Trapezoidality and the Alexander Polynomial for Alternating Links

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Abstract

This undergraduate thesis serves to fulfill the thesis requirement for the Dean's Scholars Honors in Mathematics and Plan II Honors degrees [1]. Using fairly elementary tools, namely the skein relation and a theorem of Jong in [6], I show that the Fox conjecture about the trapezoidality of the Alexander polynomial holds for a small class of Montesinos links. This result is of the same theme as the results in Murasugi's paper on the same conjecture [7], but I show that the conjecture is true for a few cases that are not covered in Murasugi's paper, and I use different methods.

1 Introduction

This section is a broad introduction to the subject of knot theory; it should be accessible to non-mathematicians and is not necessary for the rest of this thesis.

Knot theory is exactly what it sounds like; it is the mathematical study of knots. The knots that mathematicians study are idealized forms of the knots that we deal with in every day life: you can think of a knot as a piece of string with a normal knot tied in it and with the ends joined together. A link is just several knots tangled together. When studying a knot, you are allowed to move and stretch the strands all you like, but you are not allowed to cut a string or have one pass through itself. Knots are creatures of three-dimensional space, but we usually represent them by drawing diagrams on paper, where the diagram indicates in a neighborhood of each crossing which strand goes over and which goes under.

Although knots have been around for millennia, modern mathematicians became interested in the subject in the 1800s, when Lord Kelvin postulated that atoms were knotted bits of aether. This inspired Peter Tait to begin writing down all the possible knots he could come up with, and this is heralded the beginning of knot theory. Although Kelvin was wrong about knots making up atoms, questions about knots helped motivate the development of topology in the early 20th century, and knots are still very much a part of the vanguard

[1] I would like to thank my advisor Dr. Cameron Gordon for teaching me about the beautiful subject of knot theory and for patiently helping me with the research that led to this thesis. It has been a real honor to work with him. I would also like to thank my second readers Dr. James Vick and Dr. John Luecke.

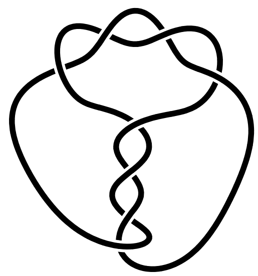

Figure 1: A knot

of mathematical research. Additionally, recently knots and knot theory have found applications in biology, chemistry, theoretical physics, and quantum computing. An interesting survey of the early history of knot theory is found in [8].

As Tait was tabulating knots, he and others observed that although they could produce many examples of knots that looked distinct, they had no way of proving that this was the case; in fact they had no way of proving that there were any non-trivial knots at all. This might sound silly, and indeed it is a not a great leap of faith to imagine that the trefoil (overhand knot) is not equivalent to a circle, but this question can be very hard in general. This is a common theme of knot theory: questions that are easy to ask, such as "Is a knot different from a circle?" are often hard to answer.

In order to prove that there were different types of knots, in the early 20th century mathematicians began to develop invariants of knots. A knot invariant is something that can be associated to knots to tell if two knots are different; if the invariant is different for two knots then those knots are actually distinct. All currently known invariants can only tell if knots are different; the only way to know that two knots are the same is to move them around so that they have identical diagrams. Here is an example of a knot invariant: the crossing number of a knot, $c(K)$, is defined as the minimum number of crossings in a diagram of K. This is a difficult invariant to compute because it is hard to show that the number of crossings in a diagram cannot be reduced, but if you succeed in doing so and get a positive number then you have proved that the knot in question is not trivial, because an unknotted circle clearly has crossing number zero.

Crossing number is an invariant that is easy to define but hard to calculate, as are most other invariants that are geometric in nature. Therefore, mathematicians sought to create invariants that were harder to describe but that could be computed algorithmically. One such invariant is the Alexander polynomial, which associates to a knot an integral polynomial. This invariant was developed by James Alexander in the 1920s, and is constructed by considering the topological space that arises when an embedded knot is removed from normal 3-dimensional space. Over forty years later, John Conway discovered that this polynomial could also be described axiomatically with a skein relation [2]. In the 1980s Vaughan Jones discovered another polynomial that was defined by a slightly different skein relation, and soon there were many different knot polynomials. Now there is a zoo of knot invariants that use the latest algebraic and topological technology, but this thesis concerns a conjecture about one of the first algebraic invariants, the Alexander polynomial.

A knot is called alternating if it has a diagram where the crossing alternate going over and under as each strand is traced. For example, the knot in Figure 1 is alternating. Alternating knots are particularly nice to study, for example if a knot has an alternating diagram where no obvious simplifications are possible, then it is non-trivial. In the 1950s Ralph Fox conjectured that for the Alexander polynomial of an alternating knot, the absolute values of the coefficients form a trapezoidal sequence. A trapezoidal sequence is one that increases, possibly plateaus, then decreases. For example 1, 2, 3, 7, 7, 3, 2, 1 is a trapezoidal sequence. Despite the fact that the Alexander polynomial has been studied for almost a century, this conjecture is still open. It has been proved for a large class of algebraic knots, genus one and two knots, pretzel knots, and probably other classes of knots and links that have escaped my attention. In this thesis I seek to verify the conjecture for some algebraic knots that were previously unstudied.

