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ANALYSIS OF A DARCY-STOKES SYSTEM MODELING FLOW THROUGH VUGGY POROUS MEDIA

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ANALYSIS OF A DARCY-STOKES SYSTEM MODELING FLOW THROUGH VUGGY POROUS MEDIA

by

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This work is dedicated to my parents and brother, Luis, the Shadduli Sufi community, and all my friends for their invaluable support.

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ANALYSIS OF A DARCY-STOKES SYSTEM MODELING FLOW THROUGH VUGGY POROUS MEDIA

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Our goal is to accurately model flow through subsurface systems composed of vuggy porous media. A vug is a small cavity in a porous medium which is large relative to the intergranular pore size. A vuggy porous medium is a porous medium with vugs scattered throughout it. While the vugs are often small, they can have a tremendous effect on the flow of fluid through the medium.

We first introduce our microscale mathematical model for flow of an incompressible, viscous fluid in vuggy porous media. Our next step is to obtain a homogenized macroscale model. In order to do so, we assume periodicity of the medium. We obtain necessary existence and uniqueness results, error estimates, and slight generalizations of two-scale convergence results for bi-modal media. First using formal homogenization and then the rigorous two-scale convergence method, we show that our microscale model homogenizes to give a much simpler modified Darcy's law macroscale model. In this homogenized

model, the permeability tensor is modified to capture the effects of the vugs on the flow through the medium.

In order to compute the homogenized permeability tensor, we essentially compute our microscale system on a (much smaller) representative cell. Toward this end, we introduce two numerical methods for the microscale model. We combine a discontinuous Galerkin method with a low order Raviart-Thomas element and obtain suboptimal convergence rates for the first method. The second method differs only slightly from the first, but yields optimal convergence rates. Unfortunately, it is less efficient in practical implementations.

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Chapter 1

Introduction

Our problem is to accurately model flow through subsurface systems composed of vuggy porous media. A vug is a small cavity in a porous medium which is large relative to the intergranular pore size. A vuggy porous medium is a porous medium with vugs scattered throughout it. Carbonate rock is often vuggy and is commonly found in oil reservoirs and water aquifers. While the vugs are often small, they can have a tremendous effect on the flow of fluid through the medium. In fact, preliminary results from Arbogast, et al. [2], [3] seem to indicate that when the vugs are fairly densely scattered through the medium, the flow essentially channels through the vugs. Thus the velocity of the fluid flow is not as limited by the porosity and permeability of the porous matrix as it would be in the absence of the vugs.

The applications of this model are certainly widespread. In the past, researchers have relied upon simple single and dual porosity type models to simulate flow in vuggy subsurface systems. However, these models do not capture the effects of the vugs, and it is not clear how to assign permeability to the vuggy regions. An ability to better model this type of medium could result in significant benefits toward important environmental and oil recovery problems. We could contribute to a better understanding of how to form public policy regarding pumping, disposal of contaminants, and protecting our water aquifers. It could aid toward more accurate simulations of the movement of contaminants in vuggy subsurface systems, and thus in their remediation. It could also help oil industries to more intelligently manage oil reserves. In the case of our research effort, this is a first step in deriving a model for flow and transport in water aquifers and petroleum reservoirs composed of vuggy porous media. We hope in the future to obtain large scale experimental data on a local aquifer in order to compare our theoretical results with experimental results.

In this dissertation, we first introduce our microscale mathematical model for flow of an incompressible, viscous fluid in vuggy porous media. As is obvious from our desired applications, we hope to simulate flow in vuggy media over quite a large scale, on the order of kilometers. As a result, we do not necessarily want information about the flow at the scale of the vugs, i.e., at the scale of our micro-model. We do, however, want bulk flow information, such as well yield rates.

Thus, our next step is to obtain a homogenized macroscale model. In order to do so, we assume the structure of the domain can be sufficiently captured by tiling the area with a small representative cell. Using first formal homogenization, and then the rigorous two-scale convergence method, we show that our microscale model homogenizes to give a much simpler modified Darcy's law macroscale model. In this homogenized model, the permeability tensor is modified to capture the effects of the vugs on the flow through the medium.

In order to compute the homogenized permeability tensor, we essentially compute our microscale system on our (much smaller) representative cell. Toward this end, we introduce a numerical method for the microscale model. We combine a discontinuous Galerkin method with a low order Raviart-Thomas element and obtain suboptimal convergence rates. A modification to this method is presented, and shown to be optimal in its convergence. However, the modified method is more complex in computation and implementation.

While we are not the first to study viscous fluid flow coupled with porous flow, previous analyses have focused on domains in which the viscous flow region and porous flow region are two seperate regions which meet along an interface [24], [17], [32]. In contrast, what we aim to analyze is the case in which the fluid and porous regions are highly intertwined.

Chapter 2

Mathematical Formulation of Flow in Vuggy Porous Media

2.1 The Model

As described in the introduction, our goal is to model fluid flow in vuggy porous media. Our domain consists of a porous matrix interspersed with vugs. We have chosen to use different differential equations for modelling flow in the two regions, coupling the two through appropriate interface boundary conditions. In this section, we state and support our choices of equations and conditions. The subsequent section gives the resulting mathematical formulation of flow in vuggy porous media.

We model flow in the porous rock regions with Darcy's Law. This law is named after Henry Darcy, who in the 1850's derived it experimentally by studying the average flow rate of water through a column of sand. Darcy's Law essentially states that the flow velocity is proportional to the gradient of the pressure, and holds for flow of homogenous fluids with low Reynolds number through porous media on scales above the pore size [28]. Since Darcy's experiments, his findings have been extended and supported in numerous ways. The law has been derived from first principles using basic ideas in fluid mechanics

[7], [38], using homogenization techniques [19], [33], and repeatedly supported in experiments. The boundary of the porous region has two parts. On the outer boundary of the domain, we assume no normal flux. On the interface of porous and vug regions, we ask for continuity of the normal flux and continuity of the normal stress.

We assume the fluid to be incompressible and such that the viscous force greatly outweighs the inertial force, and so has a low Reynolds number. Thus in the vugs, the Stokes equation of flow, together with an equation expressing conservation of mass flux, are adequate for modelling flow. The boundary of the vuggy region is again divided into two pieces: that which lies on the outer boundary of the domain, and that which lies along the interface with the porous region. On the outer boundary, we impose a no-slip condition. On the interface with the porous region, we impose the Beavers-Joseph-Saffman boundary condition, along with continuity of normal flux and continuity of the normal stress.

In 1967, Beavers and Joseph [8] determined experimentally that a free fluid in contact with a porous medium flows faster than a fluid in contact with a completely solid surface. Although thin boundary layers arise in both cases, the latter case is generally modeled by assuming that all components of the velocity vanish at the solid contact surface. In the former case, the experiments of Beavers and Joseph demonstrate that the tangential velocity of the fluid cannot vanish. They proposed to account for this slippage by imposing a boundary condition of the form

$$\frac{\partial U_s}{\partial y} = \alpha K^{-1/2}(U_s - U_d) \ ,$$

where $\partial/\partial y$ is the normal derivative, U_s is the tangential component of the Stokes velocity, U_d is the tangential component of the Darcy velocity, K is the permeability of the porous medium, and α is the dimensionless Beavers-Joseph slippage coefficient. Saffman [31] justified this law theoretically, and showed that the term involving U_d could be dropped (see also [20, 21]). Jones [23] reinterpreted this law so that it applies to curved boundaries and nontangential flows by formulating the boundary condition in terms of the tangential component of the fluid stress tensor (see (2.6) below). In essence, the resulting Beavers-Joseph-Saffman boundary condition implies that the sheer Stokes stress along the interface is proportional to the tangential component of the Stokes velocity, with the constant of proportionality being the Beavers-Joseph-Saffman slip coefficient times the viscosity divided by $K^{1/2}$.

We note that across the interface we have asked that the normal flux and the normal stress be continuous. The additional requirement on the tangential Stokes velocity (from the Beavers-Joseph-Saffman boundary condition) implies a discontinuous tangential velocity across the interface. This fact is cause for most of the difficulty in this problem and motivates our choice of a nonconforming numerical method in the development of our numerical scheme.

2.2 Notation and Equations

As noted in the previous section, we model the vuggy medium on the fine scale using Stokes equations in the vugs, Darcy's Law in the porous rock, and the Beavers-Joseph-Saffman boundary condition on the interface between the two. In order to express the conservation of mass flux consistantly throughout the domain, and match the saddle point form of the equations considered in the vug regions, we express Darcy's Law in mixed form. The portion of the domain Ω consisting of the vugs is denoted Ω_s, and that consisting of the porous rock is Ω_d. Let Γ be the interface between the two regions. Let η_s be the outer unit normal to $\partial\Omega_s$, and let τ be any unit tangent to Γ. Note that we are considering $\Omega \subseteq \mathbb{R}^2$, though the corresponding formulation for $\Omega \subseteq \mathbb{R}^3$ is clear.

Let D be the symmetric gradient, i.e., $D(\psi)$ is the matrix $\frac{1}{2}\left(\frac{\partial \psi_i}{\partial x_j} + \frac{\partial \psi_j}{\partial x_i}\right)$. Denote by $\mu > 0$ the fluid viscosity; K the permeability (scalar valued) of the porous rock matrix; and $\alpha > 0$ the Beavers-Joseph slip coefficient. The fluid velocity and pressure in the Stokes and Darcy regions are denoted u_s, p_s and u_d, p_d, respectively. These satisfy the following set of equations (wherein $q \in L^2(\Omega)$ is an external source or sink satisfying the compatibility condition that its average over Ω vanishes, and $f \in \left(L^2(\Omega)\right)^2$ is a term related to body forces such as gravitation):

Vugular region (Stokes equations)

$$-2\mu\nabla \cdot Du_s + \nabla p_s = f \qquad \text{in } \Omega_s \ , \tag{2.1}$$

$$\nabla \cdot u_s = q \qquad \text{in } \Omega_s \ , \tag{2.2}$$

Rock matrix (Darcy equations)

$$\mu(K)^{-1}u_d + \nabla p_d = f \qquad \text{in } \Omega_d \ , \tag{2.3}$$

$$\nabla \cdot u_d = q \qquad \text{in } \Omega_d \ , \tag{2.4}$$

Interface

$$u_s \cdot \eta_s = u_d \cdot \eta_s \qquad \text{on } \Gamma \ , \tag{2.5}$$

$$2\eta_s \cdot Du_s \cdot \tau = -\alpha K^{-1/2} u_s \cdot \tau \quad \text{on } \Gamma \ , \tag{2.6}$$

$$2\mu\eta_s \cdot Du_s \cdot \eta_s = p_s - p_d \qquad \text{on } \Gamma \ , \tag{2.7}$$

Outer boundary

$$u_s = 0 \qquad \text{on } \partial\Omega \cap \partial\Omega_s \ , \tag{2.8}$$

$$u_d \cdot \eta = 0 \qquad \text{on } \partial\Omega \cap \partial\Omega_d \ . \tag{2.9}$$

The interface conditions represent continuity of mass flux (2.5), the Beavers-Joseph-Saffman condition on the tangential stress (2.6), and the continuity of normal stress (2.7).

2.3 Variational Formulation

We begin by introducing some notation and spaces. Let $(\cdot,\cdot)_{\mathcal{O}}$ be the $L^2(\mathcal{O})$ or $L^2(\mathcal{O})^2$ inner product, and $\langle\cdot,\cdot\rangle_{\partial\mathcal{O}}$ represent the $L^2(\partial\mathcal{O})$ or $L^2(\partial\mathcal{O})^2$ inner product over the boundary. Set

$$
\begin{aligned}
V := &\{v \in H(\mathrm{div}, \Omega)^2 : v \in H^1(\Omega_s)^2 \ , \\
&\quad v = 0 \text{ on } \partial\Omega_s \cap \partial\Omega \ , v \cdot \eta = 0 \text{ on } \partial\Omega_d \cap \partial\Omega\} \ , \\
W := &\left\{ w \in L^2(\Omega) : \int_\Omega w \, dx = 0 \right\} \ ,
\end{aligned}
$$

where η is the outward unit normal to Ω.

In order to obtain the variational formulation of our problem, we first take a test function $v \in V$, take the dot product of it with (2.1) and (2.3), and integrate. Integration by parts yields

$$
\begin{aligned}
&(2\mu D u_s, \nabla v)_{\Omega_s} - \langle 2\mu D u_s \eta_s, v \rangle_{\partial\Omega_s} - (p_s, \nabla \cdot v)_{\Omega_s} \\
&\quad + \langle p_s \eta_s, v \rangle_{\partial\Omega_s} = (f, v)_{\Omega_s} \ , \qquad (2.10)
\end{aligned}
$$

$$
\left(\mu K^{-1} u_d, v\right)_{\Omega_d} - (p_d, \nabla \cdot v)_{\Omega_d} + \langle p_d \eta_d, v \rangle_{\partial\Omega_d} = (f, v)_{\Omega_d} \ . \qquad (2.11)
$$

First note that

$$
\begin{aligned}
(2\mu D u_s, \nabla v)_{\Omega_s} &= \left(\mu(\nabla u_s + \nabla u_s^T), \nabla v\right)_{\Omega_s} \\
&= \frac{1}{2} \left(\mu(\nabla u_s + \nabla u_s^T), \nabla v + \nabla v^T\right)_{\Omega_s} \\
&= (2\mu D u_s, D v)_{\Omega_s} \ .
\end{aligned}
$$

Since $v \in V$, $v = 0$ on $\partial\Omega_s \cap \partial\Omega$, and we have that the two boundary integrals over $\partial\Omega_s$ reduce to integration on Γ. Similarily, $v \cdot \eta_d = 0$ on $\partial\Omega_d \cap \partial\Omega$ restricts the boundary integral on $\partial\Omega_d$ to Γ. Moreover, $\eta_d = -\eta_s$ on Γ, so summing (2.10)–(2.11) yields

$$
\begin{aligned}
&(2\mu D u_s, D v)_{\Omega_s} + \left(\mu K^{-1} u_d, v\right)_{\Omega_d} - (p, \nabla \cdot v)_{\Omega} \\
&\quad + \langle (p_s - p_d)\eta_s - 2\mu D u_s \eta_s, v \rangle_\Gamma = (f, v)_\Omega \ .
\end{aligned}
$$

Note that this is equivalent to

$$\begin{aligned}&(2\mu Du_s, Dv)_{\Omega_s} + \left(\mu K^{-1}u_d, v\right)_{\Omega_d} - (p, \nabla\cdot v)_{\Omega}\\&\quad + \langle p_s - p_d - 2\mu\eta_s\cdot Du_s\cdot\eta_s, v\cdot\eta_s\rangle_\Gamma - \langle 2\mu\eta_s\cdot Du_s\cdot\tau_s, v\cdot\tau_s\rangle_\Gamma = (f, v)_\Omega \ .\end{aligned}$$

Now if we apply the interface conditions (2.6) and (2.7), we obtain

$$(2\mu Du_s, Dv)_{\Omega_s} + \left(\mu K^{-1}u_d, v\right)_{\Omega_d} + \langle\mu\alpha K^{-1/2}u_s\cdot\tau, v_s\cdot\tau\rangle_\Gamma - (p, \nabla\cdot v)_\Omega = (f, v)_\Omega \ .$$

Next we take a test function $w \in W$, multiply equations (2.2) and (2.4) by w, and integrate. Summing the resulting two equations gives

$$(\nabla\cdot u, w)_\Omega = (q, w)_\Omega \ .$$

In summary, the resulting variational formulation of (2.1)–(2.9) is to find $u \in V$ and $p \in W$ such that

$$\begin{aligned}&(2\mu Du_s, Dv)_{\Omega_s} + \left(\mu K^{-1}u_d, v\right)_{\Omega_d} + \langle\mu\alpha K^{-1/2}u_s\cdot\tau, v_s\cdot\tau\rangle_\Gamma\\&\quad - (p, \nabla\cdot v)_\Omega = (f, v)_\Omega \ , \qquad \forall v \in V \ , \end{aligned}\tag{2.12}$$

$$(\nabla\cdot u, w)_\Omega = (q, w)_\Omega \ , \qquad \forall w \in W \ . \tag{2.13}$$

Note that interface condition (2.5) is accounted for by the fact that $u \in H(\mathrm{div}, \Omega)$ and the outer boundary conditions (2.8) and (2.9) by the fact that $u \in V$.

2.4 Existence and Uniqueness

In the sequel, let $\|\cdot\|_\omega$ denote the $L^2(\omega)^N$ norm and $\|\cdot\|_{j,\omega}$ denote the $H^j(\omega)^N$, where N is one or the dimension of the space in which we are working, depending on the context. Toward proving existence and uniqueness of a solution to

(2.12)–(2.13), we first remind the reader of Korn's Inequality [10] and introduce a result related to Korn's inequality.

