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The De Giorgi's Method as Applied to The Regularity Theory for Incompressible Navier Stokes Equations

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This thesis is dedicated to Dr. Alexis Vasseur, without his insightful suggestions and constant support thoughout the last two years, the present author could have not finished his thesis in such a short period of time.

The De Giorgi's Method as Applied to The Regularity Theory for Incompressible Navier Stokes Equations

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The first part of this thesis is devoted to a regularity criterion for solutions of the Incompressible Navier-Stokes equations in terms of regularity of the solutions along the streamlines. More precisely, we prove that we can ensure the full regularity of a given suitable weak solution provided we have good control on the second derivative of the velocity along the direction of the streamlines of the fluid.

In the second part of this thesis, we will show that the global regularity of a suitable weak solution u for the incompressible Navier-Stokes equations holds under the condition that $|u|^5/(\log(1+|u|))$ is integrable in space time variables.

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Chapter 1

Introduction

1.1 The main objective of this thesis

The purpose of this thesis is to study the regularity of weak solutions for the incompressible Navier-Stokes equations by employing a method inherited from the work [8] of De Giorgi. To begin, let us say that the incompressible Navier-Stokes equations on $\mathbb{R}^3$ is the system of equations described in the following way.

$$\partial_t u - \triangle u + \operatorname{div}(u \otimes u) + \nabla p = 0, \tag{1.1}$$

$$\operatorname{div}(u) = 0, \tag{1.2}$$

where $u : (0,T) \times \mathbb{R}^3 \to \mathbb{R}^3$ is the velocity of the fluid, and $P : (0,T) \times \mathbb{R}^3 \to \mathbb{R}^1$ stands for the pressure of the fluid. At the writing of this thesis, establishing the full regularity for solutions of the incompressible Navier-Stokes equations (without any extra assumptions) is still a long standing open problem. The goal of this thesis, however, is to establish two modest regularity results under some natural assumptions imposed on the solutions under consideration. We now state the two main results obtained in this thesis.

First, we will establish the regularity of solutions for the incompressible Navier-Stokes equations under the assumption that the second derivative of the velocity of the fluid along the direction of the streamlines is regular enough. More precisely, we will prove that our (suitable) weak solution u for the incompressible Navier-Stokes equations is smooth on $(0,T)\times\mathbb{R}^3$, provided the condition $\frac{|u\cdot\nabla F|}{|u|^\gamma}\leqslant A|F|$ is satisfied on $(0,T)\times\mathbb{R}^3$, where $A\in(0,\infty)$ is some given positive constant, γ is a certain index with $\gamma\in(0,\frac{1}{3})$, and the symbol F stands for $F=\mathrm{div}(\frac{u}{|u|})$. To achieve this result, we first present some necessary background materials in Chapter 2. Then, we will eventually employ the method of De Giorgi to prove the above regularity result in Chapter 3. At this point, it is appropriate for the present author to point out that all the material in Chapter 3 of this thesis is taken directly from a piece of recent work [5] by the present author. So, the reader can consult the work [5] directly if he or she is only interested in the above regularity result.

Second, we present a piece of work [4], in which we gave a log improvement of the Prodi-Serrin criteria in the homogenuous case of $p=q=5$. More precisely, we proved that any suitable weak solution u for the incompressible Navier-Stokes equations is smooth, provided u satisfies the integrable condition $\int_0^\infty\int_{\mathbb{R}^3}\frac{|u|^5}{log(1+|u|)}dxdt<\infty$.

At this point, we should point out the fact that the two regularity results, as presented in this thesis, are obtained though the powerful De Giorgi's

Method. The De Giorgi's method was invented by De Giorgi in [8] for the purpose of establishing the regularity of solutions for linear elliptic equations with rough diffusion coefficients. A. Vasseur first employed the De Giorgi's Method in his paper [32] to give an elementary proof for the famous partial regularity theorem of Caffarelli, Kohn, and Nirenberg (see [3]). Indeed, the two regularity results of this thesis are inspired by [32].

1.2 A brief summary of the content of this disseration

In Chapter 2, we will present some often used function spaces, some classical theorems in the theory of singular integrals, the De Giorgi's Method, some important regularity criterion for solutions of the incompressible Navier-Stokes equations, and the notion of suitable weak solutions for incompressible Navier-Stokes equations. Those materials as presented in Chapter 2 are basically what we need in Chapter 3 and Chapter 4. Indeed, Chapter 2 is divided into four sections.

The purpose of section 2.1 is to collect some often used function spaces and some classical theorems in the theory of singular integrals. In particular, we first introduce the concept of weak derivatives and Sobolev's spaces. We then state the weak product rule and the weak chain rule for functions with weak derivatives, which are followed by the introduction of the Sobolev's embedding Theorem. After all these, we present the Zygmund-Calderon's theorem and the Riesz's transforms on Euclidean spaces, which are followed by the concept of BMO and the famous BMO theorem due to John and Nirenberg (see [15]). All these notions and theorems as presented in section 2.1 are considered to be classical today, and the readers can easily find them in advanced textbooks on analysis and P.D.E. .

In Section 2.2, we will state the famous De Giorgi's Theorem in the theory of linear elliptic equations and present half of it's justification, which is already good enough to illustrate those key ideas of the De Giorgi's method that we will actually employ in Chapters 3 and 4. At this point, we should mention that the present author is following the lecture notes given by L. Caffarelli in Fall 2007 closely while he is writing section 2.2. Besides this, since the present author learned the De Giorgi's method for the first time though the paper [32], the treatment given in [32] also has significant influence on the material of Section 2.2.

In Section 2.3 we describe those mathematical contributions to the regularity theory for the incompressible Navier-Stokes equations by Leray, Hopf, Prodi, Serrin, Ladyzhenskaya, and Kato. We emphasize that we are just collecting those fundamental results in the field of incompressible Navier-Stokes equations that are directly related to the subject matter of this thesis. So, it is not suprising that many other important results in the field will simply not get mentioned here.

In Section 2.4, we first introduce the important notion of suitable weak solutions of incompressible Navier-Stokes equations. Then, we will say something about the famous partial regularity theorem of Caffarelli, Kohn, and Nirenberg. Here, we explicity point out that all weak solutions of the incompressible Navier Stokes equations appearing in Chapters 3 and 4 are assumed to be suitable weak solutions. Because of this, it is better for us to clarify the concept of suitable weak solutions before it is employed in Chapters 3 and 4.

Sections 2.5 is devoted to the discussion for the regularity criteria we are going to establish in Chapter 4, while in section 2.6, we will discuss the regularity criteria we are going to establish in Chapter 3. In these two sections, the significance of the two regularity criterion in this thesis and the key ideas behind their proofs are discussed in some detail.

Finally, we mention that the whole Chapter 3 is devoted to establish the regularity criteria which ensures the full regularity of a suitable weak solution u, provided u satisfies the hypothesis that $|\frac{u\cdot\nabla F}{|u|^{\gamma}}| \leqslant A|F|$, in which $F = \mathrm{div}(\frac{u}{|u|})$, $A \in (0,\infty)$, and $\gamma \in (0,\frac{1}{3})$. Here, we also want to say that all the material of Chapter 3 are directly taken from a recent paper [5] by the present author.

On the other hand, Chapter 4 is devoted to the establishment of the second regularity criteria of this thesis which ensures the full regularity of a suitable weak solution u, provided u verifies the hypothesis that $\int_0^\infty \int_{\mathbb{R}^3} \frac{|u|^5}{log(1+|u|)} < \infty$. All the material in Chapter 4 is taken directly from the paper [**?**], which is a joint work of the present author and A. Vasseur.

Chapter 2

Background materials

2.1 Function spaces and some basic harmonic analysis

In this section, we will introduce the notion of Sobolev's spaces, and some basic results in the theory of singular operators. Many of these notions and results are classical and well-known. We intend in this chapter to present those concepts and results that are either closely related or used directly in the main portion of the thesis.

As usual, we start with the very classical definition for weak derivatives of locally integrable functions on $\mathbb{R}^n$.

Definition. Let U be an open subest of $\mathbb{R}^n$. Let $u \in L^1_{loc}(U)$, $v \in L^1_{loc}(U)$ be two given locally integrable functions defined on U. Let α be a given multi-index. We say that v is a α -th weak derivative of u, that is $v = \partial_\alpha u$, if it happens that, for every choice of $\phi \in C_c^\infty(U)$, we have $\int_U u \cdot \partial_\alpha \phi = (-1)^{|\alpha|} \int_U v \cdot \phi$.

At this point, let us remark that if v_1 , $v_2 \in L^1_{loc}(U)$ are both α- th weak derivatives of the same function $u \in L^1_{loc}(U)$. Then, it follows from the very definition of weak derivatives of locally integrable functions that

$\int_U v_1 \cdot \phi = (-1)^{|\alpha|} \int_U u \cdot \partial_\alpha \phi = \int_U v_2 \cdot \phi$, is valid for every $\phi \in C_c^\infty(U)$. Hence, it must be the case that $v_1 = v_2$ almost everywhere on U. This simple argument shows that the α-th weak derivative of a locally integrable function, provided it does exist, is uniquely determined up to a set of measure zero of U.

Next, we introduce the Sobolev's Spaces $W^{k,p}(U)$, in which $p \in [1, \infty]$, and k is a given nonnegative integer.

Definition. Let U be an open set in $\mathbb{R}^n$. let $p \in [1, \infty]$, and let k be an nonnegative integer. Then, the Sobolev's space $W^{k,p}(U)$ is the function space consisting of all those functions $u \in L^p(U)$ whose α -th weak derivatives exist and belong to the class $L^p(U)$, for any multi-index α with $|\alpha| \leqslant k$. That is , we define $W^{k,p}(U)$ to be $W^{k,p}(U) = \{ \ u \in L^p(U)$ such that $\partial_\alpha u$ exists in the weak sense and that $\partial_\alpha u \in L^p(U)$, for all α with $|\alpha| \leqslant k \ \}$.

For the Sobolev's spaces $W^{k,p}(U)$ appearing in the above definition, it is well-known that $W^{k,p}(U)$ becomes a Banach Space once it is equipped with the norm $\| \cdot \|_{W^{k,p}(U)}$ defined by $\|u\|_{W^{k,p}(U)} = \{\sum_{|\alpha| \leqslant k} \|D^\alpha u\|^p_{L^p(U)}\}^{\frac{1}{p}}$. Moreover, in the case when $p = 2$, people often use the symbol $H^k(U)$ to denote the Sobolev's space $W^{k,2}(U)$, since $H^k(U) = W^{k,2}(U)$ is a Hilbert space.

In a similar way, we can also define the local Sobolev's Spaces $W^{k,p}_{loc}(U)$ in the following way.

Definition. For any nonnegative integer $k \geqslant 1$, and any $p \in [1, \infty]$, $W^{k,p}_{loc}(U)$ is defined to be $\{u \in L^p_{loc}(U)$ such that $\partial_\alpha u$ exists in the weak sense, and that $\partial_\alpha u \in L^p_{loc}(U)$ for all α with $|\alpha| \leqslant k\}$

Before we introduce the Sobolev's embedding theorem, let us first state the weak product rule and the weak chain rule for functions with weak derivatives, which are the basic tools in the thoery of Sobolev's spaces.

Proposition 2.1.1. (Weak product rule) Let p , $q \in (1, \infty)$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. Suppose that $f \in W^{1,p}(\mathbb{R}^n)$ and $g \in W^{1,q}(\mathbb{R}^n)$. Then, it follows that $f \cdot g \in W^{1,1}(\mathbb{R}^n)$. Moreover, for every positive integer j with $1 \leqslant j \leqslant n$. We have $\partial_j(f \cdot g) = (\partial_j f) \cdot g + f \cdot \partial_j g$ to be valid, where $\partial_j(f \cdot g)$, $\partial_j f$, and $\partial_j g$ stand for the j-th first ordered weak derivatives of $f \cdot g$, f, and g respectively.

Proposition 2.1.2. (Weak chain rule) let $u : U \to \mathbb{R}^n$ be a measurable function with all its vector components $u_1, u_2, \cdots, u_n$ belong to the class $W^{1,p}_{loc}(U)$. That is, we have $u = (u_1, u_2, \cdots, u_n)$, where $u_1, u_2, \cdots, u_n \in W^{1,p}_{loc}(U)$. where U is an open subest of $\mathbb{R}^n$. Suppose that $\psi : \mathbb{R}^n \to \mathbb{R}^1$ is a C^1-function whose first derivative $\nabla\psi$ is continuous and bounded on $\mathbb{R}^n$. Let us now consider the composite function $\psi(u) : U \to \mathbb{R}^1$. It then follows that $\psi(u) \in W^{1,p}_{loc}(U)$. Moreover, for each positive integer j with $1 \leqslant j \leqslant n$, we have the following weak chain rule to be valid

$$\partial_j(\psi(u)) = \sum_{k=1}^{n} (\partial_k \psi)(u) \cdot \partial_j u_k.$$

However, for the purpose of practical use, it is worthwhile for us to mention that the above weak chain rule remains valid even if the function ψ is merely lipschitz. After all these, we now state the classical Sobolev's embedding theorem on $\mathbb{R}^n$.

Theorem 2.1.3. (Sobolev's embedding theorem on $\mathbb{R}^n$) Let $n \geqslant 1$ be a given positive integer, and let p be such that $1 \leqslant p < n$. There then exists some universal constant $C_p \in (0, \infty)$, depending only on p and n, such that the following inequality is valid for every $u \in W^{1,p}(\mathbb{R}^n)$

$$\|u\|_{L^{\frac{np}{n-p}}(\mathbb{R}^n)} \leqslant C_p \|\nabla u\|_{L^p(\mathbb{R}^n)}.$$

Besides the standard Sobolev's embedding theorem for $\mathbb{R}^n$, it is sometimes useful for us to consider the following Sobolev's embedding theorem for bounded, open set U of $\mathbb{R}^n$, with smooth boundary.

Theorem 2.1.4. (Sobolev's embedding theorem for bounded open set U with smooth boundary) Let U be a bounded, open subset of $\mathbb{R}^n$ with smooth boundary. For each p with $1 \leqslant p < n$, there exists some universal constant $C_p \in (0, +\infty)$, depending only on p, n and U, such that the following inequality estimation is valid for every $u \in W^{1,p}(U)$.

$$\|u\|_{L^{\frac{np}{n-p}}(U)} \leqslant C_p \{\|\nabla u\|_{L^p(U)} + \|u\|_{L^p(U)}\}.$$

We now turn our attention to some basic notions and results in the theory of singular integrals. In particular, we will first state the Zygmund-

Calderon's theorem and then we will introduce the notion of Riesz's transforms on the Euclidean space $\mathbb{R}^n$.

Theorem 2.1.5. (Calderon-Zygmund's Theorem) Let $\Omega : \mathbb{R}^n - \{0\} \to \mathbb{R}^1$ be a L^∞-measurable function which enjoys the property that $\Omega(r \cdot x) = \Omega(x)$, for every $r > 0$, and every $x \in \mathbb{R}^n - \{0\}$. (We usually say that Ω is a homogenuous function of degree 0 on $\mathbb{R}^n$). let us suppose further that Ω also satisfies the following two properties

- $\int_{S^{n-1}} \Omega(y) dS(y) = 0$. That is , the integral of Ω over the standard unit sphere $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$ is zero.
- $\int_0^1 \frac{\omega(t)}{t} dt < \infty$, where $\omega(t) = sup\{|\Omega(x) - \Omega(y)| : x, y \in S^{n-1}, |x - y| \leqslant t\}$.

Then, for each given $\epsilon > 0$, we can define

$$T_\epsilon f(x) = \int_{|y| \geqslant \epsilon} \frac{\Omega(y)}{|y|^n} f(x - y) dy,$$

for every $f \in C_c^\infty(\mathbb{R}^n)$. It then follows that, for each $p \in (1, \infty)$, there exists some universal constant $A_p \in (1, \infty)$, depending only on p and n, such that, for every $\epsilon > 0$, and every $f \in L^p(\mathbb{R}^n)$, we have

$$\|T_\epsilon f\|_{L^p(\mathbb{R}^n)} \leqslant A_p \|f\|_{L^p(\mathbb{R}^n)}.$$

In the case when $f \in L^p(\mathbb{R}^n)$, by the virture of the bounded estimation given as above, it follows that $Tf = \lim_{\epsilon \to 0} T_\epsilon f$ exists in the sense of $L^p(\mathbb{R}^n)$ norm, and that $Tf \in L^p(\mathbb{R}^n)$ will satisify $\|Tf\|_{L^p(\mathbb{R}^n)} \leqslant A_p \|f\|_{L^p(\mathbb{R}^n)}$ with the same bound A_p.

By the virtue of the above Zygmund-Calderon's Theorem, we now define the Riesz's Transforms on the Euclidean's space $\mathbb{R}^n$. We explicitly point out that if the homogenuous function Ω of degree zero as appears in the above Zygmund-Calderon's theorem is taken to be

$$\Omega_j(y) = c_n \frac{y_j}{|y|},$$

for each positive integer j with $1 \leqslant j \leqslant n$, where $c_n = \frac{\int_0^\infty e^{-t} t^{\frac{n-1}{2}} dt}{\pi^{\frac{n+1}{2}}}$, then the resulting singular integral operator immediately becomes the j-th Riesz's Transform on the Euclidean space $\mathbb{R}^n$. More precisely, we have the following notion of Riesz's Transforms on $\mathbb{R}^n$.

Definition. Let n be a positive integer greater than or equal to 3, and let j be some positive integer for which $1 \leqslant j \leqslant n$. Then, for each $p \in (1, +\infty)$, the j-th Riesz's transform on $\mathbb{R}^n$ is the bounded linear operator $R_j : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ defined in the following way

$$R_j f(x) = \lim_{\epsilon \to 0^+} \int_{|y| \geqslant \epsilon} c_n \frac{y_j}{|y|^{n+1}} f(x-y) dy,$$

for all $f \in L^p(\mathbb{R}^n)$, where the limit as $\epsilon \to 0^+$ is taken in the sense of $L^p(\mathbb{R}^n)$-norm.

For the purpose of later use when we are doing the pressure decomposition for the incompressible Navier-Stokes equations, we will encounter the key identity $-\triangle P = \sum_{i,j} \partial_i \partial_j \{u_i u_j\}$, where u is the velocity vector field of the fluid, and P stands for the pressure associated to u. Because of this, it is worthwhile for us to mention the following well-known fact about the Riesz's Transforms on $\mathbb{R}^n$.

Proposition 2.1.6. The Riesz's transforms R_i on $\mathbb{R}^n$, for $1 \leqslant i \leqslant n$, verify the relation $R_i R_j = \partial_i \partial_j (\triangle^{-1})$ in the weak sense, in which the symbol $\triangle^{-1}$ stands for the inverse Laplace transform.

The above proposition is very useful in dealing with the pressure P appearing in the incompressible Navier-Stokes equations because P is always assumed to be L^p-integrable for some $p \in (0, \infty)$ in the spatial direction. With this type of integrable condition imposed on the pressure P, we can at once apply the above proposition to conclude that the pressure P can be characterized alternatively by the following relation via the Riez's Transforms on $\mathbb{R}^3$.

$$P = \sum_{1 \leqslant i,j \leqslant 3} R_i R_j \{u_i u_j\}.$$

It is now the time for us to introduce the concept of BMO and a classical theorem of John and Nirenberg [15].

Definition. A measurable function f on $\mathbb{R}^n$ is said to be lying in the class $BMO(\mathbb{R}^n)$, if it happens that there exists a constant $C \in (0, +\infty)$ such that the inequality $\frac{1}{|B|} \int_B |f - f_B| \leqslant C$ is valid for any open ball B with finite positive radius in $\mathbb{R}^3$, where the symbol f_B stands for the average of f over B.

Here, let us say that we are interested in functions lying in the class of $BMO(\mathbb{R}^n)$ because of the lack of continunity of R_iR_j from $L^\infty(\mathbb{R}^n)$ to $L^\infty(\mathbb{R}^n)$. As a matter of fact, R_iR_j is a generalized Zygmund-Calderon operator sending $L^\infty(\mathbb{R}^n)$ into $BMO(\mathbb{R}^n)$ in a bounded sense, and in this context, we may regard $BMO(\mathbb{R}^n)$ as a substitute of the ordinary $L^\infty(\mathbb{R}^n)$-class in the case when bounded-estimation for operator from $L^\infty(\mathbb{R}^n)$ to $L^\infty(\mathbb{R}^n)$ is lacking.

For the purpose of later use in Chapter 3, we would like to mention here the following classical BMO theorem of John and Nirenberg in [15].

Theorem 2.1.7. (John and Nirenberg) Let B be a ball with finite radius sitting in $\mathbb{R}^3$. Then, there exists some constants α, and K, with $0 < \alpha < \infty$, and $0 < K < \infty$, depending only on the ball B and n, such that for any given $f \in BMO(\mathbb{R}^n)$, we have $\int_B exp(\alpha \frac{|f-f_B|}{\|f\|_{BMO}}) \leqslant K$, where the symbol f_B stands for the mean value of f over B.

2.2 The De Giorgi's method as applied to linear elliptic equations with rough diffusion coefficients

In 1957, E. De Giorgi made a ground-breaking advance in [8] to the theory of linear elliptic equations by employing a method which later bears his name. In particular, he proved that any weak solution of a given linear elliptic equation with rough diffusion coefficients is Holder continuous over any closed balls compactly included in the defining region of the given equation. In this section, we first state a common version of the famous De Giorgi's

Theorem. Next, we present the first half of it's proof which deals with the L^∞-boundlness of weak solutions over closed balls compactly included in the defining region of the given elliptic equation. The first half of the proof for the De Giorgi's theorem is strong enough in illustrating those basic ideas that can be applied to deduce the local L^∞-boundness of weak solutions for the incompressible Navier-Stokes equations under some suitable assumptions made on the weak solutions under consideration. Once the local L^∞-boundedness of our weak solutions for the incompressible Navier-Stokes equations is established under some suitable assumption, we can at once invoke the famous theorm of Serrin[29] to deduce the C^∞-smoothness of our given weak solutions. So, we can say that the missing argument for the Holder continuity of our weak solutions for the incompressible Navier-Stokes equations has already been covered by the famous Serrin's Theorem in [29], provided we can prove the local L^∞-boundedness of our weak solutions.

Before we give the statement of the De Giorgi's Theorem and the first half of it's proof, let us mention that the present author is following closely the lecture notes of L. Caffarelli given in Fall 2007, and the treatment given below reflects what the present author has learned from the above mentioned lecture notes. Besides, the present author learned the De Giorgi's Method for the first time though the paper [32], and the treatment as presented in [32] will have an equal influence on the presentation given below.

We begin with the following definition.

Definition. Let $a_{ij} \in L^\infty(U)$ be measurable functions of class L^∞ defined over some bounded open set U of $\mathbb{R}^n$, which are symmetric (that is $a_{ij} = a_{ji}$) and enjoy the uniform elliptic condition $\frac{1}{M}|y|^2 \leqslant \sum_{i,j} a_{ij} y_i y_j \leqslant M|y|^2$ (where M is some suitable constant). We say that $u \in W^{1,2}(U)$ is a weak solution of the elliptic equation $\sum_{i,j} \partial_i(a_{ij}\partial_j u) = 0$ on U, if it happens that the following equality is valid for all test functions $v \in W_0^{1,2}(U)$.

$$\int_U \sum_{i,j} a_{ij} \partial_j u \partial_i v dx = 0.$$

With the above definition, we now state the following De Giorgi's Theorem.

Theorem 2.2.1. (De Giorgi) Consider an elliptic equation $\sum_{i,j} \partial_i(a_{ij}\partial_j u) = 0$ defined on the unit ball $B(1)$ centered at the orign of $\mathbb{R}^n$, where $a_{ij} \in L^\infty(B(1))$ are some bounded measurable functions verifying the following two conditions

- $a_{ij} = a_{ji}$, for any $1 \leqslant i, j \leqslant n$.
- the functions a_{ij} verify the uniform elliptic condition in the sense that there exists some positive constant $M \in (1, \infty)$ for which $\frac{1}{M}|y|^2 \leqslant \sum_{i,j} a_{ij} y_i y_j \leqslant M|y|^2$ is valid for all $y \in \mathbb{R}^n$.

Then, there exists some constants $C \in (0, \infty)$, and $\alpha \in (0, \infty)$, depending only on M, n, and $\|a_{ij}\|_{L^\infty(B(1))}$ such that every weak solution $u \in W^{1,2}(B(1))$

of the elliptic equation $\sum_{i,j} \partial_i(a_{ij}\partial_j u) = 0$ will satisfy the following inequality estimation

$$\|u\|_{C^\alpha(B(\frac{1}{2}))} \leqslant C\{\|u\|_{L^2(B(1))} + \|\nabla u\|_{L^2(B(1))}\},$$

where the sybol $B(\frac{1}{2})$ stands for the ball with radius $\frac{1}{2}$ centered at the origin of $\mathbb{R}^n$.

We now present the first half of the proof for the De Giorgi's Theorem. More precisely, we will show that every weak solution $u \in W^{1,2}(B(1))$ of the elliptic equation $\sum_{i,j} \partial_i(a_{ij}\partial_j u) = 0$ must satisfy the following estimation

$$\|u\|_{L^\infty(B(\frac{1}{2}))} \leqslant C\|u\|_{W^{1,2}(B(1))},$$

where $C \in (0, \infty)$ is some universal constant depending only on M, n , and $\|a_{ij}\|_{L^\infty(B(1))}$. For this purpose, let us introduce some notations and functions. For any given weak solution $u \in W^{1,2}(B(1))$ to the given linear elliptic equation $\sum_{i,j} \partial_i(a_{ij}\partial_j u) = 0$, we define

- For each integer $k \geqslant 0$, let $u_{(k)} = [u - (1 - \frac{1}{2^k})]_+$. That is, $u_{(k)}$ is the positive part of the function $u - (1 - \frac{1}{2^k})$. Or, we say that $u_{(k)}$ is the truncation of u at the level $1 - \frac{1}{2^k}$.

- for each $k \geqslant 0$, let $B_k = B(\frac{1}{2}(1 + \frac{1}{2^k}))$ to be the ball with radius $\frac{1}{2}(1 + \frac{1}{2^k})$ centered at the origin of $\mathbb{R}^n$.

