% from 2006.

consider the defaults and partial prepayments herein. Many other prepayment reasons are relocation, change of family status, job loss and etc.

The author in [53] suggested that a mortgage pool consists of two components. The *"active"* component that includes all ready-to-refinance mortgagors who will take advantage of any opportunity available in the market. In contrast, the *"passive"* component prepays at a speed generally reflecting a typical housing turnover rate and the rate for other extrinsic reasons. As refinancing opportunities emerge, active mortgagors leave the pool, and further prepayment activity gradually declines. This is the well-known "*burnout*" effect. According to [53], if we assume no migration between the two components, this decomposition could cure the burnout effect as a major source of path-dependence. We borrowed this idea to categorize the incentives of prepayment in our model (see, Section 2.4 and 2.5).

The individual borrower's prepayment decisions are, in general, unknown to the investor. Hence we model the prepayment time as a totally inaccessible stopping time. The construction of the prepayment time follows directly from [41, 42], which give us a theoretical study for the modeling of default risk. For completeness we briefly explain below the mathematical settings.

The prepayment time τ is defined as a non-negative random variable on the probability space $(\Omega, \mathcal{G}, \mathbb{P})$, satisfying: $\mathbb{P}\{\tau = 0\} = 0$ and $\mathbb{P}\{\tau > t\} > 0$. We define the process

$$H_t = \mathbf{1}_{\{\tau \leq t\}} \tag{2.3}$$

and we denote $\mathbb{H} = (\mathcal{H}_t, t \geq 0)$, where $\mathcal{H}_t = \sigma(H_u : u \leq t)$.

The filtration $\mathbb{G} = (\mathcal{G}_t, t \geq 0)$ is referred to as the full filtration. It takes into account the information on the asset prices, the non-traded state variables and the occurrence of prepayment. Therefore, $\mathbb{G} = \mathbb{F} \vee \mathbb{H}$, i.e. $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t = \sigma(\mathcal{F}_t \cup \mathcal{H}_t)$, $t > 0$. Notice that τ is an $\mathbb{H}$-stopping time, as well as a $\mathbb{G}$-stopping time. However it is not an $\mathbb{F}$-stopping time (see, for example, Section 6.5 of [8]).

Next, define the conditional cumulative distribution of the prepayment time with respect to the traded asset filtration,

$$F_t := \mathbb{P}\{\tau \leq t | \mathcal{F}_t\}, \quad t > 0.$$

We postulate that $F_\infty = 1$ and we denote by G the $\mathbb{F}$-survival process of τ, i.e.

$$G_t := 1 - F_t = \mathbb{P}\{\tau > t | \mathcal{F}_t\}, \quad t > 0.$$

The $\mathbb{F}$-hazard process of τ under $\mathbb{P}$, denoted by $\Gamma = (\Gamma_t, t \geq 0)$, is given by

$$\Gamma_t := -\ln(1 - F_t) = -\ln G_t, \quad t > 0.$$

We next recall the familiar important **H**-hypothesis.

(H) Every $\mathbb{F}$-square integrable martingale is a $\mathbb{G}$-square integrable martingale.

Therefore, the Brownian motions $(W_t, W_t^{\perp}, t \geq 0)$ are $(\mathbb{P}, \mathbb{G})$-Brownian motions. As a consequence of **(H)**, the processes F and Γ are increasing.

This hypothesis is a necessary condition for the absence of arbitrage when the default-free market is complete (in our case, the primary asset market without prepayment in Section 2.4 and 2.6) using $\mathbb{G}$-adapted self-financing strategies (see, for example, [12] and [29]).

For tractability, we also assume that the process F is absolutely continuous, that is, $F_t = \int_0^t f_u du$ for some density process $f : \mathbb{R}^+ \to \mathbb{R}^+$. Then, for $t > 0$,

$$F_t = 1 - e^{-\Gamma_t} = 1 - \exp(-\int_0^t \lambda_u du),$$

where

$$\lambda_t := \frac{f_t}{1 - F_t}.$$

The process λ is non-negative and satisfies $\int_0^\infty \lambda_u du = \infty$. It is called the *hazard rate* or the *intensity* of τ.

Remark 2.2.1. If the process F_t is a deterministic function, the hazard process and the intensity process are also deterministic. A sufficient condition for the conditional distribution F_t to be a deterministic function is to have τ independent of $\mathbb{F}$. This situation is associated with those passive mortgagors who prepay for extrinsic reasons. On the other hand, if F_t is stochastic, it best describes the prepayment time for those active mortgagors who prepay for intrinsic reasons and take advantage of the opportunities available in the market.

2.3 Preliminaries on indifference valuation of a single mortgage contract

In this section, we price a MBS using the utility-based valuation methodology. It is important to observe that, in contrast to the existing incomplete models in which indifference valuation has been applied, a mortgage contract is *not* written on an underlying asset. In this sense, it is not a standard security. Indeed, there is no payoff at the maturity time T. The investors holding a mortgage contract only receive payments up to the prepayment time τ and receive the loan's remaining balance at the prepayment time.

In both situations, with and without the prepayment risk, trading occurs in horizon $[t, T]$, $0 \leq t \leq T$, and between the risky and the riskless asset. A process π, denoting the amount invested in the risky asset is said to be an admissible strategy with respect to $\mathbb{F}$, if π is $\mathbb{F}$-predictable and satisfies $\mathbb{E}(\int_0^T \sigma^2(s)\pi_s^2 ds) < +\infty$. The set of admissible self-financing strategies with respect to $\mathbb{F}$ is denoted by $\mathcal{A}_{\mathbb{F}}$. Similarly, we can define $\mathcal{A}_{\mathbb{G}}$ the set of admissible strategies with respect to $\mathbb{G}$.

In the absence of the mortgage contract, an investor, with initial endowment x at time t, uses an admissible self-financing strategy π to dynamically rebalance his portfolio holdings. The current wealth, X_s, solves, for $t \leq s \leq T$,

$$X_s = x + \int_t^s \mu(u)\pi_u du + \int_t^s \sigma(u)\pi_u dW_u. \tag{2.4}$$

Next, we denote by $X_s^M, t \leq s \leq T$ the wealth process *with* the mortgage contract. If at time t, the contract is already bought, the buyer trades up to time $\tau \wedge T \in [t, T]$, at which he receives the remaining balance $P(\tau \wedge T)$. The buyer's wealth $X^M_{\tau\wedge T}$ then increases to $X^M_{(\tau\wedge T)-} + P(\tau \wedge T)$. The buyer will also receive the installment payment continuously at the rate c and will use these proceeds to reinvest. After time $\tau \wedge T$, the buyer continues trading until maturity T. Therefore, for $t \le s \le T$,

$$X^M_s = x + \int_t^s \mu(u)\pi_u du + \int_t^s \sigma(u)\pi_u dW_u + \int_t^s P(u)dH_u + \int_t^s c(1 - H_u)du, \tag{2.5}$$

where $H_s = \mathbf{1}_{\{\tau \le s\}}$ (cf. (2.3)). Using that,

$$\int_t^s P(u)dH_u = \mathbf{1}_{\{t<\tau\le s\}} P(\tau)$$

and

$$\int_t^s c(1 - H_u)du = \int_t^s \mathbf{1}_{\{\tau>u\}} c du = \mathbf{1}_{\{\tau>t\}} \int_t^{\tau\wedge s} c ds,$$

we deduce that the wealth process with the mortgage contract solves, for $t \le s < \tau \wedge T$,

$$X^M_s = x + \int_t^s (\mu(u)\pi_u + c)du + \int_t^s \sigma(u)\pi_u dW_u.$$

We, next, consider an individual whose risk preferences are modeled as an exponential utility function

$$U(x) = -e^{-\gamma x}, \quad \gamma > 0,$$

and introduce two optimization problems via which the indifference price will be constructed. We will refer to the first problem as the primal one.

Problem (P) Optimization without the mortgage contract.

The maximal expected utility is given by (see [55]),

$$V_t(x) = ess \sup_{\pi \in \mathcal{A}_{\mathbb{F}}} \mathbb{E}[U(X_T^x)|\mathcal{F}_t]. \tag{2.6}$$

Problem $(P_{\mathbb{G}}^M)$ Optimization with the mortgage contract using strategies belonging to $\mathcal{A}_{\mathbb{G}}$.

The investor buys the mortgage contract at time t. Note that (s)he knows whether the prepayment occurs or not since, if the single mortgage contract has already been prepaid, he receives no installment payment. In this case, he has full information and his filtration is thus $\mathbb{G} = \mathbb{F} \vee \mathbb{H}$. However, at time t, the investor has no information about when the prepayment will occur in the future, i.e. the prepayment time τ is totally inaccessible. The investor will use strategies in $\mathcal{A}_{\mathbb{G}}$ to trade in this economy and his associated maximal utility is

$$V_t^{\mathbb{G}}(x) = ess \sup_{\pi \in \mathcal{A}_{\mathbb{G}}} \mathbb{E}[U(X_T^{M,x})|\mathcal{G}_t]. \tag{2.7}$$

We are now ready to recall the definition of the indifference price of the MBS (see [7]).

Definition 2.3.1. The buyer's indifference price of the mortgage contract is defined as the amount h such that the investor is indifferent towards the following two scenarios: optimize the expected utility without employing the mortgage contract and optimize it with the contract. In other words,

$$V_t(x) = V_t^{\mathbb{G}}(x - h) \tag{2.8}$$

for $x \in \mathbb{R}$, $t \in [0,T]$, where V_t and $V_t^{\mathbb{G}}$ are defined in (2.6) and (2.7), respectively.

To construct the above price, we first recall (see [55]) that, the solution to Problem (P) is deterministic, given by

$$V_t(x) = -\exp(-\gamma x - \frac{1}{2}\int_t^T \frac{\mu^2(s)}{\sigma^2(s)}ds) \triangleq v(x,t). \tag{2.9}$$

In order to facilitate the analysis of Problem $(P_{\mathbb{G}}^M)$, we recall the following auxiliary results.

Lemma 2.3.1. *i) Let Y be a $\mathbb{G}$-measurable random variable and assume that $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t$, for $t > 0$. Then, for $s \geq t$,*

$$\mathbb{E}_{\mathbb{P}}[\mathbf{1}_{\{t<\tau\leq s\}}Y|\mathcal{G}_t] = \mathbf{1}_{\{\tau>t\}}\mathbb{E}_{\mathbb{P}}[\mathbf{1}_{\{t<\tau\leq s\}}e^{\Gamma_t}Y|\mathcal{F}_t]. \tag{2.10}$$

ii) Let Z be a bounded, $\mathbb{F}$-predictable process. Then, for $t < s \leq +\infty$,

$$\begin{aligned}\mathbb{E}_{\mathbb{P}}[\mathbf{1}_{\{t<\tau\leq s\}}Z_\tau|\mathcal{G}_t] &= \mathbf{1}_{\{\tau>t\}}e^{\Gamma_t}\mathbb{E}_{\mathbb{P}}[\int_t^s Z_u dF_u|\mathcal{F}_t] \\ &= \mathbf{1}_{\{\tau>t\}}\mathbb{E}_{\mathbb{P}}[\int_t^s Z_u e^{-\int_t^u \lambda_v dv}\lambda_u du\,|\mathcal{F}_t].\end{aligned} \tag{2.11}$$

For the proof, see Corollary 5.1.1 and Proposition 5.1.1. of [8].

We next observe that the Dynamic Programming Principle implies

$$\begin{aligned}V_t^{\mathbb{G}}(x) &= ess\sup_{\pi\in\mathcal{A}_{\mathbb{G}}} \mathbf{1}_{\{\tau>t\}}\mathbb{E}[\mathbf{1}_{\{\tau>t\}}U(X_T^{M,x})|\mathcal{F}_t] \\ &\quad + ess\sup_{\pi\in\mathcal{A}_{\mathbb{G}}} \mathbf{1}_{\{\tau\leq t\}}\mathbb{E}[\mathbf{1}_{\{\tau\leq t\}}U(X_T^{M,x})|\mathcal{F}_t] \\ &= ess\sup_{\pi\in\mathcal{A}_{\mathbb{G}}} \mathbf{1}_{\{\tau>t\}}\mathbb{E}[\mathbf{1}_{\{\tau>t\}}v(X_{(\tau\wedge T)-}^{M,x} + P(\tau\wedge T), \tau\wedge T)|\mathcal{F}_t] \\ &\quad + \mathbf{1}_{\{\tau\leq t\}}v(x,t).\end{aligned}$$

Notice that if $\tau \leq t$, i.e. when the prepayment has already occurred before the current time, the optimization problem reduces to Problem (P). On the other hand, if $\tau > t$, we shall first consider the optimization problem in $(\tau \wedge T, T]$, and then maximize the value function v, at time $\tau \wedge T$ with endowment $X^M_{(\tau\wedge T)-} + P(\tau \wedge T)$.

In view of (2.11), if $\tau > t$, we may rewrite the solution to Problem $(P^M_{\mathbb{G}})$ as follows

$$\begin{aligned} V_t^{\mathbb{G}}(x) &= ess \sup_{\pi \in \mathcal{A}_{\mathbb{G}}} \mathbb{E}[\mathbf{1}_{\{\tau > t\}} v(X^{M,x}_{(\tau\wedge T)-} + P(\tau \wedge T), \tau \wedge T) | \mathcal{F}_t] \\ &= ess \sup_{\pi \in \mathcal{A}_{\mathbb{G}}} e^{\Gamma_t} \mathbb{E}[\int_t^\infty v(X^{M,x}_{(u\wedge T)-} + P(u \wedge T), u \wedge T) dF_u \, | \mathcal{F}_t] \\ &= ess \sup_{\pi \in \mathcal{A}_{\mathbb{G}}} e^{\Gamma_t} \mathbb{E}[\int_t^T v(X^{M,x}_{u-} + P(u), u) dF_u + (1 - F_T) v(X^{M,x}_T, T) \, | \mathcal{F}_t] \\ &= ess \sup_{\pi \in \mathcal{A}_{\mathbb{F}}} e^{\Gamma_t} \mathbb{E}[\int_t^T v(X^{M,x}_{u-} + P(u), u) dF_u + (1 - F_T) v(X^{M,x}_T, T) \, | \mathcal{F}_t]. \end{aligned} \tag{2.12}$$

The last step holds because conditioned on the event $\{\tau > t\}$, we only consider the optimization problem before the prepayment time. On $\{\tau > t\}$, there is no difference between trading with $\mathbb{F}$-measurable strategies or $\mathbb{G}$-measurable strategies.

2.4 Deterministic hazard rate of prepayment

In this section, we derive the indifference price for a single mortgage contract under the assumption that the hazard rate is a deterministic function of time t. Specifically, if $f(t)$ is the density of the conditional cumulative distribution of the prepayment time, then

$$e^{\Gamma(t)} = \frac{1}{1 - F(t)} \quad \text{and} \quad \lambda(t) = \frac{f(t)}{1 - F(t)}.$$

If $\tau > t$, the optimization Problem $(P^M_{\mathbb{G}})$ becomes

$$V^{\mathbb{G}}_t(x) = v^{\mathbb{G}}(x, t),$$

where

$$\begin{aligned} v^{\mathbb{G}}(x,t) &= \sup_{\pi \in \mathcal{A}_{\mathbb{F}}} e^{\Gamma(t)} \mathbb{E}[\int_t^T v(X^M_{u-} + P(u), u) f(u) du + (1 - F(T)) v(X^M_T, T) | X^M_t = x] \\ &= e^{\Gamma(t)} v^{\mathbb{F}}(x, t). \end{aligned}$$

where we used that $\lambda(t)$ is deterministic[5].

Herein, $v^{\mathbb{F}}(x,t)$ is defined as

$$v^{\mathbb{F}}(x,t) = \sup_{\pi \in \mathcal{A}_{\mathbb{F}}} \mathbb{E}[\int_t^T v(X^M_{u-} + P(u), u) f(u) du + (1 - F(T)) v(X^M_T, T) | X^M_t = x]. \tag{2.13}$$

Proposition 2.4.1. *For deterministic hazard rate of prepayment and on the event $\tau > t$, the value functions $v^{\mathbb{F}}(x,t)$ and $v^{\mathbb{G}}(x,t)$, are given, respectively, by*

$$v^{\mathbb{F}}(x,t) = -e^{-\gamma(x+P(t)) - \frac{1}{2}\int_t^T \frac{\mu^2(s)}{\sigma^2(s)} ds} (1 - F(t)) \mathbb{E}[e^{-\gamma \int_t^{\tau \wedge T} mP(s) ds} | \mathcal{F}_t] \tag{2.14}$$

and

$$v^{\mathbb{G}}(x,t) = -e^{-\gamma(x+P(t)) - \frac{1}{2}\int_t^T \frac{\mu^2(s)}{\sigma^2(s)} ds} \mathbb{E}[e^{-\gamma \int_t^{\tau \wedge T} mP(s) ds} | \mathcal{F}_t]. \tag{2.15}$$

[5] In [10], the authors study optimization problem similar to (2.13) assuming CRRA utility, allowing for intermediate consumption and uncertain horizon.

Proof. We introduce the quantity

$$R(t) = \exp\left(-\frac{1}{2}\int_t^T \frac{\mu^2(s)}{\sigma^2(s)}ds\right) \tag{2.16}$$

and we observe that the solution to Problem (P) satisfies $v(x,t) = U(x)R(t)$. Direct arguments yield that the value function $v^{\mathbb{F}}(x,t)$ solves the Hamilton-Jacobi-Bellman (HJB) equation

$$\begin{cases} v_t^F + f(t)U(x+P(t))R(t) + \max_{\pi}\left\{(\mu(t)\pi + c)v_x^F + \frac{1}{2}\sigma^2(t)\pi^2 v_{xx}^{\mathbb{F}}\right\} = 0 \\ v^{\mathbb{F}}(x,T) = U(x)(1-F(T)). \end{cases} \tag{2.17}$$

Using the scaling properties of the utility function, we postulate a candidate solution

$$v^{\mathbb{F}}(x,t) = U(x+P(t))(1-F(t))w(t).$$

Substituting in (2.17), $w(t)$ solves

$$\begin{cases} w'(t) - \left\{\frac{1}{2}\frac{\mu^2(t)}{\sigma^2(t)} + m\gamma P(t) + \lambda(t)\right\} w(t) + \lambda(t)R(t) = 0 \\ w(T) = 1. \end{cases} \tag{2.18}$$

In turn,

$$w(t) = J(\xi(t), \lambda(t)R(t)) \tag{2.19}$$

where

$$\xi(t) = \frac{1}{2}\frac{\mu^2(s)}{\sigma^2(s)} + m\gamma P(t) + \lambda(t) \tag{2.20}$$

and $J : C[t,T] \times C[t,T] \to \mathbb{R}$ given by

$$J(a(t), b(t)) \triangleq e^{-\int_t^T a(s)ds}\left[1 + \int_t^T b(u)e^{\int_u^T a(s)ds}du\right]. \tag{2.21}$$

Moreover,

$$
\begin{aligned}
&J(\xi(t), \lambda(t)R(t)) \\
&= e^{-\int_t^T \frac{1}{2}\frac{\mu^2(s)}{\sigma^2(s)} + m\gamma P(s) + \lambda(s) ds} + \int_t^T \lambda(u)R(u)e^{-\int_t^u \frac{1}{2}\frac{\mu^2(s)}{\sigma^2(s)} + m\gamma P(s) + \lambda(s) ds} du \\
&= \mathbb{E}[\mathbf{1}_{\{\tau > T\}} e^{-\int_t^T \frac{1}{2}\frac{\mu^2(s)}{\sigma^2(s)} + m\gamma P(s) ds} + \mathbf{1}_{\{\tau \leq T\}} R(\tau) e^{-\int_t^\tau \frac{1}{2}\frac{\mu^2(s)}{\sigma^2(s)} + m\gamma P(s) ds} | \mathcal{F}_t]
\end{aligned}
$$

where we used Lemma 2.3.1. From (2.16), we obtain

$$
J(\xi(t), \lambda(t)R(t)) = e^{-\frac{1}{2}\int_t^T \frac{\mu^2(s)}{\sigma^2(s)} ds} \mathbb{E}[e^{-\gamma \int_t^{\tau \wedge T} mP(s) ds} | \mathcal{F}_t]. \tag{2.22}
$$

and we, easily, conclude. □

Combining (2.9), (2.15), and Definition 2.3.1, we deduce the following result.

Proposition 2.4.2. *Under deterministic hazard rate of prepayment, the buyer's indifference price of a mortgage contract is given by*

$$
h(t) = \begin{cases} P(t) - \frac{1}{\gamma} \ln \mathbb{E}_{\mathbb{P}}[e^{-\gamma \int_t^{\tau \wedge T} mP(s) ds} | \mathcal{F}_t] & t < \tau \wedge T \\ 0 & t \geq \tau \wedge T. \end{cases} \tag{2.23}
$$

The above pricing formula contains a linear and a nonlinear part. Due to the specific form of terminal utility, neither price part depends on the initial endowment x. The term $P(t)$ is linear with respect to the remaining outstanding balance of the mortgage contract, the part which the investor is guaranteed to receive. The nonlinear part resembles to the traditional static insurance-type certainty equivalent. Indeed, it is given by the conditional expectation of the distorted total interest payment generated by the mortgage contract from the current time up to the prepayment time. Herein, the nonlinear expectation is taken under the historical measure $\mathbb{P}$.

Remark 2.4.1. Intuitively, we expect that when $t \to (\tau \wedge T)_-$, the indifference price $h(t)$ will converge to $P(\tau \wedge T)$, with $P(\tau \wedge T)$ being the remaining outstanding balance of the mortgage contract. However, this is not the case. Because the prepayment time is totally unpredictable to the investor, (s)he will encounter an instantaneous loss at $\tau \wedge T$, i.e. $\lim_{s \to (\tau \wedge T)_-} h(s) \geq P(\tau \wedge T)$. Equality holds only when $\tau \geq T$.

We conclude with the following price bounds.

Proposition 2.4.3. *The indifference price h, cf. (2.23), satisfies, for $t < \tau \wedge T$,*

$$P(t) \leq h(t) \leq c(T - t).$$

Proof. The lower bound $h(t) \geq P(t)$ follows trivially since the nonlinear part is nonnegative. For the upper bound, we use the dynamics for the outstanding principle, (cf. (2.2)), to obtain

$$\begin{aligned} h(t) &= P(t) - \frac{1}{\gamma} \ln \mathbb{E}_{\mathbb{P}}[e^{-\gamma \int_t^{\tau \wedge T} mP(s)ds} | \mathcal{F}_t] \\ &\leq P(t) - \frac{1}{\gamma} \ln \mathbb{E}[e^{-\gamma \int_t^T mP(s)ds}] = P(t) + \int_t^T mP(s)ds \\ &= P(t) + \int_t^T (P'(s) + c)ds = c(T - t). \end{aligned}$$

□

Notice that the quantity $c(T - t)$ is the scheduled total installment payment the investor will receive *without* prepayment. Recall that in this model, the short-rate is zero. This could also be considered as the current value of the mortgage contract without prepayment.

2.5 Stochastic hazard rate of prepayment

We now extend the analysis to the more realistic case, namely, when the prepayment rate is stochastic. We allow for correlation between prepayment and the risky asset. For this, we consider, for $|\rho| < 1$, the Brownian motion W^λ defined by

$$W_s^\lambda = \rho W_s + \sqrt{1-\rho^2} W_s^\perp, \tag{2.24}$$

with W and $W^\perp$ as in Section 2.2. The Brownian motion W^λ models the influence of the market on the prepayment behavior of mortgagor. Next, it is assumed that the hazard process Γ_s has stochastic intensity process $\lambda_s = \lambda(Y_s)$, where the non-traded state variable Y_s, solves, for $t \le s \le T$,

$$dY_s = b(Y_s, s)ds + a(Y_s, s)dW_s^\lambda \tag{2.25}$$

with $Y_t = y \in \mathbb{R}$.

Assumptions on the diffusion and drift coefficients $a, b : \mathbb{R} \times [0, T] \to \mathbb{R}$ are such that the above equation has a unique strong solution (see, for example Theorem 5.2.1 in [64]). In order to prevent pathological situations in which λ takes negative values, we assume that the function $\lambda : \mathbb{R} \to [0, \infty)$.

For $t \in [0, T]$, $y \in \mathbb{R}$ and $C_i > 0$, $i = 1, 2, 3, 4, 5$, the model coefficients satisfy the following assumptions.

(A1) Uniformly with respect to t, the coefficients a, b are Lipschitz continuous functions in y , and satisfy the linear growth condition,

$$|a(y,t)| + |b(y,t)| \leq C_1(1 + |y|).$$

(A2) For $\alpha \in (0, 1]$, $\alpha' \in (0, 1]$, the coefficient a satisfies

$$|a(y_1, t_1) - a(y_2, t_2)| \leq C_2(|y_1 - y_2|^{\alpha} + |t_1 - t_2|^{\alpha/2})$$

for $t_1, t_2 \in [0, T]$ and $y_1, y_2 \in \mathbb{R}$. The coefficient b and the function λ are continuous in $\mathbb{R} \times [0, T]$ and $\mathbb{R}$, respectively. Uniformly with respect to t, b and λ are locally Hölder continuous in y with exponent α', and satisfy the growing condition,

$$|b(y,t)| + \lambda(y) \leq C_3(1 + |y|^{\beta})$$

with $\beta \in (0, \alpha)$.

(A3) The coefficients for the risky asset are such that, μ is continuous and bounded, σ is continuous, bounded and invertible. There also exist $\varepsilon_1, \varepsilon_2 > 0$ such that

$$\varepsilon_1 < \sigma(t) < \varepsilon_2 \quad \text{and} \quad C_4 \leq a^2(y,t) \leq C_5.$$

We continue with the analysis of the solution to Problem $(P^M_{\mathbb{G}})$.

