Numerical investigation of a one-parameter family of triple close approaches occurring in stellar systems

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This is the second part of a trilogy dealing with the effect of triple close approaches on the evolution of stellar systems. The first part appeared in the *Astron. J.* Vol. 79, p. 981 (1974). The results of numerical integrations of stellar systems slightly perturbed from the condition of complete collapse are presented and it is shown that binary formations and escapes occur with systematic regularity and form a well-established family. The time variations of the pertinent parameters of triple close approaches are examined in minute detail. Three different numerical methods of integration are described. The thesis, that triple close approaches are of fundamental importance to the evolution of stellar systems, seems to be supported by the results. The astronomical applications will be offered in the sequel.

INTRODUCTION

ONE of the early references by Ambartsumyan (1937) points out the astronomical significance of triple close approaches. Analytical and precise numerical investigation of this phenomenon appears to be essential in the understanding of the dynamic behavior of multiple stellar systems. The study of triple systems instead of two-body systems is more meaningful, but at the same time much more difficult for analytical as well as numerical reasons. The fact that two-body systems with negative total energy are always bounded but triple systems are not, offers one of the important reasons for replacing local two-body considerations with three-body dynamics when gravitational n -body systems are studied. The modest aim here is to study numerically such building blocks formed of three bodies with masses of the same order of magnitude. When an asymmetric triple close approach occurs in such a system, the probability of a binary formation is high, as have been shown by Agekian (1967), Szebehely (1967), and others. According to the Agekian-Szebehely classification of the states of motion in the general problems of three bodies, the family presented in this paper belongs to class O, or "triple close approach." [For these classifications, see Agekian (1973) and Szebehely (1971)].

The analytical aspects of this work were described in a previous paper which appeared in these pages (Szebehely 1974) and which henceforth will be referred to as Paper I.

Only the novice in natural sciences will be surprised by the fact that the physically important phenomenon of triple close approach in three-body dynamics also represents the most severe problem for numerical and analytical treatment. Without dwelling on the generally nonregularizable nature of triple collisions, a theorem usually associated with Siegel (1941) but first discussed by Block (1909), we mention this as the basic reason for the difficulties encountered during numerical integrations of triple configurations. If the total energy of the system is positive, instability always occurs and

the system either explodes (all three bodies depart on perturbed hyperbolic orbits) or a binary is formed and the third body escapes.

On the other hand, in the astronomically more important case of negative total energy, one or several triple close approaches proceed an eventual escape, if it occurs at all. Therefore, the detailed behavior during triple close approaches might be considered relevant to stellar dynamics.

DESCRIPTION OF THE SYSTEM

The analytical description of the initial conditions are given in Paper I and the general arrangement is shown in Fig. 1 of this paper. The three masses are equal, $m_1 = m_2 = m_3 = 1$, and the equilateral triangle at $t=0$ has sides of unit length. The initial velocities are selected so that the center of mass is at rest: $v_{1x} = v_{2x} = -v_0/2$, $v_{3x} = v_0$, $v_{1y} = v_{2y} = v_{3y} = 0$. Because of the asymmetry occurring when $v_0 \neq 0$, the triple collision ($v_0 = 0$) is replaced by a triple close approach, provided v_0 is small (see Paper I). In fact, the moment of inertia of the system, $I = \sum m_i r_i^2$, initially is one with $\dot{I}(0) = 0$, independently of the value of v_0 . The second derivative initially depends on v_0 and it is given by $\ddot{I}(0) = 3(v_0^2 - 2)$. Consequently, the second derivative at $t=0$ is positive (negative) for $v_0 > \sqrt{2}$ (for $v_0 < \sqrt{2}$), the curve $I = I(t)$ is convex (concave) from below and the virial coefficient is larger (smaller) than unity. The total energy of the system is conserved and is given by $E_t = \frac{3}{4}(v_0^2 - 4)$. Consequently, the energy is positive (negative) if $v_0 > 2$ (if $v_0 < 2$). The angular momentum of the system is proportional to v_0 , and since the motion is planar, $c = \sqrt{3}v_0/2$. The arrangement shows symmetry with respect to the y axis (Fig. 1), therefore, only positive values of v_0 are considered. For small values of v_0 the total energy and $\ddot{I}(0)$ are negative and the virial coefficient is less than one. This is certainly true for $0 \leq v_0 < \sqrt{2}$, consequently in this range the motion begins with a contraction.