Besides all these, in the process of localizing our weak solution u, we also need a sequence $\{\phi_{(k)}\}_{k=1}^{\infty}$ of test functions in $C_c^{\infty}(B(1))$ which verifies the following properties.

- for each $k \geqslant 1$, we have $0 \leqslant \phi_{(k)} \leqslant 1$ on $B(1)$.
- for each $k \geqslant 1$, we have $\phi_{(k)} = 1$ on B_k, and that $\phi_{(k)} = 0$ on $B(1) - B_{k-1}$.
- for each $k \geqslant 1$, we have $|\nabla \phi_{(k)}| \leqslant C 2^k$ on $B(1)$, where C is some constant depending only on the dimension of $\mathbb{R}^n$.

Let us now study the energy of the truncation of our weak solution u at the level $(1 - \frac{1}{2^k})$. More precisely, we consider the quantity A_k which is defined as

$$A_k = \int_{B_k} |u_{(k)}|^2 + \int_{B_k} |\nabla u_{(k)}|^2,$$

for every $k \geqslant 1$. The main point of the De Giorgi's method is to build up a nonlinear recurrence relation for the sequence A_k. More precisely, we need to consturct some universal constant $C^* \in (1, \infty)$ and some positive index $\beta > 1$ such that the nonlinear recurrence relation $A_k \leqslant C^{*k} A_{k-1}^{\beta}$ is valid for any given weak solution $u \in W^{1,2}(B(1))$ of the elliptic equation $\sum_{i,j} \partial_i (a_{ij} \partial_j u) = 0$. This will eventually lead to the conclusion that $\lim_{k \to \infty} A_k = 0$, provided the given initial value A_0 is small enough (see, for example, lemma 4 in Vasseur [32]). We will clarify this point later. For now, let us concentrate on establishing the nonlinear recurrence relation for A_k. To begin, let $u \in W^{1,2}(B(1))$ be any given weak solution for the elliptic equation $\sum_{i,j} \partial_i(a_{ij}\partial_j u) = 0$. For any test function $v \in W_0^{1,2}(B(1))$, we have the following equality to be valid

$$\int_{B(1)} \sum_{i,j} a_{ij}\partial_i u \partial_j v = 0.$$

Let us now replace the symbol v by the function $\phi_{(k)}^2 \cdot u_{(k)}$, which is indeed an element in $W_0^{1,2}(B(1))$. We then obtain the following equality

$$\int_{B(1)} \sum_{i,j} a_{ij}\partial_i u \partial_j (\phi_{(k)}^2 u_{(k)}) = 0,$$

which is the same as saying that

$$\int_{B(1)} \sum_{i,j} a_{ij}(\partial_j u)(\partial_i u_{(k)})\phi_{(k)}^2 = -2\int_{B(1)} \sum_{i,j} a_{ij}(\phi_{(k)}\partial_j u)(u_{(k)}\partial_i \phi_{(k)}).$$

However, by the weak chain rule as stated in the first section of Chapter 2, we have $\nabla u_{(k)} = (\nabla u)\chi_{\{u>1-\frac{1}{2^k}\}}$, and $u_{(k)} = u_{(k)}\chi_{\{u>1-\frac{1}{2^k}\}}$. So, we can rewrite the above inequality as

$$\int_{B(1)} \sum_{i,j} a_{ij}(\partial_j u_{(k)})(\partial_i u_{(k)})\phi_{(k)}^2 = -2\int_{B(1)} \sum_{i,j} a_{ij}(\phi_{(k)}\partial_j u_{(k)})(u_{(k)}\partial_i \phi_{(k)}).$$

Then, it follows that

$$\int_{B(1)} \sum_{i,j} a_{ij}(\partial_j u_{(k)})(\partial_i u_{(k)})\phi_{(k)}^2$$

$$= 2\left|\int_{B(1)} \sum_{i,j} a_{ij}(\phi_{(k)}\partial_j u_{(k)})(u_{(k)}\partial_i \phi_{(k)})\right|$$

$$\leqslant \epsilon \int_{B(1)} \sum_{i,j} a_{ij}(\partial_j u_{(k)})(\partial_i u_{(k)})\phi_{(k)}^2$$

$$+ \frac{1}{\epsilon}\int_{B(1)} \sum_{i,j} a_{ij}\partial_i\phi_{(k)}\partial_j\phi_{(k)}u_{(k)}^2.$$

Take $\epsilon = \frac{1}{2}$, the above inequality will reduce to become

$$\frac{1}{2}\int_{B(1)} \sum_{i,j} a_{ij}(\partial_j u_{(k)})(\partial_i u_{(k)})\phi_{(k)}^2 \leqslant 2\int_{B(1)} \sum_{i,j} a_{ij}\partial_i\phi_{(k)}\partial_j\phi_{(k)}u_{(k)}^2.$$

Now, by taking the definition of $\phi_{(k)}$ into account, and by employing the uniform elliptic condition, we can now deduce from the above inequality that

$$\int_{B_k} |\nabla u_{(k)}|^2 \leqslant \int_{B(1)} \phi_{(k)}^2 |\nabla u_{(k)}|^2$$

$$\leqslant M\int_{B(1)} \sum_{i,j} a_{ij}(\partial_j u_{(k)})(\partial_i u_{(k)})\phi_{(k)}^2$$

$$\leqslant 4M\int_{B(1)} \sum_{i,j} a_{ij}\partial_i\phi_{(k)}\partial_j\phi_{(k)}u_{(k)}^2$$

$$\leqslant 4M^2\int_{B(1)} |\nabla\phi_{(k)}|^2 u_{(k)}^2$$

$$\leqslant 4M^2C^2 4^k\int_{B_{k-1}} u_{(k)}^2.$$

That is, we have,

$$\int_{B_k} |\nabla u_{(k)}|^2 \leqslant CM^2 4^k \int_{B_{k-1}} u_{(k)}^2. \tag{2.1}$$

By invoking the Sobolev's embedding theorem for a given bounded open region with smooth boundary, we can find a constant C_2, depending only on the dimension of $\mathbb{R}^n$, such that, for every $k \geqslant 0$, and every $f \in W^{1,2}(B_k)$, we have

$$\|f\|_{L^{\frac{2n}{n-2}}(B_k)} \leqslant C_2\{\|f\|_{L^2(B_k)} + \|\nabla f\|_{L^2(B_k)}\}.$$

We remark that such a universal costant C_2 exists because $B(\frac{1}{2})$ is included in B_k, for every $k \geqslant 0$,and so the family of open balls $B_{(k)}$ won't shrink down to zero, as k becomes large.

Now, by an simple application of the Chebyshev's inequality and the above Sobolev's inequality, we can at once carry out the following estimation for all $k \geqslant 1$.

$$
\begin{aligned}
\int_{B_{k-1}} u_{(k)}^2 = \int_{B_{k-1}} u_{(k)}^2 \chi_{\{u_{(k)}>0\}} \\
&\leqslant \int_{B_{k-1}} u_{(k)}^2 \chi_{\{u_{(k-1)}>\frac{1}{2^k}\}} \\
&\leqslant 2^{k(\frac{2n}{n-2}-2)} \int_{B_{k-1}} u_{(k)}^2 u_{(k-1)}^{\frac{2n}{n-2}-2} \\
&\leqslant 16^{\frac{k}{n-2}} \int_{B_{k-1}} u_{(k-1)}^{\frac{2n}{n-2}} \\
&\leqslant 16^{\frac{k}{n-2}} C_2 \{\|u_{(k-1)}\|_{L^2(B_{k-1})}^{\frac{2n}{n-2}} + \|\nabla u_{(k-1)}\|_{L^2(B_{k-1})}^{\frac{2n}{n-2}}\} \\
&\leqslant 2C_2 16^{\frac{k}{n-2}} A_{k-1}^{\frac{n}{n-2}}.
\end{aligned}
$$

We remark that in the above estimation, we have used the facts that $\{u_k > 0\}$ is totally included in $\{u_{k-1} > \frac{1}{2^k}\}$, and $0 \leqslant u_{(k)} \leqslant u_{(k-1)}$, which are obviously true for every $k \geqslant 1$. In the same way, we can now deduce from inequality 2.1 that we also have the following estimation

$$
\int_{B_k} |\nabla u_{(k)}|^2 \leqslant CM^2 4^k \int_{B_{k-1}} u_{(k)}^2 \leqslant 2C_2 CM^2 4^k 16^{\frac{k}{n-2}} A_{k-1}^{\frac{n}{n-2}}.
$$

By combining our last two inequalities, we may deduce that, for every $k \geqslant 1$, we have

$$
A_k \leqslant C_0^k A_{k-1}^{\frac{n}{n-2}},
$$

where C_0 stands for some universal constant depending only on M and n. To conclude the argument, we need to invoke the following technical lemma in the theory of recurrence relation. (this lemma is exactly lemma 4 in [32]).

Lemma 2.2.2. For any positive constant $C > 0$ and any positive $\beta > 1$, there exists some sufficiently small $\delta_0 > 0$ so that whenever we have a sequence $\{a_k\}_{k \geqslant 0}$ of nonnegative numbers verifying $0 < a_0 \leqslant \delta_0$ and $a_k \leqslant C^k a_{k-1}^{\beta}$, for all $k \geqslant 1$, we will have $\lim_{k\to\infty} a_k = 0$.

By applying the above lemma in the case when $\beta = \frac{n}{n-2}$. we deduce that there exists some sufficiently small $\delta_0 > 0$, depending only on M and n, such that for any weak solution $u \in W^{1,2}(B(1))$ of the elliptic equation $\sum_{i,j} \partial_i(a_{ij}\partial_j u) = 0$ which verifies $\|u\|^2_{L^2(B(1))} + \|\nabla u\|^2_{L^2(B(1))} = A_0 \leqslant \delta_0$, we have $\lim_{k\to\infty} A_k = 0$, which in particular tells us that $\int_{B(\frac{1}{2})} [u-1]^2_+ \leqslant \lim_{k\to\infty} A_k = 0$, and hence $u \leqslant 1$, almost everywhere on $B(\frac{1}{2})$.

The above argument shows that if our weak solution u to the given linear elliptic equation satisfies $\|u\|_{W^{1,2}(B(1))} \leqslant \delta_0$, then we must have $u \leqslant 1$ almost everywhere on $B(\frac{1}{2})$. If we multiply our weak solution u by -1, we know that $-u$ is just another weak solution to the same elliptic equation $\sum_{i,j} \partial_i(a_{ij}\partial_j u) = 0$. So, it is easy to see that, if u is a weak solution of $\sum_{i,j} \partial_i(a_{ij}\partial_j u) = 0$ which verifies the condition that $\|u\|_{W^{1,2}(B(1))} \leqslant \delta_0$, then we will also have $(-u) \leqslant 1$ almost everywhere on $B(\frac{1}{2})$, which means the same thing as $u \geqslant -1$ almost everywhere on $B(\frac{1}{2})$.

So, in conclusion, we have shown that, every weak solution $u \in W^{1,2}(B(1))$ with $\|u\|_{W^{1,2}(B(1))} \leqslant \delta_0$ must satisfies $\|u\|_{L^\infty(B(\frac{1}{2}))} \leqslant 1$. Since the equation under consideration is linear, this is just the same thing as saying that the following inequality is valid for any weak solution $u \in W^{1,2}(B(1))$ to our given linear elliptic equation

$$\|u\|_{L^\infty(B(\frac{1}{2}))} \leqslant \frac{1}{\delta_0}\|u\|_{W^{1,2}(B(1))}.$$

So, we are done in giving the first half of the proof of the famous De Giorgi's Theorem.

2.3 Some important theorems in the regularity theory for incompressible Navier-Stokes equations

In this section, we will state some important theorems in the regularity theory for the incompressible Navier-Stokes equations. We remark that those basic theorems we choose to present here are the conerstones in the regularity theory for the incompressible Navier-Stokes equations. These fundamental results, which orginated from the works of Leray, Hopf, Prodi, Serrin, Ladyzhenskaya, and Kato, had influenced the subject profoundly. Of course, besides the achievements of those mathematicians mentioned above, we also need to mention the famous partial regularity theorem for incompressible Navier-Stokes equations which was due to L. Caffarelli, R. Kohn, and L. Nirenberg in their paper [3]. However, we will postpone our discussion for the partial regularity theorem to the next section since the formulation of the partial regularity theorem would require the notion of suitable weak solutions.

Before we start our discussion here, two points should be made clear in advance. First, the presentation given here is largely based on those introductary materials appearing in the important paper [9] by L. Escauriaza, G. Seregin, and V. Sverak. In fact, all the basic results that we are going to discuss are taken directly from the paper [9]. So, if the readers are looking for a more detailed discussion about these results, they can directly consult [9]. Second, we should point out that it is not our intention to give a survey of all the important results in the regularity theory for the incompressible Navier-Stokes equations. So, it is not suprising that many other important results will not get mentioned here. Our goal is to state those basic regularity results which are both important and closely related to the subject matter of this thesis.

We now begin with the following fundamental contribution of Leray and Hopf ([20],[13]), in which they introduced the notion of weak solutions for the incompressible Navier-Stokes equations, and proved that there exists at least one weak solution lying in the class $L^\infty(0,T;L^2(\mathbb{R}^3)) \cap L^2(0,T;H^1(\mathbb{R}^3))$, for each given initial datum $u_0 \in L^2(\mathbb{R}^3)$ with $\mathrm{div} u_0 = 0$.

Theorem 2.3.1. (Leray and Hopf) For every given initial datum $u_0 \in L^2(\mathbb{R}^3)$ with $\mathrm{div} u_0 = 0$ to be valid in the sense of distribution. Let $T \in (0,\infty]$ be any given positive number. Then there exists at least a pair of functions $u : (0,T) \times \mathbb{R}^3 \to \mathbb{R}^3$, and $P : (0,T) \times \mathbb{R}^3 \to \mathbb{R}^1$ satisfying $u \in L^\infty(0,T;L^2(\mathbb{R}^3)) \cap L^2(0,T;H^1(\mathbb{R}^3))$, such that the following conditions are satisfied by the pair

(u, P)

- $\text{div}u = 0$ holds in the sense of distribution.
- For every choice of $\psi \in C_c^\infty((0,T)\times\mathbb{R}^3)$ with $\text{div}\psi = 0$, we have $\int_0^T\int_{\mathbb{R}^3} -u\cdot\partial_t\psi + \sum_{i,j}\partial_j\psi_i\partial_j u_i - \sum_{i,j}\partial_j\psi_i(u_iu_j)dxdt = 0$.
- u satisfies the energy inequality in the sense that, for every $t \in (0,T)$, we have $\frac{1}{2}\int_{\mathbb{R}^3}|u(t,\cdot)|^2dx + \int_0^T\int_{\mathbb{R}^3}|\nabla u|^2dxds \leqslant \frac{1}{2}\int_{\mathbb{R}^3}|u_0|^2dx$.
- $u(0,\cdot) = u_0$ is valid in the sense that $\lim_{t\to 0^+}\|u(t,\cdot)-u_0\|_{L^2(\mathbb{R}^3)} = 0$.
- the pressure P belongs to the class $L^{\frac{5}{3}}((0,T)\times\mathbb{R}^3)$ and verifies the equation $-\triangle P = \sum_{i,j}\partial_i\partial_j\{u_iu_j\}$ in the sense of distribution.

For a detailed proof of this fundamental result, the readers are advised to consult the original papers [20], [13] by Leray and Hopf. However, a more readable proof can be found in the excellent text [31] by R. Temam. The Leray-Hopf theorem stated as above remains valid even if $\mathbb{R}^3$ is replaced by any possible open subsets of $\mathbb{R}^3$, and Leray and Hopf actually proved the above result for any open subsets of $\mathbb{R}^3$. So, the readers can regard theorem 2.3.1 as a special case of the Leray-Hopf Theorem. However, we should point out that, if $\mathbb{R}^3$ is replaced by an arbritary open subset Ω of $\mathbb{R}^3$, then we should replace $H^1(\mathbb{R}^3)$ by $H_0^1(\Omega)$ in the formulation of the Leray-Hopf Theorem. Next, we would like to mention that any weak solution $u \in L^\infty(0,T;L^2(\mathbb{R}^3))\cap L^2(0,T;H^1(\mathbb{R}^3))$ satisfying the conditions appearing in Theorem 2.3.1 is called a Leray-Hopf weak solution for the incompressible Navier-Stokes equations, and we will use this notion of Leray-Hopf weak solutions thoughout the whole thesis without explicit mention.

Next, we would like to state the following fundamental theorem of James Serrin [29], which is an important regularity criteria for solutions lying in the Leray-Hopf class. As usual, the following theorem of Serrin was actually proved in [29] for any arbitrary open subset Ω of $\mathbb{R}^3$. However, we will state it in the case of $\mathbb{R}^3$ for the sake of simplicity.

Theorem 2.3.2. (Serrin) Suppose that u is a Leray-Hopf weak solution for the incompressible Navier-Stokes equations on $(0,T) \times \mathbb{R}^3$ which satisfies the following condition that $u \in L^p(0,T;L^q(\mathbb{R}^3))$, for some $1 \leqslant p,q \leqslant \infty$ verifying $\frac{2}{p} + \frac{3}{q} < 1$, then, u is smooth on $(0,T) \times \mathbb{R}^3$.

The conclusion of Theorem 2.3.2 was later extended to include the case of $\frac{2}{p} + \frac{3}{q} = 1$, with $2 \leqslant p < \infty$ in the work [30] by M.Struwe, and the work [10] by E.B.Fabes, B.F. Jones, and N. M. Riviere. However, it is quite striking that the remaining case of $L^\infty(0,T;L^3(\mathbb{R}^3))$ was covered much later in [9] by L. Escauriaza, G. Seregin, and V. Sverak.

We have already presented the above Serrin's criteria in the direction of establishing the regularity of weak solutions for the incompressible Navier-Stokes equations. We now turn our attention to the problem of uniqueness of weak solutions for the incompressible Navier-Stokes equations. We simply state the following theorem, which was established though the works of Prodi [23], James Serrin [29], and Ladyzhenskaya [18] (For a more detailed discussion about this theorem, please see the introduction of the paper [9] by L. Escauriaza, G. Seregin, and V. Sverak).

Theorem 2.3.3. (Prodi, Serrin, Ladyzhenskaya) Let u, v be two given Leray-Hopf weak solutions on $(0,T) \times \mathbb{R}^3$ satisfying the same initial datum $u_0 \in L^2(\mathbb{R}^3)$ (That is, $u(0,\cdot) = v(0,\cdot) = u_0$ is valid in the sense of $L^2(\mathbb{R}^3)$). Suppose that u satisfies the following additional condition that $u \in L^p(0,T; L^q(\mathbb{R}^3))$, for some $1 \leqslant p, q \leqslant \infty$ with $\frac{2}{p} + \frac{3}{q} = 1$. Then, it follows that u must coincides almost everywhere with v on $(0,T) \times \mathbb{R}^3$. Moreover, $u = v$ is smooth on $(0,T) \times \mathbb{R}^3$.

The conclusion of Theorem 2.3.3 was later extended to the case of $u \in L^p(0,T; L^q(\Omega))$, with $\frac{2}{p} + \frac{3}{q} < 1$, and Ω is chosen to be an open bounded subset of $\mathbb{R}^3$ (This historical remark is directly taken from the introduction of the paper [9] by L. Escauriaza, G. Seregin, and V. Sverak). Theorem 2.3.3 is truely remarkable since it tells us that any given Leray-Hopf weak solution u lying in the class $L^p(0,T; L^q(\mathbb{R}^3))$ with $\frac{2}{p} + \frac{3}{q} = 1$ must coincides with any other Leray-Hopf weak solutions which satisify the same initial datum as u. The basic principle we learn from Theorem 2.3.3 is that we can ensure uniqueness of our Leray-Hopf weak solution provided if it is regular enough.

Next, we would like to quote the following theorem which was due to Kato in [16], in which he established the local existence of smooth solutions for the incompressible Navier-Stokes equations under the assumption that the given initial datum is regular enough (that is, $u_0 \in L^3(\mathbb{R}^3)$). We remark that the following version of the Kato's Theorem is taken directly from the introduction of the paper [9].

Theorem 2.3.4. (Kato, 1982) Suppose that we are given an initial datum u_0 with $\text{div} u = 0$ which satisfies the condition that $u_0 \in L^2(\mathbb{R}^3) \cap L^3(\mathbb{R}^3)$. Then, for some suitable $T^* \in (0, \infty]$, there exists a pair of functions $u : (0, T^*) \times \mathbb{R}^3 \to \mathbb{R}^3$, and $P : (0, T^*) \times \mathbb{R}^3 \to \mathbb{R}^1$, which solves the incompressible Navier-Stokes equations in the weak sense and satisfies the following conditions

- $u \in C([0, T^*]; L^2(\mathbb{R}^3)) \cap L^2(0, T^*; H^1(\mathbb{R}^3))$.
- $u \in C([0, T^*]; L^3(\mathbb{R}^3)) \cap L^5((0, T^*) \times \mathbb{R}^3) \cap L^4((0, T^*) \times \mathbb{R}^3)$.
- $P \in C([0, T^*]; L^{\frac{3}{2}}(\mathbb{R}^3)) \cap L^{\frac{5}{2}}((0, T^*) \times \mathbb{R}^3) \cap L^2((0, T^*) \times \mathbb{R}^3)$.
- $\lim_{t \to 0^+} \|u(t, \cdot) - u_0\|_{L^3(\mathbb{R}^3)} = 0$.

We now apply the Kato's Theorem (Theorem 2.3.4) and the Prodi-Serrin-Ladyzhenskaya's uniqueness Theorem (Theorem 2.3.3) to carry out a small argument in the following way. Let u be a given Leray-Hopf weak solution for the incompressible Navier-Stokes equations on $(0, T) \times \mathbb{R}^3$. By applying the Sobolev's embedding theorem, we deduce that $\|u(t, \cdot)\|_{L^6(\mathbb{R}^3)} \leqslant$

$C_6\|\nabla u(t,\cdot)\|_{L^2(\mathbb{R}^3)}$, for almost every $t \in (0,T)$, where C_6 is some universal constant depending only on the dimension of $\mathbb{R}^3$. Hence it follows that $u \in L^\infty(0,T;L^2(\mathbb{R}^3)) \cap L^2(0,T;L^6(\mathbb{R}^3))$, with $\|u\|_{L^2(0,T;L^6(\mathbb{R}^3))} \leqslant C_6\|u\|_{L^2(0,T;H^1(\mathbb{R}^3))}$. So, we can now deduce from $u \in L^\infty(0,T;L^2(\mathbb{R}^3)) \cap L^2(0,T;L^6(\mathbb{R}^3))$ that we must have $u(t,\cdot) \in L^2(\mathbb{R}^3) \cap L^6(\mathbb{R}^3)$,for almost every $t \in (0,T)$. Since $2 < 3 < 6$, we can deduce that $u(t,\cdot) \in L^3(\mathbb{R}^3)$, for almost every $t \in (0,T)$. Now, just pick up any one of these $t_0 \in (0,T)$, with $u(t_0,\cdot) \in L^3(\mathbb{R}^3)$. We can then apply the Kato's Theorem (Theorem 2.3.4) to deduce that there exists some $\delta \in (0,\infty]$, and some locally defined solution $v \in C([t_0,t_0+\delta];L^3(\mathbb{R}^3)) \cap L^5((t_0,t_0+\delta)\times\mathbb{R}^3) \cap L^4((t_0,t_0+\delta)\times\mathbb{R}^3)$ for the Incompressible Navier Stokes equations such that $v(t_0,\cdot) = u(t_0,\cdot)$ is valid in both $L^2(\mathbb{R}^3)$-sense and $L^3(\mathbb{R}^3)$-sense. Since v in particular lies in $L^5((t_0,t_0+\delta)\times\mathbb{R}^3)$, we immediately invoke the Prodi-Serrin-Ladyzhenskaya's uniqueness theorem (Theorem 2.3.3) to deduce that our original Leray-Hopf weak solution u coincides with v on $(t_0,t_0+\delta)\times\mathbb{R}^3$, and that u is smooth on $(t_0,t_0+\delta)\times\mathbb{R}^3$. As a summary of what we have done, we have obtained the following corollary of Theorem 2.3.3 and Theorem 2.3.4.

Corollary 2.3.5. Let $u \in L^\infty(0,T;L^2(\mathbb{R}^3)) \cap L^2(0,T;H^1(\mathbb{R}^3))$ be any given Leray-Hopf weak solution for the incompressible Navier-Stokes equations. Then, for almost every $t_0 \in (0,T)$, there exists some positive number $\delta \in (0,\infty]$, depending on the choice of t_0, such that u is indeed smooth on $(t_0,t_0+\delta)\times\mathbb{R}^3$.

From now on, we will take the advantage of Corollary 2.3.5 whenever it is need without explicit mention.

2.4 The notion of suitable weak solutions and the partial regularity theorem

The purpose of this section is to introduce the notion of suitable weak solutions and say something about the famous partial regularity theorem of Caffarelli, Kohn , and Nirenberg [3]. The definition of suitable weak solutions for incompressible Navier-Stokes equations was first introduced by L.Caffarelli, R. Kohn, and L. Nirenberg in their famous paper [3]. Now, let us just make it precise in the following way.

Definition. We say that a Leray-Hopf weak solution $u \in L^{\infty}(0,T;L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega))$ is a suitable weak solution for the incompressible Navier-Stokes equations, provided u satisfies the following inequality in the sense of distribution on $(0,T)\times\Omega$.

$$\partial_t(\frac{|u|^2}{2}) + |\nabla u|^2 - \triangle(\frac{|u|^2}{2}) + \operatorname{div}(\frac{|u|^2}{2}u) + u\cdot\nabla P \leqslant 0.$$

Besides the definition of suitable weak solutions stated as above, it is of course possible to give the definition in a more direct way. That is, whenever we say a given Leray-Hopf weak solution u is a suitable weak solution for the incompressible Navier-Stokes equations, it means that the following inequality is valid for every test function $\phi \in C_c^{\infty}((0,T)\times\Omega)$ with $\phi \geqslant 0$.

$$2\int_0^T\int_\Omega |\nabla u|^2\phi dxdt \leqslant \int_0^T\int_\Omega |u|^2(\partial_t\phi+\triangle\phi)dxdt+\int_0^T\int_\Omega(|u|^2+2P)u\cdot\nabla\phi dxdt.$$

In their famous paper [3], Caffarelli, Kohn, and Nirenberg proved that the set of all possible singular points of any given suitable weak solution is of measure zero with respect to the one-dimensional Hausdoff's measure. Today, there are at least two simplified proofs available for the partial regularity theorem, which were due to Lin, and A. Vasseur in [21], and [32] respectively. The simplified proof given by Lin [21] appeared in 1998, and the simplified proof given by Vasseur [32] employed a method inherted from De Giorgi for the first time. Now, for the purpose of later use, let us state one key proposition of [3].