Theorem 2.4.1 (Korn's Inequality). *If* $V = \{v \in H^1(\mathcal{O}) : \ v|_\Lambda = 0, \ \Lambda \subset \partial\mathcal{O}, \ \text{meas}(\Lambda) \neq 0\}$*, then*

$$\|Dv\|_0 \geq C\|v\|_1 \qquad\qquad \forall v \in V$$

Lemma 2.4.2. *There exists* $C > 0$ *such that for all* $v \in V$*,*

$$\|v_s\|_{\Omega_s} + \|\nabla v_s\|_{\Omega_s} \leq C\left(\|Dv_s\|_{\Omega_s} + \|v_s \cdot \tau\|_\Gamma + \|v_d\|_{\Omega_d} + \|\nabla \cdot v\|_\Omega\right) \ .$$

Proof. Suppose not. Then there exists a sequence of $v_n \in V$ such that

$$\|v_{n,s}\|_{\Omega_s} + \|\nabla v_{n,s}\|_{\Omega_s} = 1 \tag{2.14}$$

and

$$\|Dv_{n,s}\|_{\Omega_s} + \|v_{n,s} \cdot \tau\|_\Gamma + \|v_{n,d}\|_{\Omega_d} + \|\nabla \cdot v_n\|_\Omega \leq \frac{1}{n} \tag{2.15}$$

This implies that some subsequence of $v_{n,s}$ converges weakly to some $v_s \in H^1(\Omega_s)$ and that $v_{n,d} \to 0$ in $H(\text{div}, \Omega_d)$. Let v be the extension by zero to Ω of v_s.

Note now that v_n is bounded in $H(\text{div}, \Omega)$, so that it converges weakly to some $\hat{v}$ in $H(\text{div}, \Omega)$. Moreover, it must be the case that $\hat{v} = v$. Since

$$\|v_{n,d} \cdot \eta\|_{(H_{00}^{1/2}(\Gamma))^*} \leq C\|v_{n,d}\|_{H(\text{div},\Omega_d)} \to 0 \ ,$$

where the norm on the left-hand side is the norm of the dual space of $H_{00}^{1/2}(\Gamma)$ (see [25]), we conclude that $v_d \cdot \eta = v_s \cdot \eta = 0$ on Γ. On the other hand,

$\|v_{n,s} \cdot \tau\|_\Gamma \to 0$, so that $v_s \cdot \tau = 0$ and further that $v_s = 0$ on Γ. Now Korn's inequality 2.4.1 can be applied on Ω_s to show that

$$\|v_{n,s}\|_{\Omega_s} + \|\nabla v_{n,s}\|_{\Omega_s} \le C\|Dv_{n,s}\|_{\Omega_s} \; .$$

But the left-hand side is one and the right-hand side tends to zero, contradicting the assumption that the desired inequality fails to hold. $\square$

The existence and uniqueness of a solution (u, p) to (2.12)–(2.13) now follows from the inf-sup theory of saddle point problems [6, 10, 11, 12]. For $u, v \in V$ and $w \in W$, we have the bilinear forms

$$\begin{aligned} a(u, v) &= 2\mu(Du_s, Dv)_{\Omega_s} + \langle \mu\alpha K^{-1/2} u_s \cdot \tau, v_s \cdot \tau\rangle_\Gamma + (\mu K^{-1} u_d, v)_{\Omega_d} \; , \\ b(v, w) &= (w, \nabla \cdot v)_\Omega \; . \end{aligned}$$

Then (2.12)–(2.13) can be rewritten as: Find $(u, p) \in V \times W$ such that

$$\begin{aligned} a(u, v) - b(v, p) &= (f, v), \quad v \in V \; , \\ b(u, w) &= (q, w), \quad w \in W \; . \end{aligned}$$

We endow V with the norm

$$|||u||| = (\|u\|^2_\Omega + \|\nabla \cdot u\|^2_\Omega + \|\nabla u_s\|^2_{\Omega_s})^{1/2} \; ,$$

for which it is complete. We can easily show

$$\begin{aligned} \langle \mu\alpha K^{-1/2} u_s \cdot \tau, v_s \cdot \tau\rangle_\Gamma &\le C\|u_s \cdot \tau\|_{0,\Gamma}\|v_s \cdot \tau\|_{0,\Gamma} \\ &\le C \left(\|u_s\|_{0,\Omega_s}\|u_s\|_{1,\Omega_s}\right)^{1/2} \left(\|v_s\|_{0,\Omega_s}\|v_s\|_{1,\Omega_s}\right)^{1/2} \\ &\le C \left(\|u_s\|^2_{0,\Omega_s} + \|u_s\|^2_{1,\Omega_s}\right)^{1/2} \left(\|v_s\|^2_{0,\Omega_s} + \|v_s\|^2_{1,\Omega_s}\right)^{1/2} \\ &\le C \; |||\, u \,||| \; |||v||| \; , \end{aligned}$$

so that that both a and b are bounded (i.e., continuous). Coercivity of a on $V \cap \{v \in V | \nabla \cdot v = 0\}$ is a direct result of the previous lemma.

It remains to show the inf-sup condition, but this follows from the corresponding condition known for the Stokes system on Ω; that is,

$$\begin{aligned}\inf_{w \in W} \sup_{v \in V} \frac{(\nabla \cdot v, w)}{|||v|||\ \|w\|} &\geq \inf_{w \in W} \sup_{v \in (H_0^1)^2} \frac{(\nabla \cdot v, w)}{|||v|||\ \|w\|} \\ &\geq \inf_{w \in W} \sup_{v \in (H_0^1)^2} \frac{(\nabla \cdot v, w)}{2\|v\|_{(H^1(\Omega))^2} \|w\|} \geq \gamma > 0 \ ,\end{aligned}$$

for some γ, since $(H_0^1)^2 \subset V$ and $2\|v\|_{(H^1(\Omega))^2} \geq |||v|||$.

Now the inf-sup theory provides the existence and uniqueness of a solution to our system (2.12)–(2.13) [12].

Chapter 3

Homogenization

3.1 Introduction

As was discussed in the introduction, our aim is to model flow through vuggy porous media over domains whose scale is on the order of kilometers. While our system of equations should accurately model flow, it is somewhat large and complicated. Moreover, this system of equations models the flow at the scale of the vugs. This level of detail is not needed, and for computations over such large domains, is likely to be computationally infeasable. Our desire is merely to obtain bulk properties of the flow, and so we look to upscale our model.

Homogenization is a mathematical method which provides a means for upscaling systems of differential equations. The essential idea of homogenization is to average inhomogeneous media in some way in order to capture global properties of the medium. The use of this idea dates at least as far back as 1822 [29], and has since been used extensively in science and engineering.

Formal homogenization, which is the technique used in §3.3, was used for a long time by engineers and scientists. Mathematically it is a two-scale asymptotic analysis of a system of equations over a periodic structure, where we allow the period ϵ to shrink to zero and obtain in the limit a system of equations which depend only on the macroscale variables. Suppose that our domain is tiled by a representative cell Y (see Figure 3.1). In formal homogenization, the assumption is made that the solution u to our system of differential equations can be expanded in the form

$$u^{\epsilon}(x) = \sum_{j=0}^{\infty} \epsilon^{j} u_{j}\left(x, \frac{x}{\epsilon}\right) . \tag{3.1}$$

Here $u_j(x, y)$ depends on the macroscopic variable x, representing a point in our domain, and on the microscopic variable y, representing a point in Y. It is also important to hypothesize that $u_j(x, y)$ is Y-periodic in y. Although formal homogenization has been a reliable means of obtaining insight, it is not rigorous, as our assumpution (3.1) is not justified.

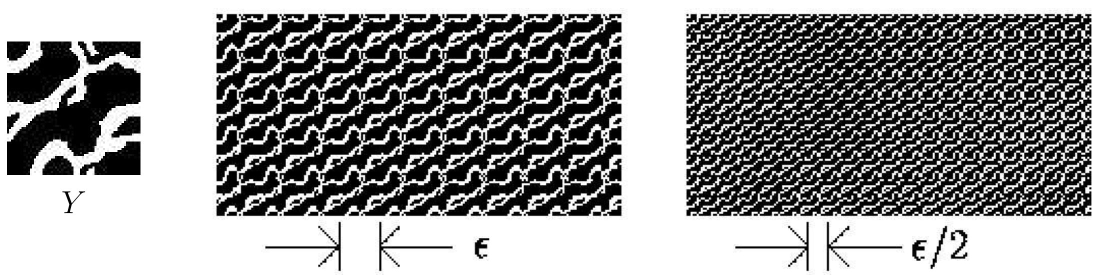

Figure 3.1: A sample representative cell and two tilings of a domain.

A rigorous mathematical theory of homogenization only began to develop in the late 1960's. For our purposes, it is pertinent only to discuss the method of two-scale convergence originally developed by Nguetseng [27] and Allaire [1]. This particular method applies only to periodic homogenization problems, as opposed to some other more general methods [19]. It was developed with the purpose of rigorously justifying at least the first two terms of the assumption (3.1) and yielding the homogenized problem all in one breath.

The two-scale convergence method consists of testing our family of solutions $u^\epsilon(x)$ against a periodically oscillating test function $\phi(x, x/\epsilon)$. In passing to the limit as $\epsilon \to 0$, we obtain the two-scale limit $u(x, y)$, that is,

$$\int_\Omega u^\epsilon(x)\, \phi(x, x/\epsilon)\, dx \to \int_\Omega \int_Y u(x, y)\phi(x, y)\, dydx\ .$$

Thus we have effectively seperated out the dependence of u on the micro and macro scale. Moreover, multiplying our (appropriately ϵ-scaled) system of equations by an oscillating test function and passing to the two-scale limit yields a homogenized system of equations which model the macroscale problem.

In this chapter, we assume periodicity of the medium, and obtain as our homogenized limit a macroscopic Darcy's Law governing the system over large scales. To illustrate the ideas, we first derive this macroscopic model formally in §3.3. The next three sections are devoted to the rigorous two-scale convergence derivation method [1, 4, 19, 27]. In §3.4 we obtain the existence, uniqueness, and energy estimates needed in the analysis. In §3.5 we develop the needed generalizations of the two-scale convergence theory needed for our bi-modal medium. The convergence of the homogenization is demonstrated in §3.6. The final section of this chapter presents a simple analytical solution to illustrate the results.

3.2 Assumptions and Notation

Although our results would extend easily to $\mathbb{R}^3$, for ease of presentation we assume that the domain Ω is Lipschitz and bounded in $\mathbb{R}^2$. For the purposes of homogenization, we now assume that the geometric vug and pore structure of Ω is periodic of period ϵY, where Y is a reference cell for the periodic tiling of unit volume $|Y|$. The portion of the domain consisting of the vugs is denoted Ω_s^ϵ, and that consisting of the porous rock is Ω_d^ϵ. Let Γ^ϵ be the interface between the two regions. Let η_s be the outer unit normal to $\partial\Omega_s^\epsilon$, and let τ be a unit tangent to Γ^ϵ, where we suppress the ϵ dependence of these vectors for notational ease.

Let $K(x/\epsilon)$ be the Y-periodic, bounded, and positive permeability of the porous rock matrix. The fluid velocity and pressure in the Stokes and Darcy regions are denoted u_s^ϵ, p_s^ϵ and u_d^ϵ, p_d^ϵ, respectively. These satisfy the following scaled equations (wherein μ, α, q, and f are as in (2.1)–(2.9)):

Vugular region (Stokes equations)

$$-2\mu\epsilon^2\nabla\cdot Du_s^\epsilon+\nabla p_s^\epsilon=f \qquad \text{in } \Omega_s^\epsilon\ , \tag{3.2}$$

$$\nabla\cdot u_s^\epsilon=q \qquad \text{in } \Omega_s^\epsilon\ , \tag{3.3}$$

Rock matrix (Darcy equations)

$$\mu(K^\epsilon)^{-1}u_d^\epsilon+\nabla p_d^\epsilon=f \qquad \text{in } \Omega_d^\epsilon\ , \tag{3.4}$$

$$\nabla\cdot u_d^\epsilon=q \qquad \text{in } \Omega_d^\epsilon\ , \tag{3.5}$$

Interface

$$u_s^\epsilon \cdot \eta_s = u_d^\epsilon \cdot \eta_s \qquad \text{on } \Gamma^\epsilon , \tag{3.6}$$

$$2\eta_s \cdot D u_s^\epsilon \cdot \tau = -\alpha \epsilon^{-1} (K^\epsilon)^{-1/2} u_s^\epsilon \cdot \tau \quad \text{on } \Gamma^\epsilon , \tag{3.7}$$

$$2\mu\epsilon^2 \eta_s \cdot D u_s^\epsilon \cdot \eta_s = p_s^\epsilon - p_d^\epsilon \qquad \text{on } \Gamma^\epsilon , \tag{3.8}$$

Outer boundary

$$u_s^\epsilon = 0 \qquad \text{on } \partial\Omega \cap \partial\Omega_s^\epsilon , \tag{3.9}$$

$$u_d^\epsilon \cdot \eta = 0 \qquad \text{on } \partial\Omega \cap \partial\Omega_d^\epsilon . \tag{3.10}$$

The homogenization problem is to determine the behavior of the system as $\epsilon \to 0$. Note that in the equations we have scaled both the viscosity μ and the permeability K^ϵ by ϵ^2. This is the usual scaling for deriving Darcy's Law from Stokes flow (see [35]), since as $\epsilon \to 0$, flow paths (in our case vugs) become constricted, so a corresponding decrease in viscosity is required to maintain flow rates. Moreover, when homogenizing heterogeneity (see, e.g., [9, 33]), the ratio of permeability to viscosity should be fixed, forcing a similar scaling of the permeability. These considerations then imply the stated scaling of the Beavers-Joseph boundary condition.

Below we will need to distinguish the geometry of the reference cell Y, so let Y_s denote the Stokes region, Y_d the Darcy region, and Γ the interface between the two. We assume that both Y_s and Y_d have positive measure, and thus also the 1-dimensional measure of Γ is positive. As usual, x will represent a point in Ω and y a point in Y.

3.3 Formal Homogenization

We proceed to formally homogenize our system of equations in the usual manner [9, 19, 22, 33]. We make the ansatz that we can expand u_ℓ^ϵ and p_ℓ^ϵ (for $\ell = s, d$) as

$$u_\ell^\epsilon = \sum_{j=0}^{\infty} \epsilon^j u_{\ell,j}\left(x, \frac{x}{\epsilon}\right) \quad \text{and} \quad p_\ell^\epsilon = \sum_{j=0}^{\infty} \epsilon^j p_{\ell,j}\left(x, \frac{x}{\epsilon}\right) ,$$

where the $u_{\ell,j}(x,y)$ and $p_{\ell,j}(x,y)$ are Y-periodic functions in y.

Substituting the above expressions into our system of equations (3.2)–(2.7), and recognizing that $\nabla = \nabla_x + \epsilon^{-1}\nabla_y$, we obtain the following equations. From the ϵ^{-1} terms of (3.2) and (3.4), and the ϵ^0 terms of (3.8), we see that

$$\nabla_y p_s^0 = 0 \quad \text{in } \Omega \times Y_s , \tag{3.11}$$

$$\nabla_y p_d^0 = 0 \quad \text{in } \Omega \times Y_d , \tag{3.12}$$

$$p_s^0 - p_d^0 = 0 \quad \text{on } \Omega \times \Gamma . \tag{3.13}$$

It follows immediately that p_s^0 and p_d^0 are independent of y and equal, so let

$$p^0(x) = p_s^0(x) = p_d^0(x) \qquad \text{on } \Omega .$$

Now the ϵ^0 terms of (3.2), (3.4) and (3.6), the ϵ^{-1} terms of (3.3), (3.5) and (3.7), and the ϵ^1 terms of (3.8) imply

$$
\begin{aligned}
-2\mu\nabla_y \cdot D_y u_s^0 + \nabla_x p^0(x) + \nabla_y p_s^1(x,y) &= f && \text{in } \Omega \times Y_s \ , \\
\nabla_y \cdot u_s^0 &= 0 && \text{in } \Omega \times Y_s \ , \\
\mu K(y)^{-1} u_d^0 + \nabla_x p^0(x) + \nabla_y p_d^1(x,y) &= f && \text{in } \Omega \times Y_d \ , \\
\nabla_y \cdot u_d^0 &= 0 && \text{in } \Omega \times Y_d \ , \\
u_s^0 \cdot \eta_s &= u_d^0 \cdot \eta_s && \text{on } \Omega \times \Gamma \ , \\
2\eta_s \cdot D_y u_s^0 \cdot \tau &= -\alpha (K(y))^{-1/2} u_s^0 \cdot \tau && \text{on } \Omega \times \Gamma \ , \\
2\mu\eta_s \cdot D_y u_s^0 \cdot \eta_s &= p_s^1 - p_d^1 && \text{on } \Omega \times \Gamma \ .
\end{aligned}
$$

With e_j being the standard Cartesian basis vector in the jth direction, let (ω_j, Φ_j) be the periodic solution of the following auxiliary or cell problem

$$-2\nabla \cdot D\omega_j^s + \nabla \Phi_j^s = e_j \quad \text{in } Y_s \ , \tag{3.14}$$

$$\nabla \cdot \omega_j^s = 0 \quad \text{in } Y_s \ , \tag{3.15}$$

$$K^{-1}\omega_j^d + \nabla \Phi_j^d = e_j \quad \text{in } Y_d \ , \tag{3.16}$$

$$\nabla \cdot \omega_j^d = 0 \quad \text{in } Y_d \ , \tag{3.17}$$

$$\omega_j^s \cdot \eta_s = \omega_j^d \cdot \eta_s \quad \text{on } \Gamma \ , \tag{3.18}$$

$$2\eta_s \cdot D\omega_j^s \cdot \tau = -\alpha K^{-1/2} \omega_j^s \cdot \tau \quad \text{on } \Gamma \ , \tag{3.19}$$

$$2\eta_s \cdot D\omega_j^s \cdot \eta_s = \Phi_j^s - \Phi_j^d \quad \text{on } \Gamma \ . \tag{3.20}$$