A solution to this conjecture would have few if any practical implications, but it is interesting to me because it might shed light on why alternating knots are so well behaved. The property of being an alternating diagram is in some sense a two dimensional property, because it is only evident in a two dimensional diagram. Knots however are three dimensional, and studying them via two dimensional pictures is done simply because it is hard for humans to visualize knots in three dimensional space. Consequently, it is surprising to me that the seemingly artificial property of being alternating actually has im-

[2]Actually this result was essentially in Alexander's original paper on the polynomial, but neither Alexander nor his contemporaries recognized its significance.

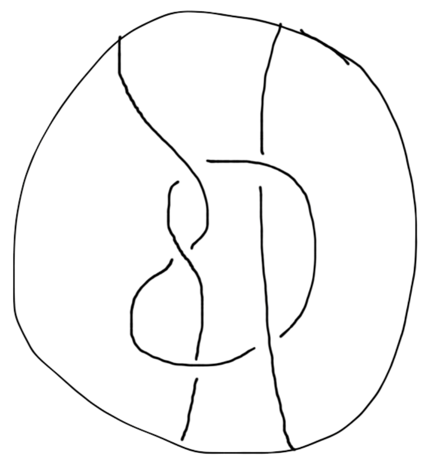

Figure 2: A (nonrational) tangle

portant consequences. Perhaps a solution to the Fox conjecture would help explain why this is.

2 Rational Tangles

A tangle is like a knot where the ends are not closed. Therefore, most of the "knots" that we encounter in our everyday lives are actually tangles. Rational tangles are the building blocks for algebraic knots, so here we briefly discuss tangles, rational tangles, and some of their properties. A good self-contained introduction to rational tangles is found in [5].

2.1 Definition. *An n-tangle is a proper embedding of the disjoint union of n closed arcs into B^3.*

We will be solely concerned with 2-tangles, and will usually just write tangle instead of 2-tangle.

Just as we represent knots with planar diagrams, indicating whether a crossing is an over-crossing or an under-crossing, so we do with tangles. In a diagram of a tangle, we will sometimes refer to the endpoints as the NE, NW, SE, and SW ends in the obvious way.

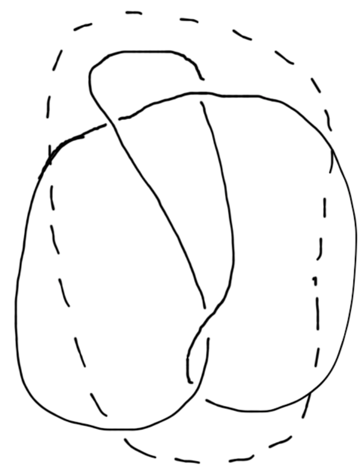

Figure 3: A tangle made from a knot

Also just like knots, two tangles are the same if they can be moved around inside of B^3 (with the endpoints fixed) to give diagrams that look the same.

2.2 Definition. *Two tangles T_1, T_2 are equivalent if there is a boundary fixing homeomorphism $h : B^3 \to B^3$ such that $h(T_1) = T_2$.*

2.3 Definition. *A trivial tangle is a 2-tangle with no crossings.*

With this definition, the arcs are allowed to knot in the interior of B^3 as in Figure 2, and any knot can be made into a tangle as in Figure 3. In order for tangles to give us a nice structure, we need a less general definition. This leads to the idea of a rational tangle as a tangle that can be untwisted.

2.4 Definition. *A rational tangle is a 2-tangle T where there there is an ambient isotopy $h_t : B^3 \to B^3$ taking T to the trivial tangle. Equivalently, a tangle is rational if there is a way to twist the ends so that the final result is the trivial tangle.*

With this definition, it should be intuitively clear that all rational tangles can be constructed by adding together tangles that just consist

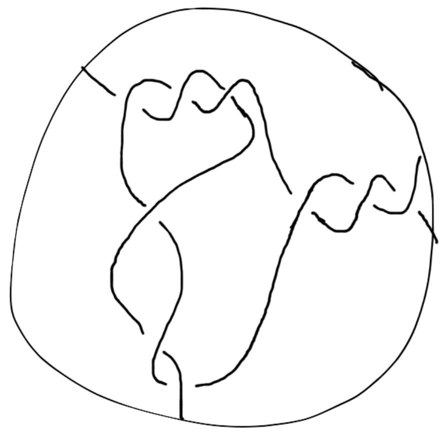

Figure 4: A rational tangle

of horizontal or vertical twists as in Figure 5, because adding these twists corresponds to twisting the strands on a trivial tangle.

These tangles are called rational because this construction gives a bijection $\{rational\ tangles\} \to \mathbb{Q} \cup 1/0$. A detailed discussion of this bijection is in [5]. From any tangle, there is a natural way to make a knot by joining some of the ends together.

2.5 Definition. *A rational link is a link obtained by joining the north and south ends of a rational tangle.*

An example is given in Figure 6. We are mainly interested in rational links because they are also described by plumbing according to a weighted arc-tree, as we describe in Section 4.

