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A Study on the Use of History in Middle School Mathematics: The Case of Connected Mathematics Curriculum

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A Study on the Use of History in Middle School Mathematics: The Case of Connected Mathematics Curriculum

by

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Dedication

To my brother Amanuel Kiflemariam Haile

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This dissertation explores the use of history of mathematics in middle school mathematics. A rationale for the importance of the incorporation of historical dimensions (HD) of mathematics is provided through a review of the literature. The literature covers pedagogical, philosophical, psychological, and social issues and provides arguments for the use of history. The central argument is that history can help reveal significant aspects regarding the origins and evolutions of ideas that provide contexts for understanding the mathematical ideas. History can be used as a means to reflect on significant aspects—errors, contractions, challenges, breakthroughs, and changes—of mathematical developments. Noting recent NCTM (2000) calls for school math to include so-called process standards, I contend that incorporating the history of mathematics can be considered as part of this standard. This study examines how HD is addressed in a contemporary mathematics curriculum. Specifically, the study examines the Connected Mathematics Project (CMP) as a case. This curriculum has some historical references which triggered further exploration on how seriously the historical aspects are incorporated. The analysis and discussion focus on four CMP units and interviews with three curriculum experts, eight teachers, and 11 middle school students. The analysis of textbooks and interviews with the experts explore the nature and purpose of historical references in the curriculum. The interviews with teachers and students focus on their perspectives on the importance of HD in learning mathematics. This study examines specifically historical incorporations of the concepts of fractions, negative numbers, the Pythagorean Theorem, and irrational numbers. The analysis reveals that CMP exhibits some level of historical awareness, but the incorporation of HD was not systematically or seriously considered in the development of the curriculum. The interviews suggest that the teachers did not seriously use the limited historical aspects available in the textbooks. The experts' and teachers' interviews suggest skepticism about the relevance of HD for middle school mathematics. The teachers' accounts indicate that students are most interested in topics that are related to their experience and to future applications. The students' accounts do not fully support the teachers' assessment of students' interest in history. I contend that incorporating HD can complement instruction in ways that relate to students' experiences and to applications besides adding an inquiry dimension to instruction.

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Chapter One: Introduction

> The history of mathematics is exhilarating, because it unfolds before us the vision of an endless series of victories of the human mind.
>
> – George Sarton (Swetz, 2001, p. 316)

George Sarton (1884-1956), known as the father of the history of science, frequently singled out mathematics as a special discipline, which requires a historical background in order to achieve a true understanding (Swetz, 2001). Numerous educators (e.g., Calinger, 1996; ICMI study, 2002; Furinghetti & Radford, 2002) have advocated incorporating the history of mathematics into mathematics education. The core arguments for incorporation of history of mathematics into school mathematics revolve around the idea that such an approach can help teachers and students develop better views and understanding about mathematics (e.g., Barbin, 1996) because of the background knowledge history can bring to fore. In this dissertation study, I examine how the history of mathematics is addressed in a contemporary school mathematics curriculum. Through this, I open a dialogue on the importance of HD in middle school mathematics.

Based on my experience with school mathematics as a student and my readings of the literature on the history of mathematics, I believe that the history of mathematics can be a useful tool in learning mathematics. I was very curious about learning the background of math ideas as a student in pre-college education or college majoring in math. I often wondered about issues pertaining to the origins and evolution of mathematical ideas. I found the few historical instances I encountered to be interesting and enduring. I thought such instances provided supporting context for understanding mathematics. Unfortunately, most of school mathematics I experienced was detached from any meaningful history.

The ahistorical nature of school mathematics somehow reinforces a certain view of what mathematics is about. The lack of historical dimensions of mathematical ideas gives the impression that historical matters do not belong in school mathematics even if they are directly about the mathematics concepts found in the curriculum. In fact, the established school subjects—math, history, physics, etc.—seem to have imaginary boundaries for knowledge. These boundaries subconsciously may reinforce the belief that certain subject matters belong to certain school subjects (i.e., historical matters about math may belong to the school subject history, not to mathematics). At least that was the impression I had. Looking back, though I was interested, I did not make a serious effort to study the history of mathematics during my math courses.

The typical delivery of math lessons effectively excluded the historical background. Our typical instruction used presentation, demonstration, and practice focusing in learning the established knowledge/concepts/procedures. This kind of experience seems common in traditional school mathematics. Ball, Lubienski, and Mewborn (2001) observe that despite various reform efforts in mathematics education, much about U.S. school mathematics instruction has remained the same. According to them, often times "the curriculum and teaching methods used in the school inundate students with skills and procedures without allowing them to develop an appreciation for the power of mathematics as a system of human thought" (p. 435).

In recent mathematics education reform literature (e.g., NCTM, 2000), we find the importance of process-related standards spelled out. The National Council of Teachers of Mathematics (NCTM) (2000, p. 15) states, "Mathematics topics can be considered important for different reasons, such as their utility in developing other mathematical ideas, in linking different areas of mathematics, or in deepening students' appreciation of mathematics as a discipline as a human creation". This document also argues, “Mathematics is one of the greatest cultural and intellectual achievements of humankind and citizens should develop an appreciation and understanding of that achievement, including its aesthetic and even recreational aspects” (p. 4). What better way to help students see mathematics as cultural heritage than incorporating the history of mathematics? Exploring how the history of mathematics is used in school mathematics can be one way to assess this component of the standard. HD offers situations that can encourage view of mathematics as a continuous process of reflection and improvement over time rather than a complete structure composed of irrefutable and unchangeable truths (ICMI study, 2002).

I do not advocate that historical perspective be the sole approach that drives instruction. Rather, I contend that history can add dimensions that are normally lacking in traditional school mathematics. Focusing only on the established knowledge gives the perception of mathematical concepts as static and mathematical activity as a clean pursuit in which one solely finds correct answers to problems. However, the study of the history can help highlight the human dimension of mathematical endeavor (ICMI, 2002). That is, history can help reveal such human attributes as the motivations, doubts, errors, contradictions, and changes involved in the evolution a mathematical concept. Furthermore, history opens opportunities to learn about the life of the people (the socio-cultural context) who developed mathematics; this may present opportunities to address multicultural issues in mathematics education. To exploit the pros and reduce the cons of using history, of course, will require teacher expertise, time, and other resources. There may be potential disadvantages that need understanding. Some of the crucial concerns are that use of history can be potentially confusing instead of useful because past events can be torturous and that history may “breed cultural chauvinism and parochial nationalism” (ICMI, 2002, p. 203).

What is meant by historical incorporation in this dissertation? The ICMI study (2002) offers a good deal of ideas on this. Broadly, it means providing relevant history of core concepts as part of the instruction in order to help students understand mathematics. Relevant history may include historical facts such as biographical sketches of key people, dates, and places (as well as some socio-cultural thinking) that affected the development of mathematical ideas. Incorporation of history may also take implicit form. That is, using problems inspired by history—problems that address the foundations of the ideas. Various strategies of using history in mathematics instruction can be thought of: Presenting the historical origins of mathematical concepts, procedures, and terms; providing brief anecdotes about the lives and work of prominent mathematicians; discussing actual historical problems; and carrying out small-scale research projects on a particular accomplishment or mathematician (Swetz, 2001, p. 319). A number of factors may affect choice and use of these strategies: the mathematical topic, instructional time, teacher's expertise, appropriateness to students, and availability of resources, among others.

By exploring the incorporation of HD in a contemporary school mathematics curriculum, the purpose of this study is to open a dialogue on the importance of history for learning mathematics in middle school. Though I believe using history is important in learning mathematics, the question is: Is history really important in contemporary middle school mathematics? I address this question partly by exploring the perspectives of those who are directly involved (curriculum experts, teachers, and students). I carry out the exploration by examining a selected case of curriculum, namely the Connected Mathematics Project (CMP). Although a general survey on the importance of HD in learning mathematics seems intriguing, pilot interviews with teachers and their professional training providers suggested that teachers did not have much familiarity or any serious experience with incorporating the history of mathematics in their teaching. Thus, I think a case study using interviews and textbook analysis allows a deeper exploration of the issues.

The CMP is a middle school mathematics curriculum developed at Michigan State University with its first edition published in 1998. It is in use in many school districts in the United States. One of the main reasons this curriculum was selected for this study is that it features some historical references that I consider as a platform for further exploration. Using these historical aspects in the curriculum, this study addresses the following questions in relation to this curriculum: What are the nature and purpose of the historical references? How are they used in the classroom? What are the perspectives of teachers and students on the importance of incorporating history?

This dissertation study is organized into six chapters. Chapter two covers a review of the literature. The literature review focuses on the rationale for the incorporation of HD by drawing on pedagogical, philosophical, psychological, and socio-cultural grounds; ways of incorporation; the history of selected topics; and associated conceptual challenges. Chapter three presents research assumptions, a conceptual framework, and methods employed in carrying out this dissertation study. Chapters four and five constitute analyses and discussion of the reviewed curriculum and the interviews. Chapter four presents analysis of the historical aspects in the CMP curriculum by focusing on selected units of the textbooks and interviews of curriculum experts. The main thrust of analysis in chapter four is the nature and purpose of the historical aspects in the CMP units. Chapter five discusses both the teachers' and students' perspectives about the incorporation of the history of mathematics in the classroom. The issues focused on in the analysis in chapter five include how concepts are introduced, how historical inquiry is talked about, how the historical aspects in the text are used, and what the perspectives of the teachers and the students are on the importance of HD in learning mathematics. Chapter six presents reflections, conclusions, implications, limitations, and recommendations of the study.

Chapter Two: Review of the Literature

This review focuses on the rationale for incorporating the history of mathematics into school mathematics and the ways the incorporation can be done. Pedagogical, philosophical, psychological, and social grounds for the incorporation of history are explored. The review stresses the connection of these grounds as a broader conceptual framework for the incorporation of history into school mathematics. With this framework in mind, the final section will present a review of the conceptual challenges and the history of concepts selected for this study (fractions, negative numbers, the Pythagorean Theorem and irrational numbers). The aim of this last section is to identify the conceptual challenges met when learning the selected concepts and to examine how history could be useful in meeting these challenges. In other words, this last section provides particular instances of why the incorporation of history could be beneficial in learning mathematics.

RATIONALE FOR INCORPORATION OF HISTORICAL DIMENSION IN SCHOOL MATHEMATICS

Pedagogical Grounds

The idea of using the history of mathematics in mathematics instruction is not a new idea (Calinger, 1996; ICMI Study, 2002; Furinghetti & Radford, 2002). In their work entitled *History and Philosophy of Modern Mathematics*, Kitcher and Aspray (1988) observe that scholarly work reflecting history as a pedagogic tool sprouted in the early decades of the 20th century in the United States. Specifically, works published during this time saw history as a way to bring a more human dimension into mathematics teaching. Furinghetti and Radford (2002, p. 631) capture the longstanding interest in the area:

> More than a century ago, Hieronymus Georg Zeuthen wrote about the history of mathematics (Zeuthen, 1902). Of course, this was not the first book in the topic, but what made Zeuthen's book was that it was intended for teachers. Zeuthen proposed that the history of mathematics should be part of teachers' general education. His humanistic orientation fitted well with the work of Cajori, 1894 who, more or less by the same time, saw in the history of mathematics an inspiring source of information for teachers. Since then, mathematics educators have increasingly made use of the history of mathematics in their lesson plans, and the spectrum of its uses has widened.

Further, Furinghetti and Radford note, "History of mathematics has been used as a powerful tool to counter teachers' and students' widespread perception that mathematical truths and methods have never been disputed" (p. 632).

The International Commission on Mathematics Instruction (ICMI) study (2002), edited by Fauvel and van Maanen, states, "Mathematicians, historians and educators in many countries have long thought about whether mathematics education can be improved through incorporating the history of mathematics in some ways" (p. xii). This study presents compelling arguments for the role of history of mathematics in mathematics education by taking political, philosophical, social, and practical issues into account. "The ICMI study is posited on the experience of many mathematics teachers across the world that its history makes a difference: that having history of mathematics as resource for the teacher is beneficial" (p. xvii). Mathematicians, mathematics educators, and historians of mathematics from around the globe contributed to this ICMI study.

Chapter three of the ICMI study contains a summary of nine (qualitative) case studies made by teachers (published in France between 1991 and 1998) on their incorporation of historical dimensions in mathematics teaching. The level of incorporation varied: some teachers explicitly used history (e.g., using problems from the history of mathematics and reading of historical texts) and some implicitly used history (e.g., problems inspired by history). Although the extent and kinds of use differ, the report notes the following results could be identified as being associated to using history, as reported by the teachers. It can bring changes in the teacher's mathematical conceptions and the role of the teacher, the student's mathematical conceptions and the students' view of mathematics, and the students' learning and understanding of mathematics. The report provides teachers' accounts that the use of historical texts helped them and their students view mathematics as more alive, not a "finished product but something that is in continuous evolution," and as an object of inquiry and controversy that contains mistakes and methods of trial and error. Reading historical texts "excites the curiosity of students and encourages them to question". Some of the accounts also refer specifically to how history helped teachers change their views on students' learning processes and on their own teaching roles. The report also notes that the teachers reported that the historical approach helped them in allowing students more time to construct ideas and identify "moments of misunderstanding" and in becoming aware of the use of the intuitive approach in teaching. The report also notes two cases that suggest limitations of historical dimensions. In one, the teacher concluded that using historical texts seemed to disadvantage the "better students;" and in the other case the teacher expressed his distrust for any historical dimensions imposed on curriculum, fearing such a move would result in a "teaching of history which would create a screen in front of mathematics".

Calinger (1996), like the ICMI study, contains a collection of papers written by scholars from around the globe on topics covering historiography and integration of history of mathematics within mathematical pedagogy. Specifically, the articles "History of Mathematics and the Teacher" (by Heiede), "The Necessity of History in Teaching Mathematics" (by Rickey), "A History of Mathematics Course for Teachers based on

Great Quotations" (by Kleiner), and "The Role of Problems in the History and Teaching of Mathematics" (by Barbin) offer interesting empirically based accounts on the importance of the history of mathematics in mathematics education. When addressing why the history of mathematics should have a place in teaching, Heiede contends that teaching mathematics properly requires including its history. "One explanation of the fact so many people – particularly children and young people in schools and colleges – find mathematics dull, boring, uninteresting, even hateful, could be that they were taught – or are being taught mathematics without its history, that is as if it were dead" (Heiede, 1996, p. 232). Rickey (1996) makes similar points recounting his experience: "Only two of my teachers were inclined to make historical remarks and both of them influenced me significantly" (p.252). Kleiner (1996) notes that history of mathematics can increase teachers' enthusiasm for the subject, promote a sense of the importance of the subject, and encourage students to ask "why" in addition to "how".

Barbin (1996) notes that mathematics teachers who become interested in the history of their subject often report that the experiences they gain influence their views about mathematics. Barbin notes that the teachers report having a different view about the errors their students make or the obstacles the students meet and having a better understanding of certain remarks students make. Barbin states, "I believe that the study of the history of mathematics profoundly changes the epistemological concepts of mathematical knowledge: that the introduction of the history of mathematics will transform the practice of teaching mathematics" (p. 17) (see Figure 2.1).

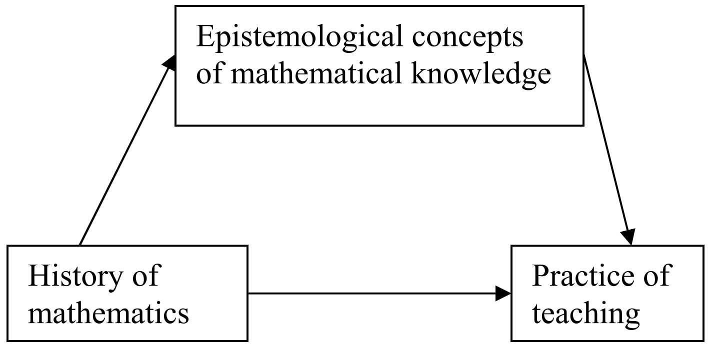


Figure 2.1. Linking History of Math to Epistemology and Practice (Source: Barbin, 1996, p.17)

The central argument here is the potential influence of the study of history can have on teacher knowledge and teaching practice. Duschl (1990) makes similar "conceptual change" in science and its implication for science education. Duschl notes that studies of the history of science have helped change the view of scientific knowledge growth as accretionary to reversionary and tentative. He argues that this principle of theory change can be applied to promote children's learning of science. Drawing from Piaget's notion of genetic epistemology, Duschl notes the similarity between the mechanisms of conceptual change in children's learning experience and scientific theory development:

> The fact that the cognitive processes children employ when learning science happen to have much in common with the epistemological frameworks for science theory development serves as partial justification for applying epistemological frameworks to teacher decision making. (Duschl, 1990, p. 81).

Duschl uses Laudan's (1984) "Triadic Network of Justification" model to describe theory change in science (Figure 2.2). Laudan (1984, p. 62) contends that there is a complex process of mutual adjustment and mutual justification among the three levels of scientific commitment: theory, method, and aim. In other words, there is interdependency among these three commitments.

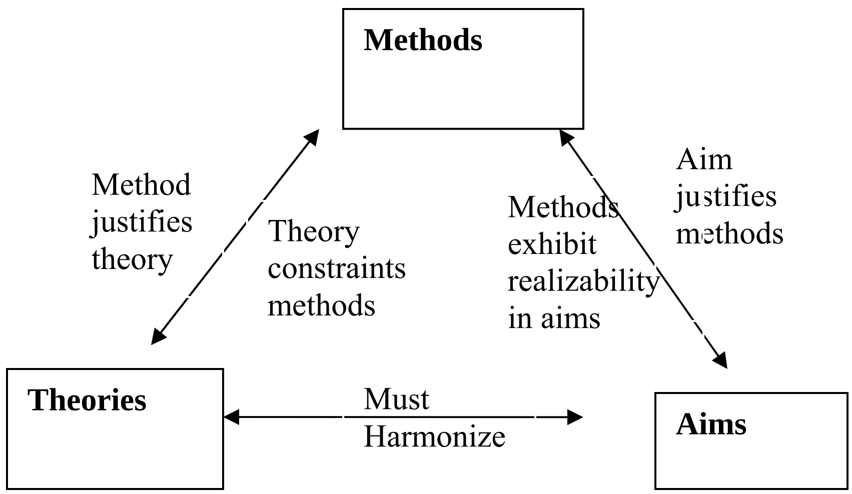


Figure 2.2. Theory Change Network (Source: Duschl, 1990, p. 87)

Duschl suggests that this model could provide a conceptual framework for science instruction, focusing lessons on theory, method, and aim. Theory lessons would focus on background knowledge with learning strategies emphasizing how to meaningfully link scientific concepts; method lessons would stress the acquisition of evidence; aim lessons would focus on what scientists and society believe to be important, cognitive goals of science.

There are similarities between Barbin's description of the role of history in mathematics teaching and Duschl's characterization of the role of the history of science in theory change in science. Both provide a similar conceptual framework linking history, philosophy (epistemology), and teaching. Both models suggest that history helps in understanding the nature of (mathematical or scientific) knowledge growth and thus can be helpful as a pedagogical tool. To elaborate the relations of the models further, it is not farfetched to consider the *theory change model* (Figure 2.2) as the epistemological component of the model in Figure 2.1. Laudan (1984) recognizes questions pertaining to the relationships in *triadic network justification* (questions such as whether the method justifies the theory choice, if the method makes sense given the aim, etc.) as epistemological. Studying the history of science can help in understanding epistemological issues of the theory development, which is similar to Barbin's point—the study of the history of mathematics can help in understanding the epistemological aspects of mathematical knowledge. Barbin's argument that such an understanding can be useful in mathematics teaching is similar to Duschl's suggestion that the notion of theory change in science (depicted in Figure 2.2) be considered for science instruction. In both Barbin and Duschl, I see a useful conceptual framework linking history, philosophy, and learning (cognitive theories).

Philosophical Grounds

> Considering the nature of mathematics from a background of the history of mathematics changes the way we conceive the epistemological problems of the development of mathematical knowledge in the individual and in society.
>
> --- The ICMI study, 2002, p. 45.

In describing the state of philosophy of mathematics, Kitcher and Aspray (1988, p. 19) identify two general programs: One focuses on the problem of the foundation of mathematics, and the other takes the central problem to be articulating the methodology of mathematics. Three schools of thought, logicism, formalism and intuitionism, emerged as dominant to repair the crisis in the foundation of mathematics in the last century (Hersh, 1997). The crisis pertains to the nature of mathematical knowledge—"What do we know in mathematics, and how do we know it?" (Davis & Hersh, 1981, p. 318). Logicism was founded by Gottlob Frege (1848-1925). The formalist program, pioneered by David Hilbert (1862-1943), sought to establish mathematics on a set of rules that are consistent and complete. Lutjens E. J. Brouwer (1882-1966) is recognized for his contribution to intuitionism. "Brouwer claimed that we construct the objects of mathematics and that our knowledge of their fundamental properties is based on an a priori intuition" (Kitcher & Aspray, 1988, p. 7). Despite their differences, these three schools of thought share a priorist view of mathematics (Kitcher, 1983)—a view of mathematical knowledge as deductive and independent of experience.

The philosophy of mathematics that takes the articulation of the methodology of mathematics as a central problem seems to focus on the ways mathematics is experienced by people. Numerous scholars have taken this philosophical position using various themes: quasi-empirical (Lakatos, 1976), Humanism (Hersh, 1997), mathematical empiricism (Kitcher, 1983), falliabilism/social constructivism (Ernest, 1991), and evolutionary perspective (Stemhagen, 2004), to name a few.

In *Proofs and Refutations*, Lakatos (1976) provides a case that challenges mathematical formalism. Lakatos criticizes the formalist philosophy for disconnecting mathematics from its methodology and its history. Lakatos states, "None of the creative periods and hardly any of the critical periods of mathematical theories would be admitted into the formalist heaven, where mathematical theories dwell like the seraphim, purged of all the impunities of earthly uncertainty" (p. 2). He advocates for what he refers to as "quasi-empirical" mathematics. "Mathematics does not grow through monotonous increase of the number of indubitably established theorems but through the incessant improvement of guesses by speculation and criticisms, by the logic of proofs and refutations," argues Lakatos (p. 5). Quasi-empirical mathematics encourages using heuristics such as trial and error and conjecturing and refutations by counter examples, methods that naturally occur in the development of mathematical knowledge. Lakatos supports his argument by constructing a dialogue that occurred in a fictional mathematics classroom seriously engaged in solving a proof problem. The problem is to find if there is a relation between the number of vertices V, the number of edges E, and the number of faces F of a polyhedra, particularly of a regular polyhedral. By trial and error the class establishes that for all regular polyhedra

$$V - E + F = 2.$$

Then the task becomes proving the conjecture that the relation expressed by the equation is true for any polyhedra. Lakatos' construction of the classroom dialogue is a good example of incorporating history into mathematics instruction. Lakatos provides a series of historical footnotes corresponding to a number of discussion points raised in the fictional classroom.

Writing on the nature of mathematical knowledge in a book titled *What is Mathematics, Really?,* Hersh (1997, p. xi) repudiates aspects of Platonism and formalism and embraces humanist philosophy.

> Repudiating Platonism and formalism, while recognizing the reasons that make them (alternatively) seem plausible, I show that *from the viewpoint of philosophy* mathematics must be understood as a human activity, a social phenomenon, part of human culture, historically evolved, and intelligible only in a social context. I call this view point "humanist".

Hersh notes the Platonist thesis that mathematical objects exist outside of time and space, in an abstract realm independent of individual or social consciousness. "Half-conscious

Platonism is nearly universal among mathematicians" (p. 11). His argument suggests that this viewpoint is reflected in school practices. He wonders, "We do not ask, how does this immaterial realm relate to material reality?" Hersh provides numerous instances that challenge the Platonist view. He argues that mathematical objects exist in a social-cultural-historical context. He makes his point by discussing mathematical ideas that represent objects we are familiar with and ideas that do not have such representations. For instance, he talks about the fact that there is less confusion with a three-dimensional idea because we may be familiar with the object it represents while there is confusion about the idea of four-dimensional object because we do not know a "real" four-cube. Thus our experience affects what we know, contrary to the Platonist view.

In *The Nature of Mathematical Knowledge*, Kitcher (1983) advocates for what he refers to as *mathematical empiricism*. He makes his case by emphasizing that mathematical knowledge is built by a community of "knowers". Unlike mathematical a priorism, mathematical empiricism recognizes the epistemological relevance of the fact that we learn from others' experiences. He maintains, "The knowledge of an individual is grounded in the knowledge of community of authorities" (p. 5), and "the knowledge of the authorities of later communities is grounded in the knowledge of the authorities of earlier communities". The mathematical knowledge of someone in the present day can be explained by reference to a chain of prior knowers. Thus, in his perspective, the historical development of mathematics is important in understanding the nature of mathematical knowledge.

In the *Philosophy of Mathematics Education*, Ernest (1991) proposes social constructivism as the philosophy of mathematics education. He first contrasts absolutist with fallibilist philosophical views of mathematics. Collectively, he considers the three schools of thought (logicism, formalism, and intuitionism) as absolutist philosophy. The general criticism labeled against absolutism is that such philosophy treats mathematical knowledge as certain truth while relying on the system that contains axioms and assumptions that do not necessarily warrant the certainty of a truth claim. Ernest notes Bertrand Russell's paradox and Gödel's incompleteness theorem as examples that shook the absolutist foundations of mathematics. Kurt Gödel's incompleteness theorem shows that the formalist requirement cannot be achieved (Kitcher & Aspray, 1988). In 1901, Russell postulated a set R that contains the set of all sets that are not members of themselves and asked whether or not R is a member of itself (Hellman, 2006; Irvine, 2003). This postulate leads to the contradiction that R can be both a member of itself and not a member of itself, revealing inconsistency in naïve set theory. Ernest (2004) argues that these kinds of contradictions are masked in absolutist philosophy of mathematics and that this reinforces the negative public image of mathematics as "rigid," "inhuman," "cold," and "abstract". Negative public image of mathematics has been noted as a contributing factor to the various problems such as low enrollment in mathematics in higher education (National Research Council, 1989; Picker & Berry, 2000; Devlin, 2001; Lim, 2002; Rensaa, 2006) and low performance in school mathematics (e.g., NRC, 1989; Lim, 2002). The alternative philosophy, he argues, should consider mathematical knowledge as fallible and open for development and change. He links fallibilist epistemology to social constructivism, which he proposes to be the philosophy of mathematics education.

The above philosophical positions—quasi-empirical of Lakatos, humanism of Hersh, mathematical empiricism of Kitcher, and falliabilism of Ernest—are discussed in stark contrast to the traditional (a priorist) view. Proponents often seem to advance these positions by rejecting the traditional view. Stemhagen (2004), taking a somewhat centrist philosophy, argues that such a dualist take does not properly portray the nature of mathematical knowledge. He proposes what he refers to as an evolutionary philosophy of mathematics education as a "way to avoid the dualist trap of being forced to understand mathematics in strictly absolutist or constructivist terms" (p. i). Stemhagen notes that the "math war," pitting traditionalists against constructionists, in contemporary mathematics education is a manifestation of such dualism. The traditionalists advocate a "back-to-basics," procedural and skills orientated mathematics while the constructionists emphasize child-centered and more applied approaches to mathematics learning. Traditionalists view mathematics as permanent and independent of human activity while constructivists believe mathematical knowledge is actively constructed by human beings. He argues that each side offers valuable insight, but each side is not complete. Evolutionary perspective, according to him, "focuses on the creation of mathematical knowledge that occurs when humans interact with their environment" (p. i). According to this perspective, mathematics is not viewed as either purely deductive, abstract, or empirical but rather as a pragmatic endeavor that possesses both the physical and mental aspects. Unlike the absolutist-constructive dualism, Stemhagen contends that the evolutionary perspective stresses the functional aspects.

The various philosophical positions provide different lenses for the importance of history in mathematics education. Traditionalist or formalist philosophy tend to, if they use it at all, portray a linear growth of knowledge that masks the complexities of the course of history. Radford (1997, p. 2) makes the following observation with respect to the historical aspect often found in traditional textbooks:

> When researchers in mathematics education turn to the history of mathematics, they often realize that the books present the history as a sequence of events that cannot answer their epistemological questions. In fact, very often, the books unfold episodic narratives implicitly underlain by an apriorist epistemology of platonic style. This leads us to see past mathematical achievement as clumsy

> efforts that always tended to the conceptual formulations that we find in our modern mathematics.

In *The Structure of Scientific Revolutions*, Kuhn (1996) makes a similar point: that the history of science found in science textbooks follows the path of "normal science" in portraying the development of science as linear. According to Kuhn (p. 136), scientific "revolutions have proven to be nearly invisible" because people get the image of science from sources of authority that "systematically disguise" science. Kuhn considers textbooks and philosophical works that modeled them among the principal sources of authority. He contends that recognizing and understanding the nature of authority is crucial if one hopes to make effective use of history. In contrast, the non-traditionalist perspectives I have noted above (e.g., quasi-empirical, humanist, fallibilist) seem to offer stronger justification for the role of history in mathematics education.

Cognitive (psychological) Grounds

Are there cognitive (psychological) theories that suggest the need for history in learning a subject matter? Some early psychological theories (drawing from Haeckel's recapitulation theory) postulated that knowledge formation with an individual follows a similar path as the historical growth of knowledge. Later theories do not support this but offer compelling explanations for the role of history in learning. Piaget and Garcia (1989) and Piaget (1970) suggest that there is some structural similarity between the historical and psychological in knowledge organization. Socio-cultural thinking (e.g., Vygotsky, 1986) challenges such structural similarity, especially if considered across cultures.

An oft-discussed idea with respect to concept formation in the mind and the history of a concept draws from recapitulation theory. This thought has its base in Haeckel's biogenic law that "ontogeny is a brief and condensed recapitulation of phylogeny" (1905, p. 413). The psychological version of Haeckel's theory "supposes that present intellectual developments are to some extent a condensed version of those of the past" (Furinghetti & Radford, 2002, p. 635).

However, this idea cannot be supported. Werner (1957) notes that "any hypothesis of recapitulation has to be rejected" because though some formal similarities can be admitted, "it would be absurd to identify the child of our own cultural sphere with the primitive man at any cultural level whatsoever" (p. 26). A collection of works of several scholars edited by Strauss (1988), *Ontogeny, Phylogeny, and Historical Development,* offer similar analyses: The psychological form of recapitulation theory is rejected, but instances of similarities between conceptual growth within an individual and historical development of a concept can be found. The book discusses similarities found between physics-naïve modern adults' thermal conception and that of 17^{th} century Florence scientists (Wiser, 1988) as well as similarities between physics-naïve modern students' conception of motion and that of impetus theorists' (medieval philosophers) (McCloskey & Kargon, 1988). On the other side, instances that challenge this notion of similarities are also available (e.g., Nesher, 1988; Radford, 1997). Nesher discusses the role of language in the construction of the number concepts. He notes the language tools the contemporary child has at her disposal that is not present in regards to the history of number concept may account for differences in the construction of the concept. Radford (1997) notes variations in epistemological obstacles due to cultural differences.

In *Psychogenesis and the History of Science*, Piaget and Garcia (1989) raise two major points in discussing the parallelism in the evolution of concepts in history and psychological development. First, they argue that there are content-wise similarities in historical and psychological developments of pre-scientific concepts. They do not make the same argument for mathematics, saying that "there we would have to go back to periods where written materials does not exist in sufficient quantity to make such analyses possible" (p. 31). In their second point, which is the focus of the book, they posit that there is a "common instrument and mechanism" underlying the development of knowledge in history and cognition. That is, there is structural similarity. The "common mechanisms" include what they refer to as transcending structure and transitional mechanism. The transcending structure is based on the idea that in "cognitive progress what gets surpassed is integrated with the new". In other words, it is in accordance with the idea of assimilation (Piaget, 1971). The transitional mechanism involves three levels of analysis–intra-object (object analysis), inter-object (analyzing relations or transformations), and trans-object (building of structure). Piaget and Garcia argue that the triad transitional mechanism exists in all domains and at all levels of development.

> That this didactical triad can be found in all domains and at all levels of development seems to us to constitute the principal result of our comparative efforts. In fact, the generality of this triplet, *intra, inter, and trans*, and its occurrence at all subsets as well as within global sequences undoubtedly constitutes the best of the arguments in favor of a constructivist epistemology. (pp. 28-29).

Piaget and Garcia do not argue that there is correspondence in development between historical and psychogenetic in terms of content. Rather, they contend that "the mechanisms mediating transitions from one historical period to the next are analogous to those mediating the transition from one psychogenetic stage to the next" (p. 28).

In his work *genetic epistemology*, Piaget (1970) offers a more specific explanation about the common mechanism (structural similarity) he conjectures to exist in how knowledge is organized in the domain and the individual. Piaget (1970, p. 13) states, "The fundamental hypothesis of genetic epistemology is that there is a parallelism between the progress made in the logical and rational organization of knowledge and the corresponding formative psychological process." For mathematics, Piaget examined whether the three “mother structures”—algebraic structure, order structure, and topological structure–that were proposed by the Bourbaki mathematicians (Bourbaki is a pen name used by a group of 20th century French mathematicians formed to reform mathematics) correspond to anything natural or psychological. He notes his observation that there is a relationship between these mathematical structures and children’s operational thinking. Piaget claims that, although the stated structures appear highly abstract, we can find structures that resemble them in the thinking of children even as young as six or seven years of age. He provides examples of the relationship: children’s operational thinking (e.g., classification to algebraic structure), simple ordering/serration of objects to order structure, and children’s pattern of drawing open and closed plane figures to topological structure. Noting topological geometry as a theoretically common source for Euclidean and projective geometries but formally as a field that was established later in time, he asks if a similar order of development can be observed in children’s thinking in geometry. As Piaget (1970, p. 31) puts it: “Will we find that Euclidean intuition and operation develop first, and topological intuitions and operations later? Or will we find that the relationship is the other way around?” He claims that we find topological intuitions first and also operations such as dividing space and ordering in space that are more similar to topological than to Euclidean operations. Piaget’s critics argue that this constructive epistemology emphasizes the individual’s construction of knowledge as if the individual is an isolated being, ignoring the influence of the sociocultural context on the individual (e.g., Abdal-Haqq, 1998).

In contrast, the sociocultural perspective brings about various factors explaining how the individual mind is situated in the social, cultural, and historical context (e.g., Wertsch, 1991; Wertsch, Del Rio, & Alvarez, 1995; Kozulin, Gindis, Ageyev, & Miller, 2003). The work of Lev Semeovich Vygotsky (1896-1934) on cultural-historical psychology often is cited within this perspective. Vygotsky (1986) maintains that cultural tools (e.g., language, social support) mediate human thought processes and actions. In his work *Thought and Language*, Vygotsky argues, "the child's intellectual growth is contingent on his mastering the social means of thought, that is, the language" (1986, p. 94). Vygotsky recognizes that thought can be manifested through use of tools other than language, which in turn are socioculturally situated. Therefore, any parallelism of knowledge construction by an individual and the history of knowledge growth would become problematic in this perspective. "The mechanism of individual developmental change is rooted in society and culture" (Vygotsky, 1978, p. 7).

However, Vygotsky distinguishes between lower and higher psychological processes in child development. Lower processes such as reflexes are regulated by biological laws (Ratner, 2004), while higher psychological processes such as speech acquisition and use of other cultural tools are regulated by social and cultural processes (Scribner, 1985). Cultural tools such as language form the bridge between the lower and high psychological developments. Scribner raises an important question: Does Vygotsky's cultural psychology imply that ontogeny recapitulates general history? According to Vygotsky's manuscript *Development of Higher Mental Functioning* (quoted in Scribner, 1985), Vygotsky argues that his position is neither recapitulationist nor parallelist.

In both the constructivist epistemology of Piaget and the socio-cultural perspective of Vygotsky, the recapitulation theory is not supported. But, they have differences in terms of how they explain any similarity of cognitive development across individuals. In Piaget, we find an argument for structural similarity of knowledge formation within an individual and in the history of a domain, whereas in Vygotsky that does not seem to be the case. Still we can find arguments from both supporting the importance of history in the child's knowledge construction. Piaget's idea of general structural parallelism between knowledge growth in history and within an individual appears as a compelling explanation to consider in some situations. Vygotsky's idea of cultural psychology that individual's higher mental functioning is shaped by the socio-cultural context serves as a reminder of how learning could be situated within a context.

Rather than looking at the dichotomy, a more nuanced explanation may come from both perspectives. Cobb (1994, p. 13) notes, "Mathematical learning should be viewed as both a process of active individual construction and a process of enculturation into the mathematical practice of wider society." Balacheff (1990, p. 139) argues that researchers need to recognize the intra/interpersonal complementarily in the learning process because

> Mathematics learning in its educational context cannot be fully interpreted intrapersonally because of its social setting. Equally, interpersonal constructs will be inadequate alone since it is always the learner who must make sense and meaning in mathematics.

The notion of *epistemological obstacle* introduced by the French philosopher Gaston Bachelard (1884-1962) may offer a unifying theme for these learning perspectives. In *The Formation of The Mind* (translated by Jones, 2002, p. 24), Bachelard states

> When we start looking for the psychological conditions in which scientific progress is made, we are soon convinced that *the problems of scientific knowledge must be posed in terms of obstacles*. This is not a matter of considering external obstacles, such as the complexity and transience of phenomenon, or indeed of incriminating the weakness of the senses or of the human mind. It is at the very heart of the act of cognition that, by some kind of functional necessity, sluggishness and disturbances arise. It is in the act of cognition that we shall show causes of stagnation and even of regression; there too we shall discern causes of inertia that we shall call epistemological obstacles.

Bachelard also notes, “When we contemplate reality, what we think we know very well casts shadow over what we ought to know” (p. 24). This shadow constitutes an obstacle. Bachelard has identified various epistemological obstacles such as the tendency to rely on intuition that could be deceptive, the tendency to generalize, language-related obstacles (Herscovics, 1989), and others. Epistemological obstacles may manifest as error that are not random but something persistent—errors that have some underlying source such as an older way of knowing that has been successful in some domain (Brousseau, 2002). Herscovics discusses several instances of seemingly common student errors in problems in which the students had basic knowledge and understood the problems. The instances include errors that suggest that students’ tendency to transport arithmetic knowledge to algebra (i.e., arithmetic knowledge poses some obstacles in learning algebra) and students’ problems with translating word problems into algebraic forms (i.e., obstacles related to language). Radford (1997) points out that epistemological obstacles were thought of as being *intrinsic difficulty of knowledge* (his emphasis), which fit in the recuptilutionistic parallelist framework, in the earlier conception. Treating epistemological obstacles as universal neglects variation in obstacles that arise due to sociocultural factors. Sierpinska (1994) notes that epistemological obstacles may occur due to cultural factors. Radford (1997, pp. 10-11) provides an example that epistemological obstacles can vary across cultures:

> If we see the difficulties the western medieval mathematicians had in facing negative numbers, and if we see the difficulties encountered by our students today, we are led to think that, effectively, positive numbers constituted an obstacle for the emergence of negative numbers. However, if we retrace negative numbers to Chinese mathematicians we see that they overcame the difficulty of handling the negative numbers through a very clever representation using colored rods. Thus the difficulty that positive numbers pose to the rise of negative numbers is not an *intrinsic* problem to knowledge.

To sum up, the recapitulationist notion that an individual's conceptualization follows the historical development of the subject matter cannot be accepted for various reasons. Historical developments of math ideas do not follow a linear path. Contemporary literature on how people learn (e.g., NRC, 1999) informs us that learning is influenced by cultural and social norms. The historical incorporation likely is accomplished using modern symbols, language, and cultural tools that are different from those of past authors (Furinghetti and Radford, 2002). Still, the reviewed works (e.g., Strauss, 1988; Piaget and Garcia, 1988; Herscovics, 1989; Bachelard, 2002) suggest we can find some shared similarities in cognitive experiences among people. Bachelard's notion of epistemological obstacles suggests some commonly shared experiences among people such as the tendency to rely on intuition, the tendency to generalize, and language-related tendencies. Human intuition may offer an intriguing conceptual frame to rationalize the incorporation of the history of mathematics because we may find shared intuition. However, intuition is a messy concept. Intuition has controversial meanings; for some, it means the source of truth, while for others it could be seen as the source of errors (Torff & Sternberg 2000; Fischbein, 2002). Use of intuition with the connotation of knowledge is likely to cause misunderstanding (Parson, 2008). Finally, it should be noted that the rationale for using history is not necessarily to search for shared experience, but to highlight the process aspect of mathematical knowledge. History can be an effective way of introducing students to the culture of mathematics (Kitcher & Aspray, 1988).

Social/Cultural Grounds

History reveals that mathematics is a product of diverse cultures. "Modern mathematics evolved to its present form as a result of centuries of cross-fertilization of ideas from different cultures" (Nelson, Joseph, & Williams, 1993, p. 20). As Nelson et al. note, there are many fascinating stories about how mathematical ideas arose and spread in different historical contexts: the story of the spread of the universal numbers system (Hindu-Arabic numeral), the controversies of the invention of calculus surrounding Newton and Leibniz, the Chinese discovery of Pascal's triangle (about 500 years before Pascal), and Greek geometry and its relations with Babylonian or Egyptian geometry, to name a few. Through the inclusion of relevant history of mathematics besides the Western contribution, students can encounter learning opportunities on the roles of other cultures in the development of mathematics. The ICMI study (2002, p. 46) captures this imagination:

> A history that shows the diversity, rather the universality, of mathematical development adds an exciting dimension to the subject. It allows the world and its history to enter the classroom in a way that works against a narrow ethnocentric view, without denying the extent to which developments have often been embedded in cultural contexts. A multicultural approach both requires and encourages us to step into a realm of thinking which challenges our valuing of different styles and branches of the activity we recognize as mathematics.

However, using history to address diversity issues presents challenges. The recorded history of mathematics predominantly reflects the legacies of Western civilization. One of the challenges is how to fairly present history (from what is known) on one hand and address diversity on the other hand. This challenge demands, besides knowledge of the history, an understanding of social dynamics. The use of a certain history may have different learning implications for different groups of students. One goal of addressing diversity issues is to promote the learning of students who may belong to underrepresented groups. However, one concern is if bringing limited segments of history serve the purpose or if that has an unintended effect of reinforcing what may be the stereotype. Studies in the social-psychological phenomenon known as stereotype threat (e.g. Steele & Aronson, 1995; Steele, 1997; Spencer, Steele, & Quinn, 1999; McGlone & Aronson, 2006) suggest that such a threat can undermine the academic performance of students who are members of negatively stereotyped groups. Steele and Aronson (1995, p. 797) define *stereotype threat* as "being at risk of confirming, as self-characteristic, a negative stereotype about one's group". Some suggestions for mitigating such threat (e.g. Steele, 1997 [Steele discusses such issues as focus on challenging tasks, use of role models, recognizing multiple perspective, etc.]) may be useful to consider in using history to promote diversity. On the other hand, I think history presents opportunities to apply these strategies. I think that, besides the recorded history of mathematics from various cultures, the diversity agenda should include issues about women in the history of mathematics and the use of mathematics in cultural activities (that may not be found in traditional mathematics texts).

The history of mathematics can provide a platform to examine factors that have contributed to the underrepresentation of women and explore ways to inspire participation in the field. Exploring the contribution of women in the past leads to the recognition of the diversity of role models they offer (Deakin, 1992). The few stories of women mathematicians before the 19th century reveal their enormous courage and sacrifice to break the cultural barriers of their times. Stories such as French mathematician Sophie Germain (1176-1831) and Russian mathematician Sonya Kovalenskaya (1850-1891) provide fascinating examples of how strong women broke societal prejudice and unfair regulation to gain access to higher learning in mathematics (e.g. Perl, 1978). Through historical analysis of women's participation in mathematics, it is necessary to educate girls about how they are not an 'other' in mathematics (Dowens, 1997). Based on her teaching at the City of London School for Girls, an independent girls' school, Perkins (1991) notes that setting mathematics in historical contexts has proven to be a successful strategy for pupils at the school, especially on issues concerning confidence and gender awareness.

The notion of using history in mathematics opens up the possibility of paying attention to the history of mathematical activities within a certain culture in addition to a general history of mathematics across cultures. D'Ambrosio (1997) suggests broadening the history of mathematics to include what he calls *ethnomathematics*. D'Ambrosio (1997, p. 13) describes ethnomathematics as the "borderline between the history of mathematics and cultural anthropology". According to his ideas, ethnomathematics includes mathematics practiced by "culturally identifiable groups, such national-tribal societies, labor groups, children of a certain age bracket, professional classes, and so on" (p. 16). Proponents of ethnomathematics (e.g., Ascher, 1991; D'Ambrosio, 1985; D'Ambrosio, 1996; Nunes, 1992) argue that everyday mathematics embedded in various cultural activities requires serious consideration in school mathematics. Studies and analyses of children's knowledge of mathematics "have shown much mathematical knowledge is acquired outside of school" (Nunes, 1992, p. 557). The everyday mathematics that students acquire can differ significantly across cultures arising from differences in conceptualization of numeration, measurement, and other similar concepts. The notion of ethnomathematics requires the understanding of mathematics and culture.

In closing this section, I would like to point out that the various grounds for incorporating history—pedagogical, philosophical, psychological, and social/cultural—are intertwined. The point is to suggest that there are various grounds from which we can make a case for the value of incorporating history in school mathematics.

WAYS OF INCORPORATING HISTORY OF MATHEMATICS IN SCHOOL MATHEMATICS

Arguing about the importance of incorporating the history of mathematics in school mathematics from a theoretical perspective is one thing. But from a practical stand point, this incorporation poses a number of challenges. Teaching is a complex enterprise because the problems to be dealt with "exist across social, temporal, and intellectual domains, and often the actions that need to be taken to solve the problem are different in different domains" (Lampert, 2001, p. 2). Often things will be lost in translation from theory to practice, from theoretically justifying that the history of mathematics is important in learning the subject to actually using it in school mathematics. "There is a magical change as mathematics, science, and social sciences move from their disciplinary spaces into the classroom" (Popkewitz, 2004, p.3). Given this complexity, how can history mediate between mathematics as a discipline and school mathematics in a productive way? Assuming we are convinced of the importance of history to learning mathematics, we would be pressed to address a series of practical questions: What, where, when, and how do we incorporate the history? Do we have the resources to do so? Do teachers or schools have the political power to do so? The bottom line is that this is not only a question of teaching, but also of curriculum, teacher education, policy, and probably more.

Cases drawn from various countries regarding the practical aspects of integrating history of mathematics into mathematics education can be found in various sources (e.g., Calinger, 1996; ICMI study, 2002; Furinghetti & Paola, 2003). Regarding curriculum issues, the ICMI study provides an overview of several countries' mathematics curricula with regards to the inclusion of the historical dimension. For the U.S. case, the ICMI report indicates that NCTM and the Mathematical Association of America have been supportive of the issue. Specifically, the ICMI study (p.18) quotes the following NCTM statement about the importance of a historical dimension in mathematics education:

> Students should have numerous and varied experience related to the cultural, historical, and scientific evolution of mathematics so they can appreciate the role of mathematics in the development of our contemporary society and explore these relationships among mathematics and the disciplines it serves…It is the interest of this goal–learning to value mathematics–to focus attention of the need for student awareness of the interaction between mathematics and historical situations from which it has developed and the impact that interaction has on our culture and our lives.

The ICMI study notes that "historical awareness has an important role to play in the construction of curricula even when there is no explicit historical content in the curriculum itself".

The book *Historical Research and Integration with Teaching* (Calinger, 1996), a collection of papers of *History and Pedagogy of Mathematics* (HPM, a group affiliated with ICMI), also discusses curricula issues with historical dimensions. For instance, Heiede's article is a case in point. Heiede discusses the changes that happened in the Danish mathematics curriculum (for grades 10-12) as a result of the regulation passed by the Danish Ministry of Education that calls for teaching all subjects with historical aspects. Heiede elaborates on how this regulation impacted mathematics textbooks and teacher education in Denmark. He notes that the regulation required textbooks to incorporate historical aspects in various forms: in some books as a separate section as an appendix; in some at the end of chapters; in some as completely integrated throughout the books; and in some cases as independent books on historical topics. The changes, according to Heiede, also prompted future teachers to take more courses in the history of mathematic.

