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ON S_1 -STRICTLY SINGULAR OPERATORS

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ON S_1 -STRICTLY SINGULAR OPERATORS

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Dedication

This thesis is dedicated to the person who has been by my side for the last 16 years. I wish to express my gratitude for all the support and love that she has showed throughout this period. But, above all, I am very glad that together we have constructed a great family, especially after the birth of our daughter, Gisele, which is my best “result” so far.

My wife Daniele, this work is dedicated to you, with all the love and admiration that one person can have.

ON S_1 -STRICTLY SINGULAR OPERATORS

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The University of Texas at Austin, 2010

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Let X be a Banach space and denote by $SS_1(X)$ the set of all S_1 -strictly singular operators $T : X \longrightarrow X$. We prove that there is a Banach space X such that $SS_1(X)$ is not a closed ideal. More specifically, we construct a space X and operators $T_1, T_2 \in SS_1(X)$ such that $T_1 + T_2 \notin SS_1(X)$. We show one example where the space X is reflexive and other where it is c_0 -saturated. We also develop some results about S_α -strictly singular operators for $0 \leq \alpha < \omega_1$.

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Chapter 1

Introduction

An important class of operators between Banach spaces are the strictly singular ones.

Definition 1.1. An operator $T : X \longrightarrow Y$, from a Banach space X to a Banach space Y , is called *strictly singular* (or *SS*), and we write $T \in SS(X, Y)$, if for every infinite dimensional subspace $Z \subset X$, its restriction $T|_Z$ is not an isomorphism into Y .

In other words, an operator T is SS if and only if for any subspace M of X the existence of a $C > 0$ such that $\|Tx\| \geq C\|x\|$ for all $x \in M$ implies that $\dim M < \infty$.

From now on, we assume that the spaces X and Y are real Banach spaces, and if X is a subspace of Y , we use $X \subseteq Y$ to mean that X is a closed linear subspace of Y .

Androulakis et al [ADST] quantified SS operators using the Schreier classes of sets $(S_\alpha)_{\alpha < \omega_1}$ (see preliminaries below, section 2.2):

Definition 1.2. Let $\alpha < \omega_1$. An operator $T : X \longrightarrow Y$ is called S_α -strictly singular

(or S_α -SS), and we write $T \in SS_\alpha(X, Y)$, if for every $\varepsilon > 0$ and for every basic sequence $(x_i)_{i=1}^\infty$, there is $F \in S_\alpha$ and $x \in \text{span}(x_i)_{i \in F}$ with $\|x\|_X = 1$ and $\|Tx\|_Y < \varepsilon$. If $X = Y$, then we write $SS_1(X, X) = SS_1(X)$.

In 1958, Kato [K] proved the following Theorem.

Proposition 1.3. *The strictly singular operators on a space X form a closed operator ideal. That is, if $T_1, T_2 : X \longrightarrow X$ are SS and $U : X \longrightarrow X$ is a bounded operator then:*

1. $T_1 \circ U$ and $U \circ T_1$ are SS.
2. $T_1 + T_2$ is SS.
3. If $(T_i)_{i=1}^\infty \subset SS(X)$ and $T_i \xrightarrow{\|\cdot\|} T$ as $i \rightarrow \infty$ then $T \in SS(X)$.

A natural question that arises is *What can be said if we replace “SS” by “ S_α -SS” on the previous result?* It is easy to see that property 3 would be still true. We shall show in Proposition 2.43 that property 1 remains valid, but property 2 is violated, in the case of S_1 -SS, by:

Theorem 1.4 (Main Theorem). *There exists a Banach space X such that the set of all S_1 -SS operators from X to X does not form an ideal. In particular, there exist $T_1, T_2 \in SS_1(X)$ so that $T_1 + T_2 \notin SS_1(X)$. Moreover, X may be chosen to be reflexive.*

Remark 1.5. have $K(X, Y) \subseteq SS_1(X, Y) \subseteq SS(X, Y)$, where $K(X, Y)$ is the set of compact operators.

The first inclusion is elementary and follows from Remark 2.36. The second inclusion is just an immediate consequence of the definitions and the fact that every infinite dimensional Banach space contains a basic sequence.

Remark 1.6. There are some Banach spaces X such that $SS_1(X)$ forms an ideal. For example, if $X = \ell_p$, $1 \leq p < \infty$, then $SS_\alpha(X) = K(X)$, the set of all compact operators on X , for all $1 \leq \alpha < \omega_1$.

In fact, we have that $SS(\ell_p) = K(\ell_p)$. This is a consequence of the fact that in ℓ_p , $1 \leq p < \infty$, any normalized basic sequence $(x_i)_{i=1}^\infty$ has a subsequence that is equivalent to the unit vector basis of ℓ_p . So, if $T : \ell_p \longrightarrow \ell_p$ is not compact, by Proposition 2.31, there is a normalized basic sequence $(x_i)_{i=1}^\infty$ such that $(Tx_i)_{i=1}^\infty$ is a seminormalized basic sequence. By passing to subsequences, we may assume that both $(x_i)_{i=1}^\infty$ and $(Tx_i)_{i=1}^\infty$ are equivalent to the unit vector basis of ℓ_p , hence T is not strictly singular.

Another space X for which $K(X) = SS(X)$ is Tsirelson's space T . The proof is similar to that of the case of ℓ_p using the fact that if $(x_i)_{i=1}^\infty$ and $(y_i)_{i=1}^\infty$ are normalized block bases of the unit vector basis $(t_i)_{i=1}^\infty$ for T and for some sequence $n_1 < n_2 < \dots$ of integers, $x_i, y_i \in \text{span}(t_j)_{j=n_i}^{n_{i+1}-1}$ for all $i \in \mathbb{N}$, then $(x_i)_{i=1}^\infty$ and $(y_i)_{i=1}^\infty$ are equivalent bases (see [CS]).

Chapter 2

Preliminaries and Further Observations

In this section, we present basic background material, and some additional observations.

We use conventional Banach space terminology. The *unit ball* of a Banach space X is denoted by $B_X = \{x \in X : \|x\|_X \leq 1\}$. The *unit sphere* of X is $S_X = \{x \in X : \|x\|_X = 1\}$.

Let c_{00} be the space of sequences of real numbers with finite support. For $x = (x_i)_{i=1}^\infty \subset \mathbb{R}$, then $x \in c_{00}$ if and only if there is $m \in \mathbb{N}$, depending on x , such that $x_n = 0$ for all $n \geq m$.

2.1 Bases

Proofs not given here can be found in [LT], unless another reference is given.

Definition 2.1. Let X be a Banach space. A sequence $(e_i)_{i=1}^\infty$ in X is called a *Schauder basis* of X if for every $x \in X$ there is a unique sequence of scalars $(a_i)_{i=1}^\infty$, called *the coordinates of x* , such that $x = \sum_{i=1}^\infty a_i e_i$.

We will use simply *basis* to mean *Schauder basis*.

Definition 2.2. A sequence $(e_i)_{i=1}^\infty$ in X is called a *basic sequence* if it is a basis of $\overline{\text{span}}(e_i)_{i=1}^\infty$.

Definition 2.3. Let $(e_i)_{i=1}^\infty$ be a basic sequence in X and let $(f_i)_{i=1}^\infty$ be a basic sequence in Y . Let $C \geq 1$. We say that $(e_i)_{i=1}^\infty$ is *C-equivalent* to $(f_i)_{i=1}^\infty$, which we denote by $(e_i)_{i=1}^\infty \stackrel{\mathcal{C}}{\sim} (f_i)_{i=1}^\infty$, if for all $n \in \mathbb{N}$ and scalars $a_1, \dots, a_n$, we have:

$$\frac{1}{C} \left\| \sum_{i=1}^n a_i e_i \right\|_X \leq \left\| \sum_{i=1}^n a_i f_i \right\|_Y \leq C \left\| \sum_{i=1}^n a_i e_i \right\|_X.$$

We say that $(e_i)_{i=1}^\infty$ is *equivalent* to $(f_i)_{i=1}^\infty$, denoted by $(e_i)_{i=1}^\infty \sim (f_i)_{i=1}^\infty$, if there is $C \geq 1$ such that $(e_i)_{i=1}^\infty$ is *C-equivalent* to $(f_i)_{i=1}^\infty$.

Definition 2.4. Let $\|\cdot\|_A$ and $\|\cdot\|_B$ be norms on a Banach space X . Let $C \geq 1$. We say that $\|\cdot\|_A$ is *C-equivalent* to $\|\cdot\|_B$, if for all $x \in X$, we have:

$$\frac{1}{C} \|x\|_A \leq \|x\|_B \leq C \|x\|_A.$$

Notation 2.5. If $(e_i)_{i=1}^\infty$ is a basis of X , the *canonical projections* $P_n : X \longrightarrow X$ are defined for $n \in \mathbb{N}$ by $P_n \left(\sum_{i=1}^\infty a_i e_i \right) = \sum_{i=1}^n a_i e_i$.

If A is a finite subset of $\mathbb{N}$, then let $P_A \left(\sum_{i=1}^\infty a_i e_i \right) = \sum_{i \in A} a_i e_i$.

Proposition 2.6. Let $(e_i)_{i=1}^\infty$ be a basis of X , then the canonical projections P_n are uniformly bounded, that is, there is a number $K \geq 1$ such that $\sup_n \|P_n\| = K < \infty$.

The value $K = \sup_n \|P_n\|$ is called the *basis constant* of $(e_i)_{i=1}^\infty$. If $K = 1$, then the basis is said to be *monotone*.

Definition 2.7. Let X be a Banach space with basis $(e_i)_{i=1}^\infty$ and X^* be its dual space. The family $(e_i^*)_{i=1}^\infty$, called the corresponding *sequence of biorthogonal functionals to the basis* $(e_i)_{i=1}^\infty$, is defined for every $i \in \mathbb{N}$ by $e_j^* (\sum_{i=1}^\infty a_i e_i) = a_j$.

By Proposition 2.6, it is easily seen that $(e_i^*)_{i=1}^\infty \subseteq X^*$, i.e., each e_i^* is continuous. Moreover, $(e_i^*)_{i=1}^\infty$ is a basic sequence in X^* .

Definition 2.8. Let $(e_i)_{i=1}^\infty$ be a basis of X , and $(e_i^*)_{i=1}^\infty$ be the corresponding sequence of biorthogonal functionals. The basis $(e_i)_{i=1}^\infty$ is called *shrinking* if $\overline{\text{span}}(e_i^*)_{i=1}^\infty = X^*$, i.e., $(e_i^*)_{i=1}^\infty$ is a basis for X^* .

Definition 2.9. Let $(e_i)_{i=1}^\infty$ be a basic sequence in X . A sequence $(f_i)_{i=1}^\infty \subset X$ is said to be a *perturbation* of $(e_i)_{i=1}^\infty$ if $\sum_{i=1}^\infty \|e_i - f_i\| \|e_i^*\| < 1$.

In this case, $(f_i)_{i=1}^\infty$ is a basic sequence which is equivalent to $(e_i)_{i=1}^\infty$.

Definition 2.10. Let $(e_i)_{i=1}^\infty$ be a basic sequence in X . A sequence of nonzero vectors $(u_j)_{j=1}^\infty$ in X of the form $u_j = \sum_{i=p_j+1}^{p_{j+1}} a_i e_i$ with scalars $a_i \in \mathbb{R}$ for all $i \in \mathbb{N}$ and integers $0 \leq p_1 < p_2 < \dots$ is called a *block basic sequence* of $(e_i)_{i=1}^\infty$. The subspace of X generate by a block basic sequence is called *block subspace*. Furthermore, $(u_j)_{j=1}^\infty$ is always basic with basis constant not exceeding that of $(e_i)_{i=1}^\infty$.

We have some equivalent statements for shrinking bases:

Proposition 2.11. Let $(e_i)_{i=1}^\infty$ be a basis for X and let $(e_i^*)_{i=1}^\infty$ be the corresponding sequence of biorthogonal functionals. Then, the following are equivalent:

1. $(e_i)_{i=1}^\infty$ is shrinking;

2. $(e_i^*)_{i=1}^\infty$ is a Schauder basis of X^* ;
3. For every $f \in X^*$, $\lim_{n \rightarrow \infty} \|f|_{\overline{\text{span}}(e_i)_{i>n}}\| = 0$.
4. Every bounded block basic sequence of $(e_i)_{i=1}^\infty$ is weakly null.

Definition 2.12. Let $(e_i)_{i=1}^\infty$ be a basis for X . The basis is called *boundedly complete* if whenever the scalars $(a_i)_{i=1}^\infty$ are such that $\sup_n \|\sum_{i=1}^n a_i e_i\| < \infty$, then $\sum_{i=1}^\infty a_i e_i$ converges.