Then by linear algebra, we can express u_s^0 and u_d^0 as

$$u_\ell^0(x,y) = \frac{1}{\mu} \sum_{j=1}^{2} \left(f_j(x) - \frac{\partial p^0}{\partial x_j}(x) \right) \omega_j^\ell(y) \ , \ \ \ell = s, d \ . \tag{3.21}$$

Define the averaging operator $\bar{v}_\ell$ by averaging v_ℓ in the sense

$$\bar{v}_\ell = \frac{1}{|Y|} \int_{Y_\ell} v_\ell(y)\, dy \ , \tag{3.22}$$

so that

$$\bar{u}_0(x) = \bar{u}_s^0(x) + \bar{u}_d^0(x) = \frac{1}{\mu} \sum_j \left(f_j(x) - \frac{\partial p^0}{\partial x_j}(x) \right) (\bar{\omega}_j^s + \bar{\omega}_j^d) \ .$$

Now let the matrix $\tilde{K}$ be defined by

$$\tilde{K}_{i,j} = \bar{\omega}_{j,i}^s + \bar{\omega}_{j,i}^d = \frac{1}{|Y|} \left(\int_{Y_d} (\omega_j^d)_i \, dy + \int_{Y_s} (\omega_j^s)_i \, dy \right) \ . \tag{3.23}$$

Then we see that

$$\mu \tilde{K}^{-1} \bar{u}_0 + \nabla p^0 = f \quad \text{in } \Omega \ . \tag{3.24}$$

Finally, using the ϵ^0 terms of (3.3) and (3.5), we obtain

$$\nabla_x \cdot u_\ell^0 + \nabla_y \cdot u_\ell^1 = q \quad \text{in } \Omega \times Y_s \ , \ \ell = s, d \ ,$$

so that if we again average over Y and sum, we obtain

$$\begin{aligned} \nabla \cdot \bar{u}_0 &+ \frac{1}{|Y|} \int_{Y_s} \nabla_y \cdot u_s^1 \, dy + \frac{1}{|Y|} \int_{Y_d} \nabla_y \cdot u_d^1 \, dy \\ &= \nabla \cdot \bar{u}_0 + \frac{1}{|Y|} \int_{\partial Y_s} u_s^1 \cdot \eta_s \, dS + \frac{1}{|Y|} \int_{\partial Y_d} u_d^1 \cdot \eta_d \, dS \\ &= q \ . \end{aligned}$$

By the periodicity of u_ℓ^1 in y, and the fact that $u_s^1 \cdot \eta_s = -u_d^1 \cdot \eta_d$ on Γ, we see that

$$\nabla \cdot \bar{u}_0 = q \quad \text{on } \Omega \ . \tag{3.25}$$

Thus we conclude from the formal analysis that $\bar{u}_0$ should satisfy a Darcy's Law on all of Ω (3.24)–(3.25), with effective permeability matrix $\tilde{K}$, independent of the fluid viscosity and given by (3.23).

Lemma 3.3.1. *The tensor $\tilde{K}$, as defined by* (3.23) *and* (3.14)–(3.20)*, is symmetric and positive definite.*

Proof. The existence and uniqueness of a weak solution to (3.14)–(3.20) follows from an analysis similar to that given above in §2.4. Note that (3.14)–(3.20) is equivalent to the variational equation

$$2(D\omega_i^s, D\psi)_{Y_s} + \langle \alpha K^{-1/2}\omega_i^s \cdot \tau, \psi \cdot \tau\rangle_\Gamma + \left(K^{-1}\omega_i^d, \psi\right)_{Y_d} = (e_i, \psi)_Y \ , \quad (3.26)$$

for ψ an infinitely differentiable and periodic vector function in Y such that $\nabla \cdot \psi = 0$. With $\psi = \omega_j^s$ on Y_s and $\psi = \omega_j^d$ on Y_d (actually a sequence approaching the same), we obtain that

$$\begin{aligned} |Y|\,\tilde{K}_{i,j} &= \left(e_i, \omega_j^s\right)_{Y_s} + \left(e_i, \omega_j^d\right)_{Y_d} \\ &= 2(D\omega_i^s, D\omega_j^s)_{Y_s} + \langle \alpha K^{-1/2}\omega_i^s \cdot \tau, \omega_j^s \cdot \tau\rangle_\Gamma + \left(K^{-1}\omega_i^d, \omega_j^d\right)_{Y_d} \ , \quad (3.27) \end{aligned}$$

and symmetry follows immediately.

To show that $\tilde{K}$ is positive definite, take any $\lambda \in \mathbb{R}^2$ and define

$$\xi^\ell(y) = \sum_i \lambda_i \omega_i^\ell(y) \quad \text{for } y \in Y_\ell \ .$$

Then, from (3.27), we conclude that

$$|Y|\,\lambda^T \tilde{K}\lambda = 2(D\xi^s, D\xi^s)_{Y_s} + \langle \alpha K^{-1/2}\xi^s \cdot \tau, \xi^s \cdot \tau\rangle_\Gamma + \left(K^{-1}\xi^d, \xi^d\right)_{Y_d} \ ,$$

and that $\tilde{K}$ is positive semi-definite. To see definiteness, suppose that $\lambda^T \tilde{K}\lambda = 0$ and conclude that each integrand above vanishes. But now (3.26) implies that

$$0 = 2(D\xi^s, D\psi)_{Y_s} + \langle \alpha K^{-1/2}\xi^s \cdot \tau, \psi \cdot \tau\rangle_\Gamma + \left(K^{-1}\xi^d, \psi\right)_{Y_d} = (\lambda, \psi)_Y \ .$$

Since λ is constant, we can take $\psi = \lambda$ and conclude that $\lambda = 0$ and, further, that $\tilde{K}$ is positive definite. $\square$

3.4 Existence and A Priori Energy Estimates

In this section, we follow the ideas in §2.4 to prove existence of solutions u^ϵ and p^ϵ to (3.2)–(3.10) for each ϵ and energy estimates for u^ϵ and p^ϵ independent of ϵ. Let

$$V^\epsilon = \{v \in H(\mathrm{div}, \Omega) : v_s = v|_{\Omega_s^\epsilon} \in H^1(\Omega_s^\epsilon),\ v \cdot \eta = 0 \text{ on } \partial\Omega \cap \partial\Omega_d^\epsilon\ ,$$
$$\text{and } v = 0 \text{ on } \partial\Omega \cap \partial\Omega_s^\epsilon\},$$

where η is the outward unit normal to Ω, and let $W = L^2(\Omega)/\mathbb{R}$.

We first recast the scaled problem (3.2)–(3.10) into a variational problem. Take test functions $v \in V^\epsilon$ and $w \in W$ and proceed as in §2.3 to obtain the variational form of the system for $u^\epsilon \in V^\epsilon$ and $p^\epsilon \in W$ satisfying

$$\begin{aligned} &2\mu\epsilon^2 \left(Du_s^\epsilon, Dv\right)_{\Omega_s^\epsilon} + \langle \epsilon\mu\alpha(K^\epsilon)^{-1/2} u_s^\epsilon \cdot \tau, v_s \cdot \tau\rangle_{\Gamma^\epsilon} \\ &\qquad - \left(p^\epsilon, \nabla \cdot v\right)_\Omega + \mu\left((K^\epsilon)^{-1} u_d^\epsilon, v\right)_{\Omega_d^\epsilon} = (f, v)_\Omega\ , \quad v \in V^\epsilon\ , \qquad (3.28) \\ &(\nabla \cdot u^\epsilon, w)_\Omega = (q, w)_\Omega\ , \qquad\qquad w \in W\ , \qquad (3.29) \end{aligned}$$

where $u_d = u|_{\Omega_d^\epsilon}$ and $u_s^\epsilon \cdot \eta_s = u_d^\epsilon \cdot \eta_s$ on Γ^ϵ is implicit from $u^\epsilon \in V^\epsilon$.

Theorem 3.4.1. *For each ϵ, there exists $(u^\epsilon, p^\epsilon) \in V^\epsilon \times W$ satisfying* (3.2)–(3.10) *weakly, i.e.,* (3.28)–(3.29)*, such that*

$$\epsilon\|\nabla u_s^\epsilon\|_{\Omega_s^\epsilon} + \sqrt{\epsilon}\|u_s^\epsilon \cdot \tau\|_{\Gamma^\epsilon} + \|u^\epsilon\|_\Omega + \|\nabla \cdot u^\epsilon\|_\Omega + \|p^\epsilon\|_\Omega \le C\left(\|f\|_\Omega + \|q\|_\Omega\right)\ , \quad (3.30)$$

with C independent of ϵ.

In order to prove this result, we first prove a lemma related to Korn's inequality as in §2.4, obtaining a bound independent of ϵ.

Lemma 3.4.2. *There exists C independent of ϵ such that for all $v \in V^\epsilon$,*

$$\|v_s\|_{\Omega_s^\epsilon} + \epsilon\|\nabla v_s\|_{\Omega_s^\epsilon} \le C\left(\epsilon\|Dv_s\|_{\Omega_s^\epsilon} + \sqrt{\epsilon}\|v_s\cdot\tau\|_{\Gamma^\epsilon} + \|v_d\|_{\Omega_d^\epsilon} + \epsilon\|\nabla\cdot v\|_\Omega\right) .$$

Proof. First we note that a similar result holds for $\hat{v} \in V_Y := \{v \in H(\mathrm{div}, Y) : v_s = v|_{Y_s} \in H^1(Y_s)\}$ by Lemma 2.4.2. That is, that there exists $\hat{C}$ such that

$$\|\hat{v}_s\|_{Y_s} + \|\nabla\hat{v}_s\|_{Y_s} \le \hat{C}\left(\|D\hat{v}_s\|_{Y_s} + \|\hat{v}_s\cdot\tau\|_\Gamma + \|\hat{v}_d\|_{Y_d} + \|\nabla\cdot\hat{v}\|_Y\right) . \tag{3.31}$$

We next use a translation and scaling argument to pass to all of Ω.

Now we have inequality (3.31) for any $\hat{v} \in V_Y$, wherein $\hat{C}$ does not depend on ϵ. We also have the inequality for an open subset $Y_p \subset Y$, as long as $Y_p \cap Y_d$ is not empty. When $Y_p \subset Y_s$, we can obtain the same inequality provided that we require the boundary condition $v = 0$ on $\partial Y_p \setminus \partial Y$.

By the structure of Ω, we can write it as $\Omega = \bigcup_{i\in\mathcal{I}} \epsilon Y^i$, where $\epsilon Y^i = \big(\epsilon(Y + \vec{n}_i)\big) \cap \Omega$, $\vec{n}_i$ is some vector whose components are integers, and $\mathcal{I}$ is some appropriate index set. Let $v \in V^\epsilon$ and define for $y \in Y$, $\hat{v}^i(y) = v(\epsilon(y + \vec{n}_i)) \in V_Y$. Then since (3.31) holds on each Y^i,

$$\|\hat{v}_s^i\|_{Y_s^i} + \|\nabla\hat{v}_s^i\|_{Y_s^i} \le \hat{C}\left(\|D\hat{v}_s^i\|_{Y_s^i} + \|\hat{v}_s^i\cdot\tau\|_{\Gamma^i} + \|\hat{v}_d^i\|_{Y_d^i} + \|\nabla\cdot\hat{v}^i\|_{Y^i}\right) ,$$

and if we sum over all i and make the change of variables $x = \epsilon(y + \vec{n}_i)$ on each Y^i, we obtain

$$\epsilon^{-1}\|v_s\|_{\Omega_s^\epsilon} + \|\nabla v_s\|_{\Omega_s^\epsilon} \le C\left(\|Dv_s\|_{\Omega_s^\epsilon} + \epsilon^{-1/2}\|v_s\cdot\tau\|_{\Gamma^\epsilon} + \epsilon^{-1}\|v_d\|_{\Omega_d^\epsilon} + \|\nabla\cdot v\|_\Omega\right) ,$$

which gives the desired result. □

Proof of Theorem 3.4.1. The theorem follows from the inf-sup theory of saddle point problems [6, 10, 11, 12], and our proof follows the argument in §2.4. For $u, v \in V^\epsilon$ and $w \in W$, we have the scaled bilinear form

$$a^\epsilon(u,v) = 2\mu\epsilon^2(Du_s, Dv)_{\Omega_s^\epsilon} + \langle \epsilon\mu\alpha(K^\epsilon)^{-1/2}u_s\cdot\tau, v_s\cdot\tau\rangle_{\Gamma^\epsilon} + (\mu(K^\epsilon)^{-1}u_d, v)_{\Omega_d^\epsilon}\ ,$$

and $b(v,w) = (w, \nabla\cdot v)_\Omega$ remains unchanged. Then (3.28)–(3.29) can be rewritten as: Find $(u^\epsilon, p^\epsilon) \in V^\epsilon \times W$ such that

$$a^\epsilon(u^\epsilon, v) - b(v, p^\epsilon) = (f, v), \quad v \in V^\epsilon\ , \tag{3.32}$$

$$b(u^\epsilon, w) = (q, w), \quad w \in W\ . \tag{3.33}$$

We endow V^ϵ with the norm

$$|||u|||_\epsilon = (\|u\|_\Omega^2 + \|\nabla\cdot u\|_\Omega^2 + \epsilon^2\|\nabla u_s\|_{\Omega_s^\epsilon}^2)^{1/2}\ ,$$

for which it is complete. We claim that

$$\langle \epsilon\mu\alpha(K^\epsilon)^{-1/2}u_s\cdot\tau, v_s\cdot\tau\rangle_{\Gamma^\epsilon} \le C\epsilon\|u_s\cdot\tau\|_{\Gamma^\epsilon}\|v_s\cdot\tau\|_{\Gamma^\epsilon} \le C\,|||\,u\,|||_\epsilon\,|||v|||_\epsilon\ , \tag{3.34}$$

so that that both a^ϵ and b are bounded (i.e., continuous) with constants independent of ϵ. To see the claim, compute

$$\begin{aligned}
\epsilon\|u_s\cdot\tau\|^2_{\Gamma^\epsilon} &= \epsilon\sum_{i\in\mathcal{I}}\|u_s\cdot\tau\|^2_{\epsilon\Gamma^i} \\
&= \epsilon\sum_{i\in\mathcal{I}}\epsilon\|\hat{u}_s\cdot\hat{\tau}\|^2_{\Gamma^i} \\
&\le \epsilon^2\sum_{i\in\mathcal{I}}\hat{C}\|\hat{u}\|_{Y^i_s}\|\hat{u}\|_{H^1(Y^i_s)} \\
&\le \hat{C}\epsilon\sum_{i\in\mathcal{I}}\|u\|_{\epsilon Y^i_s}\|u\|_{H^1(\epsilon Y^i_s)} \\
&\le \hat{C}\sum_{i\in\mathcal{I}}\left[\|u\|^2_{\epsilon Y^i_s}+\epsilon^2\|u\|^2_{H^1(\epsilon Y^i_s)}\right] \\
&= \hat{C}\left[\|u\|^2_{\Omega^\epsilon_s}+\epsilon^2\|u\|^2_{H^1(\Omega^\epsilon_s)}\right] \\
&\le C\ |||\ u\ |||^2_\epsilon\ .
\end{aligned} \tag{3.35}$$

Moreover, a^ϵ is coercive on $V^\epsilon\cap\{v\in V^\epsilon:\nabla\cdot v=0\}$ by Lemma 3.4.2, with bound independent of ϵ.

The inf-sup condition follows from the corresponding condition known for the Stokes system on Ω, so the inf-sup theory provides the existence and uniqueness of a solution to our system (3.32)–(3.33) [12]. Moreover,

$$|||u^\epsilon\ |||_\epsilon + \|p^\epsilon\| \le C(\|f\|+\|q\|), \tag{3.36}$$

where C depends on the inf-sup constant γ, the coercivity bound for a^ϵ, and the continuity bounds for a^ϵ and b, each of which is independent of ϵ. Finally, (3.36) and (3.35) imply (3.30). □

3.5 Two-Scale Convergence Results for Bimodal Media

In this section we make note of some slight extensions of the two-scale convergence results of Allaire [1, 19]. Lemmas 3.5.1 and 3.5.4 can be deduced easily from the proof of Theorem 2.7 in [1]. We include the following statements and proofs for clarity and completeness. We first recall that $\mathcal{D}(\Omega; C^\infty_\#(Y))$ is the set of infinitely differentiable functions in $\Omega \times Y$ that have compact support in Ω and are periodic in Y, and we recall the following definition.