On the question of how history can be integrated into mathematics instruction, the ICMI study (2002, pp. 208-213) discusses the following three “different but complementary” approaches.

- Learning historical facts such as biographies of key people who developed mathematical ideas in question, time and place in which the ideas were developed.
- Learning mathematical topics inspired by history. This approach involves learning not only how concepts and methods are used but also why they provide answers to mathematical problems (this approach addresses historical dimensions of ideas implicitly).
- Developing deep awareness of mathematics and the social and cultural contexts that influenced the development of mathematical concepts.

Besides these general approaches, the group presents specific ways and examples in which history can be integrated in mathematics instruction. These include using historical snippets; research projects based on an historical text; primary sources; worksheets; historical packages; taking advantage of errors, alternative conceptions, changes of perspective, revision of implicit assumptions, and intuitive arguments; historical problems; mechanical instruments; experimental mathematical activities; plays; films and other visual means; outdoors experiences; and the Internet. Swetz (2001, p. 319) outlines the following strategies: providing historical origins of the concepts, procedures, and mathematical terms; enriching lessons with brief anecdotes about the lives and work of relevant mathematicians; building lessons around actual historical activities; constructing and displaying timelines; assigning actual historical problems as a class or homework exercise; employing appropriate visual aids (i.e., posters, films, and videos); and assigning short research projects on a particular accomplishment or mathematician.

There are challenges that should be met. The ICMI report lists several arguments that object to the use of history in mathematics instruction. The challenges include lack of time, material resources, and expertise. When teachers have little time left in what they already do, how will they find time to integrate the history of mathematics? Do teachers have the necessary historical knowledge in their area? Where can they gain the needed support? The ICMI report points out that teachers' lack of knowledge regarding the history of mathematics is attributable to their teacher education programs. "Not only historical but also interdisciplinary knowledge is required, which is far beyond what mathematics teachers are equipped for" (p. 203). A more fundamental challenge relates to the view about the nature of mathematical knowledge. Often mathematical knowledge is viewed as deductive and something independent of experience. Thus history is not important to learning the subject matter. This a priorist view of mathematical knowledge is held by many people (Hersh, 1997). Thus if these challenges are not met with conviction, the idea of incorporating history in mathematics education could simply remain as rhetoric with no real value in practice.

To sum up, in a broader sense, the question of implementing historical dimensions in school mathematics should address issues of curriculum (goals of using history, core topics of various grade levels that may benefit from including historical background, in what ways curriculum materials can provide history), teacher education (how pre-service and in-service teacher education provide opportunities to learn history of mathematics), and instruction (how do teachers actually use such an approach in the classroom). This section provided a brief review of sources along these issues. Furthermore, some cases of teacher practices with this approach were explored. These sources point out that most of the accounts of the teachers favored incorporation of historical dimensions.

Conceptual Challenges and History of Selected Concepts

In this section, I present a review of the conceptual challenges and the history of the concepts of fractions, negative numbers, the Pythagorean Theorem and irrational numbers. This section explores conceptual challenges met in learning the selected topics for which their history can be useful. The review of the history of the selected concepts is not comprehensive or representative because the primary objective is pedagogical. The review focuses on literature covering ancient Egypt, Greece, Babylon, China, India, and medieval Arab and Europe because of the scarcity or non-existence of scholarly written sources in English on the history of the selected topics covering other cultures.

Conceptual Challenges Associated with Selected Concepts

Fractions

"Man's difficulty in understanding fractions has not been limited to any historic period" (Groza, 1968, p.236). This difficulty spans from the ancient Egyptians' unit fraction to the Greek's philosophical challenge of breaking the unity, to the Roman avoidance of fractions by way of dealing with subunits.

There is considerable literature concerning the conceptual challenges students in elementary/middle school encounter with formal operations of rational numbers (e.g., Behr, Harel, Post, & Lesh, 1992; Cramer, Post, & delMas, 2002; Mack, 1990; Moss & Case, 1999; Sowder, Bezuk, & Sowder, 1993; Ni & Zhou, 2005). Moss and Case (1999) cite common errors students make across all three rational number representations—fractions, decimals, percent. The errors reveal "a profound lack of conceptual understanding" of rational numbers, note Moss and Case (p. 147). Some of the main problems children have with formal operations of fractions stem from students' tendency to transfer operations of whole numbers onto rational numbers (Mack, 1990) and their failure to recognize a fraction as a single quantity (Behr, Harel, Post, & Lesh, 1993). Examples of common errors students may make are thinking $\frac{1}{5} > \frac{1}{4}$ because 5 > 4 or $\frac{1}{2} + \frac{1}{3} = \frac{2}{5}$ (Ni & Zhou, 2005) or thinking multiplication always makes a larger value and division always yields a smaller value (Greer, 1994).

Emphasis on syntactic (procedural) knowledge as opposed to semantic (conceptual meaning) knowledge is pointed out among sources for the difficulty students encounter with rational numbers (Moss, Case, 1999). Behr et al. (1993) suggest that semantic analysis on the concept of rational numbers should be done to meet the instructional challenge. They note that the concept of rational numbers is characterized by a set of subconstructs. Others (e.g., Kieren, 1993; Freudenthal, 2002) have also analyzed the concept of rational (or fractional) along various subconstructs that include quotients, measures, operators, and ratios. The purpose of such semantic analysis is not to provide a fragmented knowledge of rational numbers but to help in identifying unifying themes that can support children's learning about rational numbers (Carpenter, Fennema, & Romberg, 1993).

Carpenter et al. (1993) suggest partitioning as the unifying theme in understanding the various subconstructs of rational numbers. Mack (1990) notes that students' informal strategies of solving rational number problems involves partitioning but treating each part (partition) as if it represents a separate unit (whole number) rather than a fraction. This informal knowledge of students has some resemblance to the ancient Egyptian and also Greek conception of fractions. Ancient Egyptians mainly used parts unit fractions (e.g., Cooke, 2005), and there was use of unit fractions in ancient Greece (e.g., Fowler, 1987). Such similarity—the informal strategies of partition and the ancient fraction as “parts” of something—is intriguing in the sense that drawing from a history of fractions such as the ancient Egyptians may support the informal knowledge of students before instruction can help them build their understanding of fractions covering the various subconstructs of fractions/rational numbers. According to Pothier and Sawada (1983), “partitioning is basic to the development of rational-number concepts” (quoted in Larson [2001], p. 617). From her study on fractions, Mack (1990) concludes that students can construct meaningful algorithms by building on informal knowledge. Unit fractions may have a special role in the development of children’s thinking about fractions (Kieren, 1993).

To sum up, there is considerable literature concerned with understanding the conceptual challenges students in elementary/middle schools encounter with formal operations of rational numbers. Some of the main problems children have with formal operations of fractions are said to stem from students’ tendency to transfer operations of whole numbers onto rational numbers and their failure to recognize a fraction as a single quantity. These tendencies are reflected in some common errors students make such as thinking that fractions with larger denominators are greater than those with smaller ones when the fractions have the same numerators. Some scholars in this area suggest that semantic analysis of the concept of rational numbers needs to be employed to understand conceptual challenges that underlie the various subconstructs of rational numbers. The partitioning is noted as a possible unifying idea for understanding the conceptual meaning of the various subconstructs.

Negative Numbers

Historically, the association of numbers to quantity, and particularly negative numbers to something less than nothing, rendered a counter intuitive meaning to negative numbers.

> The notion of a *quantity* has a naturally practical connotation: it is intrinsically something more than nothing. The concept of number is naturally related to this practical meaning of quantity. The notion of negative number is, then, practically inconceivable and this fact has reappeared, time and again, in the history of mathematics. (Fischbein, 2002, p. 101).

The concept of negative numbers as less than nothing was reflected in the discourse of prominent mathematicians of not many centuries ago. According Pycior (1997, p. 192), Issac Newton wrote in 1707 in his book *Universal Arithmetick,* "*quantities are either Affirmative, or greater than nothing; or Negative, or less than nothing*" [emphasis original]. Leonhard Euler (1707-1783) similarly noted in his 1770 book *Elements of Algebra* that negative numbers could be considered less than nothing (Berlinghoff & Gouvêa, 2002).

Variants of this conceptual challenge with negative numbers seem to persist in school mathematics today. Numerous sources from research literature on the concept of negative numbers (e.g., Hefendehl-Hebeker, 1991; Gallardo, 2002; Linchevski & Williams, 1999; Thomaidis, 1993; Vlassis, 2002) have dealt with the conceptual difficulties students encounter with negative numbers. Hefendehl-Hebeker (1991) states, "One should know that the intellectual hurdles that blocked this mathematical subject [negative number] throughout its historical evolution may also block the understanding of the present-day students" (p.26). She identifies various factors that contributed to the hurdles associated with the evolution of negative numbers.

The fact that we use the number line to represent real numbers makes it apparent that the starting point—zero—can be placed *arbitrarily* and that negative numbers are *relatively* positioned to the left of zero. The concept of negative numbers was limited in meaning in the time when people had the concept of zero as 'nothing' and had no established number line, without the additional notions of zero as arbitrary and negative numbers as opposite of positive numbers. Hefendehl-Hebeker notes that, with today's students, the difficulties associated with understanding the 'relative' nature of negative numbers may be alleviated by using examples such as temperature measurement by a thermometer. The number line is also a general model by which contemporary students learn about negative numbers as directed (relative) numbers, which may help lessen the intuitive difficulty (Fischbein, 2002) inherent in the concept.

What Hefendehl-Hebeker refers to as hurdles related to *concrete viewpoint* and the *single model* highlight the difficulties associated with practically modeling negative numbers and with making sense of formal operations of negative numbers in the context of such models. Using algebraic properties of numbers, we can show the product of two negatives to be positive. But the concrete models such as credit-debt for positive-negative numbers do not seem to justify why a negative multiplied by a negative should be positive—does debt multiplied by debt make credit? Multiplication of negative numbers historically presented epistemological obstacles (ICMI study, 2002, p. 244). In earlier times, negative numbers often were associated with debt as was the case with Hindu mathematics (Gullberg, 1997). As late as the 18th century, the credit-debt model for explaining the positive-negative number concept appeared to be popular in Europe. Berlinghoff and Gouvêa (2002, p. 85), for example, quote the famous Swiss mathematician Leonhard Euler as having stated the following in his *Elements of Algebra*:

> Since negative numbers may be considered as debts, because positive numbers represent real possessions, we may say that negatives are less than nothing. Thus, when a man has nothing of his own, and owes 50 crowns, it is certain that he has 50 crowns less than nothing; for if any one were to make him a present of 50

> crowns to pay his debts, he would still be only at the point of nothing, though really richer than before.

However, when explaining why the product of two negative numbers is positive, Euler argues in a formal manner: "$-a$ times $-b$ should be the opposite of a times $-b$" (Berlinghoff & Gouvêa, 2002, p. 85). Other sources (e.g., Fischbein, 2002; Freudenthal, 2002; Thomaidis, 1993) strongly argue against the use of a concrete model in introducing negative numbers because there are no models that fully justify the formal operations of negative numbers. Freudenthal especially asserts that the origin of negative numbers is algebra and thus a formal algebraic approach (the algebraic permanence principle: $a + x = b, b < a$) should be used to introduce negative numbers. Thomaidis (1993, p. 81) observes that the emphasis on interpretive models such as credit-debt and temperature scales do not help students "remove the obstacles that hinder the acquisition of the abstract concept of the number". He argues that the roles of various empirically based models used to introduce negative numbers need to be re-examined. He goes on to suggest a formalistic approach based on a historical problem—John Napier's (1550-1617) introduction of negative numbers in the context of logarithms, which does not require a practical model for interpretation. This approach only requires an understanding of power rules of a certain positive number.

The hurdle referred to as the *Aristotelian* notion of numbers is the "subordinate" association of numbers to magnitudes. In geometry-dominated Greek mathematics, magnitudes were some sort of measures of lines, areas, volumes, and angles (Joyce, 1998). Magnitudes were different from numbers (numbers meaning only part of natural numbers), but numbers could be used, for instance, to compare magnitudes of the same kind as in ratios. In this sense, you cannot have negative magnitude. This way of thinking about magnitude is believed to have constrained the development of negative numbers, especially in Western civilization, even after the algebraization of geometry (the establishment of coordinate geometry) in the 17^{th} century (Freudenthal, 2002). After coordinate geometry was invented, negative numbers still did not have a formal status in the 18^{th} century. An influential article written by D'Alembert's (1717-1783) for "Diderot's Encyclopedia" on negative numbers indicates the confusion surrounding negative magnitudes in that era (Hefendehl-Hebeker, 1991). The sentiment of the D'Alembert article was shared by prominent mathematicians of the time such as Descartes (1596-1650), who called negative roots (magnitude) "false roots" (Gullberg, 1997). In an empirical analysis of 35 (12- and 13-year-old) students' solutions to historical word problems involving negative number solutions, Gallardo (2001) notes that various interpretations of negative numbers made by the students that are similar to those noted above. The interpretations include negative number as relative number, as a number being subtract from another (attempting to interpret negative number in terms of magnitude), and just an isolated number which as solutions to an equation.

In this section I presented a brief review of the conceptual obstacles in the development of negative numbers. Hefendehl-Hebeker (1991) identifies several obstacles in the history of negative numbers such as absence of uniform number line, the perception of absolute zero, use of a concrete model, and association of numbers to magnitudes. I inferred that some of these hurdles may manifest in students' learning experience of negative numbers in some way.

Pythagorean Theorem

The Pythagorean Theorem in its popular form (algebraic) $a^2 + b^2 = c^2$ may not be complicated to remember. If we go deeper and ask what the letters stand for and further ask about the essence of the theorem, we often might get a "blank stare" (Berlinghoff & Gouvêa, 2002). And if we go further and ask about its proof, the complexity would increase. Add the generalized version of the theorem (in which areas of similar figures must be compared instead of simply area of squares on the sides of a right triangle), and the situation could get more nuanced. Thus, the conceptual challenge may relate to grasping the concept underlying the algebraic formula and the essence of the proofs of this theorem.

There are various ways to prove the Pythagorean Theorem. Some can be simpler than others. However, as proofs, the common feature is the logical argumentation in verifying the theorem. The key role of a proof is to promote mathematical understanding (Hanna, 2000); it is also essential in establishing and communicating mathematical knowledge (Kitcher, 1984; Polya, 1981), quoted in Stylianides (2007). How does school mathematics promote students' understanding of proofs? Schoenfeld (1994, p. 74) asks, "Why do our students have so little appreciation for proof, and why do they have so little apparent aptitude for it?" The way students encounter proofs in school mathematics has "no personal meaning and explanatory power for students," notes Schoenfeld (p. 74). Stylianides (2007) quotes several sources indicating that many students' transition from elementary school mathematics to secondary school is marked by a sudden jump with regards to learning mathematical proofs. That is, most students start learning about proofs in high school geometry. Speaking from the Italian experience, Mariotti (2000) explains that students are introduced to deductive geometry in high school, while the geometry they have learned in previous grades was mostly at an "intuitive level" such as defining, naming, and describing geometry facts and figures. There is epistemological confusion, says Hanna (2000) referring to a possible conflict between the students' empirical experience and the pure deductive approach in mathematical proof. Hanna, citing

Fischbein (1982), notes that students often ask for an empirical testing of the proof of a theorem even when they say they understand it. Hanna makes an interesting point comparing students' desire for empirical testing of a proof with the experimental scientists' quest for empirical justification of a theory.

In this case, the noted conceptual challenge seems to lie in the transition from the intuitive tendency to understanding the deductive nature of mathematical proof. Some dynamic geometry software packages promise to foster an understanding of mathematical proof because they provide features that facilitate empirical justification (e.g., Mariotti, 2000; Laborde, 2000; Marrades & Gutierez, 2000). Besides such technological tools, historical aspects can be used.

Irrational Numbers

Educational research literature on the concept of irrational numbers appears to be rather slim (Sirotic & Zazkis, 2007). However, there are numerous studies on the topic (e.g., Arcavi, Bruchheimer, & Ben-zvi, 1987; Fischbein, Jehiam, & Cohen, 1995; Peled & Hershkovitz, 1999; Sirortic & Zazkis, 2007) that indicate a lack of deeper understanding about irrational numbers by students and teachers. These sources also discuss difficulties surrounding irrational numbers such as intuitive, definitional, origin, representation, and use of irrationals.

Arcavi, Bruchheimer, and Ben-zvi (1987) examined teachers' knowledge of some aspects of irrational numbers. Their subjects were teachers from junior high schools in Israel attending summer in-service. Arcavi et al. notes that teachers lack historical knowledge of irrational numbers. They report that most teachers thought decimal fractions preceded irrational numbers and most appeared to conceive the origin of irrational numbers as decimal fractions rather than geometry. They found that a number of prospective teachers confused rational approximation of irrational numbers (e.g., $\frac{22}{7}$ for π) with the irrational itself. They also note that even though most of the teachers were able to provide textbook definitions of irrationals such as "a number that cannot be expressed as a quotient of two integers" or "a number whose decimal part is not periodic and has an infinite number of digits," "only two teachers mentioned the definition by means of the Dedekind cuts" (p. 18). Arcavi, et al. did not expect junior high school teachers to have remembered this formal definition but to at least be aware that the definition based on the Dedekind-cuts exists.

As part of a research study on the understanding of irrational numbers, Zazkis (2005) analyzed the responses of 46 pre-service teachers to the following question:

> Consider 53/83. Call this number M. In performing the division, the calculator display shows 0.63855421687. Is M rational or irrational number? Explain. (p. 11).

Zazkis notes that only 60% of the participants provided correct responses with an appropriate explanation (i.e., M is rational because it is a quotient of two integers). However, the remaining percentage included those who thought $\frac{53}{83}$ is irrational since there is no pattern in its decimal representation; they thought it could be rational or irrational because they did not know if the decimal would be terminating, repeating, or neither.

Peled and Hershkovitz (1999) investigated difficulties encountered by student teachers in tasks involving the construction of an irrational length of a segment and other irrational number tasks. Their subjects consisted of two groups (one group of 55, and the other a group of 15) of student teachers who had two or three years of college mathematics. Among others, Peled and Hershkovitz analyzed responses of the student teachers to this problem:

> John gave the carpenter a 5m by 1m wooden board, asking him to use all the wood to make a square table top. The carpenter thought for a while and said: No problem. I can build your table by making a few cuts. Do you think he can make it? How? (p. 39)

According to Peled and Hershkovitz, while most students concluded that such a square should have side $\sqrt{5}$ m, some did not believe a side of a square with such length can be measured because its decimal representation involves infinite digits. The main source for this difficulty, they argue, is something that relates to the challenge to grasp the concept of limit, more like the Zeno paradox—"the notion of getting closer and closer but never reaching a certain point" (p. 45). A similar belief seems to underlie the students' claim that $\sqrt{5}$ could not be located on the number line, according to Peled and Hershkovitz. Students must rely on intuition of the number line to give meaning to the concept of irrational numbers because there is no explanation of it (the explanation is that it is not rational (Tall, 2002). But, intuition can differ based on experience; therefore, the idea that there is a clearly defined number line intuitively shared by everyone is not acceptable, notes Tall.

As part of an ongoing study of prospective teachers' understanding of irrational numbers, Sirotic and Zazkis (2007a) analyzed the responses of 46 prospective secondary school teachers to the task "show how you would find the exact location of $\sqrt{5}$ on the number line". Their report shows only 9 out of 46 the participants accurately located $\sqrt{5}$ on the number line. They claim that the vast majority of the participants perceived the number line as a rational number line. That is, these participants appeared to believe numbers such as $\sqrt{5}$ can be located on a number line only using their decimal approximations; they confuse irrational with their decimal approximation. Sirotic and

Zazkis suggest the use of geometric representation using the Pythagorean Theorem, especially for constructible length, to help teachers become aware of the distinction between irrational and its rational approximation.

Sirotic and Zazkis (2007b) investigated how participants reconciled the idea that rational and irrational numbers could fit together on a number line while both are infinitely many (dense). Among the questions, they asked participants if there is always a rational number between any two irrational numbers, if there is always an irrational between two irrational numbers, and if there is always a rational between any two rational numbers. Although most participants gave correct responses (that such numbers exist in both cases), Sirotic and Zazkis found some conflicts in the participants' ideas. They found inconsistency between prospective teachers' intuitive knowledge and formal knowledge about irrational numbers.

"There are cognitive obstacles that may account for the difficulties preventing learners from concluding that there is a rational number between any irrationals, and that there is a rational number between any two rational numbers" (p.67). One source of conflict that Sirotic and Zazkis observe is the formal knowledge that irrationals outnumber (are denser than) rational numbers. This seems to cause the thinking that there would be neighboring irrationals in between which a rational cannot be found. Fischbein, Jehiam, and Cohen (1995) also noted problems encountered by prospective teachers in comparing rational and irrational numbers in terms of the amount of elements.

Fischbein, Jehiam, and Cohen (1995) assumed that there are two intuitive obstacles associated with irrational numbers: (a) difficulty with accepting the fact that two magnitudes would be incommensurable (meaning they would not have a common unit) and (b) difficulty with accepting that rational numbers do not cover the number line. Fischbein et al. note that after the discovery of incommensurability of irrational numbers by the ancient Greeks, it took centuries until the contributions of Dedekind, Cantor, and Weierstrass to establish rigorous theory of irrational numbers. Fischbein et al. argue that this long history of change may be due to the intuitive obstacles that irrational numbers do not fit the practical models from which the number concept emerged. Fischbein et al. surveyed the irrational number knowledge of 62 high school students (grades 9 and 10) and 29 prospective teachers in Israel. They also assessed if the participants encountered the two intuitive difficulties stated previously. Their study did not confirm specifically that the subjects faced those obstacles. Fischbein et al. contend that contemplating irrational numbers in terms of the obstacles stated might require a higher level of mathematical knowledge than their subjects had. They state that many of the students had problems with classifying various numbers as rational, irrational, and real.

Although there seems to be slim research literature on the concept of irrational numbers, the studies reviewed on the topic point to a lack of deeper understanding about irrational numbers by students and teachers. These sources note conceptual challenges faced by the subjects along such aspects as intuitive, definitional, origin, representational, and use of irrationals. History can be useful to address some these conceptual challenges. Specifically, by discussing the origin of irrationals, irrational numbers can be seen intrusively as a result of the problem of measurement of magnitudes.

Historical Reviews of the Selected Topics

Fractions

The word “fraction” comes from the Latin word “fractum” meaning “to break”; and the early meaning suggests the concept of fraction as something “less than a whole” (Pothier, 2001). In earlier times, accounting for portions of objects involved breaking down the objects and counting the pieces (Berlinghoff, & Gouvêa, 2002). The idea of "counting the pieces" separately instead of in relation to the whole makes the ancient notion of parts hard to qualify as fractions in the modern sense of the concept. Cooke (2005) states that the closest term in meaning to the ancient Egyptian fractions is what we call a *part* [his emphasis]. He contends that there would be only one particular "part". For example, what we call $\frac{1}{7}$, they would call "the seventh part"— meaning the last part of a thing divided into seven.

The earliest time when fractions came into use is hard to establish. However, as Freudentthal's (2002) phenomenological analysis on the concept suggests, fractions can be viewed as a less problematic extension of the natural number concept. Freudentthal contends that we can find notions related to fractions from the first mathematical documents onward. The earliest sources that contain fractions come in the form of ancient Egyptian papyrus and Mesopotamian (Babylonian) clay tablets that date back to the second millennium B.C. The ancient Egyptian document known as Ahmes papyrus (after the scribe who wrote it around 1650 B.C.) is the oldest known source containing some sort of fractions. This document is also known as the Rhind mathematical papyrus (named after the lawyer and Egyptologist Henry Rhind, who purchased the document in 1858). This papyrus indicates that ancient Egyptians used what we may now call unit fractions–with the exception of $\frac{2}{3}$, their fraction system consisted of fractions with the numerator 1 (e.g., Berlinghoff & Gouvêa, 2002; Chace, 1979; Cooke, 2005; Roero, 1994; Groza, 1968; Ifrah, 1985; Imhausen, 2007).

Egyptians appeared to have had a number system that consisted of 1 to 1,000,000, the corresponding reciprocal numbers (unit fractions), and $\frac{2}{3}$ (Chace, 1979). Chace notes that Egyptians would put the sign for "mouth" (in hieroglyphic form) or a dot (in hieratic form) above a number to indicate a fraction. Since their number system did not use place value, writing a fraction could be confusing (Suzuki, 2002). However, as the problems were tied mostly to practical situations, the Egyptian scribe determined which value was meant from the context (Suzuki, 2002). One complication with the Egyptian fraction system is that they expressed non unit fractions in some combination of unit fractions without repetition… For instance, using modern notations, for $\frac{2}{5}$, they would have $\frac{1}{3}\ \frac{1}{15}$ (i.e., $\frac{1}{3}+\frac{1}{15}$). This task can tedious. But, Egyptians had tables for operations with fractions (Roero, 1994). One table, contained in the Ahmes papyrus, expresses doubles of odd parts into unit fractions, for odd parts up to 101. According to Roero, the table can be said to express (in modern terms) fractions of the form $\frac{2}{2n+1}$, for $1 \leq n \leq 50$, as sums of unit fractions. Tables for doubles or even parts (like $\frac{2}{2n}$) were not needed because they can be reduced easily into unit fractions. Roero indicates that the issue of how the Egyptian scribes arrived at the values of fractions in the tables—whether they used trial and error or some rules— is a matter of debated.

Questioning how the Egyptians arrived at these values and speculating about possible methods may have some instructional value for middle school math. For instance, Edwards (2005) discusses activities based on the context of Egyptian fractions in mathematics intervention programs for inner-city middle school students in Detroit, Michigan. After their introduction to Egyptian fractions, Edwards notes, the first problem students worked on required expressing proper fractions in unit fractions or sums of unit fractions without repetition. Edwards observes that some students or groups of students uncovered some general algorithms through trial and error. He concludes that the activities provided "a means to review and practice computational skills involving fractions in a way that students find exciting" (p. 229).

The Mesopotamian (Babylonian) fraction was different from the Egyptian fraction. Unlike the Egyptians' system, the Mesopotamian number system was sexagesimal (base-60) and fractions were expressed on this base (e.g., Alexander, 2001; Berlinghoff & Gouvêa, 2002; Cooke, 2005; Suzuki, 2002), and they were not unit fractions. One very important advantage of their system was that arithmetic operations of fractions were easier compared to their decimal counterpart. Adding fractions that require finding common denominators, say $\frac{1}{10}+\frac{1}{15}$, in decimal system, would become easier in sexagesimal because they already had the same denominators, $\frac{6}{60}+\frac{4}{60}$, written in their way but with modern translation as 0;6 + 0; 4 (Suzuki, 2002). Another interesting advantage of their system is that positive integers less than 10 with the exception of 7 have terminating reciprocals (Cooke, 2005). For instance $\frac{1}{3}$= 0.333…, in base 10. That would be 0;20 in their system (i.e., 0.20 in base-60). Mathematics historians usually use commas to separate numbers with different place values and semicolons as a decimal point in their transcription of the Babylonian sexagesimal numbers (Alexander, 2001).

Although the Babylonians established place value in their numbers system, they did not seem to have zero or some other placeholder in their earlier writings (Alexander, 2001; Ifrah, 1985). Thus sometimes it is ambiguous if a number stands for a fraction. From a cuneiform tablet of Babylonian text, the number equivalent to 1 might mean 1 or 60 or $\frac{1}{60}$ or any other unit fractions of powers of 60 (Suzuki, 2002; Berlinghoff & Gouvêa, 2002). Ifrah (1985, p. 379) contends, "Babylonian mathematicians and astronomers developed a genuine zero to signify the absence of sexagesimal units of a certain order" in the latter era (fourth to first century B.C). He adds that they had a special symbol for zero (as a place holder) and could write fractions without ambiguity. I find the pros and cons of the fractional system of Mesopotamia intriguing for instructional purposes in middle schools. For the arithmetic of fractions such as addition and subtraction, the idea of finding common denominators may be enriched by including this history about the sexagesimal fraction and discussing how easy it may be to find common denominators of some fractions if expressed when expressed in sexagesimal. Along the way, students also could learn about converting from one base to another base. In fact, the whole idea of the number system we use could be questioned: Why do we use the numbers system that we have now? Why not others? Discussions of time and other unit conversions could even be covered at this junction. Students can see some of the Babylonian legacy such as time and degree measurements.

The Egyptian and the Babylonian [fractional] systems were passed on to the Greeks and other Mediterranean cultures (Berlinghoff & Gouvêa, 2002). Heath (1960, v. 1, p. 41) notes, "The Greeks had a preference for expressing ordinary proper fractions as sums of two or more submultiples (unit fractions); in this way they followed the Egyptians." Heath also points out that the Greeks used sexagesimal fractions (Babylonian in origin) in their astronomical calculations. However, there are two important factors that suggest that the Greek fractional system was not only based on unit fractions and sexagesimal fractions. One is in the work of Diophantus, a Hellenistic mathematician believed to have lived in the 3rd century A.D. Numerous sources (e.g., Berlinghoff & Gouvêa, 2002; Cooke, 2005; Heath, 1960, V. II) maintain that Diophantus in his work *Arithmetica* established problems that admit both fractional and integral solutions. The fractional solutions were not necessarily unit fractions. For instance, Diophantus could express a square as the sum of two other squares such as the following (Berlinghoff & Gouvêa, 2002; Cooke, 2005):

$$16 = \left(\frac{16}{5}\right)^2 + \left(\frac{12}{5}\right)^2$$

The other factor relates to Greek view of mathematics and geometry as distinct pursuits. For the Greeks, mathematics was about numbers, and geometry was concerned with magnitudes (Freudenthal, 2002). While unit fractions might have been common in arithmetic operations, the concept of ratio in Greek geometry may qualify as one of the "subconstructs of fractional numbers" (as in Kieren, 1993). The fractional value of ratios of magnitudes in Greek geometry was not merely unit fractions, as can be inferred from Euclid's *Elements* (Book X, proposition 6). Thus, one can make the case that unit fractions might have a dominant role in practical arithmetic calculations, but there were also instances of non-unit fractions as in Diophantus' *Arithmetica* and Euclid's *Elements*.

Like the Egyptian and Babylonian systems, the Greeks' representations of fractions (in writing) could be ambiguous. Since the Greeks used the letters of their alphabet as numbers (Ifrah, 1985; Dantzig, 1954), they commonly put a horizontal bar above a letter to denote a number (Gullberg, 1997; Ifrah, 1985). Fractions were written in various ways by the ancient Greeks (Gullberg, 1997). One that appears to have been used commonly was the placement of an accent above the desired letter (letters). For instance, the fourth letter of the Greek alphabet δ represented 4 when written as $\bar{\delta}$, and δ' represented $\frac{1}{4}$. Such representation could be confusing when multiple letter numbers are involved and contextual knowledge is lacking. For example, Heath (1960) cites that Archimedes wrote $\bar{\iota}\ o\alpha'$ for $\frac{10}{71}$. This same number also could be translated to. In the later Greek civilization, Diophantus employed a more convenient method of writing fractions, something that looks like the reverse of the modern convention (Heath, 1960).

In terms of similarity to the modern handling of fractions and operations, the ancient Chinese come closer. They had a complete theory of fractions which is documented in the *Nine Chapters* (Kangshen, Crossley, & Lun, 1999). The *Nine*

Chapters on the Mathematical Art, a classic Chinese work on mathematics that dates back to 100 B.C., contains rules on reducing (simplifying), adding, subtracting, multiplying, dividing, and comparing fractions in a similar manner as we do today. For instance, from a translation of this work by Kangshen, Crossley, and Lun (1999), we find the following rule for the addition of fractions:

> Each numerator is multiplied by the denominators of the other fractions. Add them as the dividend, multiply the denominators as the divisor. Divide; if there is a reminder, let it be the numerator and the divisor be the denominator. In the case of equal denominators, the numerators are to be added directly. (p. 70)

This is basically the same as the modern addition rule:

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + cb}{bd}\text{; or } \frac{a}{b} + \frac{c}{b} = \frac{a + c}{b}.$$

The Chinese, as reflected in the *Nine chapters,* seemed to emphasize expressing the sum as a mixed fraction if it turned out to be an improper one. The rule on division of fractions also involved setting a common denominator of the dividend and divisor and then dividing:

$$\frac{a}{b} \div \frac{c}{d} = \frac{ad}{bd} \div \frac{cb}{db} = \frac{ad}{cb}.$$

This reduction to a common denominator "made the process of division natural and obvious" (Berlinghoff & Gouvêa, 2002, p. 77). Although this rule has the same effect as the modern rule (invert the divisor and multiply), it can differ in meaning. The idea of having a common denominator can mean having a common measure. Working with a common denominator in this context can be related to the less technical term "common measure". This way of dividing fractions can be more meaningful for some students. Perlwitz (2004, p. 123) interviewed six seventh graders about their understanding of the division of fractions. Perlwitz notes that two students "were able to extend their measurement interpretation of division with whole numbers to division with fractions".

Particularly, Perlwitz discusses how one of the student made sense of $\frac{3}{4} \div \frac{2}{5}$ by converting to $\frac{15}{20} \div \frac{8}{20}$, effectively establishing a common partition.

A Hindu manuscript from 7th century A.D. suggests a similar approach to that of the Chinese (Berlinghoff & Gouvêa, 2002). The Hindu mathematician Aryabhata (476-550AD) gave a "rule for addition of fractions identical to the Nine Chapters" (Kangshen, Crossley, & Lun 1999, p. 72). Brahmagupta (598-670?) also provided rules for addition and subtraction of fractions, essentially the same as the modern, in chapter 12 of his treatise, which he completed in 628 A.D. (Plofker, 2007). The Hindu system used a way of writing fractions similar to ours except it omitted the horizontal bar separating the numerator and the denominator (Groza, 1968). For instance, using modern notation, they would have $\begin{smallmatrix}3\\5\end{smallmatrix}$ instead of $\frac{3}{5}$. Arab mathematicians are believed to have introduced the horizontal bar around the 12th century and transmitted this way of writing fractions to Europe around this time.

Before the introduction of Hindu-Arab number system to Europe, computations involving fractions had been carried out using common fractions (fractions whose denominators can be any natural number) and sexagesimal systems in the case of astronomical computations (Cooke, 2005). Decimal fractions (fractions with denominators of powers of 10) started to spread around the 13th century after the Europeans had become familiar with the system through Arab contact. While decimal fractions were not common in Europe until the close of the 16th century, they commonly were used in China (Kangshen, Crossley, & Lun 1999). As Groza (1968) has noted, the need and popularity for decimal fractions gradually increased in Europe because computations with decimal fractions are relatively easier. The Flemish mathematician Simon Stevin (1548-1620) is noted for his concise and systematic explanation of decimal fractions in his 1585 book *De Thiende* (Decimals) (Cooke, 2005; Groza, 1968). His notation, however, was not handy.

As late as the 17^{th} century, the concept of a fraction as a number did not seem to exist in Europe–fractions were not seen as numbers in their own right but rather were used as a way of comparing whole numbers with each other (Pumfrey, 2004). Even a noted German mathematician from the 19^{th} century, Leopold Kronecker, regarded fractions as only possessing derivative character and only useful for notational purposes (Hellman, 2006). Hellman notes that Kronecker believed that fractions, irrational numbers, and complex numbers were illusionary ideas that had arisen from some sort of false mathematical logic. Kronecker is quoted as saying, “God makes the integers; all else is the work of man” (Swetz, 2001).

Contrary to Kronecker’s view, the 19^{th} century saw efforts to formalize fractions. Hefendehl-Hebeker (1991) cites the work of German mathematician Hermann Hankel (1839-1873) as the formalization of fractions as numbers. Hankel, according to Hefendehl-Hebeker, maintained that fractions can be introduced formally as solutions to equations of the form $x * a = b$, where a and b are natural numbers, and that $x = \frac{b}{a}$ would become part of the domain of numbers.

I think an oversimplified summary of the history of the fraction concept may be marked broadly by three episodes: the antiquity notion of “parts”, to the non-numeric notion that still appeared in 17^{th}century Europe, and to the modern number notion. The parts notion was used at least as far back as the second millennium B.C. in ancient Egypt. Sources also indicate that ancient Chinese and Hindu mathematicians employed operations on fractions with similar rules as we use today, but it is problematic to infer that their fractions were used in the similar sense as contemporary meaning. The concept of fraction as a number did not seem to exist in Europe until at least the 17^{th} century.

Fractions were not seen as numbers in their own right but rather were used as a way of comparing whole numbers with each other. Formalization of fractions as numbers was apparently established in 19th century Europe. Some of these landmarks in the meaning of fractions may be shared in some instances in school mathematics today. For instance, Larson (2001) notes students' difficulty to conceive fractions as numbers. It seems, from the history of fractions, we can uncover conceptual challenges with fractions that may be useful to consider in today's school mathematics.

Negative Numbers

We have no evidence that the ancient civilizations of Babylon, Egypt, or ancient Greece recognized negative numbers (Gullberg, 1997). Ancient civilizations from the Far East—China and India—however, appeared to have some form of negative number concept in earlier times. These early forms of negative numbers apparently were limited in scope (e.g., representing debt/loss). Some sources (e.g., Groza, 1968; Freudenthal, 2002) view the earlier development as precursor to the concept of negative number but not as a negative number. "If precursors, as in the Hindu mathematics, are disregarded, negative numbers arose about 1500, though three centuries passed before they were wholeheartedly accepted; the directed magnitudes are an invention of the 19th century" (Freudenthal, 2002, p. 432).

Freudenthal's observation may provide a rough time-reference for the development of negative numbers. What he referred to as the "precursor" includes the early development of negative numbers, like those used by ancient Chinese and Hindu mathematicians. The earliest document showing some use of negative numbers in Europe is *Ars magna* by the Italian mathematician Girolamo Cardano, which was published in

1545 (Gullberg, 1997). However, negative numbers did not have a formal status until the 19^{th} century.

The classic Chinese mathematical work known as the *Jiuzhang Suanshu* (*Nine Chapters on the Mathematical Art*) is presumably the earliest document containing discussions of positive and negative numbers (Sun, 2001). The 3^{rd} century AD commentary of Liu Hui, a noted Chinese mathematician, on the *Nine Chapters* contains a description of 'counting rods' that represent both positives and negatives. "…There are two opposite kinds of counting rods for gains and losses, let them be called positive and negative. Red counting rods are positive, black counting rods are negative" (Kangshen, Crossley, & Lun 1999, p. 404). Kangshen et al. also state that the *Nine Chapters* contains complete rules for addition and subtraction of the red-black counting rods. The "nature of the counting board and the procedures it naturally suggested led Chinese mathematicians to introduce the concept of negative numbers" (Dauben, 2007, p. 194). The use of negative numbers with specified rules is evident in chapter 8 of the *Nine Chapters*, solving problems that could be considered in the present time as systems of linear equations. For instance, problem 8 of the *Nine Chapters* (Kangshen et al., p. 409) states:

> Now sell 2 cattle [and] 5 sheep, to buy 13 pigs. Surplus 1000 cash. Sell 3 cattle [and] 3 pigs to buy 9 sheep. [There is] exactly enough cash. Sell 6 sheep, [and] 8 pigs. Then buy 5 cattle. [There is] 600 coins deficit. Tell: what is the price of a cow, a sheep and a pig, respectively?

In modern notation, this problem would be represented by the following set of linear equations:

$$2c + 5s - 13p = 1000$$
$$3c - 9s + 3p = 0$$
$$-5c + 6s + 8p = -600$$

(Where, c = price of a cow; s = price of a sheep; p= price of a pig)

The solution of this problem is provided in *Nine Chapters*. The method is known as the "Array Rule", which is like a matrix. Liu Hui's commentary on solving this problem using the "Array Rule" is much like the Gaussian elimination method. Thus, the Chinese positive-negative number conception appears to be more than simple buying-selling or gain-loss representation. The method of solving the problem suggests some level of de-contextualization of these numbers (Gallardo, 2001). That is, in the process of solving problems, the Chinese could manipulate the red-black (positive-negative) numbers without relying on the concrete meaning of the numbers. This could be a step forward along the path toward formalization than what Freudenthal seem to have acknowledged.

Similar to the ancient Chinese in general use, Hindu mathematicians also established the use of negative numbers for accounting purposes (representing debt or loss) as early as the 7^{th} century. The 7^{th} century Hindu astronomer and mathematician Brahmagupta is noted for treating negative numbers as debt and for stating rules for addition, subtraction, multiplication, and division of negative numbers (Berlinghoff & Gouvêa, 2002). As Berlinghoff and Gouvêa have noted, however, negative quantities were regarded with suspicion for a very long time by Hindu mathematicians. Some sources (e.g., Klein, 1953; Kangshen, Crossley, & Lun, 1999) maintain that the Hindu concept of negative numbers was transmitted to Europe by way of Arabic records. Since Arab mathematicians did not use negative numbers themselves, Berlinghoff and Gouvêa (2002) point out that "early European understanding of negative numbers was not directly influenced" by the Hindu work on negative numbers.

In Europe, negative numbers were used during the Renaissance (Klein, 1953). Their use must have been limited because they were not accepted fully as legitimate numbers. In fact, prominent mathematicians of this era, including Leibniz and Descartes, referred to negative numbers as "fictitious," "absurd," or "false roots" when they appeared as solutions to equations (Gullberg, 1997). According to Berlinghoff and Gouvêa, “misunderstanding” and “skepticism” about negative numbers continued during the 17th and 18th centuries in Europe. The discourse of scholars from that era suggests that misunderstandings about negative numbers appeared to be associated with interpretations of the concept beyond simple debt/loss interpretation. Berlinghoff and Gouvêa (2002, p. 84) state

> Antoine Arnauld (1612-1694) [French theologian, philosopher, and mathematician] argued that if -1 is less than 1, then the proportion -1:1 = 1: -1, which says that a smaller number is to a larger as the larger number is to the smaller, is absurd. John Wallis [1616-1703, English mathematician] claimed that negative numbers were larger than infinity. In *Arithmetica Infintorum* (1655), he argued that a ratio such as 3/0 is infinite, so when the denominator is changed to a negative (-1, say), the result must be even larger implying in this case that 3/-1, which is -3, must be greater than infinity.

D’Alembert (1717-1783), a French mathematician and co-editor of Diderot’s *Encyclopedia*, wrote an article on negative numbers in the encyclopedia reflecting the continued struggle over the conception of negative numbers during the 18th and into the 19th century (Hefendehl-Hebeker, 1991). The following is quoted in Hefendehl-Hebeker (1991, p. 29):

> The negative magnitudes are the counterparts of the positive one: the negative begins where the positive ends. …
>
> One must admit that it is not a simple matter to accurately outline the idea of negative numbers, and that some capable people have added to the confusion by their inexact pronouncements. To say that the negative numbers are below nothing is to assert an unimaginable thing. Those who say that 1 is not comparable with -1 and that the ratio of 1 and -1 is different from the ratio of -1 and 1 are doubly wrong: (1) because in algebraic operations we divide 1 by -1 every day; (2) because the equality of the product of -1 by -1 and of +1 by +1 shows that 1 is to -1 as -1 is to 1.

This excerpt suggests that D'Alembert captured some changes on how negative numbers were viewed—not something less than nothing but a magnitude opposite to positive—but failed to provide a convincing explanation for the changes of signs as a result of algebraic operations.

In the 19th century, a fundamental change was made from concrete meaning to formal extension of the number system (Hefendehl-Hebeker, 1991). Mathematicians Martin Ohm (1792-1872), George Peacock (1791-1858) and Hermann Hankel (1839-1873) were among the pioneers of this change of view. Hankel especially is credited with helping establish the formalization of negative numbers by way of "algebraic principle permanence" (Hefendehl-Hebeker, 1991; Freudenthal, 2002). Within this frame, negative numbers were invented as solutions to the equation $a + x = b$, where $b < a$ (Freudenthal, 2002). Thus, the number system is extended beyond positive numbers by including all numbers x that are solutions of this kind of equation, while the fundamental properties of the extended number system remain valid (the permanence principle). Hefendehl-Hebeker (1991, p. 31) notes, "Hankel and his supporters, giving up the fruitless search for compellingly clarifying models," or "content notions such as quantity and magnitude," extended the number systems to include negative numbers. Nonetheless, she adds that in this extended system negative numbers were not meant be completely detached of content meaning. Rather, their interpretations and applications were broadened. New areas of applications such as coordinate geometry and vector concepts were opened up.

Thus, algebra was instrumental to legitimize negative numbers as part of the extended numbers system. Freudenthal (2002) argues that the origin of negative numbers is the algebra of equations. Before the legitimization of negative numbers, negative solutions to algebraic problems would be dismissed customarily as "impossible," "absurd," or "false root". Heath (1960, v. 2), for example, notes that the Greek mathematician Diophantus described equations (that can be written) as $4x + 20= 4$ as absurd because it would yield $x = -4$. Berlinghoff & Gouvêa (2002, p. 81) cite ancient problems such as the following in which negative solution could arise: “I am 7 years old and my sister is 2. When will I be exactly twice as old as my sister?” (p. 81). This problem translates to solving for x in $7 + x = 2(2 + x)$, in modern notation. If a similar problem but with different ages, say that would result in the equation $18 + x =2(11+ x)$, the solution would be negative ($x = -4$). Berlinghoff and Gouvêa note that ancient scribes of Egypt and Mesopotamia could solve such problems but “never considered the possibility of the negative solutions”.

One noted development that can be linked, at least in the sense of signs, to negative numbers is the “rule of signs”. This rule was established in algebra long before the legitimization of negative numbers through the permanence principle (Thomaidis, 1993). Thomaidis notes that the recognition of the sign rule could be found in the work of Diophantus. He also cites the work of French mathematician Viete (1540-1603) on product rules such as $(A-B)(C-D)= AC-BC-AD+BD$. The product $(-B)(-D)= +BD$ was a recognition of the rule of signs that minus multiplied by minus becomes plus rather than the product of two negative numbers. Muhammad ibn Musa al-Khwarizmi (780-850), a medieval Islamic scholar working for the House of Wisdom in Bagdad, properly used the signs for the products of positive and negative signs (numbers) (e.g., Berggren, 2007). Klein (1953) provides possible justification for the rule of signs that the product of two minuses becomes plus drawing on a simple area calculation:

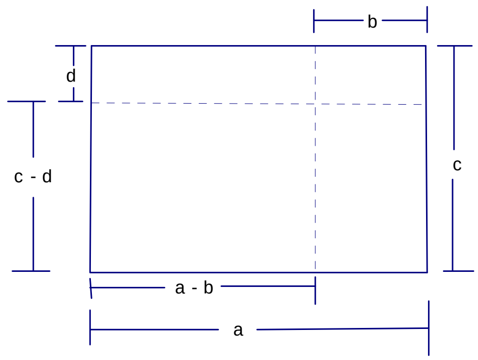


Figure 2.3. Justification of Rule of Sign Using Area (source: Klein, 1953, p. 26)

From Figure 2.3, the area of rectangle *(c-d)* by *(a-b)* can be obtained by subtracting the areas of rectangle *d* by *a*, and the one b by c from the area of rectangle *a* by c, and then adding the area of rectangles b by d (because rectangle b by d was subtracted twice as it is the common region to the rectangles a by d and b by c). Symbolically,

$(a - b)(c - d) = ac - ad - bc + bd$. From this, we can see why $(-b)(-d) = +bd$.

In sum, we have no evidence that the ancient Babylonians, Egyptians, or ancient Greek recognized negative numbers. The Chinese and Hindu mathematicians used some form of negative number concept in earlier times. These early forms of negative numbers appeared to be limited in scope (e.g., representing debt/loss). Until the 19^{th} century, negative numbers in Europe were often regarded as absurd or "false roots" when they arose as solutions to algebraic equations. The history of negative numbers suggests that people have struggled with the counterintuitive nature of the negative number concept.