In view of the Lemma 2.3.1 and equation (2.12), if $\tau > t$, the optimization Problem $(P^M_{\mathbb{G}})$ becomes,

$$V^{\mathbb{G}}_t(x) = v^{\mathbb{G}}(x, y, t)$$

where,

$$v^{\mathbb{G}}(x,y,t) = \sup_{\pi\in\mathcal{A}_{\mathbb{F}}} \mathbb{E}[\int_t^T v(X_{u-}^M + P(u), u)e^{-\int_t^u \lambda(Y_s)ds}\lambda(Y_u)du \\ + e^{-\int_t^T \lambda(Y_s)ds} v(X_T^M, T)|X_t^M = x, Y_t = y]. \tag{2.26}$$

Direct arguments yield that $v^{\mathbb{G}}$ solves the Hamilton-Jacobi-Bellman (HJB) equation in the region $E = \mathbb{R}\times\mathbb{R}\times[0,T)$,

$$\begin{cases} v_t^{\mathbb{G}} + \max_{\pi}\left\{(\mu(t)\pi + c)v_x^{\mathbb{G}} + \sigma(t)a(y,t)\rho\pi v_{xy}^{\mathbb{G}} + \frac{1}{2}\sigma^2(t)\pi^2 v_{xx}^{\mathbb{G}}\right\} + \mathcal{L}v^{\mathbb{G}} \\ \qquad +\lambda(y)U(x+P(t))R(t) - \lambda(y)v^{\mathbb{G}} = 0 \\ v^{\mathbb{G}}(x,y,T) = U(x) = -e^{-\gamma x}, \end{cases} \tag{2.27}$$

where the differential operator $\mathcal{L}$ is defined as

$$\mathcal{L} = \frac{1}{2}a^2(y,t)\frac{\partial^2}{\partial y^2} + b(y,t)\frac{\partial}{\partial y}. \tag{2.28}$$

Before providing the solution to the HJB equation (2.27), we state the following result.

Proposition 2.5.1. *Under assumptions (A1)-(A3), the semilinear parabolic equation*

$$\begin{cases} \Phi_t + \mathcal{L}^{mm}\Phi - (1-\rho^2)(\gamma m P(t) + \lambda(y))\Phi + (1-\rho^2)\lambda(y)\Phi^{-\frac{\rho^2}{1-\rho^2}} = 0 \\ \Phi(y,T) = 1, \end{cases} \tag{2.29}$$

where

$$\mathcal{L}^{mm} = \frac{1}{2}a^2(y,t)\frac{\partial^2}{\partial y^2} + (b(y,t) - \rho\frac{\mu(t)}{\sigma(t)}a(y,t))\frac{\partial}{\partial y}. \tag{2.30}$$

admits a unique positive and bounded classical solution $\Phi(y,t) \in C^{2,1}(\mathbb{R} \times [0,T)) \cap C(\mathbb{R} \times [0,T])$.

Furthermore, for $(y,t) \in \mathbb{R} \times [0,T]$, *the solution* Φ *satisfies*

$$e^{-\gamma \int_t^T (1-\rho^2) m P(s) ds} \leq \Phi(y,t) \leq 1. \tag{2.31}$$

Similar reaction-diffusion type of PDEs are studied in [80] and [77] under different assumptions. To ease the presentation, we provide a brief sketch of the proof and provide the detailed arguments in the Appendix.

Sketch of the Proof. The arguments we use to establish existence of solutions are based on the well known monotone iterative method. The main idea is to use the comparison principle (see Appendix) and work with the appropriate family of sub- and super-solutions. Comparison principle also shows that a nonnegative bounded solution is bounded away from zero. The nonlinear term in the PDE is, hence, locally Lipschitz. This fact implies, in turn, the uniqueness of solution.

It is worth noticing that the first- and zero-order coefficients for the parabolic operator in our problem are unbounded. This issue requires extra analysis even for a linear second-order parabolic equation. We employ a recent work [23] to construct the fundamental solution. Afterwards we use it for the representation of the solutions to the associated linear problems with unbounded coefficients. □

Theorem 2.5.2. *Under assumptions (A1)-(A3), the value function* $v^{\mathbb{G}}$ *is given, for* $t < \tau \wedge T$*, by*

$$v^{\mathbb{G}}(x, y, t) = -e^{-\gamma(x+P(t)) - \frac{1}{2}\int_t^T \frac{\mu^2(s)}{\sigma^2(s)} ds} \Phi(y, t)^{\frac{1}{1-\rho^2}}, \tag{2.32}$$

where Φ *as in Proposition 2.5.1. Moreover,* Φ *admits the following probabilistic representation*

$$\Phi(y, t) = \mathbb{E}_{\mathbb{Q}}[\int_t^T (1-\rho^2)\Phi(Y_u, u)^{-\frac{\rho^2}{1-\rho^2}} e^{-(1-\rho^2)\int_t^u (\gamma m P(s) + \lambda(Y_s))ds} \lambda(Y_u) du$$
$$+ e^{-(1-\rho^2)\int_t^T (\gamma m P(s) + \lambda(Y_s))ds} | Y_t = y], \tag{2.33}$$

with $\mathbb{Q}$ *being the minimal entropy measure for the risky asset process, given by*

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\left(-\int_0^T \frac{\mu(s)}{\sigma(s)} dW_s^{\lambda} - \int_0^T \frac{1}{2}\frac{\mu^2(s)}{\sigma^2(s)} ds \right). \tag{2.34}$$

Proof. We postulate a solution of the form

$$v^{\mathbb{G}}(x, y, t) = -e^{-\gamma(x+P(t)) - \frac{1}{2}\int_t^T \frac{\mu^2(s)}{\sigma^2(s)} ds} F(y, t).$$

Substituting the above in (2.27), we deduce that

$$\begin{cases} F_t + \mathcal{L}^{mm} F - \frac{1}{2}\rho^2 a^2(y, t)\frac{F_y^2}{F} - (\gamma m P(t) + \lambda(y))F + \lambda(y) = 0 \\ F(y, T) = 1, \end{cases} \tag{2.35}$$

and setting

$$F(y, t) = \Phi(y, t)^{\frac{1}{1-\rho^2}}, \tag{2.36}$$

we see that Φ satisfies (2.29). In turn, Proposition 2.5.1 shows that the classical solution for (2.29) exists. We, then, introduce the Brownian motion $W_t^{\mathbb{Q}} =$

$W_t^\lambda + \rho\frac{\mu(t)}{\sigma(t)}t$, with $\mathbb{Q}$ defined by (2.34). It is well established that the pricing measure $\mathbb{Q}$ is the minimal entropy martingale measure for the risky asset process given in (2.1). Under this measure, the dynamics for Y becomes

$$dY_s = \left\{ b(Y_s, s) - \rho\frac{\mu(s)}{\sigma(s)}a(Y_s, s) \right\} ds + a(Y_s, s)dW_s^{\mathbb{Q}}. \tag{2.37}$$

For $s \geq t$, using the terminal condition $\Phi(y, T) = 1$ and applying the integration by parts to $\Phi(Y_s, s)e^{-(1-\rho^2)\int_t^s(\gamma mP(u)+\lambda(Y_u))du}$, we deduce

$$e^{-(1-\rho^2)\int_t^T(\gamma mP(u)+\lambda(Y_u))du} = \Phi(Y_t, t) + \int_t^T e^{-(1-\rho^2)\int_t^u(\gamma mP(v)+\lambda(Y_v))dv} \times$$
$$\times \left\{ \left[\Phi_t + \mathcal{L}^{mm}\Phi - (1-\rho^2)(\gamma mP(u) + \lambda(Y_u)\Phi \right] du + b(Y_u, u)\Phi_y dW_u^{\mathbb{Q}} \right\}.$$

Taking conditional expectation under $\mathbb{Q}$ on both sides and use the equation (2.29), we get (2.33). It remains to show that the value function is given by (2.32). This follows by the Verification Theorem (see, Theorem 3.1 and Remark 3.3 of Chapter IV in [30]). □

Notice that the probabilistic representation of Φ is not given in a closed form. The current value of Φ depends on the future value of itself which admits a path-dependent feature.

Using the pricing equality (2.8) and the value function representation (2.32), we obtain the following result.

Proposition 2.5.3. *Let Φ be as in (2.33). Under assumptions (A1)-(A3), the indifference price h, is given by*

$$h(y, t) = \begin{cases} P(t) - \frac{1}{\gamma(1-\rho^2)} \ln \Phi(y, t) & t < \tau \wedge T \\ 0 & t \geq \tau \wedge T. \end{cases} \tag{2.38}$$

Similar to the case of deterministic hazard rate of prepayment, the above indifference price contains a linear and a nonlinear part. From (2.33) and (2.38), we see that,

$$h(y,t) = P(t)$$
$$-\frac{1}{\gamma(1-\rho^2)} \ln \mathbb{E}_{\mathbb{Q}}[\int_t^T (1-\rho^2)\Phi(Y_u,u)^{-\frac{\rho^2}{1-\rho^2}} e^{-(1-\rho^2)\int_t^u (\gamma m P(s)+\lambda(Y_s))ds} \lambda(Y_u) du$$
$$+ e^{-(1-\rho^2)\int_t^T (\gamma m P(s)+\lambda(Y_s))ds} | Y_t = y].$$

The two conditional expectation terms, yield, respectively, the total interest payments received when the prepayment occurs before, and after, maturity. In both terms, the mortgage rate and the stochastic intensity process are distorted by the correlation ρ.

Corollary 2.5.4. *When $\rho = 0$, the indifference price, for $\tau > t$, is given by*

$$h(y,t) = P(t) - \frac{1}{\gamma} \ln \mathbb{E}_{\mathbb{P}}[e^{-\gamma \int_t^{\tau\wedge T} mP(s)ds} | Y_t = y]. \tag{2.39}$$

This representation coincides with the deterministic hazard function case (2.23).

Proposition 2.5.5. *Before prepayment, for $y \in \mathbb{R}$, $0 \le t \le T$ and under assumptions (A1)-(A3), the indifference price h for a single mortgage contract solves*

$$\begin{cases} h_t + \mathcal{L}^{mm} h - \frac{1}{2}\gamma(1-\rho^2)a^2(y,t)h_y^2 - \frac{\lambda(y)}{\gamma}(e^{-\gamma(P(t)-h)} - 1) + c = 0 \\ h(y,T) = 0. \end{cases} \tag{2.40}$$

Moreover,

$$P(t) \le h(y,t) \le c(T-t). \tag{2.41}$$

The inequality (2.41) follows directly from the estimates (2.31) for Φ and the pricing equality (2.8).

We conclude this section by providing a payoff decomposition result for the mortgage contract (see [58], as well as, [84]). The optimal control for the optimization problem (2.26) is provided in the feedback form

$$\pi^*(x,y,t) = -\rho\frac{a(y,t)}{\sigma(t)}h_y(y,t) + \frac{\mu(t)}{\gamma\sigma^2(t)}.$$

Thus, for $t \leq s \leq \tau \wedge T$, the optimal portfolio process Π_s^*, is given by

$$\Pi_s^* = -\rho\frac{a(Y_s,s)}{\sigma(s)}h_y(Y_s,s) + \frac{\mu(s)}{\gamma\sigma^2(s)} \tag{2.42}$$

while the optimal wealth process solves

$$\begin{cases} dX_s^{M*} = (\mu(s)\Pi_s^* + c)ds + \sigma(s)\Pi_s^* dW_s \\ X_t^{M*} = x - h(y,t). \end{cases} \tag{2.43}$$

Respectively, the optimal wealth process of the classical Merton problem (2.6) is given, for $t \leq s \leq T$, by

$$\begin{cases} dX_s^* = \dfrac{\mu^2(s)}{\gamma\sigma^2(s)}ds + \dfrac{\mu(s)}{\gamma\sigma(s)}dW_s \\ X_t^* = x. \end{cases} \tag{2.44}$$

For $t \leq s < \tau \wedge T$, the indifference price process $H_s = h(Y_s,s)$, satisfies

$$\begin{aligned} dH_s =& [\frac{1}{2}\gamma(1-\rho^2)a^2(Y_s,s)h_y^2(Y_s,s) + \rho\frac{\mu(s)}{\sigma(s)}a(Y_s,s)h_y(Y_s,s) \\ &- c + \frac{\lambda(Y_s)}{\gamma}(e^{-\gamma(P(t)-h(Y_s,s))} - 1)]ds + a(Y_s,s)h_y(Y_s,s)dW_s^\lambda. \end{aligned} \tag{2.45}$$

For $t \leq s < \tau \wedge T$, we define the *residual optimal wealth process*

$$L_s = X_s^* - X_s^{M*} \quad \text{with} \quad L_t = h(Y_t, t),$$

and the residual risk process

$$R_s = H_s - L_s \quad \text{with} \quad R_t = 0.$$

The dynamics of L_s then follows from (2.43) and (2.44), namely,

$$dL_s = \rho \frac{a(Y_s, s)}{\sigma(s)} h_y(Y_s, s)(\mu(s)ds + \sigma(s)dW_s) - cds. \tag{2.46}$$

Moreover, from the above and (2.45), we deduce

$$\begin{aligned} dR_s =& \sqrt{1-\rho^2} a(Y_s, s) h_y(Y_s, s) dW_s^{\perp} \\ &+ [\frac{1}{2}\gamma(1-\rho^2) a^2(Y_s, s) h_y^2(Y_s, s) + \frac{\lambda(Y_s)}{\gamma}(e^{-\gamma(P(t)-h(Y_s,s))} - 1)]ds. \end{aligned} \tag{2.47}$$

Integrating (2.46) and (2.47) from t to $(\tau \wedge T)-$ and combining the above we obtain the following result.

Proposition 2.5.6. *The MBS payoff* $C_{\tau \wedge T} = \int_t^{\tau \wedge T} cds + P(\tau \wedge T)$ *admits the decomposition*

$$\begin{aligned} \int_t^{\tau \wedge T} cds + P(\tau \wedge T) = h(Y_t, t) &+ \int_t^{\tau \wedge T} \rho \frac{a(Y_s, s)}{\sigma(s)} h_y(Y_s, s) \frac{dS_s}{S_s} \\ &+ \left[P(\tau \wedge T) - \lim_{s \to (\tau \wedge T)_-} h(Y_s, s) \right] \\ &+ \int_t^{\tau \wedge T} \sqrt{1-\rho^2} a(Y_s, s) h_y(Y_s, s) dW_s^{\perp} \\ &+ \int_t^{\tau \wedge T} [\frac{1}{2}\gamma(1-\rho^2) a^2(Y_s, s) h_y^2(Y_s, s) + \frac{\lambda(Y_s)}{\gamma}(e^{-\gamma(P(t)-h(Y_s,s))} - 1)]ds. \end{aligned} \tag{2.48}$$

The left hand side of (2.48) is the total installment payments received by the investor up to prepayment time plus the remaining balance at that time. The right hand side consists of the indifference price and four other terms. The integrand in the first term represents the hedging amount one should invest in the risky asset. The second term is the instantaneous loss due to the totally unpredictable prepayment risk. The last two integral terms quantify the risk that cannot be hedged.

2.6 Indifference valuation of mortgage pool

In this section, we extend the previous analysis to the more realistic case when a mortgage-backed security is backed by a *pool* of underlying mortgage contracts. The prepayment behavior in such a MBS pool is, typically, heterogeneous. For model tractability, however, *homogeneity* is frequently assumed. A pool is homogeneous if it contains mortgage contracts with the same maturity, contract term and mortgage rate (see, [27, 28] for the characteristics of *GNMA mortgage-backed pass-through*).

The mortgage contracts in the pool are subject to common risk factors that may trigger prepayment events. In addition, each contract is also affected by the so-called individual risk that may, also, trigger prepayment. Intuitively, the assumption of conditional independence of prepayment times means that once the common risk factors are fixed, the individual risk factors become independent of each other.

Below we give a brief overview for the construction of conditional independence of random times (for details, see Chapter 9 of [8]). To this end, we take $\tau_1, \tau_2, \dots \tau_n$ as the prepayment times for different mortgage contracts in the pool. First we do *not* allow simultaneous prepayments, namely,

$$\mathbb{P}\{\tau_k = \tau_j\} = 0,$$

for arbitrary $k, j = 1, \dots n$ with $k \neq j$.

Definition 2.6.1. Random times $\tau_1, \tau_2, \dots \tau_n$ are said to be conditionally independent with respect to the filtration $\mathbb{F}$ and under $\mathbb{P}$, if

$$\mathbb{P}\{\tau_1 > t_1, \dots, \tau_n > t_n | \mathcal{F}_{\tilde{T}}\} = \prod_{i=1}^{n} \mathbb{P}\{\tau_i > t_i | \mathcal{F}_{\tilde{T}}\}$$

is satisfied for any $\tilde{T} > 0$ and arbitrary $t_1, \dots t_n \in [0, \tilde{T}]$.

Note that conditional independence of random times does not imply their independence. Moreover, conditional independence may not be invariant under an equivalent change of probability measure.

We associate the collection of random times $\tau_1, \tau_2, \dots \tau_n$ with its ordered sequence, namely

$$\tau^{(1)} \leq \tau^{(2)} \leq \cdots \leq \tau^{(n)}.$$

By definition, $\tau^{(1)} = \min\{\tau_1, \tau_2, \dots \tau_n\}$ and $\tau^{(i+1)} = \min\{\tau_k : k = 1, \dots n, \tau_k > \tau^{(i)}\}$. In particular, $\tau^{(n)} = \max\{\tau_1, \tau_2, \dots \tau_n\}$. Without lost of generality, we, further, assume $\tau^{(0)} = 0$.

In addition to the family $\tau_1, \tau_2, \ldots \tau_n$, we postulate that we are, also, given a reference filtration, say $\mathbb{F}$, on the probability space $(\Omega, \mathcal{G}, \mathbb{P})$. As in the single mortgage case, we need to consider the enlarged filtration $\mathbb{G}$. Herein, we set $\mathbb{G} = \mathbb{F} \vee (\bigvee_{i=1}^{n} \mathbb{H}^i)$, where $\mathbb{H}^i$ is generated by $H_t^i = \mathbf{1}_{\{\tau^i \leq t\}}$, $i = 1, 2, \ldots n$. It is convenient to denote $\mathbb{H} = \bigvee_{i=1}^{n} \mathbb{H}^i$.

We define F^i, f^i, Γ^i, and λ^i respectively as the conditional distribution, density, hazard process and hazard rate for prepayment time τ^i, $i = 1, \ldots n$.

By the assumption of conditional independence, we easily deduce the conditional distributions for the first and the last prepayment time,

$$F_t^{(1)} = 1 - \prod_{i=1}^{n} (1 - F_t^i) \quad \text{and} \quad F_t^{(n)} = \prod_{i=1}^{n} F_t^i,$$

as well as the hazard processes for all the other i-th -to-prepayment times,

$$F_t^{(i)} = \sum_{\pi \in \Pi^i} \prod_{k \in \pi} F_t^k \prod_{l \notin \pi} (1 - F_t^l),$$

where Π^i denote the family of subsets of $\{1, \cdots, n\}$ consisting of i elements.

To simplify the computational complexity of the problem, we only consider the mortgage pool subject to deterministic hazard rate of prepayment. The more realistic case, that allows for both deterministic and stochastic hazard rate of prepayment should be studied numerically.

First we note, in analogy with (2.11), that for any $\mathbb{F}$-measurable random variable Z and $s \geq t$,

$$\mathbb{E}_{\mathbb{P}}[\mathbf{1}_{\{t < \tau^{(i)} \leq s\}} Z_{\tau^{(i)}} | \mathcal{G}_t] = \mathbb{E}_{\mathbb{P}}[\mathbf{1}_{\{t < \tau^{(i)} \leq s\}} Z_{\tau^{(i)}} | \mathcal{G}_t^{(i)}]$$

for $i = 1, 2, \dots n$. Herein $\mathbb{G}^{(i)} = \mathbb{F} \vee \mathbb{H}^{(i)}$ with the filtration $\mathbb{H}^{(i)}$ being generated by $H_t^{(i)} = \mathbf{1}_{\{\tau^{(i)} \leq t\}}$.

We start the analysis of the indifference valuation of the mortgage pool with the simple example of two mortgage contracts.

2.6.1 Mortgage pool of two contracts

In this situation, we have

$$F^{(1)}(t) = 1-(1-F^1(t))(1-F^2(t)), \quad f^{(1)}(t) = f^1(t)(1-F^2(t))+f^2(t)(1-F^1(t).$$

and

$$F^{(2)}(t) = F^1(t)F^2(t), \quad f^{(2)}(t) = f^1(t)F^2(t) + f^2(t)F^1(t).$$

We next introduce the value functions $v^{\mathbb{G}}$, when $\tau^{(1)} > t$.

Firstly, we observe that when $\tau^{(1)} \wedge T < t \leq s < \tau^{(2)} \wedge T$, the wealth process $X_s^{M,2}$ is given by

$$X_s^{M,2} = x + \int_t^s (\mu(u)\pi_u + c)du + \int_t^s \sigma(u)\pi_u dW_u.$$

In the interval $(\tau^{(1)} \wedge T, \tau^{(2)} \wedge T)$, there is only one mortgage contract that has not been prepaid in the pool. We consider the auxiliary value function

$$\begin{aligned}\widehat{v}^{\mathbb{G}}(x,t) &:= \sup_{\pi \in \mathcal{A}_{\mathbb{G}}} \mathbb{E}[\mathbf{1}_{\{\tau^{(2)}>t\}} v(X^{M,2}_{(\tau^{(2)} \wedge T)-} + P(\tau^{(2)} \wedge T), \tau^{(2)} \wedge T)|\mathcal{F}_t] \\ &= \sup_{\pi \in \mathcal{A}_{\mathbb{F}}} e^{\Gamma^{(2)}(t)} \mathbb{E}[\int_t^T v(X^{M,2}_{u-} + P(u), u) f^{(2)}(u) du \\ &\qquad + (1 - F^{(2)}(T)) v(X_T^{M,2}, T) | X_t^{M,2} = x].\end{aligned}$$

When $t \leq s < \tau^{(1)} \wedge T$, the wealth process $X_s^{M,1}$, solves

$$X_s^{M,1} = x + \int_t^s (\mu(u)\pi_u + 2c)du + \int_t^s \sigma(u)\pi_u dW_u.$$

In the interval $(0, \tau^{(1)} \wedge T)$, there are two mortgage contracts in the pool. We, in turn, define

$$\begin{aligned} v^{\mathbb{G}}(x,t) &:= \sup_{\pi \in \mathcal{A}_{\mathbb{G}}} \mathbb{E}[\mathbf{1}_{\{\tau^{(1)}>t\}} \widehat{v}^{\mathbb{G}}(X^{M,1}_{(\tau^{(1)}\wedge T)-} + P(\tau^{(1)} \wedge T), \tau^{(1)} \wedge T)|\mathcal{F}_t] \\ &= \sup_{\pi \in \mathcal{A}_{\mathbb{F}}} e^{\Gamma^{(1)}(t)} \mathbb{E}[\int_t^T \widehat{v}^{\mathbb{G}}(X^{M,1}_{u-} + P(u), u) f^{(1)}(u)du \\ &\qquad + (1 - F^{(1)}(T))\widehat{v}^{\mathbb{G}}(X_T^{M,1}, T)|X_t^{M,1} = x]. \end{aligned}$$

To compute the value function $v^{\mathbb{G}}$, we observe that, for $\tau^{(1)} > t$

$$v^{\mathbb{G}}(x,t) = -e^{-\gamma(x+2P(t))} J(\xi^{(1)}, \lambda^{(1)} J(\xi^{(2)}, \lambda^{(2)} R)) \tag{2.49}$$

where

$$\xi^{(1)}(t) = \frac{1}{2}\frac{\mu^2(t)}{\sigma^2(t)} + 2m\gamma P(t) + \lambda^{(1)}(t), \qquad \xi^{(2)}(t) = \frac{1}{2}\frac{\mu^2(t)}{\sigma^2(t)} + m\gamma P(t) + \lambda^{(2)}(t), \tag{2.50}$$

and $R(t)$ defined in (2.16). The factors $\lambda^{(1)}$ and $\lambda^{(2)}$ are, respectively, the deterministic intensity functions for random times $\tau^{(1)}$ and $\tau^{(2)}$. Notice that the value function is defined iteratively by the functional J defined in (2.21). Moreover, we have

$$J(\xi^{(1)}, \lambda^{(1)} J(\xi^{(2)}, \lambda^{(2)} R)) = e^{-\frac{1}{2}\int_t^T \frac{\mu^2(s)}{\sigma^2(s)} ds} \times$$

$$\mathbb{E}_{\mathbb{P}}^{\tau^{(1)}}\left[\exp(-\gamma \int_t^{\tau^{(1)}\wedge T} 2mP(s)ds) \mathbb{E}_{\mathbb{P}}^{\tau^{(2)}}\left[\exp(-\gamma \int_{\tau^{(1)}\wedge T}^{\tau^{(2)}\wedge T} mP(s)ds) \middle| \mathcal{F}_{\tau^{(1)}} \right] \middle| \mathcal{F}_t \right].$$

Observe that in the above nested expectation, the inside expectation is taken with respect to $\tau^{(2)}$ given $\mathcal{F}_{\tau^{(1)}}$ (for fixed $\tau^{(1)}$), while the outside one is taken with respect to $\tau^{(1)}$ given $\mathcal{F}_t$.

Proposition 2.6.1. *Before any prepayment, the indifference price for a mortgage pool with two contracts and subject to deterministic hazard function of prepayment, is given by*

$$h(t) = 2P(t)$$
$$-\frac{1}{\gamma}\ln \mathbb{E}_{\mathbb{P}}^{\tau^{(1)}}\left[\exp(-\gamma\int_t^{\tau^{(1)}\wedge T} 2mP(s)ds)\mathbb{E}_{\mathbb{P}}^{\tau^{(2)}}\left[\exp(-\gamma\int_{\tau^{(1)}\wedge T}^{\tau^{(2)}\wedge T} mP(s)ds)\Big|\,\mathcal{F}_{\tau^{(1)}}\right]\Big|\,\mathcal{F}_t\right].$$

2.6.2 Mortgage pool of n contracts

The previous results can be generalized to a mortgage pool with n contracts. The value function $v^{\mathbb{G}}$ is now defined by an *n-step* procedure. The i-th step is the optimization problem in the interval $(\tau^{(n-i)} \wedge T, \tau^{(n-i+1)} \wedge T)$ with $N - i + 1$ mortgage contracts remaining in the pool.