Definition 3.5.1. *If $\{u^\epsilon\}_\epsilon \subset L^2(\Omega)$ and $u_0(x,y) \in L^2(\Omega \times Y)$ are such that*

$$\lim_{\epsilon \to 0} \int_\Omega u^\epsilon(x)\, \phi(x, x/\epsilon)\, dx = \frac{1}{|Y|} \int_\Omega \int_Y u_0(x,y)\, \phi(x,y)\, dx\, dy$$

for any function $\phi \in \mathcal{D}(\Omega; C^\infty_\#(Y))$, then $\{u^\epsilon\}_\epsilon$ is said to two-scale converge *in $\Omega \times Y$ to $u_0(x,y)$, and we write this as*

$$u^\epsilon \rightharpoonup\!\!\rightharpoonup u_0 \quad \textit{in } \Omega \times Y \textit{ as } \epsilon \to 0\ .$$

Lemma 3.5.1. *Let $\ell = s$ or d and χ^ϵ_ℓ be the characteristic function on Ω^ϵ_ℓ. If u^ϵ is such that $\|u^\epsilon\| \leq C$ for some constant C independent of ϵ, then a subsequence of $\chi^\epsilon_\ell u^\epsilon$ two-scale converges to some $\psi^\ell_0 \in L^2(\Omega \times Y)$ such that* $\operatorname{supp}(\psi^\ell_0) \subset \Omega \times \bar{Y}_\ell$. *Moreover, if u^ϵ two-scale converges to $u_0 \in L^2(\Omega \times Y)$, then $\chi^\epsilon_\ell u_\epsilon \rightharpoonup\!\!\rightharpoonup u_0|_{\Omega \times Y_\ell}$ in $\Omega \times Y$.*

Proof. Because $\|\chi^\epsilon_\ell u^\epsilon\| \leq \|u^\epsilon\| \leq C$, a subsequence of $\chi^\epsilon_\ell u^\epsilon$ two-scale converges to some $\psi^\ell_0 \in L^2(\Omega \times Y)$ [1]. Take a test function $\phi \in \mathcal{D}(\Omega; C^\infty_\#(Y))$ supported in $\Omega \times Y_k$, where $k \neq \ell$. Then $\phi^\epsilon(x) = \phi(x, x/\epsilon)$ is supported in Ω^ϵ_k and

$$0 = \lim_{\epsilon \to 0} \int_\Omega \chi^\epsilon_\ell u^\epsilon \phi^\epsilon\, dx = \frac{1}{|Y|} \int_\Omega \int_Y \psi^\ell_0 \phi\, dy\, dx = \frac{1}{|Y|} \int_\Omega \int_{Y_k} \psi^\ell_0 \phi\, dy\, dx$$

This holds for all such ϕ, so $\psi^\ell_0 = 0$ on $\Omega \times Y_k$.

Now, take a test function $\phi \in \mathcal{D}(\Omega; C^\infty_\#(Y))$ with support in $\Omega \times Y_\ell$. Then

$$\int_\Omega \chi^\epsilon_\ell u^\epsilon \phi^\epsilon \, dx = \int_\Omega u^\epsilon \phi^\epsilon \, dx \ ,$$

so, taking the limit as $\epsilon \to 0$,

$$\frac{1}{|Y|} \int_\Omega \int_{Y_\ell} \psi^\ell_0 \phi \, dy \, dx = \frac{1}{|Y|} \int_\Omega \int_Y u_0 \phi \, dy \, dx \ .$$

An application of Lusin's Theorem completes the lemma. □

Lemma 3.5.1 allows us to make the following definition and gives the following corollary. Note that a function in $\mathcal{D}(\Omega; C^\infty_\#(Y_\ell))$ is considered to be only Y-periodic in y, with no condition imposed on Γ.

Definition 3.5.2. *If $\ell = s$ or d and $\{u^\epsilon_\ell\}_\epsilon \subset L^2(\Omega^\epsilon_\ell)$ is such that, for any function $\phi(x, y)$ in $\mathcal{D}(\Omega; C^\infty_\#(Y_\ell))$,*

$$\lim_{\epsilon \to 0} \int_{\Omega^\epsilon_\ell} u^\epsilon_\ell(x) \, \phi(x, x/\epsilon) \, dx = \frac{1}{|Y|} \int_\Omega \int_{Y_\ell} u_0(x, y) \, \phi(x, y) \, dx \, dy$$

for some $u_0(x, y)$ in $L^2(\Omega \times Y_\ell)$, then $\{u^\epsilon_\ell\}_\epsilon$ is said to two-scale converge *in $\Omega \times Y_\ell$ to $u_0(x, y)$ as $\epsilon \to 0$.*

Corollary 3.5.2. *If $u^\epsilon_\ell \in L^2(\Omega^\epsilon_\ell)$ and there exists $C > 0$ such that $\|u^\epsilon_\ell\|_{\Omega^\epsilon_\ell} \leq C$ for all $\epsilon > 0$, then there exists a subsequence which two-scale converges in $\Omega \times Y_\ell$ to $u_0 \in L^2(\Omega \times Y_\ell)$.*

The following lemma is immediate and illuminates the connection between weak and two-scale convergence.

Lemma 3.5.3. *If $\{u_\ell^\epsilon\}_\epsilon$ two-scale converges to $u_0(x,y)$ in $\Omega \times Y_\ell$, and $\hat{u}_\ell^\epsilon$ denotes the extension of u_ℓ^ϵ by zero to Ω, then*

$$\hat{u}_\ell^\epsilon \rightharpoonup \frac{1}{|Y|}\int_{Y_\ell} u_0(x,y)\,dy$$

in $L^2(\Omega)$.

The next results will be needed to prove our homogenization result. As usual, $H^1_\#(Y)$ denotes the Y-periodic functions in $H^1(Y)$.

Lemma 3.5.4. *Fix $\ell = s$ or d.*

(a) *If $\|u_\ell^\epsilon\|_{\Omega_\ell^\epsilon} \leq C$ and $\|\epsilon\nabla u_\ell^\epsilon\|_{\Omega_\ell^\epsilon} \leq C$ for some constant C, then there exists $u_{0,\ell} \in \left(L^2(\Omega; H^1_\#(Y_\ell))\right)^2$ such that some subsequence of $\{u_\ell^\epsilon\}_\epsilon$ two-scale converges in $\Omega \times Y_\ell$ to $u_{0,\ell}$ and $\{\epsilon\nabla u_\ell^\epsilon\}_\epsilon$ two-scale converges in $\Omega \times Y_\ell$ to $\nabla_y u_{0,\ell}$.*

(b) *If $\|u_\ell^\epsilon\|_{\Omega_\ell^\epsilon} \leq C$ and $\|\nabla\cdot u_\ell^\epsilon\|_{\Omega_\ell^\epsilon} \leq C$ for some constant C, then there exists $u_{0,\ell} \in (L^2(\Omega; H(\mathrm{div}, Y_\ell)))^2$ such that some subsequence of $\{u_\ell^\epsilon\}_\epsilon$ two-scale converges in $\Omega \times Y_\ell$ to $u_{0,\ell}$ and $\{\nabla\cdot u_\ell^\epsilon\}_\epsilon$, extended to Ω by zero, converges weakly in $L^2(\Omega)$ to $\dfrac{1}{|Y|}\displaystyle\int_{Y_\ell} \nabla_x \cdot u_{0,\ell}\,dy$. Moreover, $\nabla_y \cdot u_{0,\ell} = 0$.*

Proof. For result (a), by Corollary 3.5.2, we have for some subsequence both

$$u_\ell^\epsilon \rightharpoonup\!\!\rightharpoonup u_{0,\ell} \quad \text{in } \Omega \times Y_\ell\ ,$$

$$\epsilon\nabla u_\ell^\epsilon \rightharpoonup\!\!\rightharpoonup \psi_{0,\ell} \quad \text{in } \Omega \times Y_\ell\ .$$

Let $\phi \in \left(\mathcal{D}(\Omega; C^\infty_\#(Y_\ell))\right)^2$ be such that $\phi|_{\Omega\times\Gamma} = 0$, and let $\phi^\epsilon(x) = \phi(x, x/\epsilon)$. Compute

$$(\epsilon\nabla u_\ell^\epsilon, \phi^\epsilon)_{\Omega_\ell^\epsilon} = -(u_\ell^\epsilon, \epsilon\nabla_x\cdot\phi^\epsilon + \nabla_y\cdot\phi^\epsilon)_{\Omega_\ell^\epsilon}\ ,$$

so that as $\epsilon \to 0$,

$$(\psi_{0,\ell}, \phi)_{\Omega\times Y_\ell} = -(u_{0,\ell}, \nabla_y \cdot \phi)_{\Omega\times Y_\ell} = (\nabla_y u_{0,\ell}, \phi)_{\Omega\times Y_\ell} - \langle u_{0,\ell} \cdot \eta, \phi\rangle_{\Omega\times(\partial Y_\ell\setminus\Gamma)} \ .$$

With $\phi|_{\Omega\times\partial Y_\ell} = 0$, we conclude that $\psi_{0,\ell} = \nabla_y u_{0,\ell}$. Then we further conclude that $u_{0,\ell}$ is periodic in $y \in Y_\ell$, i.e., that $u_{0,\ell} \in \left(L^2(\Omega; H^1_\#(Y_\ell))\right)^2$.

For (b), Corollary 3.5.2 gives us two-scale convergence of u^ϵ_ℓ to $u_{0,\ell}$, and then weak convergence of $\nabla \cdot u^\epsilon_\ell$ to $\dfrac{1}{|Y|}\displaystyle\int_{Y_\ell} \nabla_x \cdot u_{0,\ell}\, dy$ follows easily. To obtain $\nabla_y \cdot u_{0,\ell} = 0$, note that for $\phi \in \mathcal{D}(\Omega; C^\infty_0(Y))$,

$$\int_{\Omega^\epsilon_\ell} \nabla \cdot u^\epsilon_\ell\, \phi^\epsilon\, dx = -\int_{\Omega^\epsilon_\ell} u^\epsilon_\ell \cdot \left(\nabla_x \phi^\epsilon + \epsilon^{-1}\nabla_y \phi^\epsilon\right)\, dx \ ,$$

By Corollary 3.5.2, the left-hand side and the first term on the right-hand side both converge as $\epsilon \to 0$. Thus we obtain

$$\lim_{\epsilon\to 0} \int_{\Omega^\epsilon_\ell} u^\epsilon_\ell \cdot \nabla_y \phi\, dx = 0 \ ,$$

which implies that $\nabla_y \cdot u_{0,\ell} = 0$. ☐

Lemma 3.5.5. *If u^ϵ_s is such that $\|u^\epsilon_s\|_{\Omega^\epsilon_s}$ and $\|\epsilon\nabla u^\epsilon_s\|_{\Omega^\epsilon_s}$ are bounded independent of ϵ and Γ^ϵ is a smooth submanifold of Ω^ϵ, then for $\phi \in (\mathcal{D}(\Omega; C^\infty_\#(Y_s)))^2$,*

$$\lim_{\epsilon\to 0} \epsilon\langle u^\epsilon_s \cdot \tau, \phi \cdot \tau\rangle_{\Gamma^\epsilon} = |Y|^{-1}\langle u_{0,s} \cdot \tau, \phi \cdot \tau\rangle_{\Omega\times\Gamma} \ ,$$

where $u_{0,s}$ is the two-scale limit of u^ϵ_s in $\Omega \times Y_s$. Moreover,

$$\lim_{\epsilon\to 0} \epsilon\langle u^\epsilon_s \cdot \eta_s, \phi \cdot \eta_s\rangle_{\Gamma^\epsilon} = |Y|^{-1}\langle u_{0,s} \cdot \eta_s, \phi \cdot \eta_s\rangle_{\Omega\times\Gamma} \ .$$

Proof. Let $\Phi^\epsilon(x) = \Phi(x, x/\epsilon) \in (\mathcal{D}(\Omega; C^\infty_\#(Y_s)))^{2\times 2}$. Then

$$(\epsilon\nabla u^\epsilon_s, \Phi^\epsilon)_{\Omega^\epsilon_s} = -\epsilon(u^\epsilon_s, \nabla_x \cdot \Phi^\epsilon)_{\Omega^\epsilon_s} - (u^\epsilon_s, \nabla_y \cdot \Phi^\epsilon)_{\Omega^\epsilon_s} + \epsilon\langle u^\epsilon_s, \Phi^\epsilon \cdot \eta_s\rangle_{\Gamma^\epsilon} \ .$$

Taking the limit of both sides as $\epsilon \to 0$, we obtain from Lemma 3.5.4

$$|Y|^{-1}(\nabla_y u_{0,s}, \Phi)_{\Omega\times Y_s} = -|Y|^{-1}(u_{0,s}, \nabla_y \cdot \Phi)_{\Omega\times Y_s} + \lim_{\epsilon\to 0} \epsilon \langle u_s^\epsilon, \Phi^\epsilon \cdot \eta_s \rangle_{\Gamma^\epsilon} \ .$$

This implies that

$$|Y|^{-1}\langle u_{0,s}, \Phi \cdot \eta_s \rangle_{\Omega\times\Gamma} = \lim_{\epsilon\to 0} \epsilon \langle u_s^\epsilon, \Phi^\epsilon \cdot \eta_s \rangle_{\Gamma^\epsilon} \ . \tag{3.37}$$

Since Γ is smooth, the Tubular Neighborhood Theorem from topology allows us to extend the normal vector field η_s on Γ to a smooth vector field $\hat{N}(y)$ on Y. Similarly we can extend τ on Γ to a smooth vector field $\hat{T}(y)$ on Y. We can now define N^ϵ and T^ϵ on all of Ω by first setting $N(x/\epsilon) = \hat{N}(y)$ and $T(x/\epsilon) = \hat{T}(y)$ for $x \in \epsilon Y$ and then extending N^ϵ and T^ϵ to all of Ω Y-periodically.

Now take $\Phi = T^\epsilon(\phi \cdot T^\epsilon)(N^\epsilon)^T$ in (3.37), where $\phi \in (\mathcal{D}(\Omega; C^\infty_\#(Y_s)))^2$. This yields the first result. Replacing T^ϵ with N^ϵ gives the second result. □

In fact, the following more general definition makes sense and was originally stated in [26].

Definition 3.5.3. *If $\ell = s$ or d and $\{u_\ell^\epsilon\}_\epsilon \subset L^2(\Gamma^\epsilon)$ is such that, for any function $\phi(x,y)$ in $\mathcal{D}(\Omega; C^\infty_\#(\Gamma))$ which is Y-periodic in y,*

$$\lim_{\epsilon\to 0} \int_{\Gamma^\epsilon} u_\ell^\epsilon(x)\, \phi(x, x/\epsilon)\, dx = \frac{1}{|Y|} \int_\Omega \int_\Gamma u_0(x,y)\, \phi(x,y)\, dS dx$$

for some $u_0(x,y)$ in $L^2(\Omega \times \Gamma)$, then $\{u_\ell^\epsilon\}_\epsilon$ is said to two-scale converge *on $\Omega \times \Gamma$ to $u_0(x,y)$ as $\epsilon \to 0$.*

3.6 Proof of the Homogenization Result

We now prove rigorously the homogenization results obtained formally in Section 3.3.

Theorem 3.6.1. *There exists* $(u,p) \in H(\mathrm{div},\Omega)\times W$ *such that the velocity* u^ϵ *converges weakly to* u *in* $H(\mathrm{div},\Omega)$*,* p^ϵ *converges weakly to* p *in* W*, and* (u,p) *is the unique solution to the homogenized Darcy problem*

$$\mu\tilde{K}^{-1}u + \nabla p = f \qquad \text{in } \Omega\ ,$$

$$\nabla\cdot u = q \qquad \text{in } \Omega\ ,$$

$$u\cdot\eta = 0 \qquad \text{on } \partial\Omega\ ,$$

where the tensor $\tilde{K}$ *is defined by* (3.23) *and* (3.14)*–*(3.20)*.*

Proof. By our energy estimates in Theorem 3.4.1 and the two-scale convergence results of [1, 19], Corollary 3.5.2, and Lemma 3.5.4, for $\ell = s, d$, there exists $p_0(x,y) \in L^2(\Omega\times Y)$ and $u_0(x,y) \in L^2(\Omega\times Y)$ (with $u_{0,\ell} = u_0|_{\Omega\times Y_\ell}$) such that the following two-scale convergences hold:

$$p^\epsilon \rightharpoonup\rightharpoonup p_0 \qquad \text{in } \Omega\times Y\ , \tag{3.38}$$

$$u^\epsilon \rightharpoonup\rightharpoonup u_0 \qquad \text{in } \Omega\times Y\ , \tag{3.39}$$

$$u^\epsilon_\ell \rightharpoonup\rightharpoonup u_{0,\ell} \qquad \text{in } \Omega\times Y_\ell\ , \tag{3.40}$$

$$\epsilon\nabla u^\epsilon_s \rightharpoonup\rightharpoonup \nabla_y u_{0,s} \qquad \text{in } \Omega\times Y_s\ . \tag{3.41}$$

Moreover, $u_{0,s} \in \left(L^2(\Omega; H^1_\#(Y_s))\right)^2$,

$$\nabla\cdot u^\epsilon = q\ , \tag{3.42}$$

and

$$\nabla_y \cdot u_0(x,y) = 0 \; . \tag{3.43}$$

Let $\Psi(x,y) \in C_0^\infty(\Omega \times Y)^2$ be Y-periodic. Take $v(x) = \epsilon\Psi(x, x/\epsilon)$ in the variational problem (3.28), so that

$$\big(p^\epsilon, \nabla_y \cdot \Psi\big)_\Omega + O(\epsilon) = 0 \; .$$

As $\epsilon \to 0$, we obtain

$$\big(p_0, \nabla_y \cdot \Psi\big)_{\Omega\times Y} = 0 \; .$$

This implies that $\nabla_y p_0 = 0$, so that $p_0(x,y) = p_0(x)$ only.