Theorem 2.4.1. (Caffarelli, Kohn, Nirenberg) There exists two universal constants $\epsilon_0 \in (0, \infty)$, and $C_0 \in (0, \infty)$ such that, for any suitable weak solution (u, P) on $Q_1 = [-1, 0] \times B_1(0)$ satisfying $\int_{Q_1} |u|^3 + |P|^{\frac{3}{2}} \leqslant \epsilon_0$, we have $\|u\|_{L^\infty(Q_{\frac{1}{2}})} \leqslant C_0$, where $Q_{\frac{1}{2}} = [\frac{-1}{4}, 0] \times B_{\frac{1}{2}}(0)$ (Here, the symbol $B_r(0)$ stands for the ball with radius r centered at the origin).

At this point, we should point out that every Leray-Hopf weak solution appearing in the main body of this thesis is assumed to be suitable weak solution, and we will employ the notion of suitable weak solutions thoughout the whole thesis without explicit mention.

Before closing this section, we would like to establish the following technical lemma, which is crucal in applying the method of De Giorgi to suitable weak solutions (see Vasseur [32]). We should point out that the lemma we are going to state is exactly lemma 11 in the paper [32].

Lemma 2.4.2. Let u be a suitable weak solution for the Incompressible Navier Stokes equations on $(0,T)\times\mathbb{R}^3$. Let $L\in(0,\infty)$ be an arbitrary chosen positive number, and let us consider the truncation $v=\{|u|-L\}_+$. Then, it follows that the following inequality is valid in the sense of distribution.

$$\partial_t(\frac{v^2}{2})+d^2-\triangle(\frac{v^2}{2})+\text{div}(\frac{v^2}{2}u)+\frac{v}{|u|}u\cdot\nabla P\leqslant 0.$$

Here, the term d^2 is defined to be $d^2=\frac{L}{|u|}\chi_{\{|u|>L\}}|\nabla|u||^2+\frac{v}{|u|}|\nabla u|^2$.

Proof. The inequality appearing in the above lemma can be derived formally by multiplying the equation $\partial_t u-\triangle u+\text{div}(u\otimes u)+\nabla p=0$ with the term u. However, this formal calculation cannot be justified because something bad can happen when $|u|$ becomes large. the proof we are going to give here is identical to the one appearing in [32]. We will follow the steps taken in [32], and multiply the equation $\partial_t u-\triangle u+\text{div}(u\otimes u)+\nabla p=0$ by the L^∞-function $(\frac{v}{|u|}-1)u$, which is bounded by L. However, we notice that

$$(\frac{v}{|u|}-1)u=\{\frac{[|u|-L]_+}{|u|}-1\}u=\{(1-\frac{L}{|u|})_+-1\}u=\{(|u|-L)_+-|u|\}\frac{u}{|u|}.$$

This motivates us to consider the function $\Psi:\mathbb{R}^3\to\mathbb{R}^3$ which is defined by

$$\Psi(y)=\{(|y|-L)_+-|y|\}\frac{y}{|y|}.$$

Since $\nabla|y| = \frac{y}{|y|}$, we can express Ψ as

$$\Psi(y) = \{(|y| - L)_+ - |y|\}\nabla|y| = \frac{1}{2}\nabla\{(|y| - L)_+^2 - |y|^2\}.$$

So, if we write $\Phi(y) = \frac{1}{2}\{(|y|-L)_+^2 - |y|^2\}$, we immediately have $\nabla\Phi = \Psi$ on $\mathbb{R}^3$. Now, we point out that $\Psi : \mathbb{R}^3 \to \mathbb{R}^3$ is actually continuous and that $|\Psi(y)| \leqslant L$, for any $y \in \mathbb{R}^3$. So, it turns out that $\Phi : \mathbb{R}^3 \to \mathbb{R}^1$ is continuously differentiable with it's first derivative $\nabla\Phi = \Psi$ to be continuous and bounded. As a result, we can apply the weak chain rule as stated in the first section of this chapter to Φ and deduce that we have the following two identities.

- $\partial_i(\Phi(u)) = \sum_{k=1}^3 (\partial_k\Phi)(u)\partial_i u_k = \sum_{k=1}^3 \Psi_k(u)\partial_i u_k$, for $1 \leqslant i \leqslant 3$.
- $\partial_t(\Phi(u)) = \sum_{k=1}^3 \Psi_k(u)\partial_t u_k = \Psi(u) \cdot \partial_t u$.

As a result, we can carry out the following calculation.

$$
\begin{aligned}
(\frac{v}{|u|} - 1)u(u \cdot \nabla u) = \Psi(u)(u \cdot \nabla u)& \\
&= \sum_{i,j} \Psi_i(u)u_j\partial_j u_i = \sum_j \partial_j(\Phi(u))u_j \\
&= \nabla(\Phi(u)) \cdot u = \operatorname{div}(\Phi(u) \cdot u).
\end{aligned}
$$

In the above calculation, we have implicitly applied the weak product rule in the last equality, which is valid because $u(t, \cdot) \in H^1(\mathbb{R}^3)$ and $\Phi(u(t, \cdot)) \in H^1(\mathbb{R}^3)$ for almost everywhere $t \in (0, T)$. By combining what we have done, we can now deduce that we have

$$\partial_t(\frac{1}{2}(v^2 - |u|^2)) + \mathrm{div}\{\frac{1}{2}(v^2 - |u|^2)u\} - \Psi(u) \cdot \triangle u + \Psi(u) \cdot \nabla P = 0,$$

which is valid in the sense of distribution.

Before proceeding any further, we are required to compute the first derivative of Ψ. If we express Ψ in terms of it's components. That is, if we write $\Psi = (\Psi_1, \Psi_2, \Psi_3)$, we will have

$$\Psi_i(y) = \{(1 - \frac{L}{|y|})_+ - 1\}y_i,$$

for every $1 \leqslant i \leqslant 3$. By a direct computation, we can deduce that

$$(\partial_j \Psi_i)(y) = \delta_{ji}\{(1 - \frac{L}{|y|})_+ - 1\} + L\frac{y_i y_j}{|y|^3}\chi_{\{|y|>L\}},$$

for $1 \leqslant i, j \leqslant 3$. As a result, we see that Ψ itself is continuous with it's first derivative $\nabla\Psi$ exists almost everywhere on $\mathbb{R}^3$. Moreover, a simple calculation will give $|\partial_j \Psi_i| \leqslant 1 + \frac{1}{L}$, for $1 \leqslant i, j \leqslant 3$. So ,we conclude that $\Psi : \mathbb{R}^3 \to \mathbb{R}^3$ is indeed a Lipschitz function. So, we apply the weak chain rule as stated in section 2.1 to deduce that the following equality is valid for $1 \leqslant i, j \leqslant 3$.

$$\partial_i(\Psi_j(u)) = \partial_i u_j\{(1 - \frac{L}{|u|})_+ - 1\} + \sum_{k=1}^{3} \frac{L}{|u|^3}(u_k \partial_i u_k) u_j \chi_{\{|u|>L\}}.$$

Then, the above formula immediately tells us that $|\nabla\Psi| \leqslant |\nabla u|$ on $\mathbb{R}^3$, where C_L is some constant depending only on the level L. Hence, we have $\Psi(u) \in L^\infty((0,T)\times\mathbb{R}^3)\cap H^1(\mathbb{R}^3)$, with $\|\nabla\Psi\|_{L^2(\mathbb{R}^3)} \leqslant C_L\|\nabla u\|_{L^2(\mathbb{R}^3)}$. As a result, it follows that $\Psi(u(t,\cdot))\cdot\phi \in H^1(\mathbb{R}^3)$, for every $\phi \in C_c^\infty(\mathbb{R}^3)$, and almost every $t \in (0,T)$. So, we may carry out the following calculation in the sense of distribution.

$$
\begin{aligned}
&- < \Psi(u)\cdot\triangle u, \phi > \\
&\qquad\qquad\qquad = < \nabla u, \nabla[\Psi(u)\cdot\phi] > \\
&\qquad\qquad\qquad = < \nabla u, \nabla(\Psi(u))\cdot\phi > + < \nabla u, \Psi(u)\cdot\nabla\phi >,
\end{aligned}
$$

where ϕ can be any test function lying in $C_c^\infty((0,T)\times\mathbb{R}^3)$. But this is the same as saying that we have the following equality which is valid in the sense of distribution.

$$
-\Psi(u)\cdot\triangle u = -\mathrm{div}[\nabla(\Phi(u))] + \sum_{i=1}^{3}\nabla u_i\cdot\nabla[\Psi_i(u)].
$$

However, we notice that

$$
\begin{aligned}
&\sum_i \nabla u_i\cdot\nabla(\Psi_i(u)) \\
&\qquad\qquad\qquad = \sum_{i,j}(\partial_j u_i)^2\{(1-\frac{L}{|u|})_+ - 1\} \\
&\qquad\qquad\qquad + \frac{L}{|u|^3}(\partial_j u_i)u_i\partial_j(\frac{|u|^2}{2})\chi_{\{|u|>L\}} \\
&\qquad\qquad\qquad = \{\frac{v}{|u|} - 1\}|\nabla u|^2 + \frac{L}{|u|}\chi_{\{|u|>L\}}|\nabla|u||^2.
\end{aligned}
$$

That is, we have $\sum_i \nabla u_i \cdot \nabla(\Psi_i(u)) = d^2 - |\nabla u|^2$. By combining all the calculations we have done, we can conclude that the following equality is valid in the sense of distribution.

$$\partial_t\{\Phi(u)\} + \text{div}\{\Phi(u)u\} - \triangle\{\Phi(u)\} + d^2 - |\nabla u|^2 + (\frac{v}{|u|} - 1)u \cdot \nabla P = 0.$$

Finally, by coupling the above equality with the defining inequality which characterizes the notion of sutiable weak solutions, we conclude that the following inequality is valid in the sense of distribution.

$$\partial_t(\frac{v^2}{2}) + d^2 - \triangle(\frac{v^2}{2}) + \text{div}(\frac{v^2}{2}u) + \frac{v}{|u|}u \cdot \nabla P \leqslant 0.$$

So, we are done in proving Lemma 2.4.2. □

2.5 Some reflection on the mathematics of Chapter 4

It is time to discuss the second regularity criteria appearing in this thesis, which ensures the full regularity of a suitable weak solution u on $(0,T] \times \mathbb{R}^3$, provided u verifies the following condition that

$$\int_0^T \int_{\mathbb{R}^3} \frac{|u|^5}{log(1+|u|)} dxds < +\infty.$$

The regularity criteria mentioned above was established in [4] by the present author and A. Vasseur. As is mentioned in Section 4.1 of Chapter 4, Montgomery-Smith [22] had already introduced a log improvement for the

Prodi-Serrin Condition, which is similar to the one in [4]. However, we stress that the log-improvement given in [22] is in the time variable only.

Let us go back to our regularity criteria, with the imposed hypothesis to be $\int_0^T \int_{\mathbb{R}^3} \frac{|u|^5}{log(1+|u|)} dxds < +\infty$. At this point, let us recall that the famous Prodi-Serrin Criteria states that any Leray-Hopf weak solution u on $(0,T] \times \mathbb{R}^3$ verifying $u \in L^p(0,T;L^q(\mathbb{R}^3))$, with $\frac{2}{p} + \frac{3}{q} \leqslant 1$, must be smooth on $(0,T] \times \mathbb{R}^3$. We note that $p = q = 5$ is the homogenuous pair of p,q which verifies $\frac{2}{p} + \frac{3}{q} = 1$. So, we may as well say that the regularity criteria with the hypothesis $\int_0^T \int_{\mathbb{R}^3} \frac{|u|^5}{log(1+|u|)} dxds < +\infty$ can be seen as a log-improvement of the Prodi-Serrin criteria in the homogenuous case of $p = q = 5$. We want to stress that our log-improvement is in both space and time, and this is the main feature which differs our result from the one in [22].

Let us mention a recent paper [6] by Chen, M. Strain, Tsai, and yau, in which another kind of improvement of the Prodi-Serrin Criteria is provided. One of the main results obtained in [6] says that any axissymmetrical solution $v : (0,T] \times \mathbb{R}^3 \to \mathbb{R}^3$ of the incompressible Navier-Stokes equations verifying the hypothesis $|v(t,\cdot)| \leqslant \frac{C}{t^{\frac{1}{2}}}$ must be L^∞-bounded on $(0,T] \times \mathbb{R}^3$. Note that the function $\frac{1}{t^{\frac{1}{2}}}$ lies in $L^{2,\infty}(0,T)$. So, this result of [6] can be seen as an improvement from the Prodi-Serrin condition $L^2(0,T;L^\infty(\mathbb{R}^3))$ to $L^{2,\infty}(0,T;L^\infty(\mathbb{R}^3))$. Still, we note that this result of [6] relies a lot on the axissymmetrical condition imposed on the solutions. Moreover, it is worthwhile to note that the condition $|v(t,\cdot)| \leqslant \frac{C}{t^{\frac{1}{2}}}$ as considered in [6] is still invariant under the natural scaling for solutions of the incompressible Navier-Stokes equations. In contrast, the condition $\int_0^T \int_{\mathbb{R}^3} \frac{|u|^5}{log(1+|u|)} dxds < +\infty$ we consider is not scale invariant under the natural scaling $u_\lambda(t,x) = \lambda u(\lambda^2 t, \lambda x)$, and this fact may as well make our result become more interesting.

Now, let us go back to our regularity criteria with the hypothesis $\int_0^T \int_{\mathbb{R}^3} \frac{|u|^5}{log(1+|u|)} dxds < +\infty$. To give some idea of the way of proving the above regularity criteria, we will first employ the De Girogi method to establish the fact that there is a universal constant $C_0 \in (0,\infty)$ such that any suitable weak solution u on $[-1,1] \times \mathbb{R}^3$ verifying $\|u\|_{L^6([-1,1]\times\mathbb{R}^3)} \leqslant C_0$ must satisfies $\|u\|_{L^\infty([-\frac{1}{2},1]\times\mathbb{R}^3)} \leqslant 1$ (see Proposition 4.1.3 of Section 4.1). This fact, which is obtained though the De Giorgi's Method, will immediately lead to the fact that, for any sufficiently small $\lambda > 0$, we can control the L^∞ norm of u in x in an affine way by the term $\int_0^t \int_{\mathbb{R}^3} |u|^6 dxdt$, with a universal constant A_λ depending only on λ. More precisely, the above mentioned fact obtained though the De Giorgi's Method will lead to the conclusion that, for every $\lambda \in (0,2)$, there is some universal $A_\lambda \in (0,\infty)$ so that every suitable weak solution u on $(0,\infty) \times \mathbb{R}^3$ will enjoy the property that $\|u(t,\cdot)\|_{L^\infty(\mathbb{R}^3)} \leqslant A_\lambda\{1 + \int_0^t \int_{\mathbb{R}^3} |u|^6 dxdt\}$, for all $t > \lambda$ (this is proposition 4.1.2 of Section 4.1). This last conclusion (proposition 4.1.2), together with the given hypothesis $\int_0^T \int_{\mathbb{R}^3} \frac{|u|^5}{log(1+|u|)} dxds < +\infty$ will enable us to reduce the problem to become a O.D.E. type problem, which can easily be handled by a Gronwall argument (see section 4.7 for the technical details).

2.6 Some reflection on the mathematics of Chapters 3

To begin the discussion, let us recall that the regularity criteria we are going to establish in Chapter 3 ensures the full regularity of a suitable weak solution u on $(0,T] \times \mathbb{R}^3$, provided the quantity $F = \mathrm{div}(\frac{u}{|u|})$ verifies the condition $|\frac{u\cdot\nabla F}{|u|^{\gamma}}| \leqslant A|F|$ on $(0,T)\times\mathbb{R}^3$, with $A \in (0,\infty)$ to be some constant, and $\gamma \in [0,\frac{1}{3})$.

To explain why the regularity criteria stated above is interesting, we note that we can express F alternatively as $F = -\frac{u\cdot\nabla|u|}{|u|^2}$. This means that F can be seen as the first derivative of $|u|$ along the direction of the streamlines of the fluid. As a result, we may regard $\frac{u\cdot\nabla F}{|u|^{\gamma}}$ as the second derivative of $|u|$ along the direction of the streamlines. We note that the term $u \cdot \nabla u$ in the incompressible Navier-Stokes equations is the nonlinear transport term. As for transport equation in general, it is natural to consider the rate of change of a quantity dependent on u in the direction of the streamlines.

let us go back to the hypothesis $|\frac{u\cdot\nabla F}{|u|^{\gamma}}| \leqslant A|F|$ itself. With the interpretation given as above, we can say that the regularity criteria established in Chapter 3 ensures the full regularity of a suitable weak solution provided we have enough control on the second derivative of $|u|$ along the direction of the streamlines. Here, let us use this opportunity to mention that there are already some regularity criterion for Leary-Hopf solutions with conditions imposed on only one component of the velocity field u(See, for example, He [12]

,and Zhou [34])

To give some idea of the way of proving the above regularity criteria, assume that u is a suitable weak solution on $[0,1]\times\mathbb{R}^3$ verifying the condition $|\frac{u\cdot\nabla F}{|u|^\gamma}| \leqslant A|F|$, with $A\in(0,\infty)$ and $\gamma\in[0,\frac{1}{3})$. The C^∞-smoothness of u can be obtained as soon as we can prove the L^∞-boundedness of u on $[\frac{3}{4},1]\times\mathbb{R}^3$. To this end, we introduced a nested sequence $[T_k,1]$ of closed intervals, with $T_k = \frac{3}{4}-\frac{1}{4^{k+1}}$, and respectively a nested sequence of closed subsets $Q_k=[T_k,1]\times\mathbb{R}^3$ shrinking down to $[\frac{3}{4},1]\times\mathbb{R}^3$. We also need the truncations $v_k=\{|u|-R(1-\frac{1}{2^k})\}_+$ of $|u|$ at the high level $R(1-\frac{1}{2^k})$, in which R is some arbitrary chosen positive number greater than 1. Our plan is to employ the De Giorgi's Method to build up a nonlinear recurrence relation on the following sequence of nonnegative quantities

$$U_k=\|u\|^2_{L^\infty(T_k,1;L^2(\mathbb{R}^3))}+\|d_k\|^2_{L^2(Q_k)},$$

in which $d_k^2=\frac{R(1-\frac{1}{2^k})}{|u|}\chi_{\{v_k>0\}}|\nabla|u||^2+\frac{v_k}{|u|}|\nabla u|^2$. More precisely, we would like to build up the following nonlinear recurrence relation

$$U_k\leqslant\frac{C(u,A,\gamma)C_0^k}{R^\alpha}U_{k-1}^{\beta_0}, \tag{2.2}$$

with $C(u,A,\gamma)$ to be some constant depending only on u, A, and γ, $\alpha>0$,and $\beta_0>1$. Once the above nonlinear recurrence relation is established, $R>1$ will be chosen to be sufficiently large so that U_1 will become sufficiently small. As U_1 becomes small enough, the sequence U_k will converge to 0, as $k\to\infty$, which in turns will ensure the L^∞-boundness of u over $[\frac{3}{4}, 1] \times \mathbb{R}^3$.

At this point, let us say that nonlinear recurrence relation 2.2, once established, immediately gives our desired result, thanks to the term R^α appearing in the bottom of the right hand side of 2.2. However, we do encounter some difficulity in establishing 2.2 because of the fact that the pressure P depends on $|u|^2$ in a nonlocal way. We recall that, by taking divergence on the main equation of the incompressible Navier-Stokes equations, we yield the following equality which is valid in the distributional sense.

$$-\triangle P = \sum_{1 \leqslant i,j \leqslant 3} \partial_i \partial_j [u_i u_j].$$

Since, for almost every $t \in [0, T]$, $P(t, \cdot)$ is assumed to be L^p-integrable over $\mathbb{R}^3$,for some suitable $p \in (0, \infty)$, it turns out that we can characterize the pressure P alternatively via the Riesz's Transforms in the following way.

$$P = \sum_{i,j} R_i R_j [u_i u_j].$$

As is stated clearly in [32], we cannot apply the De Giorgi's Method directly on the equation $\partial_t u - \triangle u + u \cdot \nabla u + \nabla P = 0$. Instead, the De Giorgi's method can be applied on the following distributional inequality, and this is why we need the notion of suitable weak solution in the formulation of our regularity criteria (see Lemma 2.4.2 of section 2.4).

$$\partial_t(\frac{v_k^2}{2}) + d_k^2 - \triangle(\frac{v_k^2}{2}) + div(\frac{v_k^2}{2}u) + \frac{v_k}{|u|}u\nabla P \leqslant 0.$$

In the process of applying the De Giorgi's method to the above distributional inequality, we eventually recognize the main obstacle comes from the following pressure term.

$$\int_{T_{k-1}}^{1} |\int_{\mathbb{R}^3} \frac{v_k}{|u|} u \cdot \nabla P dx| dt. \tag{2.3}$$

At this point, we resolve the difficulty by using another sequence of truncations $w_k = \{|u| - R^\beta(1-\frac{1}{2^k})\}_+$ when we decompose the pressure P into $P = P_{k1} + P_{k2} + P_{k3}$, in which P_{k2} ,P_{k3} are the components of P contributed by large velocity values $|u|\chi_{\{w_k>0\}}$, while P_{k1} is the component of P contributed by small velocity values $|u|\chi_{\{w_k\leqslant 0\}}$ (For precise definitions of P_{k1},P_{k2},P_{k3}, see (3.5),(3.6),(3.7) of section 3.4). We emphasize that our way of decomposing the pressure in terms of the truncations w_k makes the pressure terms representing the effect of large velocity values become much easier to handle, and the secret lies in the fact that we are truncating $|u|$ at a higher level $R^\beta(1-\frac{1}{2^k})$ in the process of pressure decomposition. With such a decomposition of P, the pressure term appearing in 2.3 can be splitted into the following three terms.

- $\int_{Q_{k-1}} |\nabla(\frac{v_k}{|u|}) \cdot uP_{k1}| dxdt,$
- $\int_{Q_{k-1}} |(\frac{v_k}{|u|} - 1)u||P_{k2}| dxdt,$
- $\int_{Q_{k-1}} |(\frac{v_k}{|u|} - 1)u||P_{k3}| dxdt.$

Since we make the right move by truncating $|u|$ at the level $R^{\beta}(1-\frac{1}{2^k})$, with some suitable $\beta > \frac{3}{2}$, in the decomposition of the pressure, the second and the third pressure term among the above list can then be controlled by terms in the form of $\frac{C(u,A,\gamma)C_0^k}{R^{\alpha}}U_{k-1}^{\beta_0}$, with some suitable $\alpha > 0$, and $\beta_0 > 1$. To finish the picture, we also need to deal with the first pressure term in the above list, which represents the effect of those small velocity values on the whole pressure term. We remark that this is the place where our hypothesis $|\frac{u\cdot\nabla F}{|u|^{\gamma}}| \leqslant A|F|$, with $\gamma \in [0, \frac{1}{3})$, enters the picture.

Generally speaking, the component P_{k1} is the obstacle to the full regularity of any given suitable weak solution. P_{k1} makes the term $\int_{Q_{k-1}} |\nabla(\frac{v_k}{|u|}) \cdot uP_{k1}|dxdt$ difficult to control because R_iR_j, which is a Zygmund-Calderon operator, sends $L^{\infty}(\mathbb{R}^3)$ to $BMO(\mathbb{R}^3)$, instead of sending $L^{\infty}(\mathbb{R}^3)$ to $L^{\infty}(\mathbb{R}^3)$. This means that the pointwise values of P_{k1} can still be large even if $|u|\chi_{\{w_k\leqslant 0\}}$ is small.

In our present situation, the treatment we give in Chapter 3 is to control the pressure term contributed by P_{k1} by a weighted $|F|log^{+}|F|$ norm of $F = \text{div}(\frac{u}{|u|})$, and this will be achieved with the aid of a classical theorem of John and Nirenberg [15] (For further technical details, see Step 4 of Section 3.4). Finally, in the last part of Chapter 3 (Step 5 of section 3.4), we will use the hypothesis $|\frac{u\cdot\nabla F}{|u|^{\gamma}}| \leqslant A|F|$, with $\gamma \in [0, \frac{1}{3})$, to handle the above mentioned weighted $|F|log^{+}|F|$ norm of $\text{div}(\frac{u}{|u|})$.