To facilitate the exposition, we introduce the iterative functions $\mathcal{E}^i$: $[\tau^{(i-1)} \wedge T, \tau^{(i)} \wedge T) \rightarrow \mathbb{R}$, $i = 1, 2, \cdots n$ with

$$\begin{cases} \mathcal{E}^i(u) = \mathbb{E}_{\mathbb{P}}^{\tau^{(i)}}\left[exp(-\gamma\int_u^{\tau^{(i)}\wedge T}(n-i+1)mP(s)ds\;\mathcal{E}^{i+1}(\tau^{(i)}\wedge T)|\mathcal{F}_u\right] \\ \mathcal{E}^n(u) = \mathbb{E}_{\mathbb{P}}^{\tau^{(n)}}\left[exp(-\gamma\int_u^{\tau^{(n)}\wedge T} mP(s)ds\;|\mathcal{F}_u\right]. \end{cases} \tag{2.51}$$

Proposition 2.6.2. *Under the assumption of deterministic hazard rate of prepayment, the value function $v^{\mathbb{G}}$ for a mortgage pool with n contracts is* *given by*

$$v^{\mathbb{G}}(x,t) = -e^{-\gamma(x+nP(t))} J(\xi^{(1)}, \lambda^{(1)} J(\xi^{(2)}, \lambda^{(2)} J(\cdots, \lambda^{(n-1)} J(\xi^{(n)}, \lambda^{(n)} R) \cdots))),$$

where

$$\xi^{(i)}(t) = \frac{1}{2}\frac{\mu^2(t)}{\sigma^2(t)} + (n+1-i)m\gamma P(t) + \lambda^{(i)}(t), \quad i = 1, \ldots n,$$

and $\lambda^{(i)}$, $i = 1, \ldots n$ *is the deterministic intensity function for the random times* $\tau^{(i)}$.

Moreover, the indifference price for a mortgage pool with n *contracts before any prepayment occurs is given by*

$$h(t) = nP(t) - \frac{1}{\gamma} \ln \mathcal{E}^1(t)$$

where $\mathcal{E}^1$ *is the nested expectation as in (2.51).*

For the pool of n contracts, due to the homogeneity assumption, the linear part of the indifference price is n times the remaining outstanding balance for a single contract. The nonlinear part has a nested structure. Intuitively, $\mathcal{E}^i(u)$ denotes the expectation of the total distorted interest payments generated by the pool after time u, for $u \in [\tau^{(i-1)} \wedge T, \tau^{(i)} \wedge T]$. In that horizon, the first $(i-1)$ mortgage contracts have left the pool and the i-th prepayment has not yet occurred. Therefore, the investor receives $(n-i+1)$ times the interest payments generated by a single mortgage contract corresponding to the number of contracts remaining in the pool.

The iterative form (2.51) implies that the expected total interest payments generated after time u, i.e. $\mathcal{E}^i(u)$ is the expected sum of the amount $exp(-\gamma\int_u^{\tau^{(i)}\wedge T}(n-i+1)mP(s)ds$ generated from u to $\tau^{(i)}\wedge T$ and the amount $\mathcal{E}^{i+1}(\tau^{(i)}\wedge T)$ generated after $\tau^{(i)}\wedge T$.

2.7 Conclusion and future study

We have studied the valuation problem of mortgage-backed securities in the presence of totally unpredictable risk. These securities, especially the fixed-rate mortgage contract is a simple example for a rich class of securities which are subject to either default or prepayment risk. We solve the valuation problem from the point of view of an investor who is going to buy the MBS. We demonstrate our method for the cases of deterministic and stochastic hazard rate of prepayment. We thereby extend the usefulness of the indifference pricing for incomplete markets and provide new insights for pricing and hedging for mortgage-backed securities.

In real markets, investors holding mortgage-backed securities usually hedge them by trading Treasury securities (for example, Treasury Note Futures). This is mainly due to the fact that, in various aspects, the investment opportunities for mortgage-backed securities are similar to the ones for Treasury securities. Moreover, yields on 10-year and 30-year Treasury securities are typically used to set long-term mortgage rates. Some empirical work, see, for example, [14] shows that the correlation between the 10-year Treasury note yield and the average 30-year mortgage rate over the period January 1987 to

May 1994 is 0.980, where the mortgage rate is the main incentive for the prepayment to those "active" mortgagors. For the above reasons, on practical side, we should assume that the traded risky securities are Treasury bonds. The assumption of constant short-rate would no longer be appropriate. We plan to consider the problem within certain term structure model in the future. We also plan to look at mortgage pools with more complicated structure, for example, *collateralized mortgage obligation* (CMO).

Chapter 3

Investment performance measurement under asymptotically linear local risk tolerance

3.1 Introduction

This chapter is a contribution to optimal portfolio management using the forward investment performance approach. This approach, developed by M. Musiela and T. Zariphopoulou(see, [57, 62]), is based on the martingale properties of the so-called forward performance process which combines the investor's preferences with market related inputs. In many aspects, it is similar to the traditional maximal expected utility methodology where the martingality of the solution (value function) is a consequence of the dynamic programming principle. It differs, however, in that the forward performance process is defined endogenously to the market environment and for all times. A direct consequence of these properties is that the forward solution follows the market movements "path-by-path" and, moreover, can be constructed without references to a specific trading horizon.

Constructing the forward performance process and the associated with it optimal portfolio strategies poses many difficulties due to the fact that the implicit stochastic optimization problem is posed "forward" in time. A class of such processes was recently constructed in [60, 62] using the compilation of differential and stochastic input. The inputs are given, respectively, by the solution of a fully nonlinear PDE and a triple of stochastic processes representing a benchmark, alternative market views and (random) time-rescaling. The optimal policies are given as a linear combination of the investor's optimal wealth and the time-rescaled risk tolerance processes. An important result is that these two processes solve an autonomous system of stochastic differential equations.

The local risk tolerance function plays pivotal role in the above analysis. It is constructed from the investor's initial risk preferences and the solution of an equation of fast-diffusion type. It is, then, used to solve the aforementioned system and, in turn, to explicitly specify the optimal investment processes in a feedback form. We note that such optimal policies come as a surprise given the non-Markovian nature of the market model.

Motivated by the emerging modelling importance of the local risk tolerance, we concentrate herein on a specific class of such functions. The family we consider corresponds to a dynamic generalization of the popular utilities used in academic works of portfolio management, namely, the power, logarithmic and exponential ones. However, in contrast to the power and logarithmic cases, the risk tolerances we consider are globally defined (i.e. for positive and negative wealth levels).

The chapter is organized as follows. In section 3.2, we introduce the model and review the definition of forward performance process and the main results of [62]. In section 3.3, we focus on a two-parameter family of risk tolerance functions and construct the related forward performance process. In section 3.4, we provide an explicit construction of the associated optimal allocations and wealth processes. We conclude in section 3.5 where we concentrate on special limiting choices of the two risk tolerance parameters.

3.2 The model and its investment performance measurement

The market environment consists of one riskless and k risky securities. The risky securities are stocks and their prices are modelled as positive and continuous Ito processes. Namely, for $i = 1, ..., k$, the price S^i of the i^{th} risky asset solves

$$dS_t^i = S_t^i \left(\mu_t^i dt + \sum_{j=1}^{d} \sigma_t^{ji} dW_t^j \right) \tag{3.1}$$

with $S_0^i > 0$. The process $W = \left(W^1, ..., W^d\right)$ is a standard $d-$dimensional Brownian motion, defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$. For simplicity, it is assumed that the underlying filtration, $\mathcal{F}_t$, coincides with the one generated by the Brownian motion, that is $\mathcal{F}_t = \sigma\left(W_s : \ 0 \leq s \leq t\right)$.

The coefficients μ^i and σ^i, $i = 1, ..., k$, follow $\mathcal{F}_t$-adapted processes with values in $\mathbb{R}$ and $\mathbb{R}^d$, respectively. For brevity, we use σ_t to denote the volatility matrix, i.e. the $d \times k$ random matrix $\left(\sigma_t^{ji}\right)$, whose i^{th} column represents the volatility σ_t^i of the i^{th} risky asset. We may, then, alternatively write (3.1) as

$$dS_t^i = S_t^i \left(\mu_t^i dt + \sigma_t^i \cdot dW_t \right).$$

The riskless asset, the savings account, has the price process B satisfying

$$dB_t = r_t B_t dt$$

with $B_0 = 1$, and for a nonnegative, $\mathcal{F}_t$-adapted interest rate process r_t. The market coefficients, μ, σ and r are taken to be bounded.

It is postulated that there exists an $\mathcal{F}_t$-adapted process λ, known as the market price of risk, taking values in $\mathbb{R}^d$ and such that the equality

$$\mu_t^i - r_t = \sum_{j=1}^{d} \sigma_t^{ji} \lambda_t^j = \sigma_t^i \cdot \lambda_t$$

is satisfied for $t \geq 0$, for all $i = 1, ..., k$. Using vector and matrix notation, the above becomes

$$\mu_t - r_t \mathbf{1} = \sigma_t^T \lambda_t \tag{3.2}$$

where σ^T stands for the transpose matrix of σ, and $\mathbf{1}$ denotes the d-dimensional vector with every component equal to one. It is assumed that, for all $t \geq 0$, $\mathbb{E}_{\mathbb{P}} \int_0^t |\sigma_s \sigma_s^+ \lambda_s|^2 ds < \infty$, where σ^+ denotes the Moore-Penrose pseudoinverse of the volatility matrix (see [66]). Recall that the matrix σ^+ exists and is unique even if the market fails to be complete.

Starting at $t = 0$ with an initial endowment $x \in \mathbb{R}$, the investor invests at all future times $t > 0$ in the riskless and risky assets. The present value of the amounts invested are denoted, respectively, by π_t^0 and π_t^i, $i = 1, ..., k$.

The present value of her aggregate investment is, then, given by $X_t = \sum_{i=0}^{k} \pi_t^i$. We will refer to X as the discounted wealth. The investment strategies $\left(\pi_t^0, \pi_t^1, ..., \pi_t^k\right)$ will play the role of control processes and are taken to satisfy the standard assumption of being self-financing, i.e. for $s \geq 0$,

$$X_s = x + \sum_{i=1}^{k} \int_0^s \pi_u^i \left(\mu_u^i - r_u \right) du + \sum_{i=1}^{k} \int_0^s \pi_u^i \sigma_u^i \cdot dW_u. \tag{3.3}$$

Writing the above in differential form, yields the evolution of the discounted wealth,

$$dX_t = \sum_{i=1}^{k} \pi_t^i \sigma_t^i \cdot (\lambda_t dt + dW_t) = \sigma_t \pi_t \cdot (\lambda_t dt + dW_t), \tag{3.4}$$

where the (column) vector, $\pi_t = (\pi_t^i; i = 1, ..., k)$.

The set of admissible strategies, $\mathcal{A}$, consists of all self-financing $\mathcal{F}_t$-adapted processes π_t such that $\mathbb{E}_{\mathbb{P}} \int_0^s |\sigma_t \pi_t|^2 dt < \infty$, for $s > 0$. It is also assumed, in order to preclude arbitrage opportunities, that for each $s > 0$, the associated wealth process, X_t, $0 \leq t \leq s$, is a $\mathbb{Q}|_{\mathcal{F}_s}$-supermartingale for some equivalent martingale measure $\mathbb{Q}|_{\mathcal{F}_s} \sim \mathbb{P}|_{\mathcal{F}_s}$.

We continue with the definition of the forward performance process. We refer the reader to [61, 62] (see, also, [57]) for a detailed analysis on the motivation and modelling considerations that led to the development of the forward performance concept.

Definition 3.2.1. An $\mathcal{F}_t$-adapted process $U_t(x)$ is a forward performance if:

i) for each $t \geq 0$ and as a function of $x \in \mathbb{R}, U_t(x)$ is concave and increasing,

ii) for each $t \geq 0$ and each self-financing strategy, $\pi \in \mathcal{A}$,

$$\mathbb{E}_{\mathbb{P}} \left[U_t \left(X_t^{\pi} \right) \right]^+ < \infty,$$

iii) for each self-financing strategy, $\pi \in \mathcal{A}$,

$$\mathbb{E}_{\mathbb{P}}\left[U_s\left(X_s^{\pi}\right)|\mathcal{F}_t\right] \leq U_t\left(X_t^{\pi}\right), \quad s \geq t,$$

iv) there exists a self-financing strategy, $\pi^* \in \mathcal{A}$, for which

$$\mathbb{E}_{\mathbb{P}}[U_s(X_s^{\pi^*})\,|\mathcal{F}_t] = U_t(X_t^{\pi^*}), \quad s \geq t,$$

and

v) it satisfies the initial datum $U_0\left(x\right) = u_0\left(x\right)$, $x \in \mathbb{R}$ where $u_0 : \mathbb{R} \to \mathbb{R}$ is a concave and increasing function of wealth.

Related to our work is the recent paper [18] in which the authors considered random horizon choices, aiming at alleviating the dependence of the value function on a fixed (and deterministic) horizon. Their model is more general than ours, in terms of the assumptions on the price processes. However, the focus is on horizon effects and not on additional features affecting the form of the forward solution like, for example, numeraire choice, tracking a benchmark and alternative market views. Horizon issues were also considered in [37, 38] who proposed the so-called horizon-unbiased utilities in the context of lognormal diffusion models and constructed a deterministic class of solutions. While preparing this work, the authors came across the preprint [5] where a special case of forward processes is considered in a model similar to ours (see Corollary 3.2.2 below).

We mention that forward formulations of optimal control problems have been proposed and analyzed in the past. For deterministic models, we refer the reader, among others, to [75] and Chapter 1 in [51] (see, also, [82]). In stochastic settings, forward optimality has been studied, primarily under Markovian assumptions, in [50] via the associated controlled martingale problems and the construction of the Nisio semigroup (see, [63]).

Next, we review the results of [62]. They consist of three parts, namely, the representation of a family of forward performance processes, the specification of the associated optimal investment strategies and wealth processes and the construction of an autonomous system of stochastic differential equations that the optimal wealth and risk tolerance processes solve.

Theorem 3.2.1. *Let the processes Y and Z solve*

$$dY_t = Y_t \delta_t \cdot (\lambda_t dt + dW_t) \tag{3.5}$$

and

$$dZ_t = Z_t \phi_t \cdot dW_t \tag{3.6}$$

with $Y_0 = Z_0 = 1$, δ and ϕ being $\mathcal{F}_t$-adapted and bounded with δ such that $\sigma\sigma^+\delta = \delta$ and $\mathbb{E}_{\mathbb{P}} \int_0^t |\sigma_s \sigma_s^+ \phi_s|^2 ds < \infty$. Define the process

$$A_t = \int_0^t |\sigma_s \sigma_s^+ (\lambda_s + \phi_s) - \delta_s|^2 ds, \quad t \geq 0 \tag{3.7}$$

where λ as in (3.2).

Let $u : \mathbb{R} \times (0, \infty) \to \mathbb{R}$ be a concave and increasing function of the spatial argument with $u : \mathcal{C}^{4,1} (\mathbb{R} \times (0, \infty))$ satisfying the differential constraint

$$u_t u_{xx} = \frac{1}{2} u_x^2 \tag{3.8}$$

and the initial datum

$$u(x,0) = u_0(x) \tag{3.9}$$

with $u_0 : \mathbb{R} \to \mathbb{R}$ *be in* $\mathcal{C}^4(\mathbb{R})$. *Then, the process* $U_t(x)$ *defined by*

$$U_t(x) = u\left(\frac{x}{Y_t}, A_t\right) Z_t, \quad t \geq 0 \tag{3.10}$$

is a forward performance.

The process Y, which normalizes the wealth argument, may be thought as a benchmark (or numeraire) with regards to which the investment performance is measured. The process Z refers to changes in the historical probability measure and accommodates alternative views on anticipated market movements. We will refer to Y and Z as the *benchmark* and *market view processes* respectively.

Corollary 3.2.2. *In the special case* $\delta_t = \phi_t = 0$, $t \geq 0$, *the forward performance process deduces to*

$$U_t(x) = u\left(x, \int_0^t |\sigma_s \sigma_s^+ \lambda_s|^2 ds\right). \tag{3.11}$$

If, in addition, the market parameters are constant, the forward solution is given by the deterministic function

$$U_t(x) = u\left(x, |\sigma\sigma^+\lambda|^2 t\right). \tag{3.12}$$

Forward solutions of form (3.11) (resp. (3.12)) are the ones considered in [5] (resp. [38, 37]).

We continue with the optimal investment strategies and the wealth they generate. It is worth mentioning that despite the dimensionality and incompleteness of the model, as well as the allowed path-dependence of the coefficients, the optimal control policies are given in an explicit *feedback* form. To our knowledge this is one of the very few such examples.

For convenience and generality, we work in the benchmarked configuration, namely, we consider the processes

$$\tilde{\pi}_t^* \equiv \frac{1}{Y_t}\,\pi_t^* \quad \text{and} \quad \tilde{X}_t^* \equiv \frac{X_t^*}{Y_t} \tag{3.13}$$

denoting, respectively, the *benchmarked optimal portfolio* and *benchmarked optimal wealth*.

A quantity that will play an important role in the analysis herein is the *local risk tolerance* $r : \mathbb{R} \times [0, \infty) \to \mathbb{R}^+$, defined as

$$r\,(x, t) = -\frac{u_x\,(x, t)}{u_{xx}\,(x, t)} \tag{3.14}$$

with u as in (3.10). For its initial value, we will be using the notation

$$r_0\,(x) = r\,(x, 0) = -\frac{u_0'\,(x)}{u_0''\,(x)}. \tag{3.15}$$

The following assumption will be standing throughout.

Assumption 3.2.1. There exist constants K_1 and K_2 such that, for all $t \geq 0$ and $x, \bar{x} \in \mathbb{R}$,

$$r^2\,(x, t) \leq K_1\left(1 + x^2\right) \quad \text{and} \quad |r\,(x, t) - r\,(\bar{x}, t)\,| \leq K_2|x - \bar{x}|. \tag{3.16}$$

Next, we introduce the *risk tolerance process* (at benchmarked optimal wealth)

$$\tilde{R}_t^* = r\left(\tilde{X}_t^*, A_t\right) \tag{3.17}$$

with r as in (3.14) and A being the time-rescaling process defined in (3.7).

Theorem 3.2.3. *The optimal benchmarked portfolio $\tilde{\pi}_t^*$, $t > 0$, is given by*

$$\tilde{\pi}_t^* = \Pi_t^*\left(\tilde{X}_t^*\right)$$

with

$$\Pi_t^*\left(x\right) = x\sigma^+\delta_t + r\left(x, A_t\right)\sigma_t^+\left(\lambda_t + \phi_t - \delta_t\right) \tag{3.18}$$

where A as in (3.7) and $\tilde{X}_t^*$, $t > 0$, solving

$$d\tilde{X}_t^* = \left(\sigma_t\tilde{\pi}_t^* - \tilde{X}_t^*\delta_t\right)\cdot\left(\left(\lambda_t - \delta_t\right)dt + dW_t\right), \tag{3.19}$$

with $\tilde{\pi}^*$ being used.

Equivalently,

$$\tilde{\pi}_t^* = m_t\tilde{X}_t^* + n_t\tilde{R}_t^* \tag{3.20}$$

with $\tilde{R}_t^*$ as in (3.17) and the portfolio weights given by

$$m_t = \sigma_t^+\delta_t \quad \textit{and} \quad n_t = \sigma_t^+\left(\lambda_t + \phi_t - \delta_t\right). \tag{3.21}$$

An important consequence of the above theorem is that, under *any* choice of risk preferences, the optimal investment strategy is represented as a *linear combination* of two funds, namely,

$$\tilde{\pi}_t^{*,X} = m_t\tilde{X}_t^* \quad \text{and} \quad \tilde{\pi}_t^{*,R} = n_t\tilde{R}_t^*. \tag{3.22}$$

The portfolio $\tilde{\pi}_t^{*,X}$ depends functionally only on current wealth and not the risk tolerance. The situation, however, is reversed for the second investment strategy, $\tilde{\pi}_t^{*,R}$. Observe that the portfolio weights m_t, n_t, $t > 0$ are affected exclusively by the market. They may take the value zero in which case the relevant optimal allocation vanishes. Such cases are discussed at the end of this section.

Next, we present the autonomous system of stochastic differential equations that the processes $\tilde{X}_t^*$ and $\tilde{R}_t^*$, $t > 0$ solve. Solving this system and using the linear representation result of (3.20) enable us to explicitly construct the optimal allocation vector $\tilde{\pi}_t^*$.

Proposition 3.2.4. *Let r be the local risk tolerance function, introduced in (3.14), and A the time-rescaling process given in (3.7). Then, for $t > 0$, the processes $\tilde{X}_t^*$ and $\tilde{R}_t^*$, $t > 0$, representing the (benchmarked) optimal wealth and risk tolerance, solve the system*

$$\begin{cases} d\tilde{X}_t^* = \tilde{R}_t^* \sigma_t n_t \cdot \left(\left(\lambda_t - \delta_t \right) dt + dW_t \right) \\ d\tilde{R}_t^* = r_x \left(\tilde{X}_t^*, A_t \right) d\tilde{X}_t^* \end{cases} \tag{3.23}$$

with $\tilde{X}_0^* = x$, $\tilde{R}_0^* = r_0(x)$ and n_t, $t > 0$ as in (3.21).

From (3.23) we see that the solution $\left(\tilde{X}_t^*, \tilde{R}_t^* \right)$ is fully specified once the model is chosen and the local risk tolerance function is known. Recall that r is constructed from the function u (cf. (3.14)), obtained from the nonlinear equation (3.8) and the initial datum (3.9). The form of the above system, however, motivates us to question whether one should first model the differential input u and, then, specify r (cf. (3.14)) or go the opposite direction. Herein, we follow the second approach, namely, we first choose a family of risk tolerances and, in turn, recover the associated differential input. A fundamental result used for this construction is that r satisfies an autonomous differential equation. This rather interesting property was shown in [60].

Proposition 3.2.5. *If u satisfies (3.8), the associated local risk tolerance function r, defined in (3.14), satisfies*

$$r_t + \frac{1}{2} r^2 r_{xx} = 0. \tag{3.24}$$

In addition to the local risk tolerance, a quantity of interest is its reciprocal $\gamma : \mathbb{R} \times [0, \infty) \to \mathbb{R}^+$, defined as

$$\gamma(x,t) = \frac{1}{r(x,t)}, \tag{3.25}$$

and referred to as the *local risk aversion*. Using its definition and (3.24) we deduce the following.

Proposition 3.2.6. *The local risk aversion γ, defined in (3.25), satisfies*

$$\gamma_t = \frac{1}{2} \left(\frac{1}{\gamma} \right)_{xx}. \tag{3.26}$$

It is easy to see how the differential input, u, is recovered once the local risk tolerance is known. Indeed, choosing the initial condition $r_0(x) = r(x, 0)$

and using (3.15) yields (modulo two constants) the initial datum (3.9). In turn, equation (3.24), together with the initial condition r_0, will give the values $r(x,t)$, for $t > 0$. The function $u(x,t)$, $t > 0$, can be, then, retrieved from (3.14) by successive integration, provided certain (time-dependent) quantities are correctly specified. Related arguments are found in the proof of Proposition 3.3.2.

The reader with expertise in nonlinear partial differential equations will find the form of (3.24) and (3.26) familiar. In fact, they are nonlinear heat equations, frequently called equations of *fast-diffusion type* and *porous medium type*. There is a vast literature on these equations and we refer the reader, among others, to the book of [81]. Observe, however, that classical results might not be applicable since the equations are "ill-posed", a fact that adds various difficulties to the construction of well-defined and stable solutions.

We finish this section mentioning that there is alternative way to construct u from r which could, perhaps, provide more intuition for the evolution of the differential input. Namely, observe that (3.8) and (3.14) yield the transport equation

$$u_t + \frac{1}{2} r(x,t) \, u_x = 0. \tag{3.27}$$

Such first-order equations can be solved by the method of characteristics. In (3.27) these curves have slope equal to one half the local risk tolerance. The input u is, then, readily constructed through the initial condition u_0, computed from (3.15), and its propagation along the characteristic curves.

3.3 Asymptotically linear local risk tolerance functions

We now focus on a specific class of risk tolerance functions. To provide some motivation for our choice, let us recall that the utilities most frequently appearing in academic papers of portfolio management are the *power, logarithmic* and *exponential*[1]. In the generic problem of maximizing the expected utility of terminal wealth, these utilities are assigned at the end of the trading horizon, say $[0,T]$, and given, respectively, by

$$u^p\left(x;T\right)=\frac{1}{\gamma}x^{\gamma},\quad x\geq 0,\ \gamma<1,\ \gamma\neq 0, \tag{3.28}$$

$$u^l\left(x;T\right)=\log x,\quad x>0, \tag{3.29}$$

and

$$u^e\left(x;T\right)=-e^{-\kappa x},\quad x\in\mathbb{R},\ \kappa>0. \tag{3.30}$$

The associated risk tolerances (with a slight abuse of notation, we denote them by r but keep the argument T to emphasize their dependence on the horizon choice) are, naturally, time independent and given by

$$r^p\left(x;T\right)=\frac{1}{1-\gamma}x,\ x\geq 0\quad\text{and}\quad r^l\left(x;T\right)=x,\ x>0, \tag{3.31}$$

and

$$r^e\left(x;T\right)=\frac{1}{\kappa},\quad x\in\mathbb{R}. \tag{3.32}$$

[1]The quadratic utility deserves special attention due to its saturation properties and will be studied separately.

Notice that in the traditional setting [2] risk preferences are chosen exclusively at the single time instant, T. In the forward framework, however, they are set at initial time, $t = 0$, and then specified for *all future* times $t > 0$. For the family of forward performance processes we consider herein, the specification of the future values of r comes from the differential constraint (3.24).

Next, we introduce a rich family of solutions which, from one hand, are appropriate for the new framework and, on the other, resemble a dynamic extension of their traditional counterparts (3.31) and (3.32).

Proposition 3.3.1. *Let $\alpha, \beta > 0$ and $r_0 : \mathbb{R} \to \mathbb{R}^+$ be given by*

$$r_0(x) = \sqrt{\alpha x^2 + \beta}.$$

Then, the function $r : \mathbb{R} \times [0, \infty) \to \mathbb{R}^+$,

$$r(x, t; \alpha, \beta) = \sqrt{\alpha x^2 + \beta e^{-\alpha t}}, \tag{3.33}$$

solves (3.24).