Next take $\Psi \in (\mathcal{D}(\Omega; C_\#^\infty(Y)))^2$ with $\nabla_y \cdot \Psi = 0$, and let $v(x) = \Psi(x, x/\epsilon)$ in the variational equation (3.28), so that

$$\begin{aligned} 2\mu\epsilon^2 \left(Du_s^\epsilon, D\Psi^\epsilon\right)_{\Omega_s^\epsilon} + \langle \epsilon\mu\alpha (K^\epsilon)^{-1/2} u_s^\epsilon \cdot \tau, \Psi^\epsilon \cdot \tau\rangle_{\Gamma^\epsilon} + \big(\mu (K^\epsilon)^{-1} u_d^\epsilon, \Psi^\epsilon\big)_{\Omega_d^\epsilon} \\ - \left(p^\epsilon, \nabla_x \cdot \Psi^\epsilon\right)_\Omega = \left(f, \Psi^\epsilon\right)_\Omega \; . \end{aligned}$$

Using Lemma 3.5.5, passing to the two-scale limit gives

$$\begin{aligned} 2\mu(D_y u_{0,s}, D_y\Psi)_{\Omega\times Y_s} + \langle \mu\alpha K^{-1/2} u_{0,s} \cdot \tau, \Psi \cdot \tau\rangle_{\Omega\times\Gamma} + (\mu K^{-1} u_{0,d}, \Psi)_{\Omega\times Y_d} \\ - (p_0, \nabla_x \cdot \Psi)_{\Omega\times Y} = (f, \Psi)_{\Omega\times Y} \; . \end{aligned}$$

Integrating by parts and collecting terms, we obtain

$$\begin{aligned} (-2\mu\nabla_y \cdot D_y u_{0,s} + \nabla_x p_0 - f, \Psi)_{\Omega\times Y_s} + (\mu K^{-1} u_{0,d} + \nabla_x p_0 - f, \Psi)_{\Omega\times Y_d} \\ + \langle \mu\alpha K^{-1/2} u_{0,s} \cdot \tau + 2\mu\tau \cdot D_y u_{0,s} \cdot \eta_s, \Psi \cdot \tau\rangle_{\Omega\times\Gamma} \\ + \langle 2\mu\eta_s \cdot D_y u_{0,s} \cdot \eta_s, \Psi \cdot \eta_s\rangle_{\Omega\times\Gamma} = 0 \; . \end{aligned} \tag{3.44}$$

It is a well-known result that if $(\Phi, \Psi)_{\Omega\times Y} = 0$, with $\nabla_y \cdot \Psi = 0$ and $\Psi \in \mathcal{D}(\Omega; C_0^\infty(Y))^2$, then $\Phi = \nabla_y \psi$ for some $\psi \in L^2(\Omega; H^1(Y))$ [36]. Restrict to $\Psi \in (\mathcal{D}(\Omega; C_0^\infty(Y_s)))^2$ to obtain

$$(-2\mu\nabla_y \cdot D_y u_{0,s} + \nabla_x p_0 - f, \Psi)_{\Omega\times Y_s} = 0 \ .$$

Thus, there exists $p_1(x, y) \in L^2(\Omega; H^1(Y_s))$ such that in $\Omega \times Y_s$,

$$-2\mu\nabla_y \cdot D_y u_{0,s} + \nabla_x p_0 - f = -\nabla_y p_1 \ .$$

Likewise, we obtain $p_2(x, y) \in L^2(\Omega; H^1(Y_d))$ such that in $\Omega \times Y_d$,

$$K^{-1} u_{0,d} + \nabla_x p_0 - f = -\nabla_y p_2 \ .$$

Thus for all $\Psi \in \mathcal{D}(\Omega; C_\#^\infty(Y))^2$ satisfying the divergence constraint,

$$\begin{aligned}
&(-\nabla_y p_1, \Psi)_{\Omega\times Y_s} + (-\nabla_y p_2, \Psi)_{\Omega\times Y_d} \\
&\quad + \langle \mu\alpha K^{-1/2} u_{0,s} \cdot \tau + 2\mu\tau \cdot D_y u_{0,s} \cdot \eta_s, \Psi \cdot \tau\rangle_{\Omega\times\Gamma} \\
&\quad + \langle 2\mu\eta_s \cdot D_y u_{0,s} \cdot \eta_s, \Psi \cdot \eta_s\rangle_{\Omega\times\Gamma} = 0 \ ,
\end{aligned}$$

so integrating by parts yields

$$\begin{aligned}
&\langle \mu\alpha K^{-1/2} u_{0,s} \cdot \tau + 2\mu\tau \cdot D_y u_{0,s} \cdot \eta_s, \Psi \cdot \tau\rangle_{\Omega\times\Gamma} \\
&\quad + \langle 2\mu\eta_s \cdot D_y u_{0,s} \cdot \eta_s - p_1 + p_2, \Psi \cdot \eta_s\rangle_{\Omega\times\Gamma} = 0 \ .
\end{aligned} \tag{3.45}$$

By noting for example that there exists a weak solution (see [36] or the proof of Theorem 3.4.1) $\Psi \in L^2(\Omega; (H_\#^1(Y))^2)$, $w \in L^2(\Omega; L^2(Y)/\mathbb{R})$ to

$$\begin{aligned}
-\Delta_y \Psi + \nabla_y w &= 0 && \text{in } \Omega \times Y_s \ , \\
\nabla_y \cdot \Psi &= 0 && \text{in } \Omega \times Y_s \ , \\
\Psi \cdot \eta_s &= 0 && \text{on } \Omega \times \Gamma \ , \\
\Psi \cdot \tau &= \mu\alpha K^{-1/2} u_{0,s} \cdot \tau + 2\mu\tau \cdot D_y u_{0,s} \cdot \eta_s && \text{on } \Omega \times \Gamma \ , \\
\Psi &= 0 && \text{on } \Omega \times (\partial Y_s \setminus \Gamma) \ ,
\end{aligned}$$

and by the fact that $\mathcal{D}(\Omega; C^\infty_\#(Y))$ is dense in $L^2(\Omega; H^1_\#(Y))$, we obtain that each of the two terms in (3.45) vanishes. We then finally obtain that the two-scale variational equations (3.42) and (3.44) are equivalent to

$$\begin{aligned}
-2\mu\nabla_y \cdot D_y u_{0,s} + \nabla_x p_0(x) + \nabla_y p_1(x,y) &= f && \text{in } \Omega \times Y_s \ , \\
\nabla_y \cdot u_{0,s} &= 0 && \text{in } \Omega \times Y_s \ , \\
\mu K^{-1} u_{0,d} + \nabla_x p_0(x) + \nabla_y p_2(x,y) &= f && \text{in } \Omega \times Y_d \ , \\
\nabla_y \cdot u_{0,d} &= 0 && \text{in } \Omega \times Y_d \ , \\
2\mu\, \eta_s \cdot D_y u_{0,s} \cdot \eta_s &= p_1 - p_2 && \text{on } \Omega \times \Gamma \ , \\
2\tau \cdot D_y u_{0,s} \cdot \eta_s &= -\alpha K^{-1/2} u_{0,s} \cdot \tau && \text{on } \Omega \times \Gamma \ .
\end{aligned}$$

Let (ω^s_j, Φ^s_j) and (ω^d_j, Φ^d_j) be Y-periodic solutions to the auxiliary problem on Y_s and Y_d given in (3.14)–(3.20). Then, because the above problem has a unique solution, it is clear that we can express $u_{0,s}$ and $u_{0,d}$ as in (3.21):

$$u_{0,\ell}(x,y) = \frac{1}{\mu} \sum_{j=1}^{N} \left(f_j(x) - \frac{\partial p_0}{\partial x_j}(x) \right) \omega^\ell_j(y) \ ,$$

where $\ell = s$ or d. Averaging over Y_s and Y_d as in (3.22), we get

$$\begin{aligned}
\bar{u}_0 &= \bar{u}_{0,s} + \bar{u}_{0,d} \\
&= \frac{1}{\mu |Y|} \sum_{j=1}^{N} \Big(f_j(x) - \partial_{x_j} p_0(x) \Big) \Big(\int_{Y_s} \omega^s_j(y) + \int_{Y_d} \omega^d_j(y) \Big) \\
&= \frac{\tilde{K}}{\mu} (f - \nabla p_0) \ ,
\end{aligned} \tag{3.46}$$

where $\tilde{K}$ is as in (3.23).

By Lemma 3.5.3 and (3.42), we obtain the weak convergence results in $L^2(\Omega)$

$$p^\epsilon \rightharpoonup \frac{1}{|Y|}\int_Y p_0(x,y)\,dy = p_0(x)\ , \tag{3.47}$$

$$u^\epsilon \rightharpoonup \frac{1}{|Y|}\int_Y u_0(x,y)\,dy = \bar{u}_0\ , \tag{3.48}$$

$$q = \nabla\cdot u^\epsilon \rightharpoonup \nabla\cdot\bar{u}_0 = q\ . \tag{3.49}$$

Setting $p = p_0$ and $u = \bar{u}_0$, (3.46)–(3.49) gives the theorem, since convergence on the boundary of the domain is trivial. □

3.7 A Simple Analytical Solution of the Auxiliary Problem

It is not so easy to construct analytical solutions to the auxiliary problem (3.14)–(3.20), except at least in the following case. Let $Y = (0,\ell)\times(0,\ell)$ be a square of side length $\ell > 0$. With $Y_s = (0,\ell)\times(0,h)$ and $Y_d = (0,\ell)\times(h,\ell)$ repeated periodically, we have a horizontally layered medium. Note that Γ consists of two segments, $y_2 = h$ and, by periodicity, $y_2 = 0$ or $y_2 = \ell$.

When $j = 1$, it is easy to verify that the solution is

$$\omega_1^s(y) = \frac{1}{2}\left(-y_2^2 + hy_2 + \frac{K^{1/2}}{\alpha}h\right)e_1\ , \tag{3.50}$$

$$\omega_1^d(y) = Ke_1\ , \tag{3.51}$$

$$\Phi_1(y) = 0\ , \tag{3.52}$$

which has flow in the y_1-direction only. It follows from (3.23) that $\tilde{K}_{21} = 0$ (so $\tilde{K}$ is diagonal), and

$$\tilde{K}_{11} = \frac{1}{\ell}\left(\frac{1}{12}h^3 + \frac{\sqrt{K}}{2\alpha}h^2 + K(\ell - h)\right)\ . \tag{3.53}$$

This should be contrasted to the situation in which the porous matrix is replaced by an impermeable medium. Then $\omega_j^s = 0$ on Γ, and we have the well-known problem of Poiselle flow in a pipe. The solution is

$$\breve{\omega}_1^s(y) = -\frac{1}{2} y_2 (h - y_2) e_1 \ , \tag{3.54}$$

$$\check{\Phi}_1(y) = 0 \ . \tag{3.55}$$

In this case, we would compute the effective permeability for a unit pressure drop in the y_1-direction (which corresponds to the forcing function e_1) as

$$\tilde{K}_{11,\text{Poiselle}} = \frac{h^3}{12\ell} \ , \tag{3.56}$$

which is the first term on the right side of (3.53).

If instead we assume the vugular region is impermeable, the bulk flow would be reduced from the porous medium case by the geometric factor $(\ell - h)/\ell$:

$$\tilde{K}_{11,\text{Darcy}} = \frac{\ell - h}{\ell} K \ . \tag{3.57}$$

This is the last term on the right side of (3.53). Thus, when considering flow in the direction of the vugular channel, we have the representation

$$\tilde{K}_{11} = \tilde{K}_{11,\text{Poiselle}} + \tilde{K}_{11,\text{Beavers-Joseph}} + \tilde{K}_{11,\text{Darcy}} \ , \tag{3.58}$$

where

$$\tilde{K}_{11,\text{Beavers-Joseph}} = \frac{\sqrt{K}}{2\alpha\ell} h^2 \tag{3.59}$$

represents the Beavers-Joseph interface effect of fluid slippage. This term increases the Darcy-Stokes flow from the arithmetic average of the pure Stokes "pipe flow" and the pure Darcy flow. Note that we generally have $K \ll h^2$ and $\alpha = O(1)$. Thus $\tilde{K}_{11,\text{Poiselle}}$ is the leading term and $\tilde{K}_{11,\text{Beavers-Joseph}}$ is the next order term in the expansion.

When $j = 2$ in the auxiliary problem (3.14)–(3.20), it is easy to verify that the solution is

$$\omega_2(y) = \frac{\ell}{\ell - h} K e_2 \ , \tag{3.60}$$

$$\Phi_2^s(y) = y_2 \ , \tag{3.61}$$

$$\Phi_2^d(y) = \frac{h}{\ell - h}(\ell - y_2) \ . \tag{3.62}$$

Again $\tilde{K}_{12} = \tilde{K}_{21} = 0$ and

$$\tilde{K}_{22} = \frac{\ell}{\ell - h} K \ . \tag{3.63}$$

In this case, the flow is entirely in the y_2-direction.

It is well known and easily verified that one dimensional flow across a porous medium of permeability k_1 for distance h and k_2 for distance $\ell - h$ results in a flow rate that is the same as that in a uniform medium of permeability $k = \dfrac{\ell k_1 k_2}{h k_2 + (\ell - h) k_1}$, which is the harmonic average permeability. If we apply this result to our case, assuming that the vugular channel has infinite permeability, we obtain exactly (3.63).

In conclusion, this example suggests that the effective permeability represents an average of two extremes. When the vugs are interconnected in some direction, we have primarily Poisson flow behavior, with a low order correction term for the effective permeability related to the Beavers-Joseph slip (and an even lower order correction related to flow entirely in the porous matrix).

When the flow is along paths that alternate from vug to matrix, the fluid behaves as if it were flowing in a porous medium with the vugs having infinite permeability.

Chapter 4

Discontinuous Galerkin Algorithms and Analysis

4.1 Preliminaries and Notation

In this chapter we present two slightly different Discontinous Galerkin methods for solving our original microscale system (2.1)–(2.9). A term arises in the variational formulation involving the Darcy pressure and the normal jump across the interface between the Stokes and Darcy regions. The difference between the two methods lies in how we handle this term. In the first method, in order to control the normal jump we add a penalty term. In the second approach, we instead introduce a Lagrange multiplier.

We assume that our domain is a Lipschitz polygonal domain in $\mathbb{R}^N$, where $N = 2, 3$. Let $\mathcal{E}_h$ be a conforming family of triangulations of Ω consisting of triangles (or tetrahedra) of maximum diameter h, each of which lies entirely in either Ω_s or Ω_d. Let h_E be the diameter of an element E and ρ_E the diameter of its inscribed N-sphere. Let h_e be the diameter of the face e. We require our triangulation to be regular, meaning that there exists a $\sigma > 0$ such that

$$\frac{h_E}{\rho_E} < \sigma \qquad \forall E \in \mathcal{E}_h \ .$$

We will denote by e_i an interior $(N-1)$-dimensional face and by E_i an element of the Stokes or Darcy domain, so that $i = s$ or d, respectively. Similarily, e_Γ is an $(N-1)$-dimensional face lying along the interface Γ between the Stokes and Darcy regions. The standard L^2 inner product over a subdomain $\mathcal{O}$ is denoted by $(\cdot,\cdot)_{\mathcal{O}}$, and along a face e by $\langle\cdot,\cdot\rangle_e$.

We denote the space of $L^2(\Omega)$ functions whose average over the domain is zero by $L^2_0(\Omega)$. For an integer $t \geq 0$ and a domain $\mathcal{O}$ in $\mathbb{R}^N$, we denote the usual Sobolev norm and seminorm of $H^t(\mathcal{O}) = W^{t,2}(\mathcal{O})$ by $\|\cdot\|_{t,\mathcal{O}}$ and $|\cdot|_{t,\mathcal{O}}$, respectively. If $\mathcal{O} = \Omega$ we write $\|\cdot\|_t$ and $|\cdot|_s$ respectively. The broken j-th norm is denoted $|||\cdot|||_j$, and it is defined by,

$$|||u|||_j := \left(\sum_{E\in\mathcal{E}_h} |u|^2_{j,E}\right)^{1/2} .$$

In order to deal with what happens along an element face, we need to introduce some trace operators. Permanently associate to each interior face e a normal vector η_e ordered so that it is directed from E_1 to E_2, where e is shared by elements E_1 and E_2. Take any $q \in \prod_{E\in\mathcal{E}_h}(L^2(\partial E))^N$. We define on each interior face e the *average* $\{q\}$, the *full jump* $[q]$, and the *normal jump* $[\![q]\!]$, respectively, by

$$\begin{aligned} \{q\} &:= \frac{1}{2}(q_1 + q_2) , \\ [q] &:= q_1 - q_2 , \\ [\![q]\!] &:= q_1 \cdot \eta_e - q_2 \cdot \eta_e , \end{aligned}$$

where $q_i = q|_{\partial E_i}$. For any $w \in \prod_{E\in\mathcal{E}_h} L^2(\partial E)$, define the *average* of w on e by $\{w\} := \frac{1}{2}(w_1 + w_2)$.