The three papers forming the trilogy are restricted to low initial velocities, i.e., $0 < v_0 \leq 0.2$ since their

FIG. 1. Initial conditions.

purpose is to investigate triple close approaches and the subsequent binary formations. Escapes of m_2 and associated binary formations occur in the range of $0 < v_0 \leq 0.186$. As the initial velocity increases to $v_0 \geq 0.187$ the escape changes into the ejection of m_2 and eventually as v_0 further increases, the mode becomes what is known as interplay. The global behavior of the system, i.e., the study of the range $0.2 < v_0$ will be presented elsewhere.

The choice of the initial configurations and initial velocities is governed by the simplicity of treatment, usefulness of showing the prominent physical aspects, and applicability to stellar dynamics. In our case the model clearly shows the effect of small initial velocities, the role of triple close approaches are emphasized, the parameterization is natural, and the numerical values found are realistic. This last item will become more apparent in the third part of the trilogy.

RESULTS OF NUMERICAL INTEGRATIONS

A. Analysis of the Behavior for $v_0 = 0.001$

The initial velocity is varied from $v_0 = 10^{-10}$ to $v_0 = 0.2$ and the trajectories of the particles are obtained by numerical integration for each case using $G=1$, for the constant of gravity. The second body escapes (or is ejected) in all cases. The asymptotic hyperbolic escape velocity v_∞ , the semimajor axis of the binary formed a_∞ , its eccentricity e , the direction of escape, the minimum moment of inertia I_m , the time t_m when it is reached, the close binary approaches and the conditions for escape are evaluated for all cases. When escape does not occur, the elliptic ejection path is established. All escapes were verified by Standish's (1971) escape criteria.

A typical, representative member of the family with $v_0 = 0.001$ is discussed in some detail in the following. Figure 2 shows the critical part of the triple close approach. Because of the small initial velocities, the three bodies begin their motion with a contraction toward the origin where the center of mass is located. The system will collapse (with zero initial velocity) in $t_c = \pi/(24)^{1/2}$ time units as shown in Paper I. (Translation

FIG. 2. Orbits near triple close approach ($v_0 = 0.001$).

of the dimensionless quantities to physically significant values will be described at the end of the paper and in the sequel.) The corresponding time with nonzero initial velocity might be taken to be when the quantity Σr_{ij}^2 reaches its minimum. Since the moment of inertia with respect to the center of mass is

$$I = \frac{1}{M} \sum m_i m_j r_{ij}^2,$$

where $1 \leq i < j \leq 3$, we have for equal and unit masses,

FIG. 3. Moment of inertia versus time ($v_0 = 0.001$).

TABLE I. Times corresponding to the points shown on the orbits in Fig. 1.

Point	A_i	α_i	β_i	γ_i	B_i	δ_i	C_i	E_i	D_i
Time	0.641218	0.641234	0.641244	0.641250	0.641254	0.641274	0.641288	0.641316	0.641356
τ	218	234	244	250	254	274	288	316	356

that $I = \frac{1}{3} \sum r_{ij}^2$. Consequently, the time necessary to reach the minimum value of the moment of inertia might be compared to the above collapse time $t_c = 0.6412749151$.

The asymmetry is apparent when the bodies are at the points marked A_i in Fig. 2. The next interesting feature occurs when m_1 and m_3 experience a close approach (see points B_1 and B_3). At this time m_2 is delayed because of the directions of the initial velocities. In fact, m_2 is a latecomer for all members of the family of orbits under discussion and, consequently, it is always the escaping or the ejected body. This first close binary approach, $r_{13} = 1.5626 \times 10^{-4}$ occurs at $t_{13} = 0.641254$ and the dimensionless relative velocity between m_1 and m_3 is $v_{13} \cong 160$ velocity unit. The second body at the same time is about twice as far from the origin as m_3 and m_1 and the moment of inertia of the system is 5.134×10^{-6} . This is not the value for $I_{\min}$, since m_2 and m_3 still move toward the center of mass (while m_1 is in a relatively slow loop-orbit). The second body increases its velocity between points B_2 and C_2 since both m_1 and m_3 attract it. At $t_m = 0.641288$ a close approach occurs between m_2 and m_3 , $r_{23} = 8.64 \times 10^{-4}$, and at the same time $I_{\min} = 2.919 \times 10^{-6}$ is established (see points C_i). The relative velocity during this close approach is $v_{23} \cong 88$. The velocity of the first body m_1 , at the same time, is only about $v_1 \cong 20$ units, and it is moving about normal to the directions of m_2 and m_3 , which latter bodies move with $v_2 \cong 47$ and $v_3 \cong 41$ (see Fig. 2). The distances from m_1 to m_2 and m_3 are about $2r_{23}$. The three distances from the center of mass at