It is easy to verify that for *fixed* $t = T$, $r^p(x; T)$, $r^l(x; T)$ and $r^e(x; T)$ are *limiting* cases of (3.33) in their respective spatial domains. Indeed,

$$r^p(x; T) = \lim_{\beta \to 0} r(x, T; \alpha, \beta), \ x \geq 0, \quad \text{and} \quad \gamma = \frac{\sqrt{\alpha} - 1}{\sqrt{\alpha}}, \ \alpha \neq 1, \tag{3.34}$$

[2] We remind the reader that there is no intermediate consumption and, thus, no risk preferences are allocated to incoming consumption streams.

$$r^l(x;T) = \lim_{\beta\to 0} r(x,T;\alpha,\beta), \; x > 0 \quad \text{and} \quad \alpha = 1, \tag{3.35}$$

and

$$r^e(x;T) = \lim_{\alpha\to 0} r(x,T;\alpha,\beta) \quad \text{and} \quad \beta^2 = \kappa^{-1}. \tag{3.36}$$

It is immediate that the family $r(x,t;\alpha,\beta)$ satisfies Assumption 3.2.1. Moreover, it is *globally* defined and remains strictly positive at all positive times,

$$r(x,t;\alpha,\beta) > 0, \quad x \in \mathbb{R} \quad \text{and} \quad t > 0.$$

It has a global minimum at the origin, $(0,0)$, at which it degenerates, i.e. $r(0,0;\alpha,\beta) = 0$. The top panel of Figure 3.1 provides its graph for $\alpha = 4$ and $\beta = 0.1$.

The family (3.33) will be called *asymptotically linear* due to its limiting behavior

$$\lim_{x\to\pm\infty} \frac{r(x,t;\alpha,\beta)}{|x|} = \sqrt{\alpha}, \quad t \geq 0. \tag{3.37}$$

Remark 3.3.1. The above class can be readily generalized to the three-parameter family

$$r(x,t;x_0,\alpha,\beta) = \sqrt{\alpha(x-x_0)^2 + \beta e^{-\alpha t}}, \quad t > 0.$$

Since the arguments developed in the sequel can be easily extended, for the above case, we choose $x_0 = 0$.

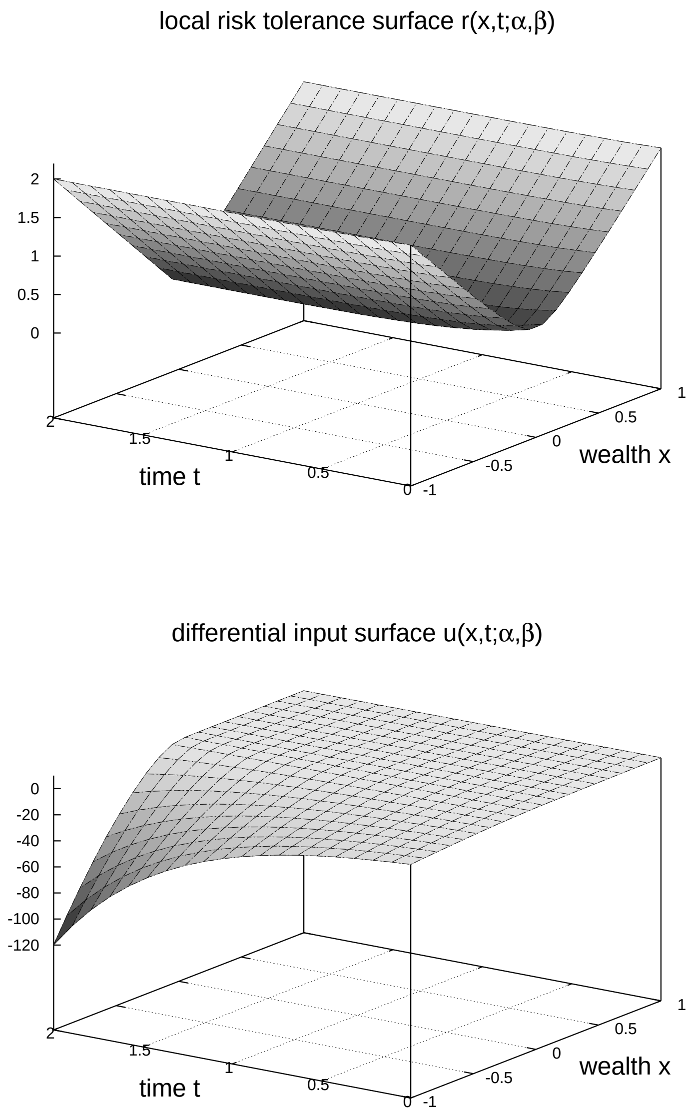

Figure 3.1: **The risk tolerance and differential input surfaces.** For parameters $\alpha = 4$ and $\beta = 0.1$, this figure presents the local risk tolerance surface $r(x, t; \alpha, \beta) = \sqrt{\alpha x^2 + \beta e^{-\alpha t}}$ (first panel) and the differential input surface $u(x, t; \alpha, \beta)$ given in (3.39), for $M = 1$ and $N = 0$ (second panel).

The rest of the chapter is dedicated to the construction of the forward performance process, the optimal investment allocations and the optimal wealth when the local risk tolerance is given by (3.33). The first step is to identify the differential input that is associated with (3.33), i.e. for an increasing and concave function $u(x,t;\alpha,\beta)$ satisfying

$$-\frac{u_x(x,t;\alpha,\beta)}{u_{xx}(x,t;\alpha,\beta)}=\sqrt{\alpha x^2+\beta e^{-\alpha t}},\quad x\in\mathbb{R}\ \text{ and }\ t\geq 0.$$

It is easy to verify that the construction is invariant under affine transformations, namely, if $u(x,t;\alpha,\beta)$ satisfies the above then, for M,N constants,

$$\bar{u}(x,t;\alpha,\beta)=Mu(x,t;\alpha,\beta)+N \tag{3.38}$$

satisfies it as well. To preserve the desired monotonicity of u we need to choose $M>0$.

As it will be clear from the proof of the next Proposition, the form of u depends on the range of the parameter α. Specifically, one needs to look at the cases $\alpha=1$ and $\alpha\neq 1$, separately.

Proposition 3.3.2. *Let r be given by (3.33) with $\alpha,\beta>0$. The following statements hold.*

i) If $\alpha\neq 1$, the associated differential input is given, for $x\in\mathbb{R}$ and $t\geq 0$, by

$$u(x,t;\alpha,\beta)$$
$$=M\frac{\left(\sqrt{\alpha}\right)^{1+\frac{1}{\sqrt{\alpha}}}}{\alpha-1}e^{\frac{1-\sqrt{\alpha}}{2}t}\frac{\left(\frac{\beta}{\sqrt{\alpha}}e^{-\alpha t}+\left(1+\sqrt{\alpha}\right)x\left(\sqrt{\alpha}x+\sqrt{\alpha x^2+\beta e^{-\alpha t}}\right)\right)}{\left(\sqrt{\alpha}x+\sqrt{\alpha x^2+\beta e^{-\alpha t}}\right)^{1+\frac{1}{\sqrt{\alpha}}}}+N. \tag{3.39}$$

ii) If, $\alpha = 1$, then, for $x \in \mathbb{R}$ and $t \geq 0$,

$$u(x,t;1,\beta) \\ = \frac{M}{2}\left(\log\left(x+\sqrt{x^2+\beta e^{-t}}\right) - \frac{e^t}{\beta}x\left(x-\sqrt{x^2+\beta e^{-t}}\right) - \frac{t}{2}\right) + N. \tag{3.40}$$

Proof. Rewriting (3.14) as $(\log u_x(x,t;\alpha,\beta))_x = -r(x,t;\alpha,\beta)^{-1}$ and integrating yields

$$u_x(x,t;\alpha,\beta) = m(t)\left(x+\sqrt{x^2+\frac{\beta}{\alpha}e^{-\alpha t}}\right)^{-\frac{1}{\sqrt{\alpha}}} \tag{3.41}$$

for some function $m : [0,\infty) \to \mathbb{R}^+$. In turn,

$$u_{xx}(x,t;\alpha,\beta) = -m(t)\frac{\left(x+\sqrt{x^2+\frac{\beta}{\alpha}e^{-\alpha t}}\right)^{-\frac{1}{\sqrt{\alpha}}}}{\sqrt{\alpha x^2+\beta e^{-\alpha t}}}.$$

From equation (3.8) we, then, deduce that

$$u_t(x,t;\alpha,\beta) = -\frac{1}{2}m(t)\left(x+\sqrt{x^2+\frac{\beta}{\alpha}e^{-\alpha t}}\right)^{-\frac{1}{\sqrt{\alpha}}}\sqrt{\alpha x^2+\beta e^{-\alpha t}}.$$

Integrating yields, for $\alpha = 1$,

$$u(x,t;1,\beta) = -\frac{1}{2\beta}m(t)\left(e^t x^2 - e^t x\sqrt{x^2+\beta e^{-t}} - \beta\log\left(x+\sqrt{x^2+\beta e^{-t}}\right)\right) + n(t)$$

while, for $\alpha \neq 1$,

$$\begin{aligned} u(x,t;\alpha,\beta) =& m(t)\frac{\sqrt{\alpha}}{\alpha-1}\left(x+\sqrt{x^2+\frac{\beta}{\alpha}e^{-\alpha t}}\right)^{-\left(1+\frac{1}{\sqrt{\alpha}}\right)} \times \\ &\times\left(\frac{\beta}{\alpha}e^{-\alpha t} + \left(1+\sqrt{\alpha}\right)x\left(x+\sqrt{x^2+\frac{\beta}{\alpha}e^{-\alpha t}}\right)\right) + n(t). \end{aligned}$$

We analyze only the latter case. Differentiating the above gives

$$u_t(x,t) = n'(t) + \left(x + \sqrt{x^2 + \frac{\beta}{\alpha} e^{-\alpha t}}\right)^{-\left(1+\frac{1}{\sqrt{\alpha}}\right)} \times$$

$$\times \left(\beta e^{-\alpha t}\left(\frac{m'(t)}{\sqrt{\alpha}(\alpha-1)} - \frac{m(t)}{2(\sqrt{\alpha}+1)}\right) + m'(t)\frac{\sqrt{\alpha}}{\sqrt{\alpha}-1} x \left(x + \sqrt{x^2 + \frac{\beta}{\alpha} e^{-\alpha t}}\right)\right).$$

Reconciling the above two expressions for $u_t(x,t)$ yields

$$m'(t) = -\frac{\sqrt{\alpha}-1}{2} m(t) \quad \text{and} \quad n'(t) = 0.$$

Thus, $m(t) = Me^{-\frac{\sqrt{\alpha}-1}{2}t}$ and $n(t) = N$, and (3.39) follows. □

The initial value u_0, derived from (3.39) and (3.40) for $t = 0$, will be needed for special cases presented in the sequel. For convenience we write it below, namely, for $x \in \mathbb{R}$, $\alpha > 0$ ($\alpha \neq 1$),

$$u_0(x;\alpha,\beta) = M\frac{(\sqrt{\alpha})^{1+\frac{1}{\sqrt{\alpha}}}}{\alpha-1} \frac{\left(\frac{\beta}{\sqrt{\alpha}} + (1+\sqrt{\alpha})\, x \left(\sqrt{\alpha}x + \sqrt{\alpha x^2 + \beta}\right)\right)}{\left(\sqrt{\alpha}x + \sqrt{\alpha x^2+\beta}\right)^{1+\frac{1}{\sqrt{\alpha}}}} + N \tag{3.42}$$

while for $\alpha = 1$,

$$u_0(x,1,\beta) = \frac{M}{2}\left(\log\left(x + \sqrt{x^2+\beta}\right) - \frac{x}{\beta}\left(x - \sqrt{x^2+\beta}\right)\right) + N. \tag{3.43}$$

Once the differential input is specified, the construction of the forward performance process is an immediate application of Theorem 4.4.8.

Proposition 3.3.3. *Let the local risk tolerance and* (Y, Z, A) *be as in (3.33) and (3.5), (3.6) and (3.7). Then, for* $x \in \mathbb{R}$ *and* $t \geq 0$*, the process*

$$U_t\left(x;\alpha,\beta\right) = u\left(\frac{x}{Y_t}, A_t;\alpha,\beta\right) Z_t, \tag{3.44}$$

with $u\left(x,t;\alpha,\beta\right)$ *given in Proposition 3.3.2, is a forward performance.*

Remark 3.3.2. It is important to notice that in the classical case, the power and logarithmic utilities u^l and u^p (cf. (3.28) and (3.29)) are not everywhere defined. This restrains the applicability of such preferences especially when we introduce derivatives and liabilities. Observe, however, that their time-dependent forward counterparts, (3.39) and (3.40), are spatially globally defined. For this reason, the above process $U_t\left(x;\alpha,\beta\right)$ is also globally defined. The situation changes, however, when the time dependence disappears which occurs when $\beta \to 0$ and/or $\alpha \to 0$. These cases deserve special attention and are discussed separately (see Section 3.5).

In the second panel of Figure 3.1, we provide the graph of the function $u\left(x,t;\alpha,\beta\right)$ (cf. (3.39)), for $\alpha = 4$ and $\beta = 0.1$. We, also, provide the cross-sections $u\left(x,t_0;\alpha,\beta\right)$ and $u\left(x_0,t;\alpha,\beta\right)$. The first panel of Figure 3.2 shows, for fixed time t_0, the monotonicity and concavity of $u\left(x,t_0;\alpha,\beta\right)$ while the second panel shows the monotonicity of $u\left(x_0,t;\alpha,\beta\right)$ in terms of time.

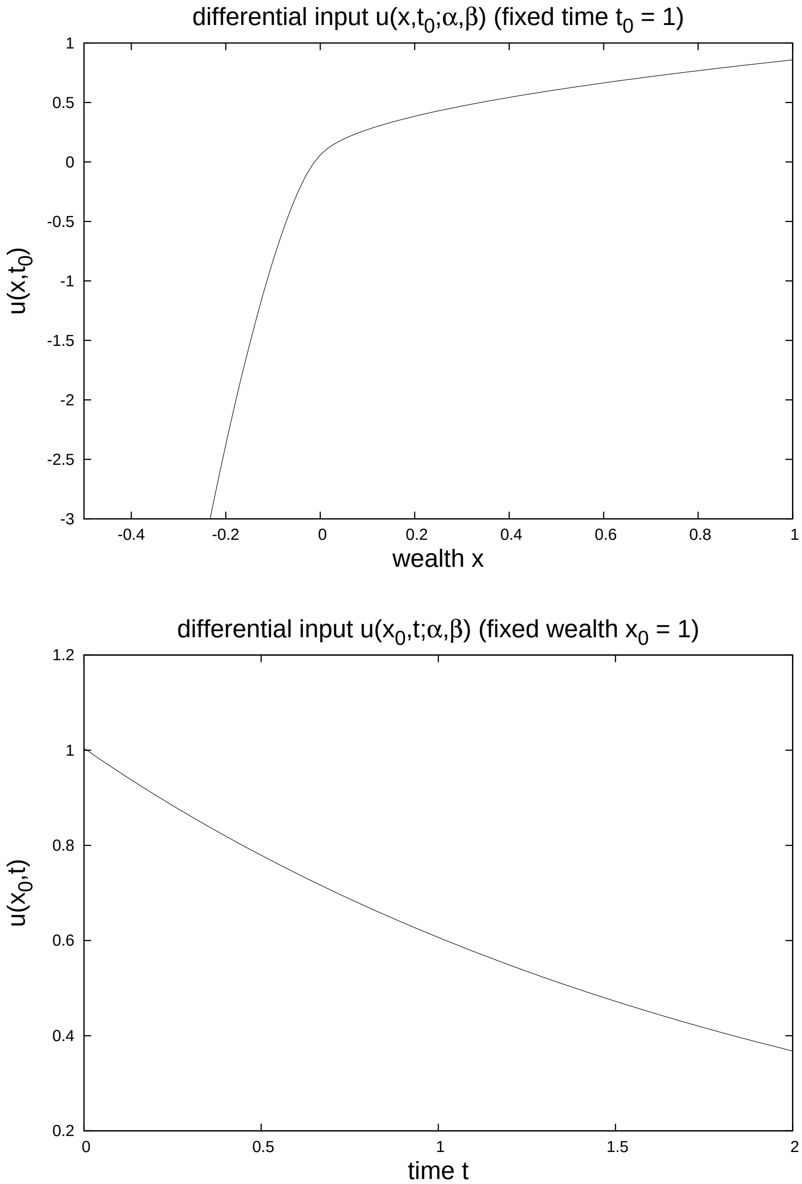

Figure 3.2: **Cross sections of the differential input.** For parameters $\alpha = 4$ and $\beta = 0.1$, this figure presents the cross sections of the differential input surface $u(x, t; \alpha, \beta)$ given in (3.39), for $M = 1$ and $N = 0$. The first panel corresponds to $u(x, t_0; \alpha, \beta)$, with $t_0 = 1$. The second panel corresponds to $u(x_0, t; \alpha, \beta)$, with $x_0 = 1$.

3.4 At the optimum

We provide explicit solutions for the optimal investment policies, the associated wealth and the optimal investment performance. The key ingredients used in the construction of these processes are the autonomous system that the optimal wealth and risk tolerance processes satisfy (cf. (3.23)) together with the specific form of the local risk tolerance function (cf. (3.33)). We remind the reader that the results are stated in the benchmarked configuration.

Theorem 3.4.1. *The processes $\tilde{X}_t^*$ and $\tilde{R}_t^*$, $t > 0$, representing the optimal (benchmarked) wealth and risk tolerance solve the system of linear stochastic differential equations*

$$\begin{cases} d\tilde{X}_t^* = \tilde{R}_t^* \sigma_t n_t \cdot ((\lambda_t - \delta_t)\, dt + dW_t) \\ \\ d\tilde{R}_t^* = \alpha \tilde{X}_t^* \sigma_t n_t \cdot ((\lambda_t - \delta_t)\, dt + dW_t) \end{cases} \tag{3.45}$$

with $\tilde{X}_0^* = x$ and $\tilde{R}_0^* = r(x, 0) = \sqrt{\alpha x^2 + \beta}$.

In turn,

$$\tilde{X}_t^* = e^{-\frac{\alpha}{2}\int_0^t |\sigma_s n_s|^2 ds} \left(x \cosh\left(\sqrt{\alpha} k_t\right) + \sqrt{x^2 + \frac{\beta}{\alpha}} \sinh\left(\sqrt{\alpha} k_t\right) \right) \tag{3.46}$$

and

$$\tilde{R}_t^* = e^{-\frac{\alpha}{2}\int_0^t |\sigma_s n_s|^2 ds} \left(\sqrt{\alpha} x \sinh\left(\sqrt{\alpha} k_t\right) + \sqrt{\alpha x^2 + \beta} \cosh\left(\sqrt{\alpha} k_t\right) \right), \tag{3.47}$$

where n_t, $t > 0$, as in (3.21) and

$$k_t = \int_0^t \sigma_s \sigma_s^+ (\lambda_s + \phi_s - \delta_s) \cdot ((\lambda_s - \delta_s)\, ds + dW_s). \tag{3.48}$$

The vector of optimal asset allocations is given by

$$\tilde{\pi}_t^* = m_t \tilde{X}_t^* + n_t \tilde{R}_t^* \tag{3.49}$$

with $\tilde{X}_t^*$, $\tilde{R}_t^*$ *as above and* m_t *as in (3.21).*

Proof. The coefficients in (3.45) follow from Theorem 3.2.3 (see (3.18) and (3.19)) and (3.33). The admissibility conditions for the optimal policy follow from the boundedness assumption on the market coefficients. Indeed, one can easily see that the integrability condition $\mathbb{E}_{\mathbb{P}} \int_0^s |\pi_t^*|^2 dt < \infty$ holds for $0 \le t \le s$ and that the wealth process X_t^*, $0 \le t \le s$, is a $\mathbb{Q}|_{\mathcal{F}_s}$-martingale where

$$\frac{d\mathbb{Q}}{d\mathbb{P}}\Big|_{\mathcal{F}_s} = \exp\{-\int_0^s \lambda_t \cdot dW_t - \frac{1}{2}\int_0^s |\lambda_t|^2 ds\}.$$

The arguments in the benchmarked configuration follow easily as well.

Adding and subtracting the equations in (3.45) yields

$$d\left(\sqrt{\alpha}\tilde{X}_t^* + \tilde{R}_t^*\right) = \sqrt{\alpha}\left(\sqrt{\alpha}\tilde{X}_t^* + \tilde{R}_t^*\right)\sigma_t n_t \cdot \left(\left(\lambda_t - \delta_t\right)dt + dW_t\right)$$

and

$$d\left(\sqrt{\alpha}\tilde{X}_t^* - \tilde{R}_t^*\right) = -\sqrt{\alpha}\left(\sqrt{\alpha}\tilde{X}_t^* - \tilde{R}_t^*\right)\sigma_t n_t \cdot \left(\left(\lambda_t - \delta_t\right)dt + dW_t\right)$$

and we easily conclude. □

For completeness, we provide the optimal allocations π_t^* and wealth X_t^* in the original (non-benchmarked) formulation. Recall (see (3.13) and (3.5)) that, for $t > 0$,

$$X_t^* = Y_t \tilde{X}_t^* \quad \text{and} \quad \pi_t^* = m_t X_t^* + n_t Y_t r\left(\frac{X_t^*}{Y_t}, A_t\right).$$

Proposition 3.4.2. *Let $x \in \mathbb{R}$ be the investor's initial endowment. Then, the optimal allocation vector and associated optimal wealth are given, respectively, by*

$$\begin{aligned}\pi_t^* = e^{\zeta_t} m_t &\left(x \cosh\left(\sqrt{\alpha} k_t\right) + \sqrt{x^2 + \frac{\beta}{\alpha}} \sinh\left(\sqrt{\alpha} k_t\right) \right) \\ &+ e^{\zeta_t} n_t \left(\sqrt{\alpha} x \cosh\left(\sqrt{\alpha} k_t\right) + \sqrt{\alpha x^2 + \beta} \sinh\left(\sqrt{\alpha} k_t\right) \right), \quad t > 0,\end{aligned} \tag{3.50}$$

and

$$X_t^* = e^{\zeta_t} \left(x \cosh\left(\sqrt{\alpha} k_t\right) + \sqrt{x^2 + \frac{\beta}{\alpha}} \sinh\left(\sqrt{\alpha} k_t\right) \right), \quad t \geq 0 \tag{3.51}$$

where

$$\zeta_t = \int_0^t \left(\delta_s \cdot \lambda_s - \frac{1}{2} |\delta_s|^2 - \frac{\alpha}{2} |\sigma_s n_s|^2 \right) ds + \int_0^t \delta_s \cdot dW_s \tag{3.52}$$

and m_t, n_t and k_t as in (3.21) and (3.48).

Next we look at the extreme cases $m_t = n_t = 0$, $t > 0$ leading, respectively, to $\tilde{\pi}_t^{*,X} = 0$ and $\tilde{\pi}_t^{*,R} = 0$. It is easy to check that they reduce to $\delta_t = 0$ and $\lambda_t + \phi_t - \delta_t = 0$, $t \geq 0$.

3.4.1 Absence of benchmark: $\delta_t = 0$

Then (3.5) yields $Y_t = Y_0 = 1$, $t \geq 0$. The first portfolio component vanishes, $\pi_t^{*,X} = 0$, while the second simplifies to

$$\begin{aligned}\pi_t^{*,R} = e^{-\frac{\alpha}{2} \int_0^t |\sigma_s \sigma_s^+ (\lambda_s + \phi_s)|^2 ds} &\sigma_t^+ (\lambda_t + \phi_t) \times \\ &\times \left(\sqrt{\alpha} x \cosh\left(\sqrt{\alpha} k_t'\right) + \sqrt{\alpha x^2 + \beta} \sinh\left(\sqrt{\alpha} k_t'\right) \right)\end{aligned}$$

with

$$k_t' = \int_0^t \sigma_s \sigma_s^+ (\lambda_s + \phi_s) \cdot (\lambda_s ds + dW_s). \tag{3.53}$$

The optimal wealth is given by

$$X_t^* = e^{-\frac{\alpha}{2}\int_0^t |\sigma_s \sigma_s^+ (\lambda_s + \phi_s)|^2 ds} \left(x \cosh\left(\sqrt{\alpha} k_t'\right) + \sqrt{x^2 + \frac{\beta}{\alpha}} \sinh\left(\sqrt{\alpha} k_t'\right) \right).$$

The (sub)case $\lambda_t + \phi_t = 0$ deserves special attention since $\pi_t^{*,R}$ also vanishes. Moreover, $A_t = 0$, $t \geq 0$, which leads to the performance process

$$U_t(x, t; \alpha, \beta) = u_0(x; \alpha, \beta) Z_t \tag{3.54}$$

with u_0 as in (3.42) or (3.43). Moreover,

$$\tilde{\pi}_t^{*,X} = \pi_t^{*,X} = 0 \quad \text{and} \quad \tilde{\pi}_t^{*,R} = \pi_t^{*,R} = 0, \quad t \geq 0$$

and, in turn,

$$\tilde{X}_t^* = X_t^* = x, \;\; t \geq 0.$$

At the optimum,

$$U_t^*(x; \alpha, \beta) = U_t(x; \alpha, \beta) = u_0(x; \alpha, \beta) Z_t.$$

The above results show that for the above choice of coefficients ($\lambda_t + \phi_t = 0$ and $\delta_t = 0$, $t \geq 0$), it is optimal for the investor to invest *zero wealth into each risky asset,* a result that comes as a surprise given the non-zero returns. Notice that such a solution seems to capture quite accurately the strategy of a *derivatives' trader* for whom the underlying objective is to hedge as opposed to the asset manager whose objective is to invest. Naturally, under this strategy, the forward performance process is not affected by the time evolution of u. This a direct consequence of the fact that the time-rescaling process A degenerates.