3 The Alexander Polynomial

The Alexander Polynomial $\Delta(t)$ is a polynomial in $\mathbb{Z}[t, t^{-1}]$, and it was one of the first invariants of knots to be studied. It classically comes from the homology of the infinite cyclic cover of the complement of a knot in S^3. A multivariable polynomial in $\mathbb{Z}[t_1, t_1^{-1}..., t_n, t_n^{-1}]$ (n is the

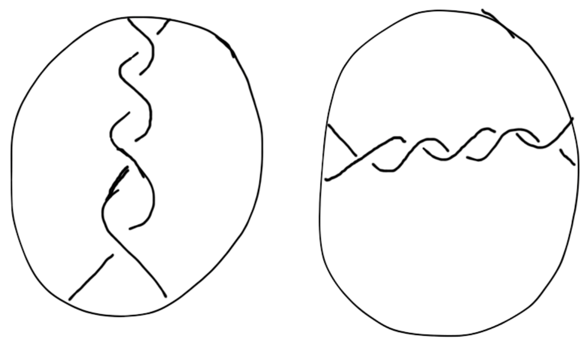

Figure 5: Building blocks for tangles

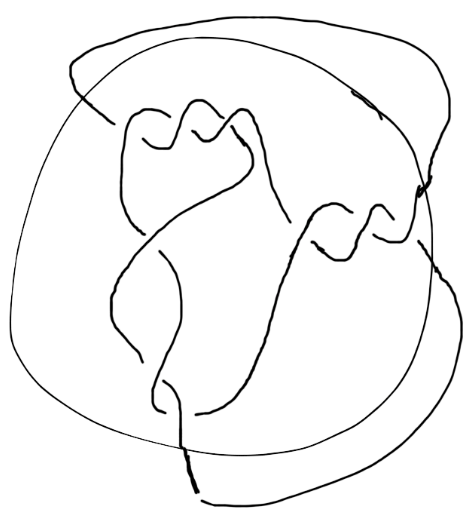

Figure 6: A rational link

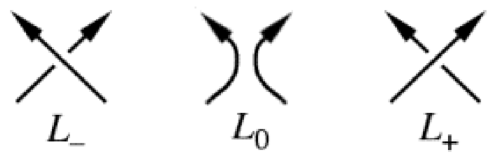

Figure 7: Diagram for the skein relation

number of components of the link) is similarly defined for links, and the reduced Alexander polynomial is obtained by setting the $t_1, ...t_n$ equal and multiplying by $(t^{-1} - t)$. We will always work with the reduced Alexander polynomial, and will usually drop the 'reduced'. Normally this polynomial is defined only up to multiplication by units in $\mathbb{Z}[t, t^{-1}]$.

We will use two different way to compute the polynomial. The first is an axiomatic formulation due to John Conway, and it defines a polynomial for oriented links.

3.1 Definition. *Given a link L in $\mathbb{R}^3$ or S^3, define a orientation on L by giving it an orientation as a $1-manifold$.*

Note that if L is the boundary of an orientable surface, then a choice of orientation on the surface defines an orientation on L.

3.2 Definition. *Define the Conway polynomial $\nabla(L)$ for an oriented link L by*

1. $\nabla(unknot) = 1$
2. $\nabla(L_+) - \nabla(L_-) = z\nabla(L_0)$

where L_+, L_-, and L_0 are diagrams differing only in a neighborhood of a crossing as depicted in Figure 7.

By setting $z = t^{-1} - t$, we recover the Alexander polynomial. When it is defined axiomatically in this way the Alexander polynomial is well defined, and is not just defined up to multiplication by powers of t. Note that L_+ and L_0 have a different number of components; this demonstrates how links naturally occur even if one is mainly interested in knots. Also, it should be noted that usually the substitution $z = t^{-1/2} - t^{1/2}$ is used instead. The substitution we use avoids non-integral powers of t, but the polynomial is different than that found in some references. This is the approach taken in [3], so polynomials obtained with this skein relation agree with the ones in that book.

Another way to compute the Alexander polynomial for alternating links is due to Richard Crowell [4] and it uses a directed graph constructed from an alternating link diagram.

3.3 Definition. *From an oriented alternating link diagram D of an alternating link L, construct a 4-valent graph called the Crowell graph as follows: put a vertex at each crossing, and an edge for each arc between crossings. Edges are directed from over crossings to under crossings. In the chosen orientation for the link, label the edge to the right of each over crossing t, and the edge to the left label 1. This is called the Crowell graph of D.*

The Crowell graph is 4-valent, and at each vertex there are two incoming edges and two outgoing edges. To compute the Alexander polynomial from a Crowell graph arbitrarily choose a vertex in the graph to be a root. Then, we write down all rooted spanning subtrees of the graph.

3.4 Definition. *A spanning subtree rooted at v_i of a directed graph is a subgraph T such $V(T) = V(G)$, every vertex except for v_i has one incoming edge, v_i has no incoming edges, and the number of edges in T is $|V(T)| - 1$.*

An example for the Whitehead link is given in Figure 8.

Informally, this can be thought of as starting at the root vertex, and traversing the graph while only crossing edges in the 'correct' direction. Writing down all possible ways to do this gives all trees rooted at a particular vertex. Crowell proved the following theorem.

3.5 Theorem (Crowell [4])**.** *Let L be an alternating link, and G the Crowell graph associated to an alternating diagram of L. Let $S(G, v_i)$ be the set of spanning subtrees of G rooted at v_i, and $w(T)$ be the number of edges labeled t in a spanning subtree T. Then*

$$\Delta_L(-t) = \sum_{T \in S(G,v_i)} t^{w(T)}, \tag{3.1}$$

where $\Delta_L(-t)$ is the Alexander polynomial, normalized so that the power of the lowest term is t^0, with the convention that the constant term has a positive coefficient, and where equality is up to multiplication by powers of t.