t_m are $r_1 = 13.0 \times 10^{-4}$, $r_2 = 6.4 \times 10^{-4}$, and $r_3 = 9.0 \times 10^{-4}$. After this time m_2 escapes, and m_1 and m_3 form a binary. The escape is a consequence of the asymmetric initial conditions as well as of three-body effects. Without m_1 's presence, m_2 and m_3 would have formed a high-eccentricity binary and no escape could have occurred since the total energy of the system is negative, forbidding unbounded two-body orbits.

The additional marks on the orbits (α_i , β_i , γ_i , δ_i , and ϵ_i) are synchronized. The times corresponding to the 27 points of Fig. 2 are listed in Table I. In what follows, the quantity τ will be used together with the time t . The relation is $t = 0.641 + \tau \times 10^{-6}$.

The variation of the moment of inertia with time is shown in Fig. 3 and 4 and $\dot{I} = \dot{I}(t)$ is presented in Fig. 5. Note the expected smoothness of $I(t)$ in the magnified plot of Fig. 4, the sharp rise after $I_{\min}$ in Fig. 3, and the asymmetric behavior of $\dot{I}(t)$ in Fig. 5. The largest change in $\dot{I}$ occurs at point B when $\dot{I}$ is maximum (see also Fig. 7) due to the close approach of m_1 and m_3 .

The virial coefficient $\alpha = 2T/F \geq 0$ and the total energy $E_t = T - F < 0$ may be used to express the kinetic energy, $T = E_t \alpha (\alpha - 2)^{-1} \geq 0$, the negative of the potential energy, $F = 2E_t (\alpha - 2)^{-1} \geq 0$, and the second derivative of the moment of inertia, $\ddot{I} = 4E_t (\alpha - 1) (\alpha - 2)^{-1}$. Since the total energy is negative for the family under discussion, we have

$$T = \frac{|E_t| \alpha}{2 - \alpha}, \quad F = \frac{2|E_t|}{2 - \alpha}, \quad \ddot{I} = \frac{4|E_t| (\alpha - 1)}{2 - \alpha}.$$

During the motion every member of the family has a fixed E_t value and T , F , and $\ddot{I}$ change with α as α varies with time. It may be shown that $\alpha \leq 2$, the equality corresponding to a binary collision, while

FIG. 4. Moment of inertia at triple close approach ($v_0 = 0.001$).

FIG. 5. First derivative of the moment of inertia at triple close approach ($v_0 = 0.001$).

FIG. 6. Virial coefficient vs time ($v_0 = 0.001$).

$\alpha = 1$ corresponds to $\ddot{I} = 0$ as also follows from the Lagrange-Jacobi equation.

Figures 6 and 7 show the variation of the virial coefficient with time for $v_0 = 0.001$. At $t = 0$ the value of the virial coefficient is $\alpha = v_0^2/2 = 5 \times 10^{-7}$. At $t \approx 0.52$ the $I(t)$ curve has an inflection point and $\alpha = 1$ (see point X in Figs. 3 and 6). At the first close binary approach (between m_1 and m_3) $\alpha = 1.99916$ as shown in Fig. 7, point B. Because of the compressed time scale in Fig. 6, no details can be seen. The high-frequency variation of α beginning at $t = 0.64$ in Fig. 6 corresponds

to the final binary formed between m_1 and m_3 . The period of this binary is $T \approx 3 \times 10^{-4}$, corresponding to $a \approx 1.6746 \times 10^{-3}$. The details may be seen in Fig. 7, where the letters correspond to the points shown in Fig. 2. Attention is directed to point B at the absolute maximum of the curve $\alpha(t)$. At this point the absolute value of the potential is maximum. At point C the moment of inertia is minimum and α has another maximum, smaller than at point B. Consequently, $F_B > F_C$, and therefore, the minimum value of I is not associated with the maximum of F . The value of F may be governed by one single close binary approach while I is governed by the closeness of all three bodies. This fact agrees with Agekian's (1969) observations and following him we introduce as a measure of the probability of escape $\eta = F/|E_t| = (1 - \alpha/2)^{-1}$ computed when $I_{\min}$ occurs. At point C, when the minimum moment of inertia occurs, $\eta = 722$. This value is due to a close triple approach, which is indeed needed for escape and which is an indication of a forthcoming escape. At point B the close binary approach gives $\eta = 2373$. In comparison, if the initial velocity is $v_0 = 0.2$ (and ejection occurs instead of escape) the critical value of the virial coefficient at I_m is $\alpha = 1.8$ and Agekian's measure becomes $\eta = 10$.