Chapter 3

Smootheness criteria for solutions of incompressible Navier-Stokes equations along the streamlines

3.1 Statement of the the problem and it's related history

In this chapter, we consider the incompressible Navier-Stokes equations on $\mathbb{R}^3$, given by

$$\partial_t u - \triangle u + \mathrm{div}(u \otimes u) + \nabla p = 0, \tag{3.1}$$

$$\mathrm{div}(u) = 0, \tag{3.2}$$

where u is a vector-valued function representing the velocity of the fluid, and P is the pressure. Note that the pressure depends in a non local way on the velocity u. It can be seen as a Lagrange multiplier associated to the incompressible condition (3.2). The initial value problem of the above equation is endowed with the condition that $u(0, \cdot) = u_0 \in L^2(\mathbb{R}^3)$. Leray [20] and Hopf [13] had already established the existence of global weak solutions for the incompressible Navier-Stokes equations. In particular, Leray introduced a notion of weak solutions for the incompressible Navier-Stokes equations, and proved that, for every given initial datum $u_0 \in L^2(\mathbb{R}^3)$, there exists a global weak solution $u \in L^\infty(0,\infty; L^2(\mathbb{R}^3)) \cap L^2(0,\infty; \dot{H}^1(\mathbb{R}^3))$ verifying the incompressible Navier-Stokes equations in the sense of distribution. From that time on, much effort has been devoted to establish the global existence and uniqueness of smooth solutions to the incompressible Navier-Stokes equations. Different Criteria for regularity of the weak solutions have been proposed. The Prodi-Serrin conditions (see Serrin [29], Prodi [23], and [30]) states that any weak Leray-Hopf solution verifying $u \in L^p(0,\infty; L^q(\mathbb{R}^3))$ with $2/p + 3/q \leqslant 1$, $2 \leq p < \infty$, is regular on $(0,\infty) \times \mathbb{R}^3$. The limit case of $L^\infty(0,\infty; L^3(\mathbb{R}^3))$ has been solved very recently by L. Escauriaza, G. Seregin, and V. Sverak (see [9]). Here, we just mention a piece of work [4] by Chi Hin Chan and Alexis Vasseur which is devoted to a log improvement of the Prodi-Serrin criteria in the case in which $p = q = 5$. Other criterions have been later introduced, dealing with some derivatives of the velocity. Beale, Kato and Majda [1] showed the global regularity under the condition that the vorticity $\omega = \text{curl}\, u$ lies in $L^\infty(0,\infty; L^1(\mathbb{R}^3))$ (see Kozono and Taniuchi for improvement of this result [17]). Beirão da Veiga show in [2] that the boundedness of ∇u in $L^p(0,\infty; L^q(\mathbb{R}^3))$ for $2/p + 3/q = 2$, $1 < p < \infty$ ensures the global regularity. In [7], Constantin and Fefferman gave a condition involving only the direction of the vorticity. Until more recently, in a short paper [33], A. Vasseur gave another regularity criteria which states that any Leray-Hopf weak solution u for the Navier-Stokes equation satisfying $div(\frac{u}{|u|}) \in L^p(0,\infty; L^q(\mathbb{R}^3))$ with $\frac{2}{p} + \frac{3}{q} \leqslant \frac{1}{2}$ is necessary smooth on $(0,\infty) \times \mathbb{R}^3$. As we can see, the regularity criteria given in [33] is the one with some integrable condition imposed on

$div(\frac{u}{|u|})$. However, the goal of this chapter is to obtain the full regularity of a suitable weak solution u under some suitable assumption about the smoothness of $div(\frac{u}{|u|})$ along the streamlines of the fluid. More precisely, the goal of this chapter is to prove the following theorem

Theorem 3.1.1. Let u be a suitable weak solution for the incompressible Navier-Stokes equations on $(0,T]\times\mathbb{R}^3$ which satisfies the condition that $|\frac{u\cdot\nabla F}{|u|^{\gamma}}| \leqslant A|F|$, in which A is some positive constant, and γ is some positive constant for which $0 < \gamma < \frac{1}{3}$. Then, it follows that u is a smooth solution on $(0,T]\times\mathbb{R}^3$.

As for Theorem 3.1.1, we note that $F = div(\frac{u}{|u|})$ can be rewritten as $F = -\frac{u\cdot\nabla|u|}{|u|^2}$, and hence is the first derivative of $|u|$ along the streamlines of the fluid. Then, the condition appearing in the hypothesis of Theorem 3.1.1 can be seen as a constraint on the second derivative of $|u|$ along the streamlines. Theorem 3.1.1 itself shows that such a constraint on the second derivative of $|u|$ along the streamlines is enough to give the full regularity of the solution.

Before we proceed any further, let us say something about the term suitable weak solution. The concept of suitable weak solutions for the incompressible Navier-Stokes equations was first introduced by Caffarelli, Kohn, and Nirenberg in [3] for the purpose of developing the partial regularity theory for solutions of Navier-Stokes equations (Here, we should mention that before the publication of the partial regularity theorem in [3], Scheffer had already begun his analysis on the possible singular points of solutions to the incompressible

Navier-Stokes equations though a series of papers [24], [25], [26], [27]). By a suitable weak solution for the Navier-Stokes equations, we mean a Leray-Hopf weak solution $u \in L^{\infty}(0,T;L^2(\mathbb{R}^3)) \cap L^2(0,T;\dot{H}^1(\mathbb{R}^3))$ which satisfies the following inequality in the sense of distribution on $(0,T) \times \mathbb{R}^3$.

$$\partial_t(\frac{|u|^2}{2}) + div(\frac{|u|^2}{2}u) + div(Pu) + |\nabla u|^2 - \triangle(\frac{|u|^2}{2}) \leqslant 0.$$

Here, we decide to work with suitable weak solutions instead of just Leray-Hopf weak solutions because suitable weak solutions enjoy some very nice properties such as the partial regularity Theorem due to Caffarelli, Kohn, and Nirenberg in their joint work [3]. Now, let us turn our attention back to Theorem 3.1.1. Indeed the conclusion for Theorem 3.1.1 will follow at once provided if we can prove the following proposition.

Proposition 3.1.2. Let u be a suitable weak solution for the incompressible Navier-Stokes equations on $(0,1] \times \mathbb{R}^3$ which satisfies the condition that $|\frac{u \cdot \nabla F}{|u|^{\gamma}}| \leqslant A|F|$, where A is some positive constant, and γ is some positive number satisfying $0 < \gamma < \frac{1}{3}$. It then follows that u is essentially bounded over the region $[\frac{3}{4},1] \times \mathbb{R}^3$. That is, we have $\|u\|_{L^{\infty}([\frac{3}{4},1]\times\mathbb{R}^3)} < \infty$.

Before we devote our effort to prove proposition 3.1.2, let us first explain why proposition 3.1.2 will lead to the conclusion of Theorem 3.1.1 as follow. Assume that proposition 3.1.2 is indeed true. Without the loss of generality, let us assume that u is a suitable weak solution for the incompressible Navier-Stokes equations on $(0,1] \times \mathbb{R}^3$ satisfying the hypothesis of Theorem 3.1.1 (we note that if our suitable weak solution u is over $(0,T]\times\mathbb{R}^3$, with T to be some positive number other than 1, we can always rescale our weak solution u). Now, proposition 3.1.2 automatically tells us that u is essentially bounded on the region $[\frac{3}{4},1]\times\mathbb{R}^3$. So, over such a region, we can apply the Serrin criterion with $p=q=\infty$ to conclude that u is smooth over $(\frac{3}{4},1)\times\mathbb{R}^3$. So, the only question remains is how to justify that u is also smooth over $(0,\frac{7}{8})\times\mathbb{R}^3$. So, to finish our job, let $\tau\in(0,\frac{7}{8})$ be arbitrary chosen and fixed, and let us consider the function $u_\lambda(t,x)=\lambda u(\lambda^2 t,\lambda x)$, with $\lambda=(\frac{8\tau}{7})^{\frac{1}{2}}$. Notice that u_λ is then another suitable weak solution on $(0,1]\times\mathbb{R}^3$, which satisfies the same hypothesis of Theorem 3.1.1(with a different constant A_λ, of course). So, we can invoke proposition 3.1.2 again to conclude that u_λ is essentially bounded over $[\frac{3}{4},1]\times\mathbb{R}^3$. However, this means the same thing as saying that our original suitable weak solution u is essentially bounded over the region $[\frac{6\tau}{7},\frac{8\tau}{7}]\times\mathbb{R}^3$, and hence u must be smooth over the region $(\frac{6\tau}{7},\frac{8\tau}{7})\times\mathbb{R}^3$. Since the number $\tau\in(0,\frac{7}{8})$ is arbitrary chosen in the above argument, we conclude that u must be smooth over $(0,1)\times\mathbb{R}^3$, provided that proposition 3.1.2 is valid. So, it is clear that the main task of the whole paper is to prove proposition 3.1.2, which is what we will do in the following sections.

3.2 Basic setting for the chapter

In order to prove proposition 3.1.2, we would like to use the method of energy decompositions with respect to a sequence of cutting functions $v_k=\{|u|-R(1-\frac{1}{2^k})\}_+$ as introduced by A. Vasseur in [32]. Indeed, A.

Vasseur was the first to use such a method of energy decompositions inherted from De Giorgi [8] to give another proof of the famous Partial Regularity Theorem of Caffarelli, Kohn and Nirenberg (see [32]). So, we would like to introduce some notation first. Then, we will state one lemma and one proposition which are related to the proof of proposition 3.1.2. So, let us fix our notation as follow.

- for each $k \geqslant 0$, let $Q_k = [T_k, 1] \times \mathbb{R}^3$, in which $T_k = \frac{3}{4} - \frac{1}{4^{k+1}}$.
- for each $k \geqslant 0$, let $v_k = \{|u| - R(1 - \frac{1}{2^k})\}_+$.
- for each $k \geqslant 0$, let $d_k^2 = \frac{R(1-\frac{1}{2^k})}{|u|}\chi_{\{|u|>R(1-\frac{1}{2^k})\}}|\nabla|u||^2 + \frac{v_k}{|u|}|\nabla u|^2$.
- for each $k \geqslant 0$, let $U_k = \frac{1}{2}\|v_k\|^2_{L^\infty(T_k,1;L^2(\mathbb{R}^3))} + \int_{T_k}^1 \int_{\mathbb{R}^3} d_k^2 dx\, dt$.

With the above setting, we are now ready to state the lemma and proposition which are related to proposition 3.1.2 as follow.

Proposition 3.2.1. Let u be a suitable weak solution for the incompressible Navier-Stokes equations on $[0, 1] \times \mathbb{R}^3$ which satisfies the condition that $|\frac{u\cdot\nabla F}{|u|^\gamma}| \leqslant A|F|$, where A is some finite-positive constant, and γ is some psoitive number satisfying $0 < \gamma < \frac{1}{3}$. Then, there exists some constant $C_{p,\beta}$, depending only on $1 < p < \frac{5}{4}$, and $\beta > \frac{6-3p}{10-8p}$,and also some constants $0 < \alpha, K < \infty$, which do depend on our suitable weak solution u, such that the following inequality holds

$$U_k \leqslant$$
$$C_{p,\beta} 2^{\frac{10k}{3}} \{ \frac{1}{R^{\beta \frac{10-8p}{3p} - \frac{2-p}{p}}} \|u\|^{2(1-\frac{1}{p})}_{L^\infty(0,1;L^2(\mathbb{R}^3))} U^{\frac{5-p}{3p}}_{k-1} +$$
$$(1+A)(1+\frac{1}{\alpha})(1+K^{1-\frac{1}{p}})(1+\|u\|_{L^\infty(0,1;L^2(\mathbb{R}^3))}) \times$$
$$[(\frac{1}{R^{\frac{10}{3}-2p\beta+1-\gamma-p}})^{\frac{1}{p}} U^{\frac{5}{3p}}_{k-1} + \frac{1}{R^{\frac{10}{3}-2\beta-\gamma}} U^{\frac{5}{3}}_{k-1}]\},$$

for every sufficiently large $R > 1$.

Here, let us make some important comments on the conclusion of proposition 3.2.1. As indicated by the inequality which appears in the conclusion of proposition 3.2.1, it is important for us to emphasis that those terms such as $R^{\beta \frac{10-8p}{3p} - \frac{2-p}{p}}$, $R^{\frac{10}{3}-2p\beta+1-\gamma-p}$, and $R^{\frac{10}{3}-2\beta-\gamma}$ should all appear in the denomerator. But unfortunately, the standard approach of carrying out decompositions on both the energy and pressure by using the same sequence of cutting functions $v_k = \{|u| - R(1-\frac{1}{2^k})\}_+$ is not powerful enough to ensure such a result as promised by proposition 3.2.1. So, in proving proposition 3.2.1, we will carry out the decomposition of the pressure P by introducing another sequence of cutting functions $w_k = \{|u| - R^\beta(1-\frac{1}{2^k})\}_+$, for $k \geqslant 1$, where $\beta > \frac{3}{2}$ should be some suitable index sufficiently close to $\frac{3}{2}$ (for more detail, see inequalities (3.5), (3.6), and (3.7)). We remark that the inequality $\|\chi_{\{w_k \geqslant 0\}}\|_{L^q(Q_{k-1})} \leqslant \frac{2^{\frac{10k}{3q}}}{R^{\beta \frac{10}{3q}}} C_q U^{\frac{5}{3q}}_{k-1}$, for $q \geqslant 1$ provides us with the term $\frac{1}{R^{\frac{10\beta}{3q}}}$ which decays to 0 in a way much faster than $\frac{1}{R^{\frac{10}{3q}}}$ as $R \to \infty$, and this is the reason why we use the cutting functions w_k instead of v_k in carrying out the decomposition of the pressure P.

Let us first show that Proposition 3.2.1 provides the result of Proposition 3.1.2. First, we show that the sequence $\{U_k\}_{k\geqslant 1}$ converges to 0, when k goes to infinity, provided R is chosen to be sufficiently large. We can use for instance the following easy lemma (see [32]):

Lemma 3.2.2. For any given constants B, $\beta > 1$, there exists some constant C_0^* such that for any sequence $\{a_k\}_{k\geqslant 1}$ satisfying $0 < a_1 \leq C_0^*$ and $a_k \leqslant B^k a_{k-1}^{\beta}$, for any $k \geqslant 1$, we have $lim_{k\to\infty} a_k = 0$.

With the assistance of lemma 3.2.2, we will derive the conclusion of proposition 3.1.2 from proposition 3.2.1 in the following way. Let u be a suitable weak solution which satisfies the hypothesis of proposition 3.1.2. Then, according to the conclusion of proposition 3.2.1, we know that if the number p with $1 < p < \frac{5}{4}$ is chosen to be sufficiently close to 1, and if the number $\beta > \frac{6-3p}{10-8p}$ is chosen to be sufficiently close to $\frac{3}{2}$, it follows that the sequence $\{U_k\}_{k=1}^{\infty}$ will satisfies the following inequality

$$U_k \leqslant \frac{D}{R^{\Phi(p,\beta,\gamma)}} 2^{\frac{10k}{3}} \{U_{k-1}^{\frac{5-p}{3p}} + U_{k-1}^{\frac{5}{3p}} + U_{k-1}^{\frac{5}{3}}\}, \tag{3.3}$$

in which D stands for some positive constant which depends on the choice of the suitable weak solution u but independent of R, and $\Phi(p,\beta,\gamma)$ is some positive index which depends only on p, β, and γ. Now, let us apply

Lemma 3.2.2 to deduce that there is some constant C_0^* , such that for any sequence $\{a_k\}_{k=1}^{\infty}$ satisfying $0 < a_1 \leqslant C_0^*$ and $a_k \leqslant 2^{\frac{10k}{3}} a_{k-1}^{\frac{5-p}{3p}}$for all $k \geqslant 1$, we have $lim_{k\to\infty} a_k = 0$. We then choose $R > 1$ to be sufficiently large, so that we have $\frac{3D}{R^{\Phi(p,\beta,\gamma)}} < 1$, and that $U_1 \leqslant min\{1, C_0^*\}$. With this suitable choice of R, we see that the sequence $\{U_k\}_{k=0}^{\infty}$ will satisfies the conditions that $U_1 \leqslant C_0^*$ and $U_k \leqslant 2^{\frac{10k}{3}} U_{k-1}^{\frac{5-p}{3p}}$, for all $k \geqslant 1$. Hence it follows that $lim_{k\to\infty} U_k = 0$. However, because for almost every $t \in [\frac{3}{4}, 1]$, we have

$$\int_{\mathbb{R}^3} |u(t,x) - R|^2 dx \leqslant 2 lim_{k\to\infty} U_k = 0.$$

It follows at once that $|u| \leqslant R$, almost everywhere over $[\frac{3}{4}, 1] \times \mathbb{R}^3$. This indicates that u is essentially bounded over $[\frac{3}{4}, 1] \times \mathbb{R}^3$. Hence, we see that the conclusion of proposition 3.1.2 follows provided that proposition 3.2.1 is indeed valid.

For this reason, the main task of this paper is to give a detailed proof of proposition 3.2.1, which is what we will achieve in the following sections. More precisely, after we have given some preliminaries in section 3.3, we will actually carry out the proof of proposition 3.2.1 in section 3.4. Moreover, the proof of proposition 3.2.1 as presented in section 3.4 will be splitted into five successive steps. In step one, we will derive the inequality of the level set energy which gives an estimate of U_k with respect to the pressure term $\int_{T_{k-1}}^1 |\int_{\mathbb{R}^3} \frac{v_k}{|u|} u \nabla P dx| ds$. In step two, we will decompose the pressure P into

$P = P_{k1} + P_{k2} + P_{k3}$ by using the cutting functions $w_k = \{|u| - R^{\beta}(1 - \frac{1}{2^k})\}_+$, with $\beta > \frac{3}{2}$ to be some sutiable index sufficiently close to $\frac{3}{2}$ (for more detail see equations (3.5), (3.6), and (3.7)). Here, we remark that P_{k2} and P_{k3} represent the effect of large velocity values $|u|\chi_{\{|u|>R^{\beta}(1-\frac{1}{2^k})\}}$ on the pressure, while P_{k1} represents the effect of those velocity values smaller than $R^{\beta}(1 - \frac{1}{2^k})$ on the pressure. Step three is didicated to the control of the two pressure terms involving big velocity values. Thanks to the introduction of the cutting functions $w_k = \{|u| - R^{\beta}(1 - \frac{1}{2^k})\}_+$ in the decomposition of the pressure, the control on these two terms can then be performed successfully. In step four and step five, we will control the pressure term $\int_{T_{k-1}}^{1} |\int_{\mathbb{R}^3} \nabla(\frac{v_k}{|u|}) u P_{k1} dx| ds$ which depends on those velocity values smaller than $R^{\beta}(1 - \frac{1}{2^k})$. In step four, we will show that such a pressure term depending on those velocity values smaller than $R^{\beta}(1 - \frac{1}{2^k})$ can be controlled by a weighted $|F| log^+ |F|$ norm of $div(\frac{u}{|u|})$. We will finally show in step five that, in some specific way, we can eventually control the pressure term $\int_{T_{k-1}}^{1} |\int_{\mathbb{R}^3} \nabla(\frac{v_k}{|u|}) u P_{k1} dx| ds$ successfully by employing the hypothesis $\frac{|u \cdot \nabla F|}{|u|^{\gamma}} \leqslant A|F|$ of proposition 3.2.1.

3.3 Preliminaries for the proof of proposition 3.2.1

Lemma 3.3.1. There exists some constant $C > 0$, such that for any $k \geqslant 1$, and any $f \in L^{\infty}(T_k, 1; L^2(\mathbb{R}^3))$ with $\nabla f \in L^2(Q_k)$, we have $\|f\|_{L^{\frac{10}{3}}(Q_k)} \leqslant C\|f\|_{L^{\infty}(T_k,1;L^2(\mathbb{R}^3))}^{\frac{2}{5}} \|\nabla f\|_{L^2(Q_k)}^{\frac{3}{5}}$.

Proof. By Sobolev-embedding Theorem, there is a constant C, depending only on the dimension of $\mathbb{R}^3$, such that

$$(\int_{\mathbb{R}^3} |f(t,x)|^6 dx)^{\frac{1}{6}} \leqslant C(\int_{\mathbb{R}^3} |\nabla f(t,x)|^2 dx)^{\frac{1}{2}}.$$

for any $t \in [T_k, 1]$, where $k \geqslant 1$, and f is some function which verifies $f \in L^\infty(T_k, 1; L^2(\mathbb{R}^3))$, and $\nabla f \in L^2(Q_k)$. By taking power 2 on both sides of the above inequality and then taking integration along the variable $t \in [T_k, 1]$, we yield

$$\int_{T_k}^1 (\int_{\mathbb{R}^3} |f|^6 dx)^{\frac{1}{3}} dt \leqslant C^2 \int_{T_k}^1 \int_{\mathbb{R}^3} |\nabla f|^2 dx\, dt.$$

On the other hand, by Holder's inequality, we have

$$\begin{aligned} \|f\|_{L^{\frac{10}{3}}(Q_k)}^{\frac{10}{3}} &= \int_{T_k}^1 \int_{\mathbb{R}^3} |f|^2 |f|^{\frac{4}{3}} dx\, dt \\ &\leqslant \int_{T_k}^1 (\int_{\mathbb{R}^3} |f|^6 dx)^{\frac{1}{3}} (\int_{\mathbb{R}^3} |f|^2 dx)^{\frac{2}{3}} dt \\ &\leqslant \|f\|_{L^\infty(T_k,1;L^2(\mathbb{R}^3))}^{\frac{4}{3}} \|f\|_{L^2(T_k,1;L^6(\mathbb{R}^3))}^2. \end{aligned}$$

By taking the advantage that $\|f\|_{L^2(T_k,1;L^6(\mathbb{R}^3))} \leqslant C\|\nabla f\|_{L^2(Q_k)}$, we yield

$$\|f\|_{L^{\frac{10}{3}}(Q_k)}^{\frac{10}{3}} \leqslant C^2 \|f\|_{L^\infty(T_k,1;L^2(\mathbb{R}^3))}^{\frac{4}{3}} \|\nabla f\|_{L^2(Q_k)}^2.$$

Hence, we have

$$\|f\|_{L^{\frac{10}{3}}(Q_k)} \leqslant C \|f\|_{L^\infty(T_k,1;L^2(\mathbb{R}^3))}^{\frac{2}{5}} \|\nabla f\|_{L^2(Q_k)}^{\frac{3}{5}}.$$

so, we are done □

Lemma 3.3.2. For any $1 < q < \infty$, we have $\|\chi_{\{v_k>0\}}\|_{L^q(Q_{k-1})} \leqslant \frac{2^{\frac{10k}{3q}}}{R^{\frac{10}{3q}}} C^{\frac{1}{q}} U_{k-1}^{\frac{5}{3q}}$

.

Proof. First, we have to notice that $\{v_k > 0\}$ is a subset of $\{v_{k-1} > \frac{R}{2^k}\}$, hence we have

$$\int_{Q_{k-1}} \chi_{\{v_k>0\}} \leqslant \int_{Q_{k-1}} \chi_{\{v_{k-1}>\frac{R}{2^k}\}} \leqslant \frac{2^{\frac{10k}{3}}}{R^{\frac{10}{3}}} \int_{Q_{k-1}} |v_{k-1}|^{\frac{10}{3}}.$$

By our previous Lemma, we have

$$\begin{aligned} \|v_{k-1}\|_{L^{\frac{10}{3}}(Q_{k-1})}^{\frac{10}{3}} \leqslant & \\ & C^2 \|v_{k-1}\|_{L^\infty(T_{k-1},1;L^2(\mathbb{R}^3))}^{\frac{4}{3}} \|\nabla v_{k-1}\|_{L^2(Q_{k-1})}^2 \\ & \leqslant C^2 (U_{k-1}^{\frac{1}{2}})^{\frac{4}{3}} \|d_{k-1}\|_{L^2(Q_{k-1})}^2 \\ & \leqslant C^2 U_{k-1}^{\frac{2}{3}} U_{k-1} \\ & = C^2 U_{k-1}^{\frac{5}{3}}. \end{aligned}$$

So, it follows that $\int_{Q_{k-1}} \chi_{\{v_k>0\}} \leqslant \frac{2^{\frac{10k}{3}}}{R^{\frac{10}{3}}} C^2 U_{k-1}^{\frac{5}{3}}$, and hence we have $\|\chi_{\{v_k>0\}}\|_{L^q(Q_{k-1})} \leqslant \frac{2^{\frac{10k}{3q}}}{R^{\frac{10}{3q}}} C^{\frac{1}{q}} U_{k-1}^{\frac{5}{3q}}$, where C is some universal constant. So, we are done. □

Just as we have said before, we will need to decompose the pressure by employing the sequence of cutting functions $w_k = \{|u| - R^\beta(1 - \frac{1}{2^k})\}_+$, for $k \geqslant 1$. We also said that we prefer to do this because the cutting functions w_k satisfies the following inequality which can be justified in the same way as lemma 3.3.2.

Lemma 3.3.3. For every $q \geqslant 1$, we have $\|\chi_{\{w_k>0\}}\|_{L^q(Q_{k-1})} \leqslant \frac{1}{R^{\frac{10\beta}{3q}}} 2^{\frac{10k}{3q}} C_q U_{k-1}^{\frac{5}{3q}}$, for all $k \geqslant 1$, in which C_q is some constant depending only on q.

Indeed, in dealing with the pressure terms, we will invoke the lemma 3.3.3 without explicit mention.

In the proof of Lemma 3.3.2, we have used the fact that $|\nabla v_k| \leqslant d_k$, whose justification will be given immediately in the following paragraph. Before we leave this section, we also want to list out some inequalities which will often be used in the proof of proposition 3.2.1 as follow:

- $|(1 - \frac{v_k}{|u|})u| \leqslant R(1 - \frac{1}{2^k})$.
- $\frac{v_k}{|u|}|\nabla u| \leqslant d_k$.
- $\chi_{\{v_k>0\}}|\nabla|u|| \leqslant d_k$.
- $|\nabla v_k| \leqslant d_k$.
- $|\nabla(\frac{v_k}{|u|}u)| \leqslant 3d_k$.

Now, we first want to justify the validity of $|(1 - \frac{v_k}{|u|})u| \leqslant R(1 - \frac{1}{2^k})$. In the case in which the point (t, x) satisfies $|u(t, x)| \leqslant R(1 - \frac{1}{2^k})$, we have $v_k(t, x) = 0$, and hence it follows that

$$|\{1 - \frac{v_k(t,x)}{|u(t,x)|}\}u(t,x)| = |u(t,x)| \leqslant R(1 - \frac{1}{2^k}).$$

In the case in which (t,x) satisfies $|u(t,x)| > R(1-\frac{1}{2^k})$, we have $v_k(t,x) = |u(t,x)| - R(1-\frac{1}{2^k})$, and hence it follows that

$$|\{1-\frac{v_k}{|u|}\}u(t,x)| = |1-\frac{|u|-R(1-\frac{1}{2^k})}{|u|}||u| = R(1-\frac{1}{2^k}).$$

So, no matter in which case, we always have the conclusion that $|(1-\frac{v_k}{|u|})u| \leqslant R(1-\frac{1}{2^k})$.