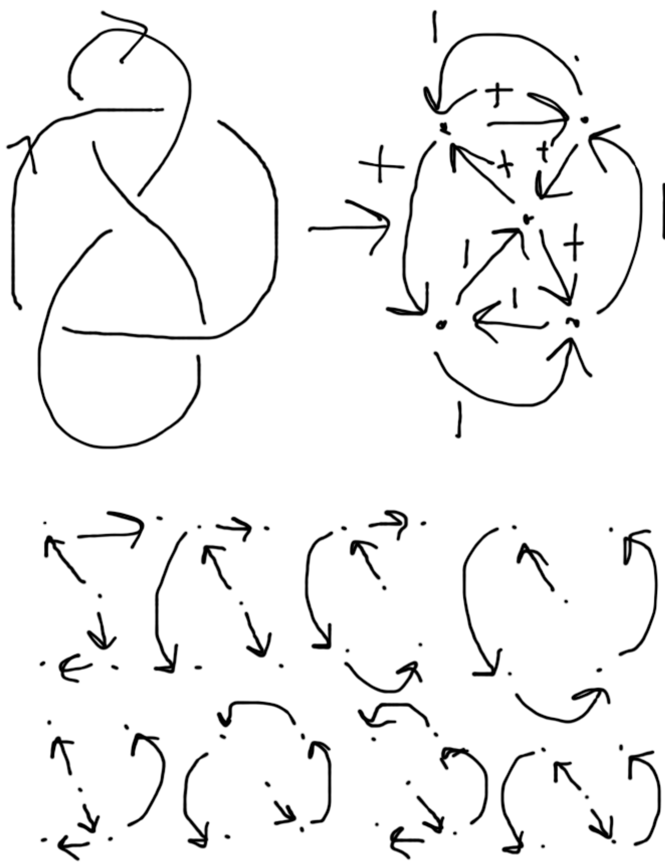

Figure 8: The Crowell graph and spanning trees for the Whitehead link, the spanning trees are rooted at the central vertex.

We will call the polynomial obtained from the skein relation the symmetrized polynomial, and the polynomial coming from the Crowell graph the normalized polynomial. Also, depending on our intended emphasis, we will denote the Alexander polynomial of a link L as $\Delta_L(x)$ or $\Delta(L)$. Now, we note a few important properties of the Alexander polynomial, these properties are phrased in terms of the symmetrized polynomial.

3.6 Theorem. *Let $\Delta_L(x)$ be the Alexander polynomial for a link L. $\mu(L)$ is the number of components of L and $g(L)$ is the minimal genus of an oriented spanning surface.*

1. *$\Delta_L(x)$ is skew-symmetric, that is $\Delta_L(-x) = (-1)^{\mu(L)-1}\Delta_L(x)$, and $\Delta_L(x^{-1}) = (-1)^{\mu(L)-1}\Delta_L(x)$.*
2. *If L^* is L with all the crossings changed, then $\Delta(L^*) = (-1)^{\mu(L)-1}\Delta(L)$.*
3. *$L_1\#L_2 = \Delta(L_1)\Delta(L_2)$, where $\#$ is connected sum.*
4. *If L is alternating, then the coefficients of $\Delta_L(x)$ alternate in sign, and the span of $\Delta_L(x) = 4g(L) + 2\mu(L) - 2$.*

Proofs of these facts can be found in [3]. We are almost ready to state the Fox conjecture; first we need to define a trapezoidal sequence.

3.7 Definition. *A sequence of natural numbers $\{a_i\}$ is trapezoidal if $\exists n$ such that $\forall i \leq n$, $a_i \leq a_{i+1}$, and $\forall i \geq n$, $a_i \geq a_{i+1}$.*

Here are some easy properties of trapezoidal polynomials:

3.8 Theorem. *If g and f are trapezoidal symmetrized knot polynomials, then gf is as well. If g and f are positive trapezoidal polynomials of the same degree, then $g + f$ is trapezoidal. If g and f are symmetrized trapezoidal knot polynomials and they agree in sign at one coefficient of the same power, then $g + f$ is trapezoidal.*

3.9 Conjecture (Fox)**.** *If L is an alternating link, then the absolute values of the coefficients of $\Delta(L)$ form a trapezoidal sequence.*

3.10 Definition. *Polynomials are always assumed to be written so that the degrees of the terms are decreasing, so the first coefficient of a polynomial is the coefficient of the highest degree term. If the absolute values of the coefficients of a polynomial form a trapezoidal sequence, we will call that polynomial trapezoidal. If L is a link and $\Delta(L)$ is trapezoidal, we will say that L is trapezoidal.*

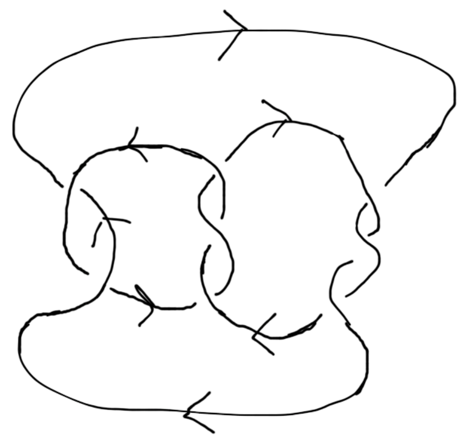

Figure 9: A non-alternating knot, usually labeled 8_{19}

Here is a heuristic for why the Fox conjecture should be true: Crowell's theorem roughly says that the coefficient of the first term of the polynomial is the number of paths that traverse a knot while only traveling on the t edges, and that for the second term it is the number of paths that travel on t edges except for one 1 edge. Intuitively, if you are allowed more choices for what edges you can use in a path, there should be more paths. If this could be rigorously proven, then we could solve the Fox conjecture.

