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Causality in Quantum Physics, the Ensemble of Beginnings of Time, and the Dispersion Relations of Wave function

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Causality in Quantum Physics, the Ensemble of Beginnings of Time, and the Dispersion Relations of Wave function

by

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This dissertation discusses a consequence of the limitation on causality originated in smallness of a quantum system. In quantum physics, disturbance due to a measurement is not negligible. Because of this fact, the time parameter t cannot be identified as a time continuum of experimenter's clock T on which observed events are recorded. Indeed, it will be shown that t represents an ensemble of time intervals on T during which a microsystem travels undisturbed. In particular $t = 0$ represents the ensemble of preparation events that we refer to as the ensemble of beginnings of time. This restricts t to range the positive real line only, but such a time evolution of quantum states cannot be achieved in the Hilbert space. Hence one needs the time asymmetric boundary condition in which only the semigroup time evolution is allowed. This boundary condition is characterized by the energy wave functions of quantum state (and of observables) satisfying the Hilbert transform, called in physics the dispersion relation.

The time asymmetric boundary condition is formulated as a pair of Hardy rigged Hilbert spaces. They are developed to incorporate Einstein's causality. Within the framework of Hardy rigged Hilbert space, decaying states are described by Gamow vectors, and they are associated to S-matrix poles in the lower-half complex energy plane. This framework provides one a non-perturbative description of a decaying particle. From the Gamow vectors, exponential decay law of a relativistic particle is derived. Finally the neutral kaon decay experiment and the Z-boson resonance are discussed as applications.

Contents

Chapter 1

Introduction

It has been nearly eighty years since Krönig [1] and Kramers [2] found a relation between causality and an analyticity of a complex refractive index. The causality condition they employed was that a signal cannot travel faster than the speed of light, and the analyticity derived from it was given as a simple integral formula relating a dispersive process to an absorption process [3]. This analyticity is generally referred to as dispersion relation. It was Krönig [4] who first gave the proof that the dispersion relation is the necessary and sufficient condition for the causality condition to be satisfied. Since then, the dispersion relation has been generalized and used in may branches of physics [3, 5]. In non-relativistic quantum physics, Schutzer and Tiomno [6] derived a dispersion relation of a S-matrix element for the scattering of particles by a finite range scatterer; their causality condition was that the out-going scattered wave must be zero for all times before the in-going incident wave hits the scattering center [7]. This work was criticized and generalized by van Kampen [8], and its even more general derivation was given later by Wigner [9]. In relativistic quantum physics, Gell-Mann, Goldberger, and Thirring [10] derived a dispersion relation of S-matrix elements from their causality condition, called microscopic causality, that the commutator (or anticommutator for fermions) of two Heisenberg field operators taken at space-like points vanish. This has been providing a non-perturbative method in quantum field theory [11].

This dissertation presents a new dispersion relation in quantum physics, a dispersion relation that the *wave functions* of state and of observable satisfy in the energy representation. The causality condition we employ here is the quantum physics version of "cause and effect" discussed in detail by Ludwig [12] and his school. This is a statement, which is now called *the preparation-registration arrow of time* [13], that a state first must be prepared by a preparation apparatus before an observable can be detected in it by the registration apparatus. On applying this causality condition, we notice a serious limitation due to smallness of a quantum system as stated by Dirac [14]: "Causality applies only to a system which is left undisturbed." We shall here accept this limitation as a phenomenological principle, in stead of considering the collapse-of-wave-function axiom in theory of measurement.

On the basis of the preparation-registration arrow of time and the smallness of quantum system, we will derive in Chapter 2 that the time evolution of a quantum state and of an observable are restricted to $0 \leq t < \infty$. This type of time evolution is called the semigroup time evolution. It will be shown in Chapter 3 that the Hilbert space boundary condition does not accommodate the semigroup time evolution, so one needs another boundary condition. Hence we introduce the time asymmetric boundary condition which has been implemented for a unified theory of resonance and decay [15]. This is a pair of Hardy rigged Hilbert spaces; our purpose for using it is twofold: to attain the semigroup time evolution and to provide Dirac's bra-ket formalism a mathematical counterpart. It is this abstract vector space that yields the dispersion relation of the energy wave function of state and of observable as Hardy functions. It should be noted that the Hardy function analytic in the upper complex semi-plane is indeed what has been known as *causal transform* [3, 5]. We also define the S-matrix elements as a transformation function between two different Hardy rigged Hilbert spaces. In Chapter 4, we incorporate the relativistic causality. In Chapter 5, we introduce the relativistic Gamow vector to discuss neutral kaon decay experiment and the mass and width determination of Z-boson resonance.

Chapter 2

Causality in Quantum Physics

We begin our discussion by defining a microsystem, a quantum system that undergoes non-trivial time evolution. After reviewing time evolution of sates and observables in Sec. 2.1, we impose the causality condition in Sec. 2.2.

2.1 Preparation and registration of microsystems – states and observables

A microsystem is a physical object that can be detected by a macroscopic measuring apparatus [12]. It is a "small" system, e.g., a particle, in the sense that disturbance due to an observation cannot be neglected[1]. In experiments, as shown in Fig. 2.1, a microsystem is first prepared by a preparation apparatus (e.g., accelerator), and they are subjected to a registration apparatus (e.g., detector). A result of experiment, namely a relation between preparation and

[1]Dirac has discussed a limit to the finiteness of one's power of observation and the smallness of the accompanying disturbance [14]. If the disturbance is not negligible, the system being observed is said to be "small".

registration, is described by quantum mechanics.

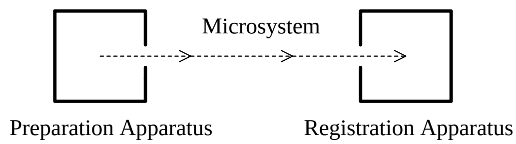

Figure 2.1: Experiment with microsystems. Microsystems are first prepared by a preparation apparatus then observed by a registration apparatus.

Performing preparation under the same condition would result in an *ensemble* of microsystems. This ensemble is represented by a quantum state which is most generally described by a density operator. For the sake of simplicity we restrict ourself here to a special case of it, a *pure state*, whose density operator is a projection operator $\Lambda = |\phi\rangle\langle\phi|$ satisfying $\Lambda^2 = \Lambda$. Its constituent vector $|\phi\rangle$ is a state vector which belongs to an abstract vector space Φ_{state}; the Hilbert space H is commonly chosen as Φ_{state} in axiomatic quantum mechanics [16, 17]. Hereafter, we call such a state described by a vector $|\phi\rangle$ the state ϕ.

Measurements of a physical quantity with the registration apparatus is represented by an observable $\mathcal{O}$. In the Hilbert space, $\mathcal{O}$ is described by a selfadjoint operator; its eigenkets form a complete set $\{|o_i\rangle\}$. If one performs a single measurement with $\mathcal{O}$ on a microsystem prepared in the state ϕ, the result one obtains is one of the eigenvalues o_i with a probability $w_i = |\langle o_i|\phi\rangle|^2$. Hence the expectation value of the measurement is given by $\langle\mathcal{O}\rangle \equiv \langle\phi|\mathcal{O}|\phi\rangle = \sum_i o_i\, w_i$.

In the experimental arrangement in Fig 2.1, every microsystem must first be prepared before it can be registered [12]; this statement is called the *preparation-registration arrow of time* [13]. According to this causality, the registration of $\mathcal{O}$ can take place anytime but only after preparation of the state ϕ has been completed. This time evolution is described by a dynamical equation. In the Schrödinger picture the state evolves in time obeying the Schrödinger equation[2],

$$i\frac{d}{dt}|\phi(t)\rangle = H\,|\phi(t)\rangle, \tag{2.1a}$$

where H is the Hamiltonian operator of the microsystem under consideration. Equivalently in the Heisenberg picture the observable evolves in time by obeying the Heisenberg equation,

$$-i\frac{d\mathcal{O}(t)}{dt} = [H, \mathcal{O}(t)]\,, \tag{2.1b}$$

and the state vector $|\phi\rangle$ is time independent. If $|\phi\rangle$ is prepared at time t_0, the solutions of Eqs. (2.1a) is given by,

$$|\phi(t)\rangle = e^{-iH(t-t_0)}\,|\phi\rangle, \tag{2.2a}$$

provided $\frac{d}{dt}H = 0$, and the solution of Eq. (2.1b) is given by

$$\mathcal{O}(t) = e^{iH(t-t_0)}\,\mathcal{O}e^{-iH(t-t_0)}. \tag{2.2b}$$

In these dynamical pictures, the expectation value of a measurement is ex-

[2]We take natural unit system in which $\hbar = 1$.

pressed as

$$\langle \mathcal{O} \rangle_t = \langle \phi(t) | \mathcal{O} | \phi(t) \rangle \text{ in the Schrödinger picture} \tag{2.3a}$$

$$= \quad \langle \phi | \mathcal{O}(t) | \phi \rangle \quad \text{in the Heisenberg picture.} \tag{2.3b}$$

On integrating the Schrödinger equation (2.1a) to obtain the solution (2.2a), a *boundary condition*, the abstract vector space $\Phi_{\rm state}$, has been tacitly assumed. If one takes the Hilbert space H for the boundary condition, i.e., $\Phi_{\rm state} = \mathsf{H}$, the time parameter t in Eqs. (2.2) and (2.3) must range the whole real line, $-\infty < t < \infty$ [18]. In the following section, we will show that this does not meet the physical requirement from causality.

2.2 The ensemble of beginnings of time and the limitation on time evolution

The preparation-registration arrow of time we have employed is a general statement on "cause and effect" whose application is not necessarily restricted to a microsystem; it could equally be applied to any "large" object which obeys the laws of classical physics. In quantum physics, however, there is a limitation on application of causality due to the smallness of a microsystem. This principle given by Dirac[3] we now quote: "Causality applies only to a system which is left undisturbed. If a system is small, we cannot observe it without producing a serious disturbance and hence we cannot expect to find any causal connexion between the results of our observations." As a result, the

[3] See section 1 of Ref [14].

best we can expect from the observations (measurements) is a probability. In accordance with this phenomenological principle, therefore, *a description of a causal connection by the dynamical equations is possible only between two successive measurements, the preparation and the registration, during which the microsystem is left undisturbed*[4] . Consequently once a microsystem is prepared in a state, any previous information about this microsystem is completely destroyed, and hence it is impossible for one to trace back with the dynamical equations.

To express this phenomenological principle in a mathematical form, let us first discuss the preparation of microsystems. For definiteness, we here consider an ensemble of N identical microsystems (e.g., electrons) all to be prepared in the same state ϕ. (Here N can be arbitrary large number.) In a laboratory, an experimenter would complete this preparation by performing measurements under the same condition, and the preparation events would be recorded referring to the experimenter's clock T. Let us denote the preparation instant of i-th microsystem by T_i. As a result of this preparation, one obtains an ensemble of preparation events at $\{T_1, T_2, \cdots, T_N\}$, that we refer to *the ensemble of beginnings of time* [19]. Since this ensemble describes the times at which the same state ϕ is prepared, all of these times must be mapped onto exactly the same point $t = t_0$ as shown in Fig. 2.2. Without loss of generality one can set t_0 for 0, and thus we denote the ensemble of beginnings of time as

$$t = 0 \; : \; \{T_1, T_2, \cdots, T_N\} \text{ for the state } \phi. \tag{2.4}$$

Note that each T_i in this ensemble is completely individual, i.e., there is no

[4]See section 27 of Ref [14].

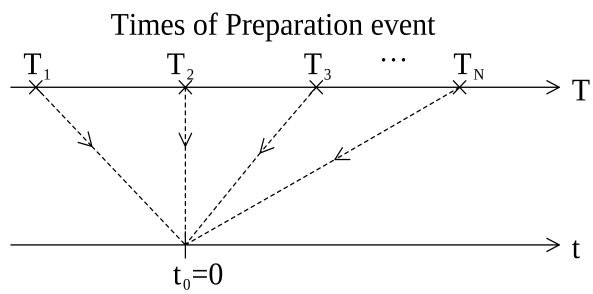

Figure 2.2: The ensemble of beginnings of time for the state ϕ.

correlation. In order to illustrate this fact, we shall here present two extremely different but completely equivalent ways of the preparation. At one extreme, one can prepare all of the N microsystems simultaneously (but in general at different places) at one instant of time on the experimenter's clock, say T^0. In this way all of the preparation times in Eq. (2.4) would be equal,

$$T^0 = T_1 = T_2 = \cdots = T_N. \tag{2.5}$$

Although this is possible in principle it is very difficult to achieve in practice. A more realistic way is then given at the other extreme: One can work with only one single microsystem at a time and repeat its preparation N times. Examples of this are the single-ion experiments [20], where the microsystem (single ion) is prepared in an unstable state for $N = 100 - 200$ times. In such an experiment, preparation events take place in chronological order as

$$T_1 < T_2 < \cdots < T_N, \tag{2.6}$$

and hence they are all *different* from one another.

Now we turn our attention to the time evolution of the states and observables. For each of the N preparation events (2.4), there corresponds a registration event of observable $\mathcal{O}$. Let us here denote the time of registration event by T'_i for the i-th individual microsystem. Then one obtains the ensemble of the "ends of time,"

$$\{T'_1, T'_2, \cdots, T'_N\} \text{ for observable } \mathcal{O}. \tag{2.7}$$

In accordance with the preparation-registration arrow of time, the registration event at T'_i must always be later than its corresponding preparation event at T_i, that is

$$T'_i \geq T_i \quad \text{for } i = 1, 2, \cdots, N \tag{2.8}$$

where equality holds when the registration follows immediately after the preparation. Finally, as shown in Fig 2.3, from the ensembles (2.4) and (2.7) one obtains an *ensemble* of time intervals $\{t_1, t_2, \cdots, t_N\}$ during which the microsystems were left undisturbed from any measurements,

$$\begin{aligned} t_1 &= T'_1 - T_1, \\ t_2 &= T'_2 - T_2, \\ &\vdots \\ t_N &= T'_N - T_N. \end{aligned} \tag{2.9}$$

Here $t_i \geq 0$ must hold because of the preparation-registration arrow of time (2.8). Now we draw a conclusion from Eq. (2.9): *the time parameter*

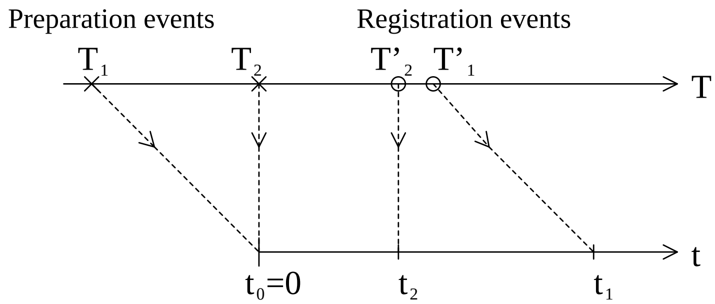

Figure 2.3: The preparations and the registrations in the experimenter's time T and the quantum mechanical parameter t. All of the preparation times (beginnings of time) are mapped onto $t_0 = 0$. Then the time intervals between the preparation and the registration are mapped onto the time parameter t.

t represents the ensemble of time-intervals during which the system evolved undisturbed,

$$t \ : \ \{t_1, t_2, \cdots, t_N\}. \tag{2.10}$$

Because each of the time intervals is always positive we have

$$0 \leq t < \infty. \tag{2.11}$$

This is a general expression of the preparation-registration arrow of time in quantum physics.