There is an equivalence for boundedly complete basis:

Proposition 2.13. Let $(e_i)_{i=1}^\infty$ be a basis for X , then the following are equivalent:

1. $(e_i)_{i=1}^\infty$ is not boundedly complete;
2. there is a seminormalized block basic sequence $(x_j)_{j=1}^\infty$ of $(e_i)_{i=1}^\infty$ such that

$$\sup_n \left\| \sum_{j=1}^n x_j \right\|_X < \infty.$$

Proof. Suppose that there is a sequence of scalars $(a_i)_{i=1}^\infty$ such that $\sup_n \|\sum_{i=1}^n a_i e_i\| = M < \infty$ but $\sum_{i=1}^\infty a_i e_i$ does not converge. Then there is $\varepsilon > 0$ such that for every $n_0 \in \mathbb{N}$ there are $n, m \in \mathbb{N}$ with $m > n \geq n_0$ such that $\|\sum_{i=n}^m a_i e_i\|_X \geq \varepsilon$. Thus, we can find a sequence $n_1 < m_1 < n_2 < m_2 < \dots$, where for all $j \in \mathbb{N}$ we have $\|\sum_{i=n_j}^{m_j} a_i e_i\|_X \geq \varepsilon$. Let $x_1 = \sum_{i=1}^{m_1} a_i e_i$ and for $j \geq 2$ let $x_j = \sum_{i=m_{j-1}+1}^{m_j} a_i e_i$. Then $\|x_1\|_X \leq M$, and for $j \geq 2$ we have $\|x_j\|_X \leq 2M$. On the other hand, $\|x_j\|_X \geq \frac{\varepsilon}{1+K}$, where K is the basis constant of $(e_i)_{i=1}^\infty$. Clearly, $(x_j)_{j=1}^\infty$ is a block basic sequence.

The reverse implication is trivial □

James [J1] proved the following characterization of reflexive Banach spaces:

Proposition 2.14. *Let X be a Banach space with a basis $(e_i)_{i=1}^\infty$. X is reflexive if and only if $(e_i)_{i=1}^\infty$ is both shrinking and boundedly complete.*

Definition 2.15. Let $(e_i)_{i=1}^\infty$ be a basis for X . If there is a constant $K \geq 1$ such that for $n \in \mathbb{N}$ and every sequence $(a_i)_{i=1}^n$ and for every choice of signs $(\theta_i)_{i=1}^n$, we have:

$$\left\| \sum_{i=1}^n \theta_i a_i e_i \right\|_X \leq K \left\| \sum_{i=1}^n a_i e_i \right\|_X,$$

then we say that $(e_i)_{i=1}^\infty$ is K -unconditional. We define the unconditional constant K_u for such bases to be the smallest K that satisfies the above inequality.

It can be shown that a basis $(e_i)_{i=1}^\infty$ is unconditional if and only if for every permutation π of $\mathbb{N}$, $(e_{\pi(i)})_{i=1}^\infty$ is a basis of X .

A natural question is whether a given Banach space X contains an isomorphic copy of some classical spaces, like ℓ_1 and c_0 . James [J2] proved the following for spaces with an unconditional basis:

Theorem 2.16. *Let X be a separable Banach space with an unconditional basis $(e_i)_{i=1}^\infty$, then:*

1. *if $(e_i)_{i=1}^\infty$ is not boundedly complete then X contains an isomorphic copy of c_0 ;*
2. *if $(e_i)_{i=1}^\infty$ is not shrinking then X contains an isomorphic copy of ℓ_1 .*

Definition 2.17. Let $(e_i)_{i=1}^\infty$ be a basis for X . If there is $K \geq 1$ such that for every sequence of scalars $(a_i)_{i=1}^\infty$ and for every $n \in \mathbb{N}$, if $F \subset \{1, \dots, n\}$, then:

$$\left\| \sum_{i \in F} a_i e_i \right\|_X \leq K \left\| \sum_{i=1}^n a_i e_i \right\|_X,$$

then we say that $(e_i)_{i=1}^\infty$ is *suppression- K unconditional*. We define the suppression unconditional constant K_{su} for such bases to be the smallest K that satisfies the above inequality.

A basis $(e_i)_{i=1}^\infty$ is unconditional if and only if it is suppression- K unconditional, for some $K > 0$. In fact $K_{su} \leq K_u \leq 2K_{su}$.

An example of a basis that is not unconditional is the *summing basis* $(s_i)_{i=1}^\infty$. This is defined, for every $n \in \mathbb{N}$ and scalars $a_1, \dots, a_n$, by:

$$\left\| \sum_{i=1}^n a_i s_i \right\| = \sup_{1 \leq m \leq p \leq n} \left| \sum_{i=m}^p a_i \right|.$$

It is not unconditional since $\|\sum_{i=1}^n s_i\| = n$, yet $\|\sum_{i=1}^n (-1)^i s_i\| = 1$. The summing basis is equivalent to the basis $(e_1, e_1 + e_2, e_1 + e_2 + e_3, \dots)$ for c_0 , where $(e_i)_{i=1}^\infty$ is its unit vector basis.

Definition 2.18. A basis $(e_i)_{i=1}^\infty$ of a Banach space X is said to be *symmetric*, if for any permutation σ of the positive integers, the sequence $(e_{\sigma(i)})_{i=1}^\infty$ is a basic sequence which is equivalent to $(e_i)_{i=1}^\infty$.

We note that every symmetric basis $(e_i)_{i=1}^\infty$ is unconditional, as a basic sequence is unconditional if and only if for each permutation σ of the positive integers $(e_{\sigma(i)})_{i=1}^\infty$ is a basic sequence.

Let $(e_i)_{i=1}^\infty$ be a symmetric basis of a Banach space X . Then for each permutation π of the positive integers, the operator $V_\pi : X \longrightarrow X$ defined by $V_\pi(\sum_{i=1}^\infty a_i e_i) = \sum_{i=1}^\infty a_i e_{\pi(i)}$ is evidently an automorphism of X . By the uniform boundedness princi-

ple, the family (V_π) , where π ranges over all permutations of the integers, is a bounded group.

For every choice of signs $\theta = (\theta_i)_{i=1}^\infty$, where $\theta_i \in \{-1, 1\}$ for all $i \in \mathbb{N}$, we define the operator $M_\theta : X \longrightarrow X$ by $M_\theta(\sum_{i=1}^\infty a_i e_i) = \sum_{i=1}^\infty \theta_i a_i e_i$. We also define the number $K_s = \sup_{\theta, \pi} \|M_\theta V_\pi\| < \infty$, which is called *symmetric constant* of $(e_i)_{i=1}^\infty$.

As examples of spaces with symmetric basis, we can cite ℓ_p for $p \geq 1$, c_0 and Lorentz sequence spaces (which will be important to us) defined on section 2.4.

Definition 2.19. Let $(y_i)_{i=1}^\infty$ be a normalized basic sequence in a Banach space X . A basic sequence $(\tilde{y}_i)_{i=1}^\infty$, possibly in another Banach space, is called a *spreading model* of $(y_i)_{i=1}^\infty$ if for some sequence $\varepsilon_i \searrow 0$ and all $n \in \mathbb{N}$, $(a_i)_{i=1}^n \subset [-1, 1]$ we have:

$$\left| \left\| \sum_{i=1}^n a_i y_{k_i} \right\| - \left\| \sum_{i=1}^n a_i \tilde{y}_i \right\| \right| \leq \varepsilon_n. \quad (2.1)$$

whenever $n \leq k_1 < \dots < k_n$.

It is well known that, using Ramsey theory (see [AT]), one can show that for every normalized basic sequence $(y_i)_{i=1}^\infty$ in a Banach space X and for every sequence $\varepsilon_i \searrow 0$, there exists a subsequence $(x_i)_{i=1}^\infty$ of $(y_i)_{i=1}^\infty$ with a spreading model $(\tilde{x}_i)_{i=1}^\infty$, satisfying (2.1).

The sequence $(\tilde{x}_i)_{i=1}^\infty$ is a suppression-1 unconditional basic sequence if $(y_i)_{i=1}^\infty$ is weakly null (see [BL]). This, in conjunction with Rosenthal's ℓ_1 Theorem (Theorem 2.20), yields that every separable infinite dimensional Banach space X admits a suppression-1 unconditional spreading model $(\tilde{x}_i)_{i=1}^\infty$. In fact, one can always find a

1-unconditional spreading model (see [R2]).

We use Rosenthal's dichotomy Theorem several times within this thesis. It was introduced in [R1], and it can be stated as:

Theorem 2.20. *Let X be a Banach space and let $(x_i)_{i=1}^\infty$ be a bounded sequence in X . Then $(x_i)_{i=1}^\infty$ has a subsequence $(x'_i)_{i=1}^\infty$ that satisfies one of the following (mutually exclusive) choices:*

1. $(x'_i)_{i=1}^\infty$ is a basic sequence, and is equivalent to the unit vector basis of ℓ_1 ;
2. $(x'_i)_{i=1}^\infty$ is weak-Cauchy, i.e., $\lim_{i \rightarrow \infty} x^*(x_i)$ exists for all $x^* \in X^*$.

Corollary 2.21. *If $(x_i)_{i=1}^\infty$ is a bounded sequence in X , then a subsequence $(\bar{x}_i)_{i=1}^\infty$ of $(x_i)_{i=1}^\infty$ satisfies either:*

1. $(\bar{x}_{2i} - \bar{x}_{2i+1})_{i=1}^\infty$ is seminormalized basic, or,
2. $\|\bar{x}_{2i} - \bar{x}_{2i+1}\|_X \rightarrow 0$, as $i \rightarrow \infty$.

Proof. Let $(x'_i)_{i=1}^\infty$ be the subsequence given by Rosenthal's Theorem. Let $z_i = x'_{2i} - x'_{2i+1}$. If $(x'_i)_{i=1}^\infty$ is equivalent to the ℓ_1 basis, then $(z_i)_{i=1}^\infty$ is seminormalized and also equivalent to the ℓ_1 basis. Now, assume that $(x'_i)_{i=1}^\infty$ is weak-Cauchy. Passing to a subsequence we either have $\|z_i\|_X \rightarrow 0$ or $(z_i)_{i=1}^\infty$ is seminormalized weakly null. In this case, it admits a further subsequence which is basic. $\square$

Other Corollaries are:

Corollary 2.22. *Let X be any weakly sequentially complete Banach space. Then, either X is reflexive or X contains a subspace isomorphic to ℓ_1 .*

Corollary 2.23. *Let X be an infinite-dimensional Banach space for which the weak and norm sequential convergence coincide (i.e., X is a Schur space). Then every closed infinite-dimensional subspace Y of X contains an isomorphic copy of ℓ_1 .*

Proposition 2.24. *Let X be a Banach space that does not contain ℓ_1 with a normalized unconditional basis $(e_i)_{i=1}^\infty$ and let $\varepsilon_i \searrow 0$. If $(x_i)_{i=1}^\infty$ is a normalized basic sequence in X , then, there is a subsequence $(y_i)_{i=1}^\infty$ of $(x_i)_{i=1}^\infty$ such that one of the following must hold:*

1. $(y_i)_{i=1}^\infty$ is a perturbation of a block basic sequence, or,
2. there are sequences $(\tilde{y}_i)_{i=1}^\infty$ and $(\hat{y}_i)_{i=1}^\infty$ in X such that:

$$\|y_i - (\tilde{y}_i + \hat{y}_i)\|_X < \varepsilon_i, \text{ for all } i \in \mathbb{N},$$

where $(\tilde{y}_i)_{i=1}^\infty$ is equivalent to the summing basis and $(\hat{y}_i)_{i=1}^\infty$ is either a block basic sequence or a null sequence.

Proof. Let $\varepsilon_i \searrow 0$ be given. By Rosenthal's Theorem, as $(x_i)_{i=1}^\infty$ is normalized and X does not contain ℓ_1 , we can extract a weak-Cauchy subsequence which, for simplicity, will also be called $(x_i)_{i=1}^\infty$. Let $e_j^*(x_i) \rightarrow \alpha_j$ as $i \rightarrow \infty$, for all $j \in \mathbb{N}$. Then $(x_i)_{i=1}^\infty$ converges weak* in X^{**} to $\alpha = w^* - \sum_{j=1}^\infty \alpha_j e_j$. Hence we have three possibilities:

1. $\alpha = 0$
2. $\alpha \in X \setminus \{0\}$,
3. $\alpha \in X^{**} \setminus X$.

Assume the first case. By the Bessaga-Pelczyński principle [LT], there is a subsequence which is a perturbation of a block basic sequence.

Now assume the second case. It implies that $(x_i)_{i=1}^\infty$ is weakly convergent to a nonzero α , which is not possible since $(x_i)_{i=1}^\infty$ is basic.

Let, now, $\alpha \in X^{**} \setminus X$ and define $\alpha(n) = \sum_{i=1}^n \alpha_i e_i$ for all $n \in \mathbb{N}$. Since $\alpha \notin X$, there is $\varepsilon > 0$ such that for all $k \in \mathbb{N}$ there are $m > n > k$ with $\|\alpha(m) - \alpha(n)\| \geq \varepsilon$. In particular, we can find a sequence $(n_i)_{i=1}^\infty$ such that if $\beta_i = \alpha(n_{i+1}) - \alpha(n_i)$, $(\beta_i)_{i=1}^\infty$ is a seminormalized block sequence of the unconditional basis $(e_i)_{i=1}^\infty$. The partial sums $(\sum_{i=1}^n \beta_i)_{n=1}^\infty$ are bounded, thus $(\beta_i)_{i=1}^\infty$ is equivalent to the unit vector basis of c_0 . We conclude that $(\alpha(n_i))_{i=1}^\infty$ is equivalent to the summing basis.