The Pythagorean Theorem

The theorem bears the name of Pythagoras of Samos, a Greek philosopher and founder of a society known as Pythagoreans. He is believed to have lived around the 6^{th}

to 5th century B.C. Pythagoras is a mystical figure in many ways. He "was one of those figures which impressed the imagination of succeeding times to such an extent that their real histories have become difficult to be discerned through the mystical haze that envelops them" (Cajori, 1919, p.17). No known record was left behind by either Pythagoras (Osserman, 1995) or his earliest followers about Pythagorean teachings (Kahn, 2001); what we know about him came from second- and third-hand sources (Schaaf, 1978). The first known book about Pythagorean philosophy was written by Philolaus, a Pythagorean who lived decades after Pythagoras (Kahn, 2001). This book, considered lost now, is believed to have been the source for a good deal of what other ancient writers reported about Pythagorean philosophy (Cooke, 1997). Most historical sources about Pythagoras note that he founded, the Pythagorean brotherhood, a group of followers devoted to learning and contemplation. The Pythagoreans were secretive and customarily gave credit to Pythagoras for their accomplishments. There are sources (e.g., Burkert, 1972; Burnyeat, 2007) that critically question whether what has been attributed to Pythagoras, including the theorem, was really his.

Forms of the Pythagorean Theorem might have been known to other civilizations before Pythagoras. Historical sources suggest that the Babylonians knew about the theorem before Pythagoras (e.g., Cooke, 1997; Alexander, 2001). The clay tablet "Plimpton 322," which dates back to about 2000 B.C., contains Pythagorean triplets (Swetz, 2001, Cooke 1997). Although no early texts of the ancient Egyptians mention any case of the theorem before 300 B.C., surveyors laid out accurate right angles and used rope measurement (Cooke, 1997). Ancient Egyptians are believed to have used a 3-4-5 right triangle as in land measurement. Construction of right triangles using cords of lengths 5, 12, and 13 was known in India by at least the 5th century B.C. (Heath, 1960, V.I, p. 145-6; Coolidge, 1963). Classical Chinese works such as the *Zhou bi suan jing* (a collection of ancient Chinese texts on astronomy and mathematics) and *Jiuzhang*

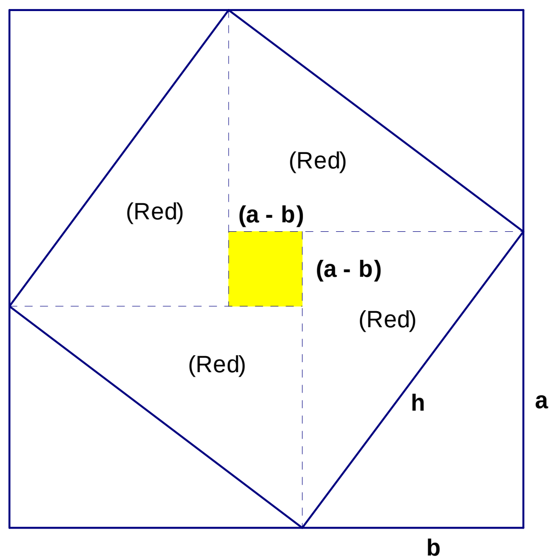


Figure 2.4. The Hypotenuse Diagram/Xian Tu (source, Dauben, 2007, p. 222)

Suanshu (Nine Chapters on the Mathematical Arts) contain the Gougu rule (Chinese version of the Pythagorean theorem) and its application in various problems (e.g., Lǐ Yǎn & Dǔ Shírán, 1987; Cullen, 1996; Kangshen, Crossley, & Lun, 1999; Dauben, 2007). The specific period in which the Gougu rule first came into use does not seem to be known. The Zhou bi, which contains the Gougu rule, probably was developed during the Han dynasty (206 B.C. – 220 A.D.) (Cullen, 1996). However, later commentaries were likely added to the Zhou bi by Zhao Shuang in the third century A.D. and by Zhen Luan in the sixth century A.D., further illustrating the Chinese use of the *gou-gu* (base-height) relation to the *xian* (hypotenuse) of a right triangle. For example, in Zhen Luan's commentary on Zhao Shuang, the diagram is referred to as xian tu, "hypotenuse" diagram. The Chinese computed the 'hypotenuse area' as the four *(red*) triangles plus the square in the center (see Figure 2.4). That is, in modern terms:

$$h^2 = 4(\frac{1}{2}ab) + (a-b)^2 = a^2 + b^2.$$

There does not seem to be solid evidence that Pythagoras or the Pythagoreans were the first to discover or prove the theorem. How did the theorem come to be known by Pythagoras' name? One speculation is that Pythagoras or his followers might have been the first to state the relationship as a general rule (Kahn, 2001). Heath (1960, V.I, p.144) notes that some writers (e.g., Plutarch around the first century A.D., Athenaeus around the third century A.D.) told stories that Pythagoras sacrificed an ox to celebrate his discovery, but the source they quoted did not mention specifically that the sacrifice was for the theorem. The story becomes even more dubious when contrasted with the alleged Pythagorean tradition of vegetarianism (Burkert, 1972). Ovid's poem *metamorphoses*, book XV, reflects Pythagoras teachings on the values of vegetarianism (Simpson, 2001; Kahn, 2001). As Kahn notes, the Pythagorean vegetarian dietary restriction perhaps was grounded on the doctrine of transmigration. Based on extensive analysis of historical sources, Burkert in his work *Lore and Science in Ancient Pythagoranism* provides a critical analysis of Pythagoras. He concludes that there is no evidence that Pythagoras proved the theorem. Speculating on the possible role Pythagoras might have had in developing the theorem that bears his name, Burkert cites Neugebauer (1928), who noted that the Babylonians routinely used the "Pythagorean theorem" before the Greeks and Pythagoras might have played an intermediary role by introducing it to the Greeks. However, he wonders how such an account alone could explain Pythagoras' fame. He further contends that Pythagoreans were more concerned with numbers than geometry and the Pythagoreans' representation of numbers in figures of pebble-like counters might have led to the observation of square number relations as can be noted for the 3, 4, and 5. The amazing thing is how ancient authorities unanimously attributed the theorem to Pythagoras based on questionable evidence (Heath, 1960). Unlike Burkert, Heath appears more sympathetic to Pythagoras when he says, “I would not go so far as to deny to Pythagoras the credit of the discovery of our proposition; I like to believe that tradition is right, and that it was really his” (p .145).

The earliest Greek recorded proof of the Pythagorean Theorem is provided in Euclid I.47. Drawing from Proclus’s appreciation of Euclid’s proof, Heath conjectures that Euclid’s proof was new. In Book I proposition 47 of the *Elements*, Euclid states, “*In right-angled triangles the square on the side opposite the right angle equals the sum of the squares on the sides containing the right angle*.” Euclid uses triangle congruence in proving this theorem. Using modern notations, the proof can be put as follows (Figure 2.5):

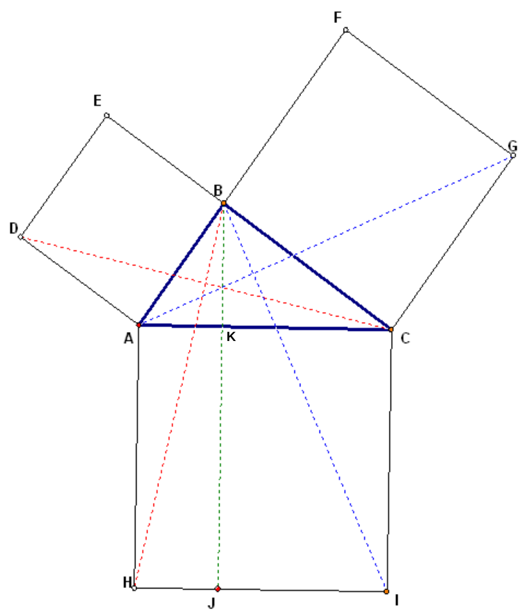


Figure 2.5. Euclid’s Proof of Pythagorean Theorem

Consider triangles ABH and ADC.

$\overline{AB} \equiv \overline{AD}$, sides of square ABED; $\overline{AH} \equiv \overline{AC}$, sides of square ACIH.

Angle HAB ≡ angle CAD, both are a right angle plus common angle BAC.

Thus, ΔABH ≡ ΔADC, by side-angle-side. ………………………………(1)

So, area of ΔABH = area of ΔADC.

But are of ΔABH = ½*AD*AB (half of area of square ABDE) and area of ΔADC = ½* AH*HJ (half of area of rectangle AHJK).

Hence, area of ABDE = area of AHJK. ……………………………………(2)

Similarly, we can show the congruence of ΔACG and ΔBCI, and that each of them is of half of square BCGF, and rectangle CIJK, respectively.

So area of BCGF = area of CIJK.……………………………………………(3)

From (2) and (3), it can be deduced that

area of ABDE + area of BCGF = area of AHJK + area of CIJK

=> area of ABDE + area of BCGF = area of AHIC

i.e., $AB^2 + BC^2 = AC^2$.

Euclid also provides a generalized Pythagorean Theorem in which areas of similar plane figures are considered on the sides of a right triangle instead of just squares. This is stated in Euclid VI.31: *In right-angled triangles the figure on opposite side of the right angle equals the sum of the similar and similarly described figures on the sides containing the right angle*. That is, the shapes on the sides do not have to be square; they can be any similar plane figures.

How the theorem became associated with Pythagoras is a mystery. The historical sources I reviewed suggest that there are no known records left by Pythagoras or his followers indicating that Pythagoreans discovered or proved the theorem. Our knowledge of him came from second- and third-hand sources. We are told that the Pythagoreans gave credit to Pythagoras for their findings. Therefore, it is conceivable that some Pythagoreans might have discovered the theorem and named it after Pythagoras.

Ancient Greek proof of this theorem is well known by way of Euclid's *Elements*. Forms of the theorem appear to have been known by other civilizations. The Babylonian clay tablet, which dates back to about 2000 B.C., and contains 'Pythagorean' triplets, has been found. Some (e.g., Cooke, 1997) suggest that ancient Egyptians used some right triangle triplets in land measurement. Classical Chinese documents on mathematics such as *Zhou bi suan jing* show they used the Gougu rule (Chinese version of the Pythagorean theorem). Some sources (Heath, 1960, Vol.I; Coolidge, 1963) mention use of Pythagorean triplets in ancient India.

Sources (Gullberg, 1997; Brown, 2003) suggest that the Pythagorean theorem may have laid the foundation for the discovery of incommensurables (later irrational numbers), particularly through the problem of finding the diagonal of a square unit.

Irrational Numbers

Arcavi, Bruckheimer, and Ben-Zvi (1987) identify three phases in the development of irrational numbers: the discovery of incommensurability, the 16^{th} century uncertainty about irrationals, and the formalization of irrational numbers.

The discovery of the incommensurable quantity is ascribed to the Pythagoreans (Coolidge, 1993). According to Pythagorean philosophy, "all is number;" they believed that all lengths (magnitudes) were commensurable (Choike, 1980). That is, they thought two segments always had a third segment that is a common measure of both (each given segment could be produced a whole number of times of the common measure). In other words, if two magnitudes are commensurable, they have a rational ratio.

According to legend, the discovery of incommensurables shocked the Pythagorean doctrine. Some stories indicate that Hippasus of Metapontum, a member of the Pythagorean brotherhood who lived around the fifth century B.C, was punished for revealing the discovery of incommensurable magnitude (e.g., Jones, 1994; Brown, 2003) that contradicted the Pythagorean beliefs about numbers. Grattan-Guinness (1998) argues that this incident may have been inaccurately interpreted by later Greek writers and thus ahistorical because earlier commentators such as Aristotle did not mention how the discovery provoked the Pythagoreans. “It is not attested in any ancient source that Hippasus discovered the irrational, or divulged this knowledge” (Burkert, 1972, p. 457). According to Burkert, there are diverse accounts that might have led some people to view Hippasus as the discoverer of irrationality. An account by Aristotle suggests that Hippasus was the first to “publish and construct” the “sphere of the twelve pentagons,” the dodecahedron, and he was punished for this (p. 457). Burkert cites Kurt von Fritz (1945), who interpreted Hippasus’ alleged discovery of the dodecahedron as an incident that led him to find the incommensurability of the diagonal of a regular pentagon to its side. Hippasus supposedly discovered that the ratio of the diagonal to the side of a regular pentagon formed the golden ratio (Choike, 1980). Other sources (e.g., Gullberg, 1997; Brown, 2003) suggest the problem of finding the diagonal of a unit square was the problem that led to the discovery of incommensurables.

However the discovery was made, the Greeks knew about incommensurable magnitudes at least around the time of Plato (Cooke, 2005). The work of Theodorus of Cyrene (fifth century B.C.) is usually cited as one of the evidences. Plato’s (trans. 1996, p. 9) dialogue *Theaetetus* mentions the work of Theodorus:

> Theodorus here was drawing diagrams to show us something about powers—namely a square of three square feet and one of five square feet aren’t commensurable, in respect of length of side, with a square of one square foot; and so on, selecting each case individually up to seventeen square feet. At that point he somehow got tied up.

This excerpt describes what is now known as the Wheel of Theodorus. This is a spiral formed by right-angled triangles starting with a unit leg of an isosceles right triangle followed by a series of right triangles whose legs are a unit and the hypotenuse of the previous triangle in the series, and ending with the triangle with hypotenuse $\sqrt{17}$ (see Figure 2.6).

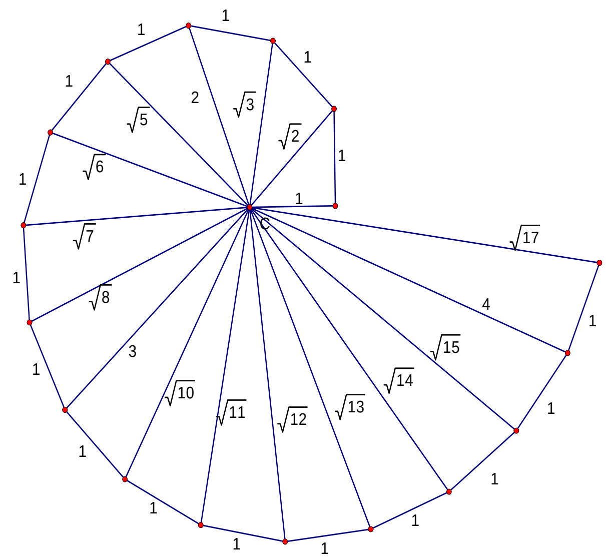


Figure 2.6. Wheel of Theodorus

Euclid's *Elements* provides more evidence that ancient Greeks knew about incommensurables. Euclid X def. 1 defines incommensurables as those magnitudes that cannot have a common measure. Further, Euclid defines the concept of *extreme and mean ratio* with respect to line segments (Book VI, definition 3) and demonstrates how to cut a given line segment into such a ratio. Although Euclid demonstrates the construction of *extreme and mean ratio* of a line segment geometrically, it is conceivable to suggest on the basis of the Euclidean algorithm (proposition XI.2), as Joyce (1998) indicates, that the Greeks could determine that the segments in extreme and mean ratio were incommensurable.

A line segment is cut into extreme and mean ratio if the ratio of the whole segment to the larger part is the same as the larger part to the smaller part. That is, $\overline{AB}$ is cut in extreme and mean ratio at C (see Figure 2.7) if

$$\frac{AB}{AC} = \frac{AC}{CB}\text{, where AC > CB.}$$

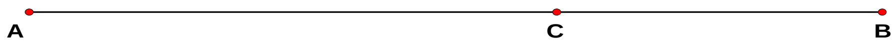


Figure.2.7. Golden Ratio

In modern terms, this ratio is known as the *golden ratio*, usually denoted by the Greek letter φ. Without loss of generality, assume AB is a unit long.

Then $\varphi = \frac{1}{AC} = \frac{AC}{1-AC}$. After some algebra, it can be shown that $\varphi^2 - \varphi - 1 = 0$.

The solution is $\varphi = \frac{1+\sqrt{5}}{2}$, which is irrational.

The golden ratio has fascinated people in various areas because of its aesthetic nature. Livio (2003, p. 1) notes, “Little did Euclid know that his innocent-looking division would preoccupy mathematicians, physicists, botanists, psychologists, and artists for the next few millennia.” The golden ratio attracted the attention of renaissance artists such as Da Vinci. Figures with a golden ratio have some inherent qualities that make them aesthetically pleasing (Huntley, 1970). Huntley cites such experimental studies as Fechner (1876), Wittmar (1894), Lalo (1908), and Thorndike (1917) suggesting a popular preference for golden rectangles over other types of rectangles. However, these findings are not generally supported. Davis and Jahnke (1991) found strong preference for figures divided into unity ratio and found no evidence of preference for ratios near or at the golden section. They concluded that symmetry seemed a more determinant factor than the golden ratio in aesthetic choice in their study.

Although the golden ratio and other irrational values such as pi were in use for a long time, their status as numbers was a late a development. Interest in irrational values grew during the 16th century after the introduction of decimal fractions (Arcavi, Bruckheimer, & Ben-Zvi, 1987; Klein, 1953). Klein notes that conversions from rational numbers to decimals led to the realization that the decimal value can be a finite or an infinite decimal but periodic (repeating). He speculates that this realization led people to think of decimals that can be "aperiodic" (neither terminating nor repeating), which are decimal forms of irrational numbers. However, until the late 19th century, the focus seemed on using them in calculation without much regard to their nature (Klein, 1953, p. 33):

> Historically, the same thing happened with irrational numbers that, as we have seen, happened with negative numbers. Calculation forced the introduction of the new concepts, and without being concerned much as to their nature or their motivation, one operated with them, the more particularly since they proved to be extremely useful.

Toward the late 19th century, the general foundation for irrational numbers was laid by German mathematicians Georg Cantor (1845-1918) and Richard Dedekind (1831-1916). The Cantor-Dedekind axiom states that points on a line can be put into one-to-one correspondence with real numbers (Boyer, 1968; Gazale, 2000). Based on this axiom, Dedekind defined irrational numbers. His idea can be summarized roughly as follows. Rational numbers alone do not cover all points on the number line. What ancient Greeks recognized as incommensurable magnitudes can be represented by points on the line that are not occupied by rational numbers. A line filled with rational points leaves "infinitely many gaps," which can be associated with irrational values. Since the notion of incommensurable as a ratio of magnitudes has inherent dependence on geometry, Dedekind's notion is to define a number that is continuous but free of the geometric restriction in order to facilitate arithmetic and other analysis. Thus, Dedekind defines irrational numbers based on the continuous nature of a straight line and the discontinuous nature of rational numbers. According to Dedekind (1901), any point *p* on a line separates the domain of rational numbers into two classes: left and right of point *p*, and call them Class A and Class B respectively. Every number *a* in A is less than every number *b* in B. The number associated with the point *p* may be in Class A, in which case it is the greatest number in that class, or if *p* is in Class B, it is the least element among the numbers in B. In either case, *p* is a rational number. Dedekind shows that p may be neither in A nor B; in this case, it is an irrational number. Dedekind calls this point of separation a *cut* (Schnitt). His definition of irrational numbers often is referred to as Dedekind-cut.

The concept of proportion formulated by Eudoxus of Cnidus (408-355 B.C.?), which is found in Euclid's Book V of the *Element*, is not far from the 19th century definition of real numbers. It separates the rational numbers $\frac{m}{n}$ into two classes, with the provision that $ma \le nb$ *or* $ma > nb$ (Boyer, 1968). This idea lies in definition 5 of Book V of Euclid, the essence of which can be stated on the basis of the equality of ratios (i.e., $\frac{a}{b} = \frac{c}{d}$, if for any given integers m, and n, whenever $ma \le nb$, $mc \le nd$; *or* when $ma \ge bn$, $mc \ge nd$). This definition includes proportionality of incommensurable magnitudes.

To sum up, the discovery of the incommensurable quantity laid the foundation for irrational numbers. The Pythagoreans are believed to have first uncovered the problem that two magnitudes may not have a common measure (may not have a rational ratio). Although irrational values such as pi and the golden were used in practical matters for a long time, their status as part of real number system was not established until the 19th century. In the latter part of this century the foundation for irrational numbers was laid through the works of German mathematicians Georg Cantor and Richard Dedekind, among others.

The interplay between geometry and arithmetic offers an intriguing explanation about the origin and development of the irrational number concept. If natural numbers arose from the need to count discrete quantities, simple operations (arithmetic) on natural numbers may have contributed to the invention of integers and rational numbers. However, geometry, specifically the need to measure magnitude, played an important role in the development of irrational numbers. The need to measure was not apparently the sole drive. If that was the case, why did the Greeks worry about incommensurables? Why not use approximation? Cooke (2005, p. 199) notes that the absence of a place-value number system in ancient Greek mathematics may have stimulated the creation of irrational numbers because place value provides an approximation of square roots for practical purposes. He argues this by citing the sexagesimal approximation used by the Babylonians for the diagonal of a square with side 1 unit as found in Yale Collection 7289. He observes that ancient Egyptian, Mesopotamian, Chinese, and Hindu texts do not seem to contain any discussion of "numbers" whose expansion is non-terminating. Not having a place-value system may have contributed to this interesting mathematical discovery.

Chapter Three: Research Methodology

In this chapter, I describe the assumptions, conceptual framework, and methods of this dissertation study. The personal assumptions and biases I brought to the study are presented under research positionality. The conceptual framework covers the qualitative research paradigm underlying the research methods employed in this study. I describe the content analysis and interviews used under the method section.

RESEARCH POSITIONALITY

In a qualitative study such as this, the researcher needs to reflect on and communicate the interests, values, experiences, and purpose he brings to the study because these factors influence the interpretation and analysis of the study (Jones et al., 2006, p. 125). In this respect, I find it important to state the reasons behind my choice to pursue this topic. I believe that the history of mathematics is very important in learning mathematics. I hold this stance based on my experience with school mathematics as a student and as a scholar reading and studying literature on the topic.

In my learning experience with school mathematics, in pre-college and at the college level, I occasionally found myself wondering about the origins and developments of the mathematical ideas I learned. Although there was not any purposeful use of the history of mathematics whether by teachers or in textbooks, I found the rare historical references in school mathematics very interesting and enduring. Unfortunately, school mathematics often did not provide the history of mathematical ideas.

School mathematics was detached from any meaningful history. From my recollection, a typical school math lesson focused on presenting the established knowledge (e.g., by defining concepts, and explaining procedures) and demonstrating how you can use the knowledge (e.g., by giving examples, and exercising on problems found in textbooks). The students and our teachers never bothered to raise questions concerning the development of ideas that we learned. It would seem irrelevant to raise the following after the teacher had introduced, "irrational numbers are numbers that cannot be written in the form of $\frac{a}{b}$, where a & b…"

- How did people think of irrational numbers in the first place?
- Did they mean the same thing to whoever thought of them as they do today?"

Regardless of my curiosity, I did not make serious effort to study the history of mathematics while I was an undergraduate student or before. I think one of the main reasons is the very ahistorical nature of school mathematics I experienced. The compartmentalization of ideas into school subjects (math, history, physics, etc.) provides some imaginary boundaries among the subjects. I think this delineation reinforces certain belief about what subject matter belongs to which school subjects. I think this reinforces the belief that historical matters of mathematics may belong in the subject of history, but not in the school mathematics. The ahistorical nature of school mathematics somehow subconsciously reinforces a certain view of what mathematics is about.

As a graduate student, reading literature covering the history and philosophy of mathematics and science (e.g., ICMI study, 2002; Calinger, 1996; Duschl, 1990; Lakatos, 1976; Kuhn, 1996), I found that many of the arguments concerning the role of history in the development of mathematical or scientific knowledge and their implications for learning these subjects spoke to my experiences. These sources reignited my interest in this topic. I agree with the thesis that the history of mathematics can be useful in developing our view of mathematics and a better understanding of concepts and theories of it (ICMI study, 2002, p. 63). In Chapter two I presented a review of the literature supporting this thesis. Arguments about how the history of mathematics can help develop views about mathematics were presented by drawing on philosophical grounds and pedagogical accounts. I also presented reviews concerning the psychological and socio-cultural explanations for the need of history.

CONCEPTUAL FRAMEWORK

The conceptual framework underlying the study provides a lens through which the research aim, method, and results are viewed (Muhall & Chenail, 2008). The set of assumptions within a particular conceptual/theoretical framework informs how the study is conceived, designed, and implemented (DeMarrais, 2004). In qualitative studies, theoretical framework explaining the phenomenon under the study should not be confused with the conceptual framework for the methodology. The former may be nonexistent, but a substantial body of work pertaining to methodology (Anfara & Mertz, 2006) is available. The conceptual framework for a research method varies depending on assumptions about the nature of knowledge, how knowledge is produced, and the role of the researcher and participants in the production of knowledge. Guba and Lincoln (2005) identify various theoretical perspectives or paradigms such as positivism, postpositivism, critical theory, and constructivism.

This dissertation study aims at exploring the incorporation of the history of mathematics into school mathematics to understand how a contemporary curriculum addresses the issues and to understand the perspectives of teachers and students about such incorporation. Understanding aspects of human activity from the perspective of those who have experienced the phenomenon (Jones, Torres, & Arminio, 2006) is consistent with the constructivist paradigm. Constructivist/interpretivist perspective/framework supports a research approach that purports to “make sense of experience, build pattern of meaning and relationship that are linked to well-described situations, and communicate what has been learned in ways that are connected to context” (Tobin, 2000, p. 510).

The nature of this study calls for a qualitative (interpretivist) method. The arguments for the incorporation of the history of mathematics into school mathematics are premised on the idea that such an approach can help students and teachers develop their views and understanding of mathematics by adding more human dimensions to school mathematics. Barbin (1996) provides qualitative accounts that mathematics teachers who become interested in the history of their subject often report that the experiences they gain influences their view of mathematics education. Barbin’s conceptual framework (Figure 2.1) linking the history of mathematics, views about the nature of mathematical knowledge, and school mathematics offers a loose framework for exploring this qualitative relationship—in exploring the teacher and student accounts about issues linking the history of mathematics with school mathematics.

Through the literature I reviewed and from my personal experience, I contend that the history of mathematics is important for learning mathematics. This leads to the main question for this study: Is history really important in contemporary middle school mathematics? I address this question partly by exploring the perspectives of those who are directly involved (curriculum experts, teachers, and students). I carry out the exploration by examining a selected case of curriculum, namely the Connected Mathematics Project (CMP). CMP is a National Science Foundation (NSF) supported middle school curriculum that was initially developed during the years 1991 to 1996 (Lappan, Fey, Fitzegrald, Friel, & Phillips, 2006). As Lappan and Phillip (2001) indicate, this curriculum is aligned to the NCTM content and process standards. CMP has been designated as exemplary by the Mathematics and Science expert panel at the U.S. Department of Education (U.S. Department of Education, 1999). This curriculum contains some historical references that help establish talking points for this study. For this study, CMP curriculum serves as what Stake (2005) refers to as an instrumental case study–a case studied to provide insight about an issue, which in this case is incorporating the history of mathematics.

METHODS OF DATA GATHERING AND ANALYSIS

The CMP textbooks and interviews with teachers, students, and experts provided the primary data for the study. Analysis of CMP textbook content focused on the historical aspects associated with four concepts (from four units). Interviews with curriculum experts, teachers, and students were conducted and analyzed focusing on issues about the nature and purpose of historical aspects in the textbook, use in the classroom, and the views of the interviewees on the importance of the historical aspects.

Content Analysis

The analysis focuses on four units from the first (1998) CMP edition. The CMP curriculum comprises 24 units, eight units for each of the grades six, seven, and eight. Based on grade level and topic, there is a certain recommended order of using the units. The four CMP units chosen include Bits and Pieces I (grade six), Covering and Surrounding (grade six), Accentuate the Negative (grade seven), and Looking for

Pythagoras (grade eight). These units focus on fractions, circumference and areas of circles, negative integers, and the Pythagorean Theorem with irrational numbers, respectively. These topics were chosen because they are among the core topics in grades 6 to 8 mathematics curriculum as indicated by the National Council of Teachers of Mathematics and Texas Essential Knowledge and Skills. They are introduced more or less for the first time at the respective grade levels indicated; therefore, if any historical references are used they probably will be found in introduction.

The purpose of the content analysis was to explore the nature of historical aspects in each unit. While there are no hard and fast rules used by qualitative researchers for analyzing written texts, methods such as Membership Categorization Analysis (MCA) were used (Peräkylä, 2005). MCA helps analyze text using selected categories based on certain conditions. For this study, three categories of historical aspects were used to describe the nature of historical incorporation in the units (see Table 3.1). The ICMI study (2002) identifies these three different but complementary ways in which history might be incorporated into math education:

- *Direct historical*—historical facts such as dates, places, names, and brief biographical sketches of key people associated with the development and/or earlier use of the topic/concept
- *Implicit historical*—mostly problems with situations that present investigation of the fundamentals of the core concept, with some cue of historical elements (but may not be obvious); in other words *problems/ideas inspired by history* of the concept in question
- *General historical*—socio-cultural history of the context in which the topic is thought to have emerged/used earlier

This classification is not mutually exclusive; a certain historical text may be classified into more than one of the above categories. However, a historical text would be put into one category on the basis of the primary emphasis implied by the content or context–whether the emphasis is on historical fact, problem solving, or socio-cultural aspects.

UNITS (TOPICS) HISTORICAL ASPECT	Bits & pieces I (Fractions)	Covering & surrounding (Circles)	Accentuate the negative(negative numbers	Looking for Pythagoras (theorem & irrational numbers)
Direct historical				
Implicit historical				
General historical				

Table 3.1. Outline for Analyzing the Historical References in Four CMP Units

Interviews

Interviewing key people–curriculum developers, teachers, and students—is important in exploring how historical issues are addressed in school mathematics. Interviewing has become one of the most common and powerful tools with which we try to understand our fellow humans (Fontana & Frey, 2005). Interviews can help us explore areas of reality such as subjective experiences that otherwise remains inaccessible (Peräkylä, 2005). Fontana and Frey note that qualitative researchers have increasingly embraced interviews not as neutral tools of data gathering but rather as active interaction between people leading to negotiated and contextualized text. In interviews, the researcher and participants tend to filter each session through unique experiences, beliefs, and assumptions about the research topic (DeMarrais, 2004). Thus, using interviews to produce useful data for a research topic is quite a challenging process. This process entails developing appropriate questions, recruiting interviewees, conducting the interview, and analyzing the data in a meaningful way.

Given the exploratory nature of this study, raising questions that allow the interviewees to provide extended responses is necessary. However, given the fact that the study focuses on historical aspects of certain concepts from a specific text, some sort of structure is imposed. Thus, the study employed semi-structure questions covering issues thought to be important in exploring the incorporation of the history of mathematics into school mathematics. Another important concern in determining and developing the interview questions is a broader definition of what historical incorporation means in this study. The ICMI study (2002) offered a good starting point about various ways of incorporating history into school mathematics. The ICMI study presents descriptions of various forms of incorporation (e.g., explicit and implicit) that can be brought through various means (such as the Internet, film, books and artifacts) into school mathematics. For this study, the definition is attuned more to incorporating the history into the (CMP) textbooks. The definition of incorporation of history here means the use of the three historical aspects (noted in the content analysis).

After developing initial questions, interviews with teachers and students were conducted. These pilot interviews suggested that historical incorporation does not seem to be a serious part of teachers' repertoires. Therefore, it was important to focus on the historical aspects in the textbooks as a ground for discussion. However, a few learning experience events were identified as issues to focus on with the teachers and students. That is, issues of how core concepts are introduced and how questions about the foundations of core mathematics concepts are addressed in instruction were identified as important for teacher and student interviews because they are a part of instructional practice that are familiar to them and also can lead to historical issues. The pilot interviews also helped attune later interviews to topics that were covered by each teacher at the time of the interview. Questions asking for the recall of specific events and experiences in detail can encourage fuller narratives (DeMarrais, 2004). Consequently, it was important to ensure that the CMP units selected for this study were covered around the time of the interview. The topics selected include fractions (grade six), negative numbers (grade seven), and Pythagoras Theorem and irrational numbers (grade eight).

The interview questions covered issues concerning the nature and purposes of the historical aspects in the CMP text and the perspectives of teachers and students on the use of these historical aspects. Interviews with the curriculum experts focused on issues dealing with the purposes and nature of the historical aspects. Questions for the experts included how they decided to include the historical aspects in the textbook, what their expectations were with regards to using such aspects, and what they think about the incorporation of the history of mathematics into the curriculum. Interviews with the teachers and students focused on exploring the use of the historical aspects in the textbook and more generally on their perspective on the importance of incorporating history in school mathematics. Specifically, the interview questions covered how concepts are introduced, how teachers and students deal with foundational issues such as the origins and development of the very ideas they cover in school mathematics, how they use historical aspects in the textbooks, and what they think about the importance of the history of ideas they learn (see interview guide Appendix A-1).

Interviews were conducted with three experts, eight teachers, and 11 students. The experts included two CMP authors and one director of consulting for the implementation of CMP. The teachers comprised one 6^{th}, two 7^{th}, four 8^{th} grade teachers, and one math specialist. The student interviewees included seven individual interviews (two grade six, one grade seven, four grade eight) and one group (of four grade seven students). I have provided brief interviewee profiles in chapter four (for the experts) and chapter five (for the teachers and students). All interviews, with the exception of the group interview, were recorded on audio tapes. The interviews with the curriculum experts were conducted via telephone. The interviews with teachers and students were conducted face-to-face. Roughly, an interview with an expert or a teacher took 45 to 50 minutes, while each interview with the students was about 20 to 25 minutes long. Interviewees were selected based on their informed position (experts with CMP or users of CMP) and their willingness to participate following my request (Appendix A-2a). For those who agreed to be interviewed, interview questions were sent via email (see Appendix A-2b for a sample) some days before the interview to give them time to reflect and come prepared.

Analysis/interpretation of the interview data is a continuous process with various steps—from field notes, to interview text, to research text, to final text (Denzin & Lincoln, 2005). The researcher has the flexibility to interpret qualitative data based on personal and professional experiences (Strauss, 1987). Qualitative literature (e.g., Gall, et al., 2003; Stake, 1995; Strauss & Corbin, 1990; Weiss, 1994) offer varying methods of analysis from more structured (e.g., categorizing/coding data into specific themes), to less structured (e.g., interpretive narratives).

Broadly, I followed what Weiss (1994) refers to as issues-focused analysis as a guide for the analysis in this study. Weiss identifies four processes in issue-focused analysis of interview material: coding, sorting, local integration, and inclusive integration. I recorded field notes for my interview encounters summarizing my impression of the interviews. I transcribed each interview mixing verbatim with occasional summaries. From the transcribed text, I made identified key points pertaining to the issues raised in the questions. Then ‘excerpt files’ containing the key points were formed. I have incorporated numerous excerpts of this type as exhibits in support of the accounts that I composed around the main issues embedded in the interview questions (presented Chapters four and five). The exhibits are used to provide an “immediately accessible” and “compact form” of the interview data so that the reader can draw meaning from what is constructed and also contrast this with her construction (Miles & Huberman, 1994).

Questions of goodness or credibility of qualitative studies such as this one are challenging but have to be addressed. The problem of assessing qualitative research has not been solved (Flick, 2002, p. 218). But, there are various ways to improve the credibility of such findings. The researcher should reveal the biases and assumptions and provide the necessary contextual information about the study in order to help the user make judgments on the goodness of the study (Jones et al., 2006). The researcher may use what Lincoln and Guba (1985) refer to as prolonged engagement, triangulation, and member checking to help reduce both misrepresentation and misinterpretations. Prolonged engagement involves the investment of sufficient time to learn about and analyze the data. I invested ample time playing and replaying recorded interviews, reading and analyzing transcripts interviews, and also reading and analyzing the CMP units. Triangulation involves using different sources and methods to verify information. I interviewed experts, teachers, and students. Most of the students interviewed were students of the teacher interviewees. I tried to pay attention to inconsistencies that transpired during common probes. I gave the experts and the teachers the opportunity to read and comment on the interview transcripts (member checking) (See Appendix A-3 for the email correspondence regarding comments I sought on the transcripts). While some commented, the majority did not respond.

THE SETTING

Interviews with teachers and students were conducted during the months of November and December, 2006 in Austin, Texas. The interviews with the experts were held in August, 2006. Expert interviews were conducted via telephone, while the interviews with teachers and students took place face-to-face in four different public middle schools in the Austin Independent School District (AISD). For confidentiality purposes the study uses the following pseudonyms for the middle schools: Cheetah, Jaguar, Leopard, and Tiger. Brief profiles of the interviewees and the schools are provided in Chapters four and five.

The teachers were identified through an informant. The main criterion was that the teachers used CMP books. A letter stating the purpose of the study and requesting participation was sent to each teacher (Appendix A-2a for a sample of the letter). The students were identified by the teachers in two of the schools. Further, the school counselor in one of the schools, from which most (five) of the student interviewees came from, helped identify student candidates. The main criterion given to the teachers and the school counselor was that the students not be shy to speak about their mathematics learning experience. Teachers, students, and parents of the student-interviewees signed written consent forms to indicate their voluntary participation.

Chapter Four

Incorporation of HD in the Curriculum:

The Case of the Connected Mathematics Project

The ways in which contemporary school mathematics curricula address the history of mathematics is a relatively unexplored area. A case study of curriculum with respect to this question can open the dialogue on this subject. This study considers the Connected Mathematics Project (CMP) to explore this issue. How does this curriculum incorporate historical aspects of core mathematics concepts? First, let me provide a brief note on why this curriculum was selected for this study before exploring the question. There are two main reasons for selecting this curriculum:

- CMP is one of the standard-based curriculum reforms (i.e., aligned to NCTM standards) that has been widely used in many school districts in the United States. CMP was designated as an exemplary program by a mathematics and science expert panel of the U.S. Department of Education (1999).
- CMP has some features such as *did you know* segments that contain historical notes. This feature provides important talking points for this study: What is the nature and purpose of the historical notes in curriculum? And, in general, how seriously was including history of mathematics considered in the curriculum development?

According to Lappan and Phillip (2001), CMP is aligned with the NCTM content and process standards. Contents are organized to enable students to develop knowledge and skills in *number operations, algebra, geometry, measurement, and data analysis and probability*. “The four overarching goals in the NCTM standard—Problem Solving, Communication, Reasoning, and Connection—serve as the major process goals for CMP" (p. 145).

CMP was developed after extensive field testing and is in use in thousands of school districts in the United States (Lappan et al., 2006; Schneider, 2000). From a research study comparing CMP and non-CMP students, Lappan et al. argue, "CMP students do as well as, or better than, non-CMP students on tests of basic skills. And CMP students outperformed non-CMP students on tests of problem solving abilities, conceptual understanding and proportional reasoning."

CMP has two editions: The first was published in 1998, and the second was published in 2006. This dissertation study focuses on the first edition, which was still in use during the academic 2006/07 in the schools (AISD) where data for this study was gathered. My preliminary examination suggests that there are not substantial changes with respect to the historical aspects between the two editions. But there some changes: Some historical notes (it appears) are eliminated from the second edition. However, some internet links that lead to historical references and other supporting information have been added to the textbooks of the second edition.

The CMP curriculum has 24 units, eight units for each grade (6, 7, and 8). CMP recommends a sequence in which the units should be taught. The overall instructional approach in each unit is the same: investigation, application, connection, extension, and mathematical reflection. Students solve problems based on a given situation, utilize the given problem as a means to understand the underlying math idea, use this idea to solve problems related to real-life problems or other math ideas, and finally reflect on the big picture—the underlying math idea.

On the basis of instructional sequence and the connections among the units, I find it justifiable to focus on the following units (with the mathematical idea) for the purpose of my study: *Bits and Pieces I* (fractions), *Covering and Surrounding* (circles), *Accentuate the Negative* (negative numbers), and *Looking for Pythagoras* (the theorem and irrational numbers). One of the main reasons for choosing these units is that the mathematics ideas are introduced for the first time within the CMP curriculum. As such, I assume that the historical aspects are likely to be found in these units. I also find these ideas intriguing because they have some elements within them that seem to defy common sense or that seem counter-intuitive, especially for middle school students. For instance, how can students appreciate fractions (often times less than 1) as a number? How would it make sense to them that two fractions (both less than 1) can have a quotient greater than 1? Negative numbers sound even more complicated: How can less than nothing be a number? How do student reconcile the notion that irrational numbers do not have exact decimal representations yet they are told they are numbers? These questions suggest that these topics present conceptual challenges for students. Given the fact that a child's world of numbers is dominated with associations to things that can be counted, fractions, negative numbers, and irrational numbers may not register naturally in their number concept. I have covered conceptual difficulty associated with these concepts in Chapter two. I think that students can benefit from learning these concepts with some historical background. Thus, an analysis of how CMP has incorporated HD pertaining to these topics will be the focus of the present chapter. In this chapter, I will be concerned with three major questions:

- What is the nature of HD incorporation in the CMP units?
- How was HD considered in the development of CMP?
- What was the expectation of the authors in regard to using HD in the units?

ANALYSIS OF HISTORICAL ASPECTS IN SELECTED CMP UNITS

The ICMI study (2002) identifies three different but complementary ways of incorporating history into math education. I adapt such characterizations in my analysis. I use the following categories to analyze and discuss the historical aspects in the CMP units:

- *Direct historical*—historical facts such as dates, places, names, and brief biographies that are associated with the development and/or earlier use of the concept in question
- *Implicit historical*—problems and situations that present an investigation of the core concept with some cues from historical elements (i.e., this category contains *problems/ideas inspired by the history* of the concept in question)
- *General historical*—socio-cultural history related to the context in which a concept is thought to have emerged and used in its history

This categorization is not mutually exclusive. I may classify a certain historical text into more than one of the above categories; however, I will limit the assignment of certain historical reference mostly to a category based on the emphasis reflected in the content—whether the emphasis is on historical fact, problem solving, or sociocultural aspects.

As noted, the CMP units contain *did you know* segments. The historical content in such segments is primarily *direct historical* and *general historical* aspects. Besides the *did you know* segments, very few historical notes are found in the introduction or body other parts of units. The *implicit historical* category is somewhat problematic because my analysis of whether contents belong to this category is based on indirect cues from the context of the problem. This analysis is subjective. By analyzing the content using the classification noted above, I do not intend to imply that CMP was purposefully designed to incorporate historical aspects along the stated categories. I impose the analysis on the content using the stated classification whether intended as a part of the curriculum or not. The content analysis along these categories also provides a general framework for what historical incorporation means in this study. I have appended copies of selected pages from the four units analyzed. I have marked the categories with brief comments on the appendices describing why certain text represents a certain category. **Table 4.1** below presents my classification of the historical aspects in the reviewed units.

CONCEPT (TOPICS) / HISTORICAL ASPECT	Fractions (Bits & pieces I)	Circle (Covering & surrounding)	Negative numbers (Accentuate the negative)	Pythagorean theorem & irrational numbers (Looking for Pythagoras)
Direct historical	B-2[1] B-3 B-4	C-4	D-2	E-2
Implicit historical	B-3	C-2 C-3	D-3 D-4	E-3 E-6 E-7 E-8 E-5
General historical	B-5			E-4

Table 4.1. The Nature of Historical References in the Selected Units of CMP

Fractions (Unit: Bits and Pieces I)

The learning goals in this unit include developing an understanding of fractions, decimals, and percents; understanding the relationships among these concepts; and

[1] The numbers in the table refer to appendices numbers, e.g., B-2 for Appendix B-2 (refer to the Appendix section for their content).

promoting problem solving using these concepts (see Appendix B-1 for overview, goals, and organization of this). The unit presents problems involving the representation of rational numbers in various forms (e.g., number lines and circular regions), changing from one form to another, and comparing and ordering of rational numbers.

For analyzing the historical content, I focus on how this unit introduces the concept of fractions, and what the nature of the historical references are (see Table 4.1).The unit's introduction contains the following *direct historical* note (Appendix B-2):

> People have been working on ways to talk about fractions and to do operations with them for a long, long time. As early as 1800 B.C. people were developing ways to communicate with each other about *parts or pieces of things* [emphasis added]. A document called the Moscow Papyrus, written in 1850 B.C., contains the first record of humans working with fractions" (Lappan et al., 1998, p. 3).

This introduction does more than simply provide historical facts. It offers a definition of the concept of fractions as "parts or pieces of things". This notion precedes the concept of fractions as a number (rational number). In a way, fractions are introduced as a part-whole notion. The introduction also provides the meaning of the word fraction as "a breaking," derived from the Latin word *fracio*.

The historical note given in Appendix B-3 can be considered direct historical, but it also provides some context for the following problems. Problems on equivalent fractions are presented immediately after this historical note. The historical note states that Hieroglyphic inscriptions of more than 4000 years ago show that Egyptian mathematicians used *unit fractions* (fractions whose numerator is 1) with the exception of $\frac{2}{3}$. As an example, it states that they would have expressed $\frac{2}{7}$ as $\frac{1}{4}+\frac{1}{28}$. Not only does this note present a historical fact that suggests that some system of fractions, although different from the contemporary conception, existed in ancient cultures but also the note is used as an introduction to equivalent fractions. However, the example of how

Egyptians expressed a unit fraction is provided in modern writing without any remark indicating what kind of writing or representation they used. Thus, we are left to wonder what students would make of this information: Do the students get the impression that ancient Egyptians used the same numbers as we do today? And, what are the long term effects on students' views of mathematics when content fails to address such issues (especially as such trends accumulate)? The first question would have been relevant to ask to the students in the study. Unfortunately, with the students I interviewed, this was not done as most said they had not covered this historical segment at all.

To be fair, the note with respect to decimals does a slightly better job of providing various ways of representation: "Throughout history mathematicians have used many different notations to represent decimal numbers"(see Appendix B-4).

This unit has some general historical aspects. These aspects are historical information about the social context not necessarily about math but may serve as motivation for learning the concept. For example, Appendix B-5 is of this type; it is part of the problem context for problems about *percents*.

The historical aspects in this unit (in the other units, too) are limited and not detailed. And, they are presented in the modern language (sense) without any remarks suggesting that differences in forms or meanings occurred in the evolution of a mathematical concept. The historical note in Appendix B-3, for instance, does not make it clear to students whether the Egyptians had a different number system and mode of representation. Since the unit has laid the ground by recognizing Egyptian unit fraction, supplanting information about how Egyptians and other civilizations wrote/represented fractions and how their meanings differed from ours can be beneficial to highlight the change over time. Barbin in the ICMI study (2002, p. 64) reminds us that

> Historical dimension encourages us to think of mathematics as a continuous process of reflection and improvement over time, rather than as a defined structure composed of irrefutable and unchangeable truths….

Circles (Unit: Covering and Surrounding)

Note that the scope of historical review of concepts in Chapter two does not cover circles. However, I decided to include analysis about circles here because the unit on circles contains some interesting historical references that asked the curriculum developers about.

This unit focuses on helping students understand perimeters and areas of plane figures, particularly polygons and circles (see Appendix C-1 for overview, goals, and organization of this unit). My analysis focuses on the circle part—how the circumference and area of a circle are introduced and if there are any historical aspects about circles.

With regards to circles, the unit has some interesting historical aspects. This unit provides interesting problem situations for finding the circumference and area of a circle that might be considered problems inspired by history. The introductory problems for finding the circumference and area of circles present opportunities for students to discover relationships that lead to the formulas for the circumference and area of a circle and the approximate value of π. Part of the problem situation for finding circumference states:

> Mathematicians have found a relationship between the diameter and circumference of a circle. You can try to discover this relationship by measuring many different circles and looking for patterns. The pattern you discover can help you develop a short cut for finding the circumference of a circle.

This problem requires students to measure diameters and circumferences of various circular objects and then present their measurements in tabular and graphic form. This allows them to investigate patterns and find relationships (see Appendix C-2).