In accordance with Eq. (2.11), the expectation value $\langle \mathcal{O} \rangle_t$ makes physical sense only for $t \geq 0$. This means that the boundary condition for the Schrödinger equation (2.1a) must be chosen such that its general solution is given by

$$|\phi(t)\rangle = e^{-iHt}\,|\phi\rangle \quad \text{for } 0 \leq t < \infty \text{ only.} \tag{2.12a}$$

Alternatively in the Heisenberg picture, the solution of Eq. (2.1b) must be given by

$$\mathcal{O}(t) = e^{iHt}\,\mathcal{O}\,e^{-iHt} \quad \text{for } 0 \leq t < \infty \text{ only.} \tag{2.12b}$$

Such a time evolution (2.12) is called *semigroup*, because the time evolution operator e^{-iHt} of the state has no inverse $(e^{-iHt})^{-1}$.

Chapter 3

Boundary Condition for the Semigroup Time Evolution

The solutions of the Schrödinger equation (2.1a) depend upon boundary condition, i.e., the conditions which space the state vectors $|\phi(t)\rangle$ belong to. As we have discussed, one has to find a boundary condition which leads to the semigroup time evolution (2.12).

3.1 Why not the Hilbert space

Under the Hilbert space boundary condition (or the Hilbert space axiom of the standard (axiomatic) quantum mechanics), the state vectors are postulated to be elements of the Hilbert space, $|\phi(t)\rangle \in \mathsf{H}$. As a consequence of this, one obtains from the Stone-von Neumann theorem [18] that the solutions of the differential equation (2.1a) with a selfadjoint Hamiltonian H are given by unitary *group* $U^\dagger(t) \equiv e^{-iHt}$ with $-\infty < t < \infty$, which means the time evolution in Eq. (2.2a) is given by

$$|\phi(t)\rangle = e^{-iHt}|\phi\rangle \quad \text{with } -\infty < t < \infty. \tag{3.1}$$

This disagrees with our phenomenological conclusion (2.11), which suggested that Eq. (2.12a) must hold.

In scattering theory, one overcomes this discrepancy between (3.1) and the phenomenological result (2.12a) by employing the propagator or retarded Green's function [7, 21] $G(t)$,

$$G(t) = \theta(t)\ e^{-iHt} = \begin{cases} e^{-iHt} & \text{for } 0 \leq t < \infty, \\ 0 & \text{for } -\infty < t < 0. \end{cases} \tag{3.2}$$

This removes the unwanted negative-time part, however, such a solution of Eq. (2.1a) no longer belongs to the Hilbert space. The impossibility of Hilbert space vectors $|\phi(t)\rangle$ with the property that

$$\int_{-\infty}^{\infty} dt\ \langle\phi(t)|\mathcal{O}|\phi(t)\rangle = 0, \tag{3.3}$$

as would be the case for $|\phi(t)\rangle = G(t)|\phi\rangle$ given by Eq. (3.2), is also the consequence of a mathematical theorem [22].

Thus, although the Hilbert space has been successful choice for eigenstates of discrete energy which have trivial time evolution (stationary states), it does not accommodate the semigroup time evolution (2.12) derived from the phenomenological causality condition.

3.2 The Hardy space for states

Now we show that one can formulate *the time asymmetric boundary condition* [15] for the time symmetric differential equation (2.1a) and that it leads to the semigroup time evolution.

Since the time asymmetric boundary condition takes the form of rigged Hilbert space (RHS) [23], we shall here develop some notations necessary for subsequent discussion. In Dirac's bra-and-ket formalism, every state vector $|\phi\rangle$ is expanded with respect to the eigenkets of an operator with a continuous as well as discrete set of eigenkets. In order for us to work with the simplest representation of the time evolution operator e^{-iHt}, we here employ the energy and angular momentum eigenkets $|E\,\ell\,\ell_3\rangle$ that satisfy the eigenvalue equations,

$$H\,|E\,\ell\,\ell_3\rangle = E\,|E\,\ell\,\ell_3\rangle, \quad 0 < E < \infty \tag{3.4a}$$

$$\boldsymbol{L}^2\,|E\,\ell\,\ell_3\rangle = \ell(\ell+1)\,|E\,\ell\,\ell_3\rangle, \quad \ell = 0, 1, 2, \cdots \tag{3.4b}$$

$$L_3\,|E\,\ell\,\ell_3\rangle = \ell_3\,|E\,\ell\,\ell_3\rangle, \quad \ell_3 = -\ell, -\ell+1, \cdots, \ell, \tag{3.4c}$$

and a normalization

$$\langle E'\ell'\ell_3'|E\,\ell\,\ell_3\rangle = \delta(E'-E)\,\delta_{\ell'\ell}\,\delta_{\ell_3'\ell_3}. \tag{3.5}$$

Assuming a spherically symmetric Hamiltonian, $[H, \boldsymbol{L}] = 0$, and neglecting the spin for simplicity, the expansion of the state vector [14] is given by

$$|\phi\rangle = \sum_{\ell\ell_3} \int_0^\infty dE\ |E\,\ell\,\ell_3\rangle\,\phi_{\ell\ell_3}(E), \tag{3.6}$$

where $\phi_{\ell\ell_3}(E) \equiv \langle E\,\ell\,\ell_3|\phi\rangle$ is the energy wave function[1]. With this expansion, one has the one-to-one correspondence between a boundary condition $\Phi_{\rm state}$ for solutions of the Schrödinger equation and *a function space for the energy wave functions*. For example, if one choses for $\phi_{\ell\ell_3}(E)$ a smooth and rapidly decreasing function of E, then its function space is the Schwartz space $\mathcal{S}$ [17], i.e., $\phi_{\ell\ell_3}(E) \in \mathcal{S}(\mathbb{R}_+)$, where $\mathbb{R}_+$ denotes the range of energy eigenvalue. In the theory of RHS, there corresponds $\mathcal{S}(\mathbb{R}_+)$ an abstract vector space Φ, which is a particular dense subspace of the Hilbert space, $\Phi \subset \mathsf{H}$. The state vector is then an element of the abstract vector space, $|\phi\rangle \in \Phi$. On the other hand, the eigenkets (3.4) with normalization (3.5) are not elements of Φ or H, but elements of the abstract vector space $\Phi^\times$ for functionals (distributions, such as the delta function [17]) over the Schwartz space $\mathcal{S}^\times$, that is $|E\,\ell\,\ell_3\rangle \in \Phi^\times$. These three abstract vector spaces all together make a rigged Hilbert space called Schwartz RHS[2],

$$\Phi \subset \mathsf{H} \subset \Phi^\times. \tag{3.7}$$

Now, in the expansion (3.6), the time asymmetric boundary condition requires of the state energy wave function not only to be a Schwartz function $\mathcal{S}(\mathbb{R}_+)$ but also to be an *analytic* function in the lower-half complex (energy)

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[1]This is more commonly called the partial wave, but to make a clear distinction between partial wave of S-matrix (scattering amplitude) and this wave function we shall consistently use a terminology "energy wave function" for the representative.

[2]Dirac has emphasized in Ref [14] "The bra and ket vectors that we now use form a more general space than a Hilbert space." The functional space $\Phi^\times$ is this general space. It follows from this that the operators in Eq. (3.4) are more general than those in the Hilbert space.

semi-plane that satisfies the following condition:

$$\int_{-\infty}^{\infty} dE\, |\phi^{+}_{\ell\ell_3}(E+iy)|^2 < \infty \text{ for any } y < 0. \tag{3.8}$$

Here the superscript "+" indicates the analyticity of the wave function[3]. Such a function is called a *Hardy function analytic in the lower complex semi-plane*, or *Hardy function from below* in short [15, 24], and we denote its function space[4] by $\mathcal{S}_{-}(\mathbb{R}_{+})$,

$$\phi^{+}_{\ell\ell_3}(E) \in \mathcal{S}_{-}(\mathbb{R}_{+}). \tag{3.9}$$

Note that in Eq. (3.8) it seems problematic that the function values on the negative real-axis (of energy), which cannot be reached by experiment, are needed to perform the integral. For Hardy functions, however, these values can be reconstructed from its boundary values on the positive-real axis[5] [25], and hence the Hardy function $\phi^{+}_{\ell\ell_3}(E)$ is completely determined by their values for the physical energies $0 < E < \infty$. It may also be noted that Eq. (3.8) (along with the required analyticity) is necessary and sufficient condition for that the

[3]This "+" sign has its origin in the scattering theory; in-state prepared by an accelerator has been conventionally denoted by $|\phi^{+}\rangle$.

[4]This is a function space for "smooth" Hardy functions. The function space $\mathcal{S}_{-}(\mathbb{R}_{+})$ is defined by the intersection of the Schwartz space $\mathcal{S}$ and the function space of the Lebesgue square-integrable functions of Hardy class $\mathcal{H}^2_{-}$ restricted to $\mathbb{R}_{+}$, i.e., $\mathcal{S}_{-}(\mathbb{R}_{+}) \equiv \left[\mathcal{S} \cap \mathcal{H}^2_{-}\right]_{\mathbb{R}_{+}}$. This allows one to obtain the Hardy RHS.

[5]This is by virtue of the van Winter theorem of Hardy functions

energy wave function fulfills the Hilbert transform,

$$\mathrm{Re}\ \phi^+_{\ell\ell_3}(E) = -\frac{1}{\pi}\ \mathrm{P}\int_{-\infty}^{\infty} d\omega\ \frac{\mathrm{Im}\ \phi^+_{\ell\ell_3}(\omega)}{\omega - E}\ , \tag{3.10a}$$

$$\mathrm{Im}\ \phi^+_{\ell\ell_3}(E) = \frac{1}{\pi}\ \mathrm{P}\int_{-\infty}^{\infty} d\omega\ \frac{\mathrm{Re}\ \phi^+_{\ell\ell_3}(\omega)}{\omega - E}\ , \tag{3.10b}$$

where P designates the Cauchy principal value. In physics, Eq. (3.10) is called *dispersion relation* [7, 26] which often appears in connection with causality.

It has been shown by Gadella that the Hardy functions provide a rigged Hilbert space, called the *Hardy RHS*, denoted by $\Phi_- \subset \mathsf{H} \subset \Phi_-^\times$ [27]. Stated generally, the time asymmetric boundary condition requires of the state ϕ to be described by a state vector $|\phi^+\rangle$ in the Hardy space Φ_-. In the Hardy RHS, there exists for every vector $|\phi^+\rangle \in \Phi_-$ a set of eigenkets $|E\,\ell\,\ell_3{}^+\rangle \in \Phi_-^\times$ [6]

$$H\,|E\,\ell\,\ell_3{}^+\rangle = E\,|E\,\ell\,\ell_3{}^+\rangle, \quad 0 < E < \infty \tag{3.11a}$$

$$\boldsymbol{L}^2\,|E\,\ell\,\ell_3{}^+\rangle = \ell(\ell+1)\,|E\,\ell\,\ell_3{}^+\rangle, \quad \ell = 0,1,2,\cdots \tag{3.11b}$$

$$L_3\,|E\,\ell\,\ell_3{}^+\rangle = \ell_3\,|E\,\ell\,\ell_3{}^+\rangle, \quad \ell_3 = -\ell, -\ell+1, \cdots, \ell, \tag{3.11c}$$

with a normalization

$$\langle{}^+E'\ell'\ell_3'|E\,\ell\,\ell_3{}^+\rangle = \delta(E'-E)\,\delta_{\ell'\ell}\,\delta_{\ell_3'\ell_3}, \tag{3.12}$$

[6]The Hamiltonian of Eq. (3.11a) should be denoted by $H_-^\times$. The Hamiltonian operators $H_\pm$ on $\Phi_\pm$ are defined as the restriction of the selfadjoint Hamiltonian on the Hilbert space, $\overline{H}$, to the dense subspaces $\Phi_\pm$ of H. The operator $H_\pm^\times$ are the uniquely defined extensions of the operator $\overline{H} = H^\dagger$ to the spaces $\Phi_\pm^\times$. In this article, however, we denote all the different Hamiltonian operators as H, for their mathematical differences are immaterial for our purpose.

such that $|\phi^+\rangle$ can be written as

$$|\phi^+\rangle = \sum_{\ell\ell_3} \int_0^\infty dE \ |E\,\ell\,\ell_3\,^+\rangle\, \phi^+_{\ell\ell_3}(E), \tag{3.13}$$

where $\phi^+_{\ell\ell_3}(E) \equiv \langle^+E\,\ell\,\ell_3|\phi^+\rangle$. This means that one has the one-to-one correspondence between the state vector $|\phi^+\rangle$ and its energy wave function $\phi^+_{\ell\ell_3}(E)$ as

$$|\phi^+\rangle \in \Phi_- \longleftrightarrow \phi^+_{\ell\ell_3}(E) \in \mathcal{S}_-(\mathbb{R}_+). \tag{3.14}$$

We shall now show that the property of the Hardy functions (3.8) leads to the semigroup time evolution. For the state ϕ prepared as $|\phi^+\rangle$ at $t = 0$, its time evolved state vector is given by $|\phi^+(t)\rangle = e^{-iHt}|\phi^+\rangle$. The basisket expansion (3.13) of this vector is then given by

$$\begin{aligned} |\phi^+(t)\rangle &= \sum_{\ell\ell_3} \int_0^\infty dE \ |E\,\ell\,\ell_3\,^+\rangle\langle^+E\,\ell\,\ell_3|e^{-iHt}|\phi^+\rangle \\ &= \sum_{\ell\ell_3} \int_0^\infty dE \ |E\,\ell\,\ell_3\,^+\rangle \left[e^{-iEt}\phi^+_{\ell\ell_3}(E)\right], \end{aligned} \tag{3.15}$$

where Eq. (3.11a) has been used to obtain the last expression. In order that the time evolved state $|\phi^+(t)\rangle$ fulfills the time asymmetric boundary condition, that is in order that $|\phi^+(t)\rangle \in \Phi_-$, its time-dependent energy wave function $e^{-iEt}\phi^+_{\ell\ell_3}(E)$ given in Eq. (3.15) must satisfy Eq. (3.8),

$$\int_{-\infty}^\infty dE \ |e^{-i(E+iy)t}\phi^+_{\ell\ell_3}(E+iy)|^2 = \int_{-\infty}^\infty dE \ e^{2ty}|\phi^+_{\ell\ell_3}(E+iy)|^2 < \infty. \tag{3.16}$$

For this integral to converge for arbitrary large negative y, the time parameter

t cannot be negative, so that only $0 \le t < \infty$ is allowed. Hence we obtain the time evolution of the state

$$|\phi^+(t)\rangle = e^{-iHt}\,|\phi^+\rangle \quad \text{for } 0 \le t < \infty \text{ only.} \tag{3.17}$$

This is exactly the desired semigroup time evolution (2.12a).