We define, as before,

$$V := \{v \in H(\mathrm{div},\Omega)^N : v|_{\Omega_s} \in H^1(\Omega_s)^N \text{ with } v = 0 \text{ on } \partial\Omega_s \cap \partial\Omega,$$
$$v \cdot \eta = 0 \text{ on } \partial\Omega_d \cap \partial\Omega\}$$
$$W := L_0^2(\Omega) \ .$$

Assuming $f \in L^2(\Omega)^2$ and $q \in L^2(\Omega)$, we have shown in §2.4 that there exists a unique weak solution $(u,p) \in V \times W$ to (2.1)–(2.9).

4.2 A First Non-Conforming Method

In this section, we explore a discontinuous Galerkin mixed method matched with the Raviart-Thomas method for solving our Darcy-Stokes system. The formulation of the discontinuous Galerkin method in the Stokes portion of the domain is based on that developed in [37] by Wheeler, Girault, and Riviere.

For $k = 1$ or 2, our finite-dimensional approximation spaces are simply taken to be

$$V_h := \{v_h \in (L^2(\Omega))^N : \forall E_s \in \mathcal{E}_h, v_h|_{E_s} \in (\mathbb{P}_k(E))^N, v_h|_{\Omega_d} \in \mathcal{RT}_{k-1}^N(\Omega_d),$$
$$v_h = 0 \text{ at the Gauss-Legendre points of each edge } e \subset \partial\Omega_s \cap \partial\Omega, \tag{4.1}$$
$$v_h \cdot \eta = 0 \text{ on } \partial\Omega_d \cap \partial\Omega\} \tag{4.2}$$

and

$$W_h := \{w_h \in L_0^2(\Omega) \mid \forall E \in \mathcal{E}_h, w_h|_E \in \mathbb{P}_{k-1}(E)\} \ , \tag{4.3}$$

where $\mathcal{RT}_{k-1}^N$ is the vector variable part of the standard Raviart-Thomas space defined in [30] with vanising normal velocity on $\partial\Omega \cap \partial\Omega_d$. On an element E, it is given by

$$\mathcal{RT}_{k-1}^{N}(E) := \vec{x}\mathbb{P}_{k-1}(E) + \mathbb{P}_{k-1}^{N}(E)$$

Recall that $h_e = \mathrm{diam}(e)$. We define the following bilinear forms on $V_h \times V_h$ and $V_h \times W_h$, respectively, as

$$\begin{aligned} a(u,v) := &\sum_{E_s} (2\mu Du, Dv)_{E_s} + \sum_{E_d} \left(\mu K^{-1}u, v\right)_{E_d} \\ &+ \sum_{e_\Gamma} \langle \alpha K^{-1/2} u_s \cdot \tau, v_s \cdot \tau \rangle_{e_\Gamma} + \sum_{e_s} \frac{1}{h_{e_s}} \langle [u], [v] \rangle_{e_s} \\ &- \sum_{e_s} \langle 2\mu \{Du\}\eta, [v] \rangle_{e_s} + \sum_{e_s} \langle 2\mu \{Dv\}\eta, [u] \rangle_{e_s} \\ &+ \sum_{e_\Gamma} \langle [\![u]\!], [\![v]\!] \rangle_{e_\Gamma} \ , \end{aligned} \tag{4.4}$$

$$b(v,p) := \sum_{E} (\nabla \cdot v, p)_E - \sum_{e_s} \langle \{p\}, [\![v]\!] \rangle_{e_s} - \sum_{e_\Gamma} \langle p_d, [\![v]\!] \rangle_{e_\Gamma} \ . \tag{4.5}$$

Note that the sums are over interior edges, as the outer boundary conditions are imposed in V_h.

Our goal is then to find $(u_h, p_h) \in V_h \times W_h$ such that for all $v_h \in V_h$ and $w_h \in W_h$

$$a(u_h, v_h) - b(v_h, p_h) = F(v_h) \ , \tag{4.6}$$

$$b(u_h, w_h) = Q(w_h) \ , \tag{4.7}$$

where

$$F(v_h) := \int_\Omega f \cdot v_h \, dx \quad \text{and} \quad Q(w_h) := \int_\Omega q w_h \, dx \ .$$

Notice that for $(u,p) \in V \times W$ and test functions $(v,w) \in V \times W$, the above finite-dimensional formulation reduces to the variational form of our original problem obtained in §2.3.

4.3 The Interpolation Operator

In this section, we define the interpolation operators needed in the analysis of the method. The velocity projection is based on the Raviart-Thomas operator on Ω_d and the standard discontinuous operators used on Ω_s, the construction of which are included for the sake of completeness.

For the pressure, we use r_h, the L^2 projection onto $\mathbb{P}_{k-1}(E)$ on each element, so that for all $q \in \mathbb{P}_{k-1}(E)$ and $p \in W$,

$$\int_E q(r_h(p) - p)dx = 0 \ , \qquad \forall q \in \mathbb{P}_{k-1}(E) \ , \tag{4.8}$$

$$\|q - r_h(q)\|_{0,E} \leq Ch_E^s |q|_{s,E} \ , \qquad \forall q \in H^s(\Omega) \cap L_0^2(\Omega) \text{ and } s \in [0,k] \ . \tag{4.9}$$

We construct the operator for the velocity seperately on the Stokes and Darcy domains. As these Stokes and Darcy interpolation operators are defined element by element, we have no problem along the interface of the two domains. Moreover, both of these interpolation operators have the property that for $v \in H^1(\Omega)$,

$$\int_e (\Pi_h(v) - v) \cdot \eta \, dS = 0 \ , \tag{4.10}$$

assuming that the faces of our triangulation are straight. Thus, as we would hope, we have a weak average normal continuity across element faces over the entire domain. Moreover, the interpolation operator on the Stokes domain has the property that, for an interior Stokes edge e_s,

$$\int_{e_s} [\Pi_h(v)] \, dS = 0 \ ,$$

so that we additionally have weak tangential continuity along interior faces of the Stokes domain. Letting e_s again represent any face interior to the Stokes domain, e_d any interior Darcy face, and e_Γ any face lying along the interface Γ, our operator $\Pi_h \in \mathcal{L}((H^1(\Omega))^N; V_h)$ has the following properties for any $v \in (H^1(\Omega))^N$:

$$\int_E w_h \nabla \cdot (\Pi_h(v) - v)\, dx = 0 \ , \qquad \forall w_h \in \mathbb{P}_{k-1}(E) \ , \tag{4.11}$$

$$\int_{e_s} \xi_h \cdot [\Pi_h(v)] dS = 0 \ , \qquad \forall \xi_h \in (\mathbb{P}_{k-1}(e_s))^N \ , \tag{4.12}$$

$$[\![\Pi_h(v)]\!] = 0 \ , \qquad \text{across each } e_d \ , \tag{4.13}$$

$$\int_{e_\Gamma} [\![\Pi_h(v)]\!]\, dS = 0 \ , \tag{4.14}$$

$$|v - \Pi_h(v)|_{r,E_s} \le C h_{E_s}^{m-r} |v|_{m,\Delta_{E_s}} \ , \qquad \forall m \in [0, k+1], r \in [0, m] \ , \tag{4.15}$$

$$|v - \Pi_h(v)|_{0,E_d} \le C h_{E_d}^{m} |v|_{m,E_d} \ , \qquad \forall m \in [1, k] \ , \tag{4.16}$$

where Δ_{E_s} is a suitable Stokes macro-element containing E_s. If $k = 1$, one may take $\Delta_{E_s} = E_s$. On the Stokes side, for $k = 2$, we need vertex values of v in order to define Π_h, for which we use the Scott-Zhang operator [34]. The choices needed in order to define this operator can be made so that Δ_{E_s} is always contained within Ω_s, as was demonstrated in [2].

On the Darcy domain, the interpolation operator is defined by the Raviart-Thomas interpolation operator [30]. This operator is defined by

$$\int_e \Pi_h v \cdot \eta_e \, \xi \, dS = \int_e v \cdot \eta_e \, \xi \, dS \ ,$$

for all $\xi \in \mathbb{P}_{k-1}(e)$ and any face e of a Darcy element E_d. If $k = 2$, we have additionally that

$$\int_{E_d} \Pi_h v \cdot \phi \, dx = \int_{E_d} v \cdot \phi \, dx \ ,$$

for all $\phi \in \mathbb{P}_{k-2}(E_d)$. It follows directly that

$$\int_{E_d} \xi \mathrm{div}(v)\, dx = \int_{E_d} \xi \mathrm{div}(\Pi_h v)\, dx$$

for all $\xi \in \mathbb{P}_{k-1}(E_d)$.

We proceed now to outline the construction of the operator on the Stokes domain. We build an interpolation operator Π_h by defining it on Ω_s as described in [14] for $k = 1$ and $N = 2$, [16] for $k = 2$ and $N = 2$, [12] for $k = 1$ and $N = 3$, [15] for $k = 2$ and $N = 3$. The polynomial degree $k = 2$ Stokes operators require evaluation of the velocity at vertices. In the absence of sufficient regularity, we approximate this value using the Scott-Zhang operator developed in [34]. This operator allows us to define the function value at a vertex by averaging over an edge emanating from that vertex. The error bounds for the Stokes, Darcy, and pressure interpolations can be verified by the results in [13].

4.3.1 The Stokes operator for $N = 2$, $k = 1$.

In the case of dimension $N = 2$ and polynomial degree $k = 1$, Crouziex and Raviart [14] construct an interpolation operator $\Pi_h : H^1(\Omega_s)^2 \to \mathbb{P}_1(\mathcal{E}_h)^2$. We apply this operator elementwise to the Stokes domain. Let $v \in H^1(\Omega_s)$. The interpolant of v, $\Pi_h v$, is defined on a Stokes element E_s by

$$\int_e \Pi_h v\, dS = \int_e v\, dS$$

for each of the three faces e of E_s. This determines six degrees of freedom, and so defines $\Pi_h v \in \mathbb{P}_1(E_s)^2$. It follows directly that

$$\int_{E_s} \mathrm{div}(v)\, dx = \int_{E_s} \mathrm{div}(\Pi_h v)\, dx$$

on each Stokes element E_s. Note as well that this implies $\Pi_h v$ will be continuous at the midpoint of each edge. This follows from the fact that if $f \in \mathbb{P}_1([a,b])$, then

$$\int_a^b f(x)\,dx = (b-a) f(\frac{a+b}{2}) \ .$$

4.3.2 The Stokes operator for $N = 2$, $k = 2$.

In the case of dimension $N = 2$ and polynomial degree $k = 2$, we follow the construction of Fortin and Soulie [16] on the Stokes domain. We begin by constructing a conforming operator $\hat{\Pi}_h$ by requiring that

$$v(x_i) = \hat{\Pi}_h v(x_i) \ , \qquad i = 1, 2, 3,$$
$$\int_e v\,dS = \int_e \hat{\Pi}_h v\,dS \ ,$$

where x_i is a vertex and e is an edge of an element E_s. These conditions determine twelve degrees of freedom, so that an element of $\mathbb{P}_2(E_s)^2$ is determined exactly. On any given edge, we have fixed six degrees of freedom, implying that each edge polynomial is completely determined as well. Thus, the approximation is fully continuous across element edges. In order to obtain preservation of the first order moments of the divergence, we perturb this approximation by setting

$$\Pi_h v := \hat{\Pi}_h v + (\alpha_1 \phi, \alpha_2 \phi) \ ,$$

where ϕ is a quadratic function on the element which is zero at the two Gauss-Legendre points on each edge [16]. We determine α_1 and α_2 in $\mathbb{R}$ by requiring that

$$\int_{E_s} x_i \,\mathrm{div}(v)\,dx = \int_{E_s} x_i \,\mathrm{div}(\Pi_h v)\,dx \ , \qquad i = 1, 2 \ .$$

Thus, the resulting approximation has the needed preservation of the zeroth and first order moments of the divergence, and retains the property that

$$\int_e \Pi_h v \, dS = \int_e v \, dS$$

on each Stokes edge e. In fact, the Gauss-Legendre continuity of the approximation additionally implies continuity of the first order moments across interior Stokes edges,

$$\int_e [\Pi_h v] \cdot \zeta \, dS = 0 \ ,$$

with $\zeta \in \mathbb{P}_1(e)^2$ on each interior Stokes edge e.

4.3.3 The Stokes operator for $N = 3$, $k = 1$.

In dimension $N = 3$, the operators are defined analogously. If $k = 1$, on the Stokes domain the construction follows that of Brezzi and Fortin [12]. There are three degrees of freedom associated with each face, namely

$$\int_f \Pi_h v \, dS = \int_f v \, dS \ ,$$

where f is a face of a Stokes element. Again, this completely determines $\Pi_h v \in \mathbb{P}_1(E_s)^2$, since it determines twelve independent degrees of freedom. Further, we get for free that

$$\int_{E_s} \mathrm{div}(\Pi_h v) \, dx = \int_{E_s} \mathrm{div}(v) \, dx \ .$$

4.3.4 The Stokes operator for $N = 3$, $k = 2$.

If $k = 2$ and $N = 3$, on the Stokes domain the construction follows that of Fortin [15]. We begin again with a conforming operator $\hat{\Pi}_h$, such that

$$\hat{\Pi}_h v(x_i) = v(x_i) \ ,$$
$$\int_e \hat{\Pi}_h v \, dS = \int_e v dS \ ,$$

for each vertex x_i and edge e of a given element E_s. Let ϕ_l be the quadratic on face f_{ijk} (determined by vertices x_i, x_j, and x_k) which vanishes on the straight lines joining the vertex x_l to the six gaussian points on the edges of the face opposite vertex x_l, where $l \neq i, j, k$ (for the construction see [15]). Note as well that the moments up to degree one of ϕ_l vanish on the faces adjacent to x_l. This is an extension of the function ϕ constructed for the case $N = 2$ and $k = 2$ for each face. We define another intermediate operator $\tilde{\Pi}_h$ by

$$\tilde{\Pi}_h v = \hat{\Pi}_h v + \sum_{i=1}^{3} \sum_{l=1}^{4} \alpha_{l,i} \, \phi_l \, e_i \ ,$$

where e_i is the standard unit vector in the i-th direction and the $\alpha_{l,i}$'s are chosen so that

$$\int_{f_{ijk}} \tilde{\Pi}_h u \, dS = \int_{f_{ijk}} u \, dS \ .$$

Next, we need the quadratic Φ_0 which vanishes on the ellipse through the six Gauss-Legendre points on each face (see [15] for the construction of Φ_0). Moreover, the moments up to degree one of Φ_0 vanish on every face. As in $N = 2$, we adjust $\tilde{\Pi}$ by

$$\Pi_h v = \tilde{\Pi}_h v + (\alpha_1 \Phi_0, \alpha_2 \Phi_0, \alpha_3 \Phi_0) \ ,$$

where α_1, α_2, and α_3 are chosed so that

$$\int_{E_s} x_i \operatorname{div}(v)\, dx = \int_{E_s} x_i \operatorname{div}(\Pi_h u)\, dx \ , \qquad i = 1, 2, 3 \ .$$

The final approximation then has continuity of moments up to degree 1 across each face, and preservation of the zeroth and first moments of the divergence on any Stokes element.

4.4 Existence and Uniqueness

We will need the following corollary to Korn's Inequality (2.4.1).

Corollary 4.4.1. *If $v \in H^1(\Omega)$, Ω is a compact polygonal domain, and $\int_e v\, dS = 0$ for each straight line segment $e \subset \partial\Omega$, then*

$$\|Dv\|_0 \geq C\|v\|_1 \qquad \forall v \in V$$

Proof. Suppose not. Then there exists a sequence $v_n \in H^1(\Omega)$ such that

$$\|v_n\|_{1,\Omega} = 1$$
$$\|Dv_n\|_{0,\Omega} < 1/n$$

The proof of Korn's Inequality in [10] shows that this implies that v_n in fact converges to a rigid motion $v \in H^1(\Omega)$. Moreover, $Dv = 0$ and v is linear. That Ω is a closed polygonal domain, $\int_e v\, dS = 0$ on each straight line segment $e \subset \partial\Omega$, and v is linear, imply that $v = 0$ at the midpoint of at least three straight line segments of $\partial\Omega$, and that those midpoints can be taken to be non-colinear. Hence $v = 0$, which contradicts $\|v\|_{1,\Omega} = 1$. $\square$

In order to prove existence of a unique solution $(u_h, p_h) \in V_h \times W_h$ to our finite dimensional problem (4.6)–(4.7), it suffices to show that if F and Q vanish, then so also u_h and p_h vanish. So let $F = Q = 0$ and take $v_h = u_h$ and $w_h = p_h$. It follows easily that

$$Du_h = 0 \qquad \text{in } \Omega_s \ , \tag{4.17}$$

$$u_h = 0 \qquad \text{in } \Omega_d \ , \tag{4.18}$$

$$[u_h] = 0 \qquad \text{on any internal face in } \Omega_s \ , \tag{4.19}$$

$$u_{h,s} \cdot \tau = 0 \qquad \text{on } \Gamma \ , \tag{4.20}$$

$$[\![u_h]\!] = 0 \qquad \text{across } \Gamma \ . \tag{4.21}$$

By (4.17) and (4.19), u_h is a single rigid motion on each connected component of Ω_s. By the fact that $u_h \in V_h$, (4.18), (4.20), (4.21), and Corollary 4.4.1, $u_{h,s} = 0$. Hence, $u = 0$ on Ω.

In order to show that $p_h = 0$ as well, we need the following lemma.