3.4.2 Tracking the benchmark: $\lambda_t + \phi_t - \delta_t = 0$

In this case, the portfolio $\tilde{\pi}_t^{*,R}$ vanishes and, thus, any dependence on the risk tolerance dissipates. The investor invests the fraction m_t of his (benchmarked) wealth to the risky assets and puts the rest in the riskless bond. We have $A_t = 0$, $t \geq 0$ and, thus, the performance process is given by (3.54). Moreover,

$$\tilde{\pi}_t^{*,X} = m_t \tilde{X}_t^* \quad \text{and} \quad \tilde{\pi}_t^{*,R} = 0, \qquad t > 0.$$

The absolute wealth tracks the benchmark while the (benchmarked) risk tolerance process remains unchanged,

$$X_t^* = xY_t \quad \text{and} \quad \tilde{R}_t^* = \tilde{R}_0^* = \sqrt{\alpha x^2 + \beta}.$$

At the optimum,

$$U_t^* \left(x; \alpha, \beta\right) = u_0 \left(\frac{X_t^*}{Y_t}; \alpha, \beta \right) Z_t = u_0 \left(x; \alpha, \beta\right) Z_t.$$

Remark 3.4.1. The above result shows that the investor allocates in the riskless asset the amount $\tilde{\pi}_t^{*,0} = p_t X_t^*$ with $p_t = 1 - m_t \cdot \mathbf{1}$. Notice that depending on the level of the *weight process* p_t, $t \geq 0$, which is determined only by the market parameters, the investor allocates *arbitrarily* small or large proportions of the wealth in the riskless asset. In the extreme case, $p_t = 0$, $t \geq 0$, the investor allocates zero wealth in the riskless asset while in the other such case, namely when $p_t = 1$, $t \geq 0$, the optimal allocation consists of putting all wealth in the riskless asset.

3.5 Special cases: CARA and CRRA forward performance processes

We now look at the behavior of the solutions when the parameters α and β vanish. Recalling equalities (3.34), (3.35) and (3.36), we anticipate that the limiting risk tolerance and differential input must resemble their classical power, logarithmic and exponential analogues. While passing to the limit in (3.33), and (3.39) and (3.40) is not difficult from the technical point of view, the emerging limits have some noteworthy properties. To simplify the notation, we skip throughout the parameter notation and use, instead, the superscripts e, p and l in a self-evident way.

3.5.1 The case $\alpha = 0$

Passing to the limit in (3.33) and (3.39) yields, for $t \geq 0$,

$$\lim_{\alpha \to 0} r\left(x, t; \alpha, \beta\right) = \sqrt{\beta}, \quad x \in \mathbb{R} \tag{3.55}$$

and

$$u^e\left(x, t\right) = \lim_{\alpha \to 0} u\left(x, t; \alpha, \beta\right) = -e^{-\frac{x}{\sqrt{\beta}} + \frac{t}{2}}, \quad x \in \mathbb{R}, \tag{3.56}$$

where we chose, for simplicity, $M = (\sqrt{\alpha})^{-\frac{1}{\sqrt{\alpha}}}(\sqrt{\beta})^{\frac{1-\sqrt{\alpha}}{\sqrt{\alpha}}}$ and $N = 0$ [3]; Figure 3.3 demonstrates this convergence.

One, easily, sees that the limiting local risk tolerance (3.55) leads to an exponential forward performance process. This class of solutions was extensively analyzed in [60, 61] and we refer the reader therein for detailed arguments.

Proposition 3.5.1. *For $\alpha = 0$, $\beta > 0$, $t \geq 0$, $x \in \mathbb{R}$, and (Y, Z, A) as in (3.5), (3.6), and (3.7), the process*

$$U_t^e(x) = -\exp\left(-\frac{1}{\sqrt{\beta}}\frac{x}{Y_t} + \frac{A_t}{2}\right) Z_t$$

is a forward performance. Moreover, the optimal (benchmarked) investment strategy and the associated wealth are given by the processes

$$\tilde{\pi}_t^{*,e} = \left(x + \sqrt{\beta}k_t\right) m_t + \sqrt{\beta}n_t \quad \text{and} \quad \tilde{X}_t^{*,e} = x + \sqrt{\beta}k_t \tag{3.57}$$

with n_t, k_t as in (3.21) and (3.48).

At the optimum,

$$U_t^e(X_t^*) = -\exp\left(-\frac{x}{\sqrt{\beta}} - k_t + \frac{1}{2}\int_0^t |\sigma_s n_s|^2 ds\right) Z_t.$$

[3]For the second limit, we use in (3.39) that for $\beta > 0$, $x \in \mathbb{R}$,

$$\lim_{\alpha \to 0}\left(\sqrt{\frac{\alpha}{\beta}} + \sqrt{\frac{\alpha}{\beta}x^2 + 1}\right)^{-\frac{\sqrt{\alpha}+1}{\sqrt{\alpha}}} = e^{-\frac{x}{\sqrt{\beta}}}.$$

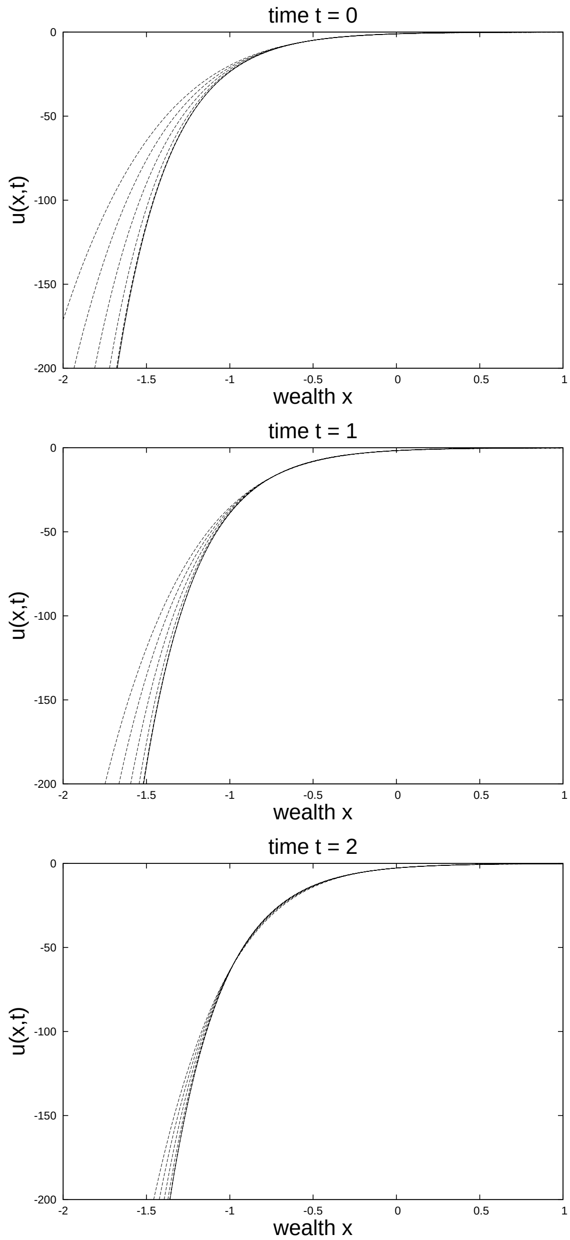

Figure 3.3: **Convergence to the exponential case.** We choose $\beta = 0.1$. For times $t = 0, 1, 2$, the three panels demonstrate the convergence, as $\alpha \to 0$, of the differential input $u(x, t; \alpha, \beta)$, given in (3.39), for $M = (\sqrt{\alpha})^{-\frac{1}{\sqrt{\alpha}}} (\sqrt{\beta})^{\frac{1-\sqrt{\alpha}}{\sqrt{\alpha}}}$ and $N = 0$. The curve of solid line corresponds to the exponential differential input $u^e(x, t) = \lim_{\alpha \to 0} u(x, t; \alpha, \beta) = -e^{-\frac{x}{\sqrt{\beta}} + \frac{1}{2}t}$. The curves of dotted lines correspond to $u(x, t; \alpha, \beta)$ for $\alpha = 1 \times 10^{-1}$, 6×10^{-2}, 3×10^{-2}, 1×10^{-2}, 1×10^{-3} and 1×10^{-4}, respectively.

Remark 3.5.1. It is interesting to observe that due to the presence of the benchmark the optimal investment policy *depends* on the current wealth. This is in contrast to the known results which yield wealth independent policies, a fact that is frequently used against the use of exponential preferences in models of investment and (indifference) valuation.

Next, we write the solutions when both the benchmark and the market view process are absent.

Corollary 3.5.2. *Let $\delta_t = \phi_t = 0$, $t \geq 0$. Then,*

$$U_t^e(x) = -\exp\left(-\frac{1}{\sqrt{\beta}}x + \frac{1}{2}\int_0^t |\sigma_s\sigma_s^+\lambda_s|^2 ds\right). \tag{3.58}$$

Moreover,

$$X_t^{*,e} = x + \sqrt{\beta}\int_0^t \sigma_s\sigma_s^+\lambda_s \cdot (\lambda_s ds + dW_s) \quad \textit{and} \quad \pi_t^{*,e} = \sqrt{\beta}\sigma_t^+\lambda_t.$$

3.5.2 The case $\beta = 0$

Passing to the limit in (3.33) yields, for $t \geq 0$,

$$\lim_{\beta\to 0} r(x,t;\alpha,\beta) = \sqrt{\alpha}|x|, \quad x \in \mathbb{R}. \tag{3.59}$$

In turn, for $\alpha > 1$ ($\alpha < 1$), (3.39) gives

$$u^p(x,t) = \lim_{\beta\to 0} u(x,t;\alpha,\beta) = \begin{cases} \dfrac{1}{\gamma}x^\gamma e^{-\frac{1}{2}\frac{\gamma}{1-\gamma}t} & \text{for } x \geq 0 \ \ (x > 0) \\ -\infty & \text{for } x < 0 \ \ (x \leq 0) \end{cases} \tag{3.60}$$

with

$$\gamma = \frac{\sqrt{\alpha}-1}{\sqrt{\alpha}}, \quad \alpha > 0, \tag{3.61}$$

and where we chose the constants $M = 2^{\frac{1}{\sqrt{\alpha}}}$ and $N = 0$.

For $\alpha = 1$, (3.40) yields

$$u^l(x,t) = \lim_{\beta \to 0} u(x,t;1,\beta) = \begin{cases} \log x - \dfrac{1}{2}t & \text{for } x > 0 \\ -\infty & \text{for } x \leq 0 \end{cases} \tag{3.62}$$

for the choice $M = 2$ and $N = -\left(\frac{1}{2} + \log 2\right)$.

The limiting behavior of the differential inputs $u(x,t;\alpha,\beta)$ and $u(x,t;1,\beta)$ when $\beta \to 0$ is shown in Figures 3.4 and 3.5.

We see that while the local risk tolerance in (3.59) is well defined for *all* $x \in \mathbb{R}$, the associated differential inputs u^p and u^l explode for non positive wealth levels. This impedes us from having globally defined forward performance processes. A well-defined problem may be formulated if we a priori constrain the set of admissible policies to strategies which generate nonnegative wealth. A modification of the proofs of Theorems 4.4.8 and 3.2.3 yields the following results.

Proposition 3.5.3. *Let the local risk tolerance be given by*

$$r(x,t;\alpha,0) = \sqrt{\alpha}x,$$

for $x \geq 0$ *when* $\alpha > 1$ *and* $x > 0$ *when* $\alpha < 1$ $(\alpha \neq 0)$. *Let, also,* (Y, Z, A) *be as in (3.5), (3.6) and (3.7). Then, for* $\alpha > 1$ $(\alpha < 1)$, *the process*

$$U_t^p(x) = \frac{1}{\gamma}\left(\frac{x}{Y_t}\right)^{\gamma} e^{-\frac{1}{2}\frac{\gamma}{1-\gamma}A_t} Z_t, \quad x \geq 0 \quad (x > 0), \tag{3.63}$$

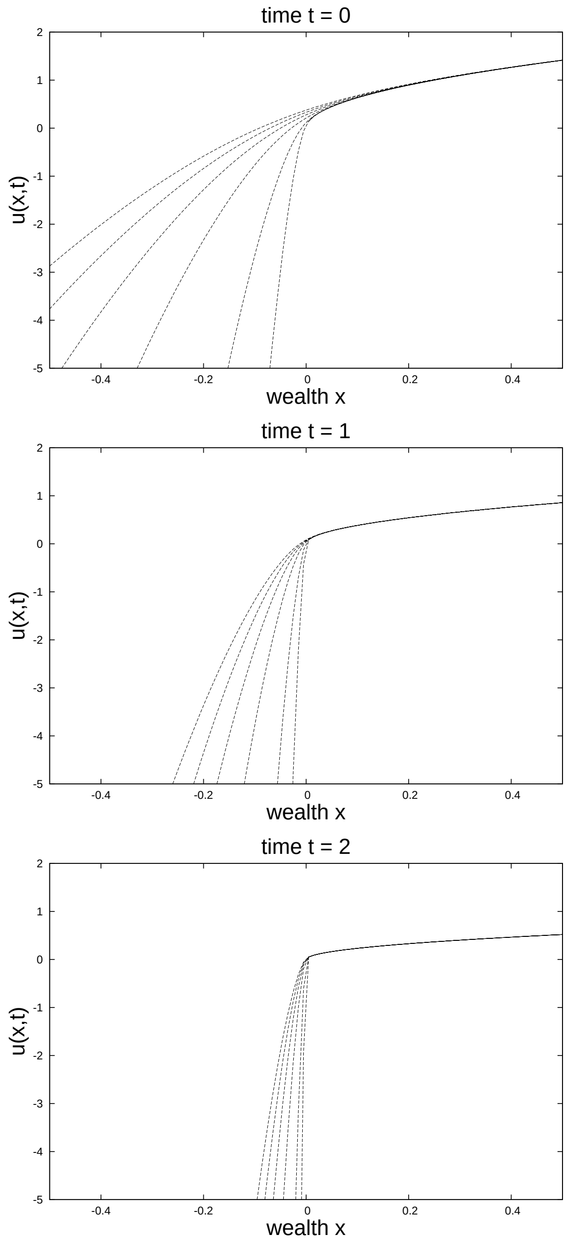

Figure 3.4: **Convergence to the power case.** We choose $\alpha = 4$. For times $t = 0, 1, 2$, the three panels demonstrate the convergence, as $\beta \to 0$, of the differential input $u(x, t; \alpha, \beta)$, given in (3.39), for $M = 2^{\frac{1}{\sqrt{\alpha}}}$ and $N = 0$. The curve of solid line corresponds to the power differential input $u^p(x, t) = \lim_{\beta \to 0} u(x, t; \alpha, \beta) = \frac{1}{\gamma} x^\gamma e^{-\frac{1}{2}\frac{\gamma}{1-\gamma}t}$. The curves of dotted lines correspond to $u(x, t; \alpha, \beta)$ for $\beta = 1 \times 10^{-1}$, 6×10^{-2}, 3×10^{-2}, 1×10^{-2}, 1×10^{-3} and 1×10^{-4}, respectively.

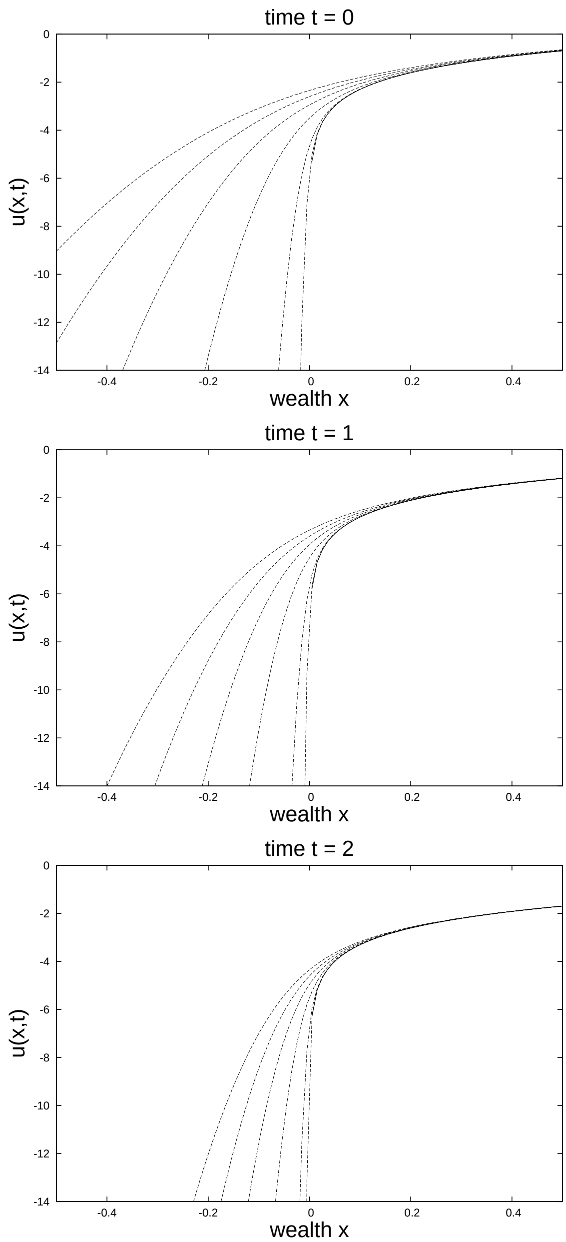

Figure 3.5: **Convergence to the logarithmic case.** For times $t = 0, 1, 2$, the three panels demonstrate the convergence, as $\beta \to 0$, of the differential input $u(x, t; 1, \beta)$, given in (3.40), for $M = 2$ and $N = -\left(\frac{1}{2} + \log 2\right)$. The curve of solid line corresponds to the logarithmic differential input $u^l(x, t) = \lim_{\beta \to 0} u(x, t; 1, \beta) = \log(x) - \frac{1}{2}t$. The curves of dotted lines correspond to $u(x, t; 1, \beta)$ for $\beta = 1 \times 10^{-1}$, 6×10^{-2}, 3×10^{-2}, 1×10^{-2}, 1×10^{-3} and 1×10^{-4}, respectively.

is a forward performance. Moreover, the optimal investment strategy and associated wealth processes are given by

$$\tilde{\pi}_t^{*,p} = x\left(m_t + \sqrt{\alpha}n_t\right)\exp(-\frac{\alpha}{2}\int_0^t |\sigma_s n_s|^2 ds + \sqrt{\alpha}k_t)$$

and

$$\tilde{X}_t^{*,p} = x\exp(-\frac{\alpha}{2}\int_0^t |\sigma_s n_s|^2 ds + \sqrt{\alpha}k_t),$$

with n_t, k_t as in (3.21) and (3.48).

At the optimum,

$$U_t^p\left(X_t^{*,p}\right)$$
$$= \frac{1}{\gamma}x^\gamma \exp\left(-\frac{\alpha-1}{2}\int_0^t |\sigma_s n_s|^2 ds + (\sqrt{\alpha}-1)k_t\right) Z_t, \quad \textit{for } x \geq 0 \quad (x>0).$$

Similar results can be obtained for the logarithmic case.

Proposition 3.5.4. *Let the local risk tolerance be given by*

$$r(x,t;1,0) = x, \quad x > 0.$$

Then, the process

$$U_t^l(x) = \left(\log\frac{x}{Y_t} - \frac{A_t}{2}\right) Z_t, \quad x > 0$$

is a forward performance. Moreover, the optimal investment strategy and associated wealth processes are given by

$$\tilde{\pi}_t^{*,l} = x(m_t + n_t)\exp(-\frac{1}{2}\int_0^t |\sigma_s n_s|^2 ds + k_t)$$

and

$$\tilde{X}_t^{*,l} = x\exp\left(-\frac{1}{2}\int_0^t |\sigma_s n_s|^2 ds + k_t\right).$$

At the optimum,

$$U_t^l\left(X_t^{*,l}\right) = \left(\log x - \int_0^t |\sigma_s n_s|^2 ds + k_t\right) Z_t$$

with n_t, k_t as in (3.21) and (3.48).

In analogy to Corollary 3.5.2, we look at the case of no benchmark and no alternative market views.

Corollary 3.5.5. *Let $\delta_t = \phi_t = 0$, $t \geq 0$ and $\beta = 0$. Then, for $\alpha > 1$ ($\alpha < 1$),*

$$U_t^p(x) = \frac{1}{\gamma} x^\gamma \exp\left(-\frac{\gamma}{2(1-\gamma)} \int_0^t |\sigma_s \sigma_s^+ \lambda_s|^2 ds\right), \quad x \geq 0 \quad (x > 0). \quad (3.64)$$

Moreover,

$$\pi_t^{*,p} = \sqrt{\alpha} x \sigma_t^+ \lambda_t \exp\left(-\frac{\alpha}{2} \int_0^t |\sigma_s \sigma_s^+ \lambda_s|^2 ds + \sqrt{\alpha} \int_0^t \sigma_s \sigma_s^+ \lambda_s \cdot (\lambda_s ds + dW_s)\right)$$

and

$$X_t^{*,p} = x \exp\left(-\frac{\alpha}{2} \int_0^t |\sigma_s \sigma_s^+ \lambda_s|^2 ds + \sqrt{\alpha} \int_0^t \sigma_s \sigma_s^+ \lambda_s \cdot (\lambda_s ds + dW_s)\right).$$

Corollary 3.5.6. *Let $\delta_t = \phi_t = 0$, $t \geq 0$ and $\beta = 0$. Then, for $\alpha = 1$,*

$$U_t^l(x) = \left(\log x - \frac{1}{2} \int_0^t |\sigma_s \sigma^+ \lambda_s|^2 ds\right), \quad x > 0. \quad (3.65)$$

Moreover,

$$\pi_t^{*,l} = x \sigma_t^+ \lambda_t \exp\left(-\frac{1}{2} \int_0^t |\sigma_s \sigma_s^+ \lambda_s|^2 ds + \int_0^t \sigma_s \sigma_s^+ \lambda_s \cdot (\lambda_s ds + dW_s)\right)$$

and

$$X_t^{*,l} = x \exp\left(-\frac{1}{2} \int_0^t |\sigma_s \sigma_s^+ \lambda_s|^2 ds + \int_0^t \sigma_s \sigma_s^+ \lambda_s \cdot (\lambda_s ds + dW_s)\right).$$

When the market coefficients are constants, the forward processes $U_t^e(x)$, $U_t^p(x)$ and $U_t^l(x)$ in (3.58), (3.64) and (3.65) reduce to deterministic functions. These special cases can be found in [37, 38].

Chapter 4

Investment performance measurement in an incomplete market driven by jump processes

4.1 Introduction

Recently, a novel concept, called forward performance has been developed in [62] (see, also [60]) to study the optimal portfolio choice problem in incomplete markets. The idea is to define a family of stochastic processes which combines both the investor's preference and the market related inputs. In contrast to the traditional framework of maximizing the expected utility at a future time, a utility is specified for today. The associated stochastic optimization problems are thus posed "forward" in time, but the solutions are still required to preserve the natural optimality properties (i.e., martingality at an optimum and supermartingality away from it). A direct consequence of this approach is that forward performance process can be constructed for all times, and without references to a specific trading horizon.

In the market environment of [62], the prices of the risky assets were modelled by multidimensional Itô processes. A class of forward performance processes was then constructed using the compilation of differential and stochastic inputs. The differential input satisfies a nonlinear differential constraint and the stochastic inputs consist of a benchmark process, a market view process and a process that rescales the time argument in the differential input. In [85], the authors studied the construction of forward performance processes and the choice of optimal portfolios based on a particular family of local risk tolerance functions which generalizes the classical exponential, power and logarithmic utilities. However, this extension remains in the market model driven by continuous processes.

A more realistic generalization is to allow jumps in the price processes for the assets. One argument in favor of this is that stock prices clearly do not move continuously and are indeed full of sudden and unpredictable jumps. Moreover, evidence from option markets indicate that Gaussian models are not appropriate in terms of calibration. We refer the readers, among others, to [21] for a full exposition of the discussion.

In most financial applications, multidimensional models with jumps are more difficult to construct than one-dimensional ones. The reason for this is that dependence in the Gaussian models could be parameterized in terms of correlation matrices while in the multidimensional models with jumps, we lack such flexibility. In this chapter, we will concentrate on a one-dimensional model driven by jump processes.

Instead of fully generalizing the asset price to a semimartingale, we focus on a more tractable family of models driven by Lévy-Itô processes. The market coefficients in our model are not necessarily Markovian. To the best of our knowledge, not many authors have studied the utility maximization problem in a Lévy market model and given explicit solutions to the optimal strategies, unless the market is of a certain simple structure or the logarithm is chosen as the utility function. Also, there is no agreement as to whether it is better to use one model or the other. For example, the authors in [22] considered a *geometric Lévy process* model and worked with HARA utilities; the author in [45] employed an *exponential Lévy process* model and provided explicit solution for power, logarithmic and exponential utility in terms of the Lévy-Khintchine triplet; the author in [4] solved the problem with a model driven by a *compound Poisson process* using the method of Bellman equation.

As our work shows, the same procedure to construct the forward performance processes in [62] can by applied to the incomplete market model driven by jump processes. In addition to the local risk tolerance function, we find that the other two functions, called the *local jump-relative marginal performance* and the *local jump-time impatience*, respectively, play pivotal roles in the representation of the optimal investment strategies and in the choice of the stochastic input that rescales the time argument. A sufficient condition for the existence of a forward performance process in a market model with jumps is to assume that the local risk tolerance functions are time-independent and linear in wealth. Consequently, under the same nonlinear constraint, the differential input only yields three types of solutions, which are exponential, power and logarithmic.

The rest of the chapter is organized as follows. In Section 4.2, we introduce the model and construct the equivalent (local) martingale measures.

In Section 4.3, we give necessary conditions for the construction of a class of forward performance processes and the associated optimal allocations and wealth processes. In Section 4.4, we focus on a family of time-independent and linear (in wealth) local risk tolerance functions and provide a sufficient condition for the existence of a forward performance process in this class. In Section 4.5, we conclude with three examples for a model with constant market coefficients.

4.2 The model and preliminary results

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a filtered probability space and $(\mathcal{F}_t)_{t \geq 0}$ the underlying filtration taken to be right-continuous and satisfying the usual conditions.

We consider a market in which two securities are traded. The riskless asset, the savings account, has the price process B satisfying

$$dB_t = r_t B_t dt \tag{4.1}$$

with $B_0 = 1$. The price of the risky asset, S, is modelled as a general Lévy-Itô process, solving

$$dS_t = S_{t-} \left(\mu_t dt + \sigma_t dW_t + \int_{\mathbb{R}} \theta(t, z) \tilde{N}(dt, dz) \right) \tag{4.2}$$

with $S_0 > 0$. The process W is an $\mathcal{F}_t$-Brownian motion while

$$\tilde{N}(dt, dz) = N(dt, dz) - \nu(dz)dt$$

is the compensated Poisson random measure[1] associated with a given pure jump $\mathcal{F}_t$-Lévy process η, taken to be independent of W. We, further, assume that η has finite first and second moments and is of finite variation. We recall that η can be written as

$$\eta_t = \int_0^t \int_{\mathbb{R}} z\tilde{N}(ds, dz),$$

where

$$\int_{\mathbb{R}} |z|\nu(dz) < +\infty \quad \text{and} \quad \int_{|z|\geq 1} |z|^2\nu(dz) < +\infty. \tag{4.3}$$

In order to simplify the notation and the presentation, we introduce the auxiliary process L defined, for $t \geq 0$, by

$$L_t = \int_0^t \mu_s ds + \int_0^t \sigma_s dW_s + \int_0^t \int_{\mathbb{R}} \theta(s, z)\tilde{N}(ds, dz). \tag{4.4}$$

We might, then, alternatively write (4.2) as the stochastic (Doléans-Dade) exponential

$$S_t = S_0 + \int_0^t S_{s-} dL_s. \tag{4.5}$$

In order to have the stochastic integral (4.4) well-defined and prevent the price process S from taking negative values, the following hypothesis will be imposed throughout.