In practice, the square-modulus of the wave function, $|\phi^+_{\ell\ell_3}(E)|^2$, is interpreted as the energy-angular distribution of the prepared microsystems, which is the characteristic of preparation apparatus. For example, if the state ϕ is prepared in such a way that its energy wave function be a Hardy function from below,

$$\phi^+_{\ell\ell_3}(E) = \frac{C_{\ell\ell_3}}{E-(a+ib/2)} \;\in \mathcal{S}_-(\mathbb{R}_+), \tag{3.18}$$

where $C_{\ell\ell_3}$ is complex in general and a, $b > 0$, then its energy distribution function $f(E)$ is a Lorentzian function,

$$f(E) \equiv \sum_{\ell\ell_3} |\phi^+_{\ell\ell_3}(E)|^2 = \frac{\sum_{\ell\ell_3}|C_{\ell\ell_3}|^2}{(E-a)^2+(b/2)^2}, \tag{3.19}$$

in which the peak energy is given by a and the FWHM by b. The coefficients $|C_{\ell\ell_3}|^2$ describe the angular distribution that satisfy the normalization condition

$$||\phi^+||^2 = \langle\phi^+|\phi^+\rangle = \int_0^\infty dE\, f(E) = 1, \tag{3.20}$$

where the definition (3.19) has been used.

3.3 Transition probability and the Hardy space for observables

In this section, we will concern with a selective measurement [28, 29, 30] (or filteration) in which a registration apparatus is designed to select only one of the eigenvectors of $\mathcal{O}$, or a particular linear combination of them. The observable that represents the measurement is given by

$$\mathcal{O}_\psi = |\psi\rangle\langle\psi|, \tag{3.21}$$

where $|\psi\rangle$ is a normalized vector which can be expanded as

$$|\psi\rangle = \sum_{\ell\ell_3} \int_0^\infty dE\, |E\,\ell\,\ell_3\rangle \psi_{\ell\ell_3}(E). \tag{3.22}$$

Here $\psi_{\ell\ell_3}(E) \equiv \langle E\,\ell\,\ell_3|\psi\rangle$ is a wave function characterizing the measurement[7]. The observable (3.21) satisfies $\mathcal{O}_\psi^2 = \mathcal{O}_\psi$; the eigenvalue of $\mathcal{O}_\psi$ is 1 ('affirmative') or 0 ('negative') only, with $|\psi\rangle$ being the eigenvector belonging to the eigenvalue 1 and all vectors orthogonal to $|\psi\rangle$ having the eigenvalue 0. For $\mathcal{O}_\psi$ is completely specified by a vector $|\psi\rangle$, we call Eq. (3.21) the *observable* ψ hereafter.

The result one obtains for a number of selective measurements is a *Born probability*, or more commonly called a *transition probability*. In the Schrödinger picture, by substituting Eq. (3.21) into Eq. (2.3), the transition

[7]For example, if the measurement is to select $\ell = 0$ eigenvector, then $\psi_{\ell\ell_3}(E)$ is nonzero only if $\ell = \ell_3 = 0$. For an observable having continuous eigenvalues, such as a Hamiltonian H, one can make an "almost eigenket". In the case of Eq. (3.22), it is given by a vector $|\psi\rangle$ in which $|\psi_{\ell\ell_3}(E)|^2$ has a very sharp peak with a finite width [31].

probability $\mathcal{P}(t)$ to detect the observable ψ in the state ϕ is the special case of expectation value given as

$$\mathcal{P}(t) \equiv \langle \mathcal{O}_\psi \rangle_t = \langle \phi^+(t)|\psi\rangle\langle\psi|\phi^+(t)\rangle = |\langle\psi|\phi^+(t)\rangle|^2 \text{ for } 0 \leq t < \infty, \tag{3.23}$$

where the time evolved state vector is given by Eq. (3.17). This quantity certainly is probability as it is bounded by the Cauchy-Schwartz inequality to range between 0 and 1,

$$0 \leq |\langle\psi|\phi^+(t)\rangle|^2 \leq \langle\psi|\psi\rangle \, \langle\phi^+(t)|\phi^+(t)\rangle = 1, \tag{3.24}$$

provided the normalization $\langle\phi^+(t)|\phi^+(t)\rangle = \langle\psi|\psi\rangle = 1$. This inequality holds as long as the scalar product (bracket) between $|\psi\rangle$ and $|\phi^+\rangle$ is defined in positive hermitian form [31].

The question of immediate concern is, Is the vector $|\psi\rangle$ in the Hardy space Φ_-? To answer it, one has to examine the Heisenberg picture. The time evolved observable $\mathcal{O}_\psi(t)$ is given by substituting Eq. (3.21) into Eq. (2.12b) as

$$\mathcal{O}_\psi(t) = e^{iHt}|\psi\rangle\langle\psi|e^{-iHt} = |\psi(t)\rangle\langle\psi(t)|, \tag{3.25}$$

where we have defined a time evolved vector

$$|\psi(t)\rangle \equiv e^{iHt}|\psi\rangle \quad \text{for } 0 \leq t < \infty. \tag{3.26}$$

This expression is similar to Eq. (2.12a) but the sign of exponent is opposite. This means that if one had taken $|\psi\rangle \in \Phi_-$ the time evolution of $|\psi(t)\rangle$ would be for $-\infty < t \leq 0$, which contradicts Eq. (3.26). Thus the vector $|\psi\rangle$ is *not* an element of Φ_-. Also, it cannot be an element of the Hilbert space either because it does not lead to the semigroup time evolution (3.26).

Now for the observable ψ one needs yet another boundary condition for solutions of the Heisenberg equation (2.1b). As can be seen in Eq. (3.16), the sign of exponent is directly connected to the domain of analyticity of the wave function, namely the lower-half energy plane, for obtaining the semigroup time evolution for $0 \leq t < \infty$. Our previous discussion about Eq. (3.26) therefore suggests that the wave function $\psi_{\ell\ell_3}(E)$ in the basisket expanstion (3.22) is to be analytic in the *uppper*-half energy plane. Following this observation, we take the other type of Hardy function, the Hardy function analytic in the *upper* complex semi-plane, or Hardy function from *above* in short [24], for this energy wave function[8]:

$$\psi^-_{\ell\ell_3}(E) \in \mathcal{S}_+(\mathbb{R}_+) \tag{3.27}$$

where the superscript "$-$" is to indicate that $\psi^-_{\ell\ell_3}(E)$ is analytic and Hardy in the *upper*-half of the complex (energy) plane. The Hardy function from above is common in physics as *causal transform* [3, 5]. The energy wave function of observable ψ then satisfies the defining criterion of the Hardy function from above,

$$\int_{-\infty}^{\infty} dE\, |\psi^-_{\ell\ell_3}(E+iy)|^2 < \infty \text{ for any } y > 0. \tag{3.28}$$

[8]Here $\mathcal{S}_+(\mathbb{R}_+) \equiv [\mathcal{S} \cap \mathcal{H}^2_+]_{\mathbb{R}_+}$

And in place of the dispersion relations (3.10), the following relation holds:

$$\mathrm{Re}\ \psi^{-}_{\ell\ell_3}(E) = \frac{1}{\pi}\ \mathrm{P}\int_{-\infty}^{\infty} d\omega\ \frac{\mathrm{Im}\ \psi^{-}_{\ell\ell_3}(\omega)}{\omega - E}\ , \tag{3.29a}$$

$$\mathrm{Im}\ \psi^{-}_{\ell\ell_3}(E) = -\frac{1}{\pi}\ \mathrm{P}\int_{-\infty}^{\infty} d\omega\ \frac{\mathrm{Re}\ \psi^{-}_{\ell\ell_3}(\omega)}{\omega - E}\ . \tag{3.29b}$$

Note that Eqs. (3.29) and (3.10) are the complex conjugate of one another; this fact expresses a mathematical symmetry that holds among the Hardy functions: the complex conjugate of a Hardy function from above is a Hardy function from below and vice versa,

$$\overline{\psi^{-}_{\ell\ell_3}}(E) \in \mathcal{S}_{-}(\mathbb{R}_{+}) \longleftrightarrow \psi^{-}_{\ell\ell_3}(E) \in \mathcal{S}_{+}(\mathbb{R}_{+}). \tag{3.30}$$

By complete analogy to the case of the state vector, one constructs for observable ψ the RHS of Hardy functions from above denoted by $\Phi_{+} \subset \mathsf{H} \subset \Phi_{+}^{\times}$ [27], and we impose the boundary condition as

$$|\psi^{-}\rangle \in \Phi_{+} \subset \mathsf{H} \quad \text{and} \quad |E\,\ell\,\ell_3{}^{-}\rangle \in \Phi_{+}^{\times} \supset \mathsf{H}. \tag{3.31}$$

Here the basiskets satisfy the eigenvalue equations[9]

$$H\,|E\,\ell\,\ell_3{}^{-}\rangle = E\,|E\,\ell\,\ell_3{}^{-}\rangle, \quad 0 < E < \infty \tag{3.32a}$$

$$\boldsymbol{L}^2\,|E\,\ell\,\ell_3{}^{-}\rangle = \ell(\ell+1)\,|E\,\ell\,\ell_3{}^{-}\rangle, \quad \ell = 0, 1, 2, \cdots \tag{3.32b}$$

$$L_3\,|E\,\ell\,\ell_3{}^{-}\rangle = \ell_3\,|E\,\ell\,\ell_3{}^{-}\rangle, \quad \ell_3 = -\ell, -\ell+1, \cdots, \ell, \tag{3.32c}$$

[9] The Hamiltonian in Eq. (3.32a) is $H_{+}^{\times}$.

with a normalization

$$\langle^{-}E'\ell'\ell_3'|E\,\ell\,\ell_3{}^{-}\rangle = \delta(E'-E)\,\delta_{\ell'\ell}\,\delta_{\ell_3'\ell_3}, \tag{3.33}$$

so that $|\psi^-\rangle$ can be written as

$$|\psi^-\rangle = \sum_{\ell\ell_3}\int_0^\infty dE\ |E\,\ell\,\ell_3{}^{+}\rangle\,\psi^-_{\ell\ell_3}(E). \tag{3.34}$$

In the Hardy RHS (3.31), the one-to-one correspondence holds between the observable ψ and its energy wave function $\psi^-_{\ell\ell_3}(E) \equiv \langle^{-}E\,\ell\,\ell_3|\psi^-\rangle$ as

$$|\psi^-\rangle \in \Phi_+ \longleftrightarrow \psi^-_{\ell\ell_3}(E) \in \mathcal{S}_+(\mathbb{R}_+). \tag{3.35}$$

From this, by following the same argument as in Eq. (3.15) to Eq. (3.17), time evolution of the observable ψ in the Heisenberg picture is obtained as

$$|\psi^-(t)\rangle = e^{iHt}\,|\psi^-\rangle \quad \text{for } 0 \le t < \infty \text{ only.} \tag{3.36}$$

With this vector, we have for the observable ψ

$$\mathcal{O}_\psi(t) = |\psi^-(t)\rangle\langle\psi^-(t)| \quad \text{for } 0 \le t < \infty, \tag{3.37}$$

that is exactly the semigroup time evolution (2.12b).

A physical interpretation of the wave function $\psi^-_{\ell\ell_3}(E)$ is such that its square-modulus, $|\psi^-_{\ell\ell_3}(E)|^2$, describes energy-and-angular resolution function of the registration apparatus. For example, if the registration apparatus is to select a microsystem within the energy resolution function $g(E)$ of Lorentzian with a peak energy at a' and the FWHM b',

$$g(E) = \frac{\sum_{\ell\ell_3} |C'_{\ell\ell_3}|^2}{(E-a')^2 + (b'/2)^2}, \tag{3.38}$$

then its corresponding energy wave function is given by

$$\psi^-_{\ell\ell_3}(E) = \frac{C'_{\ell\ell_3}}{E - (a' - ib'/2)} \ \in \mathcal{S}_+(\mathbb{R}_+), \tag{3.39}$$

where $C'_{\ell\ell_3}$ is determined to satisfy the normalization

$$||\psi^-||^2 = \langle\psi^-|\psi^-\rangle = \int_0^\infty dE\, g(E) = 1. \tag{3.40}$$

In the process of selective measurement, a new state can be prepared. It is generally assumed in quantum mechanics that after the measurement of an observable the microsystem will be in a state that has been prepared by this measurement [31]. This means that the state prepared by the observable ψ is such as to have the same energy-angular distribution as $|\psi^-_{\ell\ell_3}(E)|^2$, but its state vector is in the Hardy space Φ_-. Such a state, the *state* ψ, is uniquely obtained by the mathematical symmetry (3.30) between Hardy functions as

$$|\psi^+\rangle = \sum_{\ell\ell_3} \int_0^\infty dE\, |E\,\ell\,\ell_3\,^+\rangle \psi^+_{\ell\ell_3}(E), \tag{3.41}$$

with

$$\psi^+_{\ell\ell_3}(E) \equiv \langle^+ E\ell\,\ell_3|\psi^+\rangle = \overline{\psi^-_{\ell\ell_3}}(E) \ \in \mathcal{S}_-(\mathbb{R}_+). \tag{3.42}$$

In the case of Eq. (3.39), for example, this wave function is given by

$$\psi^+_{\ell\ell_3}(E) = \frac{\overline{C'_{\ell\ell_3}}}{E - (a' + ib'/2)}. \tag{3.43}$$

Thus, as shown in Fig. 3.1, the microsystem originally prepared in the state ϕ "jumps" into the state ψ due to the measurement of the observable ψ. This

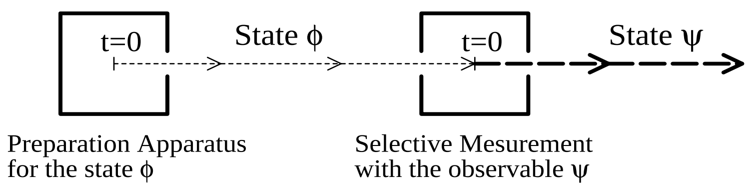

Figure 3.1: Preparation of the state ψ with the observable ψ.

preparation of state brings one a new ensemble of beginnings of time. In an idealized situation that the time it takes for a measurement is negligible, this beginnings of time would coincide with the ensemble of "ends of time" (2.7) at which the eigenvalue 1 ('affirmative') of the observable $\mathcal{O}_\psi$ is registered,

$$t = 0 \;\;:\;\; \{T'_1, T'_2, \cdots, T'_N\} \;\text{ for the state } \psi. \tag{3.44}$$

From this $t = 0$ the time evolution of $|\psi^+\rangle$ then begins.

To summarize, the time asymmetric boundary condition is a pair of Hardy RHSs, one for prepared states,

$$|\phi^+\rangle \in \Phi_- \subset \mathsf{H} \subset \Phi_-^\times \ni |E\,\ell\,\ell_3\,^+\rangle, \tag{3.45a}$$

and the other for observables of selective measurment,

$$|\psi^-\rangle \in \Phi_+ \subset \mathsf{H} \subset \Phi_+^\times \ni |E\,\ell\,\ell_3{}^-\rangle. \tag{3.45b}$$

Although the vector spaces for states and those for observables are different, a scalar product (bracket) between their vectors, such as $\langle\psi^-|\phi^+\rangle$ or even $\langle\psi^-|E\,\ell\,\ell_3{}^+\rangle$, is well defined. This is because scalar product is already defined in a linear-scalar-product space, or pre-Hilbert space, from which each of the vector spaces in RHS is obtained by completion with different topology (conversion of infinite sequence). One is thus free to take a scalar product between any of the elements among the Hardy RHS (3.45).