However, if this intuition was true, then there should be a 'Fox theorem' for any quadrivalent graph where the edges are labeled t and 1 (with the labeling subject to a few conditions), where the polynomial is defined as it is for Crowell graphs constructed for knots. This unfortunately is not the case: Sean Simmons in his undergraduate thesis found an example of a quadrivalent graph labeled in such a way so that there was a vertex in the graph where the polynomial made from considering trees rooted at that vertex was not trapezoidal [9]. Consequently, a proof of the conjecture must exploit some special properties of links.

We close this section by noting how a knot can fail to be trapezoidal by studying the first non-trapezoidal knot, 8_{19}, shown in Figure 9.

If we apply to skein relation to the top left hand crossing, we get that $L_- = 8_{19}$, $L+$ is the connect sum of two trefoils, and L_0 is the 2, -3, -3 Pretzel link. So, $\Delta(L_+) = t^4 - 2t^2 + 3 - 2t^{-2} + t^{-4}$ and $\Delta(L_0) = t^5 - t^3 + t - t^{-1} + t^{-3} - t^{-5}$. Note that both of these are trapezoidal. Therefore, $\Delta(8_{19}) = -t^4 + 2t^2 - 3 + 2t^{-2} - t^{-4} + (t - t^{-1})(t^5 - t^3 + t - t^{-1} + t^{-3} - t^{-5})$, or

$$\Delta(8_{19}) = -t^4+2t^2-3+2t^{-2}-t^{-4}+t^6-2t^4+2t^2-2+2t^{-2}-2t^{-4}+t^{-6}. \tag{3.2}$$

Consequently, the Alexander polynomial of 8_{19} is the sum of two trapezoidal polynomials, and yet $\Delta(8_{19}) = t^6-t^4+0t^2+1+0t^{-2}-t^{-4}+t^{-6}$, and is not trapezoidal. With the skein relation and the fact that the Alexander polynomials of small knots are all trapezoidal, it is easy to see that the Alexander polynomial of any link can be written as a sum of trapezoidal polynomials. The difficulty in proving the Fox conjecture is in proving that trapezoidality can be preserved when the knot is alternating, which comes down to being able to show that the polynomial is the sum of trapezoidal polynomials whose signs agree in a way such that the sum remains trapezoidal.

4 Some Graphs, Plumbing and Arborescent Links

This section describes arborescent links and contains several definitions that will be used in Section 6.

4.1 Definition. *We define an (abstract) graph G as a pair (V, E), where V is the set of vertices and E is the set of edges, an element of E being a subset $\{v_i, v_j\} \subset V$, where $i \neq j$. We will usually represent a graph by drawing the vertices and edges.*

Often we'll denote the valency of a vertex, that is the number of edges adjacent a particular vertex v, by val (v).

4.2 Definition. *An embedding of a graph is the realization of a graph as an embedded $1-$simplex, a planar graph is a graph whose geometric realization can be embedded in $\mathbb{R}^2$.*

In our discussion, different embeddings of the same abstract graph can give different links, so we just consider planar embeddings of graphs. All the graphs that will be used in this section are trees, so all are planar.

4.3 Definition. *A weighted graph is a graph G together with a function $w : V \to \mathbb{Z}$. Usually we'll specify the weights of the vertices by writing them in a picture of the graph.*

4.4 Definition. *A weighted graph is even if each weight is even.*

4.5 Definition. *An arc-tree is a tree where there is no vertex of valency greater than two. A stellar tree is a connected tree where only one vertex has valency greater than two.*

4.6 Definition. *The complement of v in T is T with v and the edges adjacent to v removed. We will note this as $T - v$.*

Weighted trees are important here because they are used to describe a surface, the boundary of which is an arborescent (sometimes called algebraic link). To understand this, we must understand how to "plumb" two twisted bands together.

4.7 Definition. *Given two twisted bands C and D, we call the following operation plumbing: select patches C' and D' on C and D, such that C' and D' are homeomorphic to closed rectangles and $C' \cap \partial C$ and $D' \cap \partial D$ are both 2 arcs. Then identity C' and D'.*

An example is given in Figure 10. The monograph [1] contains a wealth of information about plumbing and many good pictures, especially in Section 12.

4.8 Definition. *Given a weighted tree T, define a link by treating each vertex of T as a band with $w(v)$ twists, and plumb two bands together if there is an edge between them. Taking the boundary gives a link. If T is tree and L a link constructed from T, we will say that L is supported by T.*

If T is disconnected, we connect sum the different links we get from each connected component. Therefore, this is in general only well defined when T is connected. Links constructed in this way are called arborescent. This class of links was first introduced in [2], where they are called algebraic links. In that paper, the links are constructed from an algebra of rational tangles, rather than from weighted trees, but the two classes are the same. We will often interchangeably refer to T and the link constructed from T when the context is clear.

That weighted arc-trees correspond to rational links is clear after drawing a picture of what happens when two bands are plumbed together; it is the same as adding a vertical or horizontal tangle to a rational tangle. This is demonstrated in Figure 10.

