ROTOR DYNAMIC SYNCHRONOUS RESPONSE: A MORE THOROUGH TREATMENT

By:

R.N. Headifen

ASME International Gas Turbine and Aeroengine Congress and Exposition, Cincinnati, OH, May 24-27, 1993 (IGTI May 1993 turbo expo)

ASME No. 93-GT-28

PR - 175

Center for Electromechanics
The University of Texas at Austin
PRC, Mail Code R7000
Austin, TX 78712
(512) 471-4496

December 21, 1992

Rotor Dynamic Synchronous Response: A More Thorough Treatment

R.N. Headifen

The Center for Electromechanics
The University of Texas At Austin
Austin, Texas 78712

ABSTRACT

Synchronous response can be a very valuable tool for rotor dynamic analysis. It allows the user to determine rotor displacements over a wide speed range without having to perform a transient analysis at every speed increment. The method is typically used to calculate steady state displacements caused by a rotating imbalance force and the location of critical speeds. While the algorithm presented to perform linear synchronous response is straight forward, there are several modifications that can be made to provide the analyst more useful information.

SYMBOLS USED

C	the damping matrix which also can contain gyroscopic terms
K	the stiffness matrix
M	the mass matrix
Q	is a coordinate transformation matrix
F	is a force vector
c	bearing clearance
F_r	radial force
F_t	tangential force
u	is the displacement
mu	the imbalance mass multiplied by the geometric center offset from the rotor center of mass
ϵ	is the journal eccentricity in the bearing
φ	is an initial user defined angle relative to some arbitrary zero angle
ϕ	is the angle of the line of centers
θ	is the phase angle
μ	is the bearing fluid viscosity
ω	is the journal rotation speed

INTRODUCTION

The calculation of whirl orbit radius over the operating speed of a rotating machine is a critical component in its design. Depending on the complexity of the system design this can be performed in

various ways. Vance (1988) presents several analytical methods for the calculation of the whirl orbit radius due to imbalance forces for several simple lumped mass, stiffness and damping coefficients. To model more complex systems containing a rotor which has distributed mass and flexibility along with flexible supports requires a numerical solution. The two common methods employed are the Transfer Matrix Method (Vance 1988, Rao 1983) and the Finite Element Method (FEM) (Lalanne 1990, Nelson 1980, Rouch 1977).

The analysis of a system including nonlinear bearings requires further attention. Lund (1967) presented a method in which he used linearised coefficients to solve for synchronous response. He pointed out that this assumption is only valid for small eccentricity vibrations and since this is where most high speed rotor operate it is a justifiable assumption. Without this assumption, and since the bearing coefficients are displacement dependent, an iterative solution is required to calculate the final steady state displacement. McLean, (1983) discusses a method in which due to some assumptions on system symmetry they were able to partition the solution and iteratively solve for eccentricities at the damper stations. They presented results for a single damper system. Adams (1980) presents a time integration method to determine orbits due to unbalance.

For large horizontal rotors, the rotor weight can be a significant part of the load on the bearings. Therefore the orbit path of the rotor will not necessarily be about the bearing center line, but at some off center eccentricity. For linear bearings, the solution is the same as for a lighter rotor but with the off center position included. For nonlinear bearings, since the stiffness and damping coefficients are dependent on speed (spin and whirl) and eccentricity in the bearing, then even at constant speed, the coefficients will be varying within a single orbit. By separating the shaft motion into an initial zero imbalance motion and a whirling motion about the zero imbalance point, frequency response can be determined at any point in the orbit (Headifen 1991). This paper starts off covering conventional linear synchronous response for both lumped mass and distributed mass,

flexible rotor systems. The modifications for horizontal machines is shown. Following this, the modifications made to include nonlinear bearings for the same systems is presented.

CONVENTIONAL LINEAR ANALYSIS

The conventional analysis for synchronous response is derived from the following familiar equation (Rouch 1980).