Hypothesis (H1)

[1] The measure ν is defined on $\mathbb{R} \setminus \{0\}$, but, occasionally, we may alternatively extend it to the entire $\mathbb{R}$ by setting $\nu(\{0\}) = 0$.

The market coefficients r_t, μ_t and σ_t, $t \geq 0$ are $\mathcal{F}_t$-adapted, predictable and bounded. Moreover, $r_t \geq 0$ and there exists a constant $\varepsilon_1 > 0$ such that $\sigma_t \geq \varepsilon_1 \; dt \otimes \mathbb{P}$-a.e.. The process $\theta(t,z)$ is $\mathcal{F}_t$-adapted, predictable and satisfies $\theta(t,z) > -1 \; dt \otimes \nu(dz) \otimes \mathbb{P}$-a.e.. There, also, exists a constant $C_1 > 0$ such that $|\theta(t,z)| \leq C_1|z|$, uniformly in t and ω.

Given Hypothesis (H1), the stochastic exponential (4.5) has a unique càdlàg adapted solution (see [70]) given by

$$\begin{aligned}
S_t =& S_0 \exp(L_t - \frac{1}{2}[L,L]^c_t) \prod_{s \leq t} (1 + \triangle L_s) e^{-\triangle L_s} \\
=& S_0 \exp\Big(\int_0^t \sigma_s dW_s + \int_0^t (\mu_s - \frac{1}{2}\sigma_s^2) ds + \int_0^t \int_{\mathbb{R}} \log\left(1 + \theta(s,z)\right) \tilde{N}(ds,dz) \\
&+ \int_0^t \int_{\mathbb{R}} \left(\log\left(1 + \theta(s,z)\right) - \theta(s,z)\right) \nu(dz) ds \Big)
\end{aligned}$$

with L as in (4.4).

The investor starts at $t = 0$ with initial endowment $x_0 \in (c, \infty)$, where $c \in \mathbb{R} \cup \{-\infty\}$ and invests at future times $t > 0$ between the two accounts. The present value of her investment in riskless and risky assets, discounted by the saving account, are denoted by π_t^0 and π_t, respectively. We will refer to $X_{t-} = \pi_t^0 + \pi_t$ as her discounted wealth. The investment strategies (π_t^0, π_t) will play the role of control processes and are taken to satisfy the standard assumption of being self-financing, i.e. for $t \geq 0$,

$$X_t = x_0 + \int_0^t \pi_s(\mu_s - r_s) ds + \int_0^t \pi_s \sigma_s dW_s + \int_0^t \int_{\mathbb{R}} \pi_s \theta(s,z) \tilde{N}(ds,dz). \quad (4.6)$$

Writing the above in differential form yields

$$dX_t = \pi_t \left((\mu_t - r_t)dt + \sigma_t dW_t + \int_{\mathbb{R}} \theta(t,z)\tilde{N}(dt,dz) \right). \tag{4.7}$$

We continue with the definition of the forward performance process. We denote the set of admissible investment strategies by $\mathcal{A}$ and will describe it in details afterwards in Definition 4.2.2. We, also, define the effective domain of a function $f : X \to \mathbb{R} \cup \{-\infty\}$ as the set $domf := \{x \in X : f(x) > -\infty\}$.

Definition 4.2.1. An $\mathcal{F}_t$-adapted and càdlàg process $U_t(x)$ is a forward performance if:

(i) for each $t \geq 0$ and as a function of x, $U_t : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ has effective domain $\langle c_t, \infty)$[2], and U_t is increasing and concave on (c_t, ∞), where c_t is an $\mathcal{F}_t$-adapted and càdlàg process that takes values in $\mathbb{R} \cup \{-\infty\}$,

(ii) for each $t \geq 0$ and each self-financing strategy, $\pi \in \mathcal{A}$,

$$\mathbb{E}_{\mathbb{P}}[U_t(X_t)^+] < \infty,$$

(iii) for each self-financing strategy, $\pi \in \mathcal{A}$,

$$\mathbb{E}_{\mathbb{P}}[U_s(X_s)|\mathcal{F}_t] \leq U_t(X_t), \quad s \geq t,$$

(iv) there exists a self-financing strategy, $\pi^* \in \mathcal{A}$, for which

$$\mathbb{E}_{\mathbb{P}}[U_s(X_s^*)|\mathcal{F}_t] = U_t(X_t^*), \quad s \geq t,$$

[2]In this chapter, we use the notation $\langle a, b)$ to denote an interval on $\mathbb{R}$ which is either open or closed on the left side and open on the right side.

and

(v) it satisfies the initial datum $U_0(x) = u_0(x)$, where $u_0 : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ is an increasing and concave function on (c, ∞) and $c = c_0$.

We slightly modify the original definition of forward performance process (see [62]) to allow for non-globally defined initial data. For instance, in [85], the authors constructed the forward performance process for a continuous market model with classical power and logarithmic initial data, where the wealth process is restricted to be positive. This special example motivated us to define the effective domain of the forward performance process to be either the half line (i.e. $c_t \in \mathbb{R}$) or the whole line $\mathbb{R}$ (i.e. $c_t \equiv -\infty$). Notice that here the half line is not necessarily always assumed to be $\mathbb{R}^+$ (i.e. $c_t \equiv 0$) so that we could also assign initial data such as $u_0(x) = \sqrt{x-c}$ or $u_0(x) = \log(x-c)$, where $c \neq 0$.

Next, we define the set $\mathcal{A}$ of admissible policies. The minimum assumptions are the ones needed to properly define the Lévy-type stochastic integral (4.6). However, in order to rule out pathological examples such as "doubling strategies", but still make the "optimal strategy" attainable in the admissible set, we occasionally need more assumptions on π. Notice that if we let the boundary of the effective domain $c_t \equiv c$ with $c \neq -\infty$, it is equivalent to assume that the wealth process X_t^π is uniformly bounded from below by the constant c (See [25] and [35] for the classical definition of admissible set). In the literature, this definition of admissibility has been applied frequently when the price process S is locally bounded. In [6], the authors considered an enlarged class of admissible strategies such that the wealth process is bounded from below by a "sufficient integrable" random variable for the case of general unbounded price process. However, they also pointed out that, the "optimal strategy" is not necessarily attainable in the admissible set. Motivated by the above, we define the set of admissible strategies as follows.

Definition 4.2.2. A process π_t, $t \geq 0$ is an admissible portfolio if it is $\mathcal{F}_t$-adapted, predictable and satisfies $\mathbb{E}_{\mathbb{P}} \int_0^T \pi_t^2 dt < \infty$, $\forall\ T > 0$. Furthermore, for each $T > 0$, the associated wealth process X_t^π, $0 \leq t \leq T$, given by (4.6), belongs to $\langle c_t, \infty)$ and X_t^π is a $\mathbb{Q}|_{\mathcal{F}_T}$-supermartingale for some equivalent measure $\mathbb{Q}|_{\mathcal{F}_T} \sim \mathbb{P}|_{\mathcal{F}_T}$.

In the following lemma, we show that the supermartingale property is essentially a no-arbitrage condition in every interval $[0, T]$ for the wealth process.

Lemma 4.2.1. *If, for an investment strategy π_t, $0 \leq t \leq T$ and $T > 0$, there exists an equivalent measure $\mathbb{Q}|_{\mathcal{F}_T} \sim \mathbb{P}|_{\mathcal{F}_T}$ such that X_t^π is a $\mathbb{Q}|_{\mathcal{F}_T}$-supermartingale, then π is not an arbitrage portfolio.*

Proof. Let X_t^π be a wealth process with $X_0 = 0$ and $\mathbb{P}\{X_T^\pi \geq 0\} = 1$. Since X_t^π is a $\mathbb{Q}|_{\mathcal{F}_T}$-supermartingale and $\mathbb{Q}|_{\mathcal{F}_T} \sim \mathbb{P}|_{\mathcal{F}_T}$, we have $E_{\mathbb{Q}}[X_T^\pi] = 0$ and $\mathbb{Q}\{X_T^\pi \geq 0\} = \mathbb{P}\{X_T^\pi \geq 0\} = 1$. These, immediately, imply that $\mathbb{Q}\{X_T^\pi > 0\} = 0$ using Chebyshev's inequality. Hence, $\mathbb{P}\{X_T^\pi > 0\} = 0$ and π is not an arbitrage portfolio. □

For the market model at hand (cf. (4.1) and (4.2)), an important mathematical problem is to find an equivalent probability measure $\mathbb{Q} \sim \mathbb{P}$ (unique, if possible), under which the discounted price process for the risky asset $\hat{S} = B^{-1}S$ is a local martingale (or even martingale). The philosophy behind this is the fundamental theorem of asset pricing. In the seminal paper [25], it is argued that if S is a locally bounded (resp. bounded) semimartingale, there exists an equivalent local martingale measure (resp. martingale measure) if and only if S satisfies NFLVR (no free lunch with vanishing risk). This is essentially the no-arbitrage condition. Notice that the price process S defined in (4.2) is not necessarily locally bounded unless the jump sizes are bounded. Thenceforth, the authors in [26] extended the theorem to the no-locally bounded case by replacing the term equivalent local martingale measure with the term equivalent sigma-martingale measure. However, a proper mathematical statement of the theorem requires a subtle specification of the set of admissible strategies and in fact it is quite hard to give a precise version of it.

It turns out that a more detailed analysis for the construction of equivalent (local) martingale measures is necessary to help us study the optimal portfolio problem in this chapter. For this, we follow the exposition in [3] and [17] and summarize the results below.

Lemma 4.2.2. *Let ϕ_t and $\psi(t,z)$ be $\mathcal{F}_t$-adapted, predictable and bounded pro-*

cesses. Suppose $\psi \geq 0$ and $\psi(t,0) = 1$, $\forall\ t \geq 0$. Then, the exponential SDE

$$dZ_t = Z_{t-} \left(\phi_t dW_t + \int_{\mathbb{R}} (\psi(t,z) - 1)\tilde{N}(dt,dz) \right). \tag{4.8}$$

with $Z_0 = 1$, has a unique càdlàg adapted solution, which is also a nonnegative local martingale.

Next, we establish uniform integrability conditions for the stochastic exponential given in (4.8) such that it becomes a "true" martingale. We apply the so-called $I(0,1)$-condition introduced in [46] (see, also [52]) for a similar to ours setting. It turns out that this condition is quite useful throughout.

Lemma 4.2.3. *Let ϕ_t and $\psi(t,z)$ be as in Lemma 4.2.2. Moreover, suppose that there exists a constant $C_2 > 0$ such that on a z-neighborhood of zero,*

$$|\psi(t,z) - 1| \leq C_2 |z|, \tag{4.9}$$

uniformly in t and ω. Then, for any $T > 0$, $\mathbb{E}_{\mathbb{P}}[Z_T] = 1$, where Z is the unique solution to (4.8). Also, let $\mathbb{Q}$ be a measure defined on $\mathcal{F}_T$ by,

$$\frac{d\mathbb{Q}}{d\mathbb{P}}\Big|_{\mathcal{F}_T} = Z_T. \tag{4.10}$$

Then, the process

$$W_t^{\mathbb{Q}} = W_t - \int_0^t \phi_s ds \tag{4.11}$$

is a $\mathbb{Q}$-Brownian motion, while the process

$$\int_0^t \int_{\mathbb{R}} \tilde{N}^{\mathbb{Q}}(ds,dz) = \int_0^t \int_{\mathbb{R}} N(ds,dz) - \nu^{\mathbb{Q}}(ds,dz) \tag{4.12}$$

is a $\mathbb{Q}$-martingale where $\nu^{\mathbb{Q}}(dt,dz) = \psi(t,z)\nu(dz)dt$.

Proof. We only show that Z is a uniformly integrable $\mathbb{P}$-martingale on the interval $[0, T]$ and we refer the reader to Chapter III of [40] for the rest of the proof.

To this end, we first observe that the $I(0,1)$-condition (Theorem 3.2 in [46]) is, in our setting, equivalent to

$$\mathbb{E}_{\mathbb{P}}\left[\exp\left(\frac{1}{2}\int_0^T \phi_t^2 dt + \int_0^T \int_{\mathbb{R}} \left(\psi(t,z)\left(\log\psi(t,z) - 1\right) + 1\right)\nu(dz)dt\right)\right] < \infty.$$

Notice, however, that on a z-neighborhood of zero, we have

$$\psi(t,z)\left(\log\psi(t,z) - 1\right) + 1 \leq |\psi(t,z) - 1|^2 \leq C_1^2|z|^2.$$

We then easily conclude using (4.3). $\square$

We are, now, ready to give a necessary and sufficient condition for the existence of an equivalent local martingale measure.

Proposition 4.2.4. *Let ϕ_t and $\psi(t,z)$ be as in Lemma 4.2.3. Then, the discounted price process $\hat{S}_t = B_t^{-1}S_t$ is a local martingale under $\mathbb{Q}$, given by (4.10), if and only if*

$$\mu_t - r_t + \sigma_t\phi_t + \int_{\mathbb{R}} \theta(t,z)(\psi(t,z) - 1)\nu(dz) = 0 \tag{4.13}$$

for $0 \leq t \leq T$, a.s..

We note that each pair (ϕ, ψ) satisfying (4.13) specifies a market price of risk. The process ϕ characterizes the risk on the continuous part (the Brownian one) of the risky asset, while the process ψ on the jump of S. In the following proposition, we show that the specification of (ϕ, ψ) is *not* unique.

Proposition 4.2.5. *Let (H1) hold and* $\theta(t,z) \neq 0$ $dt \otimes \nu(dz) \otimes \mathbb{P}$*-a.e.. Then, the price process* $\hat{S} = B^{-1}S$ *has an infinite number of equivalent (local) martingale measures.*

Proof. The set of equivalent (local) martingale measure is not empty since we can trivially choose $\phi_t = \frac{\mu_t - r_t}{\sigma_t}$ and $\psi(t,z) \equiv 1$.

To construct another such measure, we work as follows. Let (ϕ, ψ) be a solution pair to (4.13). Choose an arbitrary $\mathcal{F}_t$-adapted, bounded and predictable process $\hat{\psi} : [0,T] \times \mathbb{R} \times \Omega \to \mathbb{R}$ such that $\hat{\psi} \geq 0$, $\hat{\psi}(t,0) = 1$, $\forall$ $t \geq 0$, and $|\hat{\psi}(t,z)| \leq C_3|z|$, for some constant $C_3 > 0$, on a z-neighborhood of zero. We easily see that $(\phi_t - \int_{\mathbb{R}} \hat{\psi}(t,z)\theta(t,z)\nu(dz), \psi(t,z) + \sigma_t\hat{\psi}(t,z))$ is also a solution pair.

Given that (ϕ, ψ) satisfies all conditions as in Proposition 4.2.4, we can construct an equivalent local martingale measure $\mathbb{Q} \sim \mathbb{P}$ for the price process $\hat{S}$, given in (4.10), under which $\hat{S}$ solves

$$d\hat{S}_t = \hat{S}_{t-} \left(\sigma_t dW_t^{\mathbb{Q}} + \int_{\mathbb{R}} \theta(t,z)\tilde{N}^{\mathbb{Q}}(dt,dz) \right).$$

Then, the $I(0,1)$-condition is equivalent to

$$\mathbb{E}_{\mathbb{Q}}\Bigg[\exp\Bigg(\frac{1}{2}\int_0^T \sigma_t^2 dt + \int_0^T \int_{\mathbb{R}} \Big((\theta(t,z)+1)\big(\log(\theta(t,z)+1) - 1\big) + 1 \Big) \psi(t,z)\nu(dz)dt \Bigg) \Bigg] < \infty.$$

Using the inequality

$$(\theta(t,z)+1)\left(\log\left(\theta(t,z)+1\right) - 1\right) + 1 \leq |\theta(t,z)|^2 \leq C_2^2|z|^2,$$

we, in turn, conclude that $\hat{S}$ is a uniformly integrable $\mathbb{Q}$-martingale in $[0, T]$.

$\square$

4.3 Necessary conditions for the construction of a class of forward performance processes

In [62], the authors considered a model in which asset prices are modelled as Itô processes and constructed a family of forward performance processes of the form

$$U_t(x) = u\left(\frac{x}{Y_t}, A_t\right) Z_t, \tag{4.14}$$

where $u(x,t)$ is a deterministic function of space and time and (Y, Z, A) are three processes affected entirely by market changes. The function u solves the fully non-linear PDE

$$u_t = \frac{1}{2}\frac{u_x^2}{u_{xx}} \tag{4.15}$$

which can be interpreted as a differential constraint between the investor's risk aversion and impatience. However, when we consider an incomplete market model driven by jump processes, we might not be able to produce a solution that "separates" the inputs u and (Y, Z, A) (cf. 4.26). Herein, we still look for processes analogous to form (4.14), but will make appropriate adjustments to accommodate the jumps in the model. Subsequently, we will derive necessary conditions for the existence of forward performance processes in this class.

Recall that the initial datum u_0 is defined on the effective domain $\langle c, \infty)$ where $c \in \mathbb{R} \cup \{-\infty\}$. Below, we summarize the assumptions on the differential input u and the initial datum u_0 we introduce throughout.

Hypothesis (U)

The differential input $u : \langle c, \infty) \times [0, \infty) \to \mathbb{R}$ is a strictly concave and strictly increasing function of the spatial argument on (c, ∞) with $u \in C^{4,1}((c, \infty) \times (0, \infty))$. Moreover, it satisfies $u(x, 0) = u_0(x)$ with $u_0 \in C^4((c, \infty))$.

Note that we do *not* impose any differential constraint on u such as (4.15) in this section.

Next, we define the following three important quantities.

Definition 4.3.1. Let u be the differential input. The *local risk tolerance* $r : (c, \infty) \times [0, \infty) \to \mathbb{R}^+$, the *local jump-relative marginal performance* $j : \mathcal{D} \times [0, \infty) \to \mathbb{R}^+$ and the *local jump-time impatience* $i : \mathcal{D} \times [0, \infty) \to \mathbb{R}$ are defined as

$$r(x,t) := -\frac{u_x(x,t)}{u_{xx}(x,t)}, \quad j(x,s,t) := \frac{u_x(x+s,t)}{u_x(x,t)} \tag{4.16}$$

and

$$i(x,s,t) := -\frac{u(x+s,t) - u(x,t)}{u_t(x,t)} \tag{4.17}$$

where $\mathcal{D} = \{(x, s) \in (c, \infty) \times \mathbb{R} : x + s > c\}$.

Notice that the local risk tolerance accounts for the individual's tolerance to risk measured at the instant it is incurred in the absence of a jump. The local jump-relative marginal performance is the ratio between the marginal performance of the differential input after and before a jump with jump size of s. The local jump-time impatience is the negative of the marginal rate of substitution between the jump in wealth and time.

Next, we consider $\mathcal{F}_t$-adapted processes, Y and Z, taken to solve, respectively,

$$dY_t = Y_{t-}\delta_t \left((\mu_t - r_t)dt + \sigma_t dW_t + \int_{\mathbb{R}} \theta(t,z)\tilde{N}(dt,dz) \right) \tag{4.18}$$

and

$$dZ_t = Z_{t-} \left(\phi_t dW_t + \int_{\mathbb{R}} (\psi(t,z) - 1)\tilde{N}(dt,dz) \right), \tag{4.19}$$

with $Y_0 = Z_0 = 1$.

Hypothesis (H2)

The coefficient δ_t, $t \geq 0$ is an $\mathcal{F}_t$-adapted, predictable and bounded process. Moreover, there exists a constant $\varepsilon_2 > 0$ such that $1 + \delta_t \theta(t,z) \geq \varepsilon_2$ $dt \otimes \nu(dz) \otimes \mathbb{P}$-a.e.[3]. The coefficients ϕ_t and $\psi(t,z)$, $t \geq 0$ satisfy all the conditions of Lemmas 4.2.2 and 4.2.3.

In analogy to the analysis in [62], one might think of the process Y as a benchmark (or a numéraire) process in relation to which we wish to measure the performance of our investment strategies. The process Z, on the other hand, plays a different role. Specifically, it refers to changes in the historical probability measure and accommodates alternative views on anticipated market movements.

[3] It is sufficient to assume $0 \leq \delta_t < 1 - \varepsilon_2$ $dt \otimes \mathbb{P}$-a.e. since $\theta(t,z) > -1$ $dt \otimes \nu(dz) \otimes \mathbb{P}$-a.e..

The third component, A, which is absolutely continuous with zero initial value, rescales the time argument in the differential input. For the time-being, we choose $A_0 = 0$ and

$$dA_t = a_t dt \tag{4.20}$$

where a is an $\mathcal{F}_t$-adapted and predictable process, to be chosen in the sequel.

We are now ready to find *necessary conditions* for the process $U_t(x)$ of the form (4.14), with Y, Z and A as in (4.18), (4.19) and (4.20), to be a forward performance process. We start with the semimartingale decomposition of $U_t(X_t)$ with X as in (4.6). In order to ease the notation, we introduce the (benchmarked) wealth and the (benchmarked) portfolio processes,

$$\tilde{X}_t = \frac{X_t}{Y_t} \quad \text{and} \quad \tilde{\pi}_t = \frac{\pi_t}{Y_{t-}}. \tag{4.21}$$

Lemma 4.3.1. *The process $U_t(X_t)$ admits the semimartingale decomposition*

$$U_t(X_t) = M_t + V_t, \tag{4.22}$$

with the processes M and V defined as

$$\begin{aligned} M_t = & \int_0^t Z_{s-} \left(u(\tilde{X}_{s-}, A_s)\phi_s + u_x(\tilde{X}_{s-}, A_s)(\tilde{\pi}_s - \delta_s \tilde{X}_{s-})\sigma_s \right) dW_s \\ & + \int_0^t Z_{s-} \int_{\mathbb{R}} \left(\psi(s,z) u(\tilde{X}_{s-} + \xi(s,z), A_s) - u(\tilde{X}_{s-}, A_s) \right) \tilde{N}(ds, dz) \end{aligned} \tag{4.23}$$

and

$$V_t = u_0(x_0) + \int_0^t v_s Z_{s-} ds, \tag{4.24}$$

where

$$\xi(t,z) = \big(\tilde{\pi}_t - \delta_t \tilde{X}_{t-}\big)\frac{\theta(t,z)}{1+\delta_t\theta(t,z)} \tag{4.25}$$

and

$$\begin{aligned} v_t =& u_t(\tilde{X}_{t-}, A_t)a_t + \frac{1}{2}u_{xx}(\tilde{X}_{t-}, A_t)(\tilde{\pi}_t - \delta_t\tilde{X}_{t-})^2\sigma_t^2 \\ &+ u_x(\tilde{X}_{t-}, A_t)(\tilde{\pi}_t - \delta_t\tilde{X}_{t-})\big(\mu_t - r_t - \sigma_t^2\delta_t + \sigma_t\phi_t - \int_{\mathbb{R}}\theta(t,z)\nu(dz)\big) \\ &+ \int_{\mathbb{R}}\psi(t,z)\big(u(\tilde{X}_{t-} + \xi(t,z), A_t) - u(\tilde{X}_{t-}, A_t)\big)\nu(dz). \end{aligned} \tag{4.26}$$

Proof. Applying Itô's formula, we obtain

$$\begin{aligned} d\tilde{X}_t =& X_{t-}d\left(\frac{1}{Y_t}\right) + \frac{1}{Y_t-}dX_t + d\Big[X, \frac{1}{Y}\Big]_t \\ =& (\tilde{\pi}_t - \delta_t\tilde{X}_{t-})\Big(\big(\mu_t - r_t - \sigma_t^2\delta_t - \int_{\mathbb{R}}\frac{\delta_t\theta^2(t,z)}{1+\delta_t\theta(t,z)}\nu(dz)\big)dt \\ &+ \sigma_t dW_t + \int_{\mathbb{R}}\frac{\theta(t,z)}{1+\delta_t\theta(t,z)}\tilde{N}(dt,dz)\Big). \end{aligned} \tag{4.27}$$

Moreover, using the regularity of u and Itô's lemma yields

$$\begin{aligned} du(\tilde{X}_t, A_t) =& u_t(\tilde{X}_{t-}, A_t)a_t dt + \frac{1}{2}u_{xx}(\tilde{X}_{t-}, A_t)(\tilde{\pi}_t - \delta_t\tilde{X}_{t-})^2\sigma_t^2 dt \\ &+ u_x(\tilde{X}_{t-}, A_t)(\tilde{\pi}_t - \delta_t\tilde{X}_{t-})\Big(\big(\mu_t - r_t - \sigma_t^2\delta_t - \int_{\mathbb{R}}\frac{\delta_t\theta^2(t,z)}{1+\delta_t\theta(t,z)}\big)dt + \sigma_t dW_t\Big) \\ &+ \int_{\mathbb{R}}\Big(u(\tilde{X}_{t-} + \xi(t,z), A_t) - u(\tilde{X}_{t-}, A_t)\Big)\tilde{N}(dt,dz) \\ &+ \int_{\mathbb{R}}\Big(u(\tilde{X}_{t-} + \xi(t,z), A_t) - u(\tilde{X}_{t-}, A_t) - u_x(\tilde{X}_{t-}, A_t)\xi(t,z)\Big)\nu(dz)dt, \end{aligned}$$

and

$$\begin{aligned} d[u(\tilde{X}, A), Z]_t =& Z_{t-}u_x(\tilde{X}_{t-}, A_t)(\tilde{\pi}_t - \delta_t\tilde{X}_{t-})\sigma_t\phi_t dt \\ &+ \int_{\mathbb{R}} Z_{t-}(\psi(t,z) - 1)\Big(u(\tilde{X}_{t-} + \xi(t,z), A_t) - u(\tilde{X}_{t-}, A_t)\Big)N(dt,dz). \end{aligned}$$

Finally, we deduce that

$$dU_t(X_t) = u(\tilde{X}_{t-}, A_t)dZ_t + Z_{t-}du(\tilde{X}_t, A_t) + d[u(\tilde{X}, A), Z]_t$$

and using that M is a local martingale and V is of finite variation, we conclude.