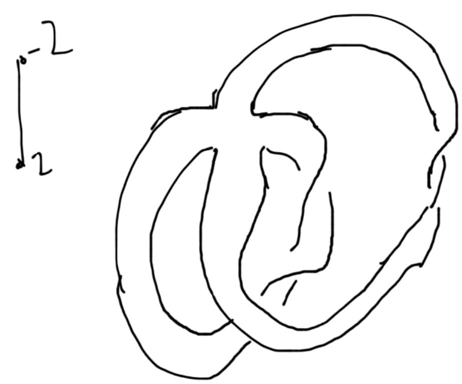

Figure 10: A link constructed from an arc tree

[7] studies the trapezoidality of some even arborescent trees. To state the main theorem of that paper, first we define notions of the uniform decomposition of a tree and an excessive tree.

4.9 Definition. *The uniform decomposition of a a weighted graph T is the (in general) disconnected graph obtained by deleting all the edges joining positive and negative vertices. An excessive graph is a graph where the weight of each vertex is greater than or equal to* $\max\{2, \mathrm{val}\,(v)\}$.

4.10 Theorem (Murasugi [7])**.** *If in the uniform decomposition of an even tree T every connected component is excessive, then the link associated to T is alternating and trapezoidal.*

Note that if a tree is even, then the surface constructed from it is orientable, so there is a natural orientation induced on the link. We will use this natural orientation and the results of the next section to extend the Fox conjecture to a subclass of links constructed from stellar trees.

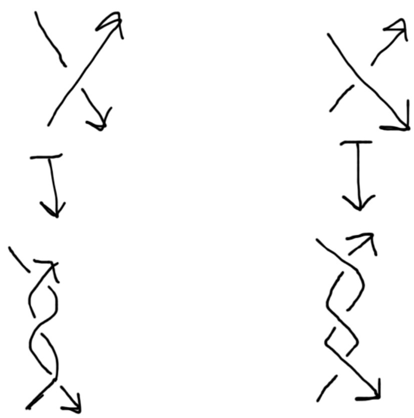

Figure 11: The t_2 move

5 Adding Twists

This section is mainly a discussion of the results of [6]. I present a sketch of a key lemma in that paper, and then prove a corollary that reduces studying even trees to studying ± 2-trees, that is those trees where the only weights are plus and minus two.

5.1 Definition. *A t_2 move on a crossing c replaces c with three crossings, as in Figure* 11*. If D is a diagram and c a crossing of D, we will write D^c for the diagram we get by applying a t_2 move to c.*

5.2 Lemma (Jong, [6])**.** *Let D be a reduced alternating diagram with crossing c, and D/c the diagram obtained by smoothing at c. Then*

$$\text{span}\,\Delta(D/c) + 1 = \text{span}\,\Delta(D). \qquad (5.1)$$

This is Lemma 4.3 in [6], it essentially comes down to counting bounded white regions in a checkerboard coloring of these two diagrams. It is important to observe that the Alexander polynomial in this section is different than the Alexander polynomial defined by the skein relation, if this polynomial is $\Delta(x)$, then $\Delta(x^2)$ gives the polynomial defined by the skein relation. In other words, this is the polynomial obtained from computing with a Crowell graph, so it is normalized so that there is a positive constant term.

In particular, in the next lemma we are working with this polynomial.

5.3 Lemma (Jong, [6]). *Let L be a nonsplit alternating link, D a reduced alternating diagram of L, and c a crossing of the diagram D. Then we have the following formula:*

$$\Delta_{D^c}(-t) = \Delta_D(-t) + (1+t)\Delta_{D/c}(-t). \tag{5.2}$$

Note that this implies that if D and D/c are trapezoidal, then D^c is trapezoidal. Also, note that this is essentially the skein relation. The key difference is that by studying the Crowell graph of D^c, the skein relation can be rephrased in terms of positive polynomials, and the sum of two positive trapezoidal polynomials of the same degree is trapezoidal.

Proof. The key is to observe that the spanning trees for D^c can be broken up into spanning trees of D and D/c, multiplied by appropriate terms. In particular, we get that $P(D^c; c) = tP(D; c) + (1 + t)P(D^c/b, c)$, where $P(D, c)$ is the graph polynomial of D rooted at vertex c, and the crossings are as labels in Figure 12. This is because all the spanning trees for D^c are in either group 1, 2, or 3. We can see that group 1 is just the spanning trees of D, multiplied by an extra t term. Groups 2 and 3 are both the same as spanning trees of the diagram we get by smoothing at the middle vertex, except in group 3 we multiply by an extra t term.

After we have this equation, because t_2 moves preserve genus of a canonical Seifert surface and the number of link components, and these determine the span of the Alexander polynomial, we know that span $(\Delta(D^c))$ = span $(\Delta(D))$, and from Lemma 5.2 span $(1 + t)\Delta_{D/c}(t)$ = span $\Delta_D(t)$. Because the graph polynomial equals the Alexander polynomial only up to multiplication by powers of t, we get $\Delta_{D^c}(-t) = \Delta_D(-t) + (1+t)\Delta_{D/c}(-t)$.

□

Immediately, we get that we can study the trapezoidality of even algebraic links by studying algebraic links with all weights equal to ± 2.