Lemma 4.4.2. *Given $p_h \in M_h$, there exists a $v_h \in V_h$ such that*

$$b(v_h, p_h) \geq \frac{1}{2} \|p_h\|^2_{0,\Omega} \ . \tag{4.22}$$

This finishes the question of existence, as we have already shown that if $F = Q = 0$, then

$$b(v, p_h) = 0$$

for all $v \in V_h$. By Lemma 4.4, it must follow that $p_h = 0$ as well. Hence our finite dimensional problem has a unique solution in $V_h \times W_h$.

Proof. We follow the idea of the proof of Theorem 4.5 in [37] handling domain decomposition. First split p_h into the sum of two functions

$$p_h = \tilde{p}_h + \bar{p}_h \ ,$$

so the restrictions, for $i \in \{s, d\}$, $\tilde{p}_h^i := \tilde{p}_h|_{\Omega_i}$ have zero mean value over Ω_i and $\bar{p}_h^i := \bar{p}_h|_{\Omega_i}$ are constant on Ω_i. More explicitly, we let

$$\bar{p}_h^i := \frac{1}{|\Omega_i|} \int_{\Omega_i} p_h \, dx \ , \tag{4.23}$$

$$\tilde{p}_h^i := p_h|_{\Omega_i} - \bar{p}_h^i \ , \tag{4.24}$$

where $i \in \{s, d\}$. Note that because $p_h \in L_0^2(\Omega)$,

$$|\Omega_s|\bar{p}_h^s + |\Omega_d|\bar{p}_h^d = 0 \ . \tag{4.25}$$

We now proceed to construct v_h for a given p_h, so that the conclusion of the lemma is satisfied. We will construct it in two pieces, $\tilde{v}_h$ and $\bar{v}_h$, corresponding to the two parts of p_h. For each $i \in \{s, d\}$, since $\tilde{p}_h^i \in L_0^2(\Omega_i)$, there exists $\tilde{v}^i \in H_0^1(\Omega_i)^N$ such that

$$\nabla \cdot \tilde{v}^i = \tilde{p}_h^i \ , \tag{4.26}$$

$$\|\tilde{v}^i\|_{1,\Omega_i} \leq C\|\tilde{p}_h^i\|_{0,\Omega_i} \ , \tag{4.27}$$

for some $C > 0$ [10]. Let $\tilde{v}_h^i := \Pi_h^i(\tilde{v}_p^i)$ and $\tilde{v}_h|_{\Omega_i} := \tilde{v}_h^i$, where Π_h^i is our interpolation operator restricted to Ω_i. Thus, $\tilde{v}_h$ has the properties

$$\sum_E \int_E \nabla \cdot \tilde{v}_h \, \tilde{p}_h \, dx = \int_\Omega (\tilde{p}_h)^2 \, dx \ , \tag{4.28}$$

$$\|\tilde{v}_h\|_{1,\Omega_s} \le C \|\tilde{p}_h\|_0 \ , \tag{4.29}$$

$$\int_{e_\Gamma} q \, \tilde{v}_h^i \cdot \eta \, dS = 0 \qquad \forall q \in \mathbb{P}_{k-1}(e_\Gamma) \ , \tag{4.30}$$

$$\int_{e_s} q \cdot [\tilde{v}_h^s] \, dS = 0 \qquad \forall q \in (\mathbb{P}_{k-1}(e_s))^N \ . \tag{4.31}$$

Moreover, these properties imply that

$$b(\tilde{v}_h, \tilde{p}_h) = \|\tilde{p}_h\|_0^2 \ \text{ and } b(\tilde{v}_h, \bar{p}_h) = 0. \tag{4.32}$$

Define $\bar{v} := |\Omega_s| \bar{p}_h^s \rho$, where $\rho \in C^2(\Omega)^N$ has compact support and $\int_\Gamma \rho \cdot \eta_s = 1$. Then by (4.25), for any $\bar{q}_h$ such that $\bar{q}_h^i = \bar{q}_h|_{\Omega_i}$ is constant for each $i \in \{s, d\}$,

$$\int_\Omega \nabla \cdot \bar{v} \bar{q}_h \, dx = \int_\Gamma |\Omega_s| \hat{p}_h^s \rho \cdot \eta_s \, (\bar{q}_h^s - \bar{q}_h^d) \, dS = \int_\Omega \bar{p}_h \bar{q}_h \, dx \ . \tag{4.33}$$

Let $\Pi_{h,k}$ be the interpolation operator into our discontinous polynomial space of degree k as defined in the previous section. Define a new interpolation operator $\hat{\Pi}$ by $\hat{\Pi} v|_{\Omega_d} = \Pi_{h,k} v|_{\Omega_d}$ and $\hat{\Pi} v|_{\Omega_s} = \Pi_{h,1} v|_{\Omega_s}$, so that $\hat{\Pi}$ is equal to Π_h when $k = 1$, but is altered if $k = 2$. This unusual definition is needed later in the proof. Thus, if we set $\bar{v}_h := \hat{\Pi} \bar{v}$, then $\bar{v}_h|_{E_s} \in \mathbb{P}_1(E_s)^N$, $\bar{v}_h|_{E_d} \in \mathbb{P}_k(E_d)^N$, and we still have that

$$\int_{e_\Gamma} (\bar{v}_h - \bar{v})|_{\Omega_i} \cdot \eta \, dS = 0 \ , \tag{4.34}$$

for $i \in \{s, d\}$, by the definitions of the interpolation operators on each side. However, the analogous equations for the higher order moments,

$$\int_{e_\Gamma} q (\bar{v}_h - \bar{v})_{\Omega_d} \cdot \eta \, dS = 0 \ , \qquad \forall q \in \mathbb{P}_{k-1}(e_\Gamma) \ , \tag{4.35}$$

hold only from the Darcy side. The definition of $\bar{v}$ ensures that $b(\bar{v}, \bar{p}_h) = \|\bar{p}_h\|_{0,\Omega}^2$. Since $\bar{p}_h$ is piecewise constant, the properties of the interpolation operator $\hat{\Pi}_h$ imply that $b(\bar{v}_h - \bar{v}, \bar{p}_h) = 0$. In conclusion, we have

$$b(\bar{v}_h, \bar{p}_h) = \|\bar{p}_h\|_0^2 \ . \tag{4.36}$$

Set $v_h := \delta\tilde{v}_h + \bar{v}_h$ for some $\delta > 0$ to be determined later. From (4.32) and (4.36), we can easily see that

$$b(v_h, p_h) = \delta\|\tilde{p}_h\|^2 + \|\bar{p}_h\|_{0,\Omega}^2 + b(\bar{v}_h - \bar{v}, \tilde{p}_h) + b(\bar{v}, \tilde{p}_h) \ . \tag{4.37}$$

While the final two terms do not simplify, we can bound them. Begin by expanding the third term of (4.37) as

$$\begin{aligned} b(\bar{v}_h - \bar{v}, \tilde{p}_h) = & \sum_E (\nabla \cdot (\bar{v}_h - \bar{v}), \tilde{p}_h)_E - \sum_{e_s} \langle \{\tilde{p}_h\}, [\![\bar{v}_h - \bar{v}]\!] \rangle_{e_s} \\ & - \sum_{e_\Gamma} \langle \tilde{p}_{h,d}, [\![\bar{v}_h - \bar{v}]\!] \rangle_{e_\Gamma} \ . \end{aligned}$$

Since $\hat{\Pi}$ is equal to the Raviart-Thomas interpolation operator onto $\mathbb{P}_k$ on the Darcy domain and $\tilde{p}_h|_{E_d} \in \mathbb{P}_{k-1}(E_d)$ we have that

$$\sum_{E_d} (\nabla \cdot (\bar{v}_h - \bar{v}), \tilde{p}_h)_{E_d} = 0$$

for the Darcy elements. Let $K_i := |\rho|_{i,\Omega}$. By Holder's inequality, properties of $\hat{\Pi}$, and the definition of $\bar{v}$, we can see that

$$\begin{aligned} \left| \sum_{E_s} (\nabla \cdot (\bar{v}_h - \bar{v}), \tilde{p}_h)_{E_s} \right| & \leq \|\tilde{p}_h\|_{0,\Omega} \, ||| \, \nabla \cdot (\bar{v}_h - \bar{v}) |||_{0,\Omega_s} \\ & \leq \|\tilde{p}_h\|_{0,\Omega} \sqrt{2} |\bar{v}_h - \bar{v}|_{1,\Omega_s} \\ & \leq \|\tilde{p}_h\|_{0,\Omega} Ch |\bar{v}|_2 \\ & \leq ChK_2 |\Omega_s|^{1/2} \|\tilde{p}_h\|_{0,\Omega} \|\bar{p}_h^s\|_{0,\Omega_s} \ , \end{aligned}$$

where in this case $|v|_{1,\Omega_s}$ represents the broken H^1 semi-norm on Ω_s.

By Remark 4.3 pertaining to Lemma 4.2 in [37] and the properties of $\hat{\Pi}$, we see

$$\begin{aligned}\left|\sum_{e_s}\langle\{\tilde{p}_h\},[\![\bar{v}_h-\bar{v}]\!]\rangle_{e_s}\right| &\le C\|\tilde{p}_h\|_{0,\Omega}\,|||\,\nabla(\bar{v}_h-\bar{v})|||_{0,\Omega_s}\\ &\le ChK_2|\Omega_s|^{1/2}\|\tilde{p}_h\|_{0,\Omega}\|\bar{p}_h^s\|_{0,\Omega_s}\ ,\end{aligned}$$

and since we have (4.35), then it similarily follows that

$$\begin{aligned}\left|\sum_{e_\Gamma}\langle\tilde{p}_h,[\![\bar{v}_h-\bar{v}]\!]\rangle_{e_\Gamma}\right| = \left|\sum_{e_\Gamma}\langle\tilde{p}_h,(\bar{v}_h-\bar{v})_s\cdot\eta_s\rangle_{e_\Gamma}\right| &\le C\|\tilde{p}_h\|_{0,\Omega}\,|||\,\nabla(\bar{v}_h-\bar{v})|||_{0,\Omega_s}\\ &\le ChK_2|\Omega_s|^{1/2}\|\tilde{p}_h\|_{0,\Omega}\|\bar{p}_h^s\|_{0,\Omega_s}\ .\end{aligned}$$

In summary, we have shown that

$$|b(\bar{v}_h-\bar{v},\tilde{p}_h)|\le ChK_2|\Omega_s|^{1/2}\|\tilde{p}_h\|_{0,\Omega}\|\bar{p}_h^s\|_{0,\Omega_s}\ .$$

Finally, since $\bar{v}$ is continuous,

$$\begin{aligned}|b(\bar{v},\tilde{p}_h)| = \left|\sum_E\nabla\cdot\bar{v}\,\tilde{p}_h\right| &\le \sqrt{2}\|\tilde{p}_h\|_{0,\Omega}^2|\bar{v}|_{1,\Omega}\\ &\le \sqrt{2}K_1|\Omega_s|^{1/2}\|\tilde{p}_h\|_{0,\Omega}^2\|\bar{p}_h^s\|_{0,\Omega_s}\ .\end{aligned}$$

Putting it all together, we have

$$\begin{aligned}b(v_h,p_h)\ge \delta\|\tilde{p}_h\|_{0,\Omega}^2+\|\bar{p}_h\|_{0,\Omega}^2-\sqrt{2}K_1|\Omega_s|^{1/2}\|\tilde{p}_h\|_{0,\Omega}\|\bar{p}_h^s\|_{0,\Omega_s}\\ -ChK_2|\Omega_s|^{1/2}\|\tilde{p}_h\|_{0,\Omega}\|\bar{p}_h^s\|_{0,\Omega_s}\ .\end{aligned}$$

If we choose $\delta=2K_1^2|\Omega_s|+C^2h^2K_2^2|\Omega_s|+\frac{1}{2}$, then it follows that

$$b(v_h,p_h)\ge\frac{1}{2}\|p_h\|_{0,\Omega}^2\ .$$

□

4.5 Error Estimates

Fix the polynomial degree $k \in \{1, 2\}$. We now set out to prove the following main theorem.

Theorem 4.5.1. *Let* (u_h, p_h) *and* (u, p) *solve* (4.6)–(4.7) *and* (2.1)–(2.9) *respectively. Then*

$$|||u-u_h|||_{1,E_s}+\|u-u_h\|_{0,\Omega_d} \leq Ch^k \left(|u|_{k+1,\Omega_s} + +|p|_{k,\Omega_s} + h^{-1/2}(|p|_{k,\Omega_d} + |u|_{k,\Omega_d})\right) .$$

Because of the interface terms, we have a suboptimal convergence rate. In order to prove this theorem we need the following three lemmas. We begin by recalling the following well-known lemma [5].

Lemma 4.5.2. *The following estimates hold:*

$$\begin{aligned} &\|v\|_{0,e}^2 \leq C(h_e^{-1}\|v\|_{0,E}^2 + h_e|v|_{1,E}^2) && \forall v \in H^1(E)^N \\ &\|\frac{\partial v}{\partial \eta}\|_{0,e}^2 \leq C(h_e^{-1}|v|_{1,E}^2 + h_e|v|_{2,E}^2) && \forall v \in H^2(E)^N \end{aligned}$$

where $h_e = \mathrm{diam}(e)$*.*

We use this lemma to prove the following useful result.

Lemma 4.5.3. *Let* (u_h, p_h) *and* (u, p) *solve* (4.6)–(4.7) *and* (2.1)–(2.9) *respectively. Let* $u_I = \Pi_h(u)$ *and* $p_I = r_h(p)$*. Then*

$$\begin{aligned} &\sum_{E_s} \|D(u_h - u_I)\|_{0,E_s}^2 + \sum_{E_d} \|u_h - u_I\|_{0,E_d}^2 + \sum_{e_s} \frac{1}{h_{e_s}} \|[u_h - u_I]\|_{0,e_s}^2 \\ &\quad + \sum_{e_\Gamma} \|(u_h - u_I)_s \cdot \tau\|_{0,e_\Gamma}^2 + \sum_{e_\Gamma} \|[\![u_h - u_I]\!]\|_{0,e_\Gamma}^2 \\ &\leq Ch^{2k} \left(|u|_{k+1,\Omega_s}^2 + |p|_{k,\Omega_s}^2 + h^{-1}(|p|_{k,\Omega_d}^2 + |u|_{k,\Omega_d}^2)\right) , \end{aligned} \tag{4.38}$$

for some $C \geq 0$*.*

Proof. As (u_h, p_h) and (u, p) both satisfy (4.6)–(4.7), we obtain easily that

$$a(u - u_h, v_h) - b(v_h, p - p_h) = 0 \qquad \forall v \in V_h \ , \tag{4.39}$$

$$b(u - u_h, w_h) = 0 \qquad \forall w_h \in W_h \ . \tag{4.40}$$

Letting $v_h = u_h - u_I$ in (4.39) and adding the result to $a(u_h - u_I, u_h - u_I)$ we obtain

$$a(u_h - u_I, u_h - u_I) = a(u - u_I, u_h - u_I) - b(u_h - u_I, p - p_h) \ . \tag{4.41}$$

Let $w_h = p_h - p_I$ in (4.40) and combine the result with the fact that, by properties (4.11), (4.12), and (4.14) of the interpolation operator,

$$b(u - u_I, p_h - p_I) = 0$$

to obtain

$$b(u_h - u_I, p_h - p_I) = 0 \ . \tag{4.42}$$

Together, (4.41) and (4.42) yield

$$a(u_h - u_I, u_h - u_I) = a(u - u_I, u_h - u_I) - b(u_h - u_I, p - p_I) \ . \tag{4.43}$$

Note that we can bound the left-hand side of (4.43) from below by

$$\begin{aligned} a(u_h - u_I, u_h - u_I) \geq & \sum_{E_s} 2\mu \|D(u_h - u_I)\|^2_{0,E_s} + \sum_{E_d} \mu C_K \|u_h - u_I\|^2_{0,E_d} \\ & + \sum_{e_s} \frac{1}{h_{e_s}} \|[u_h - u_I]\|^2_{0,e_s} + \sum_{e_\Gamma} \hat{\alpha} \|(u_h - u_I)_s \cdot \tau\|^2_{0,e_\Gamma} \\ & + \sum_{e_\Gamma} \|[\![u_h - u_I]\!]\|^2_{0,e_\Gamma} \ , \end{aligned} \tag{4.44}$$

for some constants $C_K, \hat{\alpha} > 0$. On the right-hand side of (4.43), we have

$$\begin{aligned}
&a(u-u_I, u_h-u_I) - b(u_h-u_I, p-p_I) \\
&\quad = \sum_{E_s} \left(2\mu D(u-u_I), D(u_h-u_I)\right)_{E_s} + \sum_{E_d} \left(\mu K^{-1}(u-u_I), u_h-u_I\right)_{E_d} \\
&\quad + \sum_{e_s} \frac{1}{h_{e_s}} \langle [u-u_I], [u_h-u_I] \rangle_{e_s} + \sum_{e_\Gamma} \langle [\![u-u_I]\!], [\![u_h-u_I]\!] \rangle_{e_\Gamma} \\
&\quad + \sum_{e_s} \langle 2\mu\{D(u_h-u_I)\}\eta, [u-u_I] \rangle_{e_s} - \sum_{e_s} \langle 2\mu\{D(u-u_I)\}\eta, [u_h-u_I] \rangle_{e_s} \\
&\quad + \sum_{e_\Gamma} \langle \alpha K^{-1/2}(u-u_I)_s \cdot \tau, (u_h-u_I)_s \cdot \tau \rangle_{e_\Gamma} - \sum_{E} \left(\nabla \cdot (u_h-u_I), p-p_I\right)_E \\
&\quad + \sum_{e_s} \langle \{p-p_I\}, [u_h-u_I] \rangle_{e_s} + \sum_{e_\Gamma} \langle (p-p_I)_d, [\![u_h-u_I]\!] \rangle_{e_\Gamma} \ . \qquad (4.45)
\end{aligned}$$