After point D, the curve in Fig. 7 reaches a minimum, rises again, and will oscillate because of the binary formation as mentioned before.

B. Characteristics of the Family

As the initial velocity changes from 10^{-1} to 10^{-10} the previously described numerical values change, but not the shape of the orbits. We first discuss the behavior of the close binary approach r_{23}^* at t_m , i.e., at the time when $I_{\min}$ occurs. Figure 8 shows that in the range $10^{-2} \leq v_0 \leq 10^{-10}$ the function $r_{23}^*(v_0)$ is linear on a log-log scale, i.e., $r_{23}^* = A v_0^n$ with $A \approx 0.38$ and $n \approx 0.87$.

FIG. 7. Virial coefficient at triple close approach ($v_0 = 0.001$).

As the initial velocity becomes larger, the curve flattens out and the above relation is not valid. In fact, while the topology of the orbits remains the same in the complete range of the investigation, the role of the two close binary approaches, r_{23}^* and r_{13}^* , changes at approximately $v_0 = 10^{-1}$. The first close binary approach is always between m_1 and m_3 and the second between m_2 and m_3 , but r_{12}^* is smaller than r_{23}^* only in the range $10^{-2} \leq v_0 \leq 10^{-10}$.

In the same range we find the following inequalities persisting:

$$a_\infty \geq (I_m/3)^{\frac{1}{2}} \geq r_{23}^* \geq a_\infty(1-e) \geq r_{13}^*.$$

It is essential to realize that only two of the quantities shown in this inequality occur simultaneously. The minimum moment of inertia I_m occurs at t_m , simultaneously with r_{23}^* , but r_{13}^* occurs earlier and the binary is formed later. The quantity $(I_m/3)^{\frac{1}{2}}$ is also known as the radius of gyration, the total mass being three. This quantity is used by Birkhoff (1927) and Sundman (1912) instead of the moment of inertia.

The numerical results not only give the above inequalities, but also show that the ratios of the quantities involved are approximately constant in the (more limited) range of $10^{-3} \geq v_0 \geq 10^{-10}$. Dividing by r_{23}^* we have

$$\begin{aligned} \frac{a_\infty}{r_{23}^*} &\cong 1.818 \geq \frac{(I_m/3)^{\frac{1}{2}}}{r_{23}^*} \cong 1.055 \geq 1 \geq \frac{a_\infty(1-e)}{r_{23}^*} \\ &\cong 0.727 \geq \frac{r_{13}^*}{r_{23}^*} \cong 0.154. \end{aligned}$$

From this it follows that the eccentricity is the same for all members of the family, $e = 0.6$.

Regarding the escape velocity generated, v_∞ , it was shown in Paper I that in the limit $v_\infty^2 a_\infty \rightarrow \frac{2}{3}$ as $v_0 \rightarrow 0^+$ and, in general, for the members of the family,

$$v_\infty^2 = 2(3a_\infty)^{-1} + v_0^2 - 4.$$

This relation shows good agreement with the numerical results in the (less limited) range of $0.186 \geq v_0 \geq 10^{-10}$. Figure 8 shows the escape velocities versus the parameter v_0 when $v_0 \leq 10^{-2}$, $a_\infty \leq 0.012$, $v_\infty \geq 7.18$, and the above equation is already close to its limiting form. The slope of the line $r_{23}^*(v_0)$ is twice that of the line $v_\infty(v_0)$ in these log-log presentations, since $v_\infty^2 r_{23}^* = v_\infty^2 a_\infty / 1.818 \cong 0.3667$ for $v_0 \leq 10^{-2}$.

The maximum velocity between m_1 and m_3 occurring at their close approach v_{13} is related to the escape velocity according to the numerical results by $v_{13} = 8v_\infty$ in the range $10^{-2} \leq v_0 \leq 10^{-10}$.