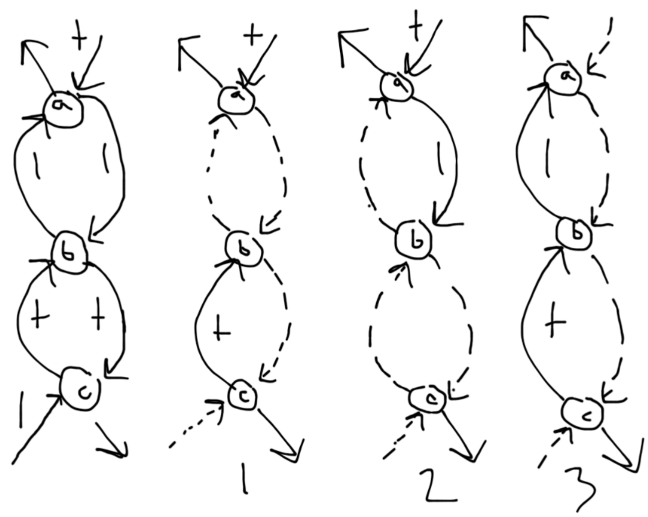

Figure 12: Breaking up the spanning trees of D^c

5.4 Definition. *If L is an even algebraic link constructed from a weighted tree T, define the 2-reduction of T, or T^2, as the link obtained by replacing $w(v)$ with* sign $w(v)(2)$ $\forall v \in V$.

5.5 Corollary. *If L is an alternating even arborescent link constructed from T with the induced orientation, L is trapezoidal if $\forall v \in V$, T^2 and $\Delta(T^2 - v)$ are trapezoidal*

Proof. This is just Lemma 5.3 and the fact that smoothing a crossing in an algebraic link is the same thing as deleting a vertex, as we explain further in Section 6. □

6 Even Alternating Montesinos Links

6.1 Definition. *A link constructed from a stellar tree is called Montesinos. This stellar tree is called alternating if the weights alternate in sign. We define the number of arms of a stellar tree as the valency of the central vertex, and the length of an arm as the number of vertices in the component of $T - v$ corresponding to that arm, where v is the central vertex.*

Note that this corresponds to the link being an obviously alternating link.

6.2 Definition. *A pretzel link is a Montesinos link where all the arms have length one and the central vertex has weight 0.*

We'll denote plumbing on a vertex in the following way:

6.3 Definition. *If T is a tree and v is a vertex, we will call T, v, n the tree obtained by attached a vertex of valency n to v.*

Note that in the case of Montesinos links, if v is a vertex of valency 1, then $T - v$ is just a smaller Montesinos link. Therefore, if we could prove that even alternating Montesinos links with each arm of length one are trapezoidal, and adding on a ± 2 vertex in a way that preserves alternation also preserves trapezoidality, then we could prove that even alternating Montesinos links are trapezoidal.

For an algebraic link constructed from an even tree, the plumbing surface is orientable, which means that there is a natural induced orientation on the link. In the rest of this section, we'll be working with the induced orientation.

First, we want to show that 2-twist pretzel links are trapezoidal. Then, using the results of section 5, we will show that all even twist pretzel links are trapezoidal.

Let P_n denote the pretzel link with n 2-twists. This corresponds to a stellar tree with $n+1$ vertices, a central vertex of weight zero and valency n, and a n twigs of weight two. Because the Alexander polynomial doesn't change under changing all crossings, we may assume that the twists are all positive [3], which means that each crossing is positive. The following lemma follows directly from the skein relation.

6.4 Lemma. $\Delta(P_n) = (t^{-1} - t)^{n-1} + (t^{-1} - t)\Delta(P_{n-1})$.

Proof. We work at the far right end of the link. Changing a positive crossing to a negative one lets us unlink that strand, which gives an n component link, where each component is linked to at most two others as in Figure 13, this n component link is the connected sum of $n-1$ positive Hopf links, so the L_- term has polynomial $(t^{-1} - t)^{n-1}$. The L_0 term lets us gets rid of the last two twists, giving us P_{n-1}, as in Figure 13. Appealing to the skein relation completes the proof. □

[3] The convention for positive and negative twists does not matter since the Alexander polynomial is essentially invariant (up to a sign change) under changing crossings and we have a coherent way to orient our link.

Figure 13: The skein relation change on P_n

Phrased in terms of the Conway polynomial, this says that $\nabla(P_n) = z^{n-1} + z\nabla(P_{n-1})$. Using the above lemma, we have the following Corollary:

6.5 Corollary. $\nabla(P_n) = nz^{n-1}$

Proof. When $n = 2$, we have that P_2 is equal to $2z$. Assuming the theorem is true for up to $k = n-1$, we have that $P_n = z^{n-1} + z\nabla(P_{n-2})$, which gives us that $P_n = z^{n-1} + z(n-1)z^{n-2}$, so we are done by induction. □

Returning to the Alexander polynomial, this shows us that $\Delta(P_n)$ is trapezoidal.

6.6 Corollary. $\Delta(P_n) = n(t^{-1} - t)^{n-1}$ *and this is trapezoidal.*

Now that we've proven this for alternating 2-twist pretzel links, by Corollary 5.5 we know that all even pretzel links are trapezoidal. We also see that if we allow the central vertex to have a negative even weight, then the resulting link is still trapezoidal.