□

Next, we introduce the risk tolerance process (at optimal benchmarked wealth)

$$\tilde{R}_t^* = r(\tilde{X}_t^*, A_t),$$

the jump-relative marginal performance and the jump-time impatience functionals (at optimality)

$$\tilde{J}^*(t, z)[\zeta_t] = j\left(\tilde{X}_{t-}^*, \sigma_t^{-1}\zeta_t\tilde{R}_{t-}^*\frac{\theta(t, z)}{1 + \delta_t\theta(t, z)}, A_t\right) \tag{4.28}$$

and

$$\tilde{I}^*(t, z)[\zeta_t] = i\left(\tilde{X}_{t-}^*, \sigma_t^{-1}\zeta_t\tilde{R}_{t-}^*\frac{\theta(t, z)}{1 + \delta_t\theta(t, z)}, A_t\right), \tag{4.29}$$

with r, j and i as in Definition 4.3.1, $\tilde{X}^*$ and A being, respectively, the optimal benchmarked wealth process and the time-rescaling process, and ζ being any $\mathcal{F}_t$-adapted and predictable process.

We are now ready to give necessary conditions for the existence of the forward performance process.

Proposition 4.3.2 (Necessary Conditions)**.** *Let (H1), (H2) and (U) hold. Let, also, the stochastic inputs* Y_t*,* Z_t *and* A_t *be as in (4.18), (4.19) and (4.20)* *and the wealth process X_t be as in (4.6), for all $t \geq 0$. If $U_t(x)$, given by (4.14), with effective domain $\langle cY_t, \infty)$ is a forward performance process, then the optimal benchmarked portfolio $\tilde{\pi}_t^*$ is of the form*

$$\tilde{\pi}_t^* = \delta_t \tilde{X}_{t-}^* + \sigma_t^{-1} \zeta_t^* \tilde{R}_{t-}^* \tag{4.30}$$

with ζ_t^*, $dt \otimes \mathbb{P}$-a.e., solving

$$\begin{aligned} \zeta_t =& \sigma_t^{-1}(\mu_t - r_t) - \sigma_t \delta_t + \phi_t \\ & - \sigma_t^{-1} \left(\int_{\mathbb{R}} \theta(t,z)\nu(dz) - \int_{\mathbb{R}} \tilde{J}^*(t,z)[\zeta_t]\rho(t,z)\nu(dz) \right). \end{aligned} \tag{4.31}$$

Moreover, the process A_t given in (4.20) should be chosen such as

$$\begin{aligned} a_t =& \frac{1}{2} \frac{u_x^2}{u_t u_{xx}} (\tilde{X}_{t-}^*, A_t) \left((\zeta_t^*)^2 - 2\sigma_t^{-1} \zeta_t^* \int_{\mathbb{R}} \tilde{J}^*(t,z)[\zeta_t^*]\rho(t,z)\nu(dz) \right) \\ & + \int_{\mathbb{R}} \psi(t,z) \tilde{I}^*(t,z)[\zeta_t^*]\nu(dz), \end{aligned} \tag{4.32}$$

where the process ρ is defined as

$$\rho(t,z) = \psi(t,z) \frac{\theta(t,z)}{1 + \delta_t \theta(t,z)}. \tag{4.33}$$

Proof. Using the semimartingale decomposition (4.22), we readily see that, the process $U_t(X_t)$ would be a supermartingale if the drift v, defined as in (4.26), remains non positive for all $\pi \in \mathcal{A}$. Moreover, $U_t(X_t)$ is a martingale if there exists $\pi^* \in \mathcal{A}$ such that the drift vanishes. Now we slightly abuse the notation v, by considering it as a function of $\tilde{\pi}$ for any fixed t and ω. Consequently, we have

$$\max_{\pi \in \mathcal{A}} v(\tilde{\pi}) = 0.$$

The first order condition implies

$$\begin{aligned}
&\frac{1}{2}u_{xx}(\tilde{X}^*_{t-}, A_t)(\tilde{\pi}_t - \delta_t \tilde{X}^*_{t-})\sigma_t^2 \\
&+ u_x(\tilde{X}^*_{t-}, A_t)\Big(\mu_t - r_t - \sigma_t^2\delta_t + \sigma_t\phi_t - \int_{\mathbb{R}} \theta(t,z)\nu(dz)\Big) \\
&+ \int_{\mathbb{R}} \psi(t,z)\frac{\theta(t,z)}{1+\delta_t\theta(t,z)} u_x\left(\tilde{X}^*_{t-} + (\tilde{\pi}^*_t - \delta_t\tilde{X}^*_{t-})\frac{\theta(t,z)}{1+\delta_t\theta(t,z)}, A_t\right)\nu(dz) = 0.
\end{aligned} \tag{4.34}$$

The maximum is reached due to the concavity of the differential input u. To obtain the above equation, we need to differentiate under an integral sign. This step is justified by the generalized Leibniz Rule (See Theorem 24.5 in [2]). In fact, it is sufficient to check that for fixed t and ω, the function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined as

$$f(\tilde{\pi}, z) = \psi(t,z)\frac{\theta(t,z)}{1+\delta_t\theta(t,z)} u_x\left(\tilde{X}_{t-} + (\tilde{\pi} - \delta_t\tilde{X}_{t-})\frac{\theta(t,z)}{1+\delta_t\theta(t,z)}, A_t\right)$$

is locally uniformly integrably bounded, i.e. for every $\tilde{\pi} \in \mathbb{R}$, there is a non-negative $L^1(\mathbb{R}, \nu)$ function $g : \mathbb{R} \to \mathbb{R}$, and a neighborhood of $\tilde{\pi}$ such that $|f(\tilde{\pi}, z)| \leq g(z)$ on that neighborhood. This is guaranteed by our assumptions on ψ, θ and δ and the differentiability of u. Finally, we postulate that $\tilde{\pi}^* = \delta\tilde{X}^* + \sigma^{-1}\zeta\tilde{R}^*$ and use this together with (4.34) to derive equation (4.31). Applying equation (4.30) and (4.31) to $v(\tilde{\pi}^*) = 0$, yields the representation (4.32). □

Equation (4.30) yields a remarkable but very intuitive representation for the optimal benchmarked portfolio $\tilde{\pi}^*$. The first term, depends functionally only on the benchmarked wealth process $\tilde{X}^*$, while the second term depends explicitly on the risk tolerance process $\tilde{R}^*$ and implicitly on the jump-relative marginal performance functional $\tilde{J}^*$.

Notice that, here, $\tilde{J}^*(t,z)[\zeta_t]$ depends functionally on ζ for any fixed t, z and ω. Hence, we do not have a closed form solution to equation (4.31). We can think of the last term on the right hand side of (4.31) as the expectation of the jump-relative marginal performance functional with different possible sizes of jump under the new measure $\rho(t,z)\nu(dz)$.

In general, processes ζ_t^* and a_t depend functionally on the optimal benchmarked wealth process $\tilde{X}_t^*$ and the process A_t in the feedback form. Consequently, using (4.27) and (4.30) yields the following system for $(\tilde{X}_t^*, A_t)$ which is usually impossible to solve:

$$\begin{cases} d\tilde{X}_t^* = \sigma_t^{-1}\zeta_t^*(\tilde{X}_{t-}^*, A_t)r(\tilde{X}_{t-}^*, A_t)dm_t, \\ dA_t = a_t(\tilde{X}_{t-}^*, A_t)dt, \end{cases} \tag{4.35}$$

herein the process m is given by

$$\begin{aligned} m_t = \int_0^t \big(\mu_s - r_s - \sigma_s^2\delta_s - \int_{\mathbb{R}} \frac{\delta_s\theta^2(s,z)}{1+\delta_s\theta(s,z)}\big)ds + \int_0^t \sigma_s dW_s \\ + \int_0^t \int_{\mathbb{R}} \frac{\theta(s,z)}{1+\delta_s\theta(s,z)}\tilde{N}(ds,dz). \end{aligned}$$

This observation leads us to study the problem under certain weaker assumptions in the following section.

4.4 Sufficient conditions: the case of time-independent linear local risk tolerance

The form of (4.32) suggests that the process A will be considerably simplified if the u-dependent coefficient in (4.32) is constant, specially, if u satisfies

$$u_t = \frac{1}{2}\frac{u_x^2}{u_{xx}} \tag{4.36}$$

for all $x \in (c, \infty)$, $t \geq 0$.

In this section, we assume that this is the case. The following PDE can be found in [62].

Proposition 4.4.1. *The local risk tolerance, defined in (4.16), satisfies*

$$r_t + \frac{1}{2}r^2 r_{xx} = 0 \quad \text{and} \quad r(x,0) = -\frac{u_0'(x)}{u_0''(x)}. \tag{4.37}$$

We then have the following lemma.

Lemma 4.4.2. *The local risk tolerance, defined in (4.16) is time-independent if and only if it is linear, i.e.*

$$r(x,t) = \alpha x + \beta, \quad (x,t) \in (c,\infty) \times [0,\infty), \tag{4.38}$$

for $\alpha > 0$, $\beta \in \mathbb{R}$ *or* $\alpha = 0$, $\beta > 0$. *If* $\alpha > 0$, *we let* $c = -\beta/\alpha$ *and if* $\alpha = 0$, *we simply let* $c = -\infty$.

The following result can be easily proved.

Lemma 4.4.3. *Let the local risk tolerance function r be as in (4.38). Then, it yields only three types of solutions to the differential input u :*

i) If $\alpha = 0$ and $\beta > 0$, then $u^e : \mathbb{R} \times [0, \infty) \to \mathbb{R}$ is given by

$$u^e(x,t) = -e^{-\frac{x}{\beta}+\frac{t}{2}}, \tag{4.39}$$

ii) If $\alpha \neq 1$ ($\alpha > 0$) and $\beta \in \mathbb{R}$, then $u^p : \langle c, \infty) \times [0, \infty) \to \mathbb{R}$ is given by

$$u^p(x,t) = \frac{1}{\gamma}(x-c)^\gamma e^{-\frac{1}{2}\frac{\gamma}{1-\gamma}t}, \quad \textit{for} \quad \gamma = \frac{\alpha-1}{\alpha}, \tag{4.40}$$

iii) If $\alpha = 1$ and $\beta \in \mathbb{R}$, then $u^l : (c, \infty) \times [0, \infty) \to \mathbb{R}$ is given by

$$u^l(x,t) = \log(x-c) - \frac{t}{2} \tag{4.41}$$

where $c = -\beta/\alpha$.

The solutions in (4.39), (4.40) and (4.41) are unique up to an affine transformation.

Lemma 4.4.4. *Let the local risk tolerance function r be as in (4.38). Let, also, the jump size of the benchmarked wealth s be proportional to the local risk tolerance r, namely, $s = \kappa r(x,t)$, for some constant κ such that $x + \kappa r(x,t) > c$. Then, the local jump-relative marginal performance function j and the local jump-time impatience function i, defined in (4.16) and (4.17), are independent of (x,t), namely,*

$$j(x, \kappa r(x,t), t) := J(\kappa) \quad \textit{and} \quad i(x, \kappa r(x,t), t) := I(\kappa).$$

Furthermore,

$$J(\kappa) = \begin{cases} j^e(x, \kappa r(x,t), t) = e^{-\kappa} & \kappa \in \mathbb{R}, \\ j^p(x, \kappa r(x,t), t) = (1+\alpha\kappa)^{\gamma-1} & 1+\alpha\kappa > 0, \\ j^l(x, \kappa r(x,t), t) = \dfrac{1}{1+\kappa} & 1+\kappa > 0, \end{cases} \tag{4.42}$$

and $I'(\kappa) = \frac{1}{2}J(\kappa)$.

As a consequence, the jump-relative marginal performance and the jump-time impatience functionals (at optimality), defined as in (4.28) and (4.29) reduce to

$$\tilde{J}^*(t,z)[\zeta_t] = J\left(\frac{\sigma_t^{-1}\zeta_t\theta(t,z)}{1+\delta_t\theta(t,z)}\right) \quad \text{and} \quad \tilde{I}^*(t,z)[\zeta_t] = I\left(\frac{\sigma_t^{-1}\zeta_t\theta(t,z)}{1+\delta_t\theta(t,z)}\right).$$

In turn, the solution ζ_t^* to (4.31) and the process a_t given by (4.32) are no longer depending on $(\tilde{X}_t^*, A_t)$. Therefore, the system (4.35) for $(\tilde{X}_t^*, A_t)$ simplifies and the underlying mathematical problem is considerably easier.

Naturally, the next question that arises is the existence and uniqueness of the solution to (4.31). Notice that (4.31) is posed pointwise. We can, alternatively, formulate (4.31) as a fixed point problem. To this end, let $\mathcal{E} \subseteq \mathbb{R}$ and define the mapping $w : \mathbb{R}^+ \times \Omega \times \mathcal{E} \to \mathbb{R}$ by

$$\begin{aligned} w(t,\omega,\zeta) = \sigma_t^{-1}(\mu_t - r_t) - \sigma_t\delta_t + \phi_t - \sigma_t^{-1}\Bigg[\int_{\mathbb{R}} \theta(t,z)\nu(dz) \\ - \int_{\mathbb{R}} J\left(\frac{\sigma_t^{-1}\zeta\theta(t,z)}{1+\delta_t\theta(t,z)}\right)\frac{\psi(t,z)\theta(t,z)}{1+\delta\theta(t,z)}\nu(dz)\Bigg] \end{aligned} \tag{4.43}$$

where J is as in (4.42).

Note that, solving (4.31) pointwise is now equivalent to finding, for fixed (t, ω), a fixed point for (4.43).

We recall that, $dt \otimes \mathbb{P}$-a.e., ζ_t^* has to be chosen so that the optimal (benchmarked) wealth process $\tilde{X}_t^*$ remains in the effective domain of the differential input u, i.e. $\tilde{X}_t^* > c$ (equivalently, $X_t^* > c_t$ with $c_t = cY_t$). In the case with exponential differential input (cf. (4.39)), where $c = -\infty$, this is always the case. However, in the case with power and logarithmic inputs (cf. (4.40) and (4.41)), where $c \in \mathbb{R}$, we require that $dt \otimes \nu(dz) \otimes \mathbb{P}$-a.e.,

$$1 + \frac{\alpha \sigma_t^{-1} \zeta_t \theta(t, z)}{1 + \delta_t \theta(t, z)} > 0.$$

Given the assumptions on σ_t, δ_t and θ_t, we find that only nonnegative and bounded solution of ζ_t^* will satisfy this.

We need the following hypothesis for the existence and uniqueness of the fixed point. Let constants C and D be, respectively,

$$C = \operatorname*{ess\,inf}_{(t,\omega) \in \mathbb{R}^+ \times \Omega} w(t, \omega, 0) \quad \text{and} \quad D = \operatorname*{ess\,sup}_{(t,\omega) \in \mathbb{R}^+ \times \Omega} w(t, \omega, 0)$$

with w as in (4.43).

Hypothesis (H3)

The essential infimum $C > 0$. Furthermore, in the case with power and logarithmic differential inputs, the essential supremum $D < \alpha^{-1} \varepsilon_1 \varepsilon_2$.

Proposition 4.4.5. *Let (H1)-(H3) hold. The mapping w given by (4.43) admits a unique fixed point ζ^*, $dt \otimes \mathbb{P}$-a.e.. Furthermore, ζ^* is nonnegative and bounded.*

Proof. We first show that for fixed (t, ω), w is decreasing as a function of ζ. This can be easily seen since

$$w'(\zeta) = \frac{1}{\sigma^2} \int_{\mathbb{R}} J' \left(\frac{\sigma^{-1}\zeta\theta(z)}{1 + \delta\theta(z)} \right) \frac{\psi(z)\theta^2(z)}{(1 + \delta\theta(z))^2} \nu(dz) \leq 0,$$

where we used that $\psi \geq 0$ and J is decreasing. In the above equation, we skip all t subscripts and t, ω variables in order to simplify notations.

Note that, due to the boundedness and integrability conditions for the market coefficients (cf. (H1) and (H2)), C and D are finite. Next, $dt \otimes \mathbb{P}$-a.e., we restrict w on the interval $[0, D]$. In the square of $[0, D] \times [0, D]$, the graph of w starts at the point $(0, w(0))$, where $w(0) \geq C > 0$, and decreases with respect to ζ. Therefore, it must intersect only once with the lower-left to upper-right diagonal inside of the square. The intersection yields a unique fixed point ζ^* such that $0 < \zeta^* < D$ $dt \otimes \mathbb{P}$-a.e.. Moreover, in the case with power and logarithmic differential inputs ($\alpha > 0$), the additional assumption $D < a^{-1}\varepsilon_1\varepsilon_2$ ensures that $0 < \zeta^* < \alpha^{-1}\varepsilon_1\varepsilon_2$, which is a sufficient condition to deduce $1 + \frac{\alpha\sigma^{-1}\zeta^*\theta}{1+\delta\theta} > 0$. □

So far, we have showed that there exists a unique positive and bounded *pointwise* solution to (4.31). The next corollary establishes the measurability of ζ.

Corollary 4.4.6. *There exists an $\mathcal{F}_t$-adapted and predictable modification for the process ζ^* constructed in Proposition 4.4.5.*

Proof. We define a mapping $\tilde{w} : \mathbb{R}^+ \times \Omega \times [0, D] \to \mathbb{R}$ by

$$\tilde{w}(t, \omega, \zeta) = -\left(w(t, \omega, \zeta) - \zeta\right)^2 \tag{4.44}$$

with w as in (4.43). If we endow $\mathbb{R}^+ \times \Omega$ with $\mathcal{P}$, the σ-algebra of predictable sets, then it is easy to check that $\tilde{w}$ is a Carathéodory function (measurable w.r.t $\mathcal{P}$ and continuous in variable ζ). Herein, we are using the fact that all market coefficients are predictable and the function J is continuous.

Next, let $\varphi : \mathbb{R}^+ \times \Omega \twoheadrightarrow [0, D]$ be a trivial correspondence. We define the correspondence $l : \mathbb{R}^+ \times \Omega \twoheadrightarrow [0, D]$ as the set of all maximizers of $\tilde{w}$ as in (4.44), namely,

$$l(t, \omega) = \left\{ \hat{\zeta} \in \varphi(t, \omega) : \tilde{w}(t, w, \hat{\zeta}) = \max_{\zeta \in \varphi(t,\omega)} \tilde{w}(t, w, \zeta) \right\}. \tag{4.45}$$

Using the Measurable Maximum Theorem (Theorem 17.18 in [1]), we have that, since the set $[0, D]$ is compact, the correspondence l admits a $\mathcal{P}$-measurable selector $\hat{\zeta}^*$. On the other hand, in Proposition 4.4.5, we have showed that $dt \otimes \mathbb{P}$-a.e., the pointwise solution ζ^* uniquely maximizes $\tilde{w}$. Hence ζ^* and the measurable selector $\hat{\zeta}^*$ coincide with each other a.e.. In other words, $\hat{\zeta}^*$ is a modification for ζ^*.

The adaptedness could be proved in the same fashion as long as we fix t, endow Ω with σ-algebra $\mathcal{F}_t$ and formulate a similar parametric constrained maximization problem as (4.45). □

Corollary 4.4.7. *Let (H1)-(H3) hold and the local risk tolerance function* r *be given by (4.38). Then, the process* a_t *given by (4.32) is positive* $dt \otimes \mathbb{P}$*-a.e.. Consequently, the process* A_t *given by (4.20) is an increasing and absolutely continuous process.*

Proof. We consider the process a_t as a function of ζ_t for fixed (t, ω). Using Lemma 4.4.4, we can easily show that $a'(\zeta) \geq 0$ and $a(0) = 0$. □

In the rest of this section, we will use the adapted and predictable modification for ζ^*. We denote it by ζ^* for simplicity.

Theorem 4.4.8 (Sufficient Conditions)**.** *Let (H1)-(H3) hold and the local risk tolerance function* r *be given by (4.38). Let, also, the stochastic inputs* Y_t *and* Z_t *be defined as in (4.18) and (4.19) and the wealth process* X_t *as in (4.6), for all* $t \geq 0$*. Moreover, let* ζ_t^* *be the adapted and predictable solution to (4.31) and the process* A_t *be as in (4.20) with* a_t *given in (4.32).*

Then, the process $U_t(x)$ *defined by (4.14) with effective domain* $\langle cY_t, \infty)$ *and differential input* u *as in (4.39), (4.40) and (4.41) is a forward performance. The optimal (benchmarked) portfolio* $\pi_t^* \in \mathcal{A}$ *exists and is represented by (4.30).*

Proof. We need to establish the validity of conditions (i)-(v) in Definition 4.2.1. Conditions (i), (iii) and (v) are immediate by construction. To prove the integrability of $U_t(X_t)^+$, we employ a similar argument as in Theorem 4 in [62], according to which it is sufficient to show that, for any admissible strategy $\pi \in$

$\mathcal{A}$, $\mathbb{E}_{\mathbb{P}}[(X_s^\pi)^2] < \infty$ and $\mathbb{E}_{\mathbb{P}}[(\frac{Z_t}{Y_t})^2] < \infty$. This is guaranteed by the assumption on the market coefficients. It remains to show that the optimal portfolio π^* belongs to the admissible set $\mathcal{A}$. The adaptedness and predictability of the optimal strategy is immediate due to the representations (4.21) and (4.30). The integrability again is trivial given the boundedness assumptions on the market coefficients.

In the case with exponential differential input, we first define an equivalent measure $\mathbb{Q}^1$, on $[0, T]$ by

$$\frac{d\mathbb{Q}^1}{d\mathbb{P}}\Big|_{\mathcal{F}_T} = \exp\left(-\int_0^T \frac{\mu_s - r_s}{\sigma_s} dW_s - \frac{1}{2}\int_0^T \left(\frac{\mu_s - r_s}{\sigma_s}\right)^2 ds \right).$$

Notice that $\mathbb{Q}^1$ is a martingale measure for the discounted price process $\hat{S} = B^{-1}S$, and under which the optimal benchmarked wealth process $\tilde{X}^*$ (cf. (4.27) and (4.35)) solves

$$\begin{aligned} d\tilde{X}_t^* = \beta\sigma_t^{-1}\zeta_t\Big(\Big(-\sigma_t^2\delta_t - \int_{\mathbb{R}} \frac{\delta_t\theta^2(t,z)}{1+\delta_t\theta(t,z)}\nu(dz)\Big)dt + \sigma_t dW_t^{\mathbb{Q}^1} \\ + \int_{\mathbb{R}} \frac{\theta(t,z)}{1+\delta_t\theta(t,z)}\tilde{N}^{\mathbb{Q}^1}(dt,dz)\Big) \end{aligned}$$

where

$$W_t^{\mathbb{Q}^1} = W_t + \int_0^T \frac{\mu_s - r_s}{\sigma_s} ds \quad \text{and} \quad \tilde{N}^{\mathbb{Q}^1}(dt,dz) = \tilde{N}(dt,dz).$$

Recall that the benchmark process Y_t is a positive martingale under $\mathbb{Q}^1$ with $Y_0 = 1$. We therefore define another equivalent measure $\mathbb{Q}^2$, on $[0, T]$ by

$$\frac{d\mathbb{Q}^2}{d\mathbb{Q}^1}\Big|_{\mathcal{F}_T} = Y_T.$$

Under the measure $\mathbb{Q}^2$, $\tilde{X}^*$ solves

$$d\tilde{X}_t^* = \beta\sigma_t^{-1}\zeta_t\Big(\sigma_t dW_t^{\mathbb{Q}^2} + \int_{\mathbb{R}} \frac{\theta(t,z)}{1+\delta_t\theta(t,z)}\tilde{N}^{\mathbb{Q}^2}(dt,dz)\Big)$$

where

$$W_t^{\mathbb{Q}^2} = W_t^{\mathbb{Q}^1} - \int_0^T \delta_s\sigma_s ds \quad \text{and} \quad \tilde{N}^{\mathbb{Q}^2}(dt,dz) = \tilde{N}^{\mathbb{Q}^1}(dt,dz) - \delta_t\theta(t,z)\nu(dz)dt.$$

We can easily show that $\tilde{X}^*$ is a $\mathbb{Q}^2$-martingale. Observe that, for $0 \leq s < t \leq T$,

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}^1}[X_t^*|\mathcal{F}_s] &= \mathbb{E}_{\mathbb{Q}^1}[\tilde{X}_t^* Y_t|\mathcal{F}_s] = Y_s\mathbb{E}_{\mathbb{Q}^2}[\tilde{X}_t^*|\mathcal{F}_s] \\ &= Y_s\tilde{X}_s^* = X_s^*.\end{aligned}$$

This implies that X^* is a $\mathbb{Q}^1$-(super)martingale.

In the case with power and logarithmic differential inputs, we have $\tilde{X}_t^* > c$ or $X_t^* + cY_t > 0$, $dt \otimes \mathbb{P}$-a.e.. Under any equivalent martingale measure $\mathbb{Q}$ for the discounted price process, Y_t is a $\mathbb{Q}$-martingale and X_t^* is a $\mathbb{Q}$-local martingale. Therefore, the process $X_t^* + cY_t$ is a positive $\mathbb{Q}$-local martingale, and also a $\mathbb{Q}$-supermartingale. Removing the martingale part cY_t, yields that X_t^* is a $\mathbb{Q}$-supermartingale. □

4.5 Examples: the case of constant market coefficients

In this section, we present examples of forward performance processes when the market coefficients are taken to be constants, namely, $r_t \equiv r$, $\mu_t \equiv \mu$, $\sigma_t \equiv \sigma$ and $\theta(t,z) = z$ for all $t \geq 0$. Without loss of generality, we also take $Y_t = Z_t \equiv 1$. In this case, (4.14) yields that the forward performance process $U_t(x)$ simplifies to a deterministic function.