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{C}\dot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{F}(t) \quad (1)$$

where

$$\mathbf{u} = \begin{Bmatrix} u_{x1} \cos(\omega t + \theta_1) \\ u_{y1} \sin(\omega t + \theta_1) \\ \vdots \\ u_{xn} \cos(\omega t + \theta_n) \\ u_{yn} \sin(\omega t + \theta_n) \end{Bmatrix} \quad (2)$$

and

θ_i = phase at degree of freedom i
 u_y = lagging u_x by one quarter revolution.

The $\mathbf{M}$, $\mathbf{C}$, and $\mathbf{K}$ matrices are either lumped parameter matrices or distributed mass and flexibility systems that are constructed by the FEM method.

Since the displacement is periodic for each revolution, then defining $\mathbf{U}$ as (Ruhl 1970)

$$\mathbf{U} = \begin{Bmatrix} U_1 e^{j\theta} \\ -jU_1 e^{j\theta} \\ \vdots \\ U_n e^{j\theta} \\ -jU_n e^{j\theta} \end{Bmatrix} \quad (3)$$

equation (2) can be rewritten as

$$\mathbf{u} = \text{Re}(\mathbf{U} e^{j\omega t}) \quad (4)$$

The imbalance force magnitude, F_0 is equal to $m\omega^2$. The force vector can be written as

$$\mathbf{F} = \begin{Bmatrix} F_{x1} \\ F_{y1} \\ \vdots \\ F_{xn} \\ F_{yn} \end{Bmatrix} = \begin{Bmatrix} F_{01} \cos(\omega t + \phi_1) \\ F_{01} \sin(\omega t + \phi_1) \\ \vdots \\ F_{0n} \cos(\omega t + \phi_n) \\ F_{0n} \sin(\omega t + \phi_n) \end{Bmatrix} \quad (5)$$

Which can be written as

$$\mathbf{F} = \text{Re} \left\{ \begin{Bmatrix} F_{01} e^{j\phi_1} \\ -jF_{01} e^{j\phi_1} \\ \vdots \\ F_{0n} e^{j\phi_n} \\ -jF_{0n} e^{j\phi_n} \end{Bmatrix} e^{j\omega t} \right\} \quad (6)$$

Substitution of (4) and (6) back into equation (1) and canceling out common terms, the equation can be rewritten as

$$[-\omega^2 \mathbf{M} + j\omega \mathbf{C} + \mathbf{K}] \mathbf{U} = \mathbf{F} \quad (7)$$

or

$$[\mathbf{H}(\omega)] \mathbf{U} = \mathbf{F} \quad (8)$$

This is a complex equation which requires a complex solver. Within the $\mathbf{U}$ vector, for a vertical X axis and horizontal Y axis, at each X DOF, the real part is the displacement in the X direction and the imaginary part is the displacement in the Y direction. The converse applies to a Y DOF. The phase angle for that node is the arc tangent of the imaginary component divided by the real component. For linear bearings whose coefficients are speed dependent, then the $\mathbf{K}$ and $\mathbf{C}$ matrices can be updated at every speed increment (Lalanne 1990). This requires of course, an independent calculation of the coefficients at every speed increment (Headifen 1991).

INCLUSION OF GRAVITY

For horizontal machines that have the weight of the rotor affecting the rotor position, then the gravity loads must be included in the force vector. This is particularly important for bearings that have a clearance with the rotor shaft such as magnetic bearings or fluid film bearings. The clearance at the bottom of the revolution can be significantly smaller than at the top. Inclusion of the gravity vector now means that a user supplied angle θ needs to be incorporated that defines the orientation of the gravity vector relative to the imbalance force vector. If θ is zero degrees, then the imbalance force and gravity loads will be added together and the displacement will be calculated at the bottom of the revolution. If θ is equal to 180° , then the gravity load will subtract from the imbalance force vector and the displacement calculated at the top of the revolution. In general;

$$\mathbf{F} = \text{Re} \left\{ \begin{Bmatrix} F_{01} e^{j\phi_1} \\ -jF_{01} e^{j\phi_1} \\ \vdots \\ F_{0n} e^{j\phi_n} \\ -jF_{0n} e^{j\phi_n} \end{Bmatrix} + \begin{Bmatrix} m_1 g \\ -jm_1 g \\ \vdots \\ m_n g \\ -jm_n g \end{Bmatrix} e^{j\theta} \right\} \quad (9)$$

NON LINEAR ANALYSIS

With nonlinear bearings, the stiffness and damping coefficients are dependent on the solution to the synchronous equation (8). Thus an iterative technique is required to approach to the final solution at each speed increment. Nonlinear analysis is separated into two parts. Lumped single mass such as a mass whirling in a bearing and the full machine analysis which includes a distributed masses, flexible shafts and stators supported by more than one bearing.

Lumped Mass Analysis

For this analysis, the equation of motion is

$$\mathbf{M}_B \ddot{\mathbf{U}} = \mathbf{F}(\dot{\epsilon}, M_B, \text{geom}, \mu, \dot{\phi}, \omega, U) \quad (10)$$

where

$\mathbf{F}$ = the vector summation of the bearing applied forces and the reaction forces that depend on the bearing geometry and fluid, whirl speed, shaft speed and position. (Tolk 1980)

$\mathbf{M}_B$ = a 2x2 mass matrix for the X and Y planes.

$\mathbf{U}$ = a vector containing the X and Y displacements

As in the linear case, the force balance for synchronous response has the unknown eccentricity and the phase angle in it. The phase angle results in the imbalance load being separated into vertical and horizontal components. Figure 1 shows the force components for a whirling shaft. Applied forces are due to gravity, imbalance, and inertia. The reaction forces are due to the oil film or magnetic pressure in the bearing. Prior to discussing the calculation of ϵ and θ further, the effect of a horizontal machine that includes gravity must be presented.

Figure 1. Dynamic force balance on whirling shaft ϵ and θ are variables that are iterated upon to balance forces in the two directions.

Effect of Gravity Loading. Inclusion of the gravity loading for a horizontal orientation machine into the equations adds a further complication. Now the shaft executes whirl about a point that is different from the center of the bearing. This point must be calculated before any other calculations are performed. The point is calculated as though the shaft was spinning with the user prescribed speed but with no imbalance thus no whirl. Under these conditions the shaft adopts a steady downwards eccentricity e , and a constant attitude angle f , in the direction of rotation. A two degree of freedom Newton-Raphson solver is used to calculate these two variables. The two equations are the forces in the vertical and horizontal direction. The applied forces are due to gravity only. At each eccentricity iteration, a bearing pressure iteration scheme is called to determine the bearing reaction forces. These are then compared to the applied forces and adjustments in ϵ and θ are made if necessary. The Newton-Raphson iteration equation is

$$\begin{Bmatrix} \frac{\partial F_x}{\partial \epsilon} & \frac{\partial F_x}{\partial \phi} \\ \frac{\partial F_y}{\partial \epsilon} & \frac{\partial F_y}{\partial \phi} \end{Bmatrix} \begin{Bmatrix} \delta \epsilon \\ \delta \phi \end{Bmatrix} = \begin{Bmatrix} F_x \\ F_y \end{Bmatrix} \quad (11)$$

where

$$F_x = FB_x - mg \cos \phi$$

$$F_y = FB_y - mg \sin \phi$$

$$FB = \text{the bearing reaction force}$$

The $\partial F / \partial \epsilon$ and the $\partial F / \partial \phi$ terms are evaluated by the small displacement method.

If gravity is not included then both ϵ and ϕ equal zero. If gravity is included but the bearing journal is pinned against rotation but free to whirl such as in a damper application, then ϕ is equal to zero but ϵ is not.

Calculation of Eccentricity and Phase Angle. Once the zero imbalance position is calculated, the Newton-Raphson iteration scheme is called again to iterate on the horizontal and vertical force balance for a full whirling shaft. The solution calculated is for a steady state synchronous whirling position hence forces due to the time rate of change of ϵ are zero. The iteration variables are the eccentricity and the phase angle θ . As above, at each eccentricity iteration the bearing pressure iteration scheme is called to determine the reaction forces at that position. The Newton-Raphson iteration equation is

$$\begin{Bmatrix} \frac{\partial F_x}{\partial \epsilon} & \frac{\partial F_x}{\partial \theta} \\ \frac{\partial F_y}{\partial \epsilon} & \frac{\partial F_y}{\partial \theta} \end{Bmatrix} \begin{Bmatrix} \delta \epsilon \\ \delta \theta \end{Bmatrix} = \begin{Bmatrix} F_x \\ F_y \end{Bmatrix} \quad (12)$$

where

$$F_x = FB_x - mg \cos \phi - m c \epsilon \phi^2 - \mu \omega^2 \cos \theta$$

$$F_y = \mu \omega^2 \sin \theta - FB_y - mg \sin \phi$$

$$\mu = \text{user supplied imbalance}$$

The partial differential terms are calculated using the small displacement method.

Since the shaft is executing steady orbits around a position different from the bearing center an explanation of where the calculated eccentricities and phase angles are is necessary. All the calculations for the bearing are performed for motion about the bearing center. If there is no initial attitude angle then at the bottom of the orbit, the shaft is whirling along a tangent perpendicular to the vertical direction. If there is an initial attitude angle ϕ , then there are only two positions per revolution where the motion is purely tangential with respect to the bearing center. These two points occur when a line at the angle ϕ to the downward vertical passing through the bearing center and the initial zero imbalance position cuts the orbit path (fig. 2). With respect to the bearing center, where the line cuts the top of the orbit path, the eccentricity is at the minimum position per revolution and where it cuts the orbit path at the bottom of the orbit is the maximum eccentricity position per revolution. When at the maximum eccentricity position, the whirl angle is the same as the initial attitude angle. Hence if equation (12) is solved with gravity included in it, then the position will be at the maximum eccentricity position inclined at an angle ϕ . The calculated phase angle in this case is the angle relative to the downward vertical axis. The real phase angle is equal to $\theta - \phi$. If there is no zero imbalance attitude angle, then the position calculated from equation (12) is the vertical downward position. In this situation the calculated phase angle θ , is the real phase angle.

Figure 2. Position of maximum and minimum eccentricity with respect to bearing center

The whirling velocity with respect to the bearing center is also affected by the gravity offset. In the offset position, the shaft is whirling around the offset position (not the bearing center), at a whirl rate equal to the shaft speed. Its absolute tangential velocity is given by the shaft speed times the radius of that orbit. The radius of this orbit is equal to the maximum eccentricity above less the initial offset eccentricity all multiplied by the clearance.

$$v_t = \omega c (\epsilon_{\max} - \epsilon_0) \quad (13)$$

The whirling speed at the maximum eccentricity position is then given by

$$\dot{\phi} = \frac{v_t}{c\epsilon} = \frac{\omega(\epsilon_{\max} - \epsilon_0)}{\epsilon} \quad (14)$$

Since the eccentricity is a function of the orbit position, then the whirling speed relative to the bearing center will vary around the orbit. It will be a maximum at the minimum eccentricity position and a minimum at the maximum eccentricity position.

Distributed Mass Model Frequency Response

The distributed mass model models the entire machine. It includes the shaft flexibility and stator flexibility if required. It can include linear springs and dampers such as linear bearings or machine mounts and also nonlinear bearings.

For both linear and nonlinear bearings, the bearing stiffness reaction forces are equal to the area under the stiffness versus displacement curve. For linear bearings this is simply written as

$$F_x = K_{xx}X + K_{xy}Y \quad (15)$$

but for nonlinear bearings, the force is equal to

$$F_x = \int K_{xx}dX + \int K_{xy}dY \quad (16)$$

Hence, it is not correct to add the bearing coefficients into the FEM constructed matrices on the LHS of equation (1), but rather the bearing reaction forces must be added to the force vector on the RHS similar to the lumped mass analysis above. A nonlinear solution technique is then required. Equation (8) becomes a non-linear complex equation. Let

$$G_i(U) = HU_0 - F = 0 \quad i = 1, 2, \dots, N \quad (17)$$

then, using a Newton-Raphson iteration scheme (Press 1986) on equation (17) an equation which solves for the increments in the solution can be derived (Headifen 1991).

$$\left(\sum_{j=1}^N \frac{\partial G_i}{\partial U_j} \right) \delta U = -G_i(U) \quad (18)$$

The corrections are then added to the solution vector,

$$U_i^{\text{new}} = U_i^{\text{old}} + \delta U_i \quad i = 1, 2, \dots, N \quad (19)$$

The summation term on the left hand side of (18) is called the tangent stiffness matrix H^T . It is an $N \times N$ matrix where N is the number of DOF in model. Expanding this term gives

$$\sum_{j=1}^N \frac{\partial G_i}{\partial U_j} = \sum_{j=1}^N \frac{\partial (H U - F)}{\partial U_j} = H + \sum_{j=1}^N \frac{\partial F_i}{\partial U_j} \quad (20)$$

Hence the tangent stiffness matrix H^T is set equal to the H matrix then any terms in the F vector that have a U dependence are differentiated with respect to U and added in at the respective DOF. The only terms in the definition of F that are dependent on U are the bearing reaction terms. Therefore at a node that has a bearing on it, then

$$\frac{\partial F_x}{\partial U_x} = k_{xx}, \frac{\partial F_x}{\partial U_y} = k_{xy}, \frac{\partial F_y}{\partial U_y} = k_{yy}, \frac{\partial F_y}{\partial U_x} = k_{yx} \quad (21)$$

For all other DOF

$$\frac{\partial F_i}{\partial U_j} = 0 \quad i, j = x, y \quad (22)$$

These are the local bearing stiffness coefficients (relative to a reference frame that rotates with the rotor). The stiffness coefficients are added in to the real components of the bearing DOF.

Similarly, due to the substitution made in (2), the terms in the definition of F that are dependent on $\delta U/\delta t$ also need to be included. These are the reaction force damping terms. Therefore at a node that has a bearing on it, then

$$\frac{\partial F_x}{\partial \dot{U}_x} = C_{xx}, \frac{\partial F_x}{\partial \dot{U}_y} = C_{xy}, \frac{\partial F_y}{\partial \dot{U}_y} = C_{yy}, \frac{\partial F_y}{\partial \dot{U}_x} = C_{yx} \quad (23)$$

For all other DOF

$$\frac{\partial F_i}{\partial \dot{U}_j} = 0 \quad i, j = x, y \quad (24)$$

Since equation (8) is complex, with the terms in the C matrix being the imaginary components, then the damping coefficients are added to the imaginary components of the bearing DOF. They are first multiplied by the speed w according to equation (7).

Hence H^T is set equal to the H matrix, then any terms in the F vector that have a U or a dU/dt dependence are differentiated with respect to U or dU/dt and added in.

$$H^T = H + \sum_{j=1}^N \frac{\partial F_i}{\partial K U_j} = j\omega \sum_{j=1}^N \frac{\partial F_i}{\partial K U_j} = H + K_{nm} + j\omega C_{nm} \quad n, m = x, y \quad (25)$$

Thus for every node in the model that has a bearing on it, the stiffness and damping coefficients are added to their respective degrees of freedom in the tangent stiffness matrix. Equation (17) is then solved for the increments in the solution for the displacement. Then the tangent stiffness matrix and the RHS of (17) are reformed with the updated coefficients and equation (17) is resolved. This process is continued until either the increments in the displacements is less than a specified convergence or the RHS of (17) is less than a specified convergence.

Reference Frame Transformation. Care must be taken to ensure that the bearing coefficients and the model have the same reference frame. Normally the bearing coefficients are expressed in a rotating reference frame that rotates with the shaft. If the model uses a stationary global reference frame (for example X positive down and Y positive to the right) then the transformation from the rotating frame to the stationary frame is given by

$$Q U_G = U_R \quad (26)$$

where

$$Q = \begin{bmatrix} \cos \phi & \sin \phi \\ -\sin \phi & \cos \phi \end{bmatrix} \quad (27)$$

and

ϕ = the angle between the two frames.

The relationship between the coefficients in the stationary frame and the rotating frame is given by (Headifen 1991)

$$K_G = Q^T K_R Q \quad (28)$$

This transformation has to be performed prior to forming the tangent stiffness matrix.

CONCLUSION

A more in depth discussion of rotor dynamic frequency response has been presented. A discussion of including the gravity load for a horizontal rotor was shown. This is important for rotors that use clearance bearings. At the bottom of the orbit, the eccentricity can be significantly greater than at the top. If this is not accounted for, a bearing rub could be possible where one was not predicted.

For nonlinear bearings, a method was shown how to include their coefficients into the displacement calculation. This is an iterative method as the bearing coefficients depend on the displacement in the bearing. The inclusion of gravity loads for these systems is more complicated. A zero imbalance position must be solved for first because, for the steady state whirl, the whirl is centered about this zero imbalance position.

A procedure for the transformation of the coefficients from their local rotating system to the global system was given so that synchronous response at any angle in the orbit could be given.

REFERENCES

- Adams M., "Non-Linear Dynamics of Flexible Multi-Bearing Rotors," *J. Sound and Vibration* (1980) 71(1), 129-144.
- Headifen R.N., 1991, "Rotor Dynamic Modeling of Flexible Rotor Machines Supported with Hydrostatic Bearings," Ph.D. Dissertation, The University of Texas at Austin, Austin, TX.
- Lalanne M., Ferraris G., *Rotordynamics Prediction in Engineering*, John Wiley & Sons, New York, New York 1990.
- Lund J., Orcutt F., "Calculations and Experiments on the Unbalance Response of a Flexible Rotor," *J. Eng for Power*, Nov 1967, pp 785-796.
- McLean L., Hahn, E., "Unbalance Behavior of Squeeze Film Multi-Mass Flexible Rotor Bearing Systems," *J. Lubrication Technology*, vol. 105, pp 22-28, Jan. 1983.
- H. Nelson, "The Dynamics of Rotor-Bearing Systems Using Finite Elements," *Journal of Mechanical Design*, October 1980, vol. 102, pp 793-799.
- Press W, Flannery B, *Numerical Recipes, The Art of Scientific Computing*, Cambridge University Press, Cambridge, 1986
- Rao J.S., *Rotor Dynamics*, John Wiley & Sons, New York, New York 1983
- Rouch K.E., Kao J.S., 1980, "Dynamic Reduction in Rotor Dynamics by the Finite Element Method," *Journal of Mechanical Design*, vol 102, April, pp 360-368
- Rouch K.E., "Finite-Element Analysis of Rotor-Bearing Systems with Matrix Reduction," Ph.D Dissertation, Marquette University, 1977.
- Ruhl R. L., 1970, "Dynamics of Distributed Parameter Rotor Systems: Transfer Matrix and Finite Element Techniques," Ph.D Dissertation, Cornell University, Cornell, NY.
- Tolk K. E., 1980, "A Modeling Technique for Design and Simulation of Hydrostatic Journal Bearings," Ph.D Dissertation, The University of Texas at Austin, Austin, TX.
- Vance J., *Rotordynamics of Turbomachinery*, Texas A&M University, 1988 John Wiley & Sons, New York, New York.