We now pass to a subsequence of $(x_i)_{i=1}^\infty$, labeled $(y_i)_{i=1}^\infty$, also choosing n_{j_i} 's here, so that given a suitable $\delta_i \searrow 0$, $\|P_{n_{j_i}}(y_i) - \alpha(n_{j_i})\| < \delta_i$, for all $i \in \mathbb{N}$ and if $\tilde{y}_i = P_{n_{j_i}}(y_i)$ and $\hat{y}_i = y_i - \tilde{y}_i$, then $(\hat{y}_i)_{i=1}^\infty$ is a perturbation of a block basis. Since $(\tilde{y}_i)_{i=1}^\infty$ is a perturbation of $(\alpha(n_{j_i}))_{i=1}^\infty$, the result follows. $\square$

2.2 Schreier Sets

The Schreier sets S_α were introduced by Alspach and Odell in [AO] for $\alpha < \omega$ and by Alspach and Argyros in [AA] for $\omega \leq \alpha \leq \omega_1$.

Let us start by setting notation.

If A and B are two finite subsets of $\mathbb{N}$, write $A < B$ if $\max A < \min B$. For $n \in \mathbb{N}$ and $A \subset \mathbb{N}$, write $n < A$ if $\{n\} < A$. By convention, let $\emptyset < A$ and $A < \emptyset$ for all finite subsets A of $\mathbb{N}$, where $\emptyset$ is the empty set.

Let C be an infinite subset of $\mathbb{N}$, and $k \in \mathbb{N}$. We denote by $[C]^k$ the set of all k -tuples $(c_i)_{i=1}^k$ such that for each $1 \leq i \leq k$, $c_i \in C$ and $c_1 < \cdots < c_k$. Also, we denote by $[C]^{<\omega}$ the union $\bigcup_{k=1}^{\infty} [C]^k$. We also define the set $[C]^\omega = \{(c_i)_{i=1}^{\infty} : c_i \in C \text{ and } c_i < c_{i+1} \text{ for all } i \in \mathbb{N}\}$.

Definition 2.25. Let $\mathcal{F}$ be a family of finite subsets of $\mathbb{N}$.

1. The family $\mathcal{F}$ is said to be *compact* if $\mathcal{F}$ is a compact subset of $2^{\mathbb{N}}$, given the product topology of the discrete topology on $\mathbb{N}$.
2. The family $\mathcal{F}$ is said to be *hereditary* if for every $F \in \mathcal{F}$ and every $G \subset F$ we have that $G \in \mathcal{F}$.
3. The family $\mathcal{F}$ is said to be *spreading* if for every $F = \{n_1 < \cdots < n_k\} \in \mathcal{F}$ and every $G = \{m_1 < \cdots < m_k\}$ with $n_i \leq m_i$, for all $i \in \{1, \dots, k\}$ we have that $G \in \mathcal{F}$.
4. The family $\mathcal{F}$ is said to be *regular* if $\mathcal{F}$ is compact, hereditary, spreading and contains singletons $\{n\}$ for all $n \in \mathbb{N}$.

If $\mathcal{F}$ and $\mathcal{G}$ are two regular families of sets, then let

$$\mathcal{F}[\mathcal{G}] = \left\{ \bigcup_{i=1}^n G_i : n \in \mathbb{N}, G_1 < \cdots < G_n, G_i \in \mathcal{G} \text{ for } i \leq n, (\min G_i)_{i=1}^n \in \mathcal{F} \right\}.$$

If $N = \{n_1, n_2, \dots\}$ is an increasing sequence in $\mathbb{N}$ and $\mathcal{F}$ is a set of finite subsets of $\mathbb{N}$, then we define $\mathcal{F}(N) = \{(n_i)_{i \in F} : F \in \mathcal{F}\}$.

For any ordinal number $0 \leq \alpha < \omega_1$, the Schreier family S_α is collection of finite subsets of $\mathbb{N}$ defined by the following transfinite recursive process:

Let $S_0 = \{\{n\} : n \in \mathbb{N}\} \cup \{\emptyset\}$.

Assuming α is a successor ordinal and S_β has been defined for $\beta + 1 = \alpha$ let,

$$S_\alpha = \left\{ \bigcup_{i=1}^n F_i : n \geq 1, n \leq F_1 < \dots < F_n, \text{ and } F_i \in S_\beta \text{ for } 1 \leq i \leq n \right\}.$$

In particular, $S_1 = \{F \subset \mathbb{N} : |F| \leq \min F\}$. If $\alpha < \omega_1$, $S_{\alpha+1} = S_1[S_\alpha]$. If $\alpha < \omega_1$, let $S_\alpha^{[n,\infty)} = \{F \in S_\alpha : n \leq F\}$. Assume S_β has been defined for all $\beta < \alpha$, where α is a limit ordinal. We fix an increasing sequence $(\beta_n)_{n=1}^\infty$ such that $\lim_{n \rightarrow \infty} \beta_n = \alpha$ and define:

$$S_\alpha = \bigcup_{n=1}^\infty S_{\beta_n}^{[n,\infty)}.$$

Remark 2.26. (see for example [AGR] or [OTW]) Let F and G be finite subsets of $\mathbb{N}$.

1. S_α is *regular* for all $\alpha < \omega_1$;
2. $S_\alpha \subseteq S_{\alpha+1}$ for every $\alpha < \omega_1$. However, $\alpha < \beta$ does not generally imply $S_\alpha \subset S_\beta$;
3. Let $1 \leq \alpha < \beta < \omega_1$. Then there exists $n \in \mathbb{N}$ such that $S_\alpha^{[n,\infty)} \subset S_\beta$.
4. For $\alpha, \beta \in \mathbb{N}$, we have that $S_\alpha[S_\beta] = S_{\beta+\alpha}$. This fails for infinite ordinals.
However, the following is true: for all $1 \leq \alpha, \beta < \omega_1$ there exists subsequences M and N of $\mathbb{N}$ such that $S_\alpha[S_\beta](N) \subset S_{\beta+\alpha}$ and $S_{\beta+\alpha}(M) \subseteq S_\alpha[S_\beta]$.
5. For all $1 \leq \beta < \alpha < \omega_1$, $\varepsilon > 0$ and M a subsequence of $\mathbb{N}$, there exists a finite set $F \subset M$ and $(a_j)_{j \in F} \subset \mathbb{R}^+$ such that $F \in S_\alpha(M)$, $\sum_{j \in F} a_j = 1$ and if $G \subset F$ with $G \in S_\beta$, then $\sum_{j \in G} a_j < \varepsilon$.

Before stating the next result, we set some notation.

Let $M = (m_i)_{i=1}^\infty$ and $N = (n_i)_{i=1}^\infty$ be elements of $[\mathbb{N}]^\omega$. We write $N \preceq M$ if $n_i \leq m_i$ for all $i \in \mathbb{N}$.

If $F \in [\mathbb{N}]^{<\omega}$, then we define the set $\widehat{F} = \{2i : i \in F\} \cup \{2i + 1 : i \in F\}$.

Proposition 2.27. *Let $1 \leq \alpha < \omega_1$ and $N_0 \in [\mathbb{N}]^\omega$. Then there exists $N \in [N_0]^\omega$ so that for all $N \preceq M$, $M = (m_i)_{i=1}^\infty$ and $F \in S_\alpha$ then $(m_i)_{i \in \widehat{F}} \in S_\alpha$.*

Proof. We do induction on α . For $\alpha = 1$, let $N = N_0$. Let $N \preceq M$. If $|F| = k$ then $F \in S_1$ gives $|\widehat{F}| = 2k \leq \min \widehat{F}$, so $\widehat{F} \in S_1$. Since S_1 is spreading, $(m_i)_{i \in \widehat{F}} \in S_1$.

Assume true for α . Choose N by the induction hypothesis for α and let $N \preceq M$, let $F = \bigcup_{i=1}^k F_i$, with $F_1 < \dots < F_k$, $F_i \in S_\alpha$ for all $i \in \mathbb{N}$ and $\min F_1 \geq k$.

Then $\widehat{F} = \bigcup_{i=1}^k \widehat{F}_i$ and $(m_j)_{j \in \widehat{F}_i} \in S_\alpha$ for $i \leq k$, by the induction hypothesis.

We have k sets in S_α starting at $m_{\min \widehat{F}_1} \geq 2k$ so $(m_j)_{j \in \widehat{F}} \in S_{\alpha+1}$.

Finally, let α be a limit ordinal with $\alpha = \lim_{n \rightarrow \infty} \beta_n$, where the sequence $\beta_n \neq \alpha$ defines S_α .

Given N_0 we can choose for each $n \in \mathbb{N}$ a sequence $N_n = (n_i^n)_{i=1}^\infty \in [N_0]^\omega$ satisfying the induction hypothesis for β_n , so that $N_1 \supset N_2 \supset \dots$ and $\min N_n \geq n$ for all $n \in \mathbb{N}$.

Choose a diagonal sequence $\bar{N}$ so that $\bar{N} = (n_i)_{i=1}^\infty$ and for all $j \in \mathbb{N}$ we have $(n_i)_{i=j}^\infty \subset (n_i^j)_{i=j}^\infty$. Let $F \in S_\alpha$ and choose j_0 so that $F \in S_{\beta_{j_0}}^{[j_0, \infty)}$.

Let $\bar{N} \preceq M$, $M = (m_i)$. Then $(m_i)_{i \in \widehat{F}} \in S_{\beta_{j_0}}$ by the induction hypothesis.

So $(m_i)_{i \in \widehat{F}} \in S_\alpha$. □

Observe that the previous result is invalid for S_0 sets, which consists only of the singletons and the empty set.

2.3 Operators between Banach spaces

Let X and Y be Banach spaces. We will denote by $B(X, Y)$ the space of all bounded linear operators between X and Y , equipped with its natural norm. In the case $X = Y$, we simply write $B(X)$ instead of $B(X, X)$.

Definition 2.28. A linear operator $T : X \longrightarrow Y$ is called *compact* if the image $T(B)$ of any bounded set $B \subset X$ is relatively compact in Y . Recall that $K(X, Y)$ denotes the set of all such operators. If $X = Y$, then we denote $K(X, X)$ simply by $K(X)$.

Definition 2.28 easily yields:

Proposition 2.29. $K(X)$ forms a closed ideal in $B(X)$.

Proposition 2.30. *The following are equivalent:*

1. $T : X \longrightarrow Y$ is compact;
2. the image of the unit ball in X under T is relatively compact in Y ;
3. there exists a neighborhood of 0, $U \subset X$, and compact set $V \subset Y$ such that $T(U) \subset V$;
4. for any bounded sequence $(x_i)_{i=1}^{\infty}$ in X , the sequence $(Tx_i)_{i=1}^{\infty}$ contains a convergent subsequence.

Another characterization is given by:

Proposition 2.31. *An operator $T : X \longrightarrow Y$ is noncompact if and only if there exists a normalized basic sequence $(x_i)_{i=1}^{\infty} \in X$ such that its image $(Tx_i)_{i=1}^{\infty}$ is a seminormalized basic sequence in Y .*

In Definition 1.1, we defined strictly singular operators (SS). Now, we state a well known characterization using basic sequences:

Lemma 2.32. *An operator $T : X \longrightarrow Y$ is SS if and only if for every $\varepsilon > 0$ and for every normalized basic sequence $(x_i)_{i=1}^\infty \in X$, there is $x \in \text{span}(x_i)_{i=1}^\infty$ with $\|x\|_X = 1$ and $\|Tx\|_Y < \varepsilon$.*

Let $\alpha < \omega_1$. For S_α -strictly singular operators, we need a simple result to ensure that we can pass to subsequences. That is:

Lemma 2.33. *Let $\alpha < \omega_1$. An operator $T : X \longrightarrow Y$ is S_α -SS if and only if for every $\varepsilon > 0$ and for every basic sequence $(x_i)_{i=1}^\infty$, there is a subsequence $(y_i)_{i=1}^\infty$ of $(x_i)_{i=1}^\infty$, a set $F \in S_\alpha$ and $y \in \text{span}(y_i)_{i \in F}$ with $\|y\|_X = 1$ and $\|Ty\|_Y < \varepsilon$.*

Proof. This follows easily from the fact that if $M \in [\mathbb{N}]^\omega$, $S_\alpha(M) \subset S_\alpha$, since S_α is spreading. $\square$

The class of S_0 -strictly singular operators has some interesting properties. From Definition 1.2, $T \in SS_0(X, Y)$ if and only if for every $\varepsilon > 0$ and every normalized basic sequence $(x_i)_{i=1}^\infty$, $\lim_{n \rightarrow \infty} \|Tx_n\| = 0$.