6.7 Observation. *Let L be such a link, a Montesinos link with central vertex of weight -2, and attached to n valence 1 vertices, each*

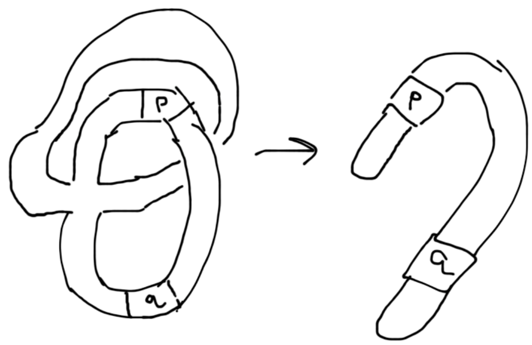

Figure 14: Plumbing on a zero band

of weight 2*. By applying the skein relation to the central vertex we see that* $\Delta(L)$ *is trapezoidal, so by Lemma* 5.5 *all alternating even Montesinos links with each arm of length one are trapezoidal.*

Now, we turn our attention to alternating even Montesinos links in general. In the rest of this section, unless otherwise stated v is a terminal vertex, and $T, v, \pm 2$ is shortened to $T, \pm 2$.

6.8 Lemma. *Let* T *be a stellar tree. Then* $\Delta(T, 2) - \Delta(T - v) = (t^{-1} - t)\Delta(T)$ *and* $-\Delta(T, -2) + \Delta(T - v) = (t^{-1} - t)\Delta(T)$*.*

Proof. This is just an application of the skein relation. A positive twist gives a positive crossing, and a negative twist gives a negative crossing. Changing the sign of a crossing results in a link that is the same as the boundary obtained by plumbing on a zero band, which is the same as deleting a vertex, which gives us $T - v$. This is illustrated in Figure 14. In this Figure, q represents a twist box, and p is a plumbing patch. The smoothing move breaks the terminal band in two, giving T. □

If we can prove that plumbing a 2 or -2 weighted vertex to the end of a branch on a stellar tree preserves trapezoidality, then by Lemma

5.5 we will have proved the Fox Conjecture for even Montesinos links supported by an alternating tree. Presently, we do so.

6.9 Lemma. *Let T be a stellar alternating even tree, and v a vertex of valency 1 and weight ± 2. Then* $\text{span}\,\Delta(T-v)+2 = \text{span}\,\Delta(T)$.

Proof. We induct on the length of arms of T. If all arms have length one, this is a pretzel link, and $T-v$ is a pretzel link with one fewer twist box, so by Corollary 6.6 this is true. If T is an arbitrary stellar tree and v is on an arm longer than length one, by Lemma 6.8, $\Delta(T)-\Delta(T') = (t^{-1}-t)\Delta(T-v)$ where T' is the tree obtained by deleting v and the vertex adjacent to it (w.l.o.g., we assume that $w(v)=2$: if $w(v)=-2$ there is a negative sign that does not affect anything). We can assume by induction that $\text{span}\,\Delta(T')+2 = \text{span}\,\Delta(T-v)$, so the span of $\Delta(T)$ is $\text{span}\,\Delta(T-v)+2$, and we're done.

□

6.10 Theorem. *Even Montesinos links supported by alternating trees are trapezoidal.*

Proof. By Lemma 6.8 and Lemma 5.5, we just need to show that we can add together $(t^{-1}-t)\Delta(T)$ and $\Delta(T-v)$ in a way that preserves trapezoidality. By Lemma 6.9, $\text{span}\,(t^{-1}-t)\Delta(T) = \text{span}\,\Delta(T-v)+4$. Therefore, if the first term of $\Delta(T-v)$ agrees in sign with the second term of $(t^{-1}-t)(\Delta(T)$, then their sum will be trapezoidal (we can assume that $\Delta(T-v)$ and $\Delta(T)$ themselves are trapezoidal).

The key to proving this final step is observing that that if v is a terminal vertex of weight 2, then $\Delta(T)$ and $\Delta(T-v)$ have the opposite leading sign, and that if v' is a terminal vertex of weight -2, then $\Delta(T)$ and $\Delta(T-v')$ have the same leading sign. Therefore, when plumbing on a new vertex of weight opposite in sign to v, by Lemma 6.8 we'll have $\Delta(T,v,-2) = \Delta(T-v)+(t^{-1}-t)\Delta(T)$ and $-\Delta(T,v',2) = -\Delta(T-v)+(t^{-1}-t)\Delta(T)$, and in both of these equations, if the first term of the first summand is the same as the second term of the second summand, then their sum is trapezoidal. So, the theorem will be complete when we prove the following Lemma. □

6.11 Lemma. *If v is a terminal vertex of weight 2, then $\Delta(T)$ and $\Delta(T-v)$ have the opposite leading sign, and if v' is a terminal vertex of weight -2, then $\Delta(T)$ and $\Delta(T-v')$ have same leading sign.*

Proof. If T has only length one arms, then by our assumption the terminal vertices have weight 2, and from our earlier work $\Delta(T)$ and

$\Delta(T-v)$ have the opposite leading sign. We just need to show that this is preserved when we plumb on more vertices. Look at $T, v, 2$, where v is a terminal vertex. $\Delta(T, v, 2) = \Delta(T-v) + (t^{-1}-t)\Delta(T)$ and the leading term of $\Delta(T, v, 2)$ is $-t$ times the leading term of $\Delta(T)$, so the theorem is true in this case.

If we are plumbing on a negative vertex, then we have $\Delta(T, v, -2) = \Delta(T-v) - (t^{-1}-t)\Delta(T)$. Again, the leading coefficient of $\Delta(T, v, -2)$ is t times the leading coefficient of $\Delta(T)$, so they have the same sign and we are done. □

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