Next, according to the definition of d_k^2, we can carry out the following estimation

$$d_k^2 \geqslant \frac{v_k}{|u|}|\nabla u|^2 \geqslant \{\frac{v_k}{|u|}|\nabla u|\}^2.$$

Hence, by taking square root, it follows at once that $d_k \geqslant \frac{v_k}{|u|}|\nabla u|$.

We now turn our attention to the inequality $\chi_{\{|u|>R(1-\frac{1}{2^k})\}}|\nabla|u|| \leqslant d_k$. To justify it, we recall that $|\nabla u| \geqslant |\nabla|u||$. Hence, it follows from the definition of d_k^2 that

$$d_k^2 \geqslant \frac{R(1-\frac{1}{2^k})}{|u|}\chi_{\{|u|>R(1-\frac{1}{2^k})\}}|\nabla|u||^2 + \{1-\frac{R(1-\frac{1}{2^k})}{|u|}\}\chi_{\{|u|>R(1-\frac{1}{2^k})\}}|\nabla|u||^2.$$

So, by simplifying the right-hand side of the above inequality, we can deduce that $d_k^2 \geqslant \chi_{\{|u|>R(1-\frac{1}{2^k})\}}|\nabla|u||^2$. Hence, we have $d_k \geqslant \chi_{\{|u|>R(1-\frac{1}{2^k})\}}|\nabla|u||$.

In addition, since it is obvious to see that $\nabla v_k = \chi_{\{|u|>R(1-\frac{1}{2^k})\}}\nabla|u|$, we also have the result that $|\nabla v_k| \leqslant d_k$.

Finally, we want to justify the inequality that $|\nabla(\frac{v_k}{|u|}u)| \leqslant 3d_k$. So, we notice that, by applying the product rule, we have

$$\nabla(\frac{v_k}{|u|}u) = \nabla(v_k)\frac{u}{|u|} + \frac{v_k}{|u|}\nabla u - \frac{v_k}{|u|^2}u\nabla|u|.$$

However, since $\frac{v_k}{|u|}|\nabla u| \leqslant d_k$, and $|\frac{v_k}{|u|^2}u\nabla|u|| \leqslant \chi_{\{|u|>R(1-\frac{1}{2^k})\}}|\nabla|u|| \leqslant d_k$, it follows at once from the above expression that $|\nabla(\frac{v_k}{|u|}u)| \leqslant 3d_k$.

3.4 proof of proposition 3.2.1

Step one

To begin the argument, we recall that, by multiplying the equation $\partial_t u - \triangle u + div(u \otimes u) + \nabla P = 0$ by the term $\frac{v_k}{|u|}u$, we yield the following inequality formally, which is indeed valid in the sense of distribution

$$\partial_t(\frac{v_k^2}{2}) + d_k^2 - \triangle(\frac{v_k^2}{2}) + div(\frac{v_k^2}{2}u) + \frac{v_k}{|u|}u\nabla P \leqslant 0.$$

Next, let us consider the variables σ , t verifying $T_{k-1} \leqslant \sigma \leqslant T_k \leqslant t \leqslant 1$. Then, we have

- $\int_\sigma^t \int_{\mathbb{R}^3} \partial_t(\frac{v_k^2}{2})dx\,ds = \int_{\mathbb{R}^3} \frac{v_k^2(t,x)}{2}dx - \int_{\mathbb{R}^3} \frac{v_k^2(\sigma,x)}{2}dx.$
- $\int_\sigma^t \int_{\mathbb{R}^3} \triangle(\frac{v_k^2}{2})dx\,ds = 0.$
- $\int_\sigma^t \int_{\mathbb{R}^3} div(\frac{v_k^2}{2}u)dx\,ds = 0.$

So, it is straightforward to see that

$$\int_{\mathbb{R}^3} \frac{v_k^2(t,x)}{2}dx + \int_\sigma^t \int_{\mathbb{R}^3} d_k^2 dx\, ds \leqslant \int_{\mathbb{R}^3} \frac{v_k^2(\sigma,x)}{2}dx + \int_\sigma^t |\int_{\mathbb{R}^3} \frac{v_k}{|u|} u\nabla P dx| ds,$$

for any σ, t satisfying $T_{k-1} \leqslant \sigma \leqslant T_k \leqslant t \leqslant 1$. By taking the average over the variable σ, we yield

$$\int_{\mathbb{R}^3} \frac{v_k^2(t,x)}{2}dx + \int_{T_k}^t \int_{\mathbb{R}^3} d_k^2 dx\, ds \leqslant \frac{4^{k+1}}{6} \int_{T_{k-1}}^{T_k} \int_{\mathbb{R}^3} v_k^2 + \int_{T_{k-1}}^t |\int_{\mathbb{R}^3} \frac{v_k}{|u|} u\nabla P dx| ds.$$

By taking the sup over $t \in [T_k, 1]$. the above inequality will give the following

$$U_k \leqslant \frac{4^{k+1}}{6} \int_{Q_{k-1}} v_k^2 + \int_{T_{k-1}}^1 |\int_{\mathbb{R}^3} \frac{v_k}{|u|} u\nabla P dx| ds.$$

But, from Lemma 3.3.2 and Holder's inequality, we have

$$\begin{aligned}
\int_{Q_{k-1}} v_k^2 = \int_{Q_{k-1}} v_k^2 \chi_{\{v_k>0\}} & \\
&\leqslant \left(\int_{Q_{k-1}} v_k^{\frac{10}{3}}\right)^{\frac{3}{5}} \|\chi_{\{v_k>0\}}\|_{L^{\frac{5}{2}}(Q_{k-1})} \\
&\leqslant \|v_k\|^2_{L^{\frac{10}{3}}(Q_{k-1})} \frac{2^{\frac{4k}{3}}}{R^{\frac{4}{3}}} C^{\frac{2}{5}} U_{k-1}^{\frac{2}{3}} \\
&\leqslant \|v_{k-1}\|^2_{L^{\frac{10}{3}}(Q_{k-1})} \frac{2^{\frac{4k}{3}}}{R^{\frac{4}{3}}} C^{\frac{2}{5}} U_{k-1}^{\frac{2}{3}} \\
&\leqslant C U_{k-1}^{\frac{5}{3}} \frac{2^{\frac{4k}{3}}}{R^{\frac{4}{3}}}.
\end{aligned}$$

As a result, we have the following conclusion

$$U_k \leqslant \frac{2^{\frac{10k}{3}}}{R^{\frac{4}{3}}} C U_{k-1}^{\frac{5}{3}} + \int_{T_{k-1}}^{1} |\int_{\mathbb{R}^3} \frac{v_k}{|u|} u \nabla P dx| ds. \tag{3.4}$$

Step two

Now, in order to estimate the term $\int_{T_{k-1}}^{1} |\int_{\mathbb{R}^3} \frac{v_k}{|u|} u \nabla P dx| ds$, we would like to carry out the following computation

$$\begin{aligned} -\triangle P = & \sum \partial_i \partial_j (u_i u_j) \\ & = \sum \partial_i \partial_j \{(1 - \frac{w_k}{|u|}) u_i (1 - \frac{w_k}{|u|}) u_j\} \\ & + 2 \sum \partial_i \partial_j \{(1 - \frac{w_k}{|u|}) u_i \frac{w_k}{|u|} u_j\} \\ & + \sum \partial_i \partial_j \{\frac{w_k}{|u|} u_i \frac{w_k}{|u|} u_j\}, \end{aligned}$$

in which w_k is given by $w_k = \{|u| - R^{\beta}(1 - \frac{1}{2^k})\}_+$, and β is simply the arbitrary index involved in proposition 3.2.1. This motivates us to decompose P as $P = P_{k1} + P_{k2} + P_{k3}$, in which

$$-\triangle P_{k1} = \sum \partial_i \partial_j \{(1 - \frac{w_k}{|u|}) u_i (1 - \frac{w_k}{|u|}) u_j\}, \tag{3.5}$$

$$-\triangle P_{k2} = \sum \partial_i \partial_j \{2(1 - \frac{w_k}{|u|}) u_i \frac{w_k}{|u|} u_j\} \tag{3.6}$$

$$-\triangle P_{k3} = \sum \partial_i \partial_j \{\frac{w_k}{|u|} u_i \frac{w_k}{|u|} u_j\}. \tag{3.7}$$

Here, we have to remind ourself that the cutting functions which are used in the decomposition of the pressure are indeed $w_k = \{|u| - R^{\beta}(1 - \frac{1}{2^k})\}_+$, for all $k \geqslant 0$, in which β is some suitable index strictly greater than $\frac{3}{2}$. With respect to the cutting functions w_k, we need to define the respective D_k as follow:

$$D_k^2 = \frac{R^{\beta}(1 - \frac{1}{2^k})}{|u|}\chi_{\{w_k > 0\}}|\nabla|u||^2 + \frac{w_k}{|u|}|\nabla u|^2.$$

Then, just like what happens to the cutting functions v_k, we have the following assertions about the cutting functions w_k, which are easily verified.

- $|\nabla w_k| \leqslant D_k$, for all $k \geqslant 0$.
- $|\nabla(\frac{w_k}{|u|}u_i)| \leqslant 3D_k$, for all $k \geqslant 0$, and $1 \leqslant i \leqslant 3$.
- $|\nabla(\frac{w_k}{|u|})u_i| \leqslant 2D_k$, for any $k \geqslant 0$, and $1 \leqslant i \leqslant 3$.

Besides these, we also need the following lemma which links D_k to d_k.

Lemma 3.4.1. There is some sufficiently large $R_0 > 1$, such that whenever $R > R_0$ and $k \geqslant 1$, we have $D_k \leqslant 5^{\frac{1}{2}} d_k$.

Proof. since $\frac{R^{\beta} - R}{R^{\beta}}$ trends to the limiting value 1, as R trends to ∞. So, there is some sufficiently large $R_0 > 1$ for which $(R^{\beta} - R) > \frac{R^{\beta}}{2}$, for all $R > R_0$. Now, notice that $\{w_k > 0\}$ is a subset of $\{v_k > (R^{\beta} - R)(1 - \frac{1}{2^k})\}$, for all $k \geqslant 0$. Hence, it follows that $\{w_k > 0\}$ is a subset of $\{v_k > \frac{R^{\beta}}{4}\}$, for all $k \geqslant 1$ and $R > R_o$. As a result, we can carry out the following computation

$$
\begin{aligned}
D_k^2 = \frac{R^\beta(1-\frac{1}{2^k})}{|u|}\chi_{\{w_k>0\}}|\nabla|u||^2 + \frac{w_k}{|u|}|\nabla u|^2 \\
&\leqslant \frac{R^\beta}{|u|}\chi_{\{w_k>0\}}|\nabla|u||^2 + \frac{v_k}{|u|}|\nabla u|^2 \\
&\leqslant \frac{4v_k}{|u|}\chi_{\{w_k>0\}}|\nabla|u||^2 + \frac{v_k}{|u|}|\nabla u|^2 \\
&\leqslant \frac{5v_k}{|u|}|\nabla u|^2 \leqslant 5d_k^2,
\end{aligned}
$$

for any $k \geqslant 1$, and $R > R_0$. Hence, we have $D_k \leqslant 5^{\frac{1}{2}} d_k$, for all $k \geqslant 1$, and all $R > R_0$. So, we are done. □

Now, let us recall that we have already used the cutting functions w_k to obtain the decomposition $P = P_{k1} + P_{k2} + P_{k3}$, in which P_{k1}, P_{k2}, andP_{k3} are described in equations (3.5), (3.6), and (3.7) respectively.

Hence, we have

$$
\begin{aligned}
\int_{T_{k-1}}^1 | \int_{\mathbb{R}^3} \frac{v_k}{|u|} u \nabla P dx| dt \\
&\leqslant \int_{T_{k-1}}^1 | \int_{\mathbb{R}^3} \nabla(\frac{v_k}{|u|}) u P_{k1} dx| dt \\
&+ \int_{Q_{k-1}} (1 - \frac{v_k}{|u|})|u||\nabla P_{k2}| \\
&+ \int_{Q_{k-1}} (1 - \frac{v_k}{|u|})|u||\nabla P_{k3}|.
\end{aligned}
$$

Step 3

We are now ready to deal with the term $\int_{Q_{k-1}}(1-\frac{v_k}{|u|})|u||\nabla P_{k2}|$. For this purpose, let p be such that $1<p<\frac{5}{4}$, and let $q=\frac{p}{p-1}$, so that $2<q<\infty$. By applying Holder's inequality, we find that

$$\begin{aligned}
&\|(1-\frac{v_k}{|u|})u\|_{L^q(\mathbb{R}^3)} \\
&\qquad \leqslant \|(1-\frac{v_k}{|u|})u\|_{L^2(\mathbb{R}^3)}^{\frac{2}{q}}\|(1-\frac{v_k}{|u|})u\|_{L^\infty(\mathbb{R}^3)}^{1-\frac{2}{q}} \\
&\qquad \leqslant R^{1-\frac{2}{q}}\|(1-\frac{v_k}{|u|})u\|_{L^2(\mathbb{R}^3)}^{\frac{2}{q}} \\
&\qquad \leqslant R^{\frac{2}{p}-1}\|u\|_{L^\infty(0,1;L^2(\mathbb{R}^3))}^{2(1-\frac{1}{p})}
\end{aligned}$$

Hence, it follows from Holder's inequality that

$$\int_{\mathbb{R}^3}(1-\frac{v_k}{|u|})|u||\nabla P_{k2}|dx \leqslant R^{\frac{2}{p}-1}\|u\|_{L^\infty(0,1;L^2(\mathbb{R}^3))}^{2(1-\frac{1}{p})}\{\int_{\mathbb{R}^3}|\nabla P_{k2}|^p dx\}^{\frac{1}{p}}.$$

Hence, we have

$$\int_{Q_{k-1}}(1-\frac{v_k}{|u|})|u||\nabla P_{k2}| \leqslant R^{\frac{2}{p}-1}\|u\|_{L^\infty(0,1;L^2(\mathbb{R}^3))}^{2(1-\frac{1}{p})}\|\nabla P_{k2}\|_{L^p(Q_{k-1})}. \tag{3.8}$$

But, we recognize that ∇P_{k2} is equal to the following expression.

$$\begin{aligned}
\nabla P_{k2} = \sum R_iR_j\{2(1-\frac{w_k}{|u|})u_i\nabla[\frac{w_k}{|u|}u_j] \\
+ 2(1-\frac{w_k}{|u|})u_j[\frac{w_k}{|u|}\nabla u_i] \\
- 2\nabla[\frac{w_k}{|u|}]u_i\frac{w_k}{|u|}u_j\}.
\end{aligned}$$

Moreover, it is straightforward to see that for any $1 \leqslant i, j \leqslant 3$, we have

- $|2(1-\frac{w_k}{|u|})u_i\nabla[\frac{w_k}{|u|}u_j] + 2(1-\frac{w_k}{|u|})u_j[\frac{w_k}{|u|}\nabla u_i]| \leqslant 8R^{\beta}D_k$.
- $|2\nabla[\frac{w_k}{|u|}]u_i\frac{w_k}{|u|}u_j| \leqslant 8w_k D_k$.

So, we can decompose ∇P_{k2} as $\nabla P_{k2} = G_{k21} + G_{k22}$, where G_{k21} and G_{k22} are given by

- $G_{k21} = \sum R_i R_j\{2(1-\frac{w_k}{|u|})u_i\nabla[\frac{w_k}{|u|}u_j] + 2(1-\frac{w_k}{|u|})u_j[\frac{w_k}{|u|}\nabla u_i]\}$.
- $G_{k22} = -\sum R_i R_j\{2\nabla[\frac{w_k}{|u|}]u_i\frac{w_k}{|u|}u_j\}$.

In order to use inequality (3.8), we need to estimate $\|G_{k21}\|_{L^p(Q_{k-1})}$ and $\|G_{k22}\|_{L^p(Q_{k-1})}$ respectively, for p with $1 < p < \frac{5}{4}$. Indeed, by applying the Zygmund-Calderon Theorem, we can deduce that

- $\|G_{k21}\|_{L^p(Q_{k-1})} \leqslant C_p R^{\beta}\|D_k\|_{L^p(Q_{k-1})}$,
- $\|G_{k22}\|_{L^p(Q_{k-1})} \leqslant C_p\|w_k D_k\|_{L^p(Q_{k-1})}$,

where C_p is some constant depending only on p. But it turns out that

$$
\begin{aligned}
\|D_k\|^p_{L^p(Q_{k-1})} = \int_{Q_{k-1}} D_k^p \chi_{\{w_k>0\}} \\
&\leqslant \{\int_{Q_{k-1}} D_k^2\}^{\frac{p}{2}} \|\chi_{\{w_k>0\}}\|_{L^{\frac{2}{2-p}}(Q_{k-1})} \\
&\leqslant \frac{5^{\frac{p}{2}}}{R^{\frac{5}{3}\beta(2-p)}} \|d_k\|^p_{L^2(Q_{k-1})} 2^{\frac{5k}{3}(2-p)} C_p U_{k-1}^{\frac{5}{6}(2-p)} \\
&\leqslant \frac{1}{R^{\frac{5}{3}\beta(2-p)}} C_p U_{k-1}^{\frac{5-p}{3}} 2^{\frac{5(2-p)k}{3}}.
\end{aligned}
$$

That is , we have

$$\|D_k\|_{L^p(Q_{k-1})} \leqslant \frac{1}{R^{\frac{5}{3p}\beta(2-p)}} C_p U_{k-1}^{\frac{5-p}{3p}} 2^{\frac{5(2-p)k}{3p}}.$$

Hence, it follows that

$$\|G_{k21}\|_{L^p(Q_{k-1})} \leqslant \frac{1}{R^{\beta(\frac{10-8p}{3p})}} C_p U_{k-1}^{\frac{5-p}{3p}} 2^{\frac{5(2-p)k}{3p}}. \qquad (3.9)$$

On the other hand, we have

$$\begin{aligned}\|w_k D_k\|^p_{L^p(Q_{k-1})} = \int_{Q_{k-1}} w_k^p D_k^p \\ &\leqslant \{\int_{Q_{k-1}} w_k^{\frac{2p}{2-p}}\}^{\frac{2-p}{2}} \{\int_{Q_{k-1}} D_k^2\}^{\frac{p}{2}} \\ &\leqslant C_p \{\int_{Q_{k-1}} w_k^{\frac{2p}{2-p}}\}^{\frac{2-p}{2}} U_{k-1}^{\frac{p}{2}}.\end{aligned}$$

Now, let us recall that $1 < p < \frac{5}{4}$, and put $r = \frac{2p}{2-p}$. we then recognize that $2 < r = \frac{2p}{2-p} < \frac{10}{3}$, if $1 < p < \frac{5}{4}$. So, we can have the following estimation

$$\begin{aligned}\int_{Q_{k-1}} w_k^{\frac{2p}{2-p}} = \int_{Q_{k-1}} w_k^r \chi_{\{w_k > 0\}} \\ &\leqslant \int_{Q_{k-1}} w_k^r \chi_{\{w_{k-1} > \frac{R^\beta}{2^k}\}} \\ &\leqslant \frac{1}{R^{\beta(\frac{10}{3}-r)}} 2^{k(\frac{10}{3}-r)} \int_{Q_{k-1}} w_{k-1}^{\frac{10}{3}} \\ &\leqslant \frac{1}{R^{\beta\frac{20-16p}{3(2-p)}}} 2^{\frac{k(20-16p)}{3(2-p)}} U_{k-1}^{\frac{5}{3}}.\end{aligned}$$

Hence, it follows that

$$\|G_{k22}\|_{L^p(Q_{k-1})} \leqslant C_p \|w_k D_k\|_{L^p(Q_{k-1})} \leqslant C_p \frac{2^{k\frac{10-8p}{3p}}}{R^{\beta\frac{10-8p}{3p}}} U_{k-1}^{\frac{5-p}{3p}}. \tag{3.10}$$

By combining inequalities (3.8), (3.9), (3.10), we deduce that

$$\begin{aligned}\int_{Q_{k-1}} (1-\frac{v_k}{|u|})|u||\nabla P_{k2}| \\ \leqslant \frac{1}{R^{\beta\frac{10-8p}{3p}-\frac{2-p}{p}}} C_p \|u\|_{L^\infty(0,1;L^2(\mathbb{R}^3))}^{2(1-\frac{1}{p})} U_{k-1}^{\frac{5-p}{3p}} 2^{\frac{10-5p}{3p}k}.\end{aligned} \tag{3.11}$$

Notice that $\beta(\frac{10-8p}{3p}) - (\frac{2-p}{p}) > 0$ if and only if $\beta > \frac{6-3p}{10-8p}$. Moreover, we know that the term $\frac{6-3p}{10-8p}$ is always positive, for $1 < p < \frac{5}{4}$. In addition, we know that as p trends to $1, \frac{6-3p}{10-8p}$ trends to $\frac{3}{2}$. This means that even though β cannot be exactly $\frac{3}{2}$, $\beta > \frac{3}{2}$ can be adjusted to be as close to $\frac{3}{2}$ as we desire.

As for the term $\int_{Q_{k-1}} (1-\frac{v_k}{|u|})|u||\nabla P_{k3}|$. We first notice that

$$P_{k3} = \sum R_i R_j \{\frac{w_k}{|u|} u_i \frac{w_k}{|u|} u_j\}.$$

So, we know that

$$\nabla P_{k3} = \sum R_i R_j \{\nabla[\frac{w_k}{|u|} u_i] \frac{w_k}{|u|} u_j + \frac{w_k}{|u|} u_i \nabla[\frac{w_k}{|u|} u_j]\},$$

with

$$|\nabla[\frac{w_k}{|u|}u_i]\frac{w_k}{|u|}u_j + \frac{w_k}{|u|}u_i\nabla[\frac{w_k}{|u|}u_j]| \leqslant 6w_kD_k.$$

So, it follows again from the Risez's theorem in the theory of singular operator that $\|\nabla P_{k3}\|_{L^p(\mathbb{R}^3)} \leqslant C_p\|w_kD_k\|_{L^p(\mathbb{R}^3)}$, in which C_p is some constant depending only on p. So, we see that we can repeat the same type of estimation, just as what we have done to the term $\int_{Q_{k-1}}(1-\frac{v_k}{|u|})|u||\nabla P_{k2}|$, to conclude that

$$\begin{aligned}
&\int_{Q_{k-1}}(1-\frac{v_k}{|u|})|u||\nabla P_{k3}| \\
&\qquad\qquad \leqslant R^{\frac{2}{p}-1}\|u\|^{2(1-\frac{1}{p})}_{L^\infty(0,1;L^2(\mathbb{R}^3))}\|\nabla P_{k3}\|_{L^p(Q_{k-1})} \\
&\qquad\qquad \leqslant \frac{C_p\|u\|^{2(1-\frac{1}{p})}_{L^\infty(0,1;L^2(\mathbb{R}^3))}}{R^{\beta\frac{10-8p}{3p}-\frac{2-p}{p}}}U_{k-1}^{\frac{5-p}{3p}}2^{\frac{(10-5p)k}{3p}}.
\end{aligned} \tag{3.12}$$

Step four

Now, let us turn our attention to the term $\int_{T_{k-1}}^1|\int_{\mathbb{R}^3}\nabla(\frac{v_k}{|u|})uP_{k1}dx|ds$. Before we deal with the term written as above, let us recall that the weak solution u that we are dealing with now is the one verifying the following condition

$$\frac{|u\cdot\nabla F|}{|u|^\gamma} \leqslant A|F|,$$

where $F = -\frac{u\cdot\nabla|u|}{|u|^2}$, and γ is some index with $0<\gamma<\frac{1}{3}$. We need to introduce the following classical Theorem of harmonic analysis which is due to John and Nirenberg [15].

Theorem 3.4.2. Let B be a ball with finite radius sitting in $\mathbb{R}^3$. Then, there exists some constants α, and K, with $0 < \alpha < \infty$, and $0 < K < \infty$, depending only on the ball B and n, such that for any given $f \in BMO(\mathbb{R}^n)$, we have $\int_B exp(\alpha \frac{|f - f_B|}{\|f\|_{BMO}}) \leqslant K$, where the symbol f_B stands for the mean value of f over B.

We now need to establish the following lemma by using the theorem quoted as above.

Lemma 3.4.3. Let B be a ball with finite radius sitting in $\mathbb{R}^3$. There exists some finite positive constants α and K,depending only on B, such that for every $\mu \geqslant 0$, every $f \in BMO(\mathbb{R}^3)$ with $\int_B f dx = 0$, and p with $1 < p < \frac{5}{4}$, we have $\int_B \mu|f| \leqslant \frac{2p}{\alpha(p-1)}\{1 + K^{1-\frac{1}{p}}\}\|f\|_{BMO}\{(\int_B \mu)^{\frac{1}{p}} + \int_B \mu log^+ \mu\}$.

Proof. For any given $\mu \geqslant 0$, and any $f \in BMO(\mathbb{R}^3)$ with $\int_B f dx = 0$, we do the following splitting

$$\int_B \mu|f| = \int_B \mu|f|\chi_{\{\mu \leqslant exp(\frac{\alpha|f|}{2\|f\|_{BMO}})\}} + \int_B \mu|f|\chi_{\{\mu > exp(\frac{\alpha|f|}{2\|f\|_{BMO}})\}}.$$

Given p be such that $1 < p < \frac{5}{4}$, and let $q = \frac{p}{p-1}$ be the conjugate exponent of p. So, it follows from Holder's inequality that

$$\int_B \mu|f|\chi_{\{\mu\leqslant exp(\frac{\alpha|f|}{2\|f\|_{BMO}})\}}$$

$$\leqslant \{\int_B \mu\chi_{\{\mu\leqslant exp(\frac{\alpha|f|}{2\|f\|_{BMO}})\}}\}^{\frac{1}{p}}$$

$$\times \{\int_B \mu|f|^q\chi_{\{\mu\leqslant exp(\frac{\alpha|f|}{2\|f\|_{BMO}})\}}\}^{\frac{1}{q}}$$

Since $t < exp(t)$, for all $t \in \mathbb{R}$, we have $\frac{\alpha|f|}{2q\|f\|_{BMO}} < exp(\frac{\alpha|f|}{2q\|f\|_{BMO}})$. Hence, we have

$$\int_B \mu|f|\chi_{\{\mu\leqslant exp(\frac{\alpha|f|}{2\|f\|_{BMO}})\}}$$

$$\leqslant \frac{2q}{\alpha}\|f\|_{BMO}\{\int_B \mu\}^{\frac{1}{p}}\{\int_B exp(\alpha\frac{|f|}{\|f\|_{BMO}})\}^{\frac{1}{q}} \quad (3.13)$$

$$\leqslant K^{1-\frac{1}{p}}\frac{2q}{\alpha}\|f\|_{BMO}(\int_B \mu)^{\frac{1}{p}},$$

But, on the other hand, we have

$$\int_B \mu|f|\chi_{\{\mu> exp(\frac{\alpha|f|}{2\|f\|_{BMO}})\}} \leqslant \int_B \frac{2}{\alpha}\|f\|_{BMO}\mu log^+\mu. \quad (3.14)$$

By combining inequalities (3.13), and (3.14), we conclude that

$$\int_B \mu|f| \leqslant \frac{2p}{\alpha(p-1)}\{1+K^{1-\frac{1}{p}}\}\|f\|_{BMO}\{(\int_B \mu)^{\frac{1}{p}} + \int_B \mu log^+\mu\}.$$

□

We are now ready to work with the term $\int_{T_{k-1}}^{1} |\int_{\mathbb{R}^3} \nabla(\frac{v_k}{|u|}) u P_{k1} dx| ds$.