A scalar product of physical importance is the transformation function between basiskets of $\Phi_+^\times$ and $\Phi_-^\times$:

$$\langle{}^-E'\ell'\ell_3'|E\,\ell\,\ell_3{}^+\rangle = S_{\ell\ell_3}(E)\,\delta(E'-E)\,\delta_{\ell'\ell}\,\delta_{\ell_3'\ell_3}, \tag{3.46}$$

where the function $S_{\ell\ell_3}(E)$ is the S-matrix element [32]. It is this function that characterizes the experiment with the state ϕ and the observable ψ. The transition probability (3.23) with the time asymmetric boundary condition (3.45) is given by

$$\begin{aligned}
\mathcal{P}(t) = \langle\phi^+(t)|\mathcal{O}_\psi|\phi^+(t)\rangle &= |\langle\psi^-|\phi^+(t)\rangle|^2 \quad \text{in the Schrödinger picture,}\\
= \ \langle\phi^+|\mathcal{O}_\psi(t)|\phi^+\rangle &= |\langle\psi^-(t)|\phi^+\rangle|^2 \quad \text{in the Heisenberg picture,}\\
\equiv \ |a(t)|^2 &\quad \text{for } 0 \le t < \infty,
\end{aligned} \tag{3.47}$$

where $a(t) \equiv \langle\psi^-|\phi^+(t)\rangle = \langle\psi^-(t)|\phi^+\rangle$ is the time-dependent transition amplitude. With the S-matrix element defined by Eq. (3.46), the transition amplitude is given by

$$a(t) = \sum_{\ell\ell_3} \int_0^\infty dE\; e^{-iEt}\, \overline{\psi^-_{\ell\ell_3}}(E) \phi^+_{\ell\ell_3}(E)\, S_{\ell\ell_3}(E), \tag{3.48}$$

where Eqs. (3.11)–(3.13) and Eqs. (3.32)–(3.34) have been used. Due to the exponential factor e^{-iEt} bounded for $0 \leq t < \infty$, this integral can converge in the lower-half complex (energy) plane, where the wave functions, $\overline{\psi^-_{\ell\ell_3}}(E)$ and $\phi^+_{\ell\ell_3}(E)$, have no singularities (by Eq. (3.30) they are both the Hardy function from below). The transition amplitude $a(t)$ is therefore determined by singularities, e.g., poles, of the S-matrix element $S_{\ell\ell_3}(E)$ in the lower-half plane. Thus the S-matrix elements are responsible for the dynamics of microsystem [15].

Chapter 4

Relativistic Causality and Boundary Condition

In Chapter 2, we have discussed the preparation-registration arrow time in non-relativistic quantum physics. In relativistic physics, there is a limitation on speed of propagation of signal. We will now incorporate this limitation with the preparation-registration arrow of time and discuss the boundary condition for relativistic causality.

4.1 Relativistic Causality

Since the relativistic causality involves not only time T but also spatial position of an event $\boldsymbol{X} = (X_1, X_2, X_3)$, it is necessary to discuss the preparation and the registration events in space-time. As a fundamental assumption of the special relativity, the time T is a reading of a synchronized clock fixed in an inertial frame and $\boldsymbol{X}$ the position of this clock [33]. For N preparation and registration events, i-th preparation event is recorded as a space-time point denoted by $X_i^{\mathrm{P}} = (T_i^{\mathrm{P}}, \boldsymbol{X}_i^{\mathrm{P}})$ and the corresponding space-time point of i-th registration event by $X_i^{\mathrm{R}} = (T_i^{\mathrm{R}}, \boldsymbol{X}_i^{\mathrm{R}})$. The space-time interval $x_i = (t_i, \boldsymbol{x}_i)$ between i-th pair of preparation and registration events is given as shown in Fig. 4.1 by

$$t_i \equiv T_i^{\mathrm{R}} - T_i^{\mathrm{P}}, \tag{4.1a}$$

$$\boldsymbol{x}_i \equiv \boldsymbol{X}_i^{\mathrm{R}} - \boldsymbol{X}_i^{\mathrm{P}} \quad \text{for } i = 1, 2, \cdots, N. \tag{4.1b}$$

This means that the space-time parameter in relativistic quantum physics is

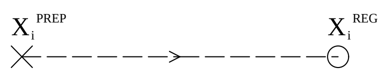

Figure 4.1: i-th preparation space-time point X_i^{P} and i-th registration space-time point X_i^{R}.

an ensemble of space-time intervals,

$$x = (c\,t, \boldsymbol{x}) = \{(c\,t_1, \boldsymbol{x}_1), (c\,t_2, \boldsymbol{x}_2), \cdots, (c\,t_N, \boldsymbol{x}_N)\} = \{x_1, x_2, \cdots, x_N\}. \tag{4.2}$$

Also, all of the N preparation events are mapped onto the same point $x^\mu = 0$, so in place of the ensemble of beginnings of time they all together form the ensemble of beginnings of space-time,

$$x^\mu = 0 \ : \ \left\{X_1^{\mathrm{P}}, X_2^{\mathrm{P}}, \cdots, X_N^{\mathrm{P}}\right\}. \tag{4.3}$$

Now the relativistic causality states that no signal can travel faster than the speed of light c. This means that an inequality holds between i-th preparation and registration event as

$$c\,(T_i^{\mathrm{R}} - T_i^{\mathrm{P}}) \geq |\boldsymbol{X}_i^{\mathrm{R}} - \boldsymbol{X}_i^{\mathrm{P}}|, \tag{4.4}$$

or in terms of space-time interval

$$c\,t_i \geq |\boldsymbol{x}_i| \quad \text{for } i = 1, 2, \cdots, N, \tag{4.5}$$

where the equality holds when the microsystem concerned is a photon. This leads to the space-time parameter of quantum physics to hold

$$c\,t - |\boldsymbol{x}| \geq 0, \tag{4.6}$$

and consequently all of the space-time intervals for the relativistic causality must be time-like,

$$(c\,t)^2 - \boldsymbol{x}^2 \geq 0. \tag{4.7}$$

Also, one has the preparation-registration arrow of time $0 \leq t < \infty$ to be satisfied in non-trivial time evolution. To incorporate the relativistic causality in time evolution of states or observables, therefore, we have

$$x^2 = (ct)^2 - \boldsymbol{x}^2 \geq 0 \text{ with } t \geq 0. \tag{4.8}$$

Thus the quantum physical space-time evolution is restricted in the forward-light-cone as shown in Fig. 4.2.

The coordinate transformation that leaves Eq. (4.8) invariant between

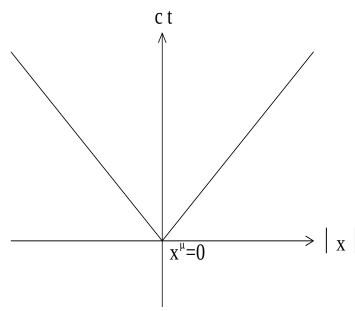

Figure 4.2: The space-time parameter is restricted in the forward-light-cone in which every space-time interval between preparation and registration event satisfies Eq. (4.8).

inertial frames is the proper-orthochronous Lorentz transformation $\Lambda^{\mu}{}_{\nu}$ in which $\det \Lambda = 1$ and $\Lambda^{0}{}_{0} \geq 1$ [17]. With this transformation, the ensemble of space-time intervals x^{μ} between preparation and registration event obtained in one inertial frame S is seen from another inertial frame S' as

$$x^{\mu} \longrightarrow x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu}. \tag{4.9}$$

On the other hand, the space-time evolution (4.8) is a space-time translation from $x^{\mu} = (0, \mathbf{0})$ to $x^{\mu} = (t \geq 0, \boldsymbol{x})$ that took place within the same inertial frame S. Combining the Lorentz transformation by Λ and the translation by x, we obtain the Poincaré semigroup into the forward-light-cone, *the causal Poincaré semigroup* [34], denoted by

$$\mathcal{P}_{+} \equiv \left\{ (\Lambda, x) \,:\, \det \Lambda = 1, \Lambda^{0}{}_{0} \geq 1, x^{2} \geq 0, t \geq 0 \right\}. \tag{4.10}$$

This is a relativistic generalization of the semigroup time evolution we have obtained in Chapter 2.

4.2 Relativistic quantum dynamics

In relativistic quantum physics, not only the time parameter t but also the position $\boldsymbol{x}$ be a parameter rather than an observable. We thus have space-time parameter x that labels state vector or observable in a dynamical picture. This suggests that the space-time evolution in S frame is described by a dynamical equation which is a relativistic generalization of non-relativistic dynamical equation (2.1). We take the Heisenberg picture and consider the following relativistic generalization of Eq. (2.1b):

$$-i\frac{\partial}{\partial x^{\mu}}\mathcal{O}(x) = [P_{\mu}, \mathcal{O}(x)], \tag{4.11}$$

where P_{μ} for $\mu = 0, 1, 2, 3$ are the momentum operators[1]. The momentum operators constitute four generators of translation subgroup of the Poincaré group, satisfying the commutation relation

$$[P_{\mu}, P_{\nu}] = 0. \tag{4.12}$$

A general solution to Eq. (4.11) is given by

$$\mathcal{O}(x) = e^{ix\cdot P}\, \mathcal{O}(x_0)\, e^{-ix\cdot P^{\dagger}}, \tag{4.13}$$

provided $\mathcal{O}(x_0)$ is known.

Covariance under the Lorentz transformation is described by an unitary

[1]Note that this is the equation of motion one has in quantum field theory [11].

operator $\mathcal{U}(\Lambda)$ defined by

$$\mathcal{U}(\Lambda) = e^{\frac{i}{2}\,\omega^{\mu\nu}\,J_{\mu\nu}}, \tag{4.14}$$

where $\omega^{\mu\nu}$ is the parameter that specifies the Lorentz transformation, and $J_{\mu\nu} = -J_{\nu\mu}$ are the six generators of Lorentz group that satisfy the following commutation relation [17]:

$$[J_{\mu\nu}, J_{\rho\sigma}] = -i\left(g_{\mu\rho}J_{\nu\sigma} + g_{\nu\sigma}J_{\mu\rho} - g_{\nu\rho}J_{\mu\sigma} - g_{\mu\sigma}J_{\nu\rho}\right). \tag{4.15}$$

The commutation relation between P_σ and $J_{\mu\nu}$ is given by

$$[J_{\mu\nu}, P_\sigma] = i\left(g_{\nu\sigma}P_\mu - g_{\mu\sigma}P_\nu\right), \tag{4.16}$$

and these ten generators altogether form the Poincaré algebra. Under the Lorentz transformation (4.9), the observable $\mathcal{O}$ transforms by $\mathcal{U}(\Lambda)$ and its inverse operator $\mathcal{U}^{-1}(\Lambda)$ as

$$\mathcal{O}'(x') = \mathcal{U}(\Lambda)\,\mathcal{O}(x)\,\mathcal{U}^{-1}(\Lambda), \tag{4.17}$$

while the state vector remains the same,

$$|\phi'\rangle = |\phi\rangle. \tag{4.18}$$

If the observable transforms as a scalar, vector, or spinor, we write the observable as $\mathcal{O}_a(x)$ and describe its covariance as

$$\begin{aligned}\mathcal{O}'_a(x') &= \mathcal{U}(\Lambda)\,\mathcal{O}_a(x)\,\mathcal{U}^{-1}(\Lambda) \\ &= \sum_b \mathcal{D}_{ab}(\Lambda)\,\mathcal{O}_b(x),\end{aligned} \tag{4.19}$$

where $D_{ab}(\Lambda) = \delta_{ab}$ for a scalar, $D_{ab}(\Lambda) = \Lambda_a{}^b$ for a vector, and $D_{ab}(\Lambda) = S_{ab}(\Lambda)$ for a spinor. In particular, the generators transform as

$$P'_\mu = \mathcal{U}(\Lambda)\,P_\mu\,\mathcal{U}^{-1}(\Lambda) = \Lambda^\nu{}_\mu P_\nu, \tag{4.20}$$

$$J'_{\mu\nu} = \mathcal{U}(\Lambda)\,J_{\mu\nu}\,\mathcal{U}^{-1}(\Lambda) = \Lambda^\rho{}_\mu \Lambda^\sigma{}_\nu J_{\rho\sigma}. \tag{4.21}$$

The expectation value of an observable $\mathcal{O}_a(x)$ in a (pure) state $|\phi\rangle$ is defined by

$$\langle\phi|\mathcal{O}_a(x)|\phi\rangle. \tag{4.22}$$

This quantity transforms under the Lorentz transformation as

$$\begin{aligned}\langle\phi'|\mathcal{O}'_a(x')|\phi'\rangle &= \langle\phi|\mathcal{U}(\Lambda)\mathcal{O}_a(x)\mathcal{U}^{-1}(\Lambda)|\phi\rangle \\ &= \sum_b \mathcal{D}_{ab}(\Lambda)\langle\phi|\mathcal{O}_a(x)|\phi\rangle.\end{aligned} \tag{4.23}$$

For a selective measurement, one takes a scalar operator that satisfies $\mathcal{O}^2_\psi(x) = \mathcal{O}_\psi(x)$ as

$$\mathcal{O}_\psi(x) \equiv |\psi(x)\rangle\langle\psi(x)|. \tag{4.24}$$

The transition probability is given by

$$\mathcal{P}(x) = \langle\phi|\mathcal{O}_\psi(x)|\phi\rangle = |\langle\psi(x)|\phi\rangle|^2. \tag{4.25}$$

This transition probability transforms as a scalar under the Lorentz transformation,

$$\begin{aligned}
\mathcal{P}'(x') &= \langle\phi'|\mathcal{O}'_\psi(x')|\phi'\rangle \\
&= \langle\phi|\mathcal{U}(\Lambda)\mathcal{O}_\psi(x)\mathcal{U}^{-1}(\Lambda)|\phi\rangle \\
&= \langle\phi|\mathcal{U}(\Lambda)|\psi(x)\rangle\langle\psi(x)|\mathcal{U}^{-1}(\Lambda)|\phi\rangle \\
&= |\langle\mathcal{U}(\Lambda)\psi(x)|\phi\rangle|^2 \\
&= \langle\phi|\mathcal{O}_\psi(x)|\phi\rangle \\
&= \mathcal{P}(x).
\end{aligned} \tag{4.26}$$

4.3 Hardy Rigged Hilbert Spaces for the causal Poincaré semigroup

It has been shown that one can construct Hardy RHSs that lead to the causal Poincaré semigroup (4.10) [34].