Proposition 2.34. *We always have the inclusion $SS_0(X, Y) \subset K(X, Y)$. More precisely, if X is reflexive then $SS_0(X, Y) = K(X, Y)$, and if X is not reflexive then $SS_0(X, Y) = \{0\}$.*

Proof. By Proposition 2.31, if T is not compact there would be a normalized basic sequence $(x_i)_{i=1}^\infty$ whose image is seminormalized. Then T would not be SS_0 .

Let X be reflexive and let $(x_i)_{i=1}^\infty$ be normalized basic in X . Then $(x_i)_{i=1}^\infty$ is weakly null as is $(Tx_i)_{i=1}^\infty$. If $\overline{\lim_{i \rightarrow \infty}} \|Tx_i\| > 0$ then there exists a subsequence $(Tx_{i_j})_{j=1}^\infty$ which is seminormalized basic in Y . Hence T is not compact.

Suppose that X is not reflexive and there exists $T \in SS_0(X, Y) \setminus \{0\}$. Assume that $Tx_0 = y_0 \neq 0$. And write $X = \langle x_0 \rangle \oplus X_1$. Since X_1 is not reflexive, by [J1] it must contain a normalized basic sequence $(x_i)_{i=1}^\infty$ that dominates the summing basis. Note that the sequence $(x_1 + x_i)_{i=1}^\infty$ is basic in X , since $(x_i)_{i=1}^\infty$ dominates the summing basis. Since $T \in SS_0(X, Y)$ then $\|Tx_i\|_Y \rightarrow 0$ as $i \rightarrow \infty$, thus $\|T(x_0 + x_i)\|_Y \rightarrow \|y_0\|_Y \neq 0$ as $i \rightarrow \infty$. Thus $T \notin SS_0(X, Y)$. $\square$

Lemma 2.35. *Let $1 \leq \alpha < \omega_1$. The following are equivalent:*

- a. *an operator $T : X \longrightarrow Y$ is S_α -SS,*
- b. *for all basic sequences $(x_i)_{i=1}^\infty \subset X$ there exists a subsequence $(y_i)_{i=1}^\infty$ such that if $z_i = y_{2i} - y_{2i+1}$ then for every $\varepsilon > 0$ there exists $F \in S_\alpha$, $x \in \text{span}(z_i)_{i \in F}$ with $\|x\|_X = 1$ and $\|Tx\|_Y < \varepsilon$.*

Proof. Suppose T is S_α -SS. Let $(x_i)_{i=1}^\infty$ be a basic sequence. If $z_i = x_{2i} - x_{2i+1}$ for all $i \in \mathbb{N}$, then $(z_i)_{i=1}^\infty$ is also a basic sequence. Hence we have the second condition.

Let $(x_i)_{i=1}^\infty$ be a basic sequence. Let $N \subset \mathbb{N}$ be the set given for α by Proposition 2.27. Let $(x_i)_{i \in N} = (w_i)_{i=1}^\infty$. Apply the second condition to get $(y_i)_{i=1}^\infty$ a subsequence of $(w_i)_{i=1}^\infty$, a set $F \in S_\alpha$, $z_i = y_{2i} - y_{2i+1}$ and $x \in \text{span}(z_i)_{i \in F}$ with $\|x\|_X = 1$ and $\|Tx\|_Y < \varepsilon$. Then $x \in \text{span}(x_i)_{i \in \bar{F}}$, where $\bar{F} \in S_\alpha$. $\square$

Remark 2.36. Let $1 \leq \alpha < \omega_1$. If $T \in K(X, Y)$ and $U \in SS_\alpha(X, Y)$ then $T + U \in SS_\alpha(X, Y)$.

Indeed, let $\varepsilon > 0$ and a normalized basic sequence $(x_i)_{i=1}^\infty \subset X$ with basis constant K be given. By Lemma 2.35 we may assume that $\lim_{i \rightarrow \infty} \|Tx_i\| = 0$. Choose a sequence $\varepsilon_i \searrow 0$, with $\sum_{i=1}^\infty \varepsilon_i < \varepsilon/2$. Passing to a subsequence, we may assume

that $\|Tx_i\|_Y < \varepsilon_i/(2K)$ for all $i \in \mathbb{N}$. Let $x = \sum_{i \in F} a_i x_i \in S_X$ with $F \in S_\alpha$ and $\|Ux\|_Y < \varepsilon/2$. Then, using that for each $i \in \mathbb{N}$, $|a_i| \leq 2K$, we have

$$\|(T+U)x\|_Y \leq \left\| T \left(\sum_{i \in F} a_i x_i \right) \right\|_Y + \|Ux\|_Y < 2K \sum_{i \in F} \frac{\varepsilon_i}{2K} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

We also give a characterization for S_1 -SS operators using spreading models.

Lemma 2.37. *An operator $T : X \longrightarrow Y$ is S_1 -SS if and only if for every normalized basic sequence $(x_i)_{i=1}^\infty$ with spreading model $(\tilde{x}_i)_{i=1}^\infty$ such that $Tx_i = y_i$, $\inf \|y_i\|_Y > 0$ and $(y_i)_{i=1}^\infty$ has spreading model $(\tilde{y}_i)_{i=1}^\infty$, then $(\tilde{x}_i)_{i=1}^\infty$ is not equivalent to $(\tilde{y}_i)_{i=1}^\infty$.*

Proof. Assume that T is S_1 -SS and there is a normalized basic sequence $(x_i)_{i=1}^\infty$ with spreading model $(\tilde{x}_i)_{i=1}^\infty$, let $Tx_i = y_i$ with $\inf \|y_i\| > 0$ and let $(\tilde{y}_i)_{i=1}^\infty$ be the assumed spreading model for $(y_i)_{i=1}^\infty$. Let $\varepsilon_k \searrow 0$. Let K be the basis constant of $(x_i)_{i=1}^\infty$. Passing to subsequences, we may assume that for all $k \in \mathbb{N}$:

$$\left| \left\| \sum_{j=1}^k a_j x_{i_j} \right\| - \left\| \sum_{j=1}^k a_j \tilde{x}_j \right\| \right| \leq \varepsilon_k \quad (2.2)$$

and

$$\left| \left\| \sum_{j=1}^k a_j y_{i_j} \right\| - \left\| \sum_{j=1}^k a_j \tilde{y}_j \right\| \right| \leq \varepsilon_k. \quad (2.3)$$

for all scalars $(a_j)_{j=1}^k \subset [-2K, 2K]$ and all $k \leq i_1 < \dots < i_k$.

By Lemma 2.33, for all k , there exists $F \in S_1$ with $\min F \geq k$ and $x = \sum_{i \in F} a_i x_i \in S_X$ with $\|Tx\| < \varepsilon_k$. Thus if $F = \{i_1, \dots, i_n\}$ and $\tilde{x} = \sum_{i=1}^n a_i \tilde{x}_i$, then, by (2.2), $\|x - \tilde{x}\| \leq \varepsilon_k$. So, $\|\tilde{x}\| \geq 1 - \varepsilon_k$. Also, by (2.3), for $\tilde{y} = \sum_{i=1}^n a_i \tilde{y}_i$, $\|\tilde{y}\| \leq 2\varepsilon_k$. Since $\varepsilon_k \searrow 0$, $(\tilde{x}_i)_{i=1}^\infty$ and $(\tilde{y}_i)_{i=1}^\infty$ are not equivalent.

To prove the converse implication, let $(x_i)_{i=1}^\infty$ be a normalized basic sequence and

let $\varepsilon > 0$. Let $Tx_i = y_i$.

We use Corollary 2.21 to obtain, after passing to a subsequence, and relabeling as $(x_i)_{i=1}^\infty$ and $(y_i)_{i=1}^\infty$, that either $(y_{2i} - y_{2i+1})_{i=1}^\infty$ is seminormalized basic or norm null. In latter case, we are done by Proposition 2.27. In former case, passing again to a subsequence, we may assume that $(x_{2i} - x_{2i+1})_{i=1}^\infty$ has a spreading model $(\tilde{v}_i)_{i=1}^\infty$ and $(y_{2i} - y_{2i+1})_{i=1}^\infty$ has a spreading model $(\tilde{w}_i)_{i=1}^\infty$. We may also assume that analogues of inequalities (2.2) and (2.3) hold. Since $(x_{2i} - x_{2i+1})_{i=1}^\infty$ is automatically seminormalized basic, using that $(\tilde{v}_i)_{i=1}^\infty$ is not equivalent to $(\tilde{w}_i)_{i=1}^\infty$ and the inequalities, we conclude T is S_1 -SS, by Lemma 2.35. $\square$

The next two Propositions were proved in [ADST].

Proposition 2.38. *Let $1 \leq \alpha, \beta < \omega_1$, if $T_1 \in SS_\alpha(X, Y)$ and $T_2 \in SS_\beta(X, Y)$ then $T_1 + T_2 \in SS_{\alpha+\beta}(X, Y)$. In particular, the sum of two S_1 -SS operators will always be an S_2 -SS operator.*

Proof. Let $(x_n)_{n=1}^\infty \subset X$ be a normalized basic sequence with basis constant C , and let $\varepsilon > 0$. By Remark 2.26, there exists $N = (n_i)_{i=1}^\infty$ a subsequence of $\mathbb{N}$ such that $S_\beta[S_\alpha](N) \subseteq S_{\alpha+\beta}$. Let $(v_i)_{i=1}^\infty = (x_{n_i})_{i=1}^\infty$, which is also a basic sequence with basis constant at most C .

Choose $F_1 < F_2 < \dots$ in S_α and $w_i \in \text{span}(v_j)_{j \in F_i}$ such that $\|w_i\|_X = 1$ and $\|T_1 w_i\|_Y < \frac{\varepsilon}{2^{i+2}C}$ for all $i \in \mathbb{N}$.

Then $(w_i)_{i=1}^\infty$ is a normalized block basic sequence, with basis constant at most C . Since T_2 is S_β -SS, we can find $G = \{k_1, \dots, k_m\} \in S_\beta$, $z = \sum_{i=1}^m a_i w_{k_i}$ such that $\|z\|_X = 1$ and $\|T_2 z\|_Y < \frac{\varepsilon}{3}$. Write $z = \sum_{i \in H} b_i x_{n_i}$, where $H \in S_\beta[S_\alpha]$.

Now $(n_i)_{i \in H} \in S_{\alpha+\beta}$, since $S_\beta[S_\alpha](N) \subset S_{\alpha+\beta}$. Also,

$$\|T_1 z\|_Y = \left\| T_1 \left(\sum_{i=1}^m a_i w_{k_i} \right) \right\|_Y \leq \sum_{i=1}^m |a_i| \|T_1 w_{k_i}\|_Y \leq 2C \frac{\varepsilon}{4C} \leq \frac{\varepsilon}{2}.$$

We conclude that $\|(T_1 + T_2)z\|_Y \leq \|T_1 z\|_Y + \|T_2 z\|_Y < \varepsilon$. $\square$

Ordinal addition is not commutative, hence this Proposition may be restated to give a better estimate if we write that if $T_1 \in SS_\alpha(X, Y)$ and $T_2 \in SS_\beta(X, Y)$ then $T_1 + T_2 \in SS_\gamma(X, Y)$ where $\gamma = \min\{\alpha + \beta, \beta + \alpha\}$. Thus, if $\alpha = \omega$ and $\beta = \omega^2$, we obtain that $T_1 + T_2 \in SS_{\omega^2}(X, Y)$ since $\min\{\omega + \omega^2, \omega^2 + \omega\} = \omega^2$.

And as a consequence of the proof of Proposition 2.38, we have:

Corollary 2.39. *Let $(x_i)_{i=1}^\infty$ be a normalized basic sequence in X . Let $S \in SS_\alpha(X, Y)$, $T \in SS_\beta(X, Y)$ and $\varepsilon > 0$, then there exists $z = \sum_{i \in F} a_i x_i \in S_X$, where $F \in S_{\alpha+\beta}$, such that both:*

1. $\|Sz\|_Y < \varepsilon$, and
2. $\|Tz\|_Y < \varepsilon$.

Proposition 2.40. *Let X and Y be Banach spaces.*

1. *If $1 \leq \alpha < \beta < \omega_1$ then $SS_\alpha(X, Y) \subseteq SS_\beta(X, Y)$;*
2. *if X and Y are separable, then*

$$\bigcup_{\alpha < \omega_1} SS_\alpha(X, Y) = SS(X, Y).$$

Proof. 1. Indeed, for $1 \leq \alpha < \beta < \omega_1$, by Remark 2.26 there exists $N \in \mathbb{N}$ such that if $S_\alpha^{[N, \infty)} \subset S_\beta$. Now if $T \in SS_\alpha(X, Y)$, $\varepsilon > 0$ and $(x_n)_{n=1}^\infty$ is a normalized

basic sequence in X , then consider the basic sequence $(y_n)_{n=1}^\infty$ where $y_i = x_{N+i}$ for all $i \in \mathbb{N}$. There exists $F = \{m_1, \dots, m_k\} \in S_\alpha$ and $z \in \text{span}(y_i)_{i \in F}$ with $\|z\| = 1$ such that $\|Tz\| \leq \varepsilon$. Since $F \in S_\alpha$, the set $\tilde{F} = \{m_1 + N, \dots, m_k + N\}$ is also S_α . Furthermore, $\tilde{F} \subset \{N, N+1, \dots\}$ and so we have that $\tilde{F} \in S_\beta$. Hence $T \in SS_\beta(X, Y)$.