The general shape of the orbits is the same for all members of the family as mentioned before, but some interesting details change. The direction of the escape of m_2 is along the positive y axis when $v_0 \cong 0.08$, and the angle decreases with decreasing v_0 , approaching 30°

FIG. 8. Distance at the first close binary approach r_{23}^* , escape velocity v_∞ , and time to triple close approach Δt , vs the parameter v_0 .

as $v_0 \rightarrow 0$. When v_0 is high enough to prevent escape, ejection occurs along the negative x axis. This change of the direction of departure of m_2 occurs in a continuous manner.

The final characteristics analyzed for the family is the time t_m to reach $I_{\min}$. Introducing $\Delta t = t_m - t_c \geq 0$, where $t_c = \pi/(24)^{\frac{1}{2}}$ as mentioned before, we may show the monotonically increasing function $\Delta t(v_0)$ in Fig. 8.

NUMERICAL METHODS USED

Most of the members of the family were established by three different computer programs using the University of Texas CDC-6600. The purpose was not to compare the efficiency of the various numerical techniques applied, but rather to obtain consistent and reliable numerical results.

The numerical integration was performed in the first program by a 15th-degree polynomial using Steffenson's (1956) method. Levi-Civita's (1903) two-dimensional regularization was applied to the smallest of the three distances at all times. This program is described by Szebehely and Peters (1967). The second program used a Runge-Kutta-Fehlberg 7(8) integrator with the Kustaanheimo-Stiefel (1965) regularization. For details of this program see Szebehely and Bettis (1971). The third program employed the same integrator as the second with the Aarseth-Zare (1974) regularization applied simultaneously to the two smallest distances at all times.

The major problem in the numerical integration is the proper treatment of the triple close approaches since even binary collisions can be regularized without difficulty. When $v_0 = 10^{-10}$, the smallest distance r_{12}^* encountered during the first close binary approach is of the order of 10^{-10} . The second close approach r_{23}^* occurring with $I_{\min}$ is of the order of 10^{-9} and the other two distances at the same time are of the same

order of magnitude. The programs were able to handle this case ($v_0=10^{-10}$) without difficulty. In fact, the Aarseth-Zare program failed only at $v_0=10^{-13}$, which was expected since a 14-digit single-precision arithmetic was used.

As long as the initial velocity was larger than 10^{-11} , the total energy was preserved to one part in a thousand during the numerical integration. It is a characteristic phenomenon occurring in such problems that the computation of the energy constant may contain significant errors since it uses actual (nonregularized quantities) while the integration is performed using regularized variables. In other words, the orbit is computed with high accuracy in the regularized system, but various transformed printouts using physical (nonregularized) variables might be in error. This explains the spurious fact that the energy constant "recovers" after close approaches. The well-known dictum and principle of Ellis Strömberg regarding the accuracy of periodic orbits in the Copenhagen problem is applicable to our case, i.e., all orbits are computed with high enough accuracy to establish the family.

CONCLUDING REMARKS AND APPLICATIONS

This paper, the second of three, is dedicated to the numerical aspects of triple close approaches. All results are presented in nondimensional form to facilitate any applications. The nondimensional parameter is $GMT^2L^{-3}=1$ in the numerical work. Here, G is the constant of gravity and M , T , L are the units of mass, time, and length. The third paper of the trilogy will discuss the applications of this work to stellar dynamics. At this point an example might be mentioned concerning the case of $v_0=0.001$ which is treated in considerable detail in the text. For characteristic units we use $L=1$ pc and $M=M_\odot$ giving $T\simeq 1.5\times 10^7$ y, and $V=0.0655$ km/sec for the unit of time and unit of

velocity. At the first close approach $r_{13}^*\simeq 1.6\times 10^{-4}=32$ AU $= 6400R_\odot$, and the relative velocity is $v_{13}=160=10.5$ km/sec. The minimum moment of inertia is reached in $t_m\simeq 0.64129=9.6\times 10^6$ y from the beginning of the motion when the three Sun-like stars are 1 pc apart with initial velocities of the order of 0.1 m/sec. The binary formed has a semimajor axis $a_\infty\simeq 1.9\times 10^{-3}=380$ AU and eccentricity $e\simeq 0.5$ with periastron distance of $a_\infty(1-e)\simeq 186$ AU. The escape velocity is $v_\infty\simeq 20\simeq 1.3$ km/sec.

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