One of the characteristics of the Hardy RHSs relativistic quantum physics is that they use so-called velocity basis [35]. From a momentum operator P_μ, one defines a velocity operator $\hat{P}_\mu$ through the following relation:

$$P_\mu = M\,\hat{P}_\mu, \tag{4.27}$$

where $M \equiv \sqrt{M^2}$ is the invariant mass operator defined by

$$P_\mu P^\mu \equiv M^2. \tag{4.28}$$

This means that in relativistic quantum physics the mass of a microsystem is a quantity to be observed as an eigenvalue of the operator, rather than a given constant. From the ten generators of the Poincaré group $J_{\mu\nu}$ and P_μ, one can take the following complete set of commuting observables (CSCO),

$$\left\{ M^2, W, \hat{\boldsymbol{P}}, S_3, \mathcal{N} \right\}, \tag{4.29}$$

where W is another Lorentz invariant operator defined by

$$W \equiv -w^2 = -w_\mu w^\mu, \text{ with } w_\mu \equiv \frac{1}{2}\epsilon_{\mu\nu\rho\sigma} P^\nu J^{\rho\sigma}, \tag{4.30a}$$

$\hat{\boldsymbol{P}}$ and S_3 are defined by

$$\hat{\boldsymbol{P}} \equiv (\hat{P}_1, \hat{P}_2, \hat{P}_3), \tag{4.30b}$$

$$S_3 \equiv M^{-1} \mathcal{U}\left(L(p)\right) w_3 \, \mathcal{U}^{-1}\left(L(p)\right), \tag{4.30c}$$

with $L(p)$ the Lorentz boost matrix, and $\mathcal{N}$ the particle species operator (e.g., charge, isospin). Basis kets $|[\mathsf{s}, j]\hat{\boldsymbol{p}}\, j_3\, n^{\pm}\rangle \in \Phi_{\mp}^{\times}$ of the functional spaces $\Phi_{\mp}^{\times}$

are defined to satisfy the eigenket equations [34],

$$M^2 \,|[\mathsf{s}, j]\hat{\boldsymbol{p}}\, j_3\, n^{\,\pm}\rangle = \mathsf{s}\, |[\mathsf{s}, j]\hat{\boldsymbol{p}}\, j_3\, n^{\,\pm}\rangle,\ \ 0 < \mathsf{s}_0 \leq \mathsf{s} < \infty \tag{4.31a}$$

$$W \,|[\mathsf{s}, j]\hat{\boldsymbol{p}}\, j_3\, n^{\,\pm}\rangle = j(j+1)\, |[\mathsf{s}, j]\hat{\boldsymbol{p}}\, j_3\, n^{\,\pm}\rangle,\ \ j = -|\ell - s|, \cdots, \ell + s \tag{4.31b}$$

$$\hat{P}_\mu \,|[\mathsf{s}, j]\hat{\boldsymbol{p}}\, j_3\, n^{\,\pm}\rangle = \hat{p}_\mu\, |[\mathsf{s}, j]\hat{\boldsymbol{p}}\, j_3\, n^{\,\pm}\rangle, \tag{4.31c}$$

$$\hat{S}_3 \,|[\mathsf{s}, j]\hat{\boldsymbol{p}}\, j_3\, n^{\,\pm}\rangle = j_3\, |[\mathsf{s}, j]\hat{\boldsymbol{p}}\, j_3\, n^{\,\pm}\rangle,\ \ j_3 = -j, -j+1, \cdots, j-1, j, \tag{4.31d}$$

$$\mathcal{N} \,|[\mathsf{s}, j]\hat{\boldsymbol{p}}\, j_3\, n^{\,\pm}\rangle = n\, |[\mathsf{s}, j]\hat{\boldsymbol{p}}\, j_3\, n^{\,\pm}\rangle. \tag{4.31e}$$

Here the velocity eigenvalue is given by

$$\hat{p} = p/\sqrt{\mathsf{s}} = (\hat{p}^0, \hat{\boldsymbol{p}}) = (\sqrt{1 + \hat{\boldsymbol{p}}^2}, \hat{\boldsymbol{p}}) = (\gamma(\boldsymbol{v}), \gamma(\boldsymbol{v})\, \boldsymbol{v}), \tag{4.32}$$

where $\boldsymbol{v}$ is the three-velocity and $\gamma(\boldsymbol{v}) \equiv 1/\sqrt{1 - \boldsymbol{v}^2}$,

$$\hat{\boldsymbol{p}} = \gamma(\boldsymbol{v})\boldsymbol{v} = \frac{\boldsymbol{v}}{\sqrt{1 - \boldsymbol{v}^2}}. \tag{4.33}$$

The basis kets transform under the Poincaré group as

$$\mathcal{U}^\times(\Lambda, x)\, |[\mathsf{s}, j]\hat{\boldsymbol{p}}\, j_3\, n^{\,\pm}\rangle = e^{-i\sqrt{\mathsf{s}}\,\hat{p}_\mu x^\mu} \sum_\sigma \mathcal{D}^j_{\sigma\, j_3}\left(W(\Lambda^{-1}, p)\right) |[\mathsf{s}, j]\hat{\boldsymbol{p}}\, j_3\, n^{\,\pm}\rangle, \tag{4.34}$$

for $x^2 \geq 0$ and $t \geq 0$ ($-$ sign) or $t \leq 0$ ($+$ sign). With the basis kets, one has the Hardy RHSs

$$|\phi^+\rangle \in \Phi_- \subset \mathsf{H} \subset \Phi_-^\times \ni |[\mathsf{s}, j]\hat{\boldsymbol{p}}\, j_3\, n^{\,+}\rangle \text{ for states,} \tag{4.35a}$$

$$|\psi^-\rangle \in \Phi_+ \subset \mathsf{H} \subset \Phi_+^\times \ni |[\mathsf{s}, j]\hat{\boldsymbol{p}}\, j_3\, n^{\,-}\rangle \text{ for observables,} \tag{4.35b}$$

in which basis ket expansions for a state vector and for an observable vector are given by

$$|\phi^+\rangle = \sum_{jj_3n} \int_{\mathsf{s}_0}^{\infty} d\mathsf{s} \int \frac{d^3\hat{p}}{2\hat{p}^0}\ |[\mathsf{s},j]\hat{\boldsymbol{p}}\,j_3\,n^{\,+}\rangle\ \phi^+_{jj_3n}(\mathsf{s},\hat{\boldsymbol{p}}), \tag{4.36a}$$

$$|\psi^-\rangle = \sum_{jj_3n} \int_{\mathsf{s}_0}^{\infty} d\mathsf{s} \int \frac{d^3\hat{p}}{2\hat{p}^0}\ |[\mathsf{s},j]\hat{\boldsymbol{p}}\,j_3\,n^{\,-}\rangle\ \psi^-_{jj_3n}(\mathsf{s},\hat{\boldsymbol{p}}), \tag{4.36b}$$

where the wave functions are defined by

$$\phi^+_{jj_3n}(\mathsf{s},\hat{\boldsymbol{p}}) \equiv \langle^+\,[\mathsf{s},j]\hat{\boldsymbol{p}}\,j_3\,n|\phi^+\rangle \in \tilde{\mathcal{S}}_-(\mathbb{R}_{\mathsf{s}_0}) \otimes \mathcal{S}(\mathbb{R}^3), \tag{4.37a}$$

$$\psi^-_{jj_3n}(\mathsf{s},\hat{\boldsymbol{p}}) \equiv \langle^-\,[\mathsf{s},j]\hat{\boldsymbol{p}}\,j_3\,n|\psi^-\rangle \in \tilde{\mathcal{S}}_+(\mathbb{R}_{\mathsf{s}_0}) \otimes \mathcal{S}(\mathbb{R}^3). \tag{4.37b}$$

These wave functions satisfy dispersion relations for state wave function,

$$\mathrm{Re}\ \phi^+_{jj_3n}(\mathsf{s},\hat{\boldsymbol{p}}) = -\frac{1}{\pi}\ \mathrm{P}\int_{-\infty}^{\infty} d\mathsf{s}'\ \frac{\mathrm{Im}\ \phi^+_{jj_3n}(\mathsf{s}',\hat{\boldsymbol{p}})}{\mathsf{s}'-\mathsf{s}}\ , \tag{4.38a}$$

$$\mathrm{Im}\ \phi^+_{jj_3n}(\mathsf{s},\hat{\boldsymbol{p}}) = \frac{1}{\pi}\ \mathrm{P}\int_{-\infty}^{\infty} d\mathsf{s}'\ \frac{\mathrm{Re}\ \phi^+_{jj_3n}(\mathsf{s}',\hat{\boldsymbol{p}})}{\mathsf{s}'-\mathsf{s}}\ , \tag{4.38b}$$

and for observable wave function,

$$\mathrm{Re}\ \psi^-_{jj_3n}(\mathsf{s},\hat{\boldsymbol{p}}) = \frac{1}{\pi}\ \mathrm{P}\int_{-\infty}^{\infty} d\mathsf{s}'\ \frac{\mathrm{Im}\ \psi^-_{jj_3n}(\mathsf{s}',\hat{\boldsymbol{p}})}{\mathsf{s}'-\mathsf{s}}\ , \tag{4.39a}$$

$$\mathrm{Im}\ \psi^-_{jj_3n}(\mathsf{s},\hat{\boldsymbol{p}}) = -\frac{1}{\pi}\ \mathrm{P}\int_{-\infty}^{\infty} d\mathsf{s}'\ \frac{\mathrm{Re}\ \psi^-_{jj_3n}(\mathsf{s}',\hat{\boldsymbol{p}})}{\mathsf{s}'-\mathsf{s}}. \tag{4.39b}$$

The Hardy RHSs (4.35) completely fulfill a requirement from the Poincaré semigroup (4.8). In the Heisenberg picture, time evolution of a microsystem in S frame is described by a time evolved operator,

$$\mathcal{O}_\psi(x) = |\psi^-(x)\rangle\langle\psi^-(x)|, \tag{4.40}$$

where

$$|\psi^-(x)\rangle = \mathcal{U}(I,x)\,|\psi^-\rangle \quad \text{for } x^2 \geq 0 \text{ and } t \geq 0. \tag{4.41}$$

The wave function of this space-time dependent observable vector is given by

$$\psi^-_{jj_3n}(\mathsf{s},\hat{\boldsymbol{p}};x) \equiv \langle^-[\mathsf{s},j]\hat{\boldsymbol{p}}\,j_3\,n|\psi^-(x)\rangle \tag{4.42a}$$

$$= \langle^-[\mathsf{s},j]\hat{\boldsymbol{p}}\,j_3\,n|\mathcal{U}(I,x)\psi^-\rangle \tag{4.42b}$$

$$= \overline{\langle\psi^-|\mathcal{U}^\times(I,x)|[\mathsf{s},j]\hat{\boldsymbol{p}}\,j_3\,n^-\rangle} \tag{4.42c}$$

$$= \overline{e^{-ix\cdot\hat{p}\sqrt{\mathsf{s}}}\langle\psi^-|[\mathsf{s},j]\hat{\boldsymbol{p}}\,j_3\,n^-\rangle} \tag{4.42d}$$

$$= e^{ix\cdot\hat{p}\sqrt{\mathsf{s}}}\,\psi^-_{jj_3n}(\mathsf{s},\hat{\boldsymbol{p}}). \tag{4.42e}$$

For this function to be the Hardy function from above, it has to be bounded by

$$\int_{-\infty}^{\infty} d\mathsf{s}\;|e^{ix\cdot\hat{p}\sqrt{\mathsf{s}}}\,\psi^-_{jj_3n}(\mathsf{s},\hat{\boldsymbol{p}})|^2 = \int_{-\infty}^{\infty} d\mathsf{s}\;e^{-2x\cdot\hat{p}\,\mathrm{Im}\sqrt{\mathsf{s}}}\,|\psi^-_{jj_3n}(\mathsf{s},\hat{\boldsymbol{p}})|^2 < \infty. \tag{4.43}$$

This integral converges only when

$$x\cdot\hat{p}\,\mathrm{Im}\sqrt{\mathsf{s}} \geq 0. \tag{4.44}$$

Since s is in the upper-half s plane, $\mathrm{Im}\sqrt{\mathsf{s}} \geq 0$, Eq. (4.44) requires that

$$t' \equiv x \cdot \hat{p} = \gamma(t - \boldsymbol{v} \cdot \boldsymbol{x}) \geq 0. \tag{4.45}$$

Here t' is the time-interval between preparation and registration seen from the instantaneous "rest frame" S' of a microsystem moving with a velocity $\boldsymbol{v}$ relative to S frame. Since this Lorentz boost leaves invariant both the time order and the space-time interval, one concludes that the space-time evolution (4.45) is possible only within the forward-light-cone.

The transition probability is given by

$$\mathcal{P}(x) = |\langle \psi^-_{n'}(x)|\phi^+_n\rangle|^2 \equiv |a^{n'n}(x)|^2 \text{ for } x^2 \geq 0 \text{ and } t \geq 0. \tag{4.46}$$

where we have defined the transition amplitude by $a^{n'n}(x) \equiv \langle \psi^-_{n'}(x)|\phi^+_n\rangle$. Now we define the S-matrix element $S^{n'n}_{jj_3}(s, \hat{\boldsymbol{p}})$ as follows:

$$\langle^-[\mathsf{s}', j']\, \hat{\boldsymbol{p}}'\, j_3'\, n'|[\mathsf{s}, j]\, \hat{\boldsymbol{p}}\, j_3\, n\, ^+\rangle \equiv S^{n'n}_{jj_3}(s, \hat{\boldsymbol{p}})\delta(\mathsf{s}' - \mathsf{s})\, 2\hat{p}^0 \delta^3(\hat{\boldsymbol{p}}' - \hat{\boldsymbol{p}})\, \delta_{j'j}\, \delta_{j_3'j_3}. \tag{4.47}$$

By the basis-ket expansion, we obtain an expression of the transition amplitude in terms of the S-matrix element,

$$a^{n'n}(x) = \sum_{jj_3} \int_{\mathsf{s}_0}^{\infty} d\mathsf{s} \int \frac{d^3\hat{p}}{2\hat{p}^0}\, e^{-i\sqrt{\mathsf{s}}\, x\cdot\hat{p}}\, \overline{\psi^-_{jj_3n'}}(\mathsf{s}, \hat{\boldsymbol{p}})\, \phi^+_{jj_3n}(\mathsf{s}, \hat{\boldsymbol{p}})\, S^{n'n}_{jj_3}(\mathsf{s}, \hat{\boldsymbol{p}}). \tag{4.48}$$

This is a relativistic generalization of Eq. (3.48).

Chapter 5

Decaying State

One of the great advantages of using the Hardy spaces is that the functional space $\Phi_+^\times$ contains the Gamow vectors. G. Gamow [36] explained α decay of a nucleus by introducing a "state vector" $|\phi^G\rangle$ whose time evolution would be

$$|\phi^G(t)\rangle = e^{-i(E_R - i\Gamma/2)t}\,|\phi^G\rangle, \tag{5.1}$$

where $E_R > 0$ and $\Gamma > 0$. The survival probability then diminishes obeying the exponential decay law,

$$|\langle\phi^G|\phi^G(t)\rangle|^2 = e^{-\Gamma\,t}\,|\langle\phi^G|\phi^G\rangle|^2. \tag{5.2}$$

In spite of this attractive feature, the Gamow state has been considered only as a heuristic tool for the description of decaying phenomena. There are two reasons for this. First, the Gamow state has a complex energy eigenvalue prohibited from the selfadjointness of a Hamiltonian on the Hilbert space. Second, if the time t is taken to extend over $-\infty < t < \infty$, as is the case of the Hilbert space, the time evolution (5.1) leads to the "exponential catastrophe" [37]. In what follows, we shall show the functional space $\Phi_+^\times$ solves these problems.

5.1 Complex eigenvalues, Gamow ket, and exponential decay law

In Chapter 3, (generalized) eigenvalue equations for a Hamiltonian in the Hardy RHS $\Phi_\pm \subset \mathsf{H} \subset \Phi_\pm^\times$ has been introduced by Eqs. (3.11) and (3.32). We here discuss their possible (generalized) eigenvalues.