Indeed, by a simple application of the partial regularity theorem due to Caffarelli, Kohn, and Nirenberg, it can be shown that, if B is a sufficiently large open ball centered at the origin of $\mathbb{R}^3$(we will choose B to be large enough so that it will satisfy $|B| > 1$), then it follows that

- $[\frac{1}{2}, 1] \times \mathbb{R}^3 \cap \{v_k > 0\}$ is a subset of $[\frac{1}{2}, 1] \times B$, for all $k \geqslant 1$, and if R is sufficiently lage.

On the other hand, since $\nabla(\frac{v_k}{|u|}) u = -R(1 - \frac{1}{2^k}) F \chi_{\{v_k > 0\}}$. So, we have

$$
\begin{aligned}
&|\int_{\mathbb{R}^3} \nabla(\frac{v_k}{|u|}) u P_{k1} dx| \\
&\qquad = |\int_B R(1 - \frac{1}{2^k}) F \chi_{\{v_k > 0\}} P_{k1} dx| \\
&\qquad \leqslant R \int_B |F| \chi_{\{v_k > 0\}} |P_{k1} - (P_{k1})_B| dx \\
&\qquad + R \int_B |F| \chi_{\{v_k > 0\}} |(P_{k1})_B| dx,
\end{aligned}
$$

for all $k \geqslant 1$, and all $\frac{1}{2} < t < 1$, provided that R is sufficiently large (here, the symbol $(P_{k1})_B$ stands for the average value of P_{k1} over the ball B).

Now, since $P_{k1} = \sum R_i R_j \{(1 - \frac{w_k}{|u|}) u_i (1 - \frac{w_k}{|u|}) u_j\}$, it follows from the Risez's Theorem in the theory of singular integral that $\|P_{k1}(t, \cdot)\|_{L^2(\mathbb{R}^3)} \leqslant C_2 R^\beta \|u(t, \cdot)\|_{L^2(\mathbb{R}^3)}$, for all $t \in [0, 1]$, in which C_2 is some constant depending only on 2. So, we can use the Holder's inequality to carry out the following estimation

$$
\begin{aligned}
|(P_{k1})_B(t)| &\leqslant \frac{1}{|B|}\int_B |P_{k1}(t,x)|dx \\
&\leqslant \frac{1}{|B|^{\frac{1}{2}}}\|P_{k1}(t,\cdot)\|_{L^2(B)} \\
&\leqslant \frac{1}{|B|^{\frac{1}{2}}}C_2R^\beta\|u(t,\cdot)\|_{L^2(\mathbb{R}^3)} \\
&\leqslant C_2R^\beta\|u\|_{L^\infty(0,1;L^2(\mathbb{R}^3))}.
\end{aligned}
$$

We remark that the last line of the above inequality holds since our open ball B is sufficiently large so that $|B| > 1$. As a result, it follows that

$$
\begin{aligned}
&|\int_{\mathbb{R}^3}\nabla(\frac{v_k}{|u|})uP_{k1}dx| \\
&\leqslant R\int_B |F|\chi_{\{v_k>0\}}|P_{k1}-(P_{k1})_B|dx \\
&+ C_2R\|u\|_{L^\infty(0,1;L^2(\mathbb{R}^3))}\int_B R^\beta|F|\chi_{\{v_k>0\}}.
\end{aligned}
\tag{3.15}
$$

Indeed, the operator R_iR_j is indeed a Zygmund- Calderon operator, and so R_iR_j must be a bounded operator from $L^\infty(\mathbb{R}^3)$ to $BMO(\mathbb{R}^3)$. Hence we can deduce that

$$
\begin{aligned}
&\|P_{k1}(t,\cdot)-(P_{k1})_B(t)\|_{BMO} \\
&= \|P_{k1}(t,\cdot)\|_{BMO} \\
&\leqslant C_0\|(1-\frac{w_k}{|u|})u_i(1-\frac{w_k}{|u|})u_j\|_{L^\infty(\mathbb{R}^3)} \\
&\leqslant C_0R^{2\beta},
\end{aligned}
$$

for all $t \in (0,1)$, in which C_0 is some constant depending only on $\mathbb{R}^3$. So, we now apply Lemma 3.4.3 with $\mu = |F|\chi_{\{v_k>0\}}$, and $f = P_{k1} - (P_{k1})_B$ to deduce that

$$\begin{aligned}
&\int_B |F|\chi_{\{v_k>0\}}|P_{k1} - (P_{k1})_B|dx \\
&\qquad\qquad \leqslant \frac{2pC_0}{\alpha(p-1)}\{1 + K^{1-\frac{1}{p}}\}\times \\
&\qquad\qquad \{(\int_B R^{2p\beta}|F|\chi_{\{v_k>0\}})^{\frac{1}{p}} \\
&\qquad\qquad + \int_B R^{2\beta}|F|log^+|F|\chi_{\{v_k>0\}}\},
\end{aligned}$$

in which the symbol $(P_{k1})_B$ stands for the mean value of P_{k1} over the open ball B. Since we know that $\{v_k > 0\}$ is a subset of $\{|u| > \frac{R}{2}\}$, for all $k \geqslant 1$, so it follows from the above inequality that

$$\begin{aligned}
&\int_B |F|\chi_{\{v_k>0\}}|P_{k1} - (P_{k1})_B|dx \\
&\qquad\qquad \leqslant \frac{2C_0}{\alpha}\frac{p}{p-1}4^{p\beta}\{1 + K^{1-\frac{1}{p}}\}\times \\
&\qquad\qquad \{(\int_B |u|^{2p\beta}|F|\chi_{\{v_k>0\}})^{\frac{1}{p}} \\
&\qquad\qquad + \int_B |u|^{2\beta}|F|log^+|F| \cdot \chi_{\{v_k>0\}}\}.
\end{aligned}$$

So, we can conclude from inequality (3.15), and the above inequality that

$$
\begin{aligned}
&\int_{T_{k-1}}^{1} |\int_{\mathbb{R}^3} \nabla(\frac{v_k}{|u|}) u P_{k1} dx| dt \\
&\qquad \leqslant R\frac{2C_0}{\alpha}\frac{p}{p-1}4^{p\beta}(1+K^{1-\frac{1}{p}})\times \\
&\qquad \{(\int_{Q_{k-1}} |u|^{2p\beta}|F|\chi_{\{v_k>0\}})^{\frac{1}{p}} \\
&\qquad + \int_{Q_{k-1}} |u|^{2\beta}|F|log(1+|F|)\chi_{\{v_k>0\}}\} \\
&\qquad + C_2 2^{\beta} R\|u\|_{L^\infty(0,1;L^2(\mathbb{R}^3))} \int_{Q_{k-1}} |u|^{\beta}|F|\chi_{\{v_k>0\}}.
\end{aligned} \tag{3.16}
$$

Now, notice that

$$
\begin{aligned}
&\int_{Q_{k-1}} |u|^{2\beta}|F|log(1+|F|)\chi_{\{v_k>0\}} \\
&\qquad \leqslant \int_{Q_{k-1}} |u|^{2\beta}|F|log(1+|F|)\chi_{\{|F|\leqslant\frac{1}{R}\}}\chi_{\{v_k>0\}} \\
&\qquad + \int_{Q_{k-1}} |u|^{2\beta}|F|log(1+|F|)\chi_{\{|F|>\frac{1}{R}\}}\chi_{\{v_k>0\}} \\
&\qquad \leqslant \frac{log2}{R}\int_{Q_{k-1}} |u|^{2\beta}\chi_{\{v_k>0\}} \\
&\qquad + \int_{Q_{k-1}} |u|^{2\beta}|F|log(1+|F|)\chi_{\{|F|>\frac{1}{R}\}}\chi_{\{v_k>0\}}.
\end{aligned} \tag{3.17}
$$

Step five

To deal with the second term in the last line of inequality (3.17), we consider the sequence $\{\phi_k\}_{k=1}^{\infty}$ of nonnegative continuous functions on $[0,\infty)$, which are defined by

- $\phi_k(t) = 0$, for all $t \in [0, C_k]$.

- $\phi_k(t) = t - C_k$, for all $t \in (C_k, C_k + 1)$.

- $\phi_k(t) = 1$, for all $t \in [C_k + 1, +\infty)$.

where the symbol C_k stands for $C_k = R(1 - \frac{1}{2^k})$, for every $k \geqslant 1$. Moreover, we also need a smooth function $\psi : \mathbb{R} \to \mathbb{R}$ satisfying the following conditions that:

- $\psi(t) = 1$, for all $t \geqslant \frac{1}{R}$.

- $0 < \psi(t) < 1$, for all t with $0 < t < \frac{1}{R}$.

- $\psi(0) = 0$.

- $-1 < \psi(t) < 0$, for all t with $-\frac{1}{R} < t < 0$.

- $\psi(t) = -1$, for all $t \leqslant -\frac{1}{R}$.

- $0 \leqslant \frac{d}{dt}\psi \leqslant 2R$, for all $t \in \mathbb{R}$.

With the above preperation, let λ be such that $2 < \lambda < \frac{10}{3} + 1 - \gamma$. We can then carry out the following calculation

$$
\begin{aligned}
&div\{|u|^{\lambda-1}u\psi(F)log(1+|F|)\phi_k(|u|)\} \\
&\qquad = -(\lambda-1)|u|^{\lambda}F\psi(F)log(1+|F|)\phi_k(|u|) \\
&\qquad - |u|^{\lambda+1}F\psi(F)log(1+|F|)\chi_{\{C_k<|u|<C_k+1\}} \\
&\qquad + |u|^{\lambda-1}\frac{d\psi}{dt}(F)(u\nabla F)log(1+|F|)\phi_k(|u|) \\
&\qquad + |u|^{\lambda-1}\psi(F)\frac{u\cdot\nabla|F|}{1+|F|}\phi_k(|u|).
\end{aligned} \tag{3.18}
$$

Since our weak solution u on $(0,1]\times\mathbb{R}^3$ satisfies $\frac{|u\cdot\nabla F|}{|u|^{\gamma}} \leqslant A|F|$, it follows that

- $|u\cdot\nabla F|(t,x) \leqslant \frac{A}{R}|u(t,x)|^{\gamma}$, if it happens that (t,x) satisfies $|F(t,x)| \leqslant \frac{1}{R}$.
- $|\frac{u\cdot\nabla|F|}{1+|F|}| \leqslant \frac{|u\cdot\nabla|F||}{|F|} = \frac{|u\cdot\nabla F|}{|F|} \leqslant A|u|^{\gamma}$.

So, it follows from inequality (3.18) that

$$
\begin{aligned}
&\Lambda_1 + \Lambda_2 \\
&\qquad \leqslant \int_{Q_{k-1}} |u|^{\lambda-1} |\frac{d\psi}{dt}(F)| \cdot |u \cdot \nabla F| log(1+|F|)\phi_k(|u|) \\
&\qquad + \int_{Q_{k-1}} |u|^{\lambda-1} |\psi(F)| \cdot |\frac{u \cdot \nabla |F|}{1+|F|}|\phi_k(|u|) \\
&\qquad \leqslant \int_{Q_{k-1}} |u|^{\lambda-1}(2R)(\frac{A}{R}|u|^{\gamma}) log(1+\frac{1}{R})\phi_k(|u|) \\
&\qquad + \int_{Q_{k-1}} |u|^{\lambda-1} \cdot A \cdot |u|^{\gamma}\phi_k(|u|) \\
&\qquad \leqslant A(1+2log2)\int_{Q_{k-1}} |u|^{\lambda-1+\gamma}\phi_k(|u|) \\
&\qquad \leqslant A(1+2log2)\int_{Q_{k-1}} |u|^{\lambda-1+\gamma}\chi_{\{v_k>0\}},
\end{aligned}
\tag{3.19}
$$

in which the terms Λ_1, and Λ_2 are given by

- $\Lambda_1 = (\lambda-1)\int_{Q_{k-1}} |u|^{\lambda} F\psi(F) \cdot log(1+|F|)\phi_k(|u|)$.
- $\Lambda_2 = \int_{Q_{k-1}} |u|^{\lambda+1}(\psi(F)F) \cdot log(1+|F|)\chi_{\{C_k<|u|<C_k+1\}}$.

We then notice that

- Since $\lambda > 2$, we have $\Lambda_1 \geqslant \int_{Q_{k-1}} |u|^{\lambda}(F\psi(F))log(1+|F|)\chi_{\{|u|\geqslant C_k+1\}}$.
- $\Lambda_2 \geqslant \frac{R}{2}\int_{Q_{k-1}} |u|^{\lambda}F\psi(F)log(1+|F|)\chi_{\{C_k<|u|<C_k+1\}}$, for every $k \geqslant 1$. Notice that this is true because $C_k = R(1-\frac{1}{2^k})$, and that $(1-\frac{1}{2^k}) \geqslant \frac{1}{2}$, for every $k \geqslant 1$.

Hence, it follows from inequality (3.19) that

$$\int_{Q_{k-1}} |u|^\lambda F\psi(F)log(1+|F|)\chi_{\{v_k>0\}}$$

$$\begin{aligned} &= \int_{Q_{k-1}} \{(\chi_{\{C_k<|u|<C_k+1\}} + \chi_{\{|u|\geqslant C_k+1\}}) \\ &\times |u|^\lambda F\psi(F)log(1+|F|)\}dxdt \\ &\leqslant \frac{2}{R}\Lambda_2 + \Lambda_1 \\ &\leqslant 3C\cdot A\int_{Q_{k-1}} |u|^{\lambda-1+\gamma}\chi_{\{v_k>0\}}. \end{aligned} \tag{3.20}$$

As a matter of fact, inequality (3.20) leads us to raise up the index for the term $\int_{Q_{k-1}} |u|^\theta\chi_{\{v_k>0\}}$, for any θ with $0<\theta<\frac{10}{3}$, in the following way

$$\int_{Q_{k-1}} |u|^\theta\chi_{\{v_k>0\}}$$

$$\begin{aligned} &= \int_{Q_{k-1}} \{R(1-\frac{1}{2^k})+v_k\}^\theta\chi_{\{v_k>0\}} \\ &\leqslant C_\theta\{R^\theta\int_{Q_{k-1}}\chi_{\{v_k>0\}} + \int_{Q_{k-1}} v_k^\theta\chi_{\{v_k>0\}}\} \\ &\leqslant \frac{C_\theta}{R^{\frac{10}{3}-\theta}}\{2^{\frac{10k}{3}} + 2^{(\frac{10}{3}-\theta)k}\}\int_{Q_{k-1}} v_{k-1}^{\frac{10}{3}} \\ &\leqslant \frac{C_\theta}{R^{\frac{10}{3}-\theta}}2^{\frac{10k}{3}}U_{k-1}^{\frac{5}{3}}. \end{aligned}$$

for every θ with $0<\theta<\frac{10}{3}$, where C_θ is some positive constant depending only on θ. Hence it follows from inequalities(3.17), (3.20), and our last inequality that

$$
\begin{aligned}
&\int_{Q_{k-1}} |u|^{2\beta}|F| \cdot log(1+|F|)\chi_{\{v_k>0\}} \\
&\qquad \leqslant \frac{log2}{R}\int_{Q_{k-1}} |u|^{2\beta}\chi_{\{v_k>0\}} \\
&\qquad + \int_{Q_{k-1}} |u|^{2\beta}|F|log(1+|F|)\chi_{\{|F|>\frac{1}{R}\}}\chi_{\{v_k\geqslant 0\}} \\
&\qquad \leqslant \frac{log2}{R}\frac{C_{2\beta}2^{\frac{10k}{3}}}{R^{\frac{10}{3}-2\beta}}U_{k-1}^{\frac{5}{3}} \\
&\qquad + 3C\cdot A\int_{Q_{k-1}} |u|^{2\beta-1+\gamma}\chi_{\{v_k>0\}} \\
&\qquad \leqslant C_{\beta,\gamma}(1+A)\cdot 2^{\frac{10k}{3}}U_{k-1}^{\frac{5}{3}}\{\frac{1}{R^{\frac{10}{3}-2\beta+1}} \\
&\qquad + \frac{1}{R^{\frac{10}{3}-2\beta+1-\gamma}}\},
\end{aligned}
\tag{3.21}
$$

in which $\beta > \frac{3}{2}$, and that β is sufficiently close to $\frac{3}{2}$, and $C_{\beta,\gamma}$ is some constant depending only on β, andγ.

Before we can finish our job, we also need to deal with the term $(\int_{Q_{k-1}} |u|^{2p\beta}|F|\chi_{\{v_k>0\}})^{\frac{1}{p}}$, and the term $\int_{Q_{k-1}} |u|^{\beta}|F|\chi_{\{v_k>0\}}$, which appear in inequality (3.16). For this purpose, we will consider λ again to be $\frac{3}{2} < \lambda < \frac{10}{3} + 1 - \gamma$, and let us carry out the following computation, in which ψ and ϕ_k etc are just the same as before.

$$
\begin{aligned}
&div\{|u|^{\lambda-1}u\psi(F)\phi_k(|u|)\} \\
&\qquad = -(\lambda-1)|u|^{\lambda}F\psi(F)\phi_k(|u|) \\
&\qquad + |u|^{\lambda-1}\frac{d\psi}{dt}(F)(u\cdot\nabla F)\phi_k(|u|) \\
&\qquad - |u|^{\lambda+1}F\psi(F)\chi_{\{C_k<|u|<C_k+1\}}.
\end{aligned}
$$

Hence, we have

$$
\begin{aligned}
&(\lambda-1)\int_{Q_{k-1}}|u|^{\lambda}F\psi(F)\phi_k(|u|) \\
&\qquad + \int_{Q_{k-1}}|u|^{\lambda+1}F\psi(F)\chi_{\{C_k<|u|<C_k+1\}} \\
&\qquad \leqslant \int_{Q_{k-1}}|u|^{\lambda-1}|\frac{d\psi}{dt}(F)|\cdot|u\cdot\nabla F|\phi_k(|u|) \\
&\qquad \leqslant \int_{Q_{k-1}}|u|^{\lambda-1}(2R)(\frac{A}{R}|u|^{\gamma})\chi_{\{v_k>0\}} \\
&\qquad \leqslant 2A\int_{Q_{k-1}}|u|^{\lambda-1+\gamma}\chi_{\{v_k>0\}}.
\end{aligned}
$$

So, it follows that

$$
\begin{aligned}
\int_{Q_{k-1}} |u|^{\lambda} F\psi(F)\chi_{\{v_k>0\}} \\
&= \int_{Q_{k-1}} |u|^{\lambda} F\psi(F)\chi_{\{C_k<|u|<C_k+1\}} \\
&+ \int_{Q_{k-1}} |u|^{\lambda} F\psi(F)\chi_{\{|u|\geqslant C_k+1\}} \\
&\leqslant \frac{2}{R}\int_{Q_{k-1}} |u|^{\lambda+1} F\psi(F)\chi_{\{C_k\leqslant|u|\leqslant C_k+1\}} \\
&+ \int_{Q_{k-1}} |u|^{\lambda} F\psi(F)\phi_k(|u|) \\
&\leqslant 3\{\int_{Q_{k-1}} |u|^{\lambda+1} F\psi(F)\chi_{\{C_k<|u|<C_k+1\}} \\
&+ (\lambda-1)\int_{Q_{k-1}} |u|^{\lambda} F\psi(F)\phi_k(|u|)\} \\
&\leqslant 6A\int_{Q_{k-1}} |u|^{\lambda-1+\gamma}\chi_{\{v_k>0\}},
\end{aligned}
$$

in which λ satisfies $\frac{3}{2} < \lambda < \frac{10}{3} + 1 - \gamma$. Now, put $\lambda = 2p\beta$, with $\beta > \frac{3}{2}$ to be sufficiently close to $\frac{3}{2}$, and $1 < p < \frac{5}{4}$ to be sufficiently close to 1, it follows from our last inequality that

$$
\begin{aligned}
&\int_{Q_{k-1}} |u|^{2p\beta}|F|\chi_{\{v_k>0\}} \\
&\quad = \int_{Q_{k-1}} |u|^{2p\beta}|F|\chi_{\{|F|\leqslant\frac{1}{R}\}}\chi_{\{v_k>0\}} \\
&\quad + \int_{Q_{k-1}} |u|^{2p\beta}\chi_{\{|F|>\frac{1}{R}\}}\chi_{\{v_k>0\}}|F| \\
&\quad \leqslant \frac{1}{R}\int_{Q_{k-1}} |u|^{2p\beta}\chi_{\{v_k>0\}} \\
&\quad + 6A\int_{Q_{k-1}} |u|^{2p\beta-1+\gamma}\chi_{\{v_k>0\}} \\
&\quad \leqslant C(1+A)\{\frac{1}{R^{\frac{10}{3}-2p\beta+1}} + \frac{1}{R^{\frac{10}{3}-2p\beta+1-\gamma}}\}2^{\frac{10k}{3}}U_{k-1}^{\frac{5}{3}}.
\end{aligned}
\tag{3.22}
$$

In exactly the same way, by setting λ to be β, with $\beta > \frac{3}{2}$ to be sufficiently close to $\frac{3}{2}$, it also follows that

$$
\begin{aligned}
&\int_{Q_{k-1}} |u|^{\beta}|F|\chi_{\{v_k>0\}} \\
&\quad = \int_{Q_{k-1}} |u|^{\beta}|F|\chi_{\{|F|\leqslant\frac{1}{R}\}}\chi_{\{v_k>0\}} \\
&\quad + \int_{Q_{k-1}} |u|^{\beta}|F|\chi_{\{|F|>\frac{1}{R}\}}\chi_{\{v_k>0\}} \\
&\quad \leqslant \frac{1}{R}\int_{Q_{k-1}} |u|^{\beta}\chi_{\{v_k>0\}} + 6A\int_{Q_{k-1}} |u|^{\beta-1+\gamma}\chi_{\{v_k>0\}} \\
&\quad \leqslant C_{\beta,\gamma}(1+A)\{\frac{1}{R^{\frac{10}{3}-\beta+1}} + \frac{1}{R^{\frac{10}{3}-\beta+1-\gamma}}\}2^{\frac{10k}{3}}U_{k-1}^{\frac{5}{3}}.
\end{aligned}
\tag{3.23}
$$

By combining inequalities (3.16), (3.21), and (3.22),and (3.23) we now conclude that

$$\int_{Q_{k-1}} |\int_{Q_{k-1}} \nabla(\frac{v_k}{|u|}) u P_{k1} dx| ds$$
$$\leqslant (1+A)(1+\frac{1}{\alpha}) C_{p,\beta} (1+K^{1-\frac{1}{p}}) \times$$
$$(1+\|u\|_{L^\infty(0,1;L^2(\mathbb{R}^3))}) \times \tag{3.24}$$
$$\{ (\frac{1}{R^{\frac{10}{3}-2p\beta+1-\gamma-p}})^{\frac{1}{p}} 2^{\frac{10k}{3p}} U_{k-1}^{\frac{5}{3p}}$$
$$+ \frac{1}{R^{\frac{10}{3}-2\beta-\gamma}} 2^{\frac{10k}{3}} U_{k-1}^{\frac{5}{3}} \}.$$

Notice that if $p \to 1^+$, and $\beta \to \frac{3}{2}^+$, then,we have $(\frac{10}{3} - 2p\beta + 1 - p - \gamma) \to (\frac{1}{3} - \gamma) > 0$, and that $(\frac{10}{3} - 2\beta - \gamma) \to (\frac{1}{3} - \gamma) > 0$.

So, finally, we recognize that by combining inequalities (3.11), (3.12), and (3.24), we conclude that we are done in proving proposition 3.2.1 .

Chapter 4

Log improvement of the Prodi-Serrin criteria for the incompressible Navier-Stokes equations

4.1 Statement of the problem and it's related history

This chapter is devoted to the following log improvement of the Prodi-Serrin criterion corresponding to $p = q = 5$:

Theorem 4.1.1. Suppose that u is a sutiable weak solution of the incomprssible Navier-Stokes equations satisfying

$$\int_0^\infty \int_{\mathbb{R}^3} \frac{|u|^5}{\log(1+|u|)} dx\, ds < \infty,$$

then, $u \in C^\infty((0,\infty) \times \mathbb{R}^3)$.

Montgomery-Smith introduced the following criterium in [22]:

$$\int_0^\infty \frac{\|u(t)\|^p_{L^q(\mathbb{R}^3)}}{1 + \log^+ \|u(t)\|_{L^q(\mathbb{R}^3)}}\, dt < \infty.$$

Notice that the log improvement is, here, in time only. This can be seen as a natural Gronwall type extension of the Prodi-Serrin conditions. So we can see it as a one dimension ODE type extension.

The goal of our result is to extend this log improvement also in x. For this purpose we focused on the homogeneous case $p = q = 5$, even though extension to the Prodi-Serrin range $2 \le p < \infty$ should be doable.