2. This would take us too far astray to present here. The proof uses descriptive set theory. We refer the interested reader to [ADST]. $\square$

Beanland [B] proved that if X and Y are totally incomparable, that is there is no infinite dimensional subspace of X isomorphic to a subspace of Y , then there is $\alpha < \omega_1$ such that $SS_\alpha(X, Y) = SS(X, Y)$.

Suppose we have an S_1 -SS operator $T : X \longrightarrow Y$. We can construct an S_1 -SS operator $\bar{T}$ whose domain and co-domain are the same, by the following Lemma. In $X \oplus Y$ we use the norm $\|(x, y)\|_{X \oplus Y} = \|x\|_X \vee \|y\|_Y$, i.e., we consider $(X \oplus Y)_\infty$. Hence to prove the main Theorem, it suffices to show that there are two Banach spaces X and Y such that $SS_1(X, Y)$ is not closed under sums.

Lemma 2.41. *Let $0 \leq \alpha < \omega_1$. If $T : X \longrightarrow Y$ is S_α -SS, then, if we define $\bar{T} : X \oplus Y \longrightarrow X \oplus Y$ by $\bar{T}(x, y) = (0, Tx)$, we have that $\bar{T}$ is S_α -SS.*

Proof. First, assume $\alpha \geq 1$. Let $z_i = (x_i, y_i)_{i=1}^\infty$ be a normalized basic sequence in $(X \oplus Y)_\infty$ and $\varepsilon > 0$. By Corollary 2.21 and Lemma 2.35, we can assume that $(x_i)_{i=1}^\infty$ is seminormalized basic in X . The result then follows easily. Choose $F \in S_\alpha$, and $x \in \text{span}(x_i)_{i \in F}$ with $\|x\| = 1$ and $\|Tx\| < \varepsilon$. If $x = \sum_{i \in F} a_i x_i$, we let $z = \sum_{i \in F} a_i (x_i, y_i)$, and $\bar{z} = \frac{z}{\|z\|}$. Then, $\|\bar{T}\bar{z}\| < \frac{\varepsilon}{\|z\|}$. Since $\|z\| \geq 1$, we are done.

If $\alpha = 0$, the result follows easily from Proposition 2.34. $\square$

It is easy to see that if $S : X \longrightarrow Y$ is not S_α -SS, then $\bar{S} : X \oplus Y \longrightarrow X \oplus Y$ is not S_α -SS.

Hence:

Corollary 2.42. *Let $0 \leq \alpha < \omega_1$. Then $S : X \longrightarrow Y$ is S_α -SS if and only if $\bar{S} : X \oplus Y \longrightarrow X \oplus Y$ is S_α -SS.*

Proposition 1.3 yields that for any Banach spaces X and Y , the set $SS(X, Y)$ has three important properties that make it a closed ideal over the $B(X, Y)$, namely: (1) closure with respect to composition with bounded linear operators, (2) closure with respect to sum and (3) norm closure. It is easy to see that for any $\alpha < \omega_1$, the set of S_α -strictly singular operators is a closed subset of $B(X, Y)$. Our main Theorem (Theorem 1.4) states that the set of S_1 -SS operators fails to have the second property. Our next result shows that the set of S_α -SS operators have the first property for every $0 \leq \alpha < \omega_1$.

Proposition 2.43. *Let $0 \leq \alpha < \omega_1$. Let X , Y and Z be Banach spaces. If $T : X \longrightarrow Y$ is S_α -SS and $U_1 : Z \longrightarrow X$ and $U_2 : Y \longrightarrow Z$ are bounded linear operators, then both $T \circ U_1$ and $U_2 \circ T$ are S_α -SS.*

Proof. First, let us show that $U_2 \circ T$ is S_α -SS. Let $\varepsilon > 0$ and a normalized basic sequence $(x_i)_{i=1}^\infty \subset X$ be given. As T is S_α -SS, there is a set $F \in S_\alpha$ and an element $x \in \text{span}(x_i)_{i \in F}$ with $\|x\| = 1$ such that $\|Tx\| < \frac{\varepsilon}{\|U_2\|}$. Hence $\|(U_2 \circ T)(x)\| \leq \|U_2\| \cdot \|Tx\| < \varepsilon$.

Now, to prove that $T \circ U_1$ is S_α -SS, let $\varepsilon > 0$ and a normalized basic sequence $(x_i)_{i=1}^\infty \subset X$ be given. By Corollary 2.21, we may assume that $(U_1(x_{2i} - x_{2i+1}))_{i=1}^\infty$ is seminormalized basic or norm null. In either case, the result follows easily, using Lemma 2.35, when $\alpha \geq 1$. If $\alpha = 0$, the result follows from Proposition 2.34. $\square$

Definition 2.44. A Banach space X is said to have the *approximation property*, if, for every compact set $K \subset X$ and every $\varepsilon > 0$, there is an operator $T : X \longrightarrow X$ of finite rank (bounded operator such that its range is finite dimensional) so that $\|Tx - x\| \leq \varepsilon$, for every $x \in K$.

The general problem of identifying the closed ideals in $B(X)$, the Banach algebra of bounded linear operators on X , is extremely difficult and the answer is only known for a very few spaces such as ℓ_p , for $1 \leq p < \infty$, where the only nontrivial ideal is $K(\ell_p)$, by the sliding hump argument. The question whether $SS_1(X)$ is such was posed to us by Dr. Thomas Schlumprecht. We should also notice that if X is a space with a basis, or more generally with the approximation property, then any nonzero closed ideal in $B(X)$ must contain $K(X)$. Indeed, let I be a closed ideal of X , with $I \neq \{0\}$. Then it can be shown that I must contain all the finite rank operators, and since X is closed, it can be shown to contain the compact operators as well.

2.4 Lorentz sequence spaces

Let $1 \leq p < \infty$, and a sequence $\omega = (\omega_1, \omega_2, \dots) \in c_0$, with $1 = \omega_1 \geq \omega_2 \geq \dots > 0$ and $\sum_{i=1}^{\infty} \omega_i = \infty$. We define the space $d(\omega, p) = \{x = (x_1, x_2, \dots) \in c_0 : \sup_{\sigma \in \pi} \sum_{i=1}^{\infty} |x_{\sigma(i)}|^p \omega_i < \infty\}$ where π is the set of all permutations of $\mathbb{N}$. Then $d(\omega, p)$ with the norm $\|x\| = \left(\sup_{\sigma \in \pi} \sum_{i=1}^{\infty} |x_{\sigma(i)}|^p \omega_i\right)^{1/p}$ is a Banach space, called a *Lorentz sequence space*.

For any $x = (x_1, x_2, \dots) \in d(\omega, p)$, let $x^\circ = (x_{(1)}, x_{(2)}, \dots)$ where $(x_{(i)})_{i=1}^{\infty}$ is an enumeration of the nonzero elements of $(|x_i|)_{i=1}^{\infty}$ such that $|x_{(1)}| \geq |x_{(2)}| \geq \dots$. Then it can be proved that $\|x\| = \left(\sum_{i=1}^{\infty} |x_{(i)}|^p \omega_i\right)^{1/p}$.

The canonical unit vector basis $(\alpha_i)_{i=1}^{\infty}$ of $d(\omega, p)$ is clearly 1-symmetric, and

Garling [G1] proved that if $1 < p < \infty$ then $d(\omega, p)$ is reflexive.

Lemma 2.45. [ACL] *If $(\alpha_i)_{i=1}^\infty$ is the canonical unit vector basis of $d(\omega, p)$ then for all normalized block basic sequences $(\beta_i)_{i=1}^\infty$ of $(\alpha_i)_{i=1}^\infty$ we have:*

$$\left\| \sum_{i=1}^{\infty} a_i \beta_i \right\|_{d(\omega, p)} \leq \left(\sum_{i=1}^{\infty} |a_i|^p \right)^{1/p}.$$

for all sequence of scalars $(a_i)_{i=1}^\infty$.

Definition 2.46. Let $\omega_1 = (\omega_1^i)_{i=1}^\infty$ and $\omega_2 = (\omega_2^i)_{i=1}^\infty$ be sequences of nonnegative numbers in $c_0 \setminus \ell_1$, with $1 = \omega_j^1 \geq \omega_j^2 \geq \dots > 0$. Let $W_1 = d(\omega_1, p)$ and $W_2 = d(\omega_2, p)$ be the two correspondent Lorentz sequence spaces. For each $n \in \mathbb{N}$, let $\lambda(n) = \omega_1^1 + \dots + \omega_1^n$ and $\mu(n) = \omega_2^1 + \dots + \omega_2^n$.

The sequences ω_1 and ω_2 are called the *weights* of Lorentz sequence spaces W_1 and W_2 , respectively.

For $C \geq 1$, we say that W_1 has *C-submultiplicative weight* if:

$$\lambda(nm) \leq C\lambda(n)\lambda(m).$$

for all $n, m \in \mathbb{N}$.

We say that W_1 and W_2 have *incomparable weights* if:

$$\inf_n \frac{\lambda(n)}{\mu(n)} = 0 \quad \text{and} \quad \sup_n \frac{\lambda(n)}{\mu(n)} = \infty.$$

Proposition 2.47. [ACL]. *A Lorentz sequence space $d(\omega, p)$ contains exactly two nonequivalent symmetric basic sequences, one equivalent to the unit vector basis of ℓ_p and other equivalent to the unit vector basis of $d(\omega, p)$, if and only if it has a C-submultiplicative weight, for some $C > 0$. Moreover, if we have C-submultiplicative*

weight the following is true:

Given a normalized block basic sequence $(x_i)_{i=1}^\infty$, let $(x_{(i)})_{i=1}^\infty$ be the sequence of decreasing rearrangements of the $|x_i|$'s. Suppose a subsequence $(x_{(i_j)})_{j=1}^\infty$ converges pointwise to an element α , which must lie in $B_{d(\omega,p)}$. Then

1. if $\|\alpha\| = 1$ then a subsequence of $(x_{i_j})_{j=1}^\infty$ is equivalent to the unit vector basis of $d(\omega, p)$;
2. if $\|\alpha\| < 1$ then a subsequence of $(x_{i_j})_{j=1}^\infty$ is equivalent to the unit vector basis of ℓ_p .

The “moreover” statement, which will prove to be quite useful to us, follows from the proof in [ACL].

Remark 2.48. Let $C > 0$, $1 < p < \infty$ and $X = d(\omega, p)$ be a Lorentz sequence space having a C -submultiplicative weight. Then $SS_1(X) = SS(X) \neq K(X)$.

Indeed, let $T : X \longrightarrow X$ be strictly singular. Let $(x_i)_{i=1}^\infty$ be a normalized basic sequence. Since X is reflexive, $(x_i)_{i=1}^\infty$ is weakly null, hence by perturbation we may assume $(x_i)_{i=1}^\infty$ is a block basis. By Proposition 2.47, it has a subsequence that is either equivalent to $(\alpha_i)_{i=1}^\infty$, the canonical unit vector basis of X , or to the unit vector basis of ℓ_p , both being symmetric bases.

If there is a subsequence $(y_i)_{i=1}^\infty$ of $(x_i)_{i=1}^\infty$ such that $\|Ty_i\|_X \rightarrow 0$, then $T \in SS_1(X)$. If not, then there is a subsequence $(y_i)_{i=1}^\infty$ whose image $(Ty_i)_{i=1}^\infty$ is equivalent to either $(\alpha_i)_{i=1}^\infty$ or to the unit vector basis of ℓ_p .

Since T is SS and using Lemma 2.45, the only possibility is having $(y_i)_{i=1}^\infty$ equivalent to the unit vector basis of ℓ_p and $(Ty_i)_{i=1}^\infty$ equivalent to $(\alpha_i)_{i=1}^\infty$. Since these bases are symmetric and not equivalent, $T \in SS_1(X)$.

Consider in X a block sequence $(u_i)_{i=1}^\infty$ formed by long normalized averages, i.e., $u_i = \frac{1}{[\lambda(n_{i+1} - n_i)]^{1/p}} \sum_{j=n_i+1}^{n_{i+1}} \alpha_j$, where $n_{i+1} - n_i \rightarrow \infty$ as $i \rightarrow \infty$. Then, by Proposition 2.47, a subsequence of $(u_i)_{i=1}^\infty$, denoted again by $(u_i)_{i=1}^\infty$, is equivalent to the unit vector basis of ℓ_p . Since the basis is symmetric, there exists an averaging projection $P : X \rightarrow \overline{\text{span}}(u_i)_{i=1}^\infty$, see [LT]. By Lemma 2.45, the unit vector basis of ℓ_p dominates $(\alpha_i)_{i=1}^\infty$. Define $T : \overline{\text{span}}(u_i)_{i=1}^\infty \rightarrow X$ by for all $i \in \mathbb{N}$, $T(u_i) = \alpha_i$. Then $T \circ P : X \rightarrow X$ is not compact, but T is S_1 -SS, hence so is $T \circ P$.