In the functional space $\Phi_+^\times$ the generalized eigenvalue equation is defined by the following equation [31]:

$$\langle H\psi^-|z\,\ell\,\ell_3{}^-\rangle = \langle\psi^-|H^\times|z\,\ell\,\ell_3{}^-\rangle = z\,\langle\psi^-|z\,\ell\,\ell_3{}^-\rangle = z\,\overline{\psi^-_{\ell\ell_3}}(z)\ . \tag{5.3}$$

Now the only restriction on Eq. (5.3), i.e., on $H^\times$, is the boundary condition (3.35). To determine the eigenvalue z of this $H^\times$, we use the following analyticity of the Hardy functions: as implied in Eqs. (3.10) and (3.29), *the complex conjugate of a Hardy function from below is a Hardy function from above and vice versa,*

$$\psi^-_{\ell\ell_3}(z) \in \mathcal{S}_+(\mathbb{R}_+) \text{ for } z \in \mathbb{C}_+ \longleftrightarrow \overline{\psi^-_{\ell\ell_3}}(z) \in \mathcal{S}_-(\mathbb{R}_+) \text{ for } z \in \mathbb{C}_-\,. \tag{5.4}$$

From this follows that since $|H\psi^-\rangle \in \Phi_+$ the very left-hand side of Eq. (5.3), $\langle H\psi^-|z^-\rangle = \overline{\langle^-z|H\psi^-\rangle}$, is the Hardy function from *below* $\mathcal{S}_-(\mathbb{R}_+)$, i.e., it is analytic in the *lower*-half plane, and so is the very right-hand side,

$$z\,\overline{\psi^{-}_{\ell\ell_3}}(z) \in \mathcal{S}_{-}(\mathbb{R}_{+}). \tag{5.5}$$

This allows the generalized eigenvalue z to extend over the whole *lower*-half complex plane $\mathbb{C}_{-}$. Thus we obtain the eigenket and the generalized eigenvalue in $\Phi_{+}^{\times}$ as

$$H^{\times}\,|z\,\ell\,\ell_3{}^{-}\rangle = z\,|z\,\ell\,\ell_3{}^{-}\rangle \quad \text{for } z = E + iy \text{ with } y < 0 \text{ and } -\infty < E < \infty. \tag{5.6}$$

In the same manner as from Eq. (5.3) to Eq. (5.6) for a vector in the Hardy space $|\phi^{+}\rangle \in \Phi_{-}$, we obtain the eigenket in $\Phi_{-}^{\times}$ with the generalized eigenvalues extending over the whole *upper*-half plane $\mathbb{C}_{+}$,

$$H^{\times}\,|z\,\ell\,\ell_3{}^{+}\rangle = z\,|z\,\ell\,\ell_3{}^{+}\rangle \quad \text{for } z = E + iy \text{ with } y > 0 \text{ and } -\infty < E < \infty. \tag{5.7}$$

Hence in the space $\Phi_{\pm}^{\times}$ one has the eigenkets $|z\,\ell\,\ell_3{}^{\mp}\rangle \in \Phi_{\pm}^{\times}$ not only with continuous real but also with *complex* eigenvalues $z \in \mathbb{C}_{\mp}$. From these eigenkets, the eigenkets (3.11a) and (3.32a) are obtained as

$$|E\,\ell\,\ell_3{}^{\pm}\rangle = |z\,\ell\,\ell_3{}^{\pm}\rangle \quad \text{for } y \to \pm 0. \tag{5.8}$$

As the special case of Eq. (5.6), the ket representing a decaying microsystem called *Gamow ket* is defined in $\Phi_+^\times$ by

$$H^\times\,|z_R\,\ell\,\ell_3{}^-\rangle = (E_R - i\Gamma/2\ )\ |z_R\,\ell\,\ell_3{}^-\rangle, \tag{5.9}$$

where $E_R > 0$ and $\Gamma > 0$[1]. The time evolution of the Gamow ket is given by

$$|z_R\,\ell\,\ell_3\,(t)^-\rangle = e^{-iH^\times t}\,|z_R\,\ell\,\ell_3{}^-\rangle = e^{-i(E_R-i\Gamma/2)t}\,|z_R\,\ell\,\ell_3{}^-\rangle, \tag{5.10}$$

that is the same as Eq. (5.1) of the decaying "state" that Gamow has envisioned. In the Hardy RHSs (3.45) one can see that $\Psi_{\text{state}}\ (=\Phi_-)$ being a dense subspace of $\Phi_+^\times$ via the Hilbert space, $\Psi_{\text{state}} \subset \Phi_+^\times$. This shows the possibility that the Gamow ket be a constituent of a state vector.

A pure state that represents a decaying microsystem prepared by an experimental apparatus needs to be described by a vector $|\phi^+\rangle \in \Phi_-$. It has been shown that with association to resonance poles of a S-matrix element, a time dependent state vector is expanded by Gamow kets as [15]

$$|\phi^+(t)\rangle = \sum_{n=1}^{N_R} |z_{R_n}{}^-\rangle e^{-iz_{R_n}t} + |B^-(t)\rangle, \tag{5.11}$$

where N_R is the number of resonance poles and where $|B^-\rangle$, called the background term, is defined by

$$|B^-(t)\rangle \equiv \sum_{\ell\ell_3}\int_0^{-\infty} dE\,|E\,\ell\,\ell_3{}^-\rangle\langle{}^-E\,\ell\,\ell_3|\phi^+(t)\rangle. \tag{5.12}$$

(Since a resonance or a decay occurs in a certain value of angular momentum, the labels ℓ and ℓ_3 are suppressed in Gamow kets in Eq. (5.11).) The

[1]As we will see shortly, Gamow ket is associated with an S-matrix pole

expansion (5.11) is called the complex basis ket expansion [13].

The decay phenomenon of a microsystem is a transition from a microsystem to the other microsystem, the decay products. This transition is described by a yes-or-no observable $|\psi^-\rangle \in \Phi_+$ which represents the decay products. For simplicity, if only one Gamow ket appears in the complex basis ket expansion (5.11) (with $N_R = 1$), the transition probability that the unstable state has decayed into the decay products is given by substituting Eq. (5.11) into Eq. (3.23) as

$$\begin{aligned}\mathcal{P}(t) &= |e^{-i(E_R - i\Gamma/2)t} \langle\psi^-|z_R{}^-\rangle + \langle\psi^-|B^-(t)\rangle|^2 \\ &= e^{-\Gamma t}\, |\langle\psi^-|z_R{}^-\rangle|^2 + |\langle\psi^-|B^-(t)\rangle|^2 + 2\mathrm{Re}\left\{\langle\psi^-|z_R{}^-\rangle\langle B^-(t)|\psi^-\rangle\right\}.\end{aligned} \tag{5.13}$$

where in the last line the first term describes the exponential decay law whereas the second and the third terms describe the deviation from the exponential decay law. If the amplitude $\langle\psi^-|B^-(t)\rangle$ is too small to detect, then an approximation called Weisskopf-Wigner treatment [38] is obtained,

$$\mathcal{P}(t) \simeq e^{-\Gamma t}\, |\langle\psi^-|z_R{}^-\rangle|^2. \tag{5.14}$$

This is not suffered from "exponential catastrophe" because the transition probability (5.13) or (5.14) is given only for $t \geq 0$.

In the following section, we will use the decomposition (5.11) generalized to the relativistic case [34, 39].

5.2 Relativistic Gamow vector associated to S-matrix pole

In this section we derive relativistic Gamow vectors from S-matrix poles and discuss a space-time dependent transition probability [34]. Since our derivation concerns only with s-variable, we introduce the following simplified notations of the velocity basis ket:

$$|\mathsf{s}^{\pm}\rangle \equiv |[\mathsf{s}, j]\, \hat{\boldsymbol{p}}\, j_3\, n\, ^{\pm}\rangle. \tag{5.15}$$

The basisket expansions in this simplified notation are given by

$$|\phi^{+}\rangle = \int_{\mathsf{s}_0}^{\infty} d\mathsf{s}\ |\mathsf{s}^{+}\rangle\, \phi^{+}(\mathsf{s}), \tag{5.16a}$$

$$|\psi^{-}\rangle = \int_{\mathsf{s}_0}^{\infty} d\mathsf{s}\ |\mathsf{s}^{-}\rangle\, \psi^{-}(\mathsf{s}). \tag{5.16b}$$

The transition amplitude at $x = 0$ is expressed as

$$\begin{aligned} a(0) \equiv \langle\psi^{-}|\phi^{+}\rangle &= \int_{\mathsf{s}_0}^{\infty} d\mathsf{s}\ \overline{\psi^{-}}(\mathsf{s})\phi^{+}(\mathsf{s})\, S(\mathsf{s}) \\ &= \int_{\mathsf{s}_0}^{\infty} d\mathsf{s}\ h(\mathsf{s})\, S(\mathsf{s}), \end{aligned} \tag{5.17}$$

where we have defined the product of wave functions $h(s)$ as

$$h(\mathsf{s}) \equiv \overline{\psi^{-}}(\mathsf{s})\phi^{+}(\mathsf{s}) \in \tilde{\mathcal{S}} \cap \mathcal{H}^{1}_{-}|_{\mathsf{s}_0} \tag{5.18}$$

Suppose the S-matrix is analytic in the lower-half s-plane except at two first-order poles, s_{R_1} and s_{R_2}, on the lower-half s-plane, so that it behaves around

$\mathsf{s} \sim \mathsf{s}_{R_i}$ for $i = 1, 2$ like

$$S(\mathsf{s}) = \frac{r_i}{\mathsf{s} - \mathsf{s}_{R_i}} + S_i^{(0)} + S_i^{(1)} (\mathsf{s} - \mathsf{s}_{R_i}) + \cdots . \tag{5.19}$$

The transition amplitude is then given by

$$\langle \psi^- | \phi^+ \rangle = \left[\oint_{C_1} d\mathsf{s} \ + \oint_{C_2} d\mathsf{s} \ + \int_{s_0}^{-\infty} d\mathsf{s} \ + \int_{R_\infty} d\mathsf{s} \ \right] h(\mathsf{s})\, S(\mathsf{s}) \tag{5.20}$$

where R_∞ is a large semicircle extending in the lower-half s-plane. Since the function $h(\mathsf{s})S(\mathsf{s})$ diminishes like polynomial for R_∞, we have the last term of Eq. (5.20) vanishes,

$$\int_{R_\infty} d\mathsf{s}\ h(\mathsf{s})\, S(\mathsf{s}) = 0. \tag{5.21}$$

The third term of Eq. (5.20) is called the background term, and it can be rewritten [40] by using the van Winter theorem for Hardy functions as

$$\int_{s_0}^{-\infty} d\mathsf{s}\ h(\mathsf{s})\, S(\mathsf{s}) = \int_{\mathsf{s}_0}^{\infty} d\mathsf{s}\ h(\mathsf{s})\, b(\mathsf{s}), \tag{5.22}$$

where $b(\mathsf{s})$ is a Mellin transformation of $S(\mathsf{s})$. This integral can be further rewritten by restoring the definition (5.18) as

$$\langle \psi^- | \phi^{bg} \rangle \equiv \int_{\mathsf{s}_0}^{\infty} d\mathsf{s}\ \langle \psi^- | \mathsf{s}^- \rangle\, b(\mathsf{s}), \tag{5.23}$$

and omitting the arbitrary vector $|\psi^-\rangle \in \Phi_+$ as

$$|\phi^{bg}\rangle \equiv \int_{\mathsf{s}_0}^{\infty} d\mathsf{s}\ |\mathsf{s}^-\rangle\, b(\mathsf{s}). \tag{5.24}$$

We refer this as the background ket. The first and second terms of Eq. (5.20), that we call pole terms, can be written as

$$\oint_{C_i} d\mathsf{s}\; h(\mathsf{s})\, S(\mathsf{s}) = \oint_{C_i} d\mathsf{s}\; h(\mathsf{s})\, \frac{r_i}{\mathsf{s}-\mathsf{s}_{R_i}}. \tag{5.25}$$

The right-hand-side of Eq. (5.25) can be written in two different ways:

$$\oint_{C_i} d\mathsf{s}\; h(\mathsf{s})\, \frac{r_i}{\mathsf{s}-\mathsf{s}_{R_i}} = -2\pi i\, h(\mathsf{s}_{R_i}) \quad \text{by Cauchy integral} \tag{5.26a}$$

$$= \int_{-\infty}^{\infty} d\mathsf{s}\; h(\mathsf{s})\, \frac{r_i}{\mathsf{s}-\mathsf{s}_{R_i}} \quad \text{by Titchmash theorem} \tag{5.26b}$$

By recalling the definitions (5.18) and (5.16), and dropping the arbitrary vector $|\psi^-\rangle$, we obtain from the equality (5.26) that

$$|\mathsf{s}_{R_i}{}^-\rangle = i\int_{-\infty}^{\infty} \frac{d\mathsf{s}}{2\pi}\; |\mathsf{s}^-\rangle\, \frac{\phi^+(\mathsf{s})}{\phi^+(\mathsf{s}_{R_i})}\frac{r_i}{\mathsf{s}-\mathsf{s}_{R_i}}. \tag{5.27}$$

We call this relativistic Gamow vector [34]. Eventually, the vector $|\phi^+\rangle$ is written by Eqs. (5.27) and (5.24) as

$$|\phi^+\rangle = |\mathsf{s}_{R_1}{}^-\rangle + |\mathsf{s}_{R_2}{}^-\rangle + |\phi^{bg}\rangle. \tag{5.28}$$

The space-time dependent transition amplitude is given by

$$\begin{aligned} a(x) &= \langle\psi^-(x)|\phi^+\rangle \\ &= \langle\psi^-(x)|\mathsf{s}_{R_1}{}^-\rangle + \langle\psi^-(x)|\mathsf{s}_{R_2}{}^-\rangle + \langle\psi^-(x)|\phi^{bg}\rangle. \end{aligned} \tag{5.29}$$

Now we calculate the time evolution of a relativistic Gamow vector. Since the time evolution of Gamow vector depends upon not only the pole s_R but also the velocity $\hat{\boldsymbol{p}}$, we temporarily restore the velocity $\hat{\boldsymbol{p}}$ for this calculation.

$$\langle\psi^-(x)|\mathsf{s}_{R_i},\hat{\boldsymbol{p}}^-\rangle = i\int_{-\infty}^{\infty}\frac{d\mathsf{s}}{2\pi}\int\frac{d^3\hat{p}}{2\hat{p}^0}\,\langle\psi^-(x)|\mathsf{s},\hat{\boldsymbol{p}}^{\,-}\rangle\frac{\phi^+(\mathsf{s},\hat{\boldsymbol{p}})}{\phi^+(\mathsf{s}_{R_i},\hat{\boldsymbol{p}})}\frac{r_i}{\mathsf{s}-\mathsf{s}_{R_i}} \tag{5.30a}$$

$$= i\int_{-\infty}^{\infty}\frac{d\mathsf{s}}{2\pi}\int\frac{d^3\hat{p}}{2\hat{p}^0}\,\langle\psi^-|e^{-ix\cdot P^\times}|\mathsf{s},\hat{\boldsymbol{p}}^{\,-}\rangle\frac{\phi^+(\mathsf{s},\hat{\boldsymbol{p}})}{\phi^+(\mathsf{s}_{R_i},\hat{\boldsymbol{p}})}\frac{r_i}{\mathsf{s}-\mathsf{s}_{R_i}} \tag{5.30b}$$

$$= i\int_{-\infty}^{\infty}\frac{d\mathsf{s}}{2\pi}\int\frac{d^3\hat{p}}{2\hat{p}^0}\,e^{-i\sqrt{\mathsf{s}}\,x\cdot\hat{p}}\psi^-(\mathsf{s},\hat{\boldsymbol{p}})\frac{\phi^+(\mathsf{s},\hat{\boldsymbol{p}})}{\phi^+(\mathsf{s}_{R_i},\hat{\boldsymbol{p}})}\frac{r_i}{\mathsf{s}-\mathsf{s}_{R_i}} \tag{5.30c}$$

Here we assume the wave function $|\psi^-(\mathsf{s},\hat{\boldsymbol{p}})|^2$ or $|\phi^+(\mathsf{s},\hat{\boldsymbol{p}})|^2$ has a strong peak around a particular value of velocity $\hat{\boldsymbol{p}}_0$. Then the above expression is approximated as

$$\simeq i\int_{-\infty}^{\infty}\frac{d\mathsf{s}}{2\pi}\,e^{-i\sqrt{\mathsf{s}}\,x\cdot\hat{p}_0}\psi^-(\mathsf{s},\hat{\boldsymbol{p}}_0)\frac{\phi^+(\mathsf{s},\hat{\boldsymbol{p}}_0)}{\phi^+(\mathsf{s}_{R_i},\hat{\boldsymbol{p}}_0)}\frac{r_i}{\mathsf{s}-\mathsf{s}_{R_i}} \tag{5.30d}$$

$$= e^{-i\sqrt{\mathsf{s}}\,x\cdot\hat{p}_0}\,\langle\psi^-|\mathsf{s}_{R_i},\hat{\boldsymbol{p}}_0{}^-\rangle. \tag{5.30e}$$

With the understanding that $|\mathsf{s}_{R_i}{}^-\rangle = |\mathsf{s}_{R_i},\hat{\boldsymbol{p}}_0{}^-\rangle$ and $\hat{\boldsymbol{p}}\simeq\hat{\boldsymbol{p}}_0$, we have

$$a(x) = e^{-i\sqrt{s_{R_1}}x\cdot\hat{p}}\,\langle\psi^-|\mathsf{s}_{R_1}{}^-\rangle + e^{-i\sqrt{s_{R_2}}x\cdot\hat{p}}\,\langle\psi^-|\mathsf{s}_{R_2}{}^-\rangle + \langle\psi^-(x)|\phi^{bg}\rangle \tag{5.31}$$

Thus we have superposition of two exponential time evolutions of Gamow vector and non-exponential time evolution of the background amplitude $\langle\psi^-(x)|\phi^{bg}\rangle$.