The area of a circle is introduced with a problem called "*squaring"* a circle. In a nutshell, the problem involves estimating the area of a circle using a "radius square," a square with sides equal to the radius of the given circle. As Appendix C-3 shows, the circle and the radius square are provided on the grid drawing that makes approximating area easier in square units.

These problems are followed by the remark that students discover the "area of a circle is a *little more than 3* times the radius squared" and that "the distance around a circle is a *little more than 3* times the diameter". This provides some *direct historical* information about the number referred to as "a little more than 3" (see Appendix C-4)

These problems are definitely inspired by history. *Squaring the circle, also* known as the *quadrature of the circle*, is a famous historical problem that caught the attention of many professional and amateur mathematicians for many years (O'Connor and Robertson, 1999). I think the use of squaring a circle in this unit presents an opportunity to consider what is known as the *exhaustion method* in order to provide measurements of the circle. The famous Greek mathematician Archimedes (287-212 B.C) is believed to have employed this method to estimate pi. His method involved approximating the circle by inscribing and circumscribing it with regular polygons (hexagon, 12-gon, 24-gon,…96-gon). Some historical reference to Archimedes and his method might have been helpful because this method would provide some foundation for what students would find in higher level math concepts. The appropriateness of the inclusion of such a method for middle school mathematics (grade six) is debatable. But, simplified forms of such a method (i.e., using simpler polygons such as hexagons and octagons with the use technology, such as the Geometer's Sketchpad), allow measurements of the circle and the polygons to be obtained easily.

Negative Numbers (Unit: Accentuate the Negative)

Unit goals include: helping students develop understanding of integer operations (addition, subtraction, multiplication, and division of integers, comparing/ordering integers); modeling integers with a number line, thermometer scale, or chip board; using integers to solve problems; plotting integer points on a coordinate plane; and graphing linear equations using a calculator (see Appendix D-1 for overview, goals, and organization of this unit).

Unlike the unit on fractions, this unit does not have an explicit introductory statement containing a historical note. In fact, there is only one direct historical note (Appendix D-2**)** in the entire unit. One of my observations of the CMP units is that there is a lack of regularity regarding the inclusion of historical aspects in units. Some units have explicit historical references in the introductory section, and some do not. This irregularity may be due to not only the nature of the concept in a unit but also the emphasis—introduction or application—in a given unit. For example, the units *Bits and Pieces I* and *Bits and Pieces II* are both about fractions/rational numbers. While the first introduces the concept of fractions, the latter is about application of fractions. Not surprisingly, the first includes more historical aspects. But, the emphasis of a unit alone does not explain the discrepancy in historical notes across the units. Consider the unit *Bits and Pieces I,* which deals with fractions and the unit *Accentuate the Negative* that deals with negative integers. I do not see any reason why the unit on negative numbers would utilize less historical background than the unit on fractions. Later in this chapter I discuss the authors' accounts on how seriously they considered incorporating the historical aspects in the development of this curriculum; this may help elucidate the noted irregularities.

As noted in the beginning of this chapter, besides explicit HD, I examine implicit HD or problems inspired by historical aspects. For this unit, I focus on two facets: (a) the introductory problems and (b) the various models used in the unit to represent negative numbers. The unit poses these three introductory problems:

- If a negative number is subtracted from a negative number, then the difference is a negative number. Decide whether this statement is always true, sometimes true, or always false. Give examples to illustrate your findings.
- On Tuesday, a cold front passed through, causing the temperature to change -2^{o} F per hour from noon until 10:00 A.M. the next morning. The temperature at noon on Tuesday was 75^{o} F. What was the temperature at 4:00 P.M. Tuesday?
- In the first quarter of the big game, the Littleton Lions gain 5 yards on every play. They are now on their own 25 yard line. On what yard line were the Lions three plays ago?

Like most of the introductory problems in the other units, these (with the exception of the first one) problems focus on modern uses/application of the concept. The first question involves general operational understanding and assumes some prior knowledge of negative numbers. The second and third questions can be related to some of the early meaning of negative numbers. The fact that these problems associate negative numbers with a decrease (in temperature) or loss (of yards) has some similarity with the early concept of negatives thought to be associated with ideas of loss/debt. However, since these connections are not explicitly stated, the teacher serves as the likely agent to bring up such foundational issues if history were to be discussed. According to the teachers interviewed (discussed in the next chapter), the teachers usually did not address such foundational issues.

The unit uses various models that include a *game*, the *thermometer scale,* the *number line*, and *chip boards* to represent negative numbers. The teacher's guide for this unit (Lappan et al., 1998, p. 1a) reminds the teacher about the intended use of these models:

> To introduce students to work with integers, the context of winning points (represented by positive integers) and losing points (represented by negative integers) in a game is used. The game provides an entry point for discussing order and the comparing of integers as well as for developing the concepts of opposites, distances on a number line, and absolute value. The number line is used throughout the unit to model strategies for adding, subtracting, and multiplying integers. A board with chips of two colors is a second model students will use for addition and subtraction.

These models suggest that negative numbers may mean various things. The game uses problems that resembles the TV show *Jeopardy!*® A player earns points in a category if she answers correctly; otherwise, she loses the same amount of points. Thus, negative is introduced as *losing points*. The thermometer scale models negative numbers with measuring temperature. Here, negative numbers are introduced as numbers on the scale below zero. On the *number line* negative is represented by moving a given number of units to the left from a fixed point on the number line or as the opposite of a given positive number. The *chip board* models addition and subtraction of integers. This resembles an early Chinese representation of negatives and positives. The Chinese represented positive values using red counting rods and negative values by black counting rods (Kangshen, Crossley, & Lun 1999). In the *chip board* model used in the unit, 'black chips" are positive and "red chips" are negative (see Appendix D-3). The Chinese name "Jing-mei" in the text (see Appendix D-3) is an apt choice. I would say the way this chip board is used in the text suggests some implicit historical consideration.

But, how can such presumably historically inspired problems be useful? This earlier Chinese conception can offer an easier way to illustrate addition and subtraction of negative numbers. For example, I imagine that explaining -11- (- 5) as "taking away five red chips from 11 red chips" (demonstrated in Appendix D-3**)** would make more sense conceptually to a 7th grader than stating that -11 - (-5) = -11 + 5 = -6 (with the explanation that - (-5) becomes + 5 because a negative times a negative yields a positive). As Hefendehl-Hebeker (1991) notes, present-day students face similar hurdles to those encountered by people in the history of negative numbers. Historically inspired problems may help anticipate similar hurdles faced by students today. As I will discuss in the next chapter, the interviewed teachers did not describe using history in this manner. Most teachers noted they do not cover even the limited historical notes in their text let alone expand on the problems I identify as historically inspired (like the chip board).

A look at the historical background of negative numbers also leads to questions regarding the ways in which negative numbers are taught. For instance, the representation of negative numbers with models such as those used in this unit comes with limitations. Thomaidis (1993, p. 81) observes that an emphasis on interpretive models (e.g., credit-debt and temperature scales) does not help students "remove the obstacles that hinder the acquisition of the abstract concept of the number". The chip board model, for example, would not help explain why the product of two negative numbers is positive. In terms of the model, would red chips multiplied by red chips gives black chips make sense? The authors of CMP acknowledge this difficulty:

> Students develop rules for multiplying and dividing integers. As it is difficult to model multiplying or dividing a negative integer by a negative integer, students look for patterns and further develop their understanding of integers and of the operations of multiplication and division as a means for developing rules with integers (Lappan et al., 1998, p. 1g).

The unit avoids the model approach but instead uses operation patterns to introduce multiplication and division of two negative numbers. The multiplication pattern students learn for negative numbers is depicted in Appendix D-4**.** In the Teacher's Guide of the unit, it is stated, "The thinking about multiplying a negative by a negative is abstract and quite difficult to explain for most middle school students' mathematical understanding and language. These ideas need time and reinforcement to make sense." (Lappan, et al. 1998, p. 66d). The shift from the model approach to the operation patterns with no explanation to students leaves a hole. How do teachers and students fill this hole? This question needs further research study. I think the unit may benefit from an expansion of the historical note provided in Appendix D-2—addressing questions about the challenges with negative numbers. Moreover, offering some explanation like using geometry (such as the demonstration in Figure. 2.3) may help provide a case in which a product of two "negatives" is positive.

Some scholars (e.g., Fischbein, 2002; Freudenthal, 2002; Thomaidis, 1993) have strongly argued against the use of concrete model when introducing negative numbers because no models fully justify the formal operations of negative numbers. Freudenthal asserts that algebra is the origin of negative numbers; thus, a formal algebraic approach (the algebraic permanence principle: $a + x = b,\ b < a$) should be used to introduce negative numbers. Historically, the acceptance of negative numbers as a part of the extended number system was achieved through a fundamental shift in viewpoint from concrete interpretation to algebraic formulation during the 19th century (Hefendehl-Hebeker, 1991; Freudenthal, 2002).

The Pythagorean Theorem and Irrational Numbers (Unit: Looking for Pythagoras)

Before addressing the Pythagorean theorem and irrational numbers, this unit introduces students to plotting points on the coordinate plane and finding areas and lengths of polygons on dot-grids (for the overview, goals, and organization of problems of this unit, refer to Appendix E-1). By way of finding the areas and lengths on the grid, the unit introduces students to square roots. Midway through the final section, the unit focuses on the Pythagorean theorem and irrational numbers, the focus of my analysis.

Not surprisingly, the unit on the Pythagorean theorem has more historical aspects than the other units (see Table 4.1). The unit begins with this historically inspired problem (Appendix E-3):

> In ancient Egypt, the Nile River overflowed annually, destroying property boundaries. As a result, the Egyptian had to remeasure their land every year. Their tool to make right angles was a rope divided by knots divided into 12 segments. How do you think they used it?

This problem presents an opportunity for students to think about how the Egyptians may have found a solution. It also sets a historical precedence for the Pythagorean theorem. The unit presents a chance, if teachers/students can seize it, for a debate on how the theorem came to be known as Pythagorean and if other civilizations knew about such an approach before Pythagoras. This problem includes what I refer to as general historical: information about Pythagoras and the Pythagoreans (Appendix E-2) and a photograph of a monument to Pythagoras (Appendix E-4**).** These provide interesting historical backgrounds. But, the historical references with respect to the Pythagorean theorem fail to include any mention of the Babylonian or the Chinese use of similar ideas. Most history of mathematics sources consulted acknowledge that forms of the Pythagorean relationship were known to the ancient Egyptians and the Chinese, in some cases before the time of Pythagoras. Also, most of what was attributed to Pythagoras, including the theorem itself, was known earlier to the Babylonians (Schaaf, 1978). Pythagoras may have been responsible for introducing the theorem to the Greeks, but there is no recorded evidence that he did so (Kahn, 2001).

This unit has some problems that I have classified as historically inspired. The theorem in the unit is introduced using geometry, which is consistent with how the Greeks established the theorem—on the basis of the relationships between areas of the squares on the sides of the right triangle. The unit presents this problem with the aid of a diagram similar to Figure 4.1 below. A unit-leg isosceles right triangle is given and then three squares are drawn around the triangle. Students are asked to find the relations among the areas of the squares drawn on the sides of the triangle.

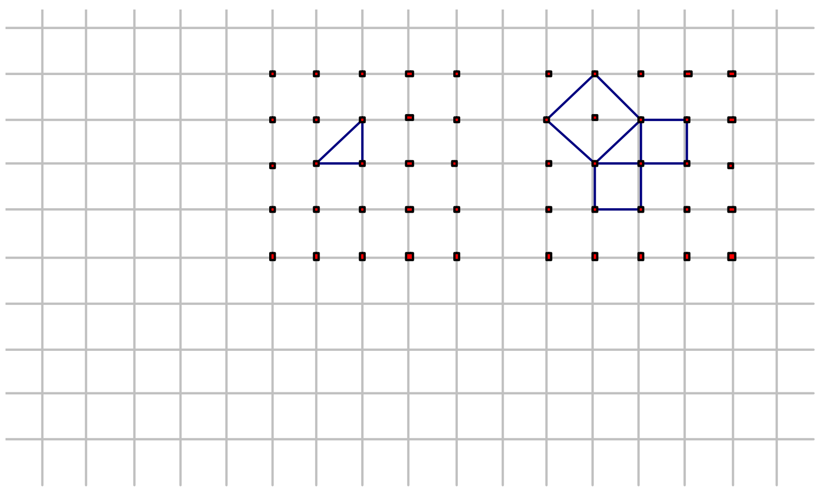


Figure 4.1. Pythagorean Theorem Using Right Triangle with Unit Legs (source: Lappan et al., 1998)

The dotted grid in the drawing (Fig 4.1) provides useful hints on how to compare the areas of the squares (see Appendix E-5 for details on how the theorem is introduced). The problem is then extended beyond the unit-leg isosceles right triangle to other right triangles with various combinations of legs such as 1, 2 and 2, 2, etc. Following these problems, the unit presents a problem on the proof of the theorem in the form of a puzzle, which is related to the Euclidean proof of theorem (Appendix E-6).

The introduction of irrational number in the unit is historically inspired. Irrational numbers are introduced following the Pythagorean theorem. The unit asks students to find the length of a square whose area is 2 square units: “The square below has an area of 2 square units. The length of a side of this square can be expressed as $\sqrt{2}$. Just how long is $\sqrt{2}$?” (Appendix E-7)**.** This problem tells students that although they can estimate $\sqrt{2}$ using a calculator, they cannot find the exact value. But, they can precisely locate it on the number line. Further, the unit uses the Wheel of Theodorus to show how segments with lengths such as $\sqrt{2}$, $\sqrt{3}$, $\sqrt{4}$ …can be drawn (Appendix E-8). The text states, “The Wheel of Theodorus is named after its creator, Theodorus of Cyrene. Theodorus was a Pythagorean and one of Plato’s teachers.” However, the unit does not provide explicit historical notes associating irrational numbers to the Pythagoreans. Whether having such an explicit history would make any difference in the understanding of irrational numbers is again debatable.

The unit does not contain an explicit history explaining the relationship between the Pythagorean theorem and irrational numbers. It refers to the Wheel of Theodorus and notes that Theodorus was both a Pythagorean and one of Plato’s teachers. However, the mere mention of these facts may suggest irrational numbers were in use at least since the time of Theodorus. According to Jones (1994, p. 176), the Pythagoreans recognized the existence of “magnitudes, areas, or lines, which did not have ratios represented by integers” but their numbers were still integers; they did not extend their numbers to include what we now call irrational numbers. Legend has it that the Pythagoreans believed in the power of exact numbers to express things so much that they punished Hippasus, a member of the brotherhood, by throwing him into the sea because after uncovering the existence of incommensurables (Jones, 1994) he revealed that the side of a square with area 2 sq. units could not be an exact number (Brown, 2003).

I will now turn to the accounts of the curriculum experts—their choice to incorporate the historical aspects into the units, their purpose in doing so, and their expectations for use.

ANALYSIS OF CURRICULUM EXPERTS' VIEWS ON THE INCORPORATION OF HD

The curriculum experts comprise three CMP authors and one person involved in teacher professional development. I will refer to them as curriculum experts or simply experts. For individual quotes or references, I have identified them using the pseudonyms Mana, Gary, Rose, and Yorda. Mana, Gary, and Rose are CMP authors, while Yorda works as a teacher professional development facilitator. A brief description of their profiles is given below:

Mana is a distinguished professor in the Department of Mathematics and the Division of Science and Mathematics Education at a major public university in the Midwest United States. She is co-director of the Connected Mathematics Project. Her professional experience in teacher education and curriculum development matters include serving as president of the National Council of Teachers of Mathematics and as a chair of the middle school writing group for the NCTM Curriculum and Evaluation Standards. My interview with Mana was conducted by phone in early August 2006.

Gary is a professor in the Department of Mathematics and the Department of Curriculum and Instruction at a major public university in the East Coast of the United States. He is co-director of the Connected Mathematics Project. Besides CMP, Gary has been involved in the development and evaluation of other curricula. Gary was interviewed by phone in early August 2006.

Rose is a specialist in the Division of Mathematics Education at a major public university in the Midwest. She is also a co-director of the CMP. Rose responded via email, was not interviewed.

Yorda is currently the Director of Consulting for the implementation of CMP in middle schools around the country. She has held this position for three years. Yorda has 14 years of teaching experience in middle schools in the Southwestern United States. As a teacher, she has used the CMP curriculum in her classroom. She has also served as a mathematics specialist for a school district in which she trained teachers to use CMP. Yorda was interviewed via phone in early August 2006.

How Was HD Considered the Development of CMP?

I have noted that the CMP units include some historical features. But, these features are limited in scope and frequency. Questions of interest for this study are as follows: What are the purposes of these features? And, how seriously (or systematically) was incorporating such historical features considered in the development of the CMP? Key excerpts from the interviews with the experts regarding these questions are presented below (**Exhibits 4.1** and **4.2**).

Expert	Excerpt	Notes
Mana	Two ways we have considered the history of math: (a) Identifying (and confirming through modern research in mathematics education) the things that have been historically a challenge for students. Often, what we find is that they were a challenge for ancient mathematician in the development of	Mana noted that the CMP curriculum group included experts with a special interest in the history

	the ideas of mathematics as well. (b) Including in the materials themselves some occasional historical references with the purpose of giving kids the notion that this thing called mathematics is something that has evolved over human time and that there is a lot of interest in looking at the ways in which mathematics was done in ancient times.	of math. Thus they were conscious about HD during development.
Mana	We did not have the specific incorporation of historical ideas as a major thread through the materials. It happened because we were interested in engaging kids, in giving them information that might hook them into mathematics. So, some of our 'did you know' boxes are historical references. But, we wanted kids to actually see mathematics as something that people are interested in and that people had been interested in, that it had a place in the world. It wasn't that we set out to make history one of CMP's main themes… The 'did you know' materials were included to interest kids in interesting historical facts that were related to things kids were studying, like the unit fractions.	Explained how they decided to include some historical aspects and in the process leave out other potentially relevant historical aspects.
Gary	We all were aware of the historical background of mathematical ideas, but we didn't feel that it was a major perspective that would be appealing to children in the middle grades. There are a few historical references here and there, but for the most part it wasn't our basic intention to use the historical background of mathematical ideas as part of the curriculum materials. I think we are all aware of particular difficulties that arose in the development of mathematics historically and so that made us sensitive to the fact that some of these ideas are certainly going to be difficult for kids. But, in terms of using the historical record and stories as part of the curriculum material, that wasn't our prominent consideration. I mean, that was our judgment.	Gary shared that he did not believe that the historical aspects or an emphasis on historical roots of mathematical ideas would be particularly engaging to American kids at that age [middle school].
Rose[2]	We selected historical information to include in the material when we thought it might be of interest to	

[2] Rose responded to the questions via email. I did not use elaborate probing with her so I only include her responses in the exhibits and not in the discussions.

	students and when we thought it might add a bit of information to aid in developing or extending students' understanding of a specific concept or skill.	

Exhibit 4.1. Excerpts from Experts' Interviews on Consideration of HD in the Development of CMP

The experts indicated that they were conscious of the history of mathematics in some respect during the development of the curriculum. However, the incorporation of history was not something they seriously or systematically considered. "We did not have the specific incorporation of historical ideas as a major thread through the materials. It happened because we were interested in engaging kids, in giving them information that might hook them into mathematics," said Mana. Gary noted that they were sensitive to some extent in recognizing historically difficult ideas that might also be difficult for students; thus, they included some historical references here and there. But, he admitted that they did not make what he termed as "prominent consideration" of the history of mathematics during the development of CMP.

Both Mana and Gary explained that the historical notes in CMP units were more a result of their intention to set math in contexts that they believed would be engaging for students rather than a deliberate incorporation of the history of mathematics. "It wasn't that we set out to make history one of CMP's main themes. Our main themes were to give kids a chance to make sense of mathematics to learn how to think mathematically…," stated Mana. She also recounted that setting mathematics in context was the emphasized:

> We set for ourselves the kind of context we wanted to use in the materials [by addressing questions]: What would be the goal of setting mathematics in context? What would be our definition of context? We described for ourselves what we meant by contextualized mathematics. So you can see that there were many decisions that we made were setting our direction before we started writing. And

> one of our main goals was to write mathematics and to write context for mathematics that was going to engage students.

Further, Mana pointed out that they conducted a survey to find out things that middle school students “want to know more about” and the things they “like to be doing”. “So with a clear articulation of what we meant by context, with a lot of information about middle school kids, and a lot of our own need to give kids information about the world of mathematics, we set out to write the materials.” Gary agreed that the emphasis was setting mathematics in a relevant context: “…we made a judgment that kids would be engaged by problems that were set in context that they experience every day. So, that’s why we did not use historical settings very extensively.” Gary expressed concerns that the incorporation of history would undermine the relevance of the material to students.

From this, two purposes can be associated with the historical aspects in the CMP materials as indicated by Mana and Gary:

(a) *To recognize historically challenging concepts*—Mana and Gary noted they were sensitive to concepts that were historically very challenging because such concepts tend to be challenging for students as well. By identifying and confirming “through modern research in mathematics education, the things that have been historically a challenge for students, often, what we find is that they were a challenge for ancient mathematician in the development of the ideas of mathematics as well,” said Mana. Gary concurred: “I think we are all aware of particular difficulties that arose in the development of mathematics historically and so that made us sensitive to the fact that some of these ideas are certainly going to be difficult for kids…” Gary also provided similar reasoning when explaining how they decided to incorporate HD for some topics (e.g., circles) while not doing the same for some other topics: “…in some sense what we were doing was (with the circle business) acknowledging that this is a very deep idea and so we started looking at the way people fundamentally tried to deal with it."

(b) *To recognize contributions of various cultures to mathematics*—Gary noted that they made some consciously efforts to reflect the contributions of various cultures to mathematics.

> One thing we were a little more sensitive to, at least a little more conscious of, was attempts to show, where possible, that contributions to mathematics had come from a variety of places on the earth—not just from Western Europe. The historical and cultural material that is available now is helpful in that score because there is a lot more information available to us in English in this country about mathematical contributions of other cultures.

Mana also made a similar point when she explained the purpose behind some of the historical references found in the material. She noted that they want to convey to kids "the notion that this thing called mathematics is something that has evolved over human time and that there is a lot of interest in looking at the ways in which mathematics was done in ancient times". As an example, she mentioned the ancient Egyptian use of 'unit fractions' found in *Bits and Pieces I.*

The experts admitted that the incorporation of the history of mathematics was not something that they considered seriously during the development of CMP. The main reasons they noted for such a lack of consideration can be briefly summarized as concerns with *relevance, time, and students' interests.*

The experts questioned the *appropriateness* (relevance) of the history of some topics to middle school grades when asked about expanding the historical references in CMP. For example, I asked Mana why they stopped at the "squaring" of the circle problem in the unit C*overing and Surrounding* (see Appendix C-3). I also questioned why they did not consider the exhaustion method; historically mathematicians such as Archimedes used this method to approximate the circumference and area of a circle. Mana's explanation was

> I guess our assumption on that was it would be hard to articulate in a way that would be of interest to 6th grade kids. So, at one level, we could say that we were required to make our best judgments about what 6th grade kids, or 7th grade kids, or 8th grade kids, first of all, could handle mathematically. What could we explain at a level that the youngsters could actually take away something?

The issue of appropriateness for a grade level is debatable. On the basis of what Gagne, Briggs, and Wager (1992) call the "systems approach" instructional design, which involves analysis of needs/goals/priorities, one can argue that certain content is not suitable for a certain grade level because prerequisite tasks/skills have to be accomplished first. In contrast, one can follow what Bruner (1960, p. 68) call "spiral curriculum" and argue any subject matter may be taught at any level with "some intellectual honesty". Specifically regarding the history of mathematics ideas, I am not sold on the argument against including the history of mathematics on the grounds of appropriateness. I tend to agree with Bruner's notion. Having said that, I can appreciate that some of the appropriateness arguments made by the experts were not connected to difficulty but rather to relevance or student interest. As Gary reflected, "We all were aware of the historical background of children in the middle grade mathematical ideas; but we didn't feel that it was a major perspective that would be appealing to children."

Time is always of the essence when it comes to school and in almost everything. "How much can you put into a unit like that in terms of the time it is going to take children to do it?" Mana queried. "Teachers were not able to get through all 8 units in any of their draft forms," she recalled from the field testing phase of the curriculum. Gary concurred:

> For instance, in [the] revision of Connected Math that is just completed, there is a fair amount of material that we have taken out that was in the first edition because there was too much there. Schools and teachers couldn't cover it all. So we asked ourselves very hard questions about how much of what is there do we absolutely have to keep, what is the core that we really have to have.

Gary put forth a compelling argument about the drawbacks of the incorporation of HD as far as school time is concerned: "…people have figured out, sometimes painfully and sometimes over long period of time…they have gained insight into ideas that allow kids to learn things at quite a young age now that took mathematicians centuries if not thousands of year to figure out."

As for *students' interest*, Gary was also very skeptical. He questioned whether incorporating HD bodes well with American middle school *students' interests*. His view about students' interest is also compelling. Gary pointed out earlier that he did not believe "an emphasis on historical roots of mathematical ideas would be particularly engaging to American kids at that age". Refer to Exhibit 4.2 below for the exchange with Gary on this issue.

T[3]	…what makes you think kids at the middle school level might not be interested?
G	Why do I think they wouldn't be interested?
T	What makes you feel that way?
G	Well, I guess it is a sense; it is our sense about American culture. It is not very conscious of history; there isn't a lot of reverence. It is a very much contemporary kind of culture and so our sense was that kids are more interested in things around them than what was around 500 years ago. I mean, to be honest with you, we did not have lengthy discussions about this issue so what I am giving is my own perspective.
T	Yeah, I understand. That's what I am interested in: your perspective, your experience and your feelings about the process.

[3] T stands for my question and G stands Gary's response

G	I mean, I find the historical evolution of mathematics interesting myself but I am a math teacher; and, you know, I have a different perspective than, I think, a lot of kids and teachers do. I think for the most part, our judgment is—and I think this is the judgment of American teachers in all subject—is that kids are interested in what is around them today, and they are interested in things that they see as having some potential usefulness in their lives. And, historical perspective in all subjects, not just mathematics, is one that is not very highly respected in this country. So, I guess that it was our judgment, that if we took a historical slant on things that it wouldn't be real appealing to kids.

Exhibit 4.2. Excerpt from Gary's Interview on Students' Interest on HD

Gary's view that students are not interested in the history of the subject is shared by most of the teachers I interviewed (see Chapter 5). It should be noted that Gary mentioned the view of other teachers unprompted (see Exhibit 4.2).

Despite their concerns, Mana and Gary acknowledged that they may have missed the advantages of HD as they had not looked seriously into incorporation. For example, one question I asked them was to comment on the fact that the unit *Looking for Pythagoras* lacks historical references describing irrational numbers in the context of the Pythagorean theorem. Particularly, I suggested that a story connecting irrational numbers with the Pythagoreans would help students appreciate the past conception related to irrational numbers. Mana admitted that they did not look into this, but they might look at these aspects more seriously in future revisions of the curriculum.

> …Anybody, with a particular interest and deep knowledge of some area or some part of mathematics, can look at any set of materials and envision other things that could enrich those materials. I think that's wonderful. And if we do a third iteration of Connected Mathematics, perhaps, we will be able to take more seriously other threads through the material that would teach kids something about our human struggle to develop mathematics more explicitly than we have. So, I celebrate your interest and ideas, and I said at the beginning of the interview that trying to stream history through the materials in a systematic way was not on our original planning list.

Commenting on idea of including more pertinent historical references to topics such Pythagoras theorem and circles, Gary stated, "It just happens that we didn't take that as our primary vehicle and we probably missed a number of interesting opportunities because our main focus was different."

What Was the Expectation of the Use of HD in CMP Units?

I asked the experts about their expectations regarding how the historical notes in the units were meant to be used and if they addressed these issues when training teachers to use the CMP materials. Exhibit 4.3 below provides key excerpts from the expert accounts on these issues along with my notes for elaboration and clarity.

Expert	Excerpt	Notes
Mana	We've not done any special sessions that focus only on the historical development of the ideas. In the professional development (PD) that we do with teachers we will often mention, just because, again, something of interest to us, the ideas that were of challenge to mathematicians over time. In the work that we do in the development of algebra from a function point of view as well as algebra from the symbolic point of view, we talk about the change in definition of *function* over time. You know, in the last 300 years mathematician have struggled with defining function…We point out historical differences in the way in which mathematicians in different areas of mathematics write mathematics. So, we haven't planned these things as much as they come as part of our own interest in knowledge of history when we are doing PD ourselves.	The 'did you know' segments are put into the text as information for both teachers and kids, noted Mana, suggesting that there was not specific expectations on how to use them. CMP II, the second edition of the curriculum, contains a live link on the publisher's website that provides more details about those "did you know" sections, added Mana.
	We tried to help the teachers think about, and encourage teachers to think 'how can I relate this	Gary said he does not know what

Gary	mathematics to something the kids can recognize?' rather than drawing on something that, while it might be significant historically, wouldn't be something the kids naturally know about or be able to identify with. So, I will be honest with you, we didn't spend a lot of time with historical perspective in the professional development.	teachers do with the HD. He recalled some teachers reporting on how things were going during the field testing, but he did not recall hearing them talk about the "did you know" notes.
Rose	They are not high on our PD needs, and their use in PD would depend on the PD provider.	Rose suggested names of other professional development people to interview
Yorda	When I help the teachers in the professional development sessions, I introduce them just like I would with the students in the different ways that I have mentioned. So, we encourage teachers to use them that way.	Yorda noted they focus on the content but occasionally touch on what is known as process standards [in TEKS] such as helping kids communicate a reason for mathematics.

Exhibit 4.3. Excerpts from Experts' Interviews on Use of HD in Teacher Training Aimed at Using CMP

There were no specific expectations with respect to how the historical aspects in the units were to be used. Other than some historical aspects that might have occurred naturally and not planned, there was no professional development (PD) that focused on the HD, as Mana pointed out.

> We've not done any special sessions that focus only on the historical development of the ideas. In the PD that we do with teachers, we will often mention, just because, again, something of interest to us, the ideas that were of challenge to mathematicians over time. In the work that we do in development of algebra,

> from a function point of view as well as algebra from the symbolic point of view, we talk about the changing definition of *function* over time.

Gary again reflected his earlier point that PD emphasized helping the teachers make math relevant to students.

> We tried to help the teachers think about, and encourage teachers to think 'how can I relate this mathematics to something the kids can recognize?' rather than drawing on something that, while it might be significant historically, wouldn't be something the kids naturally know about or be able to identify with.

While Rose agreed with Mana and Gary that there was no professional training that focused primarily on using historical aspects, she added that any specific focus such as addressing the HD is left to the PD provider. But, according to Yorda and Fred (Fred's account is discussed with the teacher interviewees in Chapter five), who were both involved in PD for teachers in their districts, historical aspects are not something that they paid serious attention to in professional development. Yorda said she used the historical aspects in the CMP during PD in a similar manner to how she used them in her own middle school classroom—as introductions and as interesting information in presentations. Yorda also noted that she finds the historical notes interesting and that she encouraged teachers to use them.

SUMMARY

In this chapter, I presented an analysis of the nature and purpose of the historical aspects in the CMP curriculum by examining four selected units (fractions, circles, negative numbers, and the Pythagorean theorem). This analysis also included the interview with the curriculum experts. I classified the nature of historical aspects in the units as direct historical, implicit historical, and general historical (see Table 4.1).

I noted several historical aspects across the units. The unit on fractions contains an interesting introduction noting that the ancient Egyptians used fractions, specifically unit fractions (Appendices B-2, B-3). It also contains some history about the notation of decimals—notations used by some European mathematician around the 16^{th} century (Appendix B-4). The unit on circles has a few historical elements: the problem introducing circumference (Appendix C-2), the problem introducing area (Appendix C-3), and "squaring" a circle, may suggest some implicit historical consideration. This unit also provides some direct historical notes on π (Appendix C-4). The unit on negative numbers has only one explicit historical note (Appendix D-2). This provides information about the use of negative numbers by Hindu mathematician in the 7^{th} century and that European mathematicians in 17^{th} century thought of negative numbers as "absurd". The unit on negative numbers also contains some instances that I consider as implicit historical aspect (Appendices D-3 & D-4). The unit on Pythagoras has the most historical aspects (Appendices E-2 through E-8) in comparison to the other units. Though CMP should be commended for including these historical elements, these aspects are limited both in number and scope. The inclusion of historical aspects across units is somewhat irregular. Some units have explicit historical information in the introduction, while some do not; some units have more historical aspects than others. Explicit historical aspects are limited and are simply provided as information. They do not provide follow-up questions for discussion. Historical notes also are presented in modern language without any indication of how they differed in form and meaning from the present. For instance, Appendix B-3 talks about ancient Egyptians' use of unit fractions without any explanation of how they might have represented this in their writing although the note does mention Hieroglyphics. I offer specific suggestions for improving the historical aspects in Chapter six. Given these findings, it is important to question the purpose of historical aspects in this curriculum.

In the interviews with curriculum experts, I explored the purpose of the historical aspect in CMP. According the experts, there was no serious consideration of incorporating HD in the development of CMP. The curriculum experts stressed that they were more concerned with setting mathematics in contexts that they thought would be engaging to students. Thus, the historical aspects appear to be the byproduct of this general intention to situate mathematics in context rather than a deliberate incorporation of HD. Still, according to accounts of the two authors, Mana and Gary, the historical aspects in the units serve two main purposes: (a) to address historically challenging concepts and (b) to recognize contributions of various cultures to mathematics. According to these authors, there were no specific expectations on how teachers would use such aspects. Most of the explicit historical references seem to reflect the purpose noted in (b), while what I have referred as implicit historical aspect seem along the purpose in (a).

As far as usage of the historical aspects is concerned, the curriculum experts noted they do not have information about how teachers use these aspects. Decisions on usage are left to the teachers or school districts. The most commonly expressed concerns of the interviewed curriculum experts included the relevance of such an approach to middle school students and the time such an approach would require.

Chapter Five:

Incorporation of HD in Instruction: Analysis of Teachers' and Students' Views

This chapter discusses the accounts of both teachers and students concerning the use of the history of mathematics in school mathematics. The accounts are supplemented by numerous exhibits containing key excerpts providing an "immediately accessible" and "compact form" (Miles & Huberman, 1994) of the interview data in order to help the reader construct meaning and contrast this with what is constructed and presented. The accounts focus on the following issues:

- Introduction of core concepts
- Inquiry about foundation of core math ideas
- The use of historical aspects in the textbook
- Interest in HD

ANALYSIS OF TEACHERS' VIEWS

All teacher interviewees work in public middle schools in the Austin independent School District, Texas. Seven of the interviewees came from four different schools, and one interviewee was from the district office. The four middle schools are identified using the pseudonyms Cheetah, Jaguar, Leopard, and Tiger. Cheetah, Jaguar, and Leopard were designated as "academically acceptable" in the district accountability rating, while Tiger was rated as "recognized" in the academic year 2006-07. Generally, the academically acceptable schools are average performing schools while the "recognized" is better than average in terms of the percentage of students passing state standardized tests. The student population of each school falls in the range of 700 to 1100. Brief profiles of the interviewees are given followed by an analysis of the interview. Note that the names of the interviewees are also pseudonyms.

Teacher	Brief profile	Concepts interview focused on
Barb	B.A. degree in general studies and an M.A. degree in educational administration; teaching certificate through summer training program. Seven years teaching experience: taught grade 6 math & science for six years, grade 8 pre-algebra for a year. At the time of the interview, she was teaching 6th grade math at Jaguar Middle School.	Fractions and circles
Eden	B.A. degree in education with emphasis on math; teaching certificate for K-8 to teach all subjects. Seven years of teaching experience: taught math for grades 6-8 at Jaguar Middle School. She was teaching 7th grade math at the time of interview.	Negative numbers
Mark	B.A. degree in psychology; became a teacher through alternative certification; certified to teach K-12th grade math. Has taught for 13 years in middle schools in various school districts in the state of Texas; He was teaching 7th grade math for regular and honors students at Tiger Middle School.	Negative numbers
Delta	B.A degree in finance; obtained teaching certificate after attending a teacher education program for a year and a half. Taught math at Cheetah Middle School for grades 8 and 7 for five years. She was teaching 8th grade at the time of interview.	Pythagorean theorem and irrational numbers.
Karen	B.A. degree in psychology and a master's degree in education; certified to teach early childhood through grade 12, special education classes. Taught special education classes for eight years. She was teaching 8th grade math for the 'resource class' at Leopard Middle School. The resource class is for kids who, according to her, are supposedly 2-3 years behind grade level.	Pythagorean theorem and irrational numbers
Laila	B.S. degree in mathematics with a minor in physics from a university in the Far-East; a teaching certificate from the U.S. Taught high school physics for seven years in her country of origin; taught middle school math and high school algebra in the United States for eight years. She was teaching 8th grade math at Leopard Middle School at the time of interview.	Pythagorean theorem and irrational numbers
Tewelde	Elementary education teaching certificate with a major in math and a minor in reading. Taught middle school math for 10 years. He was part of the group that participated in the initial implementation of CMP (pilot stage). He was teaching 8th grade math at Jaguar Middle School at the time of interview.	Pythagorean theorem and irrational numbers
Fred	B.A. degree in mathematics; teaching certificate. Taught middle school math and some summer high school algebra for 13 years. He was working as a math specialist, conducting workshops and training for teachers, for almost a year at the time of interview.	Pythagorean theorem, irrational numbers, negative number

Exhibit 5.1. Profiles of Teacher Interviewees

The teachers' interview accounts will be discussed along the following main themes and questions:

1. Teachers' accounts of their ways of introducing core concepts to students

   (a) How do teachers customarily introduce core concepts in a unit?

   (b) What connections are made to HD?

2. Teachers' accounts on students' interest in HD

   (a) What are the teachers' assessments of students' interests in the history of mathematics?

   (b) Do teachers encourage discussions pertaining to history of mathematics ideas?

3. Teachers' accounts on how they have used the historical aspects in the curriculum/textbook

   (a) How do teachers use the explicit historical aspects in the textbook?

   (b) How do teachers use the implicit historical aspects in the textbook?

4. Teachers' views about the importance of HD.

   (a) What do teachers think about the importance of HD?

   (b) What do teachers think needs to be done to incorporate HD into school mathematics?

How Do Teachers Introduce Core Concepts?

I asked the teachers to describe their customary approach in introducing *core concepts* of a unit. The core concepts referred to were fractions, negative numbers, circles, or the Pythagorean theorem (and irrational numbers) depending on the topics the teacher had taught or was teaching during the time of the interview (These concepts are the focus throughout this dissertation study; refer to Chapter four for justification). Most of the teachers had either completed or were teaching these concepts around the time of the interviews. I assumed the teachers had recent memories on how they had introduced these topics.

What are the customary ways teachers introduce new concepts in a unit?

Teacher	Excerpt
Barb	Usually…I give them a heading about what we are covering that day and then we will start with the vocabulary words that we are going to be using; write down the vocabulary and the definition and examples…I try to relate it to things they have done in the past or *something that they have experience with*[4]—money or whatever it is they have experience with. And then I just go through, you know, either we use manipulative or I do diagrams or we just write the information and talk about [it]. Sometimes I let them come up with their own idea about why it happens or how it happens. I do not have any one particular way that I do things. It depends on what the concept is; if they have seen it in the past or if it is something brand new.
Delta	It sort of depends on the unit. I would say as far as the *Looking for Pythagoras* goes,…, the *CMP book does a good job in teaching you how to launch it*. So overall I would…use the way that the book launches it. When I was a brand new teacher, this [CMP] book was really helpful; it gives guiding questions. So as a brand new teacher, this pretty much saved my life. Now, after teaching for 5 years, I tweak it a little bit; I kind of add my own flair to some of the units. Overall the books are pretty good about teaching the teacher how to launch a new unit. They are really good in teacher reference section. Unfortunately, the new ones that we are looking at to adopt take away a lot of this.
Fred	There are several ways. One…would be by *connecting to something to what they have previously learned*. Then I go to introducing it by giving them a *story*, giving them a situation that they can think about and take that situation and develop to what becomes the new information they are going to get.
Eden	Generally, *application*. Not necessarily from the historical sense but this is where you are going to see it in the real world; this is how it is used; um, this is why it is important. And, this is kind of what we are going into. And, that the *introduction on all of the CMP books*, they give previews on the types of word problems they are going to experience.
Karen	I *review what is familiar to them first*. [For] the Pythagoras theorem, I will introduce perimeter and area first…that way I have to break all that down for them literally step by step before I can go into the theorem. I couldn't just say $a^2 + b^2 = c^2$. So what I always do is review first what we have already

[4] Emphasis (in italics) added to indicate themes for describing way of introduction.

	gone over and then I will go into asking them: who knows about this? Has anybody seen this?' Once I find out, if no one has seen it, then I know I am staring fresh. If I know someone has seen, I will pair him/her with someone who has not dealt with it. Sort of like peer tutoring. So they can work together also while I am doing a lesson.
Laila	Well, the word Pythagoras is somehow different from the English language. So you kind of have some students [wondering], 'what is Pythagoras?' So, you kind of have to introduce the topic with the story, the *history of where the Pythagorean theorem comes from*. So I have to go back to the historical background of who Pythagoras was…the title of the book [is] *looking for Pythagoras*. So I need to know what they are looking for. Who is Pythagoras? Where did he come from? You know, and who actually wrote the theorem? And so, when I do that, I kind of give the students a way to grasp the idea, 'ok, we are not doing math here; we are doing history'. They kind of want that—to hear a story of the (concept) you are about to discuss. I don't just go to the blackboard and say, "Ok, Pythagorean theorem: $a^2 + b^2 = c^2$." No, it is not going to click. Based on my experience, whenever I tell a story about where this concept is coming from, it makes the student respond, more responsive.
Mark	Well, the good thing will be, although the students haven't learned negative numbers in school, they have seen *negative numbers throughout their life*—especially in sports events, weather. So we talk about like 'where they have seen negative numbers' And a lot of the time, kids will talk about the weather. "Have you seen negative numbers in the weather here in Austin [Texas]?" No way. Even though it gets cold, it doesn't get negative. So we talk about the weather in places where it gets negative and also [we talk about] sea levels. We also talk about golf. Students most of the time come up with golf on their own without me telling them. Usually, there is one or two student(s) in a class that will say 'golf'—either they have watched on TV or they actually play it. Kids found this very interesting, and they talk about golf strokes and some of the names. So that's a great way to talk about negative numbers. And football is very popular. Usually they need some help in discussing about the negative. So, it's really getting something from their *prior knowledge and experience* when we begin topics, sometime.
Tewelde	Usually… I try to bring in some type of example that we are using today to give them an idea of what we are going to be doing and why we are going to be doing it. The one good thing about the CMP *is that it does have a lot of real life examples*. …we may start the investigation and talk about what we have learned and where we could use it. Sometimes I say, "OK, we are going to look at right triangles" and, you know, we are going to do something based on that. But there are times when I say we are going to just do it and see what happens and go from there. Usually, even if it is structured, I like to let the

	kids take me where they want to go. I don't take them where I want to go because they either understand or don't understand. I try to go in the direction they need to so that we can make sure when we finish, they really do understand. The honors kids I have had in the past, they were able to read through the beginning/introductions of everything and then get started with the problem that we were doing. For my academic kids, we usually want [to] sit down and discuss with them what we were going to do and what we were going to talk about.

Exhibit 5.2. Excerpts from Teachers' Interviews on the Ways They Introduce the Core Concepts

According to the teachers' accounts, the customary ways of introduction include: *review, follow the CMP text, relate to the students' experience and real world application,* and *use history of the concept*. These themes are not mutually exclusive nor are they identical. When the teacher says he follows the CMP text to introduce a concept, it is possible the teacher may have used various ways such as making connections to previously learned concepts (review), real world applications, and historical aspects. However, it is quite possible that a teacher may have selectively focused on one or two of these aspects (i.e., review and making connections to real world applications). In fact, that appears to be the case with most of the teachers I interviewed. For example, Delta, Eden, and Tewelde noted using *the ways the CMP book introduces* the concepts. Upon further probing (as presented in Exhibit 5.2), making a *connection to students' experiences* and *real world applications* were the approaches stressed by the teachers.

Reviewing previously learned concepts and *making connections to students' life experiences and real-world applications* were cited as the most commonly used introductions. Other than Laila, none of the teachers specifically mentioned *using history* in response to the question. Laila recounted using some of the history of Pythagoras as part of her introduction to the Pythagorean theorem. When asked to explain if that was something she has done only with the Pythagorean theorem, Laila responded:

> Not necessarily. There would be some more. There would be a story of similar triangles. You know, if I could just go to the board and draw two similar triangles and tell the students these triangles are similar. It won't click. I have to tell them 'where do we use this'. If you are talking about the history of it, I am not sure that it is the history of that particular concept itself. But I have to introduce them where do use this in real life.

However, Laila did not necessarily mean history of the mathematics when she said "story". Similarly, Fred talked about using a "story" (see Exhibit 5.2):

> For example, if you are going to introduce the Pythagorean theorem, I would say something about right triangles. Or, I would say something about measuring the distance across something. OK, how would you measure the distance across the pond if you can't swim all across the pond but you can go around the edges? So, there is a way to figure out what the distance maybe using the edges. So, if you want to measure something but you couldn't directly measure, you could use something around it to measure.

Like Laila, Fred was not specifically referring to the history of the concept when he said "using a story". Rather he meant stories that relate to the students' life experiences. This partly happens because the question raised is somewhat general; I deliberately posed this general question in order to allow the teachers to describe their ways of introduction and to see if they might make any references (unprompted) to using historical aspects. Almost all the teachers did not make such a reference. With this in mind, I asked the followed up question about whether teachers had ever brought the history of a concept into the discussion. Exhibit 5.3 below presents key excerpts from their responses (excluding Laila):

Does the teacher make connections to historical aspects that address issues of 'how the concepts came to be?' during introduction/discussion about the concept?

Teacher	Excerpt
Barb	Um, not usually. Most of that may be that either some of the information is

	historically beyond their [students'] understanding or something that I just don't know. You know, I may not know, historically, where we got this or where it came from. I have had books in the past that [dealt] with some history of math. …a study of a person, and it talked about what they were renowned for. But, um, time does not permit a lot of studying historical mathematicians. When I taught science, we did more of [such] studying; we were able to study scientists, a lot of times known for math as well. But in math, we just have so many topics that we have to cover. I mean I probably could do it if I … I would say I do not think about it very much.
Delta	We introduce... I mean, as long as I think it is going to be interesting to them. Especially in the Pythagoras, we did go a little into who Pythagoras was, and some of the folklore around him. The kids overall aren't really interested in that. They are taking social studies. What I am seeing in math, and the way to teach middle school math, it's got to be something that is important for them now. So what is important for them? Money, sports… You know, as far as the historical aspects, not as much.
Fred	Yeah. Like the Pythagoras, for example, what I have done in the past…I brought in either an encyclopedia or something on the internet about Pythagoras and Pythagoras theorem and tell them who he is. You know, it is a brief thing because I don't want that to be the focus of what they do, but I get them to understand who this person was and what was he thinking or trying to do when he figured out what the Pythagorean theorem was. So give them a little bit background information of what was really going on in his mind when he figured it out.
Karen	No. The main reason I would not is it is not an honors class. They do not care, and that is not their focus. For me as the math teacher, it may be exciting but I have to know what is exciting for them so I can use them to their maximum potential. So for me to teach them, I can't start with the history of math because a lot of kids that I have, this is not their favorite subject. So I have to make it as exciting as possible for them. To talk about the history, if I go and look into the history of the Pythagorean theorem and see something interesting, then I pull that out. But if there is nothing that I know is not going to pick their curiosity, I won't deal with it. I will go straight into the topic.
Mark	Yeah, there are a few times we will bring topics into the discussion. The things about this [like] 'the did you know's are kind of helpful to bring some historical perspective. I would actually like to assign, because I have done that in the past, to do research [about] famous math historians from years but also some of the recent ones. But, if you look at the schedule, it's pretty much packed. There is no time to do that. In addition to that we have some technology that is coming in that we have to teach that is in the TEKS [state curriculum standard].
Tewelde	Realistically, we really don't; occasionally, in my pre-AP classes, we…get a

	question about how was this brought out or how this came up or something like that.