Chapter 3

Proof of the main Theorem

We recall our main Theorem:

Theorem 1.4 (Main Theorem). *There exists a Banach space X such that the set of all S_1 -SS operators from X to X does not form an ideal. In particular, there exist $T_1, T_2 \in SS_1(X)$ so that $T_1 + T_2 \notin SS_1(X)$. Moreover, X may be chosen to be reflexive.*

We will construct a space $\widetilde{W}$, Lorentz sequence spaces W_1 and W_2 and operators $T_1, T_2 : \widetilde{W} \longrightarrow W_1 \oplus W_2 \in SS_1(\widetilde{W}, W_1 \oplus W_2)$, such that the sum $T_1 + T_2 \notin SS_1(\widetilde{W}, W_1 \oplus W_2)$. By Corollary 2.41, this yields the main Theorem for $X = \widetilde{W} \oplus W_1 \oplus W_2$.

We first construct an example where the space $\widetilde{W}$ is not reflexive. Then we will alter it to get a reflexive example. The spaces W_1 and W_2 will always be reflexive. Let $2 < p < \infty$. First, we consider two Lorentz Sequence Spaces $W_1 = d(\omega_1, p)$ and $W_2 = d(\omega_2, p)$, with incomparable weights $\omega_1 = (\omega_1^i)_{i=1}^\infty$ and $\omega_2 = (\omega_2^i)_{i=1}^\infty$. As in Definition 2.46, consider the functions: $\lambda(n) = \sum_{j=1}^n \omega_1^j$ and $\mu(n) = \sum_{j=1}^n \omega_2^j$.

Also, we will require that $\sup_{n,k} \frac{\lambda(nk)}{\lambda(n)\lambda(k)} < \infty$ and $\sup_{n,k} \frac{\mu(nk)}{\mu(n)\mu(k)} < \infty$. In

fact, these sup's can be made equal to 1. The existence of such weights follows from the following Proposition (see [DOS], Proposition 2.6):

Proposition 3.1. *There are functions $\lambda(x)$ and $\mu(x)$, defined on $[1, \infty)$ that satisfy the following conditions:*

1. *Each one is continuous, strictly increasing, concave and piecewise linear;*
2. *$\lambda(x) = \mu(x) = x$ for $1 \leq x \leq 2$;*
3. *$\lambda(xy) \leq \lambda(x)\lambda(y)$ and $\mu(xy) \leq \mu(x)\mu(y)$ for all $1 \leq x, y$.*
4. *$\inf_n \frac{\lambda(n)}{\mu(n)} = 0$ and $\sup_n \frac{\lambda(n)}{\mu(n)} = \infty$*

Note that condition 4 implies that $\lambda(n) \rightarrow \infty$ and $\mu(n) \rightarrow \infty$ as $n \rightarrow \infty$.

If, we define $\lambda(0) = \mu(0) = 0$, then the weights are defined by these functions as follows:

$$\omega_1 = (\lambda(n) - \lambda(n-1))_{n=1}^{\infty} \text{ and } \omega_2 = (\mu(n) - \mu(n-1))_{n=1}^{\infty}.$$

$\widetilde{W}$ will have an 1-unconditional basis $(d_i)_{i=1}^{\infty}$ and the operators T_j , where $j = 1$ or 2 , will be the formal identities $d_j \rightarrow e_{i,j}$ where $(e_{i,j})_{j=1}^{\infty}$ is the unit vector basis of W_i for $i = 1, 2$, regarded as elements of $W_1 \oplus W_2$ in the natural way.

We define the formal identity $i : \overline{\text{span}}(e_{1,i})_{i=1}^{\infty} \rightarrow \overline{\text{span}}(e_{2,i})_{i=1}^{\infty}$ by $i(\sum_{i=1}^{\infty} a_i e_{1,i}) = \sum_{i=1}^{\infty} a_i e_{2,i}$, for all $\sum_{i=1}^{\infty} a_i e_{1,i} \in W_1$. We note that i is not bounded as follows from the next result.

From condition 4 of Proposition 3.1, we have:

Proposition 3.2. *For every $\varepsilon > 0$ there exists a set $F \in S_1$ and an element $\tilde{x} \in \text{span}(x_i)_{i \in F}$ with $\|\tilde{x}\|_{W_1} = 1$ and $\|i(\tilde{x})\|_{W_2} < \varepsilon$.*

Proof. Let $x = \sum_{i=k+1}^{2k} \frac{e_{1,i}}{\lambda(k)^{1/p}}$, where k , obtained by condition 4, satisfies $\frac{\mu(k)}{\lambda(k)} < \varepsilon^p$. $\square$

Obviously, this result remains valid if we interchange $\|\cdot\|_{W_1}$ and $\|\cdot\|_{W_2}$, and consider the inverse i^{-1} of i .

3.1 The space $\widetilde{W}$

Let W be a Lorentz sequence space with canonical basis $(e_i)_{i=1}^\infty$. If g_1 and g_2 are elements of W^* , we denote by $\text{supp } g_i$ the set $\{j \in \mathbb{N} : g_i(e_j) \neq 0\}$. We say that $g_1 < g_2$ if $\text{supp } g_1 < \text{supp } g_2$. Also, we define $\psi(n) = \frac{0.4}{n}$. And if $g_1 < \dots < g_l$ are elements of $B_{W_1^*}$ (or elements of $B_{W_2^*}$), we define:

$$\phi(g_1, \dots, g_l) = \begin{cases} 1, & \text{if } g_1 = \dots = g_l = 0 \\ \psi(\max \text{supp } \sum_{i=1}^l g_i), & \text{otherwise.} \end{cases}$$

We note the following properties of ϕ :

- ϕ takes values in $(0, 1]$;
- $\phi(g_1, \dots, g_l) = \phi(|g_1|, \dots, |g_l|)$, for all $g_1 < \dots < g_l \in B_{W_1^*}$ or $B_{W_2^*}$;
- $\phi(g_1, \dots, g_k) < \phi(h_1, \dots, h_l)$ if $\max \text{supp } (\sum_{i=1}^k g_i) > \max \text{supp } (\sum_{i=1}^l h_i)$.

Let j be 1 or 2:

$$\Gamma_j = \left\{ \frac{1}{\sqrt{n}} \sum_{i=1}^n g_i : g_i \in B_{W_j^*}, g_1 < \dots < g_n, n \in \mathbb{N} \text{ and } \|g_{l+1}\|_\infty \leq \phi(g_1, \dots, g_l), \text{ for } 1 \leq l < n \right\},$$

Note that the last functional g_n could have infinite support.

We define a norm on c_{00} by $\|x\|_{\Gamma_i} = \sup_{f \in \Gamma_i} f(x)$, $i = 1, 2$. Define $\Gamma = \Gamma_1 \cup \Gamma_2$.

The space $\widetilde{W}$ is defined to be the completion of c_{00} under:

$$\|\cdot\|_{\widetilde{W}} = \max \{ \|\cdot\|_{\Gamma_1}, \|\cdot\|_{\Gamma_2} \} = \|\cdot\|_{\Gamma}.$$

Since Γ contains the unit balls of W_1^* and W_2^* , $\|\cdot\|_{\widetilde{W}} \geq \max \{ \|\cdot\|_{W_1}, \|\cdot\|_{W_2} \}$.

Let i be 1 or 2. We observe that if $f \in \Gamma_i$ and $I \subset \mathbb{N}$ then $f|_I \in \Gamma_i$. Furthermore, the unit vector basis (d_i) is a normalized 1-unconditional basis for $\widetilde{W}$. Indeed, $f \in \Gamma_i$ if and only if $|f| \in \Gamma_i$.

By definition of $\widetilde{W}$, $\Gamma \subset B_{\widetilde{W}^*}$. Also, in Γ , pointwise convergence and weak-star convergence are the same, as shown in the next result:

Proposition 3.3. *Let $(f_i)_{i=1}^\infty$ be a sequence of elements of Γ . Then:*

1. *If $f_i \xrightarrow{w^*} f$ then $f_i \xrightarrow{ptwise} f$, i.e., $f_i(d_j) \rightarrow f(d_j)$ for all $j \in \mathbb{N}$.*
2. *$f_i \xrightarrow{ptwise} f$ then $f \in \Gamma$ and $f_i \xrightarrow{w^*} f$.*

Proof. We need only verify 2. Let $(f_i) \subset \Gamma$, $f_i \xrightarrow{ptwise} f$. To prove that $f \in \Gamma$, for each $i \in \mathbb{N}$, write $f_i = \frac{1}{\sqrt{n_i}} \sum_{j=1}^{n_i} f_j^{(i)}$, as in the definition of Γ . Without loss of generality, we assume that all $f_j^{(i)} \in B_{W_1^*}$.

If $(n_i)_{i=1}^\infty$ is unbounded $(f_i)_{i=1}^\infty$ converges pointwise to zero, which is in Γ . So, passing to a subsequence, we may assume that there exists $n \in \mathbb{N}$ such that for all $i \in \mathbb{N}$, $n_i = n$.

By passing to a further subsequence, using a diagonal argument, we may assume that for all $1 \leq j \leq n$, $f_j^{(i)} \xrightarrow{w} f_j \in B_{W_1^*}$ as $i \rightarrow \infty$, since W_1 is reflexive.

Then $F = \frac{1}{\sqrt{n}} \sum_{j=1}^n f_j \in \Gamma_1$. Indeed, one needs only verify that for $1 \leq l < n$, $\|f_{l+1}\|_\infty \leq \phi(f_1, \dots, f_l)$.

This holds, since for large enough i , $\text{supp } f_j^{(i)} \supset \text{supp } f_j$, for all $j < n$ and thus $\|f_{l+1}\|_\infty \leq \lim_{i \rightarrow \infty} \|f_{l+1}^{(i)}\|_\infty \leq \phi(f_1, \dots, f_l)$, for all $l < n$. $\square$

Thus, if $\bar{\Gamma}$ is the weak* closure of Γ in $B_{\widetilde{W}^*}$, then $\bar{\Gamma} = \Gamma$. We consider Γ as a compact metric space in $(B_{\widetilde{W}}, w^*)$.

Remark 3.4. $\widetilde{W}$ embeds isometrically into $C(\Gamma)$.

The isometry is given by $\Phi : \widetilde{W} \longrightarrow C(\Gamma)$ defined as: $\Phi(x) = (f(x))_{f \in \Gamma}$.

Proposition 3.5. *The unit vector basis $(d_i)_{i=1}^\infty$ of $\widetilde{W}$ is shrinking.*

Proof. Suppose not. Then by Remark 3.4, there is a normalized block basic sequence $(x_i)_{i=1}^\infty$ in $\widetilde{W}$, a real number $\varepsilon > 0$ and $f = \frac{1}{\sqrt{n}} \sum_{i=1}^n f_i \in \Gamma$ such that $f(x_i) > \varepsilon$ for $i \in \mathbb{N}$. Without loss of generality, we may assume $f \in \Gamma_1$. Thus, $\frac{1}{\sqrt{n}} f_n(x_i) > \varepsilon$, for all large i . But that is impossible, since the sequence (x_i) is bounded in W_1 and the space W_1 is reflexive, hence its basis is shrinking. $\square$

Remark 3.6. Hence, we conclude the following:

- if $(x_i)_{i=1}^\infty$ is a bounded sequence in $\widetilde{W}$ that converges pointwise to zero, i.e. $d_n^*(x_i) \rightarrow 0$ for all $n \in \mathbb{N}$, then it is weakly-null.
- the space $\widetilde{W}$ does not contain ℓ_1 .

Proposition 3.7. *Let $F \in S_1$ and $x = \sum_{i \in F} b_i d_i$. Then:*

$$\left\| \sum_{i \in F} b_i d_i \right\|_{\widetilde{W}} = \max \left\{ \left\| \sum_{i \in F} b_i e_{1,i} \right\|_{W_1}, \left\| \sum_{i \in F} b_i e_{2,i} \right\|_{W_2} \right\}.$$

Proof. Let us assume $F \in S_1$, and that

$$\max \left\{ \left\| \sum_{i \in F} b_i e_{1,i} \right\|_{W_1}, \left\| \sum_{i \in F} b_i e_{2,i} \right\|_{W_2} \right\} = \left\| \sum_{i \in F} b_i e_{1,i} \right\|_{W_1} = 1.$$

We only need to prove that $\|x\|_{\widetilde{W}} \leq 1$. Let $x = \sum_{i \in F} b_i d_i$, and $\min F = k$.