The construction of forward performance processes is, then, equivalent to choosing a proper constant a (cf. (4.20)) such that there exists a classical solution (not necessarily unique), which does not depend on the market inputs μ, r, σ and $\nu(dz)$, to the following integro-differential equation

$$\begin{cases} u_t a + \sup_{\pi} \Big(u_x \pi(\mu - r - \int_{\mathbb{R}} z\nu(dz)) + \frac{1}{2} u_{xx} (\pi)^2 \sigma^2 \\ \qquad\qquad + \int_{\mathbb{R}} (u(x + z\pi, t) - u(x,t))\nu(dz) \Big) = 0 \\ u(x, 0) = u_0(x). \end{cases} \tag{4.46}$$

Now we revisit hypotheses (H1)-(H3). (H1) and (H2) hold trivially while (H3) simplifies to $\mu - r > 0$. Moreover, (H3) requires $\mu - r < \alpha^{-1}\sigma^2$ in the case with power and logarithmic differential inputs.

4.5.1 Exponential initial datum

For $u_0(x) = -\exp(-\frac{x}{\beta})$, $\beta > 0$, the differential input is given by

$$u(x,t) = -\exp\left(-\frac{x}{\beta} + \frac{1}{2}t\right).$$

The optimal investment strategy is

$$\pi_t^* = \beta\sigma^{-1}\zeta^*,$$

where ζ^* solves

$$\sigma\zeta = \mu - r - \int_{\mathbb{R}} z(1 - e^{-z\sigma^{-1}\zeta})\nu(dz).$$

The constant a appearing in (4.46) should be chosen as

$$a = (\zeta^*)^2 - 2\sigma^{-1}\zeta^* \int_{\mathbb{R}} z e^{-z\sigma^{-1}\zeta^*}\nu(dz) + 2\int_{\mathbb{R}} \left(1 - e^{-z\sigma^{-1}\zeta^*}\right)\nu(dz).$$

Under assumption $\mu - r > 0$, we can easily show that $a > 0$ and $0 < \zeta^* < \frac{\mu - r}{\sigma}$ (or equivalently, $0 < \pi^* < \frac{\beta(\mu - r)}{\sigma^2}$). Hence, we optimally invest a positive amount of our wealth in the risky asset, if the excess return is positive. Furthermore, we notice that $\frac{\beta(\mu-r)}{\sigma^2}$ is the optimal amount we should invest when the risky asset is modelled by a continuous Itô process under exponential forward performance (cf. Corollary 5.1 in [85]). When jumps are allowed in the model, the corresponding optimal strategy does not exceed the amount $\frac{\beta(\mu-r)}{\sigma^2}$.

4.5.2 Power initial datum

For $u_0(x) = \frac{1}{\gamma} x^\gamma$, where $\gamma = \frac{\alpha - 1}{\alpha}$ and $\alpha \neq 1$ ($\alpha > 0$), the differential input is given by

$$u(x,t) = \frac{1}{\gamma} x^\gamma e^{-\frac{1}{2}\frac{\gamma}{1-\gamma}t}.$$

The optimal investment strategy is given in the feedback form by

$$\pi_t^* = \frac{1}{1-\gamma} \sigma^{-1} \zeta^* X_t^*,$$

where ζ^* solves

$$\sigma\zeta = \mu - r - \int_{\mathbb{R}} z \Big(1 - \Big(1 + \frac{z\sigma^{-1}\zeta}{1-\gamma} \Big)^{\gamma - 1} \Big) \nu(dz).$$

The constant a should be chosen as

$$\begin{aligned} a = (\zeta^*)^2 - 2\sigma^{-1}\zeta^* \int_{\mathbb{R}} z \Big(1 + \frac{z\sigma^{-1}\zeta^*}{1-\gamma} \Big)^{\gamma-1} \nu(dz) \\ +2 \int_{\mathbb{R}} \frac{1-\gamma}{\gamma} \Big(\Big(1 + \frac{z\sigma^{-1}\zeta^*}{1-\gamma} \Big)^{\gamma} - 1 \Big) \nu(dz). \end{aligned}$$

Under assumptions $\mu - r > 0$ and $\mu - r < \alpha^{-1}\sigma^2$, we can easily show that $a > 0$ and $0 < \zeta^* < \frac{\mu-r}{\sigma} < \alpha^{-1}\sigma$ (or equivalently, $0 < \pi^* < \frac{\mu-r}{(1-\gamma)\sigma^2}x$). In this case, $\frac{\mu-r}{(1-\gamma)\sigma^2}$ is the optimal percentage of our wealth we should invest when the risky asset is modelled by a continuous Itô process under power forward performance (cf. Corollary 5.2 in [85]).

4.5.3 Logarithmic initial datum

For $u_0(x) = \log(x)$, the differential input is given by

$$u(x,t) = \log(x) - \frac{1}{2}t.$$

The optimal investment strategy is given in the feedback form by

$$\pi_t^* = \sigma^{-1}\zeta^* X_t^*,$$

where ζ^* solves

$$\sigma\zeta = \mu - r - \int_{\mathbb{R}} \frac{z^2\zeta}{1+z\zeta}\nu(dz).$$

The constant a should be chosen as

$$a = (\zeta^*)^2 - 2\sigma^{-1}\zeta^* \int_{\mathbb{R}} \frac{z}{1+z\zeta^*}\nu(dz) + 2\int_{\mathbb{R}} \log(1+z\zeta^*)\nu(dz).$$

Under assumptions $\mu - r > 0$ and $\mu - r < \sigma^2$, we can easily show that $a > 0$ and $0 < \zeta < \frac{\mu-r}{\sigma} < \sigma$ (or equivalently, $0 < \pi^* < \frac{\mu-r}{\sigma^2}x$). In this case, $\frac{\mu-r}{\sigma^2}$ is the optimal percentage of our wealth we should invest when the risky asset is modelled by a continuous Itô process under logarithmic forward performance (cf. Corollary 5.3 in [85]).

Appendix A

Proof of Proposition 2.5.1

In this Appendix, we provide the proof of Proposition 2.5.1. We start with a well established comparison result.

Theorem (A Phragmèn-Lindelöf Principle)**.** *Let D be an unbounded subdomain of $\mathbb{R}^n$ and $\mathcal{L}$ the uniformly parabolic operator in $D \times (0, T]$*

$$\mathcal{L}u \equiv \sum_{i,j=1}^{n} a_{ij}(x,t) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x,t) \frac{\partial u}{\partial x_i} + c(x,t)u - \frac{\partial u}{\partial t}.$$

Assume that there are constants C_1, C_2, such that

$$|x|^{-2} \sum_{i,j=1}^{n} a_{ij} x_i x_j \le C_1, \quad (1 + |x|^2)^{-1} \sum_{i=1}^{n} b_i x_i \le C_2,$$

and that c satisfies $c(x,t) < K(1 + |x|^{2\delta})$, for $K > 0$ and $0 < \delta < 1$. Also, assume that $\mathcal{L}u \ge 0$ in $D \times (0, T]$ and that

$$u(x,t) \le A \exp\left[b\, |x|^2 \right], \quad \text{when } |x| \to \infty \tag{A.1}$$

for positive constants A and b. Then, if $u(x,0) \le 0$ for $x \in \overline{D}$ and $u \le 0$ on $\partial D \times (0, T]$, $u \le 0$ in $\overline{D} \times [0, T]$.

We refer the reader to Theorem 3.10 and Remark (i) following that Theorem in [69]. In analogy with the result for elliptic operators, we call this a Phragmèn-Lindelöf Principle.

In order to show the existence and uniqueness of a positive bounded classical solution for (2.29), we work, equivalently, with the following Cauchy problem

$$\begin{cases} u_t - \tilde{\mathcal{L}}^{mm} u - c(y)u = f(y,t,u) \\ u(y,0) = 1 \end{cases} \tag{A.2}$$

where

$$u(y, T-t) = \Phi(y,t) e^{\gamma \int_t^T (1-\rho^2) m P(s) ds}. \tag{A.3}$$

Herein,

$$\tilde{\mathcal{L}}^{mm} = \frac{1}{2} a^2(y, T-t) \frac{\partial^2}{\partial y^2} + (b(y,T-t) - \rho \frac{\mu(T-t)}{\sigma(T-t)} a(y,T-t)) \frac{\partial}{\partial y},$$

and

$$c(y) = -(1-\rho^2)\lambda(y),$$

$$f(y,t,u) = \lambda(y)(1-\rho^2) e^{\gamma \int_{T-t}^T m P(s) ds} u^{-\frac{\rho^2}{1-\rho^2}}. \tag{A.4}$$

Notice that since Φ (or u) will give us the solution to the optimization problem, we are only interested in the positive and bounded solution for the Cauchy problem (A.2). Hence the growth condition (A.1) in the comparison principle is automatically satisfied.

A.1 Uniqueness of solution

We start with establishing that if a positive bounded solution exists, then it is unique. Note, that, in general, the nonlinear term $f(y,t,u)$ is *not* Lipschitz continuous in the variable u, because $-\frac{\rho^2}{1-\rho^2} < 1$.

We define, for all $(y,t) \in \mathbb{R} \times [0,T]$, the smooth functions $\underline{u}(y,t) \equiv 1$ and $\overline{u}(y,t) = e^{\gamma \int_{T-t}^{T} (1-\rho^2) m P(s) ds}$. Then,

$$\begin{cases} \overline{u}_t - \tilde{\mathcal{L}}^{mm} \overline{u} - c(y)\overline{u} \geq f(y,t,\overline{u}) \\ \overline{u}(y,0) = 1 \end{cases} \tag{A.5}$$

and

$$\begin{cases} \underline{u}_t - \tilde{\mathcal{L}}^{mm} \underline{u} - c(y)\underline{u} \leq f(y,t,\underline{u}) \\ \underline{u}(y,0) = 1. \end{cases} \tag{A.6}$$

In other words, $\overline{u}$ (resp. $\underline{\mathrm{u}}$) is a supersolution (resp. subsolution) to problem (A.2). Moreover, $\underline{u} \leq \overline{u}$ for all $(y,t) \in \mathbb{R} \times [0,T]$.

Next, we assume for the moment (and establish in the sequel) that a positive classical solution, denoted by Φ, exists. Then, from (2.36), we deduce that the positive bounded classical solution F also exists for the quasilinear equation (2.35). We define $\widetilde{F}$ as the classical solution to

$$\begin{cases} \tilde{F}_t + \mathcal{L}^{mm} \tilde{F} - (\gamma m P(t) + \lambda(y))\tilde{F} + \lambda(y) = 0 \\ \tilde{F}(y,T) = 1. \end{cases}$$

Using the comparison principle, we, easily, obtain $F \leq \tilde{F} \leq 1$.

This implies that $\Phi \leq 1$ and, in turn, $u \leq \overline{u}$.

Using the latter inequality, which is equivalent to $c(y)u + f(y,t,u) \geq 0$, the comparison principle yields $\underline{u} \leq u$.

We then see that as long as the bounded classical solution exists, for $(y,t) \in \mathbb{R} \times [0,T]$, we have

$$1 \leq u(y,t) \leq e^{\gamma \int_{T-t}^{T} (1-\rho^2) m P(s) ds}. \tag{A.7}$$

Notice that the estimates (2.31) for Φ will follow. Hence $f(y,t,u)$ is locally Lipschitz in the variable u when $u \in [\underline{u}, \overline{u}]$, because u is bounded away from zero.

If there are two positive classical solutions u^1 and $u^2 \in [\underline{u}, \overline{u}]$, we define $\widehat{u} = u^2 - u^1$ and we observe that $\widehat{u}$ satisfies the following PDE

$$\begin{cases} \widehat{u}_t - \tilde{\mathcal{L}}^{mm}\widehat{u} - c(y)\widehat{u} = (f(y,t,u^2) - f(y,t,u^1)) = \widehat{c}(y,t)\widehat{u} \\ \widehat{u}(y,0) = 0 \end{cases}$$

where

$$\widehat{c}(y,t) = \int_0^1 \frac{\partial f}{\partial s}(y,t,u^1 + s(u^2 - u^1))ds.$$

Note that $\widehat{c}(y,t)$ is well defined, due to the fact that $f(y,t,u)$ is locally Lipschitz in u if $u \in [\underline{u}, \overline{u}]$. We can apply the comparison principle to easily deduce that $\widehat{u} \equiv 0$.

A.2 Existence of solution

The approach we use to establish the existence of the positive classical solution is based on the Monotone Iterative Method (see, among others, Chapter 7 in [65]).

The main difficulties for the problem at hand are that in (A.2), the coefficient $b(y,t)$ and $c(y)$ are unbounded in y and the nonlinear term $f(y,t,u)$ is not globally Lipschitz continuous in variable u. To address these difficulties, we work as follows.

We construct a monotone sequence which lies between the ordered subsolution and supersolution and show that the sequence converges uniformly to the classical solution of the original problem. Recall that $f(y,t,u)$ is locally Lipschitz in u when $u \in [\underline{u}, \overline{u}]$. As a matter of fact, herein, there exist positive constants $\underline{C}$ and $\overline{C}$ such that

$$-\underline{C}\lambda(y)(u^1 - u^2) \leq f(y,t,u^1) - f(y,t,u^2) \leq \overline{C}\lambda(y)(u^1 - u^2) \qquad \text{(A.8)}$$

for $\underline{u} \leq u^2 \leq u^1 \leq \overline{u}$ and $y \in \mathbb{R}$. Notice that the constants $\underline{C}$ and $\overline{C}$ are chosen independently of u^1, u^2 and time t.

We will outline the main steps of the proof as follows.

First, we construct the sequence $\{u^{(k)}\}$ from the iteration process,

$$\begin{cases} u_t^{(k)} - \mathcal{L}_c u^{(k)} = \tilde{f}(y,t,u^{(k-1)}) \\ u^{(k)}(y,0) = 1 \end{cases} \qquad \text{(A.9)}$$

with $u^{(0)}(y,t) = \overline{u}(y,t)$. We also define the function

$$\tilde{f}(y,t,u) \triangleq f(y,t,u) + \underline{C}\lambda(y)u \qquad \text{(A.10)}$$

and the operator

$$\mathcal{L}_c u \triangleq \tilde{\mathcal{L}}^{mm} u + [c(y) - \underline{C}\lambda(y)]\, u.$$

The above PDE (A.9) is linear. But the classical existence and uniqueness results for solution to the Cauchy problem for linear second-order parabolic equation (see, for example, the book of [31]) do not apply here. This is because for unbounded domain, we usually assume that all the coefficients of

$\mathcal{L}_c$ are bounded Hölder continuous and in certain Hölder spaces. Then, based on those assumptions, the fundamental solution can be constructed. In our problem, however, the first- and zero-order coefficients for the operator $\mathcal{L}_c$ are not bounded. Constructions of fundamental solution for equations with unbounded coefficients have been given, under various conditions in literature. We refer the reader to a recent paper [23] who generalize the classical Parametrix Method to construct the fundamental solution under the same assumptions as in our proposition.

In view of Assumptions (A2)-(A3) and Theorem 2.1 in Deck and Kruse, there exists a fundamental solution $\Gamma(y, t; \xi, s)$ associated with the parabolic operator $\frac{\partial}{\partial t} - \mathcal{L}_c$.

Moreover, (see Corollary 4.2 in the same paper), for all $\delta > 0$, there is a constant $K(\delta) > 0$ such that the fundamental solution satisfies

$$|\Gamma(y, t; \xi, s)| \leq \frac{K(\delta)}{(t-s)^{1/2}} \exp(-\frac{d|y-\xi|^2}{2(t-s)}) \exp(\delta\xi^2) \tag{A.11}$$

where $d \in (0, 1)$ chosen independently of δ.

Notice that, for $u \in [\underline{u}, \overline{u}]$, the nonhomogeneous term $\tilde{f}(y, t, u(y, t))$ is locally Hölder continuous in y, uniformly with respect to t. We also observe that, because of (A.4), $\tilde{f}(y, t, u(y, t))$ satisfies a quadratic exponential growth condition respect to y, namely,

$$|\tilde{f}(y, t, u(y, t))| \leq A \exp(b|y|^2) \tag{A.12}$$

for some constant $A > 0$ and $b < \frac{d}{2T}$.

We, then, conclude that the linear equation (A.9) admits a unique classical solution which can be represented as (see, Section 2.1 of [65])

$$u^{(k)}(y,t) = \int_{\mathbb{R}} \Gamma(y,t;\xi,0)d\xi + \int_0^t ds \int_{\mathbb{R}} \Gamma(y,t;\xi,s)\tilde{f}(u^{(k-1)})(\xi,s)d\xi. \tag{A.13}$$

Theorem 2.2 in [23] assures that the solution $u^{(k)}$ satisfies the same quadratic exponential growth condition as (A.12).

In what follows we prove that, for all $(y,t) \in \mathbb{R}\times[0,T]$, and $k = 1, 2, \cdots$,

$$\underline{u}(y,t) \le \cdots \le u^{(k)}(y,t) \le u^{(k-1)}(y,t) \le \cdots \le u^{(1)}(y,t) \le \overline{u}(y,t).$$

From (A.5) and (A.9), we have

$$\begin{cases} (\overline{u} - u^{(1)})_t - \mathcal{L}_c(\overline{u} - u^{(1)}) \ge 0 \\ (\overline{u} - u^{(1)})(y,0) = 0. \end{cases}$$

By the comparison principle, for all $(y,t) \in \mathbb{R} \times [0,T]$, $u^{(1)}(y,t) \le \overline{u}(y,t)$.

Similarly, from (A.6), (A.9) and condition (A.8), we obtain

$$\begin{cases} (u^{(1)} - \underline{u})_t - \mathcal{L}_c(u^{(1)} - \underline{u}) \ge f(y,t,\overline{u}) - f(y,t,\underline{u}) + \underline{C}\lambda(y)(\overline{u} - \underline{u}) \ge 0 \\ (u^{(1)} - \underline{u})(y,0) = 0. \end{cases}$$

Using again the comparison principle, we deduce that for all $(y,t) \in \mathbb{R} \times [0,T]$, $\underline{u}(y,t) \le u^{(1)}(y,t)$. Using induction, we can, easily, show $u^{(k-1)} \ge u^{(k)}, k = 2, 3, \cdots$ and $\underline{u} \le u^{(k)}, k = 1, 2, \cdots$.

Observe that $\{u^{(k)}\}$ is a monotone decreasing sequence, bounded below by $\underline{u}$, for every $(y,t) \in \mathbb{R} \times [0,T]$, Therefore, it converges to a function, $u(y,t)$, pointwisely. It remains to show that $u(y,t)$ belongs to $C^{2,1}(\mathbb{R} \times (0,T]) \cap C(\mathbb{R} \times [0,T])$ and is a solution to (A.2).

Next, we prove that for any compact set $B \subset \mathbb{R}$, $u^{(k)}(y,t)$ converges uniformly to $u(y,t)$ in $B \times [0,T]$. Some arguments are similar to the ones used in to Theorem 4.2.1 in [79], but we incorporate them for completeness. When $m, n \geq 1$, we have

$$|u^{(m)}(y,t) - u^{(n)}(y,t)| = \left| \int_0^t ds \int_{\mathbb{R}} \Gamma(y,t;\xi,s)(\tilde{f}(u^{(m-1)}) - \tilde{f}(u^{(n-1)}))(\xi,s) d\xi \right|.$$

From (A.11), we obtain

$$\begin{aligned}
&|u^{(m)}(y,t) - u^{(n)}(y,t)| \\
\leq\ & \int_0^t \int_{\mathbb{R}} \frac{K(\delta)}{(t-s)^{1/2}} \exp(-\frac{d|y-\xi|^2}{2(t-s)}) \exp(\delta\xi^2) |(\tilde{f}(u^{(m-1)}) - \tilde{f}(u^{(n-1)}))(\xi,s)| d\xi ds \\
\leq\ & \left(\int_0^t \int_{\mathbb{R}} \frac{K^p(\delta)}{(t-s)^{p/2}} \exp(-\frac{d|y-\xi|^2}{2(t-s)}) \right)^{\frac{1}{p}} \times \\
& \left(\int_0^t \int_{\mathbb{R}} |(\tilde{f}(u^{(m-1)}) - \tilde{f}(u^{(n-1)}))(\xi,s)|^q \exp(-\frac{d|y-\xi|^2}{2(t-s)}) \exp(q\delta\xi^2) d\xi ds \right)^{\frac{1}{q}} \\
\leq\ & C(\delta) \left(\int_0^t \frac{1}{(t-s)^{(p-1)/2}} ds \right)^{\frac{1}{p}} \times \\
& \left(\int_0^t \int_{\mathbb{R}} |(\tilde{f}(u^{(m-1)}) - \tilde{f}(u^{(n-1)}))(\xi,s)|^q \exp(-\frac{d|y-\xi|^2}{2(t-s)}) \exp(q\delta\xi^2) d\xi ds \right)^{\frac{1}{q}}
\end{aligned}$$

where p is a positive constant in $[1,3]$ and q is its conjugate ($\frac{1}{p} + \frac{1}{q} = 1$) and $C(\delta)$ is some constant depending on δ.

We, then, notice that, since $(p-1)/2 < 1$, then for all $0 \le t \le T$,

$$\int_0^t \frac{1}{(t-s)^{(p-1)/2}} ds < M_1$$

where M_1 is a constant independent of t. Moreover,

$$\begin{aligned}
& \int_0^t \int_{\mathbb{R}} |(\tilde{f}(u^{(m-1)}) - \tilde{f}(u^{(n-1)}))(\xi, s)|^q \exp(-\frac{d|y-\xi|^2}{2(t-s)}) \exp(q\delta\xi^2) d\xi ds \\
= & \int_0^T \int_{\mathbb{R}\setminus B_K} |(\tilde{f}(u^{(m-1)}) - \tilde{f}(u^{(n-1)}))(\xi, s)|^q \exp(-\frac{d|y-\xi|^2}{2(t-s)}) \exp(q\delta\xi^2) d\xi ds \\
& + \int_0^T \int_{\mathbb{R}\cap B_K} |(\tilde{f}(u^{(m-1)}) - \tilde{f}(u^{(n-1)})(\xi, s)|^q \exp(-\frac{d|y-\xi|^2}{2(t-s)}) \exp(q\delta\xi^2) d\xi ds
\end{aligned}$$

where B_K is the interval $[-K, K]$ with radius $K > 0$. Next, for any $\varepsilon > 0$, we choose a sufficiently large $K > 0$ and a sufficiently small δ, such that, for all $(y,t) \in B \times [0,T]$, we have

$$\begin{aligned}
& \int_0^T \int_{\mathbb{R}\setminus B_K} |(\tilde{f}(u^{(m-1)}) - \tilde{f}(u^{(n-1)}))(\xi, s)|^q \exp(-\frac{d|y-\xi|^2}{2(t-s)}) \exp(q\delta\xi^2) d\xi ds \\
\le & \ M_2 \int_0^T \int_{\mathbb{R}\setminus B_K} |\xi|^q \exp(-\frac{d|y-\xi|^2}{2(t-s)}) \exp(q\delta\xi^2) d\xi ds \le \frac{\varepsilon}{2}
\end{aligned}$$

where M_2 is a positive constant depending on $\underline{C}$ and $\overline{C}$. Here we are using the estimate (A.8) for f and the construction (A.10) for $\tilde{f}$. On the other hand, when n, m are sufficiently large and δ sufficiently small, the inequality

$$\begin{aligned}
& \int_0^T \int_{B_K} |\tilde{f}(u^{(m-1)}) - \tilde{f}(u^{(n-1)}))(\xi, s)|^q \exp(-\frac{d|y-\xi|^2}{2(t-s)}) \exp(q\delta\xi^2) d\xi ds \\
\le & \int_0^T \int_{B_K \cup B} |\tilde{f}(u^{(m-1)}) - \tilde{f}(u^{(n-1)}))(\xi, s)|^q d\xi ds \le \frac{\varepsilon}{2}
\end{aligned}$$

holds, uniformly for $(y,t) \in B \times [0,T]$, due to continuity in $L^q(B_K \cup B)$ space.

Thus $u^{(k)}(y,t)$ converges uniformly to $u(y,t)$ in $B \times [0,T]$. Therefore, u is a continuous function in $\mathbb{R} \times [0,T]$. Using once more the estimate (A.11) and the quadratic exponential growth condition (A.12), we have, for $|\xi|$ large enough,

$$|\Gamma(y,t;\xi,s)\tilde{f}(u^{(k)})(\xi,s)| \leq \frac{A \cdot K(\delta)}{(t-s)^{1/2}} \exp(-\frac{d|y-\xi|^2}{2(t-s)}) \exp(\delta\xi^2 + b\xi^2).$$

Again, because $b < \frac{d}{2T}$, we can choose sufficiently small δ such that

$$\int_0^t \int_{\mathbb{R}} \frac{A \cdot K(\delta)}{(t-s)^{1/2}} \exp(-\frac{d|y-\xi|^2}{2(t-s)}) \exp(\delta\xi^2 + b\xi^2) d\xi ds < +\infty.$$

Setting $k \to +\infty$ in the integral representation (A.13), the Dominated Convergence Theorem shows that the pointwise limit $u(y,t)$ satisfies the same integral equation, that is,

$$u(y,t) = \int_{\mathbb{R}} \Gamma(y,t;\xi,0) d\xi + \int_0^t ds \int_{\mathbb{R}} \Gamma(y,t;\xi,s)\tilde{f}(u)(\xi,s) d\xi.$$

Since u is continuous, then the function $\tilde{f}(u)(y,t) \in C(\mathbb{R} \times [0,T])$ is locally Hölder continuous in y, uniformly with respect to $t \in [0,T]$. By the well known volume potentials results (see, for example, Lemma 1.1 in [65]), we know that $u(y,t) \in C^{2,1}(\mathbb{R} \times (0,T])$ and $u_t - \mathcal{L}_c u = \tilde{f}(y,t,u)$. This concludes the proof that u is a classical solution to the Cauchy problem (A.2).

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Vita

Ti Zhou was born in Shanghai, China on February 9, 1981, the son of Qifeng Zhou and Shengli Zhou. In September 1999, He entered Fudan University in Shanghai and graduated four years later with a B.S. degree in Mathematics and Applied Mathematics. He continued his study from August 2003 onwards in the Department of Mathematics of the University of Texas at Austin. From January until August 2008, he worked as a quantitative analyst for BNP Paribas in London, England. He is expecting to receive the Ph.D. degree in Mathematic from the University of Texas at Austin in December 2008.

Permanent address: No. 2 Lane 460 Siping Rd
Apt. 2102
Shanghai, China 200086

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