Exhibit 5.3. Excerpts from Teachers' Interviews on Connections Made to Historical Aspects of Core Concepts

Note that Laila and Eden are not included in Exhibit 5.3 due to their response to the previous question. Laila said she used some historical aspects in her introduction particularly with respect to the Pythagorean theorem, and Eden pointed out that she did not use historical aspects.

The teachers' accounts suggest very few instances of making historical connections when they introduce core concepts in their teaching. Delta, Fred, and Mark cited making historical connections occasionally, but the nature of the historical aspects they pointed out were limited to historical facts such as talking about "who Pythagoras was" or other "famous mathematicians" and not the concept. Barb, Karen, and Tewelde said they did not make any historical references.

The main reasons cited for such limited or non-use of historical aspects in their instruction include: *lack of knowledge of history of math* (Barb), *time* (Mark and Barb), *beyond students' level* (Barb, Karen, and Tewelde), and *students are not interested in history of math* (Karen, Delta). The concern about students' level of understanding and interest as necessary to appreciate historical aspects is more commonly shared among interviewees (as we will see in the discussions to follow). They noted they usually do not have time to cover topics outside of the curriculum.

What Are Teachers' Perspectives of Students' Interest in HD?

In exploring the potential use of HD, understanding the interest of teachers and students on this issue is important. Since this topic was not a familiar part of their instruction, the following questions were seen as useful the questions relate to teachers' instructional experience.

- Do teachers encounter instances in which students wonder about where the concepts come from?
- Are there any instances in which the teacher has encouraged such inquiry?

When teaching core concepts[5], does the teacher encounter instances in which students wonder 'how the concepts came to be?', 'how people thought about the concepts?', 'how they originated?' …?

Teacher	Excerpt
Barb	They want to know more about how they are going to use it than…care about where it came from. You know what I am saying? They want to know when [they are] ever going to have to use this more than they want to know where it came from. I do not think that—maybe I am wrong—they really think about why we do the things we do. I mean, we did measurements. I have to think about talking about the history of measurement—where it came from. Where did the foot [come from]? Or the inch? We write about it; we talked about it. But…
Delta	I actually wish they would do it more. I wish kids would be more inquisitive about that. I would say some of pre-AP kids are that way. [But] most of my own grade level, no—they don't question as much [about] where something came from or why it came to be. I think maybe it is that there always has been a math teacher that has told them, "This is the way you do it." And so they don't question 'where did negative number come from?', 'when did we start thinking about …?'
Fred	Yeah. The kids look and say, "Why would somebody want to figure out the circumference of a circle?" or "Why does somebody want to do that?" or "Why they tried to figure out that number π?" The thing is that a lot of times they do not see mathematicians as real people. They see mathematicians as people that are in some environment where they are closed in and they are sitting there playing around with numbers and figures and shapes and trying to figure out about it.
Eden	Yes, in my pre-AP classes. They do. They get carious about that kind of

[5] Often times, I referred fractions, negative numbers, or the Pythagoras Theorem (and irrational numbers) as examples of core concepts to the interviewee depending on whichever topic(s) they were teaching around the time of the interview.

	stuff. I will get questions like: “Gosh, how does someone figure that out? Where is [this] coming from? Where is [this] going?” They are also the same kids that say just [teach] me how to do it. So it depends on your subject matter. The regular classes, um, they won’t ask the question but if you present the information, you know, if you present half a story and situation just kind of throw out a comment and see what their reaction is, they generally want to know more about background and that kind of stuff. They won’t ask on their own if you don’t bring it up, though.
Karen	No. I never had a student ask “Where does it come from?” or “How did this come about?” The only time I have ever seen that is in types of subjects [such as] social studies. But when it comes to math, I have never had a kid ask about the history or like you said Pythagoras Theorem, they never inquire about that.
Laila	They don’t ask that question. The students are very passive. They see the teacher as the active person in the classroom and that they are there to receive the idea. Not until you stir their brains and their imaginations that [they say] “Ok, this is how you use this, this is how it relates to the world concept”. That’s when the imagination and the wondering starts coming up. So, I have to connect it to their prior experience in order…positive response…
Mark	… Usually once in a year we have some of these questions come up. Because kids have a natural desire to question why – when we talk about exponents that are negative, “Why isn’t 5^{-2} = -25?” Things like that. Sometimes it is easy to answer. For the exponent, I show them a pattern like 5^4 …, 5^1. [Then] what is 5^0? Students always want to say 0. [I would say] well, follow this pattern—you are dividing by 5, then they would say “Oh, it’s 1”. What would 5^{-1} be? We have that type of discussion. To be honest, who invented exponent, I have no idea. [Laughs]
Tewelde	Well, the kids that I have this year won’t because I just have the academic kids. But, in the past, where I have had the honor kids that have done it. For the most part, they don’t as well but you usually have a handful. Usually I have 60-70 pre-AP kids and you may have 5-10 that actually think that way. They may be wondering, you know, “Where does this come about?” And usually it is not brought up…the most kids don’t do that or think that is a valuable question. If they are interested, they will come after the lesson is over or when we are doing tutoring or something like that, and say “Mr. Tewelde what happens” …, or “What is another way to do this?” something like that. As far as the historical aspect, there is rarely an incident.

Exhibit 5.4. Excerpts from Teachers’ Interviews on Their Encounters of Students’ Inquiries

All these teachers agree that most students do not inquire about how the concepts came to be. Barb, Karen, and Laila said their students *never* asked them such questions; Delta, Eden, and Tewelde noted they have had *some* students, but those students tend to be from the pre-AP (Advanced Placement) students. While Mark and Fred pointed out they had students who asked questions of that nature, they also referred to limited instances. Mark stated, "Usually once in a year we have some of these questions come up." The "once in a year" may not necessarily be accurate. However, his responses to a later probe suggest that he was referring to a few instances in which he had students raise questions such as "How the division sign came up" and as a result a few students did research on such topics.

Some of the reasons given by the teachers for why their students do not raise where-does-that-come-from questions are telling about the ahistoric nature of math instruction. Delta suggested that students tend to accept what the teacher tells them without questioning.

> I think maybe it is the idea [that] there always has been a math teacher that's told them this is the way you do it and so they don't question 'where did negative number come from?', 'when did we start thinking about …?'

Eden saw school mathematics as inherently dry in nature and as something that does not entertain students. Moreover, she noted students were not interested in history and that they might wonder more about what math could help them in their real lives and present interests.

> We were venting about (the other day in the teachers' room) how hard it is to teach math at this age group because there is no connection to them as a person. It [math] is just very dry. And they are very egocentric at this age—it is about me and what can I do.

Eden, Tewelde, and Karen, who all said honors/pre-AP students are more likely to raise questions pertaining to history, suggested that the differences in students' level of understanding rather than differences in students' interest in math were the main reason why some might question historical background. This belief can be a crucial factor for the teachers' decisions to encourage student inquiry. Instances of this belief at work are noted in the responses of Laila and Eden. Laila stated that her students do not ask questions (e.g., "how did this come to be?"). But, she added that they would say, "Ok, this is how you use this; this is how it relates to the world concept" when "you stir their brains". Further she noted, "That's when the imagination and the wondering starts coming up." Eden shared a similar sentiment. She said her regular classes needed some impetus from the teacher if they were to inquire about the history of ideas:

> The regular classes, um, they won't ask the question but if you present the information, you know, if you present half a story and situation just kind of throw out a comment, and see what their reaction is, they generally want to know more about background and that kind of stuff. They won't ask on their own if you don't bring up, though.

Laila and Eden observed that their students needed some motivation to inquire about the history of ideas. With such a consensus among the teachers that most students do not ask historical questions, the questions then is: Do the teachers encourage *that* type of question in their instruction?

Do teachers encourage inquiry about history of math concepts?

Barb and Karen were not probed on this question because they strongly indicated that they do not cover the historical aspects, and they expressed their belief that historical aspects would not help their students learn math. Key excerpts from the teachers' interviews with my side notes are given in Exhibit 5.5 below.

Excerpts		Notes
T :	If you recognize *that*, what do you do to make it more human? What have you done?	*that* refers to math as dry. Eden described (middle) school math as kind of dry.
Eden:	Oh, everything I can pull out of it. That creativity and anything that I can pull out to make it a little bit more human—the best thing I can,…anything I can think of. You think of a situation … like these things where [we talked] about similar figures. I named the last one thug on purpose…	She talked about instances gave human characters to some figures when they discussed similarity. She noted that her students enjoyed the discussion.
T:	Ok, let's go back to the negative numbers. Earlier you said that if there are some historical notes, you may read them or skip them. You also mentioned that putting a more human face excites them [the students].	I had already asked her about how they used the historical note in Appendix D-2 about negative numbers.
Eden:	Oh, absolutely, yes.	From the context, her agreement was to the statement that "putting a more human face to math excites students"
T:	[But] these historical notes are instances of 'putting on a human face'.	Referring to notes like Appendix D-2.
Eden :	Right.	
T	So, why don't you guys go far into digging through such notes?	She had said that she skipped the historical note in Appendix D-2.
Eden :	Most of that has to do with time. We've got a lot to cover in very short period of time. And, you don't have time to get into the entire investigation if you spend more than, say two minutes. If they would even give us 5 minutes and cut out some of the other stuff, it would help in terms of hooking some of those kids may [be] unmotivated into it.	
T:	Generally, how do classrooms address the issue so that students appreciate that math is done by humans and mathematicians are very human and they make errors?	Fred noted earlier that students "do not see mathematicians as real people" At the time of the interview, Fred was working as a

		teacher facilitator. Thus the question addresses general "classrooms", not only his.
Fred:	Normally, most of the classrooms are supposed to. In the past, they haven't because teachers haven't looked at that as important. When we went to school, it wasn't important for us to relate the mathematics to what we were doing in our daily life. It was more important to relate to things of a mathematical nature. Now, it should be happening more because there is the thing that says you must be culturally responsive to the students, which means you must make a curriculum that reaches every student no matter what their environment is.	Fred also elaborated on using the 'city of Euclid' in the CMP unit *Looking for Pythagoras* as an example. Fred said he would make this context more culturally relevant to the kids by talking about the city where the kids live.
T:	So, you are relating the concept to their immediate surroundings. On the other hand, one can argue that the choice of the city of Euclid may be on purpose to provide background context for plane geometry…	The city of 'Euclid' is used in the unit in problem context for plotting coordinate points (see Appendix E-9)
Fred:	At this age, plane geometry in itself happens in what they have seen. But it is not like we would introduce this as plane geometry…	He added that historical stuff may turn the students off because it does not relate to the things they know or have seen
T:	Besides relating it to their immediate experience, will there be a chance for the students to know who Euclid was [and the reason behind the city of Euclid]?	
Fred:	Yeah. I mean, they eventually are going to [do] this… It is just a matter to start it, to give them some sense/something to what they know. Then you go to explain to them the reason behind the city of Euclid or whatever else is in the book. But, you know, we call that a hook—to get them hooked on the concept. We find that giving something that is more relative to them, hooks their mind first.	
T:	You earlier said you have had students do research. Could you talk a little about that? How often, on what topics…?	He had mentioned that some of his students did 'research' on famous mathematicians

Mark:	It depends on the situation. If a student was excited about a topic, have him/her research the topic. I haven't really assigned as a school or a class wide project to do some historical research. I would like to, but like I said there isn't very much time.	He was referring to limited instance(s).
T:	'Irrespective of how it came or who discovered it, 'now it's here, how do I use it' sort?	I raised this question to Delta after she had stated that her students ask about how it is going to be useful to them and that rarely do they ask about the history. She had also noted: "If I bring up more historical aspects, then their answer to me is going to be that's great but this is now"
Delta:	We [spend] shorter amount of time on the historical and much more emphasis on the practical, I would say. And the thing [is], that is not built in our IPG.	IPG is Instructional Planning Guide, a general lesson plan provided by the school district
T:	In the process standard [found in NCTM or the state curriculum standard], they have something related generally to cultural and social issues.	The process standard has some general aspect that suggests HD incorporation.
Delta	For the most part our kids are not tested on that. I mean they are not going to be tested, you know, 'tell me what you know about Pythagoras?' I mean, they find it interesting for a couple of minutes and then they are ready to move on. You know… they take math at face value. So if the math teacher says this is Pythagorean theorem: "Ok, just teach me how to do it and teach me how to use". They don't question, "Ok, $a^2 + b^2 = c^2$, so the *a* and the *b* are legs. But who named the leg a...?" I do not get a lot of that.	This is a good defense. In a way, the teacher is understandably concerned with making sure her kids learn what is required for testing.
T[6] :	So, how do you provoke them to think [wonder] about…?	Laila had said that the students don't ask questions about where ideas come from (see

[6] T stands for my probes

		Exhibit 5.4).
Laila :	Questioning techniques. Today, in their warm up, there was a question of the circumference and the diameter. Well, in order for me to provoke their thinking I have to review by asking them questions that will lead to the answer of what circumference is. If I just go there and solve the problem, it isn't going to do it. If they don't know what circumference is or they have forgotten, it isn't going to click.	"No, I don't go that far. Sorry," Laila said when I asked if they go beyond talking review and application of the concepts to history of the concept.
T :	And, on your part, do you provoke students to pose these kinds of questions?	I mean questions like 'how the concepts came to be?'
Tewelde:	As far as the historical aspect of it?	
T :	Yeah.	
Tewelde:	No	

Exhibit 5.5. Excerpt from Teachers' Interviews on Teacher Initiatives in Historical Inquiry

These accounts suggest very limited instances (if any) about using inquiry in instruction. The teachers offered various reasons why they did not address inquiries dealing with the history of concepts. The reasons include lack of time, nature of the curriculum, relevance to the students, and state standardized test requirements. These reasons are intertwined in many ways.

Although some of these teachers noted raising some historical issues, they noted they do not have time to do more in this issue of using history. They felt they need to cover what is in the curriculum first and foremost. Mark noted some limited instances in which he has had students do research about famous mathematicians, but he does not assign class-wide projects. "I would like to [do more] but like I said there isn't very much time." Besides time, some said the curriculum is packed and that it does not allow them to cover historical aspects. Eden noted that she felt that the math curriculum for middle school is "dry" because its focus is more on skills such doing computations.

When I reminded her that the historical note about negative numbers in their textbook could be considered in adding more human face on math, she argued that there was not enough time to cover those aspects.

While *time* is a common excuse/reason given by the teachers, Fred and Delta talked about *relevance to students* and *students' interest* respectively as prime concerns for not using the historical aspects. "You must be culturally responsive to the students," noted Fred. He elaborated on "culturally responsive," stating that he tries to make instruction more relevant to the students. He cited an example in which he would replace the 'city of Euclid' in the text with the city where the students live. The 'city of Euclid' is used in the unit *Looking for Pythagoras* as a problem context for plotting coordinate points on a plane (Appendix E-9). He argued that using something "relative" to students' experience hooks them to the lesson better, especially when introducing a lesson. When asked if there might be any historical purpose why Euclid is chosen as a context for the coordinate plane and that if there would be any chance for students to learn about that, Fred explained that they may not rule out presenting such chances to students but noted that first and foremost instruction has to be relevant to the students.

Delta's main concern was with students' interests. She maintained that students are not interested in historical aspects. According to Delta, students are inclined to say "just teach me how to do it and teach me how to use it". They care less about the history. Moreover, Delta noted that historical aspects are "not built into" their Instructional Planning Guide. As a result, her students will not be tested on such aspects when they take the state standardized test.

How Do Teachers Use the Historical Aspects in the CMP Units?

In Chapter four (Table 4.1), I outlined three forms of historical incorporation: direct historical, implicit historical, and general historical. In the discussion that follows, I will refer direct historical information as *explicit HD* and those that are less obvious as *implicit HD*. I will discuss the teachers' accounts pertaining to how they used historical aspects found in the selected CMP units.

How do the teachers use the historical notes such as the 'did you know' segments and beginning of unit notes in the CMP books?[7]

Teacher	Excerpt
Barb	Personally, I don't really use them in class. I don't use anything other than we talked about measurement where the foot, meter…came from and things like that. I don't use any of the historical. This is my first year using this book because I came from another district and I am not a person that goes page by page, and I do not use any one book solely. I use different resources.
Delta	How do I use it within a lesson plan? It depends on the lesson. If it is something they [students] are really interested in, like the 'did you know', they loved this [referring to Pythagoras theorem]. … They still bring this up to me; it is in the little box here. This one, I brought it before the lesson. So I use it as part of my launch, as my intro to try to get their interest. So usually if it is something that sort of cultural aspect or historical aspect, I will throw it in as an intro before we jump into the concept.
Fred	Most of the time we use them as an intro because if you look at most of that, it usually is written in the beginning of the book. Oh, yeah [he agreed on using the historical note about Pythagoras theorem]. This is the kind stuff we want them to understand where this came from and why it is so. The reason it is put in this book is so that we will use them, so the kids will be exposed to what the historical facts are. We do use those. It's just not the primary focus when we start the unit. Depending where you are in terms of physically in the city, because if you go to another area of the city, this will be something you could present the way it is and will fly just fine. But not everywhere you go in [the city of] Austin could you present it

[7] During the interviews, I explained to the teachers what I meant by the explicit history and showed them from the sections of the books I was referring to. I had already taken note of the main concepts they were covering— Pythagoras Theorem (8th grade), negative numbers (7th grade), and fractions and circles (6th grade).

	that way and get the kids interested.
Eden	I actually didn't cover this 'did you know' [referring to Appendix D-2]. They have a similar one in another book about the concept of zero and where it came from, the kids love that. I have been teaching this book for seven years and honestly, I don't do this particular… I gave them the investigation and the number line, but I don't have to show them in the book. I show them actually on the overhead and so they don't open their book.
Karen	We don't or I don't. I get to the point. With my kids, they are already going to be behind. So I don't have the time to talk about the history of something when I know in a class period of 90 minutes, it is going to take me 90 minutes just to show them how to find area and perimeter. And for 90 minutes, they are going to have to go over and over again in order to get to this theorem. Whereas in another class, they may take that 15 minutes and talk about the 'did you knows'. I don't have the luxury of time.
Laila	We read them as a class. And then from the class, we start provoking questions. But, you are right; not too many of those topics have those stories. So, just like in my Pythagorean theorem [lesson], I did the Pythagoras tree and how it is conceived to be; and it becomes like this 'le-v-c' curve. That is explicit; I mean implicit historical background of how did we come up with the Pythagoras tree.
Mark	This [referring to Appendix D-2] would be explicit. We are talking about 628 AD and how long ago that was. And so this kind of makes them think about how long math has been in use. We read this and talk about it and ask a few questions and then we move on. But that's the kind of role it plays. It just takes about 5 minutes. Sometimes it's a longer discussion, as long as the discussion doesn't become silly. I like having this here and where it is located [referring to same appendix].
Tewelde	Um, I hate to say it but for the most part, my academic kids won't go into it. My pre-AP kids, they will read it. If anybody is interested anything about it, they may ask. But I don't necessarily say 'ok, this is something you have to read and go through and ask questions about anything like that'.

Exhibit 5.6. Excerpts from Teachers' Interviews on the Use of the Historical References in the Units

Overall these accounts suggest that the teachers' use of HD in the books ranges from non-use to limited use.

Delta, Fred, Laila, and Mark reported *using* the explicit historical notes in the CMP units. "Using" is a key word; the kinds of "use" these teachers referred to were of a limited nature that can be described as a *brief introduction* or as *general class reading*. None of them talked about extended discussions, class projects, research, or something involving expanded use of the historical notes. Delta, Fred, and Laila were primarily asked how they used the historical notes in the unit *Looking for Pythagoras: the Pythagoras Theorem* because they have taught this unit to 8^{th} graders, while Mark was asked about to the unit on negative numbers because he was teaching this unit to his 7^{th} grade class.

Both Delta and Fred talked about using these notes as a brief introduction to a lesson. Delta noted that her students loved the historical segment about Pythagoras. But, "it has got to be something that engages their attention; and, it should be short, and they can buy into it as 8^{th} graders," she added. Fred argued that they cannot go beyond the notes given in the book because they don't have time and "it is not a theory class". "We don't teach to the test but there is a summative test in mind," noted Fred explaining why they did not have much time for the historical aspects. Mark said, "We read this [Appendix D-2] and talk about it and ask a few questions and then we move on. But that's the kind of role it plays. It just takes about 5 minutes."

Laila, in contrast to Mark, Delta, and Fred, described a more detailed use of the historical aspects. She noted that she used the historical notes as class reading and discussion. Laila recounted showing a video on proofs of the Pythagorean theorem to her class.

> I actually showed a video…—a video of the different proofs for the Pythagorean theorem. I had to show a video. That's why I have the projector there [in the classroom]. And I had to get that primarily from the United Streamline. Gladly,

> we have all that access. If we want videos for how Euclid has proved, how the Chinese… the Chinese have their own proofs. The Greeks have their own.

Laila added that she showed this video as an introduction to the Pythagorean theorem.

The rest of the teachers (Barb, Karen, Eden, and Tewelde) described not using the explicit historical notes found in the units. These teachers gave various explanations for not using the notes. Barb explained that this was her first year using CMP textbooks because she transferred from another school district where they did not use these textbooks. Additionally, she strongly expressed her belief that the history of mathematics would not help her students academically.

> I think that the benefit is not going to be academic. …I don't think that academically they have to know who or what or why [that] came to be. I think that the benefit would be for the personal knowledge. I do not think that they have to know who came up with this or why we call it this to be able to compute, to be able to apply. I think the benefit would be personal not necessarily academic, because I do not think that they have to know the history of a mathematics concept to be able to use, and apply it and connect it.

When I asked Barb if we could make an argument for using the history of mathematics to provide the background information of a math concept and to motivate students to learn about math, she reflected: "…I think a lot of kids would have a hard time relating their current world to the ancient Egyptians, 1800 BC or whatever it is." Karen and Tewelde also shared similar sentiments that their students would not value these historical aspects. Karen asserted that "history is not something" with which her students would be impressed and say "oh, yeah, I want to know who Pythagoras was; I want to know why they decided to use [this]". Karen argued that her students have special needs and that they would require more time than the regular classes simply to cover the required material. Thus, the history of ideas would be the "last thing they are thinking about". Tewelde implied that students' interests would dictate discussing the historical aspects. But, he noted students normally would not raise questions that lead to historical discussion. "They won't see it as useful; it is not helping them do what they need to do right now. I mean you would think, considering as small as the book is, that you are going to read everything that was there. But they don't value that." Eden was not even aware that the historical notes about negative numbers (Appendix D-2) existed in their text: "I have been teaching this book for 7 years. And honestly, I don't do this particular…," she said referring to the note. However, she recalled talking about some of the history of the number zero, which she said was not taken from the CMP book.

How do Teachers Use the Implicit Historical Aspects in the Units?

There are some aspects in the CMP units that I refer to as implicit historical aspects (as discussed in Chapter four) because (a) they have historical cues (but are not obviously stated) regarding a major concept or (b) they address some fundamental issues about the concept (e.g., why they are needed). During the interviews, I emphasized implicit historical aspects related to fractions, negative numbers, and the Pythagorean theorem.

Barb was the only teacher who teaching grade six (in the term the interviews were conducted) and had covered the topic on fractions. Barb repeatedly pointed out that she did not include any historical discussion of fractions. Barb explained why students do not need history when learning about fractions:

> … in math, especially, a lot of the things in grade 6 were introduced in 2^{nd}, 3^{rd}, 4^{th} grades so it is not anything new to them. And so it is not something they would go "Oh, where does that come from?" We talk about fractions, percent: What does cent mean? What is 0.01? OK, so what does percent mean? Out of a 100; if you have one penny out of a dollar, it is one percent. I mean we talk about those types of things. But, we don't talk about where the idea of percent came from or who came up with the word fraction. You know what I am saying? We do not talk

> about why we use the base 10 system as opposed to the base 3 system or base 5. It's just not something that… I mean, maybe it would be something they would find interesting but I just never got it personally.

The unit on negative numbers has very few contextual cues about historical aspects. But, there are aspects like modeling of negative numbers with the temperature scale, number line, and chip board that can be used to provoke historical discussions. Specifically, the chip board model in the unit resembles the rod counting of the ancient Chinese. I asked Mark and Eden, who taught this concept, how they introduced the concept of negative numbers. Mark mostly talked about using strategies in the unit introduction and using measurement problems such as yardage gained or lost in football, changes in temperature, golf strokes, and changes in altitude. When asked if they discussed further the meaning of negative numbers (i.e., what is meant by a temperature below zero in Fahrenheit), Mark explained:

> Most of the book focuses on drop in temperature [like] starting in 67 degrees and changing for 2degrees/hour for 5 hours. So the drop in temp by 2 is the – 2 degrees. It doesn't really talk that much about temp below zero in this book. That is a good way to hook on negative numbers.

Similarly, Eden elaborated on how she used models when introducing the negative number concept.

> I personally supplement it with money, say owing money to the cafeteria. That I find a lot more effective in terms of really getting to internalize what a negative number is because they experience it every day, and they understand whether or not they are in debt. And, understanding things like subtracting a negative number means you are subtracting away debt.

In both cases there is nothing that suggests conscious efforts to address historical issues about how negative numbers evolved.

The unit on Pythagoras theorem contains various remarks/problems that I consider as implicit historical aspects (Appendices E-3 through E-9). I focused on how the teachers associated irrational numbers with the Pythagorean theorem because of the challenge the concept of irrational numbers presents. Further, the connection made in the unit, though implicit, is understandable. Delta, Fred, Laila, Karen, and Tewelde, all of whom taught this unit, were interviewed on this topic. Key excerpts from their interviews are presented in Exhibit 5.7 below (T stands for my probing questions).

Excerpts		**Notes**
T :	The way a problem is framed, it looks like the authors considered the historical aspect of it but without telling specifically about it. For instance, they used the Pythagorean theorem as a background for discussing irrational numbers. How do you guys talk about this?	
Delta:	With irrational numbers, it is pretty much what you just said. Since this book is primarily discovering the Pythagorean theorem, as we were discovering the theorem, we were doing different types of squares. There are squares and tilted squares, and so they ended up realizing when they took side length of squares that they weren't getting… 'wait a minute, it is not 2, it's not 2.5'. [That is] they couldn't grab a hold of what the numbers were.	When she said "it is pretty much what you just said", she was referring to using the Pythagoras theorem as background [the same as in the book].
T :	Was there any association between irrational numbers and the Pythagorean theorem?	
Delta:	Oh, yeah. That is exact. This is all… Absolutely.	She is agreeing with the fact that there was an association between the Pythagoreans and irrational numbers, but she did not communicate that in her class (see her responses that follow)
T:	Do the students understand the connection?	
Delta	I don't know if they understand or not. I think the understanding of an irrational numbers to them is something that's very vague because they can't put their hands on it. … They definitely understood the difference between rational and irrational.	

T:	But do they understand [how] irrational numbers were uncovered, you know?	
Delta:	Just by measuring the side-length the way this book will show it. But that's it.	The book does not make any explicit historical reference linking irrational to Pythagoras
T:	So you don't go into the history of irrational numbers linking to the Pythagorean theorem?	
Delta:	We are under so much pressure to teach these 8th graders math, and unfortunately (even with the double block), there is just not [enough time], and they are not written into our IPG to do that. So, our instructional planning guides do not give us a lot of room to spend that time.	
T	But is there [some] explanation to the students, for instance, that the Pythagoreans found what is called the incommensurability but they did not acknowledge [irrational numbers]? [We are told] the need for irrational numbers arose from this kind of problem.	
Fred:	We supplement *this* with things like the number π : how did that come about and why that is an irrational number… We use the Pythagorean theorem because it is a natural way for irrational numbers to be created.	By *this* he was referring to the introduction of irrational numbers he said he used—similar to that in the book (see Appendix E-7)—using right triangle with legs of 1 unit long.
T:	Exactly. So, do student become aware [of that] by presenting to them something historical.	
Fred:	…that's the closest we can do with that particular one. I mean, like you said this book doesn't say all the reasons why.	
T:	Exactly. But it is probably understandable that some connection is going on.	
Fred:	Again, we are talking about grade 8 versus somebody that is doing mathematical theory,…going into depth with something like that. What we are doing in 8th grade is give [some idea of] them irrational, touch on and highlight what they look like…	
T:	In the unit, irrational numbers are discussed in relation to the Pythagorean theorem. How do you talk	

	about irrational numbers in that sense?	
Karen	What I did was [teach] irrational numbers in relation to square roots, [such as] $\sqrt{2}, \sqrt{3}$, …	
T:	So the link between Pythagoras and irrational numbers is not discussed in your class?	I asked Karen this question by referring to the introductory diagram on irrational numbers (Appendix E-7) and recalling the brief history that the Pythagoreans did not have a number system that included irrationals
Karen:	No. My main thing is to make it understandable to them. If I try to [do that]—[bringing story of irrational numbers] –I have lost them. I am not going to say it's going to happen in every math class, but in this group, I will lose them.	Karen appeared to believe that HD is an additional work for her special education students
T:	The book introduces irrational numbers with a right angled triangle of 1 unit sides and [leads the hypotenuse $\sqrt{2}$ units long]. But it does not make [historical reference like] the ancient Greeks' struggles with irrational numbers...	
Laila:	No, it doesn't. That would be too far for them [students] to grasp… They don't care about the irrational number. They know already that "Ok, if the area is 2, the side will give me the $\sqrt{2}$ ". ... But talking outside of the $\sqrt{2}$ as an irrational number, per se, it doesn't cover that in the book, and I don't have that much time to go and search for an example [where] an irrational number come[s] from and this and that… If it were in the book, it could easily be accessible for me.	
T:	Do you think it would have been helpful had the book included *this kind of history*?	I briefly recalled that there is a story about how Pythagoreans had problems with irrational numbers.
Laila:	Exactly. It would be very helpful even for me	

	because I don't have the time to search for every historical topic that I could relate to the concept. I don't have that time.	
T:	From your experience, how do you make the link between the Pythagorean theorem and irrational numbers?	I first reminded him that their textbook introduces irrational numbers following the Pythagorean theorem—implicitly recognizing the historical link.
Tewelde	…we start from square roots and usually go from there. We go from using the square roots of perfect squares to numbers that are not perfect squares, and we start talking about the difference between the two. At this age, they want a specific answer. And what we try to get them to do, especially when they start using this format, is to leave it as [is say] $\sqrt{21}$. They want to use the calculator to find whatever it is…	He talked about an instance in which they discussed the distinctions between irrational (e.g., $\sqrt{21}$) and rational values (e.g., $\sqrt{9}$) using a calculator. Tewelde shared an interesting observation that his students would wonder why an irrational number never ends (when written in decimal form) while on the number line it is just a segment with ends.

Exhibit 5.7. Excerpts from Teachers' Interviews on Irrational Numbers and the Pythagorean Theorem

Delta, Fred, Laila, Karen, and Tewelde said they did not discuss the history of irrational numbers in relation to the Pythagorean theorem. These teachers mentioned similar reasons for not going into the history of irrational numbers like those noted earlier (i.e., no time, beyond students' level, the curriculum (IPG) does not contain such aspects). Laila accurately pointed out that the unit *Looking for Pythagoras* does not provide explicit historical notes on irrational numbers. Teachers did not exploit the historical aspect despite the fact that irrational numbers are presented in connection with the Pythagorean theorem (though no explicit history explaining the connection in the unit). The unit has some interesting cues such as using the square root of 2 (Appendix E-7) and the Wheel of Theodorus (Appendix E-8**)** to discuss irrational numbers. As revealed by the interviews, besides what the teachers claimed as reasons for not making the historical connection, most did not appear to be cognizant of the historical connection between the Pythagorean theorem and irrational numbers.

The unit also includes problems such as *Puzzling Through a Proof* (Appendix E-6), and *Measuring the Egyptian Way* (Appendix E-3) that have some historical elements. I asked some of the teachers about these problems. As noted earlier, only Laila recounted an instance in which she made use of a historical reference. She noted that she showed a video containing various proofs of the Pythagorean theorem to her class.

Teachers' Views about the Importance of HD

What do teachers think about the importance of incorporating HD in teaching?

Toward the end of the interviews, I asked teachers to share their views on the importance of incorporating HD in teaching. Key excerpts from their accounts are given in Exhibit 5.8 below.

Teacher	**Excerpt**	**Notes**
Barb	I think that the benefit is not going to be academic. I don't think that academically they have to know [the history of things] who, or what, or why it came to be. I think that the benefit would be for the personal knowledge. I do not think that they have to know who came up with this or why we call it this to be able to compute, to be able to apply. I think the benefit would be personal not necessarily academic. Because I do not think that they have to know the history of a mathematics concept to be able to use, and apply it and connect it.	Barb appeared to be the most skeptical about the importance of HD. .

Delta	I think it should be touched on, but [you can't] take away from the concept that you're teaching kids. Unfortunately, time is very valuable as a middle school teacher; I don't have time to go into 30 minutes with them every day of the historical aspect of a concept. Can I touch on it more? Yes, I probably could, and I wish I had even more time to do that. But overall, when there is such a stress and so much pressure to get through those essential knowledge & skills in the state standards, there's so much stuff to teach these kids. In 8^{th} grade especially, I think more so than in 6^{th} and 7^{th}, that there is just not a lot of time to do the historical aspects as much as I wish there would be.	In addition to time pressure to cover the curriculum, Delta noted the fact that students do not come to her with the necessary prerequisite as a limiting factor for not going into the history of ideas.
Fred	I think it would be important, but I think to a certain extent … I think what ends up happening [is]…teachers may not come out of college with a theory class; they may have learned you can teach mathematics in middle school without even having [to take] a calculus class. So, the thing is you have to look at how far people have to go to in order to get what they have as degree of being able to teach. The historical facts to them may not even be apparent or may not be important to them. So what you have, a lot of times in middle school, is that those historical facts while touched on, they are not looked at in depth. Because, the depth of something like that requires more time. But most teachers don't have…Teachers in [this state] are too bogged down with thinking about the end, which is the TAKS [state standardized] test. There are so many concepts that need to be covered to get them to that [test].	Although Fred stated HD is important, he noted lack of teacher knowledge on history of math and time coupled with system-wide requirements such as state testing as restricting factors limiting serious consideration by teachers. He also noted that the curriculum does not indicate that they need to cover HD.
Eden	I am a bit naïve. I don't really know what it would look like eventually. But, I can imagine, and I think it would be working wonders in giving a face to it [math]. I have way too many kids who love history, love language arts, that	

	come in here and they just go, "I hate math, hate math". And ask adults, 80%-90% will say I hate math. That comes from somewhere. I think something drastic has to change, and if it comes from a more humanistic approach to math, then, you know, give it a shot.	
Karen	I think it would help when it comes to interdisciplinary, when we are doing more than one subject… This goes back to the fact that we don't have enough time. I would love to talk about who is Pythagoras. The point is, as a teacher, what you have is our schedule. As far as I am concerned, if it said make sure that you connect the history of it, then I am going to because that means somewhere/somehow down the line they are going to be tested. If you are not telling me that I have to do this, the things I am going to focus on is what I have to do. The history is extra.	Karen said that she teaches a resource class (similar to special education). Besides math, she teaches other subjects like social studies to the same class. When she referred to the importance of the history of math, she apparently meant for social studies not necessarily for learning math. She insisted that HD would not help her students in learning math.
Laila	It will help stir the response from the students. From my experience, if I introduce a concept with a historical background, they are more interested where that particular concept is coming from.	Laila spoke of the importance of HD by referring to her experience. She appeared to be more conscious of HD. Laila also reflected that it would have been very helpful had she taken a history of mathematics course during her college education.
Mark	I think it is very important. I think it [would be] beneficial to the kids; but, when you look at the time, there isn't time.	Mark said the IPG [curriculum] is "packed" and thus he does not have enough time to incorporate HD.
Tewelde	…we were talking at the beginning, I think today's students are a lot different than when we were growing up. I think we were… we did things because that's what we were told to do. I think we did more things; we were interested in more things other than video games, TV, and things of that nature. Kids today are more [like], I don't say more selfish. "How is it	Tewelde argued that HD might not be that important from his students' stand point. He asserted that students (today), especially in middle school, are the "me generation"—they are more into what benefits them now

	going to affect me?", "Why do I need to know this?" If they feel that it is something that they really don't need to know, it is not something they really care to deal with. That's one of the biggest things that teachers will have to fight with because they [kids] will say, "Why do I need this?"	and in the future.

Exhibit 5.8. Excerpts from Teachers' Interviews on the Importance of HD in Teaching Math

At the surface, all these teachers' accounts suggest that they believe the history of mathematics would be important for their instruction. However, a closer look suggests that their views vary and that most of appear very skeptical about whether history has anything to do with learning mathematics. Barb and Karen remained the most skeptical. Barb thought that if there was any benefit to learning the history of mathematics for her students, it would be to advance their historical knowledge. She did not see any benefit in terms of learning math concepts. Karen noted her "resource class" students are already behind other students, and she viewed incorporating historical aspects as an additional burden. However, because Karen teaches both social studies and math, she saw a possibility for using HD as a resource for social studies. "I think it would help when it comes to interdisciplinary, when we are doing more than one subject." Karen further noted that she would incorporate HD if it was part of their required curriculum.

Although Delta, Fred, and Mark said HD could be important, they emphasized the constraints in practice. They do not see it as feasible to go beyond a brief use of history here and there. They saw time as their biggest constraint. Also, they noted that they were under time pressure to complete the curriculum, and, as Fred said, they "are too bogged down with thinking about the end, which is the TAKS [state standard] test". They also noted that a lot of students come without meeting required prerequisites, which forces them to review materials from previous grades. Fred added other legitimate reasons: lack of teacher knowledge on the history of mathematics and the fact that the curriculum (IPG) does not specifically suggest covering historical background. These teachers also noted a lack of student interest in history. Tewelde argued that today's students, especially in middle school, belong to the "me generation". Thus, they are more concerned with the present and the future rather than the past.

Laila and Eden provided a more positive view noting that incorporating HD could be important and can be accomplished even with current educational conditions if they had additional support. Laila spoke of the importance of HD on the basis of her limited experience. She said she would incorporate historical aspects if it was made more accessible, if they were provided with resources such as historical texts that could save her time. Eden shared, "I am a bit naïve. I don't really know what it would look like eventually. But, I can imagine, and I think it would be working wonders in giving a face to it [math]." Eden noted that she did not use the limited historical notes in her textbook. However, it appeared that she might have become more aware of HD after this interview.

What do the teachers think should be done to incorporate HD in their instruction?

This question targeted those teachers who gave more favorable accounts of HD in response to the last question. Delta, Eden, Fred, Mark, and Laila shared their thoughts on this question. These teachers suggested having historical aspects spelled out in the curriculum (Instructional Planning Guide), with official time allotted to teaching this material. They also wanted more professional development and training on how to use historical aspects in their teaching.

"Time is huge, um, just when you have so much to teach these kids in a one year period, and they are not coming to you prepared for 8^{th} grade math," said Delta. "So where you could spend that amount of time doing a lot more investigative work, having kids maybe do some research behind the math, we just do not. Our hands are tied." The teachers' reasons were that they have too much to cover in their current IPG and thus do not have enough time to incorporate extra information. Eden and Fred, among others, implied that if HD is incorporated in their IPG, meaning time is allotted to these aspects by the school district, then they would do it. Eden said:

> The district office needs to supplement the IPG with certain facts. The district coming in and saying 'this needs to be added'…We are almost scripted by the school district. We are told almost literally exactly what to say each day. A lot of it does fall [apart] exactly because it just does not feel natural. But many teachers do, particularly new teachers, [follow it].

Besides supplementing the IPG, Eden, Laila, and Mark suggested having textbooks with more HD so teachers who want to use the history but who do not have time to search for historical source would benefit. Laila reflected this sentiment:

> The teacher doesn't have enough time to go search for some historical background to match her/his concept that she/he will introduce in the class. There is no time. So what should be done to make it a lot easier for all of us teachers to incorporate history in our concepts, those historical backgrounds should be sufficiently, not just superficially like that, the entirety of the history of that particular concept should be added in our textbooks. So it would be accessible for all of us students and teachers alike.

The third point of emphasis was related to teacher education, particularly cited by Fred, Eden, Laila, and Mark. Fred has noted that most teachers in the middle schools come to the profession without having taken any history of mathematics course. In fact, all of the interviewed teachers agreed with Fred that they had not taken any history of mathematics in their education. Eden, Laila, and Mark specifically suggested some form of professional development in HD would be helpful if they were to incorporate HD.

> [Eden]: To my knowledge, there is not anything like that that existed. So, if you had a professional development and then present all the information and allow the teacher to incorporate into the lesson that fits best, that would work, I think, probably more so with the veteran teachers. Other than that, it is just teachers finding their own resources and incorporating into their lessons.
>
> [Laila]: The other thing is we can get a professional development that will discover 'Ok, let's say, the first six weeks, or 2nd six weeks, this is what we are covering, do we know the history of what is included in these six weeks?' We can benefit having a professional development from a history of math. That should be done. Because if you don't know those features, how could we teach them?
>
> [Mark]: If there is a night course that will get to that once in a while [it] may be exciting. Um, especially if they have something, an assignment for the kids, that may be kind of nice. I would take one. As far as teaching, if you look at [instructional] planning guides, they are just packed. There is no time. If I had time more [out of 45minutes] to do all the stuff, time for thinking and working…

From their earlier accounts, Eden noted she had not used the historical notes/aspect in the units; Mark mentioned limited use of history such brief class readings and that he once assigned about small projects on searching famous mathematicians; Laila noted showing video about Pythagoras and she was the most enthusiastic about discussing the history of mathematics.

ANALYSIS OF STUDENTS' VIEWS

For the discussions in this section, I rely largely on the accounts of seven students whose profiles are described below. I will also make some limited reference to a group interview with four 7th graders. This group interview was not recorded, and my references are based on field notes. From the individual interviews, one student's interview is partly missing (lost in the recording), so I have made references to his account with respect to the first two questions below.

Student	Brief profile	Concepts the interview focused on
Semhar	Pre-AP 6th grader at Jaguar Middle School. Mentioned math and science as her favorite subjects.	fractions and circles
Adam	Pre-AP 6th grader at Jaguar Middle School. Favorite subject: language arts.	fractions
Bella	Pre-AP 7th grader at Jaguar Middle School. Favorite subjects: math, Texas history, and language arts.	negative numbers
Brice	8th grader at Jaguar Middle School. He was taking algebra (for his math class) and noted algebra as his favorite subject.	Pythagorean theorem, irrational numbers, and circles
Gebar	8th grader at Jaguar Middle School. He was taking algebra.	Pythagorean theorem and circles
Flora	8th grader in a 'resource class' at Leopard Middle School. She noted math as her favorite.	Pythagorean theorem and irrational numbers
Kaleb	8th grader in a 'resource class' at Leopard Middle School. He mentioned math, science, and history as his favorite subjects.	Pythagorean theorem and irrational numbers
group	Four 7th graders (two boys and two girls) from an honors class at Tiger Middle.	negative numbers and circles

Exhibit 5.9. Profiles of Student Interviewees

The discussion is organized around the following interview issues/questions:

- What do students want to learn about new math ideas/concepts?
- Do students wonder about where math ideas/concepts come from?
- What do the students think about the historical notes such as the "did you know" sections in the text? Do they use them for their class?
- Do students have an interest in learning more about the history of math concepts?

What Do Students Want to Learn about Core Concepts?

I asked the students to describe what they want to know about new math concepts/ideas when learning them. This is a broad question with the intent to allow the students to say anything they wish to learn about a concept/topic. The intention is to see if there is anything from what they describe that would suggest they want to learn the historical background of a concept (see Exhibit 5.10 for responses to this question). Most of the time, in further probes, I interjected concepts such fractions, negative numbers, irrational numbers, the Pythagorean theorem, and area/circumference of a circle in place of "new concepts" depending on what the students had learned during the semester.

Student	**Excerpt**	**Note**
Adam	I guess I want to know…the easiest way to do it, different ways to do it, like use that type of equation. And, um, I guess I want to know how to do it.	
Bella	I just like to learn how to do the equation. Then I just memorize how you do the equation, and then next time I will just do it that way. But, there are some people in my class who have to understand, who want the teacher to explain why it is that way. I find that, if the teacher explains that, I just get, sort of get confused, and sometimes I stop listening; I sort of stop paying attention. And so usually I don't understand that, and when the teacher puts it simple, that is when I understand it.	Bella did not necessarily mean an equation. From the context, she also could be referring to procedures and/or concepts.
Brice	I want to know everything there is to know about that subject when I am learning. I like learning a lot of facts. I like all the niches that most people don't know. I like to know everything.	
Flora	I would like to know more like why it is and how you get the answers or, you know, what the purpose of us learning [it], or just basically other ways you can find the answer.	
Gebar		Missing this part of his interview on the tape
Kaleb	I don't know; I just want to learn how…like what different ways you can use negative numbers; how you can use them; and um…what kind of problem [you can use them]	In the probing questions I mentioned negative numbers as examples of a concept or topic.
Semhar	Well, usually I know most of it because in 4th, 5th grades, they teach you before you go to 6th grade.	The topics she alluded to when stating "most of it" included

		fractions, negative numbers, and the area and circumference of a circle.

Exhibit 5.10. Excerpts from Students' Interviews on What They Want to Learn about Core Concept

Reading the responses as they are, there are no specific statements indicating that the students want to learn about the history of the concepts. Most responses given by the students reflect sentiments such as "I want to know how to *do* it and *use* it". By "do it" they appeared to refer to being able to perform such tasks calculate, solve problems, define concepts that are needed to succeed in school mathematics, while "use it" apparently means application in their daily/future career. These responses are consistent with the teacher views about the students' primary interests: "students are more interested in knowing how they are going to use the subject matter than knowing about its history".

Nevertheless, these students' responses do not reveal any indications that the students are not interested in HD. Consider the responses of Bella and Brice, which represent two extreme cases. Bella stated, "I just like to learn the equation." "I just get sort of confused and sometimes I stop listening if the teacher explained why it is this way or that way," she added. In contrast, Brice said, "I want to know everything there is to know about that subject when I am learning." We may speculate that Brice's point may suggest an interest in history while Bella's may not. But before going further along this line of speculation, we must look for further perspective.

Do Students' Inquiry about History of Math Ideas?

I asked the students to share if they ever wondered where math concepts they learn came from. Exhibit 5.11 below presents key excerpt with my notes.

Student	Excerpt	Note
Adam	Oh, sometimes yeah like, um…division for example that could be a kind of cool to know how like the first person who thought the way to divide, and you know, to give you better understanding on like how that came. That type of math, I think if… learned it…like you learn some of how it happened, it gives you a better understanding of the subject.	He also said he wondered about who first thought of multiplication instead of repeated addition because it was "an ingenious thing".
Bella	Usually, I don't wonder that kind of thing when I am learning…. But then maybe later that night when I am doing the homework on it, I think about how people must have, you know, tried different ways to do it. And a lot of the times, I wonder how long it took someone to find out, what they want to know, and how long it took them to figure out how to do it properly. And then, why they want to figure it out or how they came across the idea of figuring it out.	Bella also noted she has pondered the following questions: -Some discoveries are accidental. But, why do people continue to use them? -Some discoveries become known after the people who made them died. How do people share discoveries?
Brice	Yeah, all the time. I mean, I like to know that information. It is really interesting to me. To see where the theory started and how it got introduced, how it was proven before…, how the mathematicians came up with all that stuff. It is very interesting to me.	He said that thought why algebra was invented instead of using simple math
Flora	Yes.	She said she could not remember what she wondered about in math. But in science, she mentioned something about snails.
Kaleb	Yeah, kind of. How people figure out what math was and how do you use the numbers and stuff…	He later appeared to give a mixed response.
Semhar	Um…yeah like where the idea math comes from and how it was invented.	