Let $f = \frac{1}{\sqrt{n}} \sum_{i=1}^n f_i \in \Gamma$, with $n \geq 2$. We need to show that $|f(x)| \leq 1$. Let $i_0 = \min\{i : f_i(x) \neq 0\}$. If no i_0 exists, then $f(x) = 0$. Now, $\max \text{supp } f_{i_0} \geq \min F = k$, and so for $i > i_0$, $\|f_i\|_\infty \leq \frac{0.4}{k}$. Since, $|F| \leq k$, $|f(x)| \leq \frac{1}{\sqrt{n}} [|f_{i_0}(x)| + \frac{0.4}{k} \|(b_i)\|_\infty k] \leq \frac{1}{\sqrt{n}} [1 + 0.4] < 1$, since $n \geq 2$ and $\|(b_i)\|_\infty \leq 1$. $\square$

Lemma 3.8. *For all $\varepsilon > 0$ there exists $N = N(\varepsilon)$ so that if $f \in B_{W_1^*}$ (respectively $f \in B_{W_2^*}$),*

$$|\{i : |f(e_{1,i})| \geq \varepsilon\}| \leq N(\varepsilon) \text{ (respectively } |\{i : |f(e_{2,i})| \geq \varepsilon\}| \leq N(\varepsilon)).$$

Proof. We present the proof for $f \in B_{W_1^*}$. Let $\varepsilon > 0$, and define $N = N(\varepsilon) = \max \left\{ n : \frac{\lambda(n)^{1/p}}{n} \geq \varepsilon \right\}$. N is finite since $\frac{\lambda(n)}{n} \rightarrow 0$ as $n \rightarrow \infty$.

We define $A_\varepsilon^{(f)} = \{i : |f(e_{1,i})| \geq \varepsilon\}$, and $M = |A_\varepsilon^{(f)}|$. Then, if we define $x = \sum_{i \in A_\varepsilon^{(f)}} \text{sign} f(e_{1,i}) e_{1,i}$. We have $1 \geq \left| f \left(\frac{x}{\|x\|} \right) \right| \geq \frac{\varepsilon M}{\lambda(M)^{1/p}}$. Hence $\frac{\lambda(M)^{1/p}}{M} \geq \varepsilon$, which implies that $M \leq N$. $\square$

Lemma 3.9. *Let $(x_i)_{i=1}^\infty$ be a normalized block basic sequence in W_1 with $\|x_i\|_\infty \rightarrow 0$.*

Then there exists a subsequence (y_i) of (x_i) and a block sequence $(g_i) \in B_{W_1^}$ with $g_i(y_i) > \frac{1}{2}$ for all $i \in \mathbb{N}$ and $\|g_i\|_\infty \rightarrow 0$.*

Proof. Choose a normalized block basic sequence $(f_i)_{i=1}^\infty$ in W_1^* with, $f_i(x_i) = 1$, for all $i \in \mathbb{N}$. Without loss of generality, $\|x_i\|_\infty < \frac{1}{2^{N_i}}$, where $N_i = N(1/i)$ as given by

Lemma 3.8. Let $g_i = f_i|_{\|f_i\| < \frac{1}{i}}$. So $g_i(x_i) \geq 1 - \frac{N_i}{2^{N_i}} \rightarrow 1$ and $\|g_i\|_\infty \rightarrow 0$. $\square$

Proposition 3.10. *Let $(x_i)_{i=1}^\infty$ be a normalized block basic sequence of elements in $\widetilde{W}$ with the properties:*

1. *There is $\delta > 0$ such that $\|x_i\|_{W_1} \geq \delta$ for all $i \in \mathbb{N}$;*
2. *$\|x_i\|_\infty \rightarrow 0$, as $i \rightarrow \infty$.*

Then there is a subsequence $(y_i)_{i=1}^\infty$ of $(x_i)_{i=1}^\infty$ such that for any finite set F :

$$\left\| \sum_{i \in F} y_i \right\|_{\widetilde{W}} \geq \frac{\delta}{2} \sqrt{|F|}.$$

Proof. By Lemma 3.9 and unconditionality, there exists a block basis $(f_i)_{i=1}^\infty \subset B_{W_1^*}$ and a subsequence $(y_i)_{i=1}^\infty$ of $(x_i)_{i=1}^\infty$ so that $f_i(y_i) > \frac{\delta}{2}$, $\text{supp}(f_i) \subseteq \text{supp}(y_i)$ and $\|f_{i+1}\|_\infty < \phi(f_1, \dots, f_i)$ for all $i \in \mathbb{N}$.

Hence:

$$\left\| \frac{1}{\sqrt{|F|}} \sum_{i \in F} y_i \right\|_{\widetilde{W}} \geq \left(\frac{\sum_{i \in F} f_i}{\sqrt{|F|}} \right) \left(\frac{\sum_{i \in F} y_i}{\sqrt{|F|}} \right) \geq \frac{|F| \delta/2}{|F|} = \frac{\delta}{2}.$$

$\square$

A similar result is valid if condition 1 is replaced by

- 1'. *There is $\delta > 0$ such that $\|x_i\|_{W_2} \geq \delta$ for all $i \in \mathbb{N}$;*

3.2 Operators T_1 and T_2

Our ultimate goal is to construct operators $T_1, T_2 : \widetilde{W} \longrightarrow W_1 \oplus W_2$ that are S₁-SS, but $T = T_1 + T_2$ is not. Recall that the operators $T_1, T_2 : \widetilde{W} \longrightarrow W_1 \oplus W_2$ are defined

as:

$$T_1(d_i) = (e_{1,i}, 0) \text{ and } T_2(d_i) = (0, e_{2,i}).$$

Lemma 3.11. $T = T_1 + T_2$ is not S_1 -SS.

Proof. Use the unit vector basis $(d_i)_{i=1}^\infty$ and Proposition 3.7. □

Proposition 3.12. T_1 and T_2 are S_1 -SS.

Proof. We give the proof for T_1 . Let $(x_i)_{i=1}^\infty$ be a normalized basic sequence in $\widetilde{W}$ and let $\varepsilon > 0$. Passing to a subsequence, by Theorem 2.20 and Remark 3.6, we may assume $(x_i)_{i=1}^\infty$ is weak-Cauchy, hence $(x_{2i} - x_{2i+1})_{i=1}^\infty$ is seminormalized weakly null. By perturbing, we may assume it is a block basis of $(d_i)_{i=1}^\infty$. Thus, by Lemma 2.35, it suffices to verify that T_1 satisfies the S_1 -SS condition for normalized block bases $(b_i)_{i=1}^\infty$ of $(d_i)_{i=1}^\infty$.

Let $(b_i)_{i=1}^\infty$ be such and let $\varepsilon > 0$. Passing to a subsequence, for simplicity also denoted by $(b_i)_{i=1}^\infty$, we may assume either:

$$1.1 \quad (T_1(b_i))_{i=1}^\infty \text{ is norm-null, that is } \|T_1(b_i)\|_{W_1} \rightarrow 0, \text{ as } i \rightarrow \infty;$$

or $(T_1(b_i))_{i=1}^\infty$ is seminormalized and

$$1.2 \quad \left(\frac{T_1 b_i}{\|T_1 b_i\|_{W_1}} \right)^\circ \xrightarrow{ptwise} b, \text{ for some } b \in W_1 \text{ with } \|b\|_{W_1} = 1. \text{ Thus, by Proposition 2.47, and passing to a further subsequence, we may assume that } (T_1(b_i))_{i=1}^\infty \sim (e_{1,i})_{i=1}^\infty, \text{ or;}$$

$$1.3 \quad \left(\frac{T_1 b_i}{\|T_1 b_i\|_{W_1}} \right)^\circ \xrightarrow{ptwise} b, \text{ for some } b \in W_1 \text{ with } \|b\|_{W_1} < 1. \text{ Thus, by Proposition 2.47, we may assume } (T_1(b_i))_{i=1}^\infty \text{ is equivalent to the unit vector basis of } \ell_p.$$

In each case, we need to guarantee the existence of an S_1 set F and $b \in \text{span}(b_i)_{i \in F}$ with $\|b\|_{\widetilde{W}} = 1$ and $\|T_1 b\|_{W_1} < \varepsilon$.

For (1.1), the conclusion is trivial.

Suppose (1.2), that $(T_1(b_i))_{i=1}^\infty$ is equivalent to the unit vector basis of W_1 . Then we look to the image $(T_2(b_i))_{i=1}^\infty$ in W_2 . It will also have a subsequence that satisfies one of the following mutually exclusive properties:

1.2.1 It is norm-null, that is $\|T_2(b_i)\|_{W_2} \rightarrow 0$, as $i \rightarrow \infty$;

1.2.2 it is equivalent to the unit vector basis of W_2 , that is $(T_1(b_i))_{i=1}^\infty \sim (e_{2,i})_{i=1}^\infty$, or;

1.2.3 it is equivalent to the unit vector basis of ℓ_p .

However, the first possibility (1.2.1) is impossible, given condition (1.2).

Consider the next possibility, (1.2.2): $(T_1(b_i))_{i=1}^\infty$ is equivalent to the unit vector basis of W_1 and $(T_2(b_i))_{i=1}^\infty$ is equivalent to the unit vector basis of W_2 . By Proposition 3.2, we can choose an element $x = \sum_{i \in F} a_i b_i$, where $F \in S_1$, such that it has W_2 -norm equal to one and W_1 -norm less than ε . Thus, the desired element is $b = \frac{x}{\|x\|_{\widetilde{W}}}$.

For the third possibility (1.2.3): $(T_1(b_i))_{i=1}^\infty$ is equivalent to the unit vector basis of W_1 and $(T_2(b_i))_{i=1}^\infty$ is equivalent to the unit vector basis of ℓ_p . We use that $\|\sum_{i=1}^n e_{1,i}\|_{W_1} = (\lambda(n))^{1/p}$ and $\|\sum_{i=1}^n e_i\|_{\ell_p} = n^{1/p}$. That is, as $\lim_{n \rightarrow \infty} \frac{\lambda(n)}{n} = 0$ we can choose an element $x = \sum_{i \in F} a_i b_i$, with $F \in S_1$, such that it has W_2 -norm equal to one and W_1 -norm less than ε . Again, the desired element is $b = \frac{x}{\|x\|_{\widetilde{W}}}$.

Now suppose (1.3). We may assume by passing to a subsequence that there exists $\delta > 0$ such that for all $i \in \mathbb{N}$, $b_i = h_i + g_i$, where $|h_i| \wedge |g_i| = 0$, $\|g_i\|_{W_1} > \delta$ for all $i \in \mathbb{N}$, and $\|g_i\|_\infty \rightarrow 0$. Then, by Proposition 3.10, again passing to a subsequence, we may assume that for all finite sets F , $\left\| \sum_{i \in F} g_i \right\|_{\widetilde{W}} \geq \frac{\delta}{2} \sqrt{|F|}$, and so $\left\| \sum_{i \in F} b_i \right\|_{\widetilde{W}} \geq$

$\left\| \sum_{i \in F} g_i \right\|_{\widetilde{W}} \geq \frac{\delta}{2} \sqrt{|F|}$. But $\left\| \sum_{i \in F} T_1 b_i \right\|_{W_1} \leq |F|^{1/p}$, by Lemma 2.45. Using that $p > 2$, choose an S_1 set F such that $\frac{|F|^{1/p}}{\sqrt{|F|}\delta/2} < \varepsilon$, and define $b = \frac{\sum_{i \in F} b_i}{\left\| \sum_{i \in F} b_i \right\|_{\widetilde{W}}}$. Clearly, $b \in S_{\widetilde{W}}$. Also,

$$\|T_1 b\|_{W_1} = \frac{\left\| \sum_{i \in F} T_1 b_i \right\|_{W_1}}{\left\| \sum_{i \in F} b_i \right\|_{\widetilde{W}}} \leq \frac{|F|^{1/p}}{\sqrt{|F|}\delta/2} < \varepsilon.$$

□

The space $\widetilde{W}$ is not reflexive. In fact, we have:

Proposition 3.13. *$\widetilde{W}$ is c_0 -saturated, i.e., c_0 embeds into every infinite dimensional subspace of $\widetilde{W}$.*

Proof. Let X be an infinite dimensional subspace of $\widetilde{W}$. We may assume X is a block subspace generated by a block basis $(x_i)_{i=1}^\infty$. For $\varepsilon > 0$, we can find $w \in \text{span}(x_i)_{i=1}^\infty$ such that $\|w\|_{\widetilde{W}} = 1$ and $\max\{\|w\|_{W_1}, \|w\|_{W_2}\} < \varepsilon$, for $i = 1, 2$. This follows from the fact that T_1 and T_2 are both S_1 -SS, and Corollary 2.39. It follows that for every $\varepsilon > 0$ and every $N \in \mathbb{N}$, there exists $w \in S_{\text{span}(x_i)_{i=1}^\infty}$ with $\|w\|_{\Gamma(N)} < \varepsilon$, where $\Gamma(N) = \{f \in \Gamma : f = \frac{1}{\sqrt{n}} \sum_{i=1}^n f_i, n \leq N\}$. Indeed, the norm $\|\cdot\|_{W_i}$ is $\sqrt{N}$ -equivalent to $\|\cdot\|_{(\Gamma_i)_{(N)}}$

Define $\Gamma^{(N_i)} = \Gamma \setminus \Gamma_{(N_i)}$ and clearly, for all $x \in \widetilde{W}$, $\lim_{N \rightarrow \infty} \|x\|_{\Gamma(N)} = 0$.