The proof of Theorem 4.1.1 is split into two parts. The first point is to show that for any time $t > \lambda$, the L^∞ norm of u in x can be bounded in a affine way by

$$\int_0^t \int_{\mathbb{R}^3} |u|^6 \, dx \, dt.$$

More precisely, we will show the following Proposition:

Proposition 4.1.2. For every λ satisfying $0 < \lambda < 2$, there exists some universal constant $A_\lambda > 0$, depending only on λ, such that, for any suitable weak solution u of the incompressible Navier-Stokes equations on $(0, \infty) \times \mathbb{R}^3$, we have $\|u(T, \cdot)\|_{L^\infty(\mathbb{R}^3)} \leqslant A_\lambda \{1 + \int_0^T \int_{\mathbb{R}^3} |u|^6 dx \, ds\}$, for any $T > \lambda$.

Then Theorem 4.1.1 follows from a Gronwall argument on $\|u(t)\|_{L^\infty(\mathbb{R}^3)}$, since:

$$\begin{aligned} \|u(t)\|_{L^\infty(\mathbb{R}^3)} &\leq A_\lambda \\ +A_\lambda \int_\lambda^t \|u(s)\|_{L^\infty(\mathbb{R}^3)} &\log(1 + \|u(s)\|_{L^\infty(\mathbb{R}^3)}) \left(\int_{\mathbb{R}^3} \frac{|u(s)|^5}{\log(1 + |u(s)|)} \, dx \right) ds \end{aligned}$$

and the Hypothesis gives that $\int_{\mathbb{R}^3} \frac{|u(s)|^5}{\log(1+|u(s)|)} \, dx$ lies in $L^1(0, \infty)$.

Notice that the inequality of Proposition 4.1.2 needs to be invariant by the scaling of the Navier-Stokes equation:

$$u_\varepsilon(t, x) = \varepsilon u(t_0 + \varepsilon^2 t, x_0 + \varepsilon x). \tag{4.1}$$

This is why the L^6 norm pops up, since it has the same scaling as that of the L^∞ norm. Taking advantage of the scaling (4.1), Proposition 4.1.2 will follow from the following rescaled Proposition:

Proposition 4.1.3. There exists a universal positive constant C^*, such that for any suitable weak solution u of the incompressible Navier-Stokes equations on $[-1,1]\times\mathbb{R}^3$ satisfying $\|u\|_{L^6(\mathbb{R}^3\times[-1,1])}\leqslant C^*$, we have $|u|\leqslant 1$ almost everywhere on $[-\frac{1}{2},1]\times\mathbb{R}^3$.

The proof of proposition 4.1.3 is in the same spirit as the proof given by A. Vasseur [32]. It relies on a method first introduced by De Giorgi to show regularity of solutions to elliptic equations with rough diffusion coefficients [8]. In this paper, the proof of proposition 4.1.3 is established though sections 4.2, 4.3, 4.4 and 4.5. In section 4.6, we will deduce proposition 4.1.2 from proposition 4.1.3. Finally, in the last section of this paper, we will use the conclusion of proposition 4.1.2, together with the fundamental result of Serrin [29], to obtain the result of Theorem 4.1.1.

4.2 Basic setting for the chapter

In order to prove proposition 4.1.3, we would like to introduce some notation first. Then, we will state two lemmas and one proposition which are related to the proof of proposition 4.1.3. So, let us fix our notation as follow.

- for each $k\geqslant 0$, let $Q_k=[T_k,1]\times\mathbb{R}^3$, in which $T_k=-\frac{1}{2}(1+\frac{1}{2^k})$.
- for each $k\geqslant 0$, let $v_k=\{|u|-(1-\frac{1}{2^k})\}_+$.
- for each $k\geqslant 0$, let $d_k^2=\frac{(1-\frac{1}{2^k})}{|u|}\chi_{\{|u|>(1-\frac{1}{2^k})\}}|\nabla|u||^2+\frac{v_k}{|u|}|\nabla u|^2$.
- for each $k\geqslant 0$, let $U_k=\frac{1}{2}\|v_k\|^2_{L^\infty(T_k,1;L^2(\mathbb{R}^3))}+\int_{T_k}^1\int_{\mathbb{R}^3}d_k^2dx\,dt$.

With the above setting, we are now ready to state the lemmas and proposition which are related to proposition 4.1.3 as follow.

Lemma 4.2.1. For any suitable weak solution u of the incompressible Navier-Stokes equations on $[-1,1]\times\mathbb{R}^3$ satisfying $\|u\|_{L^6(Q_0)}\leqslant 1$, we have $U_1\leqslant A\|u\|^6_{L^6(Q_0)}$, in which A is some universal constant strictly greater than 1.

Proposition 4.2.2. There exists some universal constants B, $\beta>1$, such that for any suitable weak solution u of the incompressible Navier-Stokes equations on $[-1,1]\times\mathbb{R}^3$ satisfying $\|u\|_{L^6(Q_0)}\leqslant\frac{1}{A^{\frac{1}{6}}}$, we have $U_k\leqslant B^kU^\beta_{k-1}$, for all $k\geqslant 1$. Here, A is the universal constant appearing in Lemma 4.2.1 .

Let us first show that Lemma 4.2.1 and Proposition 4.2.2 provide the result of Proposition 4.1.3.First we show that the sequence U_k converges to 0 when k goes to infinity. We can use for instance the following easy lemma (see [32]):

Lemma 4.2.3. For any given constants B, $\beta>1$, there exists some constant C_0^* such that for any sequence $\{a_k\}_{k\geqslant 1}$ satisfying $0<a_1\leq C_0^*$ and $a_k\leqslant B^ka^\beta_{k-1}$, for any $k\geqslant 1$, we have $lim_{k\to\infty}a_k=0$.

Indeed, let B, $\beta>1$ be the constants occurring in proposition 4.2.2, and let C_0^* be the constant associated to B,β in the sense of lemma 4.2.3. Now, take $C^*=min\{\frac{1}{A^{\frac{1}{6}}},(\frac{C_0^*}{A})^{\frac{1}{6}}\}$, in which A is the universal constant appearing in Lemma 4.2.1. Then, for any suitable weak solution u of the incompressible Navier-Stokes equations on $[-1,1]\times\mathbb{R}^3$ satisfying $\|u\|_{L^6(Q_0)}\leqslant C^*$, we have

$\|u\|_{L^6(Q_0)} \leqslant (\frac{1}{A})^{\frac{1}{6}}$. Hence, proposition 4.2.2 tells us that $U_k \leqslant B^k U_{k-1}^{\beta}$, for all $k \geqslant 1$ must be valid. On the other hand, Since $\|u\|_{L^6(Q_0)} \leqslant C^* \leqslant 1$, Lemma 4.2.1 also implies that $U_1 \leqslant A\|u\|_{L^6(Q_0)}^6 \leqslant C_0^*$. Hence, it follows from Lemma 4.2.3 that $lim_{k\to\infty} U_k = 0$. However, since we have the inequality

$$\frac{1}{2}\int_{\mathbb{R}^3} \{|u(t,x)| - 1\}_+^2 dx \leqslant \frac{1}{2} sup_{t\in[-\frac{1}{2},1]} \int_{\mathbb{R}^3} v_k^2 dx \leqslant U_k,$$

for every $t \in [-\frac{1}{2}, 1]$. As a result, $lim_{k\to\infty} U_k = 0$ immediately implies that $|u| \leqslant 1$ almost everywhere on $[-\frac{1}{2}, 1] \times \mathbb{R}^3$. This gives the result of Proposition 4.1.3.

4.3 proof of lemma 4.2.1

In this section, we will devote our effort in proving Lemma 4.2.1. Let us recall that the incompressible Navier-Stokes equations on $(-\infty, \infty) \times \mathbb{R}^3$ is

$$\partial_t u - \triangle u + div(u \otimes u) + \nabla P = 0,$$

together with the divergence free condition $div(u) = 0$. Now, by multiplying the above equation by the term $\frac{v_1}{|u|}u$, we yield the following inequality, which is valid in the sense of distribution.

$$\partial_t(\frac{1}{2}v_1^2) + d_1^2 - \triangle(\frac{1}{2}v_1^2) + div(\frac{v_1^2}{2}u) + \frac{v_1}{|u|}u\nabla P \leqslant 0.$$

Consider now the variables σ, t with $T_0 \leqslant \sigma \leqslant T_1 \leqslant t \leqslant 1$, where $T_0 = -1$, and $T_1 = -\frac{1}{2}(1 + \frac{1}{2})$. We mention that we have the following , which is valid in the sense of distribution.

- $\int_\sigma^t \int_{\mathbb{R}^3} \partial_t(\frac{1}{2}v_1^2)dx\,ds = \frac{1}{2}\int_{\mathbb{R}^3} v_1^2(t,x)dx - \frac{1}{2}\int_{\mathbb{R}^3} v_1^2(\sigma,x)dx.$
- $\int_\sigma^t \int_{\mathbb{R}^3} div(\frac{v_1^2}{2}u) - \triangle(\frac{v_1^2}{2})dx\,ds = 0.$

Hence, by taking the integral over $[\sigma,t]\times\mathbb{R}^3$ to the above inequality, we yield the following estimation.

$$\begin{aligned}
\frac{1}{2}\int_{\mathbb{R}^3} v_1^2(t,x)dx + \int_\sigma^t\int_{\mathbb{R}^3} d_k^2 dx\,ds &\leqslant \frac{1}{2}\int_{\mathbb{R}^3} v_1^2(\sigma,x)dx + \int_\sigma^t |\int_{\mathbb{R}^3} \frac{v_1}{|u|}u\nabla P dx|ds \\
&= \frac{1}{2}\int_{\mathbb{R}^3} v_1^2(\sigma,x)dx + \int_\sigma^t |\int_{\mathbb{R}^3} P\nabla(\frac{v_1}{|u|}u)dx|ds \\
&\leqslant \frac{1}{2}\int_{\mathbb{R}^3} v_1^2(\sigma,x)dx + 3\int_\sigma^t\int_{\mathbb{R}^3} d_1|P|\chi_{\{v_1>0\}} \\
&\leqslant \frac{1}{2}\int_{\mathbb{R}^3} v_1^2(\sigma,x)dx + \frac{3}{2}\int_\sigma^t\int_{\mathbb{R}^3}\alpha^2 d_1^2 dx\,ds \\
&+ \frac{3}{2}\int_\sigma^t\int_{\mathbb{R}^3}\frac{|P|^2}{\alpha^2}\chi_{\{v_1>0\}}dx\,ds,
\end{aligned}$$

in which α can be any positive constant (In the third step of the above deduction, we have used the nontrival fact that $|\nabla(\frac{v_k}{|u|}u)| \leqslant 3d_k$, whose justification will be given in the last part of Section 4). Hence we yield the following inequality which is valid for any $\alpha > 0$.

$$\int_{\mathbb{R}^3}\frac{v_1^2(t,x)}{2}dx + \int_{[\sigma,t]\times\mathbb{R}^3}\frac{(2-3\alpha^2)d_1^2}{2} \leqslant \int_{\mathbb{R}^3}\frac{v_1^2(\sigma,x)}{2}dx + \int_{[\sigma,t]\times\mathbb{R}^3}\frac{3|P|^2\chi_{\{v_1>0\}}}{2\alpha^2}.$$

If we choose $\alpha = (\frac{1}{2})^{\frac{1}{2}}$, then the inequality shown as above becomes

$$\frac{1}{2}\int_{\mathbb{R}^3} v_1^2(t,x)dx + \frac{1}{4}\int_{[\sigma,t]\times\mathbb{R}^3} d_1^2 \leqslant \int_{\mathbb{R}^3}\frac{v_1^2(\sigma,x)}{2}dx + 3\int_{[\sigma,t]\times\mathbb{R}^3}|P|^2\chi_{\{v_1>0\}}.$$

By taking average over $\sigma \in [T_0, T_1]$, we can carry out the following estiamtion

$$\int_{\mathbb{R}^3} \frac{v_1^2(t,x)}{2} dx + \int_{[T_1,t]\times\mathbb{R}^3} \frac{d_1^2}{4} \leqslant \frac{4}{2} \int_{-1}^{T_1} \int_{\mathbb{R}^3} v_1^2(\sigma, x) + 3 \int_{-1}^{t} \int_{\mathbb{R}^3} |P|^2 \chi_{\{v_1>0\}}.$$

Notice that, in the above inequality, the integer 4 appears in the first term of the right hand side because $\frac{1}{T_k - T_{k-1}} = 2^2 = 4$. Now, by taking the L^∞-norm over $t \in [T_1, 1]$, we yield

$$\frac{1}{4} U_1 \leqslant 2 \int_{-1}^{T_1} \int_{\mathbb{R}^3} v_1^2 dx\, ds + 3 \int_{Q_0} |P|^2 \chi_{\{v_1>0\}}.$$

But, we notice that

$$\begin{aligned} \int_{-1}^{T_1} \int_{\mathbb{R}^3} v_1^2 dx\, ds \leqslant \int_{Q_0} v_1^2 \chi_{\{v_1>0\}} & \\ & \leqslant \left(\int_{Q_0} v_1^6\right)^{\frac{1}{3}} \left(\int_{Q_0} \chi_{\{v_1>0\}}\right)^{\frac{2}{3}} \\ & \leqslant \|u\|_{L^6(Q_0)}^2 \left(\int_{Q_0} \chi_{\{v_0>\frac{1}{2}\}}\right)^{\frac{2}{3}} \\ & \leqslant \|u\|_{L^6(Q_0)}^2 \left(2^6 \int_{Q_0} v_0^6\right)^{\frac{2}{3}} = 2^4 \|u\|_{L^6(Q_0)}^6. \end{aligned}$$

On the other hand, since the pressure P satisfies the equation $-\triangle P = \sum \partial_i \partial_j (u_i u_j)$.

So, by the Riesz theorem in the theory of singular integral, we have $\|P\|_{L^3(Q_0)} \leqslant C_3 \|u\|_{L^6(Q_0)}^2$, in which C_3 is some universal constant . Hence, it follows that

$$\int_{Q_0} |P|^2\chi_{\{v_1>0\}} \leqslant \|P\|^2_{L^3(Q_0)}\|\chi_{\{v_1>0\}}\|_{L^3(Q_0)}$$

$$\leqslant C_3^2\|u\|^4_{L^6(Q_0)}\|\chi_{\{v_0>\frac{1}{2}\}}\|_{L^3(Q_0)}$$

$$\leqslant C_3^2\|u\|^4_{L^6(Q_0)}(2^6\int_{Q_0} v_0^6)^{\frac{1}{3}}$$

$$= 4C_3^2\|u\|^6_{L^6(Q_0)}.$$

Hence it follows that

$$\frac{1}{4}U_1 \leqslant 2\int_{Q_0} v_1^2 + 3\int_{Q_0} |P|^2\chi_{\{v_1>0\}}$$

$$\leqslant 2^5\|u\|^6_{L^6(Q_0)} + 12C_3^2\|u\|^6_{L^6(Q_0)}.$$

As a result, by taking $A = 2^7 + 48C_3^2$, we can at once deduce that

$$U_1 \leqslant A\|u\|^6_{L^6(Q_0)}.$$

So, we are done in establishing Lemma 4.2.1

4.4 Preliminaries for the proof of proposition 4.2.2

Lemma 4.4.1. There exists some constant $C > 0$, such that for any $k \geqslant 1$, and any $F \in L^\infty(T_k, 1; L^2(\mathbb{R}^3))$ with $\nabla F \in L^2(Q_k)$, we have $\|F\|_{L^{\frac{10}{3}}(Q_k)} \leqslant C\|F\|^{\frac{2}{5}}_{L^\infty(T_k,1;L^2(\mathbb{R}^3))}\|\nabla F\|^{\frac{3}{5}}_{L^2(Q_k)}$.

Proof. By Sobolev-embedding Theorem, there is a constant C, depending only on the dimension of $\mathbb{R}^3$, such that

$$(\int_{\mathbb{R}^3} |F(t,x)|^6 dx)^{\frac{1}{6}} \leqslant C(\int_{\mathbb{R}^3} |\nabla F(t,x)|^2 dx)^{\frac{1}{2}}.$$

for any $t \in [T_k, 1]$, where $k \geqslant 1$, and F is some function which verifies $F \in L^\infty(T_k, 1; L^2(\mathbb{R}^3))$, and $\nabla F \in L^2(Q_k)$. By taking the power 2 on both sides of the above inequality and then taking integration along the variable $t \in [T_k, 1]$, we yield

$$\int_{T_k}^1 (\int_{\mathbb{R}^3} |F|^6 dx)^{\frac{1}{3}} dt \leqslant C^2 \int_{T_k}^1 \int_{\mathbb{R}^3} |\nabla F|^2 dx\, dt.$$

On the other hand, by Holder's inequality, we have

$$\begin{aligned} \|F\|_{L^{\frac{10}{3}}(Q_k)}^{\frac{10}{3}} = \int_{T_k}^1 \int_{\mathbb{R}^3} |F|^2 |F|^{\frac{4}{3}} dx\, dt \\ &\leqslant \int_{T_k}^1 (\int_{\mathbb{R}^3} |F|^6 dx)^{\frac{1}{3}} (\int_{\mathbb{R}^3} |F|^2 dx)^{\frac{2}{3}} dt \\ &\leqslant \|F\|_{L^\infty(T_k,1;L^2(\mathbb{R}^3))}^{\frac{4}{3}} \|F\|_{L^2(T_k,1;L^6(\mathbb{R}^3))}^2. \end{aligned}$$

By taking the advantage that $\|F\|_{L^2(T_k,1;L^6(\mathbb{R}^3))} \leqslant C\|\nabla F\|_{L^2(Q_k)}$, we yield

$$\|F\|_{L(Q_k)^{\frac{10}{3}}}^{\frac{10}{3}} \leqslant C^2 \|F\|_{L^\infty(T_k,1;L^2(\mathbb{R}^3))}^{\frac{4}{3}} \|\nabla F\|_{L^2(Q_k)}^2.$$

Hence, we have

$$\|F\|_{L^{\frac{10}{3}}(Q_k)} \leqslant C \|F\|_{L^\infty(T_k,1;L^2(\mathbb{R}^3))}^{\frac{2}{5}} \|\nabla F\|_{L^2(Q_k)}^{\frac{3}{5}}.$$

so, we are done □

Lemma 4.4.2. For any $1 < q < \infty$, we have $\|\chi_{\{v_k > 0\}}\|_{L^q(Q_{k-1})} \leqslant 2^{\frac{10k}{3q}} C^{\frac{1}{q}} U_{k-1}^{\frac{5}{3q}}$

.

Proof. First, we have to notice that $\{v_k > 0\}$ is a subset of $\{v_{k-1} > \frac{1}{2^k}\}$, hence we have

$$\int_{Q_{k-1}} \chi_{\{v_k>0\}} \leqslant \int_{Q_{k-1}} \chi_{\{v_{k-1}>\frac{1}{2^k}\}} \leqslant 2^{\frac{10k}{3}} \int_{Q_{k-1}} |v_{k-1}|^{\frac{10}{3}}.$$

By our previous Lemma, we have

$$\begin{aligned}
\|v_{k-1}\|^{\frac{10}{3}}_{L^{\frac{10}{3}}(Q_{k-1})} \leqslant & \\
& C^2 \|v_{k-1}\|^{\frac{4}{3}}_{L^\infty(T_{k-1},1;L^2(\mathbb{R}^3))} \|\nabla v_{k-1}\|^2_{L^2(Q_{k-1})} \\
& \leqslant C^2 (U^{\frac{1}{2}}_{k-1})^{\frac{4}{3}} \|d_{k-1}\|^2_{L^2(Q_{k-1})} \\
& \leqslant C^2 U^{\frac{2}{3}}_{k-1} U_{k-1} \\
& = C^2 U^{\frac{5}{3}}_{k-1}.
\end{aligned}$$

So, it follows that $\int_{Q_{k-1}} \chi_{\{v_k>0\}} \leqslant 2^{\frac{10k}{3}} C^2 U^{\frac{5k}{3}}_{k-1}$, and hence we have $\|\chi_{\{v_k>0\}}\|_{L^q(Q_{k-1})} \leqslant 2^{\frac{10k}{3q}} C^{\frac{1}{q}} U^{\frac{5}{3q}}_{k-1}$, where C is some universal constant. So, we are done. □

In the proof of Lemma 4.4.2, we have used the fact that $|\nabla v_k| \leqslant d_k$, whose justification will be given immediately in the following paragraph.
Before we leave this section, we also want to list out some inequalities which will often be used in the proof of proposition 4.1.2 as follow:

- $|(1 - \frac{v_k}{|u|})u| \leqslant 1 - \frac{1}{2^k}$.
- $\frac{v_k}{|u|}|\nabla u| \leqslant d_k$.

- $\chi_{\{v_k>0\}}|\nabla|u|| \leqslant d_k$.
- $|\nabla v_k| \leqslant d_k$.
- $|\nabla(\frac{v_k}{|u|}u)| \leqslant 3d_k$.

Now, we first want to justify the validity of $|(1-\frac{v_k}{|u|})u| \leqslant 1-\frac{1}{2^k}$. In the case in which the point (t,x) satisfies $|u(t,x)| \leqslant 1-\frac{1}{2^k}$, we have $v_k(t,x)=0$, and hence it follows that

$$|\{1-\frac{v_k(t,x)}{|u(t,x)|}\}u(t,x)| = |u(t,x)| \leqslant 1-\frac{1}{2^k}.$$

In the case in which (t,x) satisfies $|u(t,x)| > 1-\frac{1}{2^k}$, we have $v_k(t,x) = |u(t,x)|-(1-\frac{1}{2^k})$, and hence it follows that

$$|\{1-\frac{v_k}{|u|}\}u(t,x)| = |1-\frac{|u|-(1-\frac{1}{2^k})}{|u|}||u| = 1-\frac{1}{2^k}.$$

So, no matter in which case, we always have the conclusion that $|(1-\frac{v_k}{|u|})u| \leqslant 1-\frac{1}{2^k}$.

Next, according to the definition of d_k^2, we can carry out the following estimation

$$d_k^2 \geqslant \frac{v_k}{|u|}|\nabla u|^2 \geqslant \{\frac{v_k}{|u|}|\nabla u|\}^2.$$

Hence, by taking square root, it follows at once that $d_k \geqslant \frac{v_k}{|u|}|\nabla u|$.

We now turn our attention to the inequality $\chi_{\{|u|>(1-\frac{1}{2^k})\}}|\nabla|u|| \leqslant d_k$. To justify it, we recall that $|\nabla u| \geqslant |\nabla|u||$. Hence, it follows from the definition of d_k^2 that

$$d_k^2 \geqslant \frac{1-\frac{1}{2^k}}{|u|}\chi_{\{|u|>1-\frac{1}{2^k}\}}|\nabla|u||^2 + \{1 - \frac{1-\frac{1}{2^k}}{|u|}\}\chi_{\{|u|>1-\frac{1}{2^k}\}}|\nabla|u||^2.$$

So, by simplifying the right-hand side of the above inequality, we can deduce that $d_k^2 \geqslant \chi_{\{|u|\geqslant 1-\frac{1}{2^k}\}}|\nabla|u||^2$. Hence, we have $d_k \geqslant \chi_{\{|u|\geqslant 1-\frac{1}{2^k}\}}|\nabla|u||$. In addition, since it is obvious to see that $\nabla v_k = \chi_{\{|u|\geqslant 1-\frac{1}{2^k}\}}\nabla|u|$, we also have the result that $|\nabla v_k| \leqslant d_k$.

Finally, we want to justify the inequality that $|\nabla(\frac{v_k}{|u|}u)| \leqslant 3d_k$. So, we notice that, by applying the product rule, we have

$$\nabla(\frac{v_k}{|u|}u) = \nabla(v_k)\frac{u}{|u|} + \frac{v_k}{|u|}\nabla u - \frac{v_k}{|u|^2}u\nabla|u|.$$

However, since $\frac{v_k}{|u|}|\nabla u| \leqslant d_k$, and $|\frac{v_k}{|u|^2}u\nabla|u|| \leqslant \chi_{\{|u|\geqslant 1-\frac{1}{2^k}\}}|\nabla|u|| \leqslant d_k$, it follows at once from the above expression that $|\nabla(\frac{v_k}{|u|}u)| \leqslant 3d_k$.