Exhibit 5.11. Excerpts from Students' Interviews on whether They Wondered about the History of Concepts

All six students responded affirmatively that they did wonder about where math ideas came from but with varying detail and clarity. However, what does this mean? After all, these middle school students were responding to questions such as “Have you ever wondered about where the ideas you learn in math came from?”, “Who thought about the ideas?”, and “Why they thought of them in the first place?” Bearing these limitation in mind, I attempted to probe the students on specific topics (e.g., fractions, negative numbers, irrational numbers, the Pythagorean theorem, or circle areas/circles) they had learned and asked them to give examples if possible (see Exhibits 5.12a and 5.12b).

T: Have you ever wondered where the ideas you learn in class come from…?
B: Yeah, all the time. I mean, I like to know that information. It is really interesting to me. To see where the theory started and how it got introduced, how it was proven before…, how the mathematicians came up with all that stuff. It is very interesting to me.
T: Well, do you get this information from class?
B: Well, we do *how to do it* but we usually don’t do *where it’s come from* because we have a lot of work to do. We usually don’t have much time...
T: Did you learn about irrational numbers?
B: Yeah.
T: Did you learn about the Pythagorean theorem?
B: Oh, yeah.
T: Have you ever wondered why people started thinking about irrational numbers, or negative numbers? Or, what problems lead them to think about this stuff?
B: Well, actually yeah. It is not that…because I always thought like why they invented algebra instead of doing simple math and why they made all these different equations…
T: Sometimes when you get a certain idea you might think well, it makes sense because you can use it in this way. At the same time, I don’t know if you wonder like “how the person who first thought about this come across this idea?” So, when you have this kind of questions, what do you do?
B: Well, usually I want to see if there is an answer for it. If not then I look them up. I usually don’t look up stuff right away because I have a lot of work to do. So I don’t have much time until like spring/winter/ summer breaks, those are the times I look at a lot of that stuff.

T: Did you learn about circles? Like finding the area of a circle—πr^2?

B: Yeah, I like that stuff.

T: About circumference, $2\pi r$? What is π? Where did they get that and stuff like that?

B: I have been wondering that for while. I do. π is an infinite number [referring to the number after the decimal?]; a computer has calculated it. But how do you calculate it [seems to refer to manual calculation]?
I think, in algebra we were doing the linear equation in that they graphed part of the circle and so they used that part of the graph to calculate. But I am not sure…

Exhibit 5.12a. Excerpts from Interview with Brice (B) on whether He Wondered about the Origin of Math Ideas

T: OK, tell from your experience, when you learn about math ideas, have you ever wonder about where these come from and who thought of them? Things like that?

F: Yes.

T: Have you ever wondered?

F: Yeah.

T: So when you wonder, what do you do?

F: I usually ask the teacher or just go to different [sources]… Or just look it up on the internet whenever I have time.

T: Can you give me an example of where you have done this stuff and on what topic?

F: um …I don't remember. I think I did one in science because we were doing a project. We are talking about a project. I just wonder like who figured this out or whatever. We were doing one on snails and something happens to them.

T: It was [about] what?

F: It was snails. It was a worm. You know, if you pet [it does that…]. I was wondering who just thinks about it, you know.

Exhibit 5.12b. Excerpts from Interview with Flora (F) on whether She Wondered about the Origin of Math Ideas

Although all the students indicated that they have wondered about the origins of math ideas, Brice, Bella, and Adam provided more specific and detailed responses than the rest of the group. Brice, whose teacher singled him out as a student who is very interested in the history of ideas, noted that he wondered "all the time" and "thought like why they invented algebra instead of doing simple math". Bella said she wondered about how discoveries are made —how long they take, how people share discoveries – mostly when she was working on homework. Adam shared examples of what he had wondered about: methods of division and multiplication. Semhar, Kaleb, and Flora were not specific, though I tried to probe them further. Flora, for example, noted something about snails that she wondered about in her science class project but said she could not remember any examples in math.

The fact that all the interviewed students said they wondered about origins of math ideas seems at odds with what several of the teachers reported (see Exhibit 5.4). The teachers largely suggested that most of their students do not inquire about where math came from or that they are not interested in the history of mathematics. Although students did not appear to disagree with the teachers' assessment that the students do not raise question about historical background in class, the teachers' assessment about their students' level of interest in the history of ideas is not accurate. The students may not have raised questions to the teacher about history of ideas, as the students who gave extended responses suggested. When I asked "When you have this kind of wonderment, what do you do?" Adam noted, "Sometimes I just think about it. I just think about how a person could come to that conclusion. But I don't really go too far and like asking the teacher, 'how did this originate or something?'" Responding to a similar question, Brice responded:

> Well, usually, I want to see if there is [an] answer for it, if not then I look them up. I usually don't look up stuff right away because I have a lot of work to do. So I don't have much time until like spring/winter/summer breaks, those are the times I look a lot of that stuff.

Bella stated, "Usually, I do not wonder that kind of thing when I am learning about [it]. But then maybe later that night when I am doing the home work". One of the students in the group interview also noted that she actually wondered about how people thought about some math ideas but said she did not raise her questions to the teacher. An interesting note on this issue is the fact that all the students (regardless of pre-AP or special education enrollment) said they wondered about ideas, one way or another. This is contrary to the view most teachers shared in the interviews.

To further probe the wonderment issue, I asked the students what the ideas fractions, negative numbers, irrational numbers, pi or formulas of circle (depending on what they said they had learned before) meant to them. I found this probing somewhat revealing and interesting because the students appeared to change their view about HD (Gebar, Brice, and Bella).

Responding to my rhetorical query, Semhar stated, "*Fractions* come from whole numbers." Later she gave a somewhat innocent answer that she had not figured out when people started using fractions. Semhar also said that she learned some history of math including stories about the ancient Egyptian use of fractions in her *world culture* class.

I asked Bella and one student from the group interview what *negative numbers* meant to them. Bella was quite detailed and thoughtful. Bella said they learned about negative numbers the previous year and "went more into depth" on this topic including "how to multiply and divide them" this year. When I asked if they had learned about the multiplication of negative numbers, "we learned that but I can't really remember because we learned that in the beginning of the year," replied Bella with a smirk. I told her about how there are stories about how mathematicians even struggled and resisted accepting negative numbers; some called negative numbers "absurd" (there is note about this in their textbook referring to Appendix D-2). "I was wondering if you guys talk about these kinds of things?" I asked her. Bella recounted:

> We didn't really talk about the history of it. Well, when she [the teacher] started off the project or section, the first thing she asks us as a class before she told us

> anything, she just said,…'do you guys think there is any numbers below zero?' And we had discussion on whether or not there were numbers below zero. Most of the class believed there was because we had learned a little bit about it last year but some of the class even did not remember at all. They were not paying attention last year or something and they said 'there is no number below zero. You can't have a number that's smaller than zero!'

While we were on the issue of why people struggled with the concept of negative number, I asked her to share what negative numbers mean. She responded,

> …When I think of a negative number, the first thing that I think of is it is usually if you lend someone money, how much money they owe you. But also negative numbers for temperature; I think that is definitely useful because the weather can keep going colder and colder and so you can't [still] have a zero in some place. You would have to have like zero at infinity and that… you would have to start it back really, really far. And then, to have a normal degree would be like 1000 degree will be a normal temperature because if you want to start back and be able to get all of the degrees. So I think to have a zero placed in one place and then you can go forever this way and that way is definitely useful.

Bella's understanding of negative numbers is fairly representative of the practical model approach of the textbook (i.e., owing money [loss] and temperatures below zero [relative measure]). The students from the group-interview, whom I asked similar questions, also talked about debt. One would wonder how, for instance, Bella might interpret the product of two negative numbers in the context of such a practical model. Bella noted she could not remember what they learned about the product of two negative numbers.

Another topic I discussed with some of the students was *irrational numbers*. I asked Flora in a more detailed manner about what irrational numbers meant to her. Flora serves as an interesting case because she responded somewhat assuredly at first that she understood irrational numbers but later could not provide details (see Exhibit 5.13 below).

T: Did you learn about irrational numbers?

F: Irrational numbers, um, I believe so. I think that would…

T: Like square root of 2, square root of 5…?

F: Oh, yeah, square root of 2, square root of 5 and …

T: Did you use calculators?

F: She [the teacher] wouldn't allow us to use calculators because she said the calculators wouldn't be in our test…

T: Ah, what do irrational numbers mean to you? Like do you understand it?

F: I understand it [laughs]. But me, myself I can't explain.

T: You know, numbers, for instance, 1, 2, 3… You can probably associate them with counting, right?

F: Yeah.

T: 1, 2, 3, 4 stuff like that –counting. Or fractions, you can think 'oh, part of this and that'. But [irrational] what does it mean to you? I am just curious because when I was a younger student like you, you know, there are experiences that you go through like thinking…People probably have different experiences. And I want to know if there is some…

F: I mean, to me it really…I can't explain it [more laughing].

T: Like when somebody says 'irrational number', what do people mean by that kind of stuff, if it means anything to you?

F: Uh…[sigh…more laughing]

T: Interesting, uh?

F: Yeah, kind of. What do you mean? Like when I learn it, like what I think about it?

T: Yeah.

F: Whenever… irrational numbers, I was just like kind of …weird to me. But I start getting it.

T: So do you have any idea why people thought about it and if there is any connection with…

F: Nah [no].

T: For instance it is in the same unit as the Pythagorean theorem: $a^2 + b^2$ …

F: is equal to c^2 [she completes my statement].

T: It is in the same unit as that.

F: Yes, it is.

T: Do you [see] any connection? Or did you learn any connection between [the Pythagorean theorem and irrational numbers]?

F: Um, I learned it but I don't think I so much remember it [laughs]
T: OK, don't worry.

Exhibit 5.13. Excerpt from Flora's Interview on Irrational Numbers

"I understand it. But me, myself, I can't explain," responded Flora while laughing when I asked what irrational number meant to her. Flora concurred that the Pythagorean theorem and irrational number are in the same unit in the CMP book that they had used. Not surprisingly, she did not understand the connection between the theorem and irrational numbers. According to Flora's teacher, Karen, who is on of teacher interviewees, they did not discuss the historical connection.

The other topic discussed was *circles*. I mainly asked where π (by referring to the area and circumference formulas of a circle) came from. Several of the students were familiar with π in terms of its numerical value or its association with the circle formulas, but they were not sure about what it meant to them or from where it came. "Yes, we learn about it [π]. We didn't learn about the history about it; like we did not know where it came from, who thought of this, and where area it came from, why would they need that," said Gebar. Brice also wondered, "I have been wondering that for while. I do. π is an infinite number; computer has calculated it. But how do you calculate it…?" Brice was apparently referring to the number after the decimal when he said π is "infinite" and wondered how one would calculate that manually. In talking about circumference (c=πd) a student from the group interview said she knew π was approximately 3, but she did not understand what it really was. She wondered why it is approximated as 3 and not a different value.

Students Views on How the Used the Historical Notes in the Units

What do the students say about how they used the historical notes such as the 'did you know' segments in their classes? Again, the segments that I referenced during the interviews were fractions, negative numbers, and the Pythagorean theorem. Exhibit 5.14 provides some key excerpts from the students' accounts.

Student	Excerpt	Note
Adam	Not really. No. …Not the historical aspects.	I showed him 'did you know' notes for the unit *Bits and Pieces I*. He said they do not use the book that often.
Bella	Well, I didn't even know they were in the book. Actually, I didn't know there was 'did you know' sections…	I specifically showed her the 'did you know' notes for the units *Bits and Pieces I* and *Accentuate the negative* as I was asking the question.
Brice	Well, we do how to do it but we usually don't do where it's come from because we have a lot of work to do. We usually don't have much time… to do that stuff. But not usually.	He was not asked specifically about the 'did you know' sections. He was responding generally to whether he found answers to what he wondered in his math class.
Flora	She would sometimes give us assignment like 'look at page this and this'. We will take out paper and do it…And if people had questions, the teacher would go to them and help them… …but we don't really like reading the books, we like to hear it and just take notes. It goes through… let's learn more hearing it and seeing it from a person that is his job.	Her class used the unit *Looking for Pythagoras*, but she said she did not remember learning the history of the Pythagorean theorem.

Gebar	We didn't cover the 'did you know' sections.	Gebar said they did not use the CMP books much for their algebra class.
Kaleb	No, just the triangle part.	By 'triangle part' he was referring to a right triangle and tried to describe what they learned about the Pythagorean theorem. But, he expressed that they did not use the historical parts in the book.
Semhar	[the teacher] will make us read all of these… then she will usually give us 2-3 problems out of there	I showed her the 'did you know' note of unit *Bits and Pieces I* as I was asking the question.

Exhibit 5.14. Excerpts from Students' Interviews on How the Historical Notes in the Units Were Used

Other than Semhar and Flora, who said their teachers would ask them to read the 'did you know' sections, the rest of the students pointed out that they did not use or cover the historical notes. Some of the students were not aware these notes existed, as Bella noted. One of the reasons given by some of the students is that they did not use the textbooks often. For instance, Adam said, "we've basically been doing a lot of hands-on activities instead of textbooks; we do a bunch of worksheets and stuff." Gebar pointed that they did not use the CMP book in his algebra class.

The students' descriptions of how the historical notes are used in their classes are consistent with what the teachers described (I previously termed this as *mostly–non-use to limited use*). The general consensus among the students was that their teachers did not incorporate the history of mathematics. Bella's response reflects the consensus: "…Not usually. No. they don’t explain the history of or how it was discovered if you do this. Usually, we don’t get the history on it, just how to do it and why.”

Are Students Interested in HD of Math?

Are students interested in the history of the mathematics concepts they learn? I posed this question toward the end of the interview. At this point, after a series of probing questions regarding the history of specific topics they had learned already, the students had some idea about the main issue of the interview –the use of history in learning mathematics. Still, I admit that this question is problematic in the sense that students were being asked if the history of mathematics would be helpful (or if they would be interested in such issue) without having such learning experiences. How would they know? One way I attempted to explore their interest in history was via the wonderment question. For instance, after the series of probes-responses presented in Exhibit 5.13, I asked Flora if she would be interested in knowing more about the history of irrational numbers. Each student was asked about their desire to know more history in a similar fashion.

Student	Excerpt	Note
Adam	I think it would. Because again a I think knowing… more about the subject you are learning gives a better understanding, makes it easier to do the subject. So I think it would be nice to learn more about where the history of the subject came from.	He said he would be interested in the history of mathematics.
Bella	I don’t know if it would be very useful, but I think I would find it interesting. I have always liked history a lot. And, I think the history of things we use every day is fairly interesting. Like where the names came from or how it was discovered. I think it would be interesting to understand how people had invented it.	Earlier she noted wondering about discoveries and how it would be interesting to learn such things.
Brice	Well, in my perspective, knowing the history helps	Brice mentioned at

	because then you know where they got it from. You know all the different things and just knowing those basic facts gives more knowledge or more confidence in your knowledge.	several points that he is very interested to learn more history. Toward the end of the interview, he said, "now I really wanna know".
Flora	Um…[pause]. Sometimes. Because I don't really like history, I think it is boring. But some of it…just sometimes, not all the time. It's not that important to me to bring it up.	She paused and hesitated…With encouragement to share whatever she felt, she responded [with some laughter] that she did not like history.
Gebar	…Yes, I would like to learn about the history of core concepts as you say, but maybe not the more picky things like, um …I don't know… I can't think of actually…	
Kaleb	I would like to know like where all the ideas come from, where they learn how to do, who showed them…how they used [solved?] the problems, like what the numbers were like…	He said he is interested to learn history and hoped to learn more about that in high school.
Semhar	[Nodded]	She said she studied some history of mathematics like the use of the abacus by the ancient Chinese in her World Culture class. She agreed that she would be interested if her math class included such history.

Exhibit 5.15. Excerpts from Students' Interviews on Their Interest in HD of Math

With the exception of Flora, all the students expressed interests in learning about the history of ideas. After some hesitation, Flora said, "I don't really like history, I think it is boring," when I encouraged her to share what she felt and that I was not looking for a particular answer. Brice, by contrast, appeared to be the most enthusiastic about his interest in learning about the history of mathematics. "Now, I really wanna know," said Brice. "…you like apples. You want an apple. You see an apple on the tree. You want to get a ladder, be up there to eat an apple even more," enthused Brice. The other interviewees' expressions of interest fell between Flora's and Brice's, leaning more toward Brice. While most of these students gave brief responses expressing that history would be helpful in learning mathematics, Bella gave a more nuanced explanation. She explained would be interest in the history of mathematics for the sake of knowledge about history, but she said was unsure if that would help her in learning mathematics:

> I don't know if it would be very useful, but I think I would find it interesting. I have always liked history a lot. And, I think the history of things we use every day is fairly interesting. Like where the names came from or how it was discovered. I think it would be interesting to understand how people had invented it.

This well stated concern. I was impressed that a 7^{th} grader was able to state this so eloquently. Bella's interest suggests an instance that students could be curious to learn the history of mathematics just for the sake of knowing that most of the interviewed teachers' views seem to underestimate.

SUMMARY

In this chapter I discussed the interview accounts of teachers and students on the incorporation of the history of mathematics into mathematics instruction. The accounts reveal that the notion of incorporation is an 'unfamiliar' territory for the interviewed teachers. Most have not consciously taken such an approach in teaching. Taking this fact into account, the interviews explored their perspective on HD by focusing on instructional issues that are familiar to the interviewees and issues that can lead to history/foundation of mathematics. That is, the issues raised included how concepts are introduced, how inquiries that draw on history of mathematics are addressed in instruction, how the historical aspects in the textbooks are used, and finally what their views on the general importance of this subject to school mathematics are.

Teachers noted the following customary ways of introducing core concepts: *reviewing previously learned concepts* and *making connecting to students' life experiences and to real-world applications*. Initially, none of the teachers except Laila mentioned using the history of mathematics when introducing core concepts in their classrooms. Some teachers noted making occasional historical connections during introductions when specifically asked if they had done so. But, the instances they referenced were limited to general historical facts such as "who Pythagoras was" and not about concepts (e.g., the Pythagorean theorem and negative numbers).

There was a consensus among the teachers that most of their students do not inquire about how the concepts they learn came about. Several teachers noted that some students from their advanced placement (AP) classes might raise historical questions. Those teachers explained this perceived difference as a result of differences in students' level of understanding rather than differences in students' interest on the subject matter (i.e., the teachers did not suggest that AP students as being more interested in history of math than the regular class). On their part, the teachers admit that they do very little to incorporate historical inquiry into their teaching.

Overall the teachers' use of the historical aspects in the CMP units range from non-use to very limited use. Various reasons given by the teachers for such limited/non-use of the historical aspects include: *lack of knowledge of the history of mathematics*, *lack of time*, *beyond students' level of understanding*, *lack of student interest in the history of* *mathematics*, and *historical aspects not part of Instructional Planning Guide*. Some of these concerns are shared by the curriculum experts, as noted in Chapter four.

Interestingly, most of the teachers indicated that HD could be important in their instruction in some ways (see Exhibit 5.8) when asked about the importance of HD in their instruction toward the end of interviews. Some teachers such as Laila, Eden, and Mark suggested that incorporating HD is important, and it can be done even under the current educational circumstances with some support such as professional development, or provision of historical resources. However, some (Barb and Karen) remained very skeptical.

Most of the students agreed that their teachers did not use the history of mathematics (or very little if at all) in their instruction and that they did not use the limited historical notes in their textbooks. Two students mentioned using the historical notes as class readings. Generally the interviewed students expressed their primary concerns as knowing *how to do it* and *use it* ("it" referring to the school mathematics). This is consistent with the teachers' assessments.

However, the general teachers' view that most of the students are not interested in learning the historical background of math is not supported by the students' accounts. With the exception of one student, the interviewed students indicated that they would be interested in knowing the history of math ideas, especially if the historical aspects pertain to origins and evolutions of mathematics ideas (e.g. how it came about or why people thought irrational numbers in the first). This is reflected in the students' responses to a series of probes concerning their thoughts on the meaning of concepts (e.g., fractions, negative numbers, irrational numbers, and π) and their desire to learn the history of these types of concepts.

All of the students indicated in some ways that they have wondered about where the ideas they learn came from (see Exhibit 5.11). Some of them (Bella, Brice, and Adam) managed to provide more elaborate accounts of what they wondered about. These students' accounts suggest that they did not share their wonderment with the class or the teacher. This seems consistent with the teachers' assessment that most of their students do not raise historical questions. However, this does not mean that students are not interested in or have not wondered about the history of mathematics as most of the interviewed teachers seem to believe. After all, the teachers, as they indicated themselves, rarely addressed historical inquiry. They may not have a good ground for the claim that students are not interested in or would not benefit from learning the history of mathematics concepts. This discrepancy between the accounts of the teachers and students needs further exploration.

Chapter Six: Conclusions, Implications, Limitation, and Recommendations

CONCLUSION

Based on related literature and personal interest and experience, I argue that the history of mathematics is a useful tool in learning mathematics. By considering the case of a reformed-based curriculum, the Connected Mathematics Project, this study explored the nature of the use of historical aspects in middle school mathematics. CMP was selected as case for this study because, among other reasons, it has some special features containing historical references. These aspects of the curriculum established talking points for this study: What are the nature and purposes of these aspects? What are the perspectives of those who use the curriculum? By exploring these questions, the goal is to extend the conversation (both on theoretical and empirical basis) on the need for serious incorporation of historical dimensions into school mathematics.

This dissertation study used the CMP text and interviews as primary sources. Analysis of selected CMP units and the interviews with three curriculum experts addressed questions dealing with the nature and purpose of historical aspects in the curriculum. Interviews with middle school teachers and students focused on how historical aspects are addressed in school mathematics. Specifically questions covering the following issues were raised in the interviews with the teachers and students: how concepts are introduced, how teachers and students deal with foundational issues such as origins and the development of ideas covered in school mathematics, how they use historical aspects featured in the textbooks, and what they think about the importance of the history of ideas.

The CMP curriculum reflects a general historical awareness. According to the curriculum experts' accounts, they made a conscious effort to include historical references when they saw fit. But, the references are limited and incorporation was not seriously or systematically considered. As far as the use of the historical aspects in classrooms, the interview accounts with the teachers reveal that these aspects are rarely used. In fact, the interviews with the experts and the teachers reflect a strong sense of skepticism about the practicality and relevance of the history of mathematics in school mathematics. For serious incorporation of the history into school mathematics, the roots of this skepticism need to be understood.

The Nature and Purposes of Historical References in the Curriculum

Historical awareness plays an important role in curriculum development (ICMI study, 2002). Even without including explicit historical content, historical awareness can influence the choice of problems and activities in the curriculum materials.

The CMP curriculum exhibits some level of historical awareness. Explicit historical content and some problems in the CMP units reflect a conscious effort to address history. According to the interviewed authors' accounts, there was a general consciousness of the history of mathematics in the development of the curriculum but no serious deliberation on the issue of incorporation or the use of history as a main thread in the curriculum.

The historical aspects in the CMP curriculum have two main purposes: (a) to recognize *contributions of various cultures to mathematics*—to portray mathematics as a human endeavor and (b) to address *historically challenging concepts*. The first purpose relates to addressing a multicultural agenda in mathematics education (e.g., ICMI study, 2002; Nelson, Joseph, & Williams, 1993). The second purpose can be related to the underlying notion of the epistemological obstacle (Herscovics, 1989; Radford, 1997; Furinghetti & Radford, 2002).

Generally the explicit historical content in the CMP units align with the goal of *recognizing contributions of various cultures* to mathematics—portraying mathematics as a human endeavor that has been practiced for centuries in various cultures. A number of the historical references from the reviewed units (e.g., see Appendices B-2, B-4, C-4, D-2, and E-2) serve this purpose. These historical notes make references to various cultures including ancient Egypt, Greek, Babylonia, ancient China, Hindu civilizations, and medieval Europe. These notes convey the message that the mathematics concepts covered in the units were known or used in some way for a long time.

A number of the problem situations in the reviewed units appear to address *historically challenging concepts*. The historical elements in these problems are implicit. But, the contexts of the problems suggest an important historical consideration was made to address the roots of the concepts in question. For example, problems about negative numbers (Appendix D-3), circles (Appendices C-3, C-4), and the Pythagorean theorem and irrational numbers (Appendices E-3, E-5, E-6, E-7, E-8) can be considered historically inspired. The chip board model for addition and subtraction of negatives resembles the Chinese counting-rod (Kangshen, Crossley, & Lun 1999); "squaring" the circle is used to introducing the area of a circle; and, the Pythagorean theorem is used as a context in discussing irrational numbers.

However, the historical aspects in CMP units are generally limited in number and detail. These aspects lack supporting remarks for use in instruction. They do not provide follow-up questions that provoke discussion on the history. Not surprisingly, the interview accounts suggest that the historical aspects are not used in a serious manner.

From the interviews, other than rare use as class readings, there was not a noted effort by the teachers (with the exception of one) to use historical references. All in all the interview accounts suggest that there is not much attention given to using the historical aspects in the units.

The historical elements in the CMP are best described as part of a broader agenda of situating mathematics in a context rather than a deliberate incorporation of history. According to the two authors I interviewed for this study, historical incorporation was not seriously considered as a theme in this curriculum. "We did not have the specific incorporation of historical ideas as a major thread through the materials. It happened because we were interested in engaging kids, in giving them information that might hook them into mathematics," stated Mana, one of the CMP authors. "We made a judgment that kids would be engaged by problems that were set in contexts that they experience every day. So, that's why we did not use historical settings very extensively," noted Gary, the other author interviewed for this study. Although they agreed that historical aspects might be helpful to the students, in some instances, they shared concerns if history could engage the students and if incorporating history comes at the expense of a familiar context for students.

Concerns on Practicality and Relevance of History for School Mathematics

Skepticism over the practicality and relevance of the history of mathematics in middle schools mathematics is prominent in the experts' and the teachers' accounts. These interviewees share what the ICMI (2002) report noted as practical and philosophical concerns about historical incorporation into school mathematics. The practical concerns relate to not having enough time, resources, and training and needing to meet system wide regulations such as assessments and curriculum requirements. Philosophical concerns relate to fundamental beliefs and knowledge about the importance of history for school mathematics (i.e., whether it can help students learn mathematics and whether they have an interest in such topics). I focus on the latter because I believe that the interview accounts suggest teachers' concerns are rooted more in their beliefs on and knowledge of the relevance of the history of mathematics to their teaching.

I pose and discuss three fundamental questions to challenge this skepticism:

- There seems to be a consensus among the interviewed teachers that historical aspects of mathematics are not of interest to most of their students. But, as noted, the students' accounts suggest that they are interested in learning such history. Although the credibility of these accounts should be questioned, I still find this question legitimate: Does the discrepancy between teacher and student accounts regarding student interests represent any real difference in the assessment of students' interests in learning history or not?
- Suppose the assessment of the teachers that students are not interested in the history of mathematics is more or less accurate. Should curriculum and instruction be only geared to the presumed interests of students?
- From the interview accounts and the CMP units, I noted that there is considerable emphasis on connecting mathematics to students' experience and to application. So, what is lacking in instructional approaches focused on these issues? What does historical incorporation offer that may be lacking in such approaches? I will address this question under implications.

The curriculum experts and the teachers seem to share the sentiment that middle school students would not find the history of mathematics interesting and that they would not find it relevant. They instead argue that most students at this level are interested in something that they can relate to or that they can use or apply in the future; thus, they assume students do not care about where mathematics ideas come from. Gray, one of the curriculum developers, noted that he finds the historical evolution interesting as a mathematics teacher. However, he was skeptical that middle school students and teachers share his interest. He noted that there is not a reverence for history in contemporary American culture. Most of the interviewed teachers appeared to share Gray's skepticism. This sentiment calls into question the importance of the history of mathematics in school mathematics. Although valid, this skepticism must be challenged.

The teachers' accounts suggest that they view their students as mostly interested in tasks that have "attainment value" and "utility value" (to borrow from Eccles (1983), cited in Wigfield, Ecccles, & Rodriguez, 1998). In their view, students are interested in learning tasks that can help them move to the next level in school achievement and tasks that the students view as useful for the real world. Although the students' accounts support this view help by the teachers, their accounts do not support the teachers' sentiment that students are not interested in historical aspects. When asked to describe what they would like to learn about core mathematical concepts in school mathematics, most of the students suggest that their primary concerns relate to learning and knowing how to *do* it and *use* it ("it" being math). But, they also indicated, on further probing, that they are interested in knowing the history of mathematics, especially when the historical aspects are concerned with the origin and evolution of ideas. Some of the students indicated that they have wondered about the origins of math ideas, but they did not articulate their wonderment to their teachers. Despite my efforts to elicit honest responses (such as telling the students that there are no right or wrong answers for my questions), the credibility of these accounts still should be questioned. On the other hand, the teachers' perspective that students are not interested in history of mathematics also should be questioned. This area may require further study; further research exploring students' interests and learning needs with respect to the historical background of ideas may be needed.

The second question raises the issue of the role of students' interest in determining what students need to learn. Suppose the shared view of the interviewed teachers that students are not interested in the historical aspects of math is more or less accurate. Can students be interested in things they have no relevant knowledge of or experience with? Should not education help students expand their interest in something worth learning? I think it should. Discussing the dilemma of balancing the child's interest and what the child needs to learn, Dewey (1990, p. 193) in *The Child and the Curriculum*, states that

> Interests in reality are but attitudes toward possible experience; they are not achievements; their worth is in the leverage they afford, not in the accomplishment they represent. To take the phenomena presented at a given age as in any way self-explanatory or self-contained is inevitably to result in indulgence and spoiling.

The psychological notion of interest is closely related to intrinsic motivation or a desire from within to do something (Wigfield, Ecccles, & Rodriguez, 1998; Ryan & Deci, 2000). Wigfield et al. identify two types of interest: individual interest, which is a more stable disposition that assumes knowledge and value about a topic, and situational interest, which is a desire or curiosity that arises from a given situation. These notions suggest that learning about the value of certain situations can help students expand their interests. Bruner (1960, p. 20) suggests that education should help the learner develop a positive attitude toward learning.

> Mastery of the fundamental ideas of a field involves not only the grasping of general principles, but also the development of an attitude toward learning and

> inquiry, toward guessing and hunches, toward the possibility of solving problems on one's own.

The teachers have valid reasons concerning the interests and needs of the students. But, I do not find the argument that history of mathematics is not of interest to students justifiable when they have not experienced such an approach. Teachers have responsibilities to help students expand their interests. The history of mathematics can be a helpful tool for developing interest toward learning mathematics not only for utilitarian purposes but also for learning the subject for its own sake (ICMI study, 2002).

IMPLICATIONS

In order to draw any useful lessons from this study, first and foremost, the scope of the study should be understood. This study examined how historical aspects of core concepts are addressed in a case of contemporary reform-based middle school mathematics curriculum (CMP) in the United States. Limitations of the study are noted below to help understand the scope of the study. I have also forwarded specific recommendations for research and curriculum development. I would like to focus on the implications for learning by taking into account what transpired as a dominant perspective among the interviewees in this study. The interview accounts and the CMP curriculum suggest that there is a considerable emphasis on connecting mathematics to students' experience and the potential usefulness (application) of core mathematics concepts. The teachers' interviews suggest that they often introduce core concepts by make connections to previous knowledge (review), students' experiences, and real-world applications. Making connections to prior knowledge/experience and striving for transfer of learning are very important as suggested in contemporary learning theories (e.g., National Research council (NRC), 1999). However, emphasizing such connections alone misses important aspects of developing knowledge. I argue that use of history can complement instruction to fill in these missed areas.

A historical approach can make instruction more relevant to students' experiences. There was a shared view among the interviewed teachers that history is about the past and therefore it does not offer anything relevant to learning of mathematics. I disagree with this. One of the tenets of cognitive theory suggests that learners construct their understanding by connecting new information to prior knowledge (NRC, 2001). Students come to school with a range of prior knowledge, skills, and beliefs that affect their learning of new knowledge (NRC, 1999). But, at times, prior knowledge can impede learning new knowledge—a learner's prior knowledge may reinforce conceptions that may not fit the knowledge to be learned. Lionni's *Fish is Fish* story cited in the NRC (1999) offers an interesting metaphor of how prior knowledge plays in construction of new knowledge. The story focuses on a fish who learned a lesson about land animals from his frog friend. The fish's depiction of what he learned showed that he conceived land animals as fish-like creatures, and humans as fish-like beings who could walk. The moral of the story is that prior knowledge and beliefs of the learner can lead to a conception of something that does not fit the targeted knowledge. Various examples of the effects of prior knowledge on leaning mathematics exist. Herscovics (1989) cites errors that arise from students' tendencies to transport arithmetic knowledge to algebra and errors that occur in translating word problems into algebraic forms. Children tend to transfer their knowledge of operations such as counting numbers on to fractions, which can result in errors (e.g., Mark 1990).

Conceiving in a certain manner due to influences from prior knowledge is at the heart of what Bachelard (1938) refers to as an *epistemological obstacle*. Historical perspective can provide insight into the difficulties and errors associated with the concept. This can help in anticipating similar conceptual difficulties that students face. So, when teachers say that they make connections to students' experience, it is fair to ask how they help students accommodate the new concept into their prior knowledge and how they anticipate possible "misconceptions". Historical perspective presents opportunities to investigate past obstacles and use such an investigation to anticipate similar problems students may face.

Use of history may support the transfer of learning. As noted, the teachers are very concerned with helping the students use what they learn. This is understandable because the transfer of learning (being able to use what is learned in school to life and other parts areas of study) is at the "heart of the educational process" (Bruner, 1960; Bransford & Schwartz, 1999). The teachers recounted teaching mathematics by relating it to students' experiences and to applications. The experts also pointed to the fact that CMP is contextualized mathematics. Contextualizing learning can be useful but not sufficient for the transfer of learning (e.g., Bransford & Schwartz, 1999; NRC, 1999). Transfer of learning is a complex notion (what constitutes transfer and how to assess transfer are complex subjects). Bransford and Schwartz's redefinition of the concept suggests that not only "knowing that" and "knowing how" but also "knowing with" is important for transfer of learning. "Knowing with" provides a context that guides noticing and interpretation. I would argue that incorporating historical aspects can serve as "knowing with". Through historical incorporation, various process related aspects such as errors, surprises, conflicts, changes of meaning, conventions, and representation linked to the history of mathematical ideas can be brought to the fore in the learning of the idea.

Further, historical perspective may add an inquiry dimension to the pervasive utilitarian-driven approach. Thus far, I have argued that incorporating historical aspects does not necessarily make instruction irrelevant and less applicable to students' experiences but rather can support in enriching students' experiences. The interviews and the reviewed curriculum suggest there is a high commitment to the utilitarian purpose — helping students to be good "users" of mathematical knowledge. To be fair, CMP units are problem-centered; they contain discovery oriented problems. Many of these problems, with the right support from teachers, can lead to deeper inquiry about the ideas in question. Use of history can enhance the inquiry by offering situations that provoke question but the very nature of mathematical knowledge. I present the following examples to elaborate as to how historical aspects can help add an inquiry dimension to the mostly application oriented problems:

- Consider Appendix D-2 on negative numbers. This historical note presents a background information about negative numbers that invites further inquiry about negative numbers: Why did 17^{th} century European mathematicians not accept negative numbers? Why did they think negative numbers were absurd? This can be extended to assess student views of negative numbers in general.
- Consider Appendix E-3. The emphasis on this problem is application—solving the problem using the Pythagorean triplet 3-4-5. This discussion can be complemented by further inquiry: Did ancient Egyptians know the Pythagorean relation? When? If this was before the Greeks, then why is it named after Pythagoras? Why did the Egyptians have to make plots with right angles?
- Consider irrational numbers (Appendix E-7). This investigation about irrational numbers asks how long $\sqrt{2}$ is. This example demonstrates that a segment with that length can be precisely drawn but no exact decimal value can be found for $\sqrt{2}$. This problem presents conceptual challenges. Tewelde, one of the teachers interviewed, noted that some of his students wondered why an irrational number such as this does not end (in decimal form), but it can be represented by a

line segment that has ends (i.e., while it measures a limited dimension). Peled and Hershkovitz (1999) made similar observations. In their study on irrational numbers, Peled and Hershkovitz found that some students found it difficult to believe the side of a square with an area of 5 sq. meters can be measured because its decimal representation involves infinite digits. A historical perspective can help in this respect – the Pythagorean relation and how such a relation gave rise to what we call irrational numbers.

By way of these examples I suggest that the very nature of knowledge/ideas that is tacitly accepted at times need re-examination. The history of a concept provides a context to carry out further inquiry. The incorporation of history in this manner presents an opportunity for students to learn math not only as 'users of knowledge' but also as possible mathematicians themselves and creators/developers of mathematical knowledge.

Incorporating historical aspects in the manner discussed above demands knowledge about the history of mathematics. "Even the teacher who is not a historian should have acquired historical evolution of the subject" (ICMI study, p. 209). Educators (e.g. Ball, 1991) identify three types of knowledge (substantive knowledge, pedagogical content knowledge, and knowledge of the discipline) as essential for mathematics teaching. Substantive knowledge refers to subject-matter knowledge; Shulman (1986, p. 9) describes pedagogical content knowledge as "the ways of presenting and formulating the subject that make it comprehensible to others". I want to stress knowledge of the discipline because this kind of knowledge can be considered as part of the historical aspects. Knowledge of the discipline "includes understanding of the ways in which knowledge is created and the canons of evidence that guide the inquiry" (Mewborn, 2000, p. 5); it covers knowledge about the nature and discourse of mathematics (Ball, 1991). According to Ball, this type of knowledge is rarely part of the "explicit" curriculum in school or college; students and teachers are left to “develop assumptions about the nature of mathematical knowledge and activities from experiences in mathematics classes” (p. 7). The interviewed teachers for this study acknowledged that they did not take any history of mathematics courses in their teacher education. Fred, who was working as math specialist and facilitating workshops and training, noted that most middle school teachers come to the profession without having taken any history of mathematics courses. Several of the teachers suggested that some form of professional development dealing with historical aspects of the subject would be beneficial. It is unrealistic to expect use of history when the teachers do not have the necessary training and resources.

LIMITATIONS OF THE STUDY

Recognizing the limitations is important in order to place the study in perspective and to draw out the lessons that can be learned from this study. A general limitation in making a case for the use of history in school mathematics lies in lack of evidence on the effect of this approach on students' mathematics performance. The arguments made to justify the importance of history of mathematics rely on qualitative accounts—anecdotes, case study reports, expert views, and philosophical positions on learning. Such arguments fall short of convincing those interested in finding evidence on how the history of mathematics can directly enhance student achievement in mathematics. Incorporating historical aspects in learning may have long term effects, something that is hard to assess with traditional measures. Tracking the progress of students over the years and in real situations (ICMI, 2002) may be needed.

This study relied on the content of CMP units and interviews as primary sources of data. The interviews involved curriculum experts, teachers, and students. All the teachers (eight) and the students (seven individual and a group of four students) were from one school district, and participation was voluntary. This group of interviewees is by no means representative, and I do not make any claim that their views or any findings from this study reflect the views of other teachers and students who use the Connected Mathematics curriculum. Instead, the purpose was to provide a credible case for how historical aspects are addressed in school mathematics. Because of this, informed and voluntary participants were used. The trade-off in using a smaller group of interviewees is depth over breadth. However, there is no denying the limitations that come with such an approach. In an effort to establish credibility for the accounts, I used cross-checking in the analysis of interviews of the various groups (experts, teachers, and students) and presented the experts and teachers with the chance to learn more by sending descriptions of the purpose of the study and possible interview questions before interviews. I also presented them with the chance to review and comment on the interview transcripts.

The notion of historical incorporation is not well defined. Historical incorporation in this study means making use of the relevant history of core mathematics concepts as part of instruction with the aim of helping students learn about the foundations of the concepts rather than simply focusing on mathematical knowledge and skills. Relevant history includes historical facts (e.g., biographical sketches of key figures, dates, places) and brief socio-cultural aspects that affected the development of the ideas. But, more importantly, incorporation involves presenting students with the opportunity to work on problems inspired by the history of the concept. Incorporation can be enacted in forms such as class discussions, student research, and teacher presentations However, this definition still leaves important questions unanswered, for example: How often would such historical incorporation take place and to what extent? Would there be enough instructional time to do that? Addressing these and other questions depends on various factors. Part of the purpose of this study was to explore these issues.

The history of the selected topics reviewed in this study relies on limited historical contexts. The math concepts students learn in middle school generally have a long history. So what part of the history of a concept and from what context/culture (time, place) do we incorporate? I have provided brief reviews of the history of core concepts (except circles) selected for this study. This review covered limited historical contexts such as Mesopotamia, ancients Egypt, Greek, China, ancient India, Medieval Arab, and medieval and modern Europe because of the availability of recorded sources in the English language. Although this coverage is limited, the main purpose was to highlight significant historical elements and to provide a contrast to the historical aspects found in the CMP units. Furthermore, the prime purpose is not to provide an extensive history of the topics but to offer examples of historical background for pedagogical purposes.

RECOMMENDATIONS

On the basis of this study, the following recommendations are forwarded:

For Research:

- Research on the history of mathematics concepts for educational purposes needs to be extended. Though there is ample literature on the history of mathematics, there seems to be a shortage of sources that focus on a particular topic/idea (e.g., a history of negative numbers or irrational numbers) for instructional purposes. The challenging task of doing this kind of research involves identifying sources, including educational materials/artifacts from various time periods, and synthesizing the evolution of a selected concept with special attention to the various forms of representations used and the conceptual challenges encountered.

- Research aimed at understanding conceptual difficulties faced by middle school students is important. Evidence from such research might be useful in better understanding the ways in which the history of mathematics can be used in school mathematics. The following two recommendations are in line with this.
- Research exploring the conceptual challenges faced in transition from concrete model to operational/formal approaches in learning negative numbers may be helpful. Particularly, in relation to CMP, the unit *Accentuate the Negative* introduces addition and subtraction of negative numbers using models such as the number line and chip board but uses operation patterns to introduce multiplication and division of negative numbers (Appendix D-4). There seems to be a sudden leap from the model approach to the use of operation patterns without any explanation that may ease this transition. Do these models present an epistemological obstacle to making sense of the multiplication operations of negative numbers? Research focusing on how teachers and students make sense of negative numbers as they transition from the concrete models to the operations and how teachers help students make the transitions can be worthwhile.
- Research on what irrational numbers mean to students and how teachers address the conceptual challenges associated with this concept is needed. Tewelde, one of the teachers interviewed for this study, shared an interesting observation that his students wondered why an irrational number never ends (when written in decimal form) while it can be represented on a segment (a line that has ends!) of a number

line. The conceptual difficulties associated with irrational numbers and how teachers and the curriculum address these difficulties need further investigation.

- A broader study assessing the need for learning the history of core mathematics ideas to support the learning of mathematics is important. There seems to be a gap between the teachers' views of their students' interest in learning history of ideas and that of the students. The general teachers' sentiment that most of the students are not interested in learning the historical aspects is not supported by the students' accounts in this study. With the exception of one student, the student interviewees indicated interest in knowing the history of mathematics concepts, especially when the historical aspect is concerned with the historical evolution of an idea. Further investigation on this issue involving more students and teachers would be worthwhile.

For Curriculum Development and Practice:

The CMP curriculum has some relevant and interesting historical aspects that, if expanded, could be helpful in the effort to situate mathematics in a socio-cultural context. A more serious deliberation on the incorporation of history into mathematics curriculum is desirable during curriculum development/revision. One of the main challenges would be addressing the balance between content and the historical aspects. One generally suggestion is that at least the units that introduce core concepts could add some more history. For the reviewed topics, the following specific recommendations are forwarded for consideration to expanding the historical aspects:

- Some of the historical elements in the units can be enriched by including original forms of representation used by past cultures rather than only presenting them in modern representation without any remarks. Such additions would be helpful in highlighting changes in mathematical ideas over time. Appendix B-3, for example, recognizes Egyptian unit fractions and presents it in modern notation without any remarks. As it stands, it is not clear how students should interpret notes of this nature. Supplementing what is already in the text with ancient representation such as presenting unit fractions in Egyptian heliographic (heretic) can demonstrate changes of mathematical ideas over time.

- With respect to the unit *Covering and Surrounding*, extending 'squaring a circle' (Appendix C-3) to include some simpler forms of the *exhaustion method* should be considered. Or, at the very least, an explicit historical reference to the use of such methods by mathematicians (e.g., Archimedes) can be helpful. Available computer technology (e.g. Geometer's Sketchpad) can be useful to demonstrate some forms of the exhaustion method.
- The unit *Accentuate the Negative* needs more historical information than what is currently provided. This unit contains only one explicit historical reference (Appendix D-2). The concept of negative numbers has a long history that reveals people's struggles with the concept. Today's students likely have similar difficulties (e.g., Hefendehl-Hebeker, 1991; ICMI study, 2002, p 244). The segment in Appendix D-2 can be expanded by including questions addressing why 17th century European mathematicians did not accept negative numbers and why they thought negative numbers were absurd. Discussion on these kinds of questions may help unravel conceptual difficulties students face.
- The historical aspects need to include information that does not necessarily conform to established ideas. The texts need to add some critical remarks express divergent views. The presentation of the limited historical aspects in units may give the impression that past concepts developed progressively to the modern

concepts. “Reading the past through the categories of the present,” to borrow Kitcher and Aspray’s (1988, p.47) words, takes the risk of portraying a distorted history. For example, consider the unit *Looking for Pythagoras*. This unit makes some references to the ancient Greeks and Egyptians with respect to the Pythagorean theorem, but it fails to include any references to the Babylonian or the ancient Chinese uses of forms of Pythagorean relations. There are historical sources that suggest that some forms of the Pythagorean theorem were known to these cultures before the time of Pythagoras. The unit can be enriched by not only making references to various civilizations’ use of ideas related to the theorem but also by including questions for discussion on how different cultures used similar mathematical ideas. Further, this allows students to discuss whether the theorem was known before the Greeks and why it has come to be known as the Pythagorean theorem.

- Explicit historical notes linking irrational numbers and the Pythagorean theorem can be helpful in the unit on this theorem. The unit introduces irrational numbers following the Pythagorean theorem. There is some reference to the Wheel of Theodorus (Appendix E-8). However, this seems to suggest that the Greeks used irrational numbers. The Pythagoreans recognized the existence of “magnitudes, areas, or lines, which did not have ratios represented by integers” but their numbers were still integers; they did not extend their numbers to include what we now call irrational numbers (Jones, 1994, p. 176). Explicit historical notes about the problems that led to the emergence of irrational numbers and the confusion of the Pythagoreans with those numbers can be helpful to discuss in order to help students put into perspective the problems they have with irrational numbers.

- Teacher education programs should consider providing more opportunities for future mathematics teachers to learn about the history of mathematics. Also, professional development programs should consider offering more support for teachers to learn and use the history of mathematics. All of the interviewed teachers indicated that they have not had history of mathematics in their education (in pre- or in-service education). But, most said they would be interested in such a course if it was offered in the form of professional development.
- As a guide for teachers, it can be helpful to provide explicit remarks on how to use the historical aspects in the curriculum. The CMP units do not provide remarks on how to incorporate historical aspects. The teachers see these aspects as simply informational (which mostly they are, according to the curriculum authors interviewed). But, the fact that there is no guide in the units on using such aspects does not help matters. Several of the teachers indicated that they would use the historical aspects if they were specified in their Instructional Planning Guide (IPG) provided by the school district. The teachers cited that HD is not in their IPG and thus these kinds of aspects are not going to come up in the state test. They noted that they do not have time to cover topics outside of the IPG. Allotting time for areas like the history of mathematics in such a guide can be helpful.

Appendices

APPENDIX A-1: INTERVIEW GUIDE

The following were interview guide questions used for this study. Needless to say, specific probing questions were added in the process of interviewing (example, making references to the specific historical aspects in the selected units).