The transition probability that one observes prepared n eigen-state is observed to be the observable with n' is given by

$$\begin{aligned}\mathcal{P}^{n'n}(x) &= |a^{n'n}(x)|^2 && \text{(5.32a)}\\ &= |e^{-i\sqrt{\mathsf{s}_{R_1}}x\cdot\hat{p}}\langle\psi^-|\mathsf{s}_{R_1}{}^-\rangle + e^{-i\sqrt{\mathsf{s}_{R_2}}x\cdot\hat{p}}\langle\psi^-|\mathsf{s}_{R_2}{}^-\rangle + \langle\psi^-(x)|\phi^{bg}\rangle|^2 && \text{(5.32b)}\\ &= |e^{-i\sqrt{\mathsf{s}_{R_1}}x\cdot\hat{p}}\langle\psi^-|\mathsf{s}_{R_1}{}^-\rangle|^2 + |e^{-i\sqrt{\mathsf{s}_{R_2}}x\cdot\hat{p}}\langle\psi^-|\mathsf{s}_{R_2}{}^-\rangle|^2 + \mathcal{P}^{n'n}_{\mathrm{non-exp}}(x), && \text{(5.32c)}\end{aligned}$$

where $\mathcal{P}^{n'n}_{\mathrm{non-exp}}(x)$ is the contribution to the probability from interference terms and background terms. Here if one parameterizes the S-matrix pole as

$$\mathsf{s}_{R_i} = (M_{R_i} - i\Gamma_{R_i}/2)^2 \text{ with } M_{R_i} > 0 \text{ and } \Gamma_{R_i} > 0, \tag{5.33}$$

then the time evolution of each of the amplitudes from a Gamow vector is given by

$$\begin{aligned}\langle\psi^-(x)|\mathsf{s}_{R_i}{}^-\rangle &= e^{-i(M_{R_i}-i\Gamma_{R_i}/2)\,x\cdot\hat{p}}\langle\psi^-|\mathsf{s}_{R_i}{}^-\rangle && \text{(5.34a)}\\ &= e^{-iM_{R_i}\,x\cdot\hat{p}}e^{-\Gamma_{R_i}\,x\cdot\hat{p}/2}\langle\psi^-|\mathsf{s}_{R_i}{}^-\rangle. && \text{(5.34b)}\end{aligned}$$

Substituting this into Eq. (5.32), the transition probability is given by

$$\mathcal{P}^{n'n}(t,\boldsymbol{x}) = e^{-\Gamma_{R_1}\gamma(t-\boldsymbol{v}\cdot\boldsymbol{x})}\,|\langle\psi^-|\mathsf{s}_{R_1}{}^-\rangle|^2 + e^{-\Gamma_{R_2}\gamma(t-\boldsymbol{v}\cdot\boldsymbol{x})}\,|\langle\psi^-|\mathsf{s}_{R_2}{}^-\rangle|^2 + \mathcal{P}^{n'n}_{\mathrm{non-exp}}(x) \tag{5.35}$$

This is the relativistic generalization of the exponential decay law. The transition probability is being evaluated at $\hat{\boldsymbol{p}} = \hat{\boldsymbol{p}}_0 = \gamma(\boldsymbol{v}_0)\boldsymbol{v}_0$, one can perform a Lorentz boost to the rest frame of the microsystem in which the time is the proper time τ of the microsystem,

$$\tau = \gamma_0(t - \boldsymbol{v}_0 \cdot \boldsymbol{x}). \tag{5.36}$$

In this inertial frame, the transition probability is given by

$$\mathcal{P}^{n'n}(\tau) = e^{-\tau/\tau_1} \left|\langle\psi^-|\mathsf{s}_{R_1}{}^-\rangle\right|^2 + e^{-\tau/\tau_2} \left|\langle\psi^-|\mathsf{s}_{R_2}{}^-\rangle\right|^2 + \mathcal{P}^{n'n}_{\mathrm{non-exp}}(\tau), \tag{5.37}$$

where we have defined the lifetime for each of the unstable states,

$$\tau_i \equiv 1/\Gamma_{R_i}, \text{ for } i = 1, 2. \tag{5.38}$$

We will use this formula to discuss the neutral kaon experiment and the Z-boson resonance.

5.3 The neutral kaon decay experiment

In order to illustrate the usage of the Hardy spaces we discuss the neutral kaon decay experiment. In particular, we shall focus on the lifetime measurement of the short-lived component of the neutral kaon.

5.3.1 The measurement of the lifetime of regenerated K_S

In recent years, precise measurements of the lifetime of the short-lived kaon K_S (and also other physical quantities such as the mass difference in K_S and K_L) has been made using the technique of the coherent regeneration of K_S [41].

In such an experiment a K^0 beam is first produced and then it is let coast in vacuum for many mean lifetimes of K_S ($\tau_s = 0.895 \times 10^{-10}$sec) so that only its long-lived component K_L ($\tau_L = 5.17 \times 10^{-8}$sec) is left. This pure K_L beam is impinged on a slab of material, such as boron carbide, called the regenerator (Fig. 5.1). Since by the strong interaction the $\overline{K^0}$ component of the K_L is well absorbed in the regenerator while its K^0 component almost goes free through the material, the K_S is regenerated. The remarkable feature of this process is that these K_S and K_L are *coherent* in this forward scattering and they are very much in the same angular distribution as that of the original incident K_L beam [41]. As a final product one obtains an kaon beam of coherent mixture of K_L and K_S emerging from the regenerator.

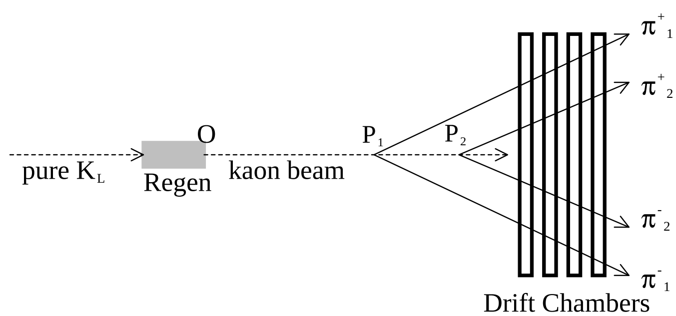

Figure 5.1: The kaon regeneration experiment. The kaon beam after the regenerator is a coherent mixture of K_S and K_L. O is the downstream edge of the regenerator, and P_i is the decay vertex of i-th kaon reconstructed from the trajectories of the $\pi_i^+ \pi_i^-$ pair.

As shown in Fig. 5.1, the regenerated kaon beam immediately proceeds to the vacuum decay volume, and its decay products are observed by detectors located downstream. For the measurement of the lifetime of K_S, the detectors are designed to observe the 2π decay modes. To be specific, let us focus on the charged 2π decay, $K_S \to \pi^+\pi^-$. The pairs of π^+ and π^- decayed from the kaon beam are detected by a series of drift chambers that register trajectories and momenta of the individual pions. The decay vertexes of kaon, $\{\mathrm{P}_1, \mathrm{P}_2, \cdots, \mathrm{P}_N\}$, are reconstructed from the obtained data set by extrapolating the trajectories of the corresponding pion pairs, and the momenta of kaon $\{\boldsymbol{p}_1, \boldsymbol{p}_2, \cdots, \boldsymbol{p}_N\}$ are determined by invoking the conservation rule,

$$\boldsymbol{p}_i = \boldsymbol{p}_i^+ + \boldsymbol{p}_i^- , \tag{5.39}$$

where $\boldsymbol{p}_i^+$ and $\boldsymbol{p}_i^-$ are the momenta of i-th π^+ and π^-, respectively. With these decay vertexes and the momenta in hand, one calculates the proper times τ_i of individual i-th kaon using the relativistic kinematics,

$$\tau_i = \frac{m\, d_i}{|\boldsymbol{p}_i|} , \tag{5.40}$$

where m is the mass of the kaon[2], and d_i is the distance from the edge of the regenerator to the i-th reconstructed decay vertex, $d_i = \overline{\mathrm{OP}_i}$. Finally, the set of proper times for N events of individual kaon decay is obtained,

$$\{\tau_1, \tau_2, \cdots, \tau_N\} , \tag{5.41}$$

with N being the order of 10^6, the order of magnitude of the number of kaons in the beam.

The lifetime τ_S for the $\pi^+\pi^-$ mode is extracted from the observed proper times (5.41) by data fitting with the phenomenological formula of the instan-

[2] There is a mass-difference Δm between K_S and K_L but it is totally negligible for this kinematics; $\Delta m/m = 0.7 \times 10^{-14}$.

taneous rate $R(\tau)$ for kaon decays [41],

$$R(\tau) \propto |\rho\, e^{-\tau/2\tau_S + i\Delta m\,\tau} + \eta\, e^{-\tau/2\tau_L}|^2, \tag{5.42}$$

where ρ is called the K_S amplitude determined by the regeneration process, η is called the K_L amplitude, and Δm is the mass difference in K_S and K_L. This formula can further be simplified. Because K_L lives about 500 times longer than K_S, the events observed close to the regenerator would be dominated by decays of K_S. On this basis, we select from Eq. (5.41) those events of small proper time, e.g., $\tau < 10\,\tau_S$, for the extraction of the K_S lifetime. In addition, the regenerator is designed so that the regeneration amplitude of $|\rho/\eta|^2 \sim \mathcal{O}(100)$ is achieved [41], meaning that K_S is more copious than K_L around the edge of regenerator. Therefore these selected events are fairly considered as only from K_S decays, then Eq. (5.42) reduces to

$$R(\tau) \propto |\rho|^2\, e^{-\tau/\tau_S} \quad \text{for small } \tau. \tag{5.43}$$

In the following section, we will reproduce the same formula by quantum mechanics with the time-asymmetric boundary conditions.

5.3.2 Preparation of the neutral kaon and registration of its decay

The analysis of the experiment just briefly described is mostly based on the account that kaons and pions behave as (classical) particles. Here we shall give a quantum mechanical account to the experiment, namely in terms of the states and observables.

Kaons are prepared by the regenerator[3] in the same quantum state $|\phi_K^+\rangle$. This preparation is completed at the edge of the regenerator (O in Fig. 5.1) from which the kaons emerge. Therefore the ensemble beginnings of time $t = 0$ of this quantum state is the ensemble of laboratory times at which the kaon start to emerge[4],

$$t = 0 \,:\, \{T_1, T_2, \cdots, T_N\} \text{ at the edge of the regenerator O.} \tag{5.44}$$

After this point O of preparation, the kaons propagate in vacuum, so they are left free of any influence besides the weak interaction in themselves.

The kaon state $|\phi_K^+\rangle$ describes a coherent mixture of K_S and K_L. This means that it is to be written, according to the superposition principle, by a linear combination of the vectors describing each of these components. This is achieved by the complex basis vector expansion with two *relativistic* Gamow kets $|K_S{}^-\rangle$ and $|K_L{}^-\rangle$ satisfying the (generalized) eigenvalue equations [34],

$$H^\times\, |K_{S,L}{}^-\rangle = \gamma\, \left(m_{S,L} - \frac{i}{2\tau_{S,L}}\right)\, |K_{S,L}{}^-\rangle, \tag{5.45}$$

where γ is the Lorentz factor, and m_S and m_L are the masses of K_S and K_L, respectively. In terms of these Gamow kets the kaon state vector is expanded

[3] To a good approximation, we have neglected the recoil of the nuclei of the regenerator in the forward scattering with K_L.

[4] Note that this $t = 0$ applies not only to a single run, but also to a whole runs of the experiment; the number of elements in the ensemble is the number of the total events that recorded in the period of the experiment performed under the same condition. (This is no longer the case, however, if any settings of beam, regenerator, or any other preparation related variables are significantly changed.)

as[5]

$$|\phi_K^+\rangle = |K_S{}^-\rangle + |K_L{}^-\rangle + |B_K^-\rangle, \tag{5.46}$$

where $|B_K^-\rangle$ is the background term [34] similar to Eq. (5.12). In the Schrödinger picture, this vector will evolve in time. If we are on the rest-frame of the kaon then $\gamma = 1$ holds and the time parameter t is taken to be the proper time τ of the kaon. Then from Eqs. (3.17), (5.45) and (5.46), we obtain the time evolution of state in the rest-frame,

$$|\phi_K^+(\tau)\rangle = e^{-i(m_S - i/2\tau_S)\tau}\,|K_S{}^-\rangle + e^{-i(m_L - i/2\tau_L)\tau}\,|K_L{}^-\rangle + |B_K^-(\tau)\rangle \text{ for } \tau \geq 0. \tag{5.47}$$

The registration of the kaon decay into $\pi^+\pi^-$ is described by the yes-or-no observable $|\psi^-_{\pi^+\pi^-}\rangle \in \Phi_+$ as whether the kaon has decayed (yes) or not (no). This observable, however, does not represent actual measurements by the series of drift chambers, because the kaon decays in vacuum without being "looked at" [13]; the decay is "observed" only afterword as a *reconstructed* vertex extracted from the data of the pions. Thus the disturbance accompanying this "observation" made on the microsystem (kaon) is not due to interaction with a measuring apparatus but due to the weak interaction responsible for the spontaneous decay of the microsystem. On this basis, the decay of the kaon state $|\phi_K^+\rangle$ is described by the registration of the observable $|\psi^-_{\pi^+\pi^-}\rangle$ at the various laboratory times

$$\{T'_1, T'_2, \cdots, T'_N\} \text{ at the respective decay vertexes } \{\mathrm{P}_1, \mathrm{P}_2, \cdots, \mathrm{P}_N\}, \tag{5.48}$$

[5] By the approximation that the momentum spread of the kaon is very small [34].

which defines the ends of time evolution of the kaon state. Thus we have the proper time intervals during which each of the kaons is left undisturbed, i.e., undecayed, as

$$\{\tau_1, \tau_2, \cdots, \tau_N\}, \tag{5.49}$$

each of which is defined by

$$\tau_i = \frac{T_i' - T_i}{\gamma_i} \quad \text{with} \quad \gamma_i = \frac{\sqrt{\boldsymbol{p}_i^2 + m^2}}{m}. \tag{5.50}$$

This set of time intervals is exactly the same as Eq. (5.41).