4.5 proof of proposition 4.2.2

To begin the argument, we recall that, by multiplying the equation $\partial_t u - \triangle u + div(u\otimes u) + \nabla P = 0$ on $(-\infty,\infty)\times\mathbb{R}^3$, we yield the following inequality formally, which is indeed valid in the sense of distribution

$$\partial_t(\frac{v_k^2}{2}) + d_k^2 - \triangle(\frac{v_k^2}{2}) + div(\frac{v_k^2}{2}u) + \frac{v_k}{|u|}u\nabla P \leqslant 0.$$

Next, let us consider the variables σ , t verifying $T_{k-1} \leqslant \sigma \leqslant T_k \leqslant t \leqslant 1$. Then, we have

- $\int_\sigma^t \int_{\mathbb{R}^3} \partial_t(\frac{v_k^2}{2}) dx\, ds = \int_{\mathbb{R}^3} \frac{v_k^2(t,x)}{2} dx - \int_{\mathbb{R}^3} \frac{v_k^2(\sigma,x)}{2} dx.$

- $\int_\sigma^t \int_{\mathbb{R}^3} \triangle(\frac{v_k^2}{2}) dx\, ds = 0.$

- $\int_\sigma^t \int_{\mathbb{R}^3} div(\frac{v_k^2}{2} u) dx\, ds = 0.$

So, it is straightforward to see that

$$\int_{\mathbb{R}^3} \frac{v_k^2(t,x)}{2} dx + \int_\sigma^t \int_{\mathbb{R}^3} d_k^2 dx\, ds \leqslant \int_{\mathbb{R}^3} \frac{v_k^2(\sigma,x)}{2} dx + \int_\sigma^t | \int_{\mathbb{R}^3} \frac{v_k}{|u|} u \nabla P dx | ds,$$

for any σ, t satisfying $T_{k-1} \leqslant \sigma \leqslant T_k \leqslant t \leqslant 1$. By taking the average over the variable σ, we yield

$$\int_{\mathbb{R}^3} \frac{v_k^2(t,x)}{2} dx + \int_{T_k}^t \int_{\mathbb{R}^3} d_k^2 dx\, ds \leqslant 2^k \int_{T_{k-1}}^{T_k} \int_{\mathbb{R}^3} v_k^2 + \int_{T_{k-1}}^t | \int_{\mathbb{R}^3} \frac{v_k}{|u|} u \nabla P dx | ds.$$

By taking the sup over $t \in [T_k, 1]$. the above inequality will give the following

$$U_k \leqslant 2^k \int_{Q_{k-1}} v_k^2 + \int_{T_{k-1}}^1 | \int_{\mathbb{R}^3} \frac{v_k}{|u|} u \nabla P dx | ds.$$

But, from Lemma 4.4.2 and Holder's inequality, we have

$$
\begin{aligned}
\int_{Q_{k-1}} v_k^2 = \int_{Q_{k-1}} v_k^2 \chi_{\{v_k>0\}} \\
&\leqslant (\int_{Q_{k-1}} v_k^{\frac{10}{3}})^{\frac{3}{5}} \|\chi_{\{v_k>0\}}\|_{L^{\frac{5}{2}}(Q_{k-1})} \\
&\leqslant \|v_k\|^2_{L^{\frac{10}{3}}(Q_{k-1})} 2^{\frac{4k}{3}} C^{\frac{2}{5}} U_{k-1}^{\frac{2}{3}} \\
&\leqslant \|v_{k-1}\|^2_{L^{\frac{10}{3}}(Q_{k-1})} 2^{\frac{4k}{3}} C^{\frac{2}{5}} U_{k-1}^{\frac{2}{3}} \\
&\leqslant C U_{k-1}^{\frac{5}{3}} 2^{\frac{4k}{3}}.
\end{aligned}
$$

As a result, we have the following conclusion

$$
U_k \leqslant 2^{\frac{7k}{3}} C U_{k-1}^{\frac{5}{3}} + \int_{T_{k-1}}^{1} | \int_{\mathbb{R}^3} \frac{v_k}{|u|} u \nabla P dx | ds.
$$

Now, in order to estimate the term $\int_{T_{k-1}}^{1} | \int_{\mathbb{R}^3} \frac{v_k}{|u|} u \nabla P dx | ds$, we would like to carry out the following computation

$$
\begin{aligned}
-\triangle P = \sum \partial_i \partial_j (u_i u_j) \\
&= \sum \partial_i \partial_j \{(1 - \frac{v_k}{|u|}) u_i (1 - \frac{v_k}{|u|}) u_j + 2(1 - \frac{v_k}{|u|}) u_i \frac{v_k}{|u|} u_j\} \\
&+ \sum \partial_i \partial_j \{\frac{v_k}{|u|} u_i \frac{v_k}{|u|} u_j\}.
\end{aligned}
$$

This motivates us to decompose P as $P = P_{k1} + P_{k2}$, in which

$$
-\triangle P_{k1} = \sum \partial_i \partial_j \{(1 - \frac{v_k}{|u|}) u_i (1 - \frac{v_k}{|u|}) u_j + 2(1 - \frac{v_k}{|u|}) u_i \frac{v_k}{|u|} u_j\},
$$

and that

$$-\triangle P_{k2} = \sum \partial_i \partial_j \{ \frac{v_k}{|u|} u_i \frac{v_k}{|u|} u_j \}.$$

First, we have to notice that:

$$\begin{aligned}
&|(1-\frac{v_k}{|u|})^2 u_i u_j + 2(1-\frac{v_k}{|u|}) u_i \frac{v_k}{|u|} u_j| \\
&\qquad \leqslant (1-\frac{1}{2^k})\{(1-\frac{v_k}{|u|})|u_j| + 2\frac{v_k}{|u|}|u_j|\} \\
&\qquad \leqslant (1-\frac{v_k}{|u|})|u_j| + 2\frac{v_k}{|u|}|u_j| \\
&\qquad \leqslant 3|u_j| \leqslant 3|u|.
\end{aligned}$$

So, by Riesz's Theorem in the theroy of singular operator, we yield

$$\|P_{k1}\|_{L^6(Q_{k-1})} \leqslant C_6 \|3u\|_{L^6(Q_{k-1})} \leqslant 3C_6 (\frac{1}{A})^{\frac{1}{6}} \leqslant 3C_6.$$

So, we have

$$\begin{aligned}
&\int_{T_{k-1}}^1 |\int_{\mathbb{R}^3} \frac{v_k}{|u|} u \nabla P_{k1} dx| ds \\
&\qquad = \int_{T_{k-1}}^1 |\int_{\mathbb{R}^3} P_{k1} \nabla (\frac{v_k}{|u|} u) dx| ds \\
&\qquad \leqslant 3 \int_{T_{k-1}}^1 \int_{\mathbb{R}^3} d_k |P_{k1}| \chi_{\{v_k>0\}} dx\, ds \\
&\qquad \leqslant 3\|d_k\|_{L^2(Q_{k-1})} \|P_{k1}\|_{L^6(Q_{k-1})} \|\chi_{\{v_k>0\}}\|_{L^3(Q_{k-1})} \\
&\qquad \leqslant 3(2^{\frac{1}{2}}) \|d_{k-1}\|_{L^2(Q_{k-1})} 3C_6 2^{\frac{10k}{9}} C^{\frac{1}{3}} U_{k-1}^{\frac{5}{9}} \\
&\qquad \leqslant 9(2^{\frac{1}{2}}) C_6 C^{\frac{1}{3}} U_{k-1}^{\frac{1}{2}} 2^{\frac{10k}{9}} U_{k-1}^{\frac{5}{9}}.
\end{aligned}$$

That is, we have the following conclusion that

$$\int_{T_{k-1}}^{1} |\int_{\mathbb{R}^3} \frac{v_k}{|u|} u \nabla P_{k1} dx| ds \leqslant C 2^{\frac{10k}{9}} U_{k-1}^{\frac{19}{18}}.$$

Next, we would like to estimate the term $\int_{T_{k-1}}^{1} |\int_{\mathbb{R}^3} \frac{v_k}{|u|} u \nabla P_{k2} dx| ds$. First, we recall that, by the very definition of P_{k2} ,we have

$$P_{k2} = \sum R_i R_j \{\frac{v_k}{|u|} u_i \frac{v_k}{|u|} u_j\}.$$

, in which R_i , R_j etc are the Riesz's Transforms. Hence, we have

$$\nabla P_{k2} = \sum R_i R_j \{\nabla(\frac{v_k}{|u|} u_i \frac{v_k}{|u|} u_j)\}.$$

Now, we notice that

$$\begin{aligned} |\nabla(\frac{v_k}{|u|} u_i \frac{v_k}{|u|} u_j)| \leqslant |\nabla(\frac{v_k}{|u|} u_i)| |\frac{v_k}{|u|} u_j| + \frac{v_k}{|u|} |u_i| |\nabla(\frac{v_k}{|u|} u_j)| \\ \leqslant 3 d_k v_k + v_k (3 d_k) \\ = 6 v_k d_k. \end{aligned}$$

So, by applying the Riesz's Theorem in the theory of Singular integral, we have

$$
\begin{aligned}
\|\nabla P_{k2}\|_{L^{\frac{3}{2}}(Q_{k-1})} \leqslant C_{\frac{3}{2}}\|v_k d_k\|_{L^{\frac{3}{2}}(Q_{k-1})} &\\
&\leqslant C_{\frac{3}{2}}\{(\int_{Q_{k-1}} v_k^6)^{\frac{1}{4}}(\int_{Q_{k-1}} d_k^2)^{\frac{3}{4}}\}^{\frac{2}{3}}\\
&= C_{\frac{3}{2}}(\int_{Q_{k-1}} v_k^6)^{\frac{1}{6}}(\int_{Q_{k-1}} d_k^2)^{\frac{1}{2}}\\
&\leqslant C_{\frac{3}{2}}\|u\|_{L^6(Q_0)}\|d_k\|_{L^2(Q_{k-1})}\\
&\leqslant C_{\frac{3}{2}}(\frac{1}{A})^{\frac{1}{6}}(2)^{\frac{1}{2}}\|d_{k-1}\|_{L^2(Q_{k-1})}\\
&\leqslant 2^{\frac{1}{2}}C_{\frac{3}{2}}U_{k-1}^{\frac{1}{2}}.
\end{aligned}
$$

So, by applying the generalized Holder's inequality with exponents $\frac{10}{3}$, 30, $\frac{3}{2}$ to the terms v_k, $\chi_{\{v_k>0\}}$, ∇P_{k2} respectively, we yield

$$
\begin{aligned}
\int_{T_{k-1}}^{1} | \int_{\mathbb{R}^3} \frac{v_k}{|u|} u\nabla P_{k2} dx| dt &\\
&\leqslant \int_{Q_{k-1}} v_k \chi_{\{v_k>0\}}|\nabla P_{k2}|dx\,dt\\
&\leqslant \|v_k\|_{L^{\frac{10}{3}}(Q_{k-1})}\|\chi_{\{v_k>0\}}\|_{L^{30}(Q_{k-1})}\|\nabla P_{k2}\|_{L^{\frac{3}{2}}(Q_{k-1})}\\
&\leqslant U_{k-1}^{\frac{1}{2}} 2^{\frac{k}{9}} C^{\frac{1}{30}} U_{k-1}^{\frac{5}{90}} 2^{\frac{1}{2}} C_{\frac{3}{2}} U_{k-1}^{\frac{1}{2}}\\
&= C 2^{\frac{k}{9}} U_{k-1}^{1+\frac{5}{90}}\\
&= C 2^{\frac{k}{9}} U_{k-1}^{\frac{19}{18}}.
\end{aligned}
$$

That is, we have

$$
\int_{T_{k-1}}^{1} | \int_{\mathbb{R}^3} \frac{v_k}{|u|} u\nabla P_{k2} dx| dt \leqslant C 2^{\frac{k}{9}} U_{k-1}^{\frac{19}{18}}.
$$

So, by combining inequalities , we yield

$$U_k \leqslant 2^{\frac{7k}{3}} C U_{k-1}^{\frac{5}{3}} + \int_{T_{k-1}}^{1} |\int_{\mathbb{R}^3} \frac{v_k}{|u|} u \nabla P dx| dt$$

$$\leqslant 2^{\frac{7k}{3}} C U_{k-1}^{\frac{5}{3}} + C\{2^{\frac{10k}{9}} + 2^{\frac{k}{9}}\} U_{k-1}^{\frac{19}{18}}$$

$$\leqslant 2^{\frac{7k}{3}} C U_{k-1}^{\frac{19}{18}}.$$

That is, we will have the result that

$$U_k \leqslant C 2^{\frac{7k}{3}} U_{k-1}^{\frac{19}{18}},$$

for any $k \geqslant 1$.

4.6 Proof of proposition 4.1.2

Now, we would like to establish proposition 4.1.2 on the foundation of proposition 4.1.3. To begin, let C^* be the positive universal constant occuring in proposition 4.1.3. First, let show the proposition in the special case $\lambda = 2$. We chose T to be an arbitrary chosen positive number greater than 2, and let u be a solution of the Navier-Stokes equation on $(0, \infty) \times \mathbb{R}^3$. In the case in which u satisfies the condition that $\int_0^T \int_{\mathbb{R}^3} |u|^6 dx\, ds \leqslant (C^*)^6$, we define the function u^* by $u^*(s, x) = u(s + (T-1), x)$,which can be regarded to be another solution of the Navier-Stokes equation on $[-1, 1] \times \mathbb{R}^3$ satsifying

$$\int_{-1}^{1} \int_{\mathbb{R}^3} |u^*|^6 dx\, ds = \int_{T-2}^{T} \int_{\mathbb{R}^3} |u|^6 dx\, ds \leqslant \int_0^T \int_{\mathbb{R}^3} |u|^6 dx\, ds \leqslant (C^*)^6.$$

Hence, we have $\|u^*\|_{L^6([-1,1]\times\mathbb{R}^3)} \leqslant C^*$. So, it follows from the conclusion of proposition 4.1.3 that $\|u(T,\cdot)\|_{L^\infty(\mathbb{R}^3)} = \|u^*(1,\cdot)\|_{L^\infty(\mathbb{R}^3)} \leqslant 1$.
So, the above argument shows that

- we have $\|u(T,\cdot)\|_{L^\infty(\mathbb{R}^3)} \leqslant 1$, if $T > 2$, and u is a solution of the Navier-Stokes equation satisfying $\int_0^T \int_{\mathbb{R}^3} |u|^6 dx\, ds \leqslant (C^*)^6$.

Next, we also need to deal with the case in which the solution u satisfies the condition that $\int_0^T \int_{\mathbb{R}^3} |u|^6 dx\, ds > (C^*)^6$. In this case, let us consider the function u_ε defined by $u_\varepsilon(t,x) = \varepsilon u(\varepsilon^2 t, \varepsilon x)$, in which $\varepsilon > 0$ is arbitrary. Then, by applying the change of variable formula, it is easy to see that

$$\int_0^{\frac{T}{\varepsilon^2}} \int_{\mathbb{R}^3} |u_\varepsilon|^6 dx\, ds = \varepsilon \int_0^T \int_{\mathbb{R}^3} |u|^6 dx\, ds.$$

So, by taking $\varepsilon = (C^*)^6 \cdot \{2\int_0^T \int_{\mathbb{R}^3} |u|^6 dx\, ds\}^{-1}$, we yield

$$\int_0^{\frac{T}{\varepsilon^2}} \int_{\mathbb{R}^3} |u_\varepsilon|^6 dx\, ds = \frac{(C^*)^6}{2} < (C^*)^6.$$

The last inequality signifies that the solution u_ε falls back to the first case in this discussion. Hence, it follows directly form the conclusion we made for the frist case that u_ε must satisfies $\|u_\varepsilon(\frac{T}{\varepsilon^2},\cdot)\|_{L^\infty(\mathbb{R}^3)} \leqslant 1$. So, we eventually have

$$\|u(T,\cdot)\|_{L^\infty(\mathbb{R}^3)} = \frac{1}{\varepsilon}\|u_\varepsilon(\frac{T}{\varepsilon^2},\cdot)\|_{L^\infty(\mathbb{R}^3)} \leqslant \frac{1}{\varepsilon} = \frac{2}{(C^*)^6}\int_0^T \int_{\mathbb{R}^3} |u|^6 dx\, ds.$$

As a result, by all the discussion we made as above, we conclude that, no matter in which case, we always have the following inequality to be valid for any $T > 2$, and any solution u of the Navier-Stokes equation on $(0, \infty) \times \mathbb{R}^3$

$$\|u(T, \cdot)\|_{L^\infty(\mathbb{R}^3)} \leqslant A\{1 + \int_0^T \int_{\mathbb{R}^3} |u|^6 dx\, ds\},$$

where A is the universal constant defined by $A = max\{1, \frac{2}{(C^*)^6}\}$. This gives the proof of Proposition 4.1.2 in the special case $\lambda = 2$.

Next, let λ be a fixed positive number satisfying $0 < \lambda < 2$. As usual, let u be a solution of the Navier-Stokes equation on $(0, \infty) \times \mathbb{R}^3$. Now, let us consider the function w which is defined by

$$w(t, x) = (\frac{\lambda}{2})^{\frac{1}{2}} u(\frac{\lambda}{2} t, (\frac{\lambda}{2})^{\frac{1}{2}} x).$$

Then, by applying the above case to w, we have the following estimation, which is valid for any $T > \lambda$.

$$\begin{aligned}\|u(T, \cdot)\|_{L^\infty(\mathbb{R}^3)} \leqslant (\frac{2}{\lambda})^{\frac{1}{2}} \|w(\frac{2T}{\lambda}, \cdot)\|_{L^\infty(\mathbb{R}^3)} & \\ & \leqslant (\frac{2}{\lambda})^{\frac{1}{2}} A\{1 + \int_0^{\frac{2T}{\lambda}} \int_{\mathbb{R}^3} |w|^6 dx\, ds\} \\ & \leqslant (\frac{2}{\lambda})^{\frac{1}{2}} A\{1 + (\frac{\lambda}{2})^{\frac{1}{2}} \int_0^T \int_{\mathbb{R}^3} |u|^6 dx\, ds\} \\ & \leqslant (\frac{2}{\lambda})^{\frac{1}{2}} A\{1 + \int_0^T \int_{\mathbb{R}^3} |u|^6 dx\, ds\}.\end{aligned}$$

This gives proposition 4.1.2, where the universal constant A_λ is chosen to be $A_\lambda = (\frac{2}{\lambda})^{\frac{1}{2}} A$.

4.7 establishment of Theorem4.1.1

Finally, we are now ready to establish the conclusion of Theorem 4.1.1 on the foundation of proposition 4.1.2. We make use of the following result due to Kato [16] (see also the book of Lemarié-Rieusset [19]).

Theorem 4.7.1. Let $p > 3$. Then, for any given initial datum $u_0 \in L^p(\mathbb{R}^3)$ satisfying $div(u_0) = 0$, there exists a positive T^* and a unique weak solution $u \in C([0, T^*); L^p(\mathbb{R}^3))$ for the Navier-Stokes equation on $(0, T^*) \times \mathbb{R}^3$ so that $u(0, \cdot) = u_0$. This solution is then smooth on $(0, T^*) \times \mathbb{R}^3$. In addition, such a unique solution will also satisfies the extra condition that $u(t, \cdot) \in C_0(\mathbb{R}^3)$, for all $t \in (0, T^*)$.

To begin, let u be a weak solution of the Navier-Stokes equation on $(0, \infty) \times \mathbb{R}^3$ satisfying the condition that $\int_0^\infty \int_{\mathbb{R}^3} \frac{|u|^5}{log(1+|u|)} dx\, ds < \infty$. Then, by using the elementary inequality $log(1 + t) \leqslant t$, which is valid for all $t \geqslant 0$, we can deduce at once that

$$\int_0^\infty \int_{\mathbb{R}^3} |u|^4 dx\, ds \leqslant \int_0^\infty \int_{\mathbb{R}^3} \frac{|u|^5}{log(1 + |u|)} dx\, ds < \infty.$$

Now, let $\lambda \in (0, 2)$ to be arbitrary chosen and fixed. Since $|u|$ is L^4-integrable over $(0, \infty) \times \mathbb{R}^3$, it follows that the quantity $\int_{\mathbb{R}^3} |u(t, x)|^4 dx$ must be finite for almost every $t \in (0, \infty)$. So, with respect to λ, we can choose some τ_0 with $0 < \tau_0 < \lambda$ in such a way that $\int_{\mathbb{R}^3} |u(\tau_0, x)|^4 dx < \infty$, or equivalently $u(\tau_0, \cdot) \in L^4(\mathbb{R}^3)$. So, by using a simple shifting technique, we may apply the

Kato's Theorem quoted as above to deduce that there exists some positive constant $T^* > \tau_0$ so that our weak solution u is smooth on $(\tau_0, T^*) \times \mathbb{R}^3$, and that $u(t, \cdot) \in C_0(\mathbb{R}^3)$, for every t with $\tau_0 < t < T^*$. Hence, we know, in particular, that our weak solution u must be lying in the space $L^\infty_{loc}(\tau_0, T^*; L^\infty(\mathbb{R}^3))$. Now, for some technical purpose, we would like to pick up two numbers τ_1 and τ_2 which verify the condition that $\tau_0 < \tau_1 < \tau_2 < min\{\lambda, T^*\}$. Once τ_1 and τ_2 are chosen, they will be fixed. Now, from our original weak solution u, we can construct another weak solution v by requiring that $v(t, x) = u(t + \tau_1, x)$. Now, by applying the conclusion of proposition 4.1.2 to the weak solution v and the number $\tau_2 - \tau_1$, we can at once deduce that we have the following inequality

$$\|v(t, \cdot)\|_{L^\infty(\mathbb{R}^3)} \leqslant A\{1 + \int_0^t \int_{\mathbb{R}^3} |v|^6 dx\, ds\},$$

to be valid for all $t > \tau_2 - \tau_1$, in which A is some universal constant depending only on $\tau_2 - \tau_1$. However, this means the same as saying that we have the following inequality

$$\|u(t + \tau_1, \cdot)\|_{L^\infty(\mathbb{R}^3)} \leqslant A\{1 + \int_{\tau_1}^{t+\tau_1} \int_{\mathbb{R}^3} |u|^6 dx\, ds\},$$

which is valid for all $t > \tau_2 - \tau_1$. Hence , it follows that we can make the following conclusion

- for every $t > \tau_2$, we have $\|u(t, \cdot)\|_{L^\infty(\mathbb{R}^3)} \leqslant A\{1 + \int_{\tau_1}^t \int_{\mathbb{R}^3} |u|^6 dx\, ds\}$, in which A is some universal constant depending only on $\tau_2 - \tau_1$.

At this stage, we are ready to apply the Gronwall's argument in the theory of ordinary differential equations as follow. For this purpose, we take $\psi(t) = t \cdot log(1+t)$, which is a strictly increasing positive valued function on $(0, \infty)$ satisfying the condition that

$$\int_1^\infty \frac{1}{\psi(t)} dt = \infty.$$

Then, it follows from our last inequality that

$$\begin{aligned} &\|u(t,\cdot)\|_{L^\infty(\mathbb{R}^3)} \\ &\qquad \leqslant A\{1 + \int_{\tau_1}^t \int_{\mathbb{R}^3} \psi(|u|) \frac{|u|^5}{log(1+|u|)} dx\, ds\} \\ &\qquad \leqslant A\{1 + \int_{\tau_1}^t \psi(\|u\|_{L^\infty(\mathbb{R}^3)}) \int_{\mathbb{R}^3} \frac{|u|^5}{log(1+|u|)} dx\, ds\}, \end{aligned}$$

which is valid for all $t > \tau_2$.

Next ,we put $F(t) = \|u(t,\cdot)\|_{L^\infty(\mathbb{R}^3)}$. Then, the above inequality can be rewritten as

$$F(t) \leqslant A\{1 + \int_{\tau_1}^t \psi(F(s)) G(s) ds\},$$

for all $t > \tau_2$, where G is the function defined by $G(s) = \int_{\mathbb{R}^3} \frac{|u|^5}{log(1+|u|)} dx$. Furthermore, we notice that by the hypothesis of Theorem 4.1.1, the function G must satisfies the condition that

$$\int_0^\infty G(s) ds = \int_0^\infty \int_{\mathbb{R}^3} \frac{|u|^5}{log(1+|u|)} dx\, ds < \infty.$$

Here, for the sake of convenience, we define

$$H(t) = A\{1 + \int_{\tau_1}^{t} \psi(F(s))G(s)ds\},$$

for all $t > \tau_1$. Then, our last inequality can be rewritten as

- $F(t) \leqslant H(t)$, for all $t > \tau_2$.

Since ψ is a strictly increasing positive valued function on $(0, \infty)$, it follows at once that

$$\frac{dH}{dt} = A\psi(F(t))G(t) \leqslant A\psi(H(t))G(t),$$

which is valid for all $t > \tau_2$. That is, we have the fact that

- for every $t > \tau_2$, we have $\frac{dH}{dt} \leqslant A\psi(H(t))G(t)$.

As a result, by taking integration in time over the interval (τ_2, t), for $t > \tau_2$, it follows at once that

$$\Psi(H(t)) - \Psi(H(\tau_2)) \leqslant A \int_{\tau_2}^{t} G(s)ds,$$

for all $t > \tau_2$, in which Ψ is the function defined by $\Psi(y) = \int_A^y \frac{1}{\psi(y)} dy$. Hence, we can deduce that

- for every $t > \tau_2$, we have $\Psi(H(t)) \leqslant \Psi(H(\tau_2)) + A \int_{\tau_2}^{t} G(s)ds$.

At this stage, in order to complete the Gronwall's argument successfully, we definitely need to show that $H(\tau_2)$ is finite. To achive this, let us recall that we have already used the Kato's Theorem to deduce that our original weak solution u must satisfies $u \in L^{\infty}_{loc}(\tau_0, T^*; L^{\infty}(\mathbb{R}^3))$, and this at once tells us that $\|u\|_{L^{\infty}([\tau_1,\tau_2]\times\mathbb{R}^3)} = sup_{t\in[\tau_1,\tau_2]}F(t) < +\infty$, because of the fact that $0 < \tau_0 < \tau_1 < \tau_2 < min\{\lambda, T^*\}$. Hence, it follows immediately that

$$H(\tau_2) \leqslant A\{1 + \psi(\|u\|_{L^{\infty}([\tau_1,\tau_2]\times\mathbb{R}^3)}) \int_{\tau_1}^{\tau_2} G(s)ds\} < +\infty.$$

So, we can now combine $H(\tau_2) < \infty$, and $\int_0^{\infty} G(s)ds < \infty$ to deduce that

- for every $t > \tau_2$, $\Psi(H(t)) \leqslant \Psi(H(\tau_2)) + \int_{\tau_2}^{t} G(s)ds < \infty$.

That is, we now know that $\Psi(H(t))$ must be finite, for every $t > \tau_2$. Since $\int_A^{+\infty} \frac{1}{\psi(y)}dy = +\infty$, this will force us to admit that $H(t) < \infty$, for all $t > \tau_2$. Hence, we eventually have the conclusion

- for every $t > \tau_2$, we have $\|u(t,\cdot)\|_{L^{\infty}(\mathbb{R}^3)} = F(t) \leqslant H(t) < \infty$. So, in particular, we now know also that $u \in L^{\infty}_{loc}(\tau_2, \infty; L^{\infty}(\mathbb{R}^3))$.

So, by applying the famous result of Serrin [29] that we mentioned in the introduction with the case in which $p = q = \infty$, $u \in L^{\infty}_{loc}((\tau_2, \infty) \times \mathbb{R}^3)$ immediately implies that $u \in C^{\infty}((\tau_2, \infty) \times \mathbb{R}^3)$, and hence we have the conclusion that u must be smooth on $(\lambda, \infty) \times \mathbb{R}^3$ (notice that $\tau_2 < \lambda$). Since $\lambda \in (0, 2)$ is arbitrary chosen in the above argument, we can finally deduce that any weak solution u satisfying the hypothesis of Theorem 4.1.1 must be smooth on $(0, \infty) \times \mathbb{R}^3$.

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