For the CMP Curriculum Developers

- Would you share about your professional experience by focusing on how you got into CMP?
- Your assessment on how seriously historical dimensions (HD) were considered in the development of CMP.

  - Probe focus on the nature and purpose of the historical aspects in the CMP units (why they selected the aspects and what purpose they were trying to achieve by including such aspects).
- What do you think of adding more historical aspects in the units?

  - Probe by referencing historical aspects related to the topics in the selected units but that are not included in the units (e.g., exhaustion method for estimating circles, explicit history of irrational numbers in connection to the Pythagorean Theorem).
- What was the expectation with regards to how the HD aspects would be used in the classroom?

  - Probe if teachers get some support from professional development concerning how they use them.

- Any information on how the HD aspects are used?

For the Teachers

- The introduction of core concepts in the units

  - How do you customarily introduce core concepts in a unit?

  -What connections are made to HD?

- Teacher's experience with any inquiry from students about history of ideas.

  - Do you the teacher encounter any students' wonderment about how math ideas came about?

  - Do you provoke discussions pertaining to history of math ideas?

- Use of the historical aspects in the CMP units

  - How do you use the explicit historical aspects in the textbook?

  (Probe by referencing to the specific aspects in the unit the teacher has covered)

  - How do you use the implicit historical aspects in the textbook?

  (Probe by referencing to the specific aspects in the unit the teacher has covered)

- Views on the importance of HD.

  - What do you think about the importance of HD?

  - What do you think needs to be done to incorporate HD into school mathematics? (for those who think it is important)

- Have you taken any training or do you have resources that might help you to address issues about historical aspect of math?

For Students

- What do you want to learn about new math ideas when you start learning them for the first time?

  -Probe by referencing to the topics they have learned in the selected units (i.e., fractions, negative numbers, the Pythagorean theorem, and irrational numbers)

- Have you ever wondered how the math ideas you learn started? How they came about?

  - Probe by referencing to the topics they have learned in the selected units (fractions, negative numbers, the Pythagorean theorem, and irrational numbers)

  - Probe by asking for an instance of what they wondered (if they say they have).

- Could you tell me how your math teacher usually introduces new topics (mention topics when needed)?
- How did you use these notes (make reference to the historical notes such as in the did you know segment in their textbooks)
- Are you interested in learning about the history of ideas such as how (refer to topics they covered) come about?
- Do you think historical aspects of core mathematics concepts such as when and how they started and evolved can help you in understanding the concept?

APPENDIX A-2A: REQUEST FOR PARTICIPATION

Re: Request for your participation in an interview on use of historical dimensions of mathematical ideas in teaching those ideas

Dear teacher:

I am a doctoral student in the department of Curriculum & Instruction at the University of Texas at Austin. I am doing my dissertation on the use of historical aspects of mathematics in the teaching and learning of the subject in middle schools. For this purpose, I would like to interview middle school mathematics teachers like you.

The interview will not last longer than an hour. The interview questions I will ask do not require special preparation time. You could respond to the questions by reflecting on your experience and perspective on the teaching and learning of mathematics.

To give you an idea about the interview, I am exploring how curriculum materials incorporate historical dimensions of mathematical ideas and how teachers address these historical aspects of mathematics in their instruction.

To be specific, consider a major concept of a unit you have taught, say, the Pythagorean Theorem. By historical dimensions of this concept, I mean information regarding context (time, place, and people), ideas, and problems that contributed to the development of the theorem, as well as other mathematical ideas/problems that may have evolved from the theorem. Depending on the grade level you have taught, you may think about historical dimensions of other major concepts in a similar manner.

The interview will center around such major questions as:

- How have the textbooks or other resources that you have used incorporated historical aspects of mathematical concepts?
- How do you address (or use) such historical dimensions in your teaching?

If you choose to participate in the study, I will send you the interview questions before we meet for an interview. It will be extremely helpful if you could reply as soon as possible if you decide to participate. Because of your professional experience as a mathematics teacher, you are a valuable resource for my study. Thank you for considering my request.

Thank you,

Tesfayohannes K. Haile
Doctoral candidate, Department of Curriculum & Instruction, The University of Texas at Austin. Email: tyohannesk@yahoo.com /Phone: 512-469-9170

APPENDIX A-2b: RE: REQUEST FOR AN INTERVIEW

Thursday, November 9, 2006 2:56 PM

From:

"Tesfayohannes Haile" <tyohannesk@yahoo.com>

To:

[redacted]@austinisd.org

Thanks for your immediate reply. Tomorrow, especially before 3:00PM, is not good for me. Can we do it on Monday (11/13) anytime convenient for you. Or I can do the interview tomorrow if you have time after 3 PM. If both of these times do not work for you, tell me what your availability times for next week.

In the mean time, have a look at the questions that would be focused during the interview:

- What are major topics/concepts you have covered in recent teaching?

- How do you usually introduce major concepts in a new unit in your instruction?

-Any observation of students' reactions during discussions of "new concepts"- Do you recall any instances of students puzzlement about how the concepts come to be?

-How are the historical notes about the mathematical idea provided in textbook used in instruction? (If you use the Connected Mathematics textbooks, how are the historical notes such as those in the "did you know" sections used in the class?)

-How has your teacher education equipped/prepared you with historical aspects of math?
-What do you think about the importance of incorporating historical dimensions in your instruction?

Thanks,
Yohannes

APPENDIX A -3 : REQUEST FOR FEEDBACK ON INTERVIEW TRANSCRIPT

Tuesday, January 2, 2007 5:13 PM

From: "Tesfayohannes Haile" <tyohannesk@yahoo.com>

To: @austinisd.org

Message contains attachments

JGInterview transcript.doc (76KB)

Dear Mrs. ,

I am attaching a copy of the transcript of the interview I conducted with you so that you could take a look/read and send comments (if you have).

The transcript is nearly verbatim. I have made some editing on "speech disfluencies" such uh, um, or repetition of words/phrases/sentence or inaudible words, etc. In some cases, I have put some words/phrases in bracket like this [.....]. These are words/phrases that can be implied from the interview but may not have been clearly stated. In all this, I have done my best not to misrepresent your views or what you said.

If you feel some of your ideas, or views, or responses are misrepresented, let me know. I will greatly appreciate it if could send your comments (if you have any) as soon as you can. In the transcript, my questions/probe are designated by T and your responses are designated by G.

Happy holidays.

Thanks again,
Tesfayohannes K. Haile,
Doctoral student,
College of Education, UT-Austin.

APPENDICES B-1 --- E-9: SELECTED CMP UNITS WITH ANALYSIS OF HISTORICAL ASPECTS

Appendix B-1: Overview of the Unit on Fractions

Overview

Rational numbers are at the heart of the middle-grades experience with number concepts. The concepts of fractions, decimals, and percents are often difficult for students. Research tells us that part of the reason for students' confusion about rational numbers is a consequence of the rush to symbol manipulation with fractions and decimals. Students need time to develop a deep understanding of fractions and decimals. The investigations in *Bits and Pieces I* ask students to make sense of fractions, decimals, and percents in different contexts.

The many different and powerful interpretations of and models for rational numbers can make grasping ideas about such numbers difficult. To gain a mature knowledge of rational numbers, students must be able to handle these various interpretations. We have carefully chosen the interpretations and models used in the unit. Some models are more powerful than others, as they contribute to developing the meaning of rational numbers and to understanding operations on rational numbers.

This unit does not teach specific algorithms for work with rational numbers. Instead, it helps the teacher create a supportive environment for students to grapple with interesting problems in which ideas of fractions, decimals, and percents are imbedded. As students work—individually, in groups, and as a class—they will develop ways of thinking about rational numbers. The teacher's role is to help students make explicit their growing ideas about this world of rational numbers. The intent of this unit is to provide a rich set of experiences that focus on developing meaning.

The Mathematics in *Bits and Pieces I*

In this unit, students will meet several interpretations and models of fractions. These have been carefully chosen so that the move between problems will add to a deepening knowledge and comfort with fractions.

Interpretations of Fractions

The major interpretations on which this unit focuses are

- fractions as parts of a whole
- fractions as measures or quantities
- fractions as indicated division
- fractions as decimals
- fractions as percents

Other interpretations—such as fractions as operators ("stretchers" or "shrinkers") and fractions as rates, ratios, or parts of a proportion—are postponed until later grades.

Fractions as Parts of a Whole

This interpretation of rational numbers is applied in situations that are continuous and in situations that consider discrete objects. The important characteristic is that this interpretation depends on partitioning an object or a set into equal-size parts and making a comparison of some of the parts to the whole object or set. For example, if there are 27 students in the class and 13 are girls, the part of the whole that is girls can be represented as $\frac{13}{27}$.

In the following diagram, two parts are shaded.

The shaded portion can be represented as $\frac{2}{3}$. The 3 tells into how many equal-size parts the whole has been divided, and the 2 tells how many of the equal-size parts have been shaded.

In the part-whole interpretation of fractions, the difficulties for students center on the following:

- determining what the whole is
- subdividing the whole into equal-size parts—not equal *shape,* but equal *size*
- recognizing how many parts are needed to represent the situation
- forming the fraction by placing the parts needed over the number of parts into which the whole has been divided

Fractions as Measures or Quantities

In this interpretation, a fraction is thought of as a number. For example, a fraction can be a measurement that is "in between" two whole measures. Students meet this every day in such references as $2\frac{1}{2}$ feet or 11.5 million people. Understanding this interpretation is important for students' mathematical development, and it leads to comparison of fractions and operations on fractions.

Fractions as Indicated Divisions

To move with flexibility between fraction and decimal representations of rational numbers, students need to understand how fractions can be thought of as indicated divisions. Sharing is a natural context in which to help students see how this interpretation is related to whole-number division. If students see that sharing 36 apples among 6 people calls for division ($36 \div 6 = 6$ apples each), then they can move to an understanding that sharing 3 apples among 8 people calls for dividing 3 by 8 to find out how many each person receives.

Fractions as Decimals

A byproduct of the division interpretation of fractions is the relationship between a fraction and decimal representation of the same quantity. For the fraction $\frac{2}{5}$, for example, we can find the decimal representation by dividing 2 by 5. Given the modern tools of calculators and computers, decimal representations are even more important today than in the past. Students need time to develop comfort and ease in moving between fractions and decimals, and they need to understand decimals in two ways:

- as special fractions with denominators of 10 and powers of 10
- as a natural extension of the place-value system for representing quantities less than 1

Fractions as Percents

Rather than treating fractions, decimals, and percents as separate topics, this unit seeks to build the connections between them. Students will see that the ideas and concepts are related and that the differences are in the symbols used to represent those ideas. Ten percent, 10%, is simply another way to represent 0.10 or 0.1, which is another way to represent $\frac{10}{100}$ or $\frac{1}{10}$. Percents are introduced as special names for parts of 100.

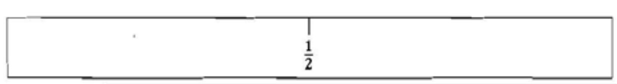
Models of Fractions

The models of rational numbers used throughout this unit were chosen because they connect directly to the interpretations of rational numbers that the unit raises. The models on which this unit focuses are

- fractions-strip models
- number-line models
- grid-area models
- partition models

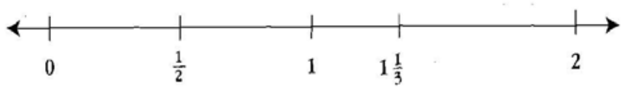
Fraction-Strip Models

Students are introduced to fractions in a situation that uses a *fraction strip* as a model. Fraction strips can be created by dividing a strip of paper into equal-size parts by folding. This is a fraction strip for halves:



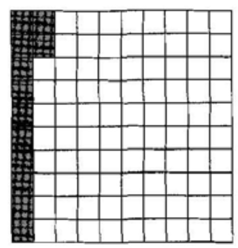
Number-Line Models

The collection of fraction strips are used to move to a number-line model of rational numbers. The ***number-line model*** helps make the connection to fractions as numbers or quantities. This is a number line for 0 to 2 with a few fractional quantities marked:



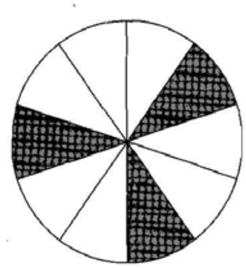
Grid-Area Models

Because 100 and powers of 10 are so useful in understanding decimals and percents, ***grid-area models*** are introduced and developed in this unit. This grid shows a shaded area of 12%.



Partition Models

Students also use a more general model of fraction situations that is based on ***partitioning an area***, such as a circle, into equal-size parts. The circle shows a shaded portion of $\frac{3}{10}$.



Mathematical and Problem-Solving Goals

Bits and Pieces I **was created to help students**

- Build an understanding of fractions, decimals, and percents and the relationships between and among these concepts and their representations
- Develop ways to model situations involving fractions, decimals, and percents
- Understand and use equivalent fractions to reason about situations
- Compare and order fractions
- Move flexibly between fraction, decimal, and percent representations
- Use 0, $\frac{1}{2}$, 1, and $1\frac{1}{2}$ as benchmarks to help estimate the size of a number or sum
- Develop and use benchmarks that relate different forms of representations of rational numbers (for example, 50% is the same as $\frac{1}{2}$ and 0.5)
- Use physical models and drawings to help reason about a situation
- Look for patterns and describe how to continue the pattern
- Use context to help reason about a situation
- Use estimation to understand a situation

The overall goals of Connected Mathematics is to help students develop sound mathematical habits. Through their work in this and other number units, students learn important questions to ask themselves about any situation that can be represented and modeled mathematically, such as: *When do we need to consider amounts that do not represent whole numbers? How can we represent concepts such as parts of a whole? Why can there be different fraction names for the same quantity? How can we tell when two names refer to the same quantity? How can we tell which of two fractions is greater? or smaller? What are some situations where fractions are commonly used? What value is there in having decimal names for fractional quantities? How can one change from a fractional name to a decimal name? Why is having a denominator of 100 such a useful tool? How is a percent like a fraction? What techniques are there for finding fractional, decimal, or percent names for the same quantity?*

Summary of Investigations

Investigation 1: Fund-Raising Fractions

Students explore three components of understanding fractions: the visual model (fraction strips), word names for fractions, and symbols for fractions. The part-whole interpretation of fractions is developed. Students make fraction strips to study the progress toward a fund-raising goal. The aim is to focus on the meaning of such phrases as, "two thirds of the goal has been reached."

Investigation 2: Comparing Fractions

The most important concept in understanding and using rational numbers is equivalence of fractions. This concept underlies operations with fractions, changing representations of fractions, and reasoning proportionally. The context of comparing fraction strips is used to motivate an investigation of equivalence and the creation of a number line that contains all of the information of the individual fraction strips. The idea of using benchmarks to estimate the size of fractions and to make comparisons is introduced.

Investigation 3: Cooking with Fractions

The context of cooking—parts of cups or other measures often called for in recipes, and the need to make multiples of a recipe, sets the stage for introducing students to different kinds of area models for fractions. The square and the rectangle are particularly useful areas because they are easy to subdivide and to shade. The circle is explored because of its use in data analysis and probability.

Investigation 4: From Fractions to Decimals

Students are introduced to decimal representations of fractions and explore the place-value interpretation of decimals. They investigate a 100-square grid and explore how it could continue to be subdivided to show 1000 parts or 10,000 parts. This process of subdividing and naming the new parts is very important mathematically; the underpinnings of the infinite process are met in this problem. The process will continue to help students understand equivalence of fraction and equivalence of decimals as well as to see the connections between fractions and decimals.

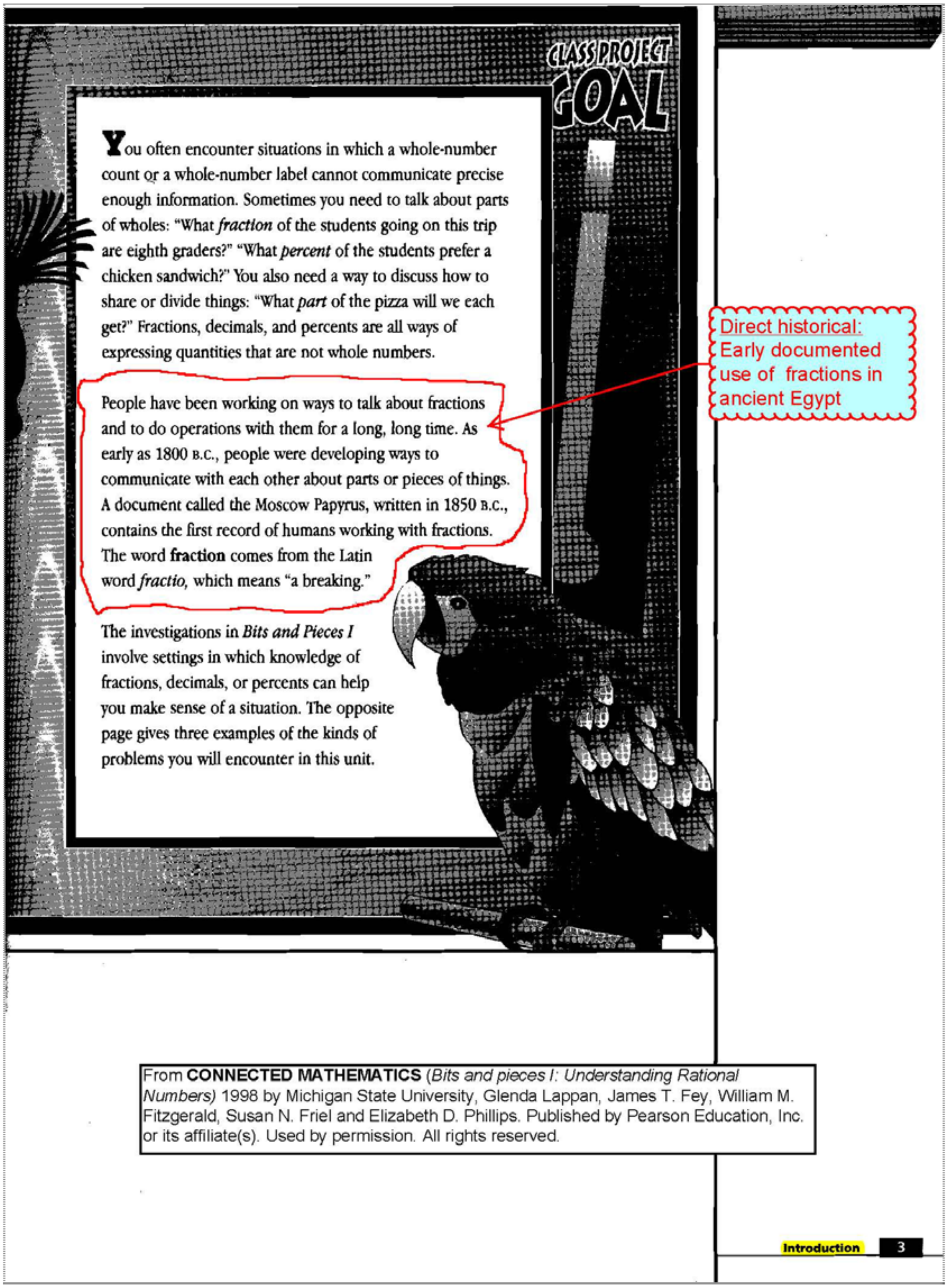
Investigation 5: Moving Between Fractions and Decimals

This investigation proposes a situation in which fractions with denominators larger than students' fraction strips show must be compared. Students find decimal estimates for fractions using the visual model. They are asked to consider whether fractions or decimals are easier to compare. Sharing is used as a context to motivate the division interpretation of fractions, leading to a strategy for changing a fraction into a decimal. Calculators are used to do the computation, providing additional evidence that the division interpretation as a way to find decimal equivalents makes sense.

Investigation 6: Out of One Hundred

By this time, students should feel comfortable with the meaning of fractions and decimals and be able to move back and forth between the two. Percents are now introduced as another form of representation. A database of information about cats is used as a context for understanding percent. Students are engaged in activities requiring them to move among fractions, decimals, and percents.

Appendix B-2: Direct Historical –Early Fraction



CLASS PROJECT

GOAL

You often encounter situations in which a whole-number count or a whole-number label cannot communicate precise enough information. Sometimes you need to talk about parts of wholes: "What *fraction* of the students going on this trip are eighth graders?" "What *percent* of the students prefer a chicken sandwich?" You also need a way to discuss how to share or divide things: "What *part* of the pizza will we each get?" Fractions, decimals, and percents are all ways of expressing quantities that are not whole numbers.

People have been working on ways to talk about fractions and to do operations with them for a long, long time. As early as 1800 B.C., people were developing ways to communicate with each other about parts or pieces of things. A document called the Moscow Papyrus, written in 1850 B.C., contains the first record of humans working with fractions. The word **fraction** comes from the Latin word *fractio,* which means "a breaking."

Direct historical:
Early documented use of fractions in ancient Egypt

The investigations in *Bits and Pieces I* involve settings in which knowledge of fractions, decimals, or percents can help you make sense of a situation. The opposite page gives three examples of the kinds of problems you will encounter in this unit.

Appendix B-3: Direct/Implicit Historical—Unit Fraction

2.2

Finding Equivalent Fractions

At a Glance

Grouping: Small Groups

Launch

- Review what students discovered about $\frac{3}{4}$ and $\frac{6}{8}$ in Problem 2.1.
- Help students understand Problem 2.2.

Explore

- Visit groups as they work, asking questions to guide them in exploring equivalent fractions.

Summarize

- Have several groups share the patterns they discovered for finding equivalent fractions.
- Ask questions to help students discover the pattern: multiply the numerator and denominator by the same number.
- Use visual tools to help students understand that when they multiply the numerator and denominator by the same number they are cutting each of the pieces into which the whole has been divided into smaller pieces.

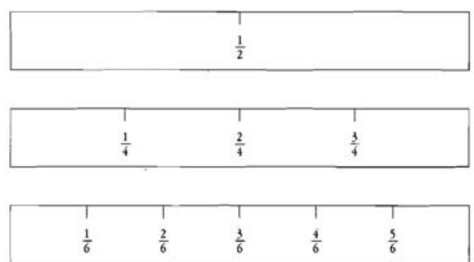
Finding Equivalent Fractions

As you worked with your fraction strips, you found that some quantities can be described by several different fractions. In fact, *any* quantity can be described by an infinite number of different fractions!

Did you know?

Hieroglyphic inscriptions from more than 4000 years ago indicate that, with the exception of $\frac{2}{3}$, Egyptian mathematicians used only fractions with 1 in the numerator. Such fractions are known as ***unit fractions.*** Other fractions were expressed as sums of these unit fractions. The fraction $\frac{2}{7}$, for example, was expressed as $\frac{1}{4} + \frac{1}{28}$.

Two fractions that name the same quantity are called **equivalent fractions.** For example, you probably know several names for the quantity $\frac{1}{2}$. As long as the whole is the same, $\frac{1}{2}$ means the same as $\frac{2}{4}$, ', $\frac{4}{8}$, $\frac{5}{10}$, and so on. You can show this with fraction strips.



Direct historical:
This "did you know" mentions use of unit fractions by ancient Egyptians.

Implicit historical (problems inspired by history):
This note also serves as motivation for the problem on equivalent fractions.

Assignment Choices

ACE questions 1–7, 19, 38–40, and unassigned choices from earlier problems

Answers to Problem 2.2

A. The fractions shown on the strips are $\frac{4}{6}$, $\frac{6}{9}$, and $\frac{8}{12}$. Additional equivalent fractions include $\frac{10}{15}$, $\frac{12}{18}$, and $\frac{14}{21}$.

B. The fractions shown on the strips are $\frac{6}{8}$, $\frac{9}{12}$, and $\frac{12}{16}$. Additional equivalent fractions include $\frac{15}{20}$, $\frac{18}{24}$, and $\frac{21}{28}$.

C. Answers will vary. Students may realize that you can find equivalent fractions by multiplying the numerator and the denominator by the same number.

Appendix B-4: Direct Historical—Representing Decimal

Problem 4.2

Look back at the original plan you drew for Justin's garden. Write each of the fractional parts for the vegetables in your plan as a decimal.

Problem 4.2 Follow-Up

1. a. What would a hundredths grid look like if each square of the grid were divided into ten equal parts? How many parts would the new grid have?
   b. What is a fraction name for the smallest part of this new grid? A decimal name?
   c. How would you shade an area of this new grid to show $\frac{1}{10}$?
   d. What fraction or decimal names could you call this shaded area?
   e. What would you call this new grid, which has every square of a hundredths grid divided into ten equal parts?
2. a. You can write $\frac{9}{100}$ as the decimal 0.09. How could you write $\frac{9}{1000}$ as a decimal?
   b. How could you write $\frac{469}{1000}$ as a decimal?
3. a. What would you need to do to the new grid you discovered in question 1 to make a grid that shows *ten thousandths?*
   b. How could you write $\frac{9}{10,000}$ as a decimal?
   c. How could you write $\frac{469}{10,000}$ as a decimal?

Direct historical:
Some history about
representation of decimals

4.3 Using Decimal Benchmarks

In Investigation 2, we developed benchmarks to help us estimate fractions. Benchmarks can also help us estimate and compare decimals. You can use what you already know about fractions to make estimating and comparing decimals easier.

Did you know?

Throughout history mathematicians have used many different notations to represent decimal numbers. For example, in 1585, Simon Stevinus would have written 2.57 as either 2, 5' 7" or 2 ⓪ 5 ①7 ②. In 1617, John Napier would have written 2/57. Other commonly-used notations included an underscore, 2<u>57</u>, and a combination of a vertical line and an underscore, 2|<u>57</u>. Even today, the notation varies from country to country. For example, in England, 2.57 is written as 2•57, and, in Germany, it is written as 2,57.

Investigation 4: From Fractions to Decimals 43

4.3

Using Decimal Benchmarks

At a Glance

Grouping: Small Groups

Launch

- As a class, develop the set of five decimal benchmarks, and use them to compare pairs of decimals.

Explore

- Circulate, asking questions to help students apply what they learned about fractions to the decimals they are now considering.

Summarize

- Discuss the answers to the problem and the follow-up.
- Extend the discussion, posing more difficult ordering problems for students and focusing on the meaning of decimal place values.

Answers to Problem 4.2 Follow-Up

1. a. 1000 parts b. $\frac{1}{1000}$; 0.001
   c. You would still shade $\frac{1}{10}$ of the grid, but now the tenth would be subdivided into 100 parts.
   d. Possible answers: $\frac{100}{1000}$, 0.100, 0.10, 0.1
   e. A thousandths grid
2. a. 0.009 b. 0.469
3. a. Subdivide each of the thousand sections into 10 parts, which would give 10,000 squares.
   b. 0.0009 c. 0.0469

Assignment Choices

ACE questions 10–36, 43, 44, and unassigned choices from earlier problems

Investigation 4 43

Appendix B-5: General Historical— Fractions

6.1

Cat	Gender	Age (yrs)	Weight (lbs)	Eye color	Pad color
Seymour	m	0.25	1.5	gold	pink/black
Shiver	m	3	12	yellow/green	pink
Simon	m	0.25	2	green/brown	peach/gray
Skeeter	m	6	13	green	black
Smokey	f	2.5	8	green	black
Smudge	m	8	10	green	gray
Snowy	m	0.5	1.5	gray	gray
Sparky	m	7	12	green	pink
Speedy	m	3	12	blue	pink
Stinky	m	0.17	3.5	yellow	pink
Sweet Pea	f	16	14.5	green	black
Tabby	f	1.5	7	green	black
Tabby Burton	m	1	10	green	black
Terra	m	3	11	green	pink
Thomas	m	4	8	green	pink
Tiger	f	5	13	green	pink
Tigger	f	4	8	yellow	brown
Ting	f	0.25	2.5	green	pink/black
Tom	m	0.25	3	green	gray
Tomadachi	m	1	6.5	gold	pink
Treasure	f	4	8	green	pink
Wally	m	5	10	green	pink/black
Weary	m	8	15	green	pink
Ziggy	f	7	10	gold	pink/black

Did you know?

Ancient Egyptians considered cats to be sacred. Bastet, the Egyptian goddess of love and fertility, was represented as having the head of a cat and the body of a woman. Punishment for harming a cat was severe, and the sentence for killing a cat was usually death. When a cat died, Egyptians shaved their eyebrows as a sign of mourning. Dead cats were often mummified and buried in cat cemeteries.

General historical: This historical note is not directly about fractions but rather about the context -- ancient Egyptian tradition.

Tips for the Linguistically Diverse Classroom

Rebus Scenario The Rebus Scenario technique is described in detail in *Getting to Know Connected Mathematics*. This technique involves sketching rebuses on the chalkboard that correspond to key words in the story or information you present orally. Example: some key words and phrases for which you may need to draw rebuses while discussing the Did you know? feature: *Ancient Egyptian* (a stick figure of an ancient Egyptian), *Baset, the Egyptian goddess* (a figure with the head of a cat and the body of a woman), *sentence for killing a cat* (a stick person next to gallows and a dead cat), *shaved* (razor), *eyebrows* (a face with eyebrows next to the same face without eyebrows), *mourning* (the face without eyebrows crying), *mummified* (a cat wrapped as a mummy), *cat cemeteries* (a tombstone with cat picture).

Appendix C-1: Overview of Unit on Circles

Overview

Throughout history we find records of the importance of measurement. In fact, in the early development of mathematics, geometry was synonymous with measurement. Today we are surrounded with ever-increasingly complex measures such as information-access rates, signal strength, and memory capacity.

The overarching goal of this unit is to help students begin to understand what it means to measure. Students study two kinds of measurements that are appropriate for grade 6: perimeter and area. Since students often have misconceptions about the effects of each of these measures on the other, it is critical to study them together and to probe their relationships. The problems in this unit are structured so that students can build a deep understanding of what it means to measure area and what it means to measure perimeter. In the process, they develop strategies for measuring perimeter and area of both rectangular and nonrectangular shapes.

The name of this unit indicates the theme that binds the investigations together: *covering* (area) and *surrounding* (perimeter). A subtheme running through the unit focuses on questions of what is the largest and what is the smallest, a precursor to the notions of maxima and minima. You will recognize connections throughout the *Covering and Surrounding* unit to all the units preceding it in the grade 6 curriculum. The connections to factors and multiples and to data gathering, organizing, and representing are especially strong.

The problems present interesting and challenging tasks while offering opportunities for meaningful progress and learning by students of different aptitudes and prior achievements. The student edition strongly supports investigative classwork. The greatest learning will occur if students conduct some exploratory work on their own, discover strategies for themselves, and then share their findings.

The Mathematics in *Covering and Surrounding*

While this unit does not explicitly focus on the more global aspects of what it means to measure, it does lay the groundwork for teachers to raise issues that help students begin to see relationships and characteristics of all measurements.

The Measurement Process

The measurement process involves several key elements.

- *A phenomenon or object is chosen, and an attribute that can be measured is identified.* This could involve such disparate properties as height, mass, time, temperature, and capacity.
- *An appropriate unit is selected.* The unit depends on the kind of measure to be made and the degree of precision needed for the measure. Units of measurement include centimeters, angstroms, degrees, minutes, volts, and decibels; instruments for measuring include rulers, calipers, scales, watches, ammeters, springs, and weights.
- *The unit is used repeatedly to "match" the attribute of the phenomenon or object in an appropriate way.* This matching might be accomplished, for example, by "covering," "reaching the end of," "surrounding," or "filling" the object.
- *The number of units is determined.* The number of units is the measure of the property of the phenomenon or object.

Measuring Perimeter and Area

Covering and Surrounding highlights two important kinds of measures—perimeter and area—that depend on very different units and measurement processes. Measuring perimeter requires *linear* units; measuring area requires *square* units. Students often confuse these, and a strong emphasis on formulas may contribute to their confusion. While students can become adept at plugging numbers into formulas, they often have a hard time remembering which formula does what. This is often because they have an incomplete fundamental understanding of what the measurement is about.

Many students think that area and perimeter are related in that one determines the other. They may think that all rectangles of a given area have the same perimeter or that all rectangles of a given perimeter have the same area. The investigations in *Covering and Surrounding* help students realize for themselves the inaccuracy of such notions.

Covering and Surrounding takes an experimental approach to developing students' understanding of measuring perimeter and area. It is assumed that students will work with tiles, transparent grids, grid paper, string, rulers, and other devices of their choice to develop a dynamic sense of "covering" and "surrounding" to find area and perimeter. Many students who engage in these kinds of investigations do invent formulas for finding area and perimeter in certain situations. This should be encouraged, but not forced. Some students need the help of a more hands-on approach to measuring for quite a while. The payoff for allowing students the time and opportunity to develop levels of abstraction with which they are comfortable is that they will eventually make sense of perimeter and area in a lasting way.

Mathematical and Problem-Solving Goals

Covering and Surrounding **was created to help students**

- Develop strategies for finding areas and perimeters of rectangular shapes and then of nonrectangular shapes
- Discover relationships between perimeter and area
- Understand how the area of a rectangle is related to the area of a triangle and of a parallelogram
- Develop formulas or procedures—stated in words or symbols—for finding areas and perimeters of rectangles, parallelograms, triangles, and circles
- Use area and perimeter to solve applied problems
- Recognize situations in which measuring perimeter or area will answer practical problems
- Find perimeters and areas of rectangular and nonrectangular figures by using transparent grids, tiles, or other objects to cover the figures and string, straight-line segments, rulers, or other objects to surround the figures
- Cut and rearrange figures—in particular, parallelograms, triangles, and rectangles—to see relationships between them and then devise strategies for finding areas by using the observed relationships
- Observe and reason from patterns in data by organizing tables to represent the data
- Reason to find, confirm, and use relationships involving area and perimeter
- Use multiple representations—in particular, physical, pictorial, tabular, and symbolic models—and verbal descriptions of data

The overall goal of Connected Mathematics is to help students develop sound mathematical habits. Through their work in this and other geometry units, students learn important questions to ask themselves about any situation that can be represented and modeled mathematically, such as: *What properties of square tiles and rectangular tiles makes them so useful for covering flat spaces? How are the perimeter and the area of a figure related? For a figure with irregular sides, how can we find perimeter? How can we find area? Can a figure have a small area yet have a large perimeter? Or can a figure exist that has a large area but a small perimeter? Are there special relationships between perimeter and area for other 4-sided figures such as parallelograms? How can these ideas be adapted for triangles? Does a circle have perimeter and area? If so, how can they be found?*

Summary of Investigations

Investigation 1: Measuring Perimeter and Area

Students build a good understanding of the difference between perimeter and area by dealing with the concepts concurrently in a concrete, manipulative setting. They use square tiles to create designs and to cover pictures of designs to find areas and perimeters, and they transform designs to fit a prescribed perimeter or area.

Investigation 2: Measuring Odd Shapes

Students consider the measuring of things with curved or irregular edges that can't be laid along a ruler. This calls for using other tools, such as string, to help find an approximation. They trace their feet on grid paper, estimate the perimeters and areas of the tracings, and discuss how these measurements might be helpful to shoe manufacturers.

Investigation 3: Constant Area, Changing Perimeter

This investigation introduces a classic maxima/minima problem, asking students to find the largest and the smallest perimeter for a given area. Students construct tables to help highlight and reveal patterns in data.

Investigation 4: Constant Perimeter, Changing Area

This investigation also focuses on maxima/minima questions, but this time perimeter is fixed and area is allowed to change. Students confront the misconception that area determines perimeter and vice versa.

Investigation 5: Measuring Parallelograms

Students cut and rearrange parallelograms to make rectangles and develop strategies for using what they know about finding the area of a rectangle to find the area of a parallelogram. Most students will develop good formulas for finding the area of a rectangle and a parallelogram. For students who need them, the use of grids and other more informal reasoning methods are encouraged.

Investigation 6: Measuring Triangles

Students are introduced to finding areas and perimeters of triangles by using grids, arranging triangles to form parallelograms, and measuring with rulers. Special triangles—such as isosceles and 30–60–90 triangles—are explored.

Investigation 7: Going Around in Circles

Questions are asked to help students see that the circumference (that is, the perimeter) of a circle is slightly more than three times its diameter, and that a circle's area is slightly more than three times the area of a square whose edges are equal to the circle's radius. These discoveries lead students to the idea of the value of pi.

Appendix C-2: Implicit Historical – Circumference

7.2 Surrounding a Circle

Mathematicians have found a relationship between the diameter and circumference of a circle. You can try to discover this relationship by measuring many different circles and looking for patterns. The patterns you discover can help you develop a shortcut for finding the circumference of a circle.

Problem 7.2

In this problem, you will work with a collection of circular objects.

A. Use a tape measure to find the diameter and circumference of each object. Record your results in a table with these column headings:

Object	Diameter	Circumference

B. Make a coordinate graph of your data. Use the horizontal axis for diameter and the vertical axis for circumference.

C. Study your table and your graph, looking for patterns and relationships that will allow you to predict the circumference from the diameter. Test your ideas on some other circular objects. Once you think you have found a pattern, answer this question: What do you think the relationship is between the diameter and the circumference of a circle?

Problem 7.2 Follow-Up

1. How can you find the circumference of a circle if you know its diameter?
2. How can you find the diameter of the circle if you know its circumference?
3. Use the relationships you discovered in the problem to calculate the circumferences of the pizzas from Problem 7.1. How do your calculations compare to your estimates?

Implicit historical:
It is believed that people discovered long ago that the ratio of circumference to diameter is constant; however, estimates for this constant may have varied. This problem presents an opportunity for students to discover this long known relationship.

7.2

Surrounding a Circle

At a Glance

***Grouping:* Pairs**

Launch

- Read the problem with the class, emphasizing that they will be looking for a shortcut for finding the circumference of a circle.
- Supply groups with circular objects, or have them use objects they have brought from home.

Explore

- Have students work in pairs to measure the circular objects.
- Do part B of the problem as a class. (*optional*)

Summarize

- As a class, look for patterns in the data.
- Discuss the relationship between diameter and circumference, and introduce the number pi.

Answers to Problem 7.2

A. Answers will vary. The circumference for each object should be a little more than 3 times the diameter.

B. Graphs will vary. Points should fall in approximately a straight line.

C. The circumference is a little more than 3 times the diameter.

Answers to Problem 7.2 Follow-Up

1. Possible answer: Add together 3 diameters and a bit more, or multiply the diameter by 3 and add a bit more.
2. Possible answer: Divide the circumference by 3 and subtract a bit from the result.
3. See page 81h.

Assignment Choices

ACE questions 1–5 (have students find the circumference at this time and the area later), 6–8, and unassigned choices from earlier problems

Appendix C-3: Implicit Historical – Squaring a Circle

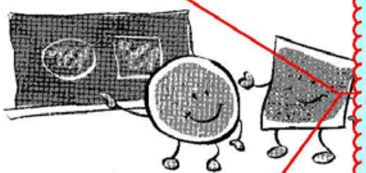
Problem 7.3 Follow-Up

Will a circle with a diameter equal to half the diameter of the circle in the problem have an area equal to half the area of that circle? Why or why not?

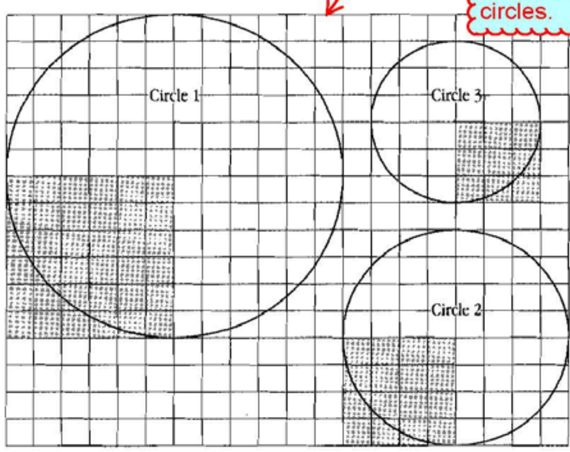
"Squaring" a Circle

In Investigations 5 and 6, you learned some things about parallelograms ... comparing them to rectangles. Now you will find out more about circles ... them to squares.

Implicit historical: Squaring a circle was a famous historical problem. The problem here presents similar strategy-- measuring the circle using a square. It offers a strategy to tackle the fundamental problem of finding the area formula for circles.



Labsheet 7.4 shows the three circles that are drawn below. A portion of e... covered by a shaded square. The sides of each shaded square are the sam... radius of the circle. We call such a square a "radius square."



Investigation 7: Going Around in Circles 73

7.4

"Squaring" a Circle

At a Glance

Grouping: Individuals

Launch

- Read the problem to the class.
- Demonstrate how to cover a circle with radius squares. (*optional*)

Explore

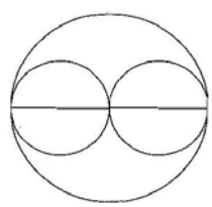
- Have students work individually to cover the circles with radius squares, then share their results with a partner.
- As you circulate, make sure students are recording results in a table.

Summarize

- Have students share their strategies for covering the circles.
- Ask questions to help them see the connection between their work with circles and the number pi.

Answers to Problem 7.3 Follow-Up

No, the area of the small circle would be less than half the area of the large circle. The two small circles in the illustration each have a diameter equal to half the diameter of the large circle. The combined area of these circles is much less than the area of the large circle.



Assignment Choices

ACE questions 1–13 and unassigned choices from earlier problems

Investigation 7 73

Appendix C-4: Direct Historical – pi

7.4

Direct historical : Some history about pi

Problem 7.4

A. For each circle, cut out several copies of the radius square from a sheet of centimeter grid paper. Find out how many radius squares it takes to cover the circle. You may cut the radius squares into parts if you need to. Record your data in a table with these column headings:

Circle	Radius of circle	Area of radius square	Area of circle	Number of radius squares needed

B. Now draw a couple of your own circles on grid paper. You can use circles from the objects you measured in Problem 7.2 and from your Shapes Set. Make radius squares for each circle, and find out how many radius squares it takes to cover each circle. Add this data to your table.

C. Describe any patterns you see in your data.

D. If you were asked to estimate the area of any circle in "radius squares," what would you report as the best estimate?

Problem 7.4 Follow-Up

1. How can you find the area of a circle if you know the diameter or the radius?
2. How can you find the diameter or radius of a circle if you know the area?

Did you know?

You have discovered that the area of a circle is a *little more than 3* times the radius squared. You have also found that the distance around a circle is a *little more than 3* times the diameter. There is a special name given to this number that is a little more than 3.

In 1706, William Jones used π (pronounced "pi"), the Greek letter for *p*, to represent this number. He used the symbol to stand for the *periphery*, or distance around, a circle with a diameter of 1 unit.

As early as 2000 B.C., the Babylonians *knew* that π was more than 3! Their estimate for π was $3\frac{1}{8}$. By the fifth century, Chinese mathematician Tsu Chung-Chi wrote that π was somewhere between 3.1415926 and 3.1415927. From 1436 until 1874, the known value of π went from 14 places past the decimal to 707 places. Computers have been used to calculate millions more digits, and today we know that the digits will never repeat and will never end. This kind of number is called *irrational*.

Answers to Problem 7.4

A. See page 81i.

B. Answers will vary.

C. Answers will vary. Ideally, students will notice that it takes a little more than three radius squares to cover any circle.

D. Possible answer: The area of any circle is a little bit more than three radius squares.

Answers to Problem 7.4 Follow-Up

See page 81i.

Appendix D-1: Overview of Unit on Negative Numbers

Overview

In the middle grades, students are introduced to fractions and decimals. The next major hurdle is building an understanding of positive and negative numbers, and in particular, integers. These kinds of numbers have been experienced by students informally in their everyday world—as temperatures in the winter drop below zero, as sports teams are said to be ahead or behind by so much. Students have intuitively used operations on integers to make sense of these situations. This unit explores situations that require representation with integers and that suggest more formal ways to add, subtract, multiply, and divide positive and negative numbers.

To introduce students to work with integers, the context of winning points (represented by positive integers) and losing points (represented by negative integers) in a game is used. The game provides an entry point for discussing order and the comparing of integers as well as for developing the concepts of opposites, distances on a number line, and absolute value. The number line is used throughout the unit to model strategies for adding, subtracting, and multiplying integers. A board with chips of two colors is a second model students will use for addition and subtraction.

The inverse relationships between addition and subtraction and between multiplication and division are investigated to help students generalize rules for these four operations. Looking at number patterns that are familiar and extending them to operations on integers is another way to attach operations on these new numbers to ideas students already have.

The Mathematics in *Accentuate the Negative*

Most of your students can add, subtract, multiply, and divide whole numbers and decimals. However, most have not been asked to think about what the operations mean and what kinds of situations call for which operation. Without the development of the disposition to seek ways of making sense of mathematical ideas and skills, students may end up with technical skills but without ways of deciding when and how those skills can be used to solve problems.

One good way to work toward creating the desire to make sense of these ideas is to model such thinking in rich classroom conversation. Asking questions about meaning, about what makes sense, as a regular, expected part of classroom discourse helps focus students on making connections. Exploring new aspects of number in a way that builds on and connects to what they already know is likely to have two good effects. First, students will deepen their understanding of familiar numbers and operations on them. Second, the new numbers—in this case, integers—will be more deeply integrated into students' own mathematical knowledge and resources.

Students find several things difficult about integers and operations on integers.

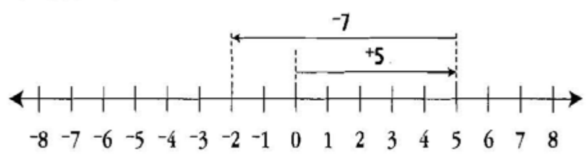
- The fact that $^{-}27$ is smaller than $^{-}12$ is contrary to students' experience with (positive) whole numbers. This understanding requires building mental images and models that allow students to visualize these new comparisons and relationships.

- The operation of subtraction, and especially subtracting a negative, is difficult for students to make sense of. In this unit, students will have several opportunities to think about what makes sense and why, and they will encounter several representations and models that will help them think more deeply about subtraction.
- The idea that subtracting a negative number gives the same result as adding the opposite of the negative number (adding a positive) is difficult for many students. This understanding must develop over time as students make observations and comparisons between subtraction and addition. Recognizing that these are inverse operations and that addition sentences are related to subtraction sentences helps students to expand their understanding of this concept.
- Multiplying two negatives and getting a positive seems like magic to most students. In fact, the usual ways of giving meaning to multiplication—such as accumulating an amount over and over (repeated addition)—seem of no help in making sense of $^{-}12 \times {}^{-}5$.

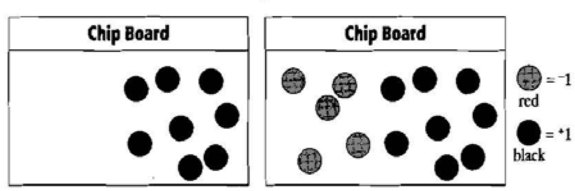
In the unit, we approach these difficult concepts through the use of two basic models and the observation of patterns. The accumulation of evidence from the models is more powerful than that from a single model. In addition, a particular model may be more useful for making sense of parts of the picture, but not the whole picture.

Modeling Addition of Integers

In this unit, students will use the number line to find a strategy for adding integers. Adding positive integers is interpreted as "going to the right"; adding negative numbers is interpreted as "going to the left." To add $^{+}5$ and $^{-}7$, for example, the strategy is to start at the point marked 0, go right 5 units to the point labeled 5, and then go left 7 units, which puts you at the point labeled $^{-}2$. Thus, $5 + {}^{-}7 = {}^{-}2$.



Colored chips can also be used to develop a strategy for adding integers. Using this model requires an understanding of opposites. Two colors of chips are used, one color (such as black) to represent positive numbers and another (such as red) to represent negative numbers. To add $^{+}8 + {}^{-}5$, the strategy is to first put 8 black chips on the chip board. Since the problem involves addition, which means to combine, 5 red chips are added to the board to represent the $^{-}5$.



Because each chip represents 1 unit, either positive or negative, a black and red chip are thought of as opposites. Two opposite chips make 0 ($^{+}1 + {}^{-}1 = 0$). In this problem, 5 chips of each color can be paired to make zeros. After the paired chips are removed, 3 black chips remain—which represent $^{+}3$, the sum.