The transition probability that the observable $|\psi^-_{\pi^+\pi^-}\rangle$ is found in the state $|\phi^+_K\rangle$ is given by

$$\mathcal{P}(\tau) = |\langle\psi^-_{\pi^+\pi^-}|\phi^+_K(\tau)\rangle|^2. \tag{5.51}$$

This reproduces the formula for the decay rate (5.43). Before we substitute the expression of the state (5.47) into Eq. (5.51), let us here make some approximations. Since the probability that K_L decays into $\pi^+\pi^-$ is negligibly small[6] compared to that of K_S, we have

$$|\langle\psi^-_{\pi^+\pi^-}|K_L\rangle|^2 \ll |\langle\psi^-_{\pi^+\pi^-}|K_S\rangle|^2. \tag{5.52}$$

We also assume that the effect of the background term is too small to be

[6]The fraction of the CP violating modes is $\mathcal{O}(10^{-3})$.

observed, namely,

$$|\langle\psi^{-}_{\pi^+\pi^-}|B^-\rangle|^2 \ll |\langle\psi^{-}_{\pi^+\pi^-}|K_S\rangle|^2. \tag{5.53}$$

By substituting Eq. (5.47) along with these approximations into the transition probability (5.51), we obtain

$$\mathcal{P}(\tau) \cong e^{-\tau/\tau_S}|\langle\psi^{-}_{\pi^+\pi^-}|K_S\rangle|^2. \tag{5.54}$$

From this, the decay rate is given by

$$R(\tau) = -\frac{d\mathcal{P}(\tau)}{d\tau} = \frac{|\langle\psi^{-}_{\pi^+\pi^-}|K_S\rangle|^2}{\tau_S}\, e^{-\tau/\tau_S}. \tag{5.55}$$

Because the K_S flux depends on the regenerator, $|\langle\psi^{-}_{\pi^+\pi^-}|K_S\rangle|^2 \propto |\rho|^2$ holds and we reproduce Eq. (5.43).

5.4 Determination of Mass and Width of Z-boson

Another application of the relativistic Gamow vector is a non-perturbative description of relativistic resonance [34, 43].

The Z-boson is observed as a resonance around $\sqrt{\mathsf{s}} = 91\,\mathrm{GeV}$ in scattering cross-section of the electron-positron beam colliding experiment [42],

$$e^+ + e^- \longrightarrow Z^0 \longrightarrow f + \bar{f}, \tag{5.56}$$

where f and $\bar{f}$ are a fermion and its anti-fermion respectively. For the standard model fit, the mass and width of the Z-boson are extracted from the scattering cross-section using an amplitude derived from the on-the-mass-shell renormalization scheme [42]. This scheme was recognized in 1991 to be gauge dependent in the next-to-the-next of the leading order of perturbation, so that the on-the-mass-shell definition of mass and width are not gauge invariant [44]. On the other hand, a model independent and a more compelling definition of mass and width of the Z-boson is given by the S-matrix pole definition [45]. This definition uses the following cross-section formula:

$$\sigma^0_{\rm tot}(\mathsf{s}) \sim \frac{g_f}{\mathsf{s}} + \frac{j_f\,(\mathsf{s} - \overline{M}_Z^2) + r_f\,\mathsf{s}}{(\mathsf{s} - \overline{M}_Z^2)^2 + \overline{M}_Z^2\overline{\Gamma}_Z^2} \tag{5.57}$$

with $f = \text{had}, e, \mu, \tau$.

Here the parameter (for the fermion f) g_f describes the photon exchange, r_f measures the Z-peak height describing the Z-exchange, and j_f describes the photon-Z-boson interference. The mass $\overline{M}_Z$ and width $\overline{\Gamma}_Z$ by this formula are given by [42]

$$\overline{M}_Z = 91.1526 \pm 0.0023 \text{ GeV}, \tag{5.58a}$$

$$\overline{\Gamma}_Z = 2.4945 \pm 0.0024 \text{ GeV}. \tag{5.58b}$$

Now the question arises: Is $1/\overline{\Gamma}_Z$ the lifetime of Z-boson?

In order to examine it, we reproduce the total cross-section formula (5.57) from our S-matrix element in Eq. (5.17). We begin with the transition amplitude at $x = 0$,

$$a(x=0) = \langle \psi^- | \phi^+ \rangle, \tag{5.59}$$

where ϕ^+ is the state prepared by the e^+e^- accelerator, and ψ^- is the observable representing $f\bar{f}$ detector. By Eqs. (5.28), (5.23), and (5.26b), the state vector $|\phi^+\rangle$ is expanded with the two relativistic Gamow vectors and expressed as

$$\begin{aligned} \langle \psi^- | \phi^+ \rangle &= \langle \psi^- | \mathsf{s}_{R_1}{}^- \rangle + \langle \psi^- | \mathsf{s}_{R_2}{}^- \rangle + \langle \psi^- | \phi^{bg} \rangle \\ &= \int_{-\infty}^{\infty} d\mathsf{s}\; h(\mathsf{s}) \left(\frac{r_1}{\mathsf{s} - \mathsf{s}_{R_1}} + \frac{r_1}{\mathsf{s} - \mathsf{s}_{R_1}} \right) + \int_{\mathsf{s}_0}^{\infty} d\mathsf{s}\; h(\mathsf{s})\, b(\mathsf{s}). \end{aligned} \tag{5.60}$$

From this, for the physical energy $\mathsf{s}_0 \le \mathsf{s} < \infty$, we have the S-matrix element

$$S(\mathsf{s}) = \frac{r_1}{\mathsf{s} - \mathsf{s}_{R_1}} + \frac{r_1}{\mathsf{s} - \mathsf{s}_{R_1}} + b(\mathsf{s}). \tag{5.61}$$

Now, let us define the scattering cross-section by the following equation:

$$\sigma(\mathsf{s}) \equiv f(\mathsf{s})\, |S(\mathsf{s})|^2, \tag{5.62}$$

where the particular form of the function $f(\mathsf{s})$ depends upon the convention for the definition of the S-matrix. By substituting Eq. (5.61) into Eq. (5.62), we obtain

$$\sigma(\mathsf{s}) = f(\mathsf{s}) \left| \frac{r_1}{\mathsf{s} - \mathsf{s}_{R_1}} + \frac{r_2}{\mathsf{s} - \mathsf{s}_{R_2}} + b(\mathsf{s}) \right|^2 . \tag{5.63}$$

Further, we take $f(\mathsf{s}) \propto \mathsf{s}$, and consider a photon γ as s_{R_1} and Z-boson as s_{R_2}, so that

$$\mathsf{s}_{R_1} = -i\epsilon, \tag{5.64a}$$

$$\mathsf{s}_{R_2} = \overline{M}_Z^2 - i\overline{M}_Z\overline{\Gamma}_Z, \tag{5.64b}$$

where ϵ is a positive infinitesimal. Neglecting the background term $b(\mathsf{s})$, we finally obtain an expression for the scattering cross-section,

$$\sigma(\mathsf{s}) \sim \frac{r_1^2}{\mathsf{s}} + \frac{2r_1r_2(\mathsf{s}-\overline{M}_Z^2) + r_2^2\,\mathsf{s}}{(\mathsf{s}-\overline{M}_Z^2)^2 + \overline{M}_Z^2\overline{\Gamma}_Z^2}, \tag{5.65}$$

for r_1 and r_2 real. If one takes $g_f \propto r_1^2$, $j_f \propto 2r_1r_2$, and $r_f \propto r_2^2$, this becomes the formula (5.57). Note that the parameterization (5.64b) of Z-boson resonance pole was crucial for the reproduction of Eq. (5.65). But this parameterization is different from Eq. (5.33) that identifies the lifetime of the unstable state with $\tau_2 \equiv 1/\Gamma_{R_2}$ in the exponential decay law (5.37). Hence we conclude that $1/\overline{\Gamma}_Z$ is *not* the lifetime of Z-boson.

The numerical values for M_R and Γ_R are, however, immediately obtained from the experimentally extracted values (5.58). This is because the two real parameters, $\overline{M}_Z$ and $\overline{\Gamma}_R$, uniquely locate the complex pole position s_{R_2} which can be correctly re-parameterized by M_R and Γ_R as

$$\mathsf{s}_{R_2} = \overline{M}_Z^2 - i\overline{M}_Z\overline{\Gamma}_Z = (M_R - i\Gamma_R/2)^2\,. \tag{5.66}$$

By solving this equation, we obtain

$$
\begin{aligned}
M_R &= \overline{M}_Z \left(\frac{\overline{\Gamma}_Z}{\Gamma_R}\right) \\
&= 91.1611 \pm 0.0023 \text{ GeV} && (5.67a) \\
\Gamma_R &= \overline{M}_Z \sqrt{2\left(\sqrt{1+(\overline{\Gamma}_Z/\overline{M}_Z)^2}-1\right)} \\
&= 2.4943 \pm 0.0024 \text{ GeV} && (5.67b)
\end{aligned}
$$

Thus according to the exponential decay law, the M_R and Γ_R must be the definitions of mass and width of the Z-boson so that the lifetime τ_2 is to be given by $\tau_Z \equiv 1/\Gamma_R$, i.e., by the inverse width as predicted by Eq. (5.37) with Eq. (5.38).

The standard parameters used for the analysis of the experimental data are neither the $(\overline{M}_Z, \overline{\Gamma}_Z)$ nor the (M_R, Γ_R) but the on-the-mass-shell values (M_Z, Γ_Z) [42] obtained from a fit of the cross-section to be

$$
\sigma^0_{\text{tot}} \sim \frac{G}{\mathsf{s}} + \frac{\mathsf{s}\,R + (\mathsf{s} - M_Z^2)\,J}{|\mathsf{s} - M_Z^2 + i\mathsf{s}\Gamma_Z/M_Z|^2}. \qquad (5.68)
$$

Thus mass value:

$$
\begin{aligned}
M_Z &= 91.1875 \pm 0.0021\,\text{GeV}, && (5.69a) \\
\Gamma_Z &= \;\;2.4939 \pm 0.0024\,\text{GeV}, && (5.69b)
\end{aligned}
$$

differs significantly from the value (5.67a) predicted by the lifetime-width relation $\tau_Z = 1/\Gamma_R$. While a numerical differences among Γ_Z, $\overline{\Gamma}_Z$, and Γ_R are within the uncertainty, the mass value M_R differs significantly from $\overline{M}_Z$ and

M_Z [43].

Chapter 6

Conclusion

From the causality condition of quantum physics, we have arrived at a non-perturbative description of decaying state by the relativistic Gamow vector. The crucial point was that the time parameter t was revealed to be an ensemble of time intervals of experimenter's clock, and because of that it needs to be lower bounded as $0 \leq t < \infty$. Such time evolution of states and observables was described not by the Hilbert space but by the time-asymmetric boundary conditions, a pair of Hardy rigged Hilbert spaces which can be characterized by energy wave functions satisfying dispersion relations. In the relativistic regime, the parameter t was extended to the space-time parameter x. The Hardy rigged Hilbert spaces were extended to incorporate the relativistic causality. They provided us with the complex eigenkets which enabled us to describe a decaying state as the relativistic Gamow vector associated to the S-matrix pole.

Appendix A

Hardy functions

In this appendix, we briefly present some of the properties of Hardy functions [24] relevant to this dissertation.

The Hardy function from above h_+ and the from below h_- are analytic functions satisfying the following square-integrability:

$$\int_{-\infty}^{\infty} d\omega \, |h_\pm(\omega \pm i\gamma)|^2 = k < \infty \quad \text{for any fixed } \gamma > 0, \tag{A.1}$$

where k depends on a particular form of the function $h_\pm$. This is necessary and sufficient condition for the Hilbert transform to hold[1],

$$\mathrm{Re}\ h_\pm(\omega) = \pm\frac{1}{\pi}\ \mathrm{P}\int_{-\infty}^{\infty} d\omega' \ \frac{\mathrm{Im}\ h_\pm(\omega')}{\omega' - \omega}\ , \tag{A.2a}$$

$$\mathrm{Im}\ h_\pm(\omega) = \mp\frac{1}{\pi}\ \mathrm{P}\int_{-\infty}^{\infty} d\omega' \ \frac{\mathrm{Re}\ h_\pm(\omega')}{\omega' - \omega}\ . \tag{A.2b}$$

This relation explicitly shows that a Hardy function cannot take a pure real or pure imaginary value except for zero on the whole real lines, but it generally

[1] See Sec. 10.2 of Ref. [7].

takes nonzero complex values. The complex conjugate of the Hardy function from above (below) is the Hardy function from below (above), i.e., $\overline{h_\pm} \in \mathcal{S}_\mp(\mathbb{R}_+)$.

Hardy functions can be generated with the Paley-Wiener theorem. Suppose there is a square-integrable function $f(t)$ which vanishes for negative values of t, i.e., $f(t) = \theta(t)f(t)$ holds,

$$||f||^2 = \int_{-\infty}^{\infty} dt\ |f(t)|^2 = \int_0^{\infty} dt\ |f(t)|^2 < \infty. \tag{A.3}$$

The Paley-Wiener theorem of Hardy function states that Hardy functions $h_\pm(\omega)$ on the real axis are obtained from the Fourier transform of $f(t)$ as

$$h_\pm(\omega) = \int_{-\infty}^{\infty} dt\ e^{\pm i\omega t} f(t) = \int_0^{\infty} dt\ e^{\pm i\omega t} f(t). \tag{A.4}$$

By the Titchmarsh theorem of Hardy function, these functions are guaranteed to be analytic, in the upper half plane for h_+ and in the lower half plane for h_-, as

$$h_\pm(\omega \pm i\gamma) = \pm\frac{1}{2\pi i}\int_{-\infty}^{\infty} d\omega'\ \frac{h_\pm(\omega')}{\omega' - (\omega \pm i\gamma)} \text{ for } \gamma > 0. \tag{A.5}$$

Thus one obtains various Hardy functions by performing the integral (A.4) for various $f(t)$. Some of the simple examples are:

- For $f(t) = \theta(t)\, e^{i(a+ib)\,t}$, one obtains

$$h_\pm(\omega) = \frac{i}{(a+ib) \pm \omega} \quad \text{for } b > 0 \text{ and } a \text{ real}. \tag{A.6}$$

The square modulus of this Hardy function is a Lorentzian function.

- For $f(t) = \theta(t)\, e^{-b\,t} \sin{(a\,t)}$, one obtains

$$h_{\pm}(\omega) = \frac{a}{a^2 + (b \mp i\omega)^2} \quad \text{for } b > 0 \text{ and } a \text{ real} \tag{A.7}$$

This Hardy function has been used in classical electrodynamics to describe the propagation of light in a dispersive medium [26], where the physics terminology "dispersion relations" for Eq. (A.2) comes from.

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