We can thus choose for $\varepsilon_i \searrow 0$, such that $\sum_{i=1}^\infty \varepsilon_i < 1$, elements $c_1 < c_2 < \dots$ in $S_{\text{span}(x_i)_{i=1}^\infty}$ and integers $N_1 < N_2 < \dots$, so that:

- $\|c_i\|_{\Gamma_{(N_{i-1})}} < \varepsilon_i$, for $i \geq 2$,
- $\|c_i\|_{\Gamma^{(N_i)}} < \varepsilon_i$, for $i \in \mathbb{N}$.

We claim that $\left\| \sum_{i=1}^n c_i \right\|_{\widetilde{W}} \leq 2$ for all n , so $(c_i)_{i=1}^\infty$ is 2-equivalent to the unit vector basis of c_0 .

Indeed, let $f = \frac{1}{\sqrt{m}} \sum_{j=1}^m f_j \in \Gamma$, then $m \in (N_{s-1}, N_s]$ for some $s \in \mathbb{N}$, with $N_0 = 0$. Thus, $|f(c_i)| < \varepsilon_i$ for all $i \in \mathbb{N}$ except at most one $|f(c_s)|$ could be greater than ε_s . Then $\left| f \left(\sum_{i=1}^n c_i \right) \right| < \sum_{i=1}^n \varepsilon_i + 1 < 2$. $\square$

Problem 3.14. *Is $SS_2(X)$ closed under addition for all Banach spaces X ? If $1 \leq \alpha, \beta < \omega_1$, and operators $T_1 \in SS_\alpha(X)$ and $T_2 \in SS_\beta(X)$ are given what is the smallest $1 \leq \alpha \leq \min \{ \alpha + \beta, \beta + \alpha \}$ such that $T_1 + T_2$ has to belong to $SS_\alpha(X)$?*

Chapter 4

A reflexive example

We now complete the proof of our Main Theorem by providing a reflexive X such that $SS_1(X)$ is not an ideal.

Let W_1 and W_2 be the Lorentz sequence spaces from the last section. We modify the sets Γ_1 and Γ_2 as follows. Let j be 1 or 2:

$$\Lambda_j = \left\{ \sum_{i=1}^{\infty} a_i g_i : g_i \in B_{W_j^*}, (a_i)_{i=1}^{\infty} \in B_{\ell_2}, g_1 < g_2 < \dots, \|g_{l+1}\|_{\infty} \leq \phi(g_1, \dots, g_l), \text{ for } 1 \leq l \right\}$$

where the function ϕ is the same as in the nonreflexive case. We define $\Lambda = \Lambda_1 \cup \Lambda_2$.

The space Ψ is defined to be the completion of c_{00} under the norm: $\|\cdot\|_{\Psi} = \|\cdot\|_{\Lambda}$.

Let $(\delta_i)_{i=1}^{\infty}$ be the canonical unit vector basis for Ψ . From the definition, we see that $(\delta_i)_{i=1}^{\infty}$ is an 1-unconditional normalized basis for Ψ .

We define the formal identity $I : \Psi \longrightarrow \widetilde{W}$ by $I(\sum_{i=1}^{\infty} a_i \delta_i) = \sum_{i=1}^{\infty} a_i d_i$, for all $\sum_{i=1}^{\infty} a_i \delta_i \in \Psi$. Since $\Gamma \subset \Lambda$, the identity is a well defined bounded linear operator:

$$\|x\|_{\Psi} \geq \|I(x)\|_{\widetilde{W}}.$$

As before we have:

Proposition 4.1. *Let $(f_i)_{i=1}^\infty$ be a sequence of elements of Λ . Then:*

1. *If $f_i \xrightarrow{w^*} f$ then $f_i \xrightarrow{ptwise} f$, i.e., $f_i(\delta_j) \rightarrow f(\delta_j)$ for all $j \in \mathbb{N}$.*
2. *$f_i \xrightarrow{ptwise} f$ then $f \in \Lambda$ and $f_i \xrightarrow{w^*} f$.*

Proof. We need only verify 2. Let $(f_i)_{i=1}^\infty \subset \Lambda$, $f_i \xrightarrow{ptwise} f$, as $i \rightarrow \infty$. To prove that $f \in \Lambda$, for each $i \in \mathbb{N}$, write $f_i = \sum_{j=1}^\infty a_j^{(i)} f_j^{(i)}$, as in the definition of Λ . Without loss of generality, we assume that all $f_j^{(i)} \in B_{W_1^*}$. By passing to a subsequence, using a diagonal argument, we may assume that for all $j \in \mathbb{N}$, $f_j^{(i)} \xrightarrow{w} f_j \in B_{W_1^*}$ as $i \rightarrow \infty$. Moreover, we may assume that $a_j^{(i)} \rightarrow a_j$ as $i \rightarrow \infty$, for all $j \in \mathbb{N}$. Then $(a_j)_{j=1}^\infty \in B_{\ell_2}$, and we also have that $b = \sum_{j=1}^\infty a_j f_j \in \Lambda_1$. Indeed, one need only verify that for $l \geq 1$, $\|f_{l+1}\|_\infty \leq \phi(f_1, \dots, f_l)$.

This holds, since if $\text{supp } f_{l+1}$ is not empty, for large enough i , $\text{supp } f_j^{(i)} \supset \text{supp } f_j$, for all $j \leq l$ and $\|f_{l+1}\|_\infty \leq \varliminf_{i \rightarrow \infty} \|f_{l+1}^{(i)}\|_\infty$. $\square$

Thus, as earlier, Λ is a w^* -compact subset of (B_{Ψ^*}, w^*) , and we have:

Remark 4.2. Ψ embeds isometrically into $C(\Lambda)$.

The isometry is given by $\Phi_2 : \Psi \longrightarrow C(\Lambda)$ defined as: $\Phi_2(x) = (f(x))_{f \in \Lambda}$.

Similar to the unit vector basis $(d_i)_{i=1}^\infty$ of $\widetilde{W}$, we have:

Lemma 4.3. *The unit vector basis $(\delta_i)_{i=1}^\infty$ of Ψ is shrinking.*

Proof. Let $f = \sum_{i=1}^\infty a_i f_i \in \Lambda$. If we define $f^{(k)} = \sum_{i=k}^\infty a_i f_i$, then $\|f^{(k)}\|_\Psi \rightarrow 0$, as $k \rightarrow \infty$. Indeed, this is clear, since we can write $f^{(k)} = \sum_{i=1}^\infty a_i^{(k)} f_i$, where $(a_i^{(k)})_{i=1}^\infty$ is a sequence in B_{ℓ_2} constructed as:

$$a_i^{(k)} = \begin{cases} 0 & \text{if } i < k \\ a_i & \text{otherwise} \end{cases}$$

The result follows from noticing that $\|(a_i^{(k)})\|_{\ell_2} \rightarrow 0$, as $k \rightarrow \infty$.

If $(x_i)_{i=1}^\infty$ is a normalized block basis of $(\delta_i)_{i=1}^\infty$, it follows that $(\Phi_2(x_i))_{i=1}^\infty$ is pointwise null on Λ and hence is weakly null. Thus $(x_i)_{i=1}^\infty$ is weakly null. $\square$

Different from the nonreflexive example, we now have:

Lemma 4.4. *The unit vector basis $(\delta_i)_{i=1}^\infty$ of Ψ is boundedly complete.*

Proof. Suppose not. By Proposition 2.13, there exists a seminormalized block basic sequence $(x_j)_{j=1}^\infty$ of $(\delta_i)_{i=1}^\infty$ such that $\sup_n \|\sum_{j=1}^n x_j\|_\Psi = M < \infty$. Without loss of generality, we may assume that $\|x_i\| \geq 1$ for all $i \in \mathbb{N}$. Hence we can find, for each i , an element $f_i = \sum_{j=1}^{m_i} a_j^{(i)} g_j^{(i)} \in \Lambda$, where each $g_j^{(i)}$ is not zero for every $i \in \mathbb{N}$ and $j \in [1, m_i]$, such that $f_i(x_i) \geq 1$. We may assume that $f_i \in \Lambda_1$ for all $i \in N$, with $N \subset \mathbb{N}$ infinite and that $\text{supp } f_i \subset \text{supp } x_i$ for all $i \in N$.

We have two cases:

1. There is an infinite set $A \subset N$ such that for each $i \in A$ there is $n_i \in \mathbb{N}$ such that $g_{n_i}^{(i)}(x_i) \geq 0.5$,
2. There is an infinite set $B \subset N$ such that for each $i \in B$, $g_n^{(i)}(x_i) < 0.5$ for each $n \leq m_i$.

The first case yields that $\|x_i\|_{W_1} \geq 1/2$ for $i \in A$, and so if $A = (p_i)_{i=1}^\infty$, $\|\sum_{i=1}^n x_{p_i}\|_\Psi \geq \|\sum_{i=1}^n x_{p_i}\|_{W_1} \rightarrow \infty$ as $n \rightarrow \infty$, since W_1 is reflexive. Thus, we have a contradiction, using the unconditionality of $(\delta_i)_{i=1}^\infty$.

Now assume the second case. Let $B = (p_i)_{i=1}^\infty$. In this case, $m_i \geq 2$ for $i \in B$. Let $M_n = \sum_{i=1}^n (m_{p_i} - 1)$. The element $h_n = \frac{1}{\sqrt{M_n}} \sum_{i=1}^n \left(g_{n_i, 2}^{(i)} + \cdots + g_{n_i, m_i}^{(i)} \right)$ belongs to Λ_1 . But $h_n \left(\sum_{j=1}^n x_{p_j} \right) \geq 0.5n$, a contradiction to the fact that $\sup_n \|\sum_{j=1}^n x_j\|_\Psi = M < \infty$, as before. $\square$

The previous two results have an important consequence:

Corollary 4.5. *The space Ψ is reflexive.*

Next, we have a result analogous to Proposition 3.7:

Proposition 4.6. *Let $F \in S_1$ and $x = \sum_{i \in F} b_i \delta_i \in \Psi$. Let*

$$M = \max \left\{ \left\| \sum_{i \in F} b_i e_{1,i} \right\|_{W_1}, \left\| \sum_{i \in F} b_i e_{2,i} \right\|_{W_2} \right\}.$$

Then:

$$\left\| \sum_{i \in F} b_i \delta_i \right\|_{\Psi} \in [M, 1.1M).$$

Proof. Let us assume $F \in S_1$, and that $M = \left\| \sum_{i \in F} b_i e_{1,i} \right\|_{W_1} = 1$. We only need

to prove that $\left\| \sum_{i \in F} b_i \delta_i \right\|_{\Psi} \leq 1.1$. Let $f = \sum_{i=1}^n a_i f_i \in \Lambda$. We need to show that $|f(x)| \leq 1.1$. Let $i_0 = \min\{i : f_i(x) \neq 0\}$. If no i_0 exists, then $f(x) = 0$. Now, $\max \text{supp } f_{i_0} \geq \min F \equiv k$, and so for $i > i_0$, $\|f_i\|_{\infty} \leq \frac{0.4}{k}$. Since, $|F| \leq k$, $|f(x)| \leq \left[|a_{i_0} f_{i_0}(x)| + \sqrt{1 - a_{i_0}^2 \frac{0.4}{k}} \|(b_i)\|_{\infty} k \right] \leq |a_{i_0}| + 0.4 \sqrt{1 - a_{i_0}^2} \leq \sqrt{29}/5 < 1.1$, since $\|(b_i)\|_{\infty} \leq 1$. $\square$

We now define operators $\tilde{T}_1, \tilde{T}_2 : \Psi \longrightarrow W_1 \oplus W_2$ as:

$$\tilde{T}_1(\delta_i) = (e_{1,i}, 0) \text{ and } \tilde{T}_2(\delta_i) = (0, e_{2,i}).$$

Note that:

$$\tilde{T}_1 = T_1 \circ I \text{ and } \tilde{T}_2 = T_2 \circ I.$$

Lemma 4.7. $\tilde{T} = \tilde{T}_1 + \tilde{T}_2$ is not S_1 -SS.

Proof. Use the unit vector basis $(\delta_i)_{i=1}^\infty$ and Proposition 4.6. $\square$

Lemma 4.8. $\tilde{T}_1$ and $\tilde{T}_2$ are S_1 -SS.

Proof. As previously noted, the identity $I : \Psi \longrightarrow \widetilde{W}$ is bounded and by Proposition 2.43 and the relations above we conclude that $\tilde{T}_1$ and $\tilde{T}_2$ are S_1 -SS. $\square$

The reflexive example asserted in the Main Theorem is given by $X = \Psi \oplus W_1 \oplus W_2$.

We have seen that, according to Propositions 2.29 and 1.3, $K(X)$ and $SS(X)$ are norm-closed ideals. For $1 \leq \alpha < \omega_1$ we have that $K(X) \subset SS_\alpha(X) \subset SS(X)$. Hence, a question arises:

Problem 4.9. *Do there exist $1 \leq \alpha < \omega_1$ and a Banach space X such that $SS_\alpha(X)$ is an ideal but different from both $K(X)$ and $SS(X)$?*

Problem 4.10. *Do there exist a uniformly convex space X and $T_1, T_2 \in SS_1(X)$ with $T_1 + T_2 \notin SS_1(X)$?*

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