Mathematical and Problem-Solving Goals

Accentuate the Negative **was created to help students**

- Develop strategies for adding, subtracting, multiplying, and dividing integers
- Determine whether one integer is greater than, less than, or equal to another integer
- Represent integers on a number line
- Model situations with integers
- Use integers to solve problems
- Explore the use of integers in real-world applications
- Compare integers using the symbols =, >, and <
- Understand that an integer and its inverse are called opposites
- Graph in four quadrants
- Set up a coordinate grid on a graphing calculator by naming the scale and maximum and minimum values of x and y
- Graph linear equations using a graphing calculator
- Informally observe the effects of opposite coefficients and adding a constant to $y = ax$
- Answer questions using equations, tables, and graphs

The overall goal of the Connected Mathematics curriculum is to help students develop sound mathematical habits. Through their work in this and other algebra units, students learn important questions to ask themselves about any situation that can be represented and modeled mathematically, such as: ***What situations in daily life can be represented by positive or negative numbers? How can a meaning be found for operations on negative numbers? Where can such operations be modeled? Is it possible to use "less than" or "greater than" concepts with integers? How are the integers different from ordinary whole numbers? How are these two sets of numbers alike? Can the coordinate grid be expanded to include negative numbers? Is it possible to make graphs on such grids using a graphing calculator? What patterns will occur in these graphs? How can these patterns be used to find and understand other patterns?***

Summary of Investigations

Investigation 1: Extending the Number Line

Students are introduced to and build on their intuitive sense of integers. They compare, order, and solve simple problems involving integers. (Operation rules are not the focus of this investigation.) A game in which points are won (+) and lost (–) introduces integers and provides an entry point for comparing integers, discussing order, and working with opposites. The representation of integers on a number line is introduced by having students examine temperature changes on a thermometer (a vertical number line). The number line is used throughout the unit to model strategies for adding, subtracting, and multiplying integers.

Investigation 2: Adding Integers

Students develop rules for adding integers. They explore two models—a number line and a board with chips of two colors, one to represent $^{+}1$ and the other to represent $^{-}1$—as ways to represent, solve, and explain addition problems involving positive and negative integers.

Investigation 3: Subtracting Integers

Students develop an understanding of subtraction as well as rules for subtracting integers. Number lines and chip boards are used to model the subtraction of positive and negative integers. Subtraction, interpreted as the opposite of addition, is modeled on a chip board as "taking away" chips and on a number line as reversing the direction of the arrow representing the second integer in a subtraction expression. Students look for patterns and find rules for subtracting positive and negative integers.

Investigation 4: Multiplying and Dividing Integers

Students develop rules for multiplying and dividing integers. As it is difficult to model multiplying or dividing a negative integer by a negative integer, students look for patterns and further develop their understanding of integers and of the operations of multiplication and division as a means for developing rules with integers. Students use a thermometer (a kind of vertical number line) to show, for example, that $3 \times {}^{-}2$ is the same as ${}^{-}2 + {}^{-}2 + {}^{-}2$. They continue to investigate the multiplication of integers by examining series of related equations and looking for patterns. They play a game with products in which the factors include positive and negative integers. Finally, they look at the relationship between multiplication and division and derive rules for dividing integers.

Investigation 5: Coordinate Grids

In previous units, students graphed in the first quadrant of a coordinate grid. In this investigation, the use of integers facilitates the introduction of the complete coordinate grid with all four quadrants. Students have used a number line to represent integers; here, the axes for coordinate grids are described as two perpendicular number lines representing both positive and negative integers. As part of their work in this unit, students spend time getting familiar with how to set up a coordinate grid and display one or more graphs on the grid using graphing calculators.

Appendix D-2: Direct Historical – Negative Number

INVESTIGATION 2

Adding Integers

The numbers 0, 1, 2, 3, 4, . . . are ***whole numbers.*** These numbers are labeled on the number line below.

0 1 2 3 4 5 6

If we extend this pattern to the left of 0, we get . . . , $^{-}4$, $^{-}3$, $^{-}2$, $^{-}1$, 0, 1, 2, 3, 4, ...

$^{-}4$ $^{-}3$ $^{-}2$ $^{-}1$ 0 1 2 3 4 5 6

Direct historical:
This note contains some history about early use of negative numbers by Hindu mathematicians and a note about some confusion about the concept in 16th and 17th century Europe.

This larger set of numbers is called the **integers**. The numbers 1, 2, 3, 4, . . . are ***positive integers,*** and the numbers . . . , $^{-}4$, $^{-}3$, $^{-}2$, $^{-}1$ are ***negative integers.*** The number 0 is neither positive nor negative.

In many situations, you need to combine integers to find a sum. In this investigation, you will use two models that will help you think about how to add positive and negative integers.

Did you know?

The Hindus were the first to use negative numbers. The Hindu mathematician Brahmagupta used negative numbers as early as A.D. 628, and even stated the rules for adding, subtracting, multiplying, and dividing with negative numbers. Many European mathematicians of the sixteenth and seventeenth centuries did not accept the idea of negative numbers, referring to them as "absurd" and "fictitious." Mathematicians of that time who did accept negative numbers often had strange beliefs about them. For example, John Wallis believed that negative numbers were greater than infinity!

Appendix D-3: Implicit Historical – Use of Chip Board Model

Problem 2.1 Follow-Up

1. You can think of the scoring in MathMania as follows: When a team answers a question correctly, a positive integer is added to their score. When a te[am answers a question] incorrectly, a negative integer is added to their score. For eac[h of the following] situations, write an addition sentence that will give the team'[s score. Assume each] team starts with 0 points.
   a. The Brainiacs answer a 200-point question correctly and [a 150-point question] incorrectly.
   b. The Aliens answer a 100-point question correctly and a [100-point question] incorrectly.
   c. The Prodigies answer a 50-point question incorrectly, a [100-point question] incorrectly, and a 250-point question correctly.
2. Illustrate each addition problem on a number line and give t[he sum.]
   a. $^{-}2 + {}^{+}2$ b. $^{+}8 + {}^{-}8$ c. $^{-}1 + {}^{+}1$
3. What happens when you add opposites? Explain how you kn[ow.]

Implicit historical: This problem presents a model of representing negative/positive numbers similar to the 'counting rods' used by ancient Chinese to represent negative/ positive with colors. See next page for problems using this model.

2.2 Inventing a New Model

In the last problem, you used the number line to help you think about adding integers. In this problem, you will explore another way to model the addition of integers.

Amber's mother is an accountant. One day, Amber heard her mother talking to a client on the phone. During the conversation, her mother used the phrases "in the red" and "in the black."

That evening at dinner, Amber asked her mother what these terms meant. Her mother said:

"When people in business talk about income and expenses, they often use colors to describe the numbers they are dealing with. Black refers to profits (or income); red refers to losses (or expenses). A company that is making money, or has money, is 'in the black'; a company that is losing money, or owes money, is 'in the red.'"

Investigation 2: Adding Integers 21

2.2

Inventing a New Model

At a Glance

***Grouping:* pairs**

Launch

- Talk with the class about any patterns they have noticed about adding positive and negative integers.
- Read the story setting with the class, and introduce the chip board model.

Explore

- Circulate as pairs work, suggesting that those having trouble consider the number line model to help make sense of the chip board model.

Summarize

- Review the problem as a class.
- Ask students to generalize the patterns they have discovered.
- Do and discuss the follow-up questions.

Answers to Problem 2.1 Follow-Up

1. a. $^{+}200 + {}^{-}150 = {}^{+}50$
   b. $^{+}100 + {}^{-}100 = 0$
   c. $^{-}50 + {}^{-}100 + {}^{+}250 = {}^{+}100$
2. See page 33i.
3. When you add opposites, the sum is 0. Possible explanation: I know this is correct, because the two numbers cancel each other out. This can be seen on the number lines for question 2.

Assignment Choices

ACE questions 8–28, 30–37, and unassigned choices from earlier problems

Assessment

It is appropriate to use Check-Up 1 after this problem.

Investigation 2 21

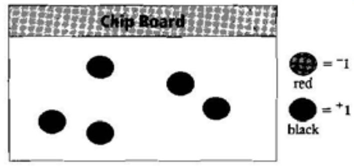
Appendix D-3 (continued)

2.2

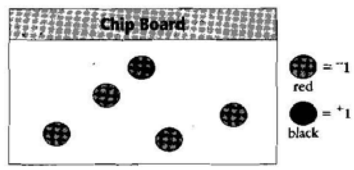
Implicit historical
In this chip board model, a red chip represents a negative value, a black chip represents a positive value.

Amber was studying integers in her math class and thought she could use these ideas of "in the black" and "in the red" to model the addition of positive and negative integers. Her model uses a chip board and black and red chips. Each black chip represents $^{+}1$, and each red chip represents $^{-}1$.

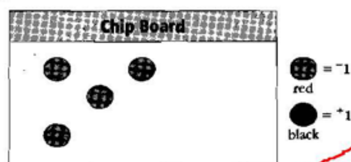
For example, this chip board shows a value of $^{+}5$:



This chip board shows a value of $^{-}5$:



To represent $^{-}4 + ^{-}3$, Amber started with an empty chip board. She represented $^{-}4$ by putting four red chips on the board.

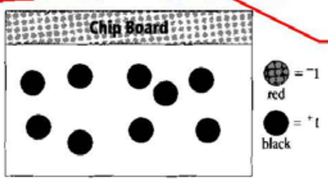


Appendix D-3 (continued)

Subtracting on a Chip Board

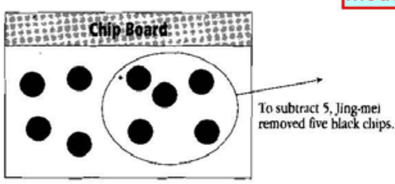
Amber's friends Jing-mei and Drew liked Amber's chip board model for adding integers. They decided to use a chip board to explore subtracting integers.

To model 9 – 5, Jing-mei started with an empty chip board and then put nine black chips on the board to represent $^{+}9$.

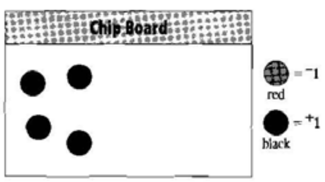


Implicit historical: The context provided here such as the use of a Chinese name may suggest some subtle recognition of the ancient Chinese model.

Jing-mei thinks about subtracting as "taking away." Therefore, to re[...] she *removed* five black chips from the board.



After removing the five black chips, four black chips remained.



Jing-mei wrote the number sentence 9 – 5 = 4 to represent her work on the chip board.

3.1

Subtracting on a Chip Board

At a Glance

Grouping: pairs

Launch

- Talk with the class about the meaning of subtraction.
- Ask the class for ways to model subtraction on a chip board.
- Discuss the model shown in the student edition.

Explore

- As pairs work on the problem, if any are having trouble re-representing amounts on the board, suggest that they ask another pair for help.

Summarize

- Carefully review each part of the problem, analyzing the various representations students used.
- Once students understand how to model subtraction on a board, let them work on the follow-up.

Assignment Choices

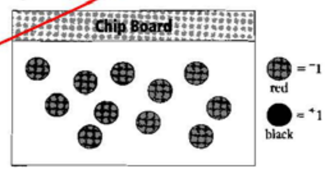
ACE questions 11–14, 34, and unassigned choices from earlier problems

Appendix D-3 (continued)

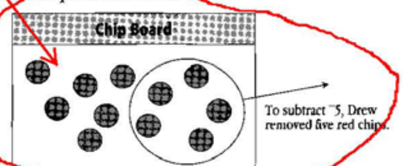
3.1

Implicit historical: This model presents -11- (-5) as taking away 5 red chips from 11 red chips.

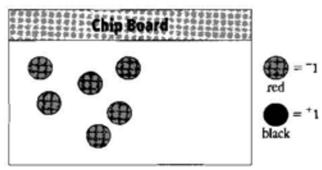
Drew tried Jing-mei's method to find $^{-}11 - {}^{-}5$. He started with an empty board and then put on 11 red chips to represent $^{-}11$.



Like Jing-mei, Drew thought of subtracting as "taking away." To represent subtracting $^{-}5$, he removed five red chips from the board.



Six red chips remained on the board.



Drew wrote the number sentence $^{-}11 - {}^{-}5 = {}^{-}6$ to represent his work on the chip board.

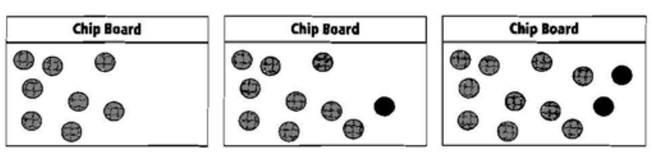
Think about this!

Why does it makes sense that the difference between 9 and 5 is 4 (that is, $9 - 5 = 4$) and the difference between $^{-}11$ and $^{-}5$ is $^{-}6$ (that is, $^{-}11 - {}^{-}5 = {}^{-}6$)?

Answers to Problem 3.1

A. 1. ⁻1 2. ⁺1 3. ⁻4 4. ⁺4

B. Possible answer:



C. See page 52I.

Appendix D-4: Implicit Historical – Multiplication of Negative Numbers

4.2

Studying Multiplication Patterns

At a Glance

Grouping: pairs

Launch

- Read the introduction and problem with the class.

Explore

- As pairs work on the problem, listen for the patterns they are noticing.
- If pairs have trouble with parts E and F, suggest that they think about what seems reasonable.

Summarize

- Discuss students' answers and thinking strategies for the problem.
- Assist the class in looking for patterns that will help predict the product of any two integers.
- Discuss the follow-up questions.

Studying Multiplication Patterns

In Investigation 3, you studied patterns to help you understand subtraction of integers. Studying patterns can also help you think about multiplication of integers. Study the equations below, and then work on the problem.

$5 \times 5 = 25$
$5 \times 4 = 20$
$5 \times 3 = 15$
$5 \times 2 = 10$
$5 \times 1 = 5$
$5 \times 0 = 0$

Implicit historical (?): The unit shifts from the concrete model to using patterns in order to introduce multiplication of negative numbers. Historically, concrete models may have presented confusion regarding the meaning of negative numbers.

A. Describe any patterns yo integers multiplied by 5 get smaller.

B. **1.** Use the patterns you observed to predict $5 \times {}^{-}1$. Explain your reasoning.

2. Write the next four equations in the pattern.

C. Complete the equations below, and use them to help you answer parts D and E.

$5 \times {}^{-}4 = ?$
$4 \times {}^{-}4 = ?$
$3 \times {}^{-}4 = ?$
$2 \times {}^{-}4 = ?$
$1 \times {}^{-}4 = ?$
$0 \times {}^{-}4 = ?$

D. Describe any patterns you observe in the way the products change as the integers multiplied by $^{-}4$ get smaller.

E. **1.** Use the patterns you observed to predict ${}^{-}1 \times {}^{-}4$. Explain your reasoning.

2. Write the next four equations in the pattern.

F. Find the following products.

1. ${}^{-}3 \times 7$ **2.** $5 \times {}^{-}8$ **3.** ${}^{-}11 \times {}^{-}12$ **4.** ${}^{-}3.6 \times 2.7$

56 **Accentuate the Negative**

Assignment Choices

ACE questions 26, 29, 32, and unassigned choices from earlier problems

Answers to Problem 4.2

A. As the size of the groups decreases, the products decrease by 5 at each step.

B. 1. $5 \times {}^{-}1 = {}^{-}5$; Possible explanation: With each equation, there is one fewer group of 5, and 5 less than 0 is $^{-}5$.

2. $5 \times {}^{-}2 = {}^{-}10$, $5 \times {}^{-}3 = {}^{-}15$, $5 \times {}^{-}4 = {}^{-}20$, $5 \times {}^{-}5 = {}^{-}25$

C. $5 \times {}^{-}4 = {}^{-}20$, $4 \times {}^{-}4 = {}^{-}16$, $3 \times {}^{-}4 = {}^{-}12$, $2 \times {}^{-}4 = {}^{-}8$, $1 \times {}^{-}4 = {}^{-}4$, $0 \times {}^{-}4 = 0$

D. The product increases each time because you are accumulating fewer negatives (one less group of $^{-}4$).

E. 1. ${}^{-}1 \times {}^{-}4 = 4$; Possible explanation: With each equation, the product increases by 4, and 4 more than 0 is 4.

2. ${}^{-}2 \times {}^{-}4 = 8$, ${}^{-}3 \times {}^{-}4 = 12$, ${}^{-}4 \times {}^{-}4 = 16$, ${}^{-}5 \times {}^{-}4 = 20$

F. 1. $^{-}21$ 2. $^{-}40$ 3. 132 4. $^{-}9.72$

56 **Investigation 4**

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Appendix E-1: Overview of Unit on Pythagorean Theorem

Overview

In *Looking for Pythagoras,* students explore two important new ideas: the Pythagorean Theorem and irrational numbers. In the process of solving the problems in this unit, students also review and make connections among the concepts of area, distance, slope, and rational numbers.

Students begin the unit by finding the distance between dots on a grid. They explore the areas of figures drawn on a dot grid, and they find relationships among the length of a side of a square, square roots, and irrational numbers. They find that the side lengths of some squares are irrational numbers. Then, students discover the Pythagorean relationship through an exploration of squares drawn on the sides of a right triangle.

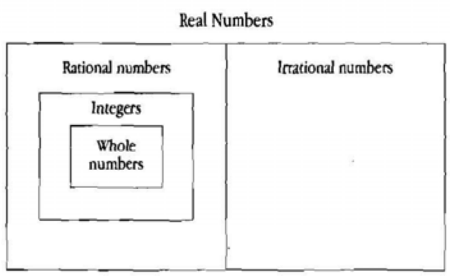
The coordinate system makes all these investigations possible. A coordinate system applied to a rectangular array of dots facilitates locating positions, calculating distances, and finding slopes of lines. We can extend the dot grid to include points between the dots that are visible and create a coordinate system on which we can locate any position in the plane. In this unit, we use only integer scales on the axes, which means that each dot has integer coordinates. The breakthrough for students comes when they can find distances on the integer coordinate system, such as $\sqrt{2}$, that are irrational.

Later in the unit, students discover that a line with an irrational slope that passes through the origin will not pass through any dot on a dot grid. This is because every dot on a grid is associated with a rational slope; irrational slopes, which cannot be expressed as a ratio of integers, are not associated with dots on the grid.

1f

Number Systems

New number systems are created when a problem arises that cannot be answered within the system currently in use, or when inconsistencies arise that can be taken care of only by expanding the domain of numbers in the system. The historical "discovery" of a new number system in response to a need is reflected in the number sets students use in grades K–12. Elementary students begin with the ***counting numbers.*** Then, zero is added to the system to create the set of whole numbers. Later, students discover that negative numbers are needed to give meaning in certain contexts, such as temperature. Now they have the number system called the ***integers.*** In elementary and middle school, students learn about fractions and situations that make fractions helpful. Their number world has been expanded to the set of ***rational numbers.*** In this unit, students encounter contexts in which the need for ***irrational numbers*** arises. The set of rational numbers and the set of irrational numbers compose the set of real numbers. The diagram below is one way to represent these sets of numbers.



Rational numbers: Numbers that can be written as a ratio of integers, such as $\frac{2}{3}$, $-\frac{7}{5}$, $\frac{4}{1}$, 0.6, and 0.3333 . . . ; in other words, integers, terminating decimals, and repeating decimals.

Integers: {. . . , –3, –2, –1, 0, 1, 2, 3, . . .}

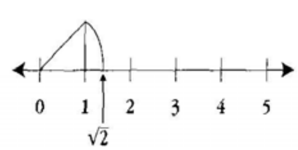
Whole numbers: {0, 1, 2, 3, . . .}

Irrational numbers: Numbers that cannot be written as a ratio of integers, such as π, $\sqrt{7}$, and 0.12131415161718 . . . ; in other words, infinite, nonrepeating decimals.

Exploring Irrational Numbers

In *Looking for Pythagoras,* students are gradually introduced to irrational numbers. They discover the need for irrational numbers by trying to measure the lengths of oblique, or tilted, lines drawn on dot grids. They find such numbers as $\sqrt{2}$, $\sqrt{3}$, and $\sqrt{5}$, which cannot be expressed as ratios of integers. They examine patterns in the decimal representations of fractions, or rational numbers, and find that the decimals either terminate or repeat. For example, $\frac{1}{5} = 0.2$ (a terminating decimal) and $\frac{1}{3} = 0.333\ldots$ (a repeating decimal).

Numbers such as $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, and π cannot be expressed as repeating decimals or terminating decimals. Students create line segments with these lengths and then locate the lengths on a number line. For example, $\sqrt{2}$ is the length of the hypotenuse of a right triangle whose legs have length 1. This procedure helps students to estimate the size of these irrational numbers.



1h

Technology

Connected Mathematics™ was developed with the belief that calculators should always be available and that students should decide when to use them. Students will need access to calculators for much of their work in this unit. Graphing calculators are required for the majority of the grade 8 Connected Mathematics units, so they are listed in this unit as well, but standard calculators that can calculate squares and square roots will suffice.

Mathematical and Problem-Solving Goals

***Looking for Pythagoras* was created to help students**

- Make connections among coordinates, slope, distance, and area
- Relate the area of a square to the length of a side
- Develop strategies for finding the distance between two dots on a dot grid or two points on a coordinate grid
- Discover and apply the Pythagorean Theorem
- Extend understanding of number systems to include irrational numbers
- Locate irrational numbers on a number line
- Represent fractions as decimals and decimals as fractions
- Determine whether the decimal representation for a fraction terminates or repeats
- Use slopes to solve interesting problems

The overall goal of the Connected Mathematics curriculum is to help students develop sound mathematical habits. Through their work in this and other geometry units, students learn important questions to ask themselves about any situation that involves the principles explored in this unit, such as: ***What is the length of a side of a square of a certain area? What is the relationship among the lengths of the sides of a right triangle? How can the Pythagorean Theorem be used to solve problems? How can knowing the slope of a line help to solve problems?***

Summary of Investigations

Investigation 1: Locating Points

Students review the concept of a coordinate grid and are introduced to finding the distance between pairs of points on a grid, setting the stage for developing strategies to find such distances. They also investigate geometric figures and their properties: given two vertices, they find other vertices that would define a square, a nonsquare rectangle, a right triangle, and a nonrectangular parallelogram.

Investigation 2: Finding Areas and Lengths

Students find areas of figures drawn on a dot grid and then explore the relationship between the area of a square and the length of its side. They are introduced to the concept of square root, and they develop a strategy for finding the distance between two points by analyzing the line segment between them: they draw a square using the segment as one side, find the area of the square, and then find the positive square root of the area. These lengths turn out to be either whole numbers or irrational numbers, such as $\sqrt{2}$.

Investigation 3: The Pythagorean Theorem

Students discover the Pythagorean Theorem and explore its implications. They collect information about the areas of the squares on the sides of right triangles and conjecture that the sum of the areas of the two smaller squares equals the area of the largest square, and they investigate a puzzle that verifies this conjecture. They apply the theorem to find the distance between two dots on a dot grid. Then, they apply the converse of the theorem to determine whether a triangle is a right triangle.

Investigation 4: Using the Pythagorean Theorem

Students use the Pythagorean Theorem to explore a variety of applications. They find distances on a baseball diamond; investigate the properties of some special right triangles, including a 30-60-90 triangle and an isosceles right triangle; and find missing lengths and angles in a group of triangles.

Investigation 5: Irrational Numbers

Students take a closer look at square roots. They express lengths as decimals, which leads to a study of decimal representations of fractions. They write fractions as terminating or repeating decimals and find fraction equivalents for terminating and repeating decimals. They are introduced to the concepts of rational numbers (those that can be represented by either terminating or repeating decimals) and irrational numbers (those that can be represented by nonterminating, nonrepeating decimals).

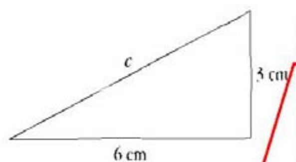
Investigation 6: Rational and Irrational Slopes

Students explore an interesting application of irrational numbers in the context of a video game in which the main character is trying to escape from a forest of trees planted in rows. Students review how to find the slope of a line and the connection between slope and points on a line and relate these concepts to rational and irrational numbers. They discover that, for a line not to pass through any grid point on an infinite grid, the line must have an irrational slope. The relationships between parallel lines (which have the same slope) and perpendicular lines (whose slopes are negative reciprocals) are explored in the ACE questions.

Appendix E-2: Direct Historical – Note about Pythagoras

Problem 3.2 Follow-Up

1. In Problems 3.1 and 3.2, you explored the Pythagorean Theorem. State this relationship as a general rule for any right triangle with legs of lengths a and b and a hypotenuse of length c.
2. A right triangle has legs of lengths 3 centimeters and 5 centimeters.
   - **a.** Use the Pythagorean Theorem to find the area of a square drawn on the hypotenuse of the triangle.
   - **b.** What is the length of the hypotenuse?
3. A right triangle has legs of lengths 5 inches and 12 inches.
   - **a.** Find the area of a square drawn on the hypotenuse of the triangle.
   - **b.** What is the length of the hypotenuse?
4. Use the Pythagorean Theorem to find the length of the hypotenuse of this triangle.



Direct historical:
Some history about Pythagoras and the Pythagoreans

5. The hypotenuse of a right triangle is 15 centimeters long, and one leg is 9 centimeters long. How long is the other leg?

Did you know?

Pythagoras, a Greek mathematician who lived in the sixth century B.C., had a devoted group of followers known as the Pythagoreans. The Pythagoreans had many rituals, and they approached mathematics with an almost religious intensity. Their power and influence became so strong that some people feared that they threatened the local political structure, so they were forced to disband. However, many Pythagoreans continued to meet in secret and to teach Pythagoras's ideas.

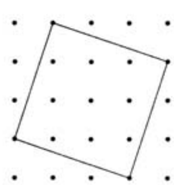
Since they held Pythagoras in such high regard, the Pythagoreans gave him credit for all of their discoveries. Much of what we now attribute to Pythagoras, including the Pythagorean Theorem, may actually be the work of his followers.

Appendix E-3: Implicit Historical – Ancient Egyptian Problem

Introducing Your Students to *Looking for Pythagoras*

Introduce this unit by asking students how they would find certain distances. "How would you calculate the distance between the classroom and the cafeteria?" "What is the distance across the classroom?" Their ideas about the first question will likely involve some sort of estimation strategy. For the second question, they may suggest pacing off the distance across the room, measuring with a meterstick, or counting ceiling or floor tiles.

Draw a square on a dot grid as shown below.



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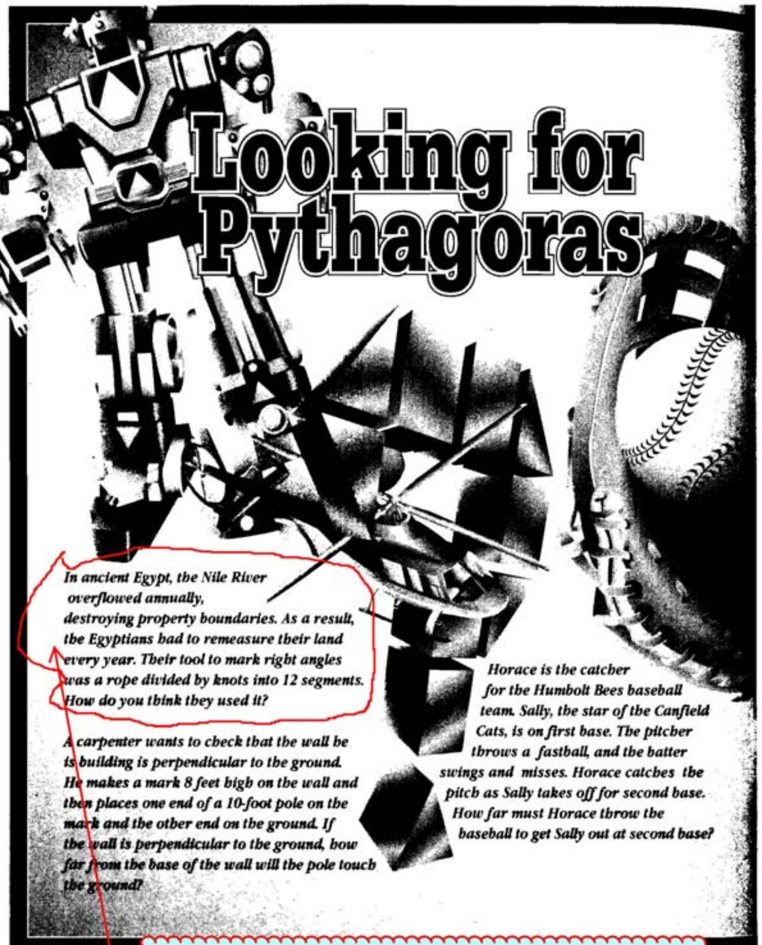


In ancient Egypt, the Nile River overflowed annually, destroying property boundaries. As a result, the Egyptians had to remeasure their land every year. Their tool to mark right angles was a rope divided by knots into 12 segments. How do you think they used it?

A carpenter wants to check that the wall he is building is perpendicular to the ground. He makes a mark 8 feet high on the wall and then places one end of a 10-foot pole on the mark and the other end on the ground. If the wall is perpendicular to the ground, how far from the base of the wall will the pole touch the ground?

Horace is the catcher for the Humbolt Bees baseball team. Sally, the star of the Canfield Cats, is on first base. The pitcher throws a fastball, and the batter swings and misses. Horace catches the pitch as Sally takes off for second base. How far must Horace throw the baseball to get Sally out at second base?

Implicit historical:
One of the introductory problems uses an ancient Egyptian practical problem that suggests use of some Pythagorean relations (see the representation of this problem on the next page. Historical sources suggest that forms of the Pythagorean theorem might have been known to ancient civilizations such as Mesopotamia and Egypt before Pythagoras.

Appendix E-3 (continued)

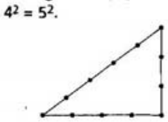
3.4

In ancient Egypt, the Nile River overflowed every year, flooding the surrounding lands and destroying property boundaries. As a result, the Egyptians had to remeasure their land every year. Since many plots of lands were rectangular, the Egyptians needed a reliable way to mark right angles. They devised a clever method involving a rope with equally spaced knots.

2. Mark off 12 segments of the same length on a piece of rope or string. Tape the ends of the string together to form a closed loop. Try to form a right triangle with side lengths that are whole numbers of segments. You may need to have a classmate hold the string in place while you check that your triangle is a right triangle. What are the side lengths of the triangle you formed? Do these side lengths satisfy the relationship $a^2 + b^2 = c^2$?

3. How do you think the Egyptians used the knotted rope?

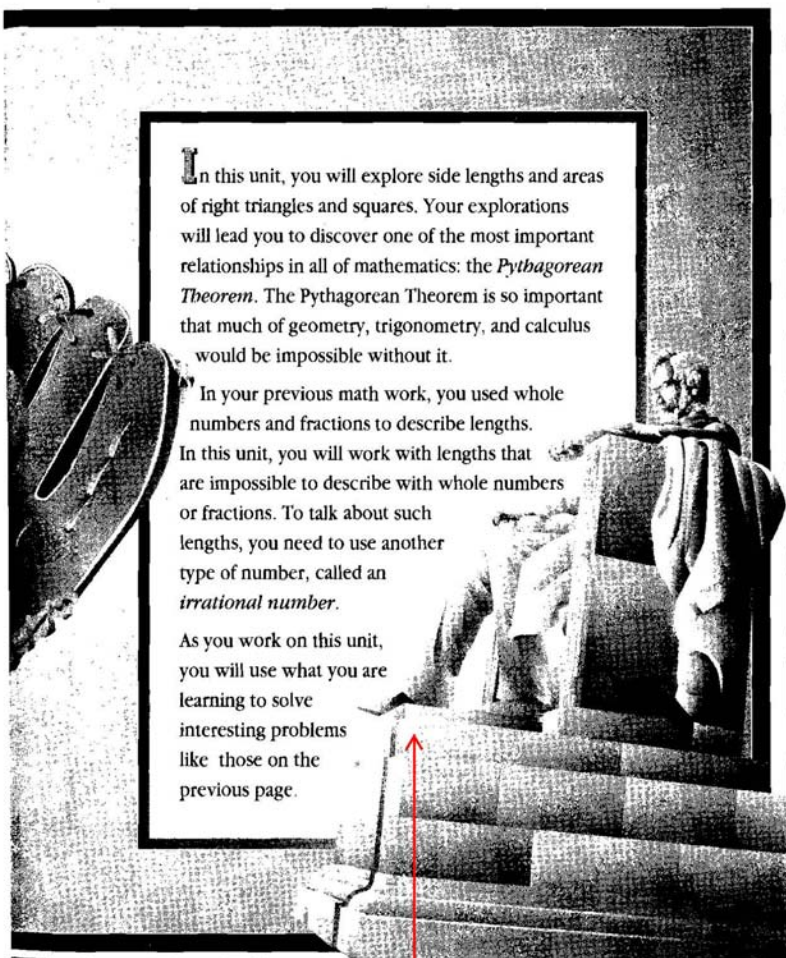
2. The side lengths of the triangle are 3, 4, and 5. They do satisfy the relationship $a^2 + b^2 = c^2$, since $3^2 + 4^2 = 5^2$.



33

3. Possible answer: The Egyptians could have used the knotted rope to re-mark the property boundaries, perhaps setting one leg of the right triangle along the river bank and putting stakes in the ground along the other leg, which extended from the river bank at a 90° angle.

Appendix E-4: General Historical – Pythagoras Monument



In this unit, you will explore side lengths and areas of right triangles and squares. Your explorations will lead you to discover one of the most important relationships in all of mathematics: the *Pythagorean Theorem*. The Pythagorean Theorem is so important that much of geometry, trigonometry, and calculus would be impossible without it.

In your previous math work, you used whole numbers and fractions to describe lengths. In this unit, you will work with lengths that are impossible to describe with whole numbers or fractions. To talk about such lengths, you need to use another type of number, called an *irrational number*.

As you work on this unit, you will use what you are learning to solve interesting problems like those on the previous page.

Let the class briefly discuss these questions: "What is the area of this figure?" "What is the length of each side of this figure?" "Is this figure a square? Why or why not?"

Explain that in this unit, students will explore the answers to these and related questions.

General historical:
This monument offers some historical context about Pythagoras.

3

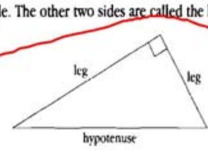
Appendix E-5: Implicit Historical – the Pythagorean Theorem

INVESTIGATION 3

The Pythagorean Theorem

In the last investigation, you found areas of figures drawn on dot paper. To find these areas, you may have used a method that involved right triangles. Recall that a right triangle is a triangle with a right, or 90°, angle. The right angle of a right triangle is often marked with a square.

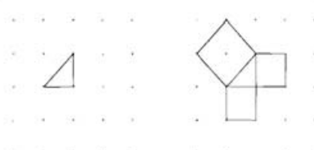
The longest side of a right triangle is the side opposite the right angle. We call this side the **hypotenuse** of the triangle. The other two sides are called the **legs**.



In this investigation, you will discover an interesting relationship among the side lengths of a right triangle.

1 Discovering the Pythagorean Theorem

Consider a right triangle with legs that each have a length of 1. Suppose you draw squares on the hypotenuse and legs of the triangle. How are the areas of these three squares related?



In this problem, you will look for a relationship among the areas of squares drawn on the sides of right triangles.

Implicit historical:
This discovery question subtly addresses the root of the theorem. Geometry (mainly relationships among areas of the squares) is used. Ancient Greeks approach on this problem relied on geometry.

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3.1 Discovering the Pythagorean Theorem

At a Glance

Grouping: *small groups*

Launch

- Introduce the idea of drawing squares on the sides of a right triangle and comparing their areas.
- Describe the problem.
- Have groups of three or four work on the problem and follow-up.

Explore

- Ask that each student make a table.
- Encourage group members to share the work.

Summarize

- Talk about the pattern in the table.
- Discuss whether the relationship works for triangles that are not right triangles.
- Go over the follow-up.

Assignment Choices

ACE questions 3–7, 12, 13, 17, 18, and unassigned choices from earlier problems

Appendix E-6: Implicit Historical – Puzzle on Proof

Puzzling Through a Proof

The pattern you discovered in Problem 3.1 is a famous theorem named after the Greek mathematician Pythagoras. A *theorem* is a general mathematical statement that has been proven true. The Pythagorean Theorem is one of the most famous and important theorems in mathematics. Over 300 different proofs have been written for this theorem. One of these proofs is based on the geometric argument that you will explore in this problem.

Labsheet 3.2 contains two square puzzle frames and 11 puzzle piec

A. Cut out the puzzle pieces. Examine a triangular piece and the t pieces. How do the side lengths of the squares compare to side triangle?

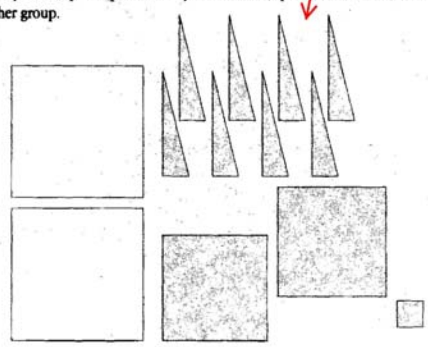
B. Arrange the 11 puzzle pieces to fit exactly into the two puzzle f four triangles in each frame.

C. Carefully study the arrangements in the two frames. What conc you draw about the relationship among the areas of the three s pieces?

D. What does the conclusion you reached in part C mean in terms lengths of the triangles?

Compare your completed puzzles and your answers to parts C and D with those of another group.

Implicit historical: This puzzle is related to Euclid's proof of the theorem. Euclid's proof is believed to be the earlier recorded proof of the theorem.



Puzzle frames Puzzle pieces

3.2

Puzzling Through a Proof

At a Glance

Grouping: ***small groups***

Launch

- Display a set of puzzle pieces, and ask what relationships students see.
- Have groups of four work on the problem; save the follow-up until after the summary of the problem.

Explore

- Have each group work with one version of the puzzle.
- Encourage groups to find more than one way to arrange the pieces.

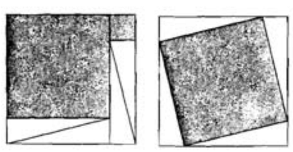
Summarize

- Ask students what relationships they notice among the puzzle pieces.
- Have a couple of groups show their completed puzzles, and help students to relate the puzzles to the theorem.
- Assign and then review the follow-up.

Answers to Problem 3.2

A. Each side length of the triangle is equal to the lengths of the sides of one of the three squares.

B. Possible arrangement:



C. The sum of the areas of the two smaller squares is equal to the area of the largest square.

D. The sum of the squares of the lengths of the legs of a right triangle is equal to the square of the hypotenuse.

Assignment Choices

ACE questions 8–11, 20, and unassigned choices from earlier problems

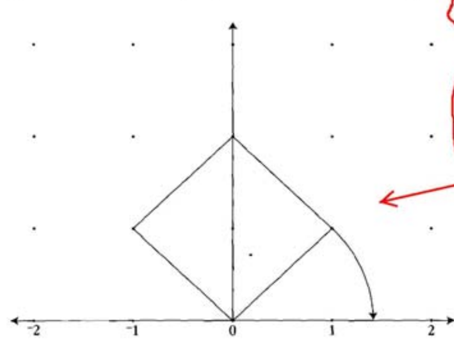
Appendix E-7: Implicit Historical – Irrational Numbers

INVESTIGATION 5

Irrational Numbers

In Investigations 2 and 3, you learned methods for finding the lengths of segments connecting dots on a grid. Sometimes you could express the lengths as whole numbers; other times, you had to use $\sqrt{\ }$ symbols or decimal approximations.

The square below has an area of 2 square units. The length of a side of this square can be expressed as $\sqrt{2}$. Just how long is $\sqrt{2}$? If we draw a number line and mark off a segment with the same length as a side of the square, we can see that the length of the segment is about 1.4.



Implicit historical: The Pythagorean theorem is used to introduce irrational numbers in this unit. Historically, Pythagoreans were believed to be the first to recognize 'incommunsuarabilty' of the side and diagonal of a square.

But $\sqrt{2} \times \sqrt{2} = 2$, while $1.4 \times 1.4 = 1.96$. So, 1.4 is too small. If we increase our estimate to 1.42, we get $1.42 \times 1.42 = 2.0164$. So, 1.42 is too large. Try squaring some other decimal numbers to see if you can get a square closer to 2.

If you use your calculator to find $\sqrt{2}$, you get something like 1.414213562, but if you multiply 1.414213562 by 1.414213562 by hand, you get 1.999999998948727844. It seems that even a calculator can't find the exact value of $\sqrt{2}$! So, although you can draw a line segment with a length of $\sqrt{2}$ and locate $\sqrt{2}$ precisely on a number line, it seems extremely difficult to find an exact decimal value for $\sqrt{2}$.

Appendix E-8: Implicit Historical – Wheel of Theodorus

5.1

Analyzing the Wheel of Theodorus

At a Glance

Grouping:
small groups

Launch

- Talk with the class about finding a decimal equivalent for $\sqrt{2}$.
- Introduce the Wheel of Theodorus.
- Have groups of two to four work on the problem and follow-up.

Explore

- Have each student label a number-line ruler.
- As students work, check on their understanding of measuring lengths and writing decimals.

Summarize

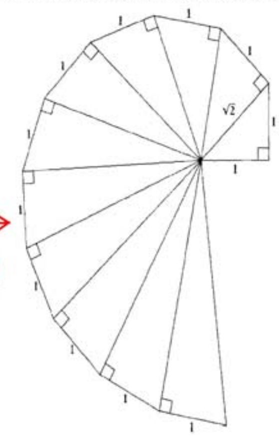
- Display the Wheel of Theodorus, and ask students for the hypotenuse lengths.
- Talk about decimal equivalents for various square roots.

5.1 Analyzing the Wheel of Theodorus

In earlier investigations, you drew segments with lengths of $\sqrt{2}$, $\sqrt{3}$, $\sqrt{4}$, and so on by drawing squares with whole-number areas. In this problem, you will investigate an interesting pattern of right triangles called the Wheel of Theodorus. The pattern will suggest another way to draw segments with lengths that are positive square roots of whole numbers.

The Wheel of Theodorus begins with a triangle with legs of length 1 and winds around counterclockwise. Each triangle is drawn using the hypotenuse of the previous triangle as one leg and a segment of length 1 as the other leg. To make the Wheel of Theodorus, you only need to know how to draw right angles and segments of length 1.

Implicit historical: This problem suggests the possibility of drawing segments of irrational length using the Pythagorean Theorem. The Wheel of Theodorus is believed to be evidence that the Pythagoreans knew incommensurable (irrational) measures.



The Wheel of Theodorus is named for its creator, Theodorus of Cyrene. Theodorus was a Pythagorean and one of Plato's teachers.

Assignment Choices

ACE questions 9–20 and unassigned choices from earlier problems

Answers to Problem 5.1

A. The lengths of the hypotenuses from least to greatest are $\sqrt{2}$, $\sqrt{3}$, 2, $\sqrt{5}$, $\sqrt{6}$, $\sqrt{7}$, $\sqrt{8}$, 3, $\sqrt{10}$, $\sqrt{11}$, and $\sqrt{12}$.

B. See page 63g.

C. The lengths $\sqrt{2}$ and $\sqrt{3}$ are between 1 and 2; $\sqrt{5}$, $\sqrt{6}$, $\sqrt{7}$, and $\sqrt{8}$ are between 2 and 3; and $\sqrt{10}$, $\sqrt{11}$, and $\sqrt{12}$ are between 3 and 4.

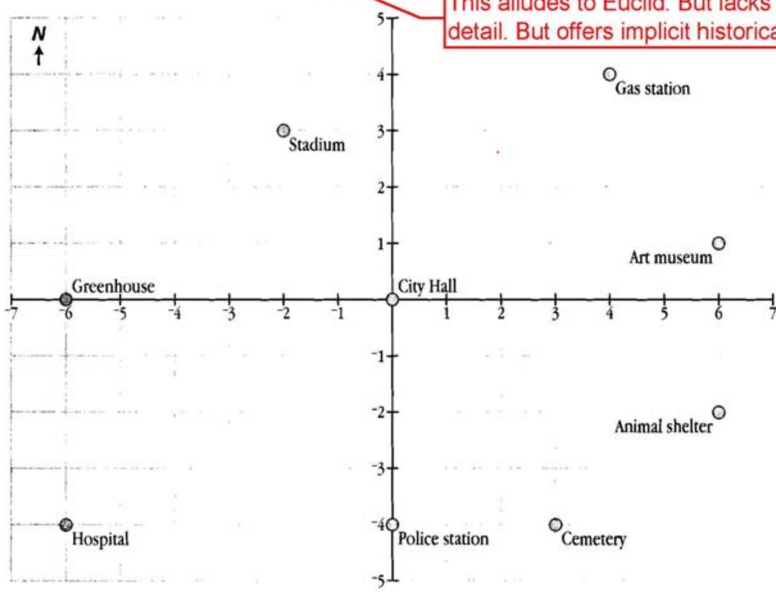
D. The length $\sqrt{2}$ is between 1.4 and 1.5; $\sqrt{3}$ is between 1.7 and 1.8; $\sqrt{5}$ is between 2.2 and 2.3; $\sqrt{6}$ is between 2.4 and 2.5; $\sqrt{7}$ is between 2.6 and 2.7; $\sqrt{8}$ is between 2.8 and 2.9; $\sqrt{10}$ is between 3.1 and 3.2; $\sqrt{11}$ is between 3.3 and 3.4; and $\sqrt{12}$ is between 3.4 and 3.5.

Appendix E-9: Implicit Historical – Euclid

Driving Around Euclid

The founders of the city of Euclid liked math so much that they named their city after a famous mathematician and designed their street system to look like a coordinate grid. A map of the city is shown below. The Euclideans describe the locations of buildings and other landmarks by giving coordinates. For example, the animal shelter is located at (6, −2).

Implicit historical:
This alludes to Euclid. But lacks historical detail. But offers implicit historical context.



[redacted]

A. Give the coordinates of each labeled landmark on the map.

B. **1.** How many blocks would a car have to travel to get from the hospital to the cemetery?

2. How many blocks would a car have to travel to get from City Hall to the police station?

3. How many blocks would a car have to travel to get from the art museum to the gas station?

C. How can you tell the distance in blocks between two points if you know the coordinates of the points?

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Vita

I was born on August 19, 1966 in Afelba, Eritrea. My mother Hiwet Mehreteab Bahta and my father Kiflemariam Haile Ghide have spent most of their lives in this village and have taught us (their children) to be disciplined and hard working. Because of my aspirations to pursue formal education, I had to live away from them for many years. I have lived with people I could call my other parents and siblings. My aunt Alem Mehreteab Bahta tops the list.

I completed my primary, secondary, and undergraduate education in Eritrea. My pre-college education was during the time of the Eritrean war for liberation from Ethiopia. Due to the war conditions, my primary education was interrupted for several years. But, in some ways, that period of my life served as a strong motivation to seriously pursue my education once I got the opportunity. After attending Addis Ababa University, Ethiopia, for the first semester of college in 1991, I completed my undergraduate education at the University of Asmara, Eritrea, in 1995. I earned a Bachelor of Science in mathematics with a minor in statistics. From 1992-1993, I taught mathematics for grades 7 and 9 as a national service teacher at Winna Technical School in Eritrea.

I worked as graduate assistant (1995-1996) and as lecturer (1997-2002) at the University of Asmara in the College of Education. During 1996-1997, I attended the University of Twente, in the Netherlands, receiving a master’s degree in educational and training systems design sponsored by the Netherlands Organization for International Cooperation in Higher Education. As a lecturer at the University of Asmara, I taught undergraduate courses in education students. My duties at this university also included serving as academic advisor for mathematics education students and as a member of the

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