We proceed to bound each term individually, using the Cauchy-Schwartz inequality, Young's inequality, and estimate (4.15). The fifth term and the eigth term of (4.45) are actually zero by (4.12) and (4.8) respectively. Now

$$\sum_{E_s} \left(2\mu D(u-u_I), D(u_h-u_I)\right)_{E_s} \leq Ch^{2k} |u|^2_{k+1,\Omega_s} + \epsilon \sum_{E_s} \|D(u_h-u_I)\|^2_{0,E_s}$$

and, using estimate (4.16),

$$\sum_{E_d} \left(\mu K^{-1}(u-u_I), u_h-u_I\right)_{E_d} \leq Ch^{2k} |u|^2_{k,\Omega_d} + \epsilon \sum_{E_d} \|u_h-u_I\|^2_{0,E_d} \ .$$

Next we bound the Stokes edge terms. Applying Lemma 4.5 and (4.15) to the third term, we obtain

$$\sum_{e_s} \frac{1}{h_{e_s}} \langle [u-u_I], [u_h-u_I] \rangle_{e_s} \leq Ch^{2k} |u|^2_{k+1,\Omega_s} + \epsilon \sum_{e_s} \frac{1}{h_{e_s}} \|[u_h-u_I]\|^2_{0,e_s} \ .$$

Similarily for the sixth term, we apply Lemma 4.5 and (4.15) to see that

$$\sum_{e_s} \langle 2\mu\{D(u-u_I)\}\eta, [u_h-u_I] \rangle_{e_s} \leq Ch^{2k} |u|^2_{k+1,\Omega_s} + \epsilon \sum_{e_s} \frac{1}{h_{e_s}} \|[u_h-u_I]\|^2_{0,e_s} \ .$$

Finally, the ninth term is bounded using Lemma 4.5 and (4.9) so that

$$\sum_{e_s} \langle \{p - p_I\}, [u_h - u_I] \rangle_{e_s} \le C h^{2k} |p|^2_{k,\Omega_s} + \epsilon \sum_{e_s} \frac{1}{h_{e_s}} \| [u_h - u_I] \|^2_{0,e_s} \; .$$

In bounding the three remaining terms, the interface terms, similar arguments to those above imply

$$\begin{aligned} &\sum_{e_\Gamma} \langle \alpha K^{-1/2} (u - u_I)_s \cdot \tau, (u_h - u_I)_s \cdot \tau \rangle_{e_\Gamma} \\ &\qquad \le C h^{2k+1} |u|^2_{k+1,\Omega_s} + \epsilon \sum \| u_h - u_I \cdot \tau \|^2_{0,e_\Gamma} \; , \\ &\sum_{e_\Gamma} \langle (p - p_I)_d, [\![u_h - u_I]\!] \rangle_{e_\Gamma} \\ &\qquad \le C h^{2k-1} |p|^2_{k,\Omega_d} + \epsilon \sum_{e_\Gamma} \| [\![u_h - u_I]\!] \|^2_{e_\Gamma} \; , \end{aligned}$$

but the fourth term requires a little more care. First expand the term into Stokes and Darcy components. By the construction of Π_h and the properties obtained thereby, namely (4.14)–(4.16), the following bound is obtained

$$\begin{aligned} &\sum_{e_\Gamma} \langle [\![u - u_I]\!], [\![u_h - u_I]\!] \rangle_{e_\Gamma} \\ &\qquad \le \sum_{e_\Gamma} \| [\![u - u_I]\!] \|^2_{0,e_\Gamma} + \epsilon \sum_{e_\Gamma} \| [\![u_h - u_I]\!] \|^2_{0,e_\Gamma} \\ &\qquad \le C \left(h^{2k+1} |u|^2_{k+1,\Omega_s} + h^{2k-1} |u|^2_{k,\Omega_d} \right) + \epsilon \sum_{e_\Gamma} \| [\![u_h - u_I]\!] \|^2_{0,e_\Gamma} \; . \end{aligned}$$

Together, the above bounds imply the desired result (4.38). □

To complete the error estimates we need one more lemma (similar to Lemma 6.2 in [37]).

Lemma 4.5.4. *For all $v \in H^1(\mathcal{E}_{h,s})$,*

$$\|v\|_{0,\Omega_s} \le C \left(\left(\sum_{E_s} \| D v \|^2_{0,E_s} \right)^{1/2} + \left(\sum_{e_s} \frac{1}{h_{e_s}} \| [v] \|^2_{0,e_s} \right)^{1/2} \right) \; .$$

Proof. Take $v \in (H^1(\mathcal{E}_{h,s}))^N$ and let $\psi \in (H_0^1(\Omega_s))^N$ be defined by

$$\int_{\Omega_s} D\psi Dw \, dx = \sum_{E_s} \int_{E_s} DvDw \, dx \qquad \forall w \in (H_0^1(\Omega_s))^N \ .$$

Then clearly $\|D\psi\|_0 \leq \big(\sum_{E_s} \|Dv\|_{0,E_s}^2\big)^{1/2}$ and by Korn's inequality on Ω_s

$$\|\psi\|_{1,s} \leq C\bigg(\sum_{E_s} \|Dv\|_{0,E_s}^2\bigg)^{1/2} \ . \tag{4.46}$$

Write

$$\|\psi - v\|_0 = \sup_{g \in (L^2(\Omega_s))^N} \frac{1}{\|g\|_0} \int_{\Omega_s} (\psi - v) \cdot g \, dx \ . \tag{4.47}$$

Fix $g \in (L^2(\Omega_s))^N$ and let ϕ solve

$$\begin{aligned} -\nabla \cdot D\phi = g \qquad & \text{in } \Omega_s \ , \\ \phi = 0 \qquad & \text{on } \partial\Omega_s \ , \end{aligned} \tag{4.48}$$

so that ϕ is in $(H^2(\Omega_s) \cap H_0^1(\Omega_s))^N$,

$$\|\phi\|_2 \leq C\|g\|_0 \ ,$$

(see [18]) and

$$\begin{aligned} \int_{\Omega_s} -(\psi - v) \cdot g \, dx &= \int_{\Omega_s} -(\psi - v)\nabla \cdot D\phi \, dx \\ &= \sum_{E_s} \int_{E_s} D\phi\nabla(\psi - v) \, dx - \sum_{E_s} \int_{\partial E_s} D\phi\eta \cdot (\psi - v) \, dS \ . \end{aligned}$$

Note that since $(D\phi, \nabla v) = (D\phi, Dv)$, by the regularity of ϕ, and the definition of ψ, we have

$$\begin{aligned} \int_{\Omega_s} (\psi - v) \cdot g \, dx &= -\sum_{E_s} \int_{\partial E_s} D\phi\eta \cdot (\psi - v) \, dS \\ &= \sum_{e_s} \int_{e_s} D\phi\eta \cdot [v] \, dS \ . \end{aligned}$$

Thus,

$$\begin{aligned}
\left|\int_{\Omega_s} (\psi - v) \cdot g \, dx\right| &= \left|\sum_{e_s} \int_{e_s} D\phi \eta [v] \, dS\right| \\
&\leq C \left(|\phi|_{1,\Omega_s} + h|\phi|_{2,\Omega_s}\right) \left(\sum_{e_s} \frac{1}{h_{e_s}} \|[v]\|^2_{0,e_s}\right)^{1/2} \\
&\leq \hat{C}\|g\|_0 \left(\sum_{e_s} \frac{1}{h_{e_s}} \|[v]\|^2_{0,e_s}\right)^{1/2} .
\end{aligned}$$

By (4.47), this implies that

$$\|\psi - v\|_0 \leq \hat{C} \left(\sum_{e_s} \frac{1}{h_{e_s}} \|[v]\|^2_{0,e_s}\right)^{1/2} ,$$

so that finally, by Poincare's inequality and (4.46),

$$\begin{aligned}
\|v\|_{0,\Omega_s} &\leq \|\psi\|_0 + \|\psi - v\|_0 \\
&\leq C\left(\|\nabla\psi\|_0 + \left(\sum_{e_s} \frac{1}{h_{e_s}} \|[v]\|^2_{0,e_s}\right)^{1/2}\right) \\
&\leq C\left(\left(\sum_{E_s} \|Dv\|^2_{0,E_s}\right)^{1/2} + \left(\sum_{e_s} \frac{1}{h_{e_s}} \|[v]\|^2_{0,e_s}\right)^{1/2}\right) .
\end{aligned}$$

□

Proof of Theorem 4.5. Applying Lemma 4.5.4 to $u_h - u_I$, we see that

$$\|u_h - u_I\|_{0,\Omega_s} \leq C\left(\left(\sum_{E_s} \|D(u_h - u_I)\|^2_{0,E_s}\right)^{1/2} + \left(\sum_{e_s} \frac{1}{h_{e_s}} \|[u_h - u_I]\|^2\right)^{1/2}\right) . \tag{4.49}$$

This result, in combination with Korn's inequality applied element by element, and estimate (4.38) gives

$$
\begin{aligned}
||| \, u_h - u_I \, |||_{1,\Omega_s}^2 + \|u_h - u_I\|_{0,\Omega_d}^2 + \sum_{e_s} \frac{1}{h_{e_s}} \|[u_h - u_I]\|_{0,e_s}^2 \\
+ \sum_{e_\Gamma} \hat{\alpha} \|(u_h - u_I)_s \cdot \tau\|_{0,e_\Gamma}^2 + \sum_{e_\Gamma} \|[\![u_h - u_I]\!]\|_{0,e_\Gamma}^2 \\
\leq C h^{2k} \left(|u|_{k+1,\Omega_s}^2 + h^{-1} |u|_{k,\Omega_d}^2 + |p|_{k,\Omega_s}^2 + h^{-1} |p|_{k,\Omega_d}^2 \right) \ .
\end{aligned}
\tag{4.50}
$$

Finally, by (4.50), the triangle inequality, and the error estimates for Π_h, we have Theorem 4.5. □

4.6 A Second Non-Conforming Method

Our second method differs from the first only in how the normal jump across the interface is handled. We again have the space V_h and W_h as in (4.1)–(4.3), and introduce an additional Lagrange multiplier space Λ_h on Γ defined by

$$
\Lambda_h(\Gamma) := \{ \lambda \in L^2(\Gamma) : \lambda|_{e_\Gamma} \in \mathbb{P}_k(e_\Gamma), \ \forall e_\Gamma \} \ .
$$

Similarly to our first method, we define the following bilinear forms on $V_h \times V_h$ and $V_h \times W_h$, respectively, as

$$
\begin{aligned}
a(u, v) := \sum_{E_s} (2\mu D u, D v)_{E_s} + \sum_{E_d} \left(\mu K^{-1} u, v \right)_{E_d} \\
+ \sum_{e_\Gamma} \langle \alpha K^{-1/2} u_s \cdot \tau, v_s \cdot \tau \rangle_{e_\Gamma} + \sum_{e_s} \frac{1}{h_{e_s}} \langle [u], [v] \rangle_{e_s} \\
- \sum_{e_s} \langle 2\mu \{D u\} \eta, [v] \rangle_{e_s} + \sum_{e_s} \langle 2\mu \{D v\} \eta, [u] \rangle_{e_s} \ ,
\end{aligned}
\tag{4.51}
$$

$$
b(v, p) := \sum_{E} (\nabla \cdot v, p)_E - \sum_{e_s} \langle \{p\}, [\![v]\!] \rangle_{e_s} \ . \tag{4.52}
$$

In addition, we define the bilinear form c on $V_h \times \Lambda_h$ to be

$$
c(v, \mu) := \sum_{e_\Gamma} - \langle \mu, [\![v]\!] \rangle_{e_\Gamma} \ . \tag{4.53}
$$

Our problem is then to find $(u_h, p_h, \lambda_h) \in V_h \times W_h \times \Lambda_h$ solving

$$a(u_h, v_h) - b(v_h, p_h) - c(v_h, \lambda_h) = F(v_h) \ , \tag{4.54}$$

$$b(u_h, w_h) = Q(w_h) \ , \tag{4.55}$$

$$c(u_h, \mu_h) = 0 \ , \tag{4.56}$$

for all $v_h \in V_h$, $w_h \in W_h$, and $\mu_h \in \Lambda_h$. Again for $(u, p, p_d) \in V \times W \times \Lambda$, $\Lambda = L^2(\Gamma)$, and test functions $(v, w, \mu) \in V \times W \times \Lambda$, the above finite-dimensional formulation reduces to the variational form of our original problem obtained in §2.3.

Existence and uniqueness of a solution to (4.54)–(4.56) follows in much the same way as that of the first method. Letting $F = Q = 0$, we obtain

$$Du_h = 0 \qquad \text{in } \Omega_s \ , \tag{4.57}$$

$$u_h = 0 \qquad \text{in } \Omega_d \ , \tag{4.58}$$

$$[u_h] = 0 \qquad \text{on any internal face in } \Omega_s \ , \tag{4.59}$$

$$u_{h,s} \cdot \tau = 0 \qquad \text{on } \Gamma \ , \tag{4.60}$$

$$\int_{e_\Gamma} [\![u_h]\!] \, dS = 0 \qquad \text{for any interface edge } e_\Gamma \ . \tag{4.61}$$

As in §4.4, we can see that $u_h \in V_h$, Corollary 4.4.1, and the above equalities imply that $u_h = 0$. Additionally, $p_h = 0$ follows again from a slight variation of the proof of Lemma 4.4. Finally, since $c(v_h, \lambda_h) = 0$ for all $v_h \in V_h$, we can chose v_h so that $[\![v_h]\!] = \lambda_h$ along Γ. Thus, $\lambda_h = 0$ as well.

The following error estimates now hold.

Theorem 4.6.1. *Let* (u_h, p_h, λ_h) *and* (u, p) *solve* (4.54)–(4.56) *and* (2.1)–(2.9) *respectively. Then*

$$|||u - u_h\,|||_{1,E_s} + \|u - u_h\|_{0,\Omega_d} \leq Ch^k \left(|u|_{k+1,\Omega_s} + |u|_{k,\Omega_d} + |p|_k \right) \ . \tag{4.62}$$

Proof. Let $\lambda := p_d|_\Gamma$. Since (u_h, p_h, λ_h) and (u, p, λ) solve (4.54)–(4.56), following the same steps as in the proof of Theorem 4.5 we have

$$a(u_h - u_I, u_h - u_I) = a(u - u_I, u_h - u_I) - b(u_h - u_I, p - p_I) - c(u_h - u_I, \lambda - \lambda_I) \ ,$$

where λ_I is L^2 projection onto Λ_h. Moreover,

$$c(u_h - u_I, \lambda - \lambda_I) = 0 \ .$$

Recall that the loss of one-half power in the convergence rate of the first method was due entirely to the presence of the interface terms. The final error estimate (4.6.1) can be obtained following exactly the steps used for our first algorithm in §4.5, as we now have

$$a(u_h - u_I, u_h - u_I) = a(u - u_I, u_h - u_I) - b(u_h - u_I, p - p_I),$$

where a and b differ from those of the first method only in the absence of the interface terms. □

Chapter 5

Conclusions and Future Work

In summary, we model the micro-scale behavior of fluid flow in vuggy porous media using Darcy's Law in the porous matrix, Stoke's equation of motion in the vugs, and the Beavers-Joseph-Saffman slip condition along the interface between the two. Two-Scale homogenization of our system, assuming a periodic representation of our medium, results in a Darcy Law macro-scale governing equation with a modified permeability tensor. The modified permeability tensor can be computed by solving the original Darcy-Stokes system over a representative cell for the periodic tiling of the domain, and is symmetric and positive definite.

In order to compute this modified permeability tensor, and therefore, numerically approximate our macro-scale governing equations, we develop a numerical scheme. Because of the need to allow for tangential discontinuity in the velocity along the Darcy-Stokes interface, we choose a Discontinuous Galerkin method developed by Wheeler, Girault, and Rivere [37] in the vug regions, coupled with a low-order Raviart-Thomas element in the porous regions. Our first method includes a term which penalizes the normal jump across the interface, and is suboptimal. In our second method, we instead introduce a Lagrange multiplier along the interface which allows us to approximate the pressure along an interface edge to the same polynomial degree as the velocity. This results in our second method being optimal, although it is potentially computationally expensive.

Possible future directions for this work include the addition of time-dependence and contaminant transport to our micro-scale model and an investigation into how to effectively upscale the resulting system. Part of the difficulty of including time dependence is the operation of two greatly different time scales – that of the flow in the vugs, and that of the flow in the rock. Considering the flow in an averaged sense may no longer be relevant. Particularly upon the addition of contaminant transport to the model, as our means of upscaling the system would likely need to be sensitive to whether or not the contaminant particles are flowing through vugs or the porous medium. We may be able to look to work done in transport in fractured media for inspiration on how to effectively handle these issues